ME 6590 Multibody Dynamics
Principal Moments of Inertia and Principal Directions

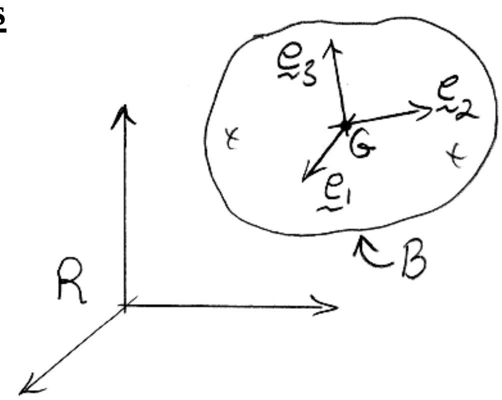
Properties of Real Symmetric Matrices

(Reference: Richard Bronson, *Matrix Methods: An Introduction*, Academic Press, 1970.)

All ***real symmetric matrices*** have the following ***properties***:

- The ***eigenvalues*** of a real symmetric matrix are ***real***.
- The ***eigenvectors*** of a real symmetric matrix can always be chosen to be ***real***.
- A real symmetric matrix is ***diagonalizable***.
- Every real symmetric matrix possesses a ***complete set of orthonormal eigenvectors***.
- For every real symmetric $n \times n$ matrix $[A]$, there exists an $n \times n$ ***real orthogonal matrix*** $[M]$ such that $[M]^T[A][M]=[D]$, where $[D]$ is a diagonal matrix.
- $[M]$ is called the ***modal matrix***. Its columns are formed by the ***eigenvectors*** of $[A]$.
- $[D]$ is a diagonal matrix whose entries are the ***eigenvalues*** of $[A]$.
- ***Eigenvalues*** appear in the ***same columns*** of $[D]$ as their ***associated eigenvectors*** appear in $[M]$.

Principal Moments of Inertia and Principal Directions



- The ***inertia dyadic*** of the body for a set of axes passing through its mass-center and parallel to the unit vector set $B:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ is defined as

$$\boxed{\underset{\approx}{I}'_G = \sum_{i=1}^{3}\sum_{j=1}^{3} I'_{ij}\underset{\sim}{e}_i\underset{\sim}{e}_j}$$

- The components of the dyadic form a ***real symmetric*** 3×3 ***matrix*** $\left[I'_G\right]$. Consequently, this matrix has all the properties listed above.
- The eigenvectors of $\left[I'_G\right]$ define the ***principal directions*** of the body for the mass-center G. Recall that the principal directions are those directions for which ***all products of inertia*** are ***zero***.
- The ***eigenvalues*** of $\left[I'_G\right]$ are the ***principal moments of inertia*** of the body for G, that is, they are the moments of inertia about the principal directions.
- Note that the ***principal directions*** and inertia ***vary*** from point to point in the body, but ***there exists only one set for any given point***.

Calculation of Principal Moments of Inertia and Principal Directions

- The principal moments of inertia of a body for a given point, say the mass-center *G*, may be calculated (as you would to find the eigenvalues of any 3×3 matrix) by setting

$$\det\begin{bmatrix}(I'_{xx}-\lambda) & -I'_{xy} & -I'_{xz}\\ -I'_{xy} & (I'_{yy}-\lambda) & -I'_{yz}\\ -I'_{xz} & -I'_{yz} & (I'_{zz}-\lambda)\end{bmatrix}=0\,. \tag{1}$$

  By expanding the determinant, the resulting characteristic equation can be written as

$$\begin{aligned}&\lambda^3+(-I'_{xx}-I'_{yy}-I'_{zz})\lambda^2+(I'_{xx}I'_{yy}+I'_{xx}I'_{zz}+I'_{yy}I'_{zz}-I'^2_{xy}-I'^2_{xz}-I'^2_{yz})\lambda\\ &\quad+(-I'_{xx}I'_{yy}I'_{zz}+I'_{xx}I'^2_{yz}+I'_{yy}I'^2_{xz}+I'_{zz}I'^2_{xy}+2I'_{xy}I'_{xz}I'_{yz})=0\end{aligned} \tag{2}$$

  The ***three roots*** to this equation are the ***three principal moments of inertia***.

- Letting $I_i\ (i=1,2,3)$ represent the three principal moments of inertia, we can find the principal directions (as you would find the eigenvectors of any 3×3 matrix) by setting

$$\begin{bmatrix}(I'_{xx}-I_i) & -I'_{xy} & -I'_{xz}\\ -I'_{xy} & (I'_{yy}-I_i) & -I'_{yz}\\ -I'_{xz} & -I'_{yz} & (I'_{zz}-I_i)\end{bmatrix}\begin{Bmatrix}a_{i1}\\ a_{i2}\\ a_{i3}\end{Bmatrix}=\{0\} \tag{3}$$

- Since the coefficient matrix is singular, these equations do not have a single solution. We can find an eigenvector to within a constant multiplier only.

- However, if we impose the further condition that $\boxed{a_{i1}^2+a_{i2}^2+a_{i3}^2=1}$, then the eigenvector is unique. The components of the eigenvector are then the components of a unit vector pointing in the principal direction. Recall that the components of a unit vector are the direction cosines for that direction.

- To solve Eq. (3), simply choose a value for one of the $a_{ij}\ (j=1,2,3)$, and then solve for the other two. Then, ***normalize*** the resulting vector.