The purpose of this document is to have a good mathematical view on deriving the jacobian used for bundle adjustment.

1 Some formulas

$$\pi(X,Y,Z) = \begin{pmatrix} \frac{f_x.X}{Z} + c_x \\ \frac{f_y.Y}{Z} + c_y \end{pmatrix}$$

$$\pi^{-1}(x,y,d) = \begin{pmatrix} \frac{x-c_x}{f_x} d \\ \frac{y-c_y}{f_y} d \\ d \end{pmatrix}$$

$$w(X,Y,Z) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

2 Problem definition

We have k cameras in Ω_k and i points in Ω_i. Each image has a position in space defined by R and t(rotation and trasnlation, $SE3$), each point has a depth,d, and x, y coords in the image in which it was first seen. We want to minimize the reprojection error of the points on the K images.

$$E = \sum_{k\in\Omega_k} \sum_{i\in\Omega_i} I_1(x_i, y_i) - I_2\Bigg(\pi(w\big[G(\boldsymbol{\xi}), \pi^{-1}(x_i, y_i, \mathbf{d})\big])\Bigg)$$

The bold ones are the variables.

So the Jacobian for two images and one point looks like this:

$$J_E = J_{I_2} J_\pi J_w J_{\pi^{-1}}$$

Lets start from back to forth.

Note: the first column and first row are respectively the functions and the variables we derive on. This is written as the jacobians are quite involved and it's easier to understand them like this.

$$J_{\pi^{-1}} = \frac{\partial \pi^{-1}(x,y,\mathbf{d})}{\partial v, w, d}\Bigg|_{x=x1,y=y1,d=d} = \begin{pmatrix} & v_1 & v_2 & v_3 & w_1 & w_2 & w_3 & d \\ X_1 & 0 & 0 & 0 & 0 & 0 & 0 & (x_1 - c_x)/f_x \\ Y_1 & 0 & 0 & 0 & 0 & 0 & 0 & (y_1 - c_y)/f_y \\ Z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$J_G = \frac{\partial G(\boldsymbol{\xi})}{\partial v, w, d}\bigg|_{\xi=\xi} = \begin{array}{cc} & \begin{array}{ccccccc} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 & d \end{array} \\ \begin{array}{c} r_{11} \\ r_{21} \\ r_{31} \\ r_{12} \\ r_{22} \\ r_{32} \\ r_{13} \\ r_{23} \\ r_{33} \\ t_x \\ t_y \\ t_z \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & r_{31} & -r_{21} & 0 \\ 0 & 0 & 0 & -r_{31} & 0 & r_{11} & 0 \\ 0 & 0 & 0 & r_{21} & -r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{32} & -r_{22} & 0 \\ 0 & 0 & 0 & -r_{32} & 0 & r_{12} & 0 \\ 0 & 0 & 0 & r_{22} & -r_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{33} & -r_{23} & 0 \\ 0 & 0 & 0 & -r_{33} & 0 & r_{13} & 0 \\ 0 & 0 & 0 & r_{23} & -r_{13} & 0 & 0 \\ 1 & 0 & 0 & 0 & t_z & -t_y & 0 \\ 0 & 1 & 0 & -t_z & 0 & t_x & 0 \\ 0 & 0 & 1 & t_y & t_x & 0 & 0 \end{pmatrix} \end{array}$$

$$J_w = \frac{\partial w(G, p)}{\partial G, p}\bigg|_{G=G(\xi), p=p} =$$

$$\begin{array}{cc} & \begin{array}{ccccccccccccccc} r_{11} & r_{21} & r_{31} & r_{12} & r_{22} & r_{33} & r_{13} & r_{23} & r_{33} & t_x & t_z & t_y & X_1 & Y_1 & Z_1 \end{array} \\ \begin{array}{c} X_2 \\ Y_2 \\ Z_2 \end{array} & \begin{pmatrix} X_1 & 0 & 0 & Y_1 & 0 & 0 & Z_1 & 0 & 0 & 1 & 0 & 0 & r_{11} & r_{12} & r_{13} \\ 0 & X_1 & 0 & 0 & Y_1 & 0 & 0 & Z_1 & 0 & 0 & 1 & 0 & r_{21} & r_{22} & r_{23} \\ 0 & 0 & X_1 & 0 & 0 & Y_1 & 0 & 0 & Z_1 & 0 & 0 & 1 & r_{31} & r_{32} & r_{33} \end{pmatrix} \end{array}$$

$$J\pi(X_2, Y_2, Z_2) = \begin{array}{cc} & \begin{array}{ccc} X_2 & Y_2 & Z_2 \end{array} \\ \begin{array}{c} x_2 \\ y_2 \end{array} & \begin{pmatrix} \frac{f_x}{Z_2} & 0 & -\frac{f_x.X_2}{Z_2^2} \\ 0 & \frac{f_Y}{Z_2} & -\frac{f_y.Y_2}{Z_2^2} \end{pmatrix} \end{array}$$

Lets multiply them back to forth again

$$J_w.J_{G\pi^{-1}} = \begin{array}{cc} & \begin{matrix} r_{11} & r_{21} & r_{31} & r_{12} & r_{22} & r_{33} & r_{13} & r_{23} & r_{33} & t_x & t_z & t_y & X_1 & Y_1 & Z_1 \end{matrix} \\ \begin{matrix} X_2 \\ Y_2 \\ Z_2 \end{matrix} & \begin{pmatrix} X_1 & 0 & 0 & Y_1 & 0 & 0 & Z_1 & 0 & 0 & 1 & 0 & 0 & r_{11} & r_{12} & r_{13} \\ 0 & X_1 & 0 & 0 & Y_1 & 0 & 0 & Z_1 & 0 & 0 & 1 & 0 & r_{21} & r_{22} & r_{23} \\ 0 & 0 & X_1 & 0 & 0 & Y_1 & 0 & 0 & Z_1 & 0 & 0 & 1 & r_{31} & r_{32} & r_{33} \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{matrix} v_1 & v_2 & v_3 & w_1 & w_2 & w_3 & d \end{matrix} \\ \begin{matrix} r_{11} \\ r_{21} \\ r_{31} \\ r_{12} \\ r_{22} \\ r_{32} \\ r_{13} \\ r_{23} \\ r_{33} \\ t_x \\ t_y \\ t_z \\ X_1 \\ Y_1 \\ Z_1 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & r_{31} & -r_{21} & 0 \\ 0 & 0 & 0 & -r_{31} & 0 & r_{11} & 0 \\ 0 & 0 & 0 & r_{21} & -r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{32} & -r_{22} & 0 \\ 0 & 0 & 0 & -r_{32} & 0 & r_{12} & 0 \\ 0 & 0 & 0 & r_{22} & -r_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{33} & -r_{23} & 0 \\ 0 & 0 & 0 & -r_{33} & 0 & r_{13} & 0 \\ 0 & 0 & 0 & r_{23} & -r_{13} & 0 & 0 \\ 1 & 0 & 0 & 0 & t_z & -t_y & 0 \\ 0 & 1 & 0 & -t_z & 0 & t_x & 0 \\ 0 & 0 & 1 & t_y & t_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (x_1 - c_x)/f_x \\ 0 & 0 & 0 & 0 & 0 & 0 & (y_1 - c_y)/f_y \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & Z_2 & -Y_2 & q_1 \\ 0 & 1 & 0 & -Z_2 & 0 & X_2 & q_2 \\ 0 & 0 & 1 & Y_2 & -X_2 & 0 & q_3 \end{pmatrix} = J_{wG\pi^{-1}}$$

Here X_2 is unprojected, scaled to depth, roteted and translated x_1 and q is unprojected and rotated only, no translation. Or whatever.. this is a very bad explanation what it is. I'll write a better one soon.

Now we continue with

$$J_{I2}.J_\pi.J_{wG\pi^{-1}} = \begin{pmatrix} \nabla Ix & \nabla Iy \end{pmatrix} \begin{pmatrix} \frac{f_x}{Z_2} & 0 & -\frac{f_x.X_2}{Z_2^2} \\ 0 & \frac{f_y}{Z_2} & -\frac{f_y.Y_2}{Z_2^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & Z_2 & -Y_2 & q_1 \\ 0 & 1 & 0 & -Z_2 & 0 & X_2 & q_2 \\ 0 & 0 & 1 & Y_2 & -X_2 & 0 & q_3 \end{pmatrix} =$$

$$= \begin{pmatrix} fx\nabla Ix & fy\nabla Iy \end{pmatrix} \begin{pmatrix} \frac{1}{Z_2} & 0 & -\frac{X_2}{Z_2^2} \\ 0 & \frac{1}{Z_2} & -\frac{Y_2}{Z_2^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & Z_2 & -Y_2 & q_1 \\ 0 & 1 & 0 & -Z_2 & 0 & X_2 & q_2 \\ 0 & 0 & 1 & Y_2 & -X_2 & 0 & q_3 \end{pmatrix}$$

$$= \begin{pmatrix} fx\nabla Ix & fy\nabla Iy \end{pmatrix} \begin{pmatrix} \frac{1}{Z_2} & 0 & -\frac{X_2}{Z_2^2} & -\frac{X_2 Y_2}{Z_2^2} & 1 + \frac{X_2^2}{Z_2^2} & -\frac{Y_2}{Z_2} & \frac{q_1}{Z_2} - \frac{q_3 X_2}{Z_2^2} \\ 0 & \frac{1}{Z_2} & -\frac{Y_2}{Z_2^2} & -1 - \frac{Y_2^2}{Z_2^2} & \frac{X_2 Y_2}{Z_2^2} & \frac{X_2}{Z_2} & \frac{q_2}{Z_2} - \frac{v_3 Y_2}{Z_2^2} \end{pmatrix}$$

$$= \begin{pmatrix} fx\nabla Ix(\frac{1}{Z_2}) \\ fy\nabla Iy(\frac{1}{Z_2}) \\ -(fx\nabla IxX_2 + f_y\nabla IyY_2)\frac{1}{Z_2^2} \\ fx\nabla Ix(-\frac{X_2Y_2}{Z_2^2}) + fy\nabla Iy(-1 - \frac{Y_2^2}{Z_2^2}) \\ fx\nabla Ix(1 + \frac{X_2^2}{Z_2^2}) + fy\nabla Iy(\frac{X_2Y_2}{Z_2^2}) \\ fx\nabla Ix(-\frac{Y_2}{Z_2}) + fy\nabla Iy(\frac{X_2}{Z_2}) \\ fx\nabla Ix(\frac{q_1}{Z_2} - \frac{q_3X_2}{Z_2^2}) + fy\nabla Iy(\frac{q_2}{Z_2} - \frac{q_3Y_2}{Z_2^2}) \end{pmatrix}$$

Now lets calculate the Hessian. I've never calcucated a Hessian before, so if this sentance is still here the hessian calculation has not been tested in code. Should happen untill 15 April 2018.

The first column is all the derivatives of $fx\nabla Ix(\frac{1}{Z_2})$

$$\frac{\delta fx\nabla Ix(\frac{1}{Z_2})}{\delta v_1} = \frac{\delta fx\nabla Ix(\frac{1}{r_{31}X_1+r_{32}Y_1+r_{33}Z_1+t_z})}{\delta v_1}$$