Probability filters as a model of belief

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Abstract

We propose a new model of uncertain belief where we model coherent beliefs by a filter, $\mathcal{F}$, on the set of probabilities. That is, it is given by a collection of sets of probabilities which are closed under supersets and finite intersections.

This can naturally model one's probabilistic opinions. When one thinks that rain is more likely than not, we will ensure that $\{p \mid p(\text{Rain}) > p(\text{No Rain})\} \in \mathcal{F}$. When one evaluates that a gamble, g, is desirable we have $\{p \mid \text{Exp}_p[g] > 0\} \in \mathcal{F}$.

This model captures all the expressive power of framework of desirable gambles. This involves some aspects of non-Archimadeanicity by permitting the filters to be non-principal, i.e., not generated by a set of probabilities. It can also capture all the expressive power of the model of choice functions, or sets of desirable gamble sets (with a mixing axiom, but no Archimadean axiom).

1 Introduction

We propose a new model of belief based on probabilities which is very expressively powerful. This models coherent beliefs by a filter, $\mathcal{F}$, on the set of probabilities. That is, it is given by a collection of sets of probabilities which are closed under supersets and finite intersections.

This directly captures one's opinions. To capture one's strict comparative judgements, such as thinking that it is more likely to rain than not, we consider whether

$$\{p \mid p(\text{Rain}) > p(\text{No Rain})\} \in \mathcal{F}.$$

To capture a judgement that a gamble, g, is desirable we put

$$\{p \mid \text{Exp}_p[g] > 0\} \in \mathcal{F}.$$

This model captures all the expressive power of framework of desirable gambles. This involves some aspects of non-Archimadeanicity by permitting the filters to be non-principal, i.e., not generated by a set of probabilities. It can also capture all the expressive power of the model of choice functions, or sets of desirable gamble sets (with a mixing axiom, but no Archimadean axiom).

Using other terms, this model was proposed and discussed in a joint paper with Jason Konek, (Campbell-Moore and Konek, 2019), using the interpretation

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of believing probabilistic contents as outlined in Moss (2018). The results of this paper were stated there without proof.

We introduce the model of probability filters in section 2. We observe how they extend the model of sets of probabilities (section 2.2) and show when and how to extend a given set of endorsements of sets of probabilities to a probability filter (section 2.4). In section 3 we show that the model of probability filters captures all the expressive power of framework of desirable gambles and in section 4 we show it captures all the expressive power of the framework of sets of desirable gamble sets or choice functions. Section 5 notes a case where it goes beyond the framework of choice functions. Full investigation of its expressive power remains future work.

2 Probability Filters

Fix Ω as a finite non-empty set. Our results are all within the context of finite sample spaces. Extending these results is future work.

Setup 2.1. A *probability function* $p : \wp(\Omega) \to \mathbb{R}$ is characterised by a probability mass function, $p : \Omega \to \mathbb{R}$, with $p(\omega) \geqslant 0$ and $\sum_{\omega\in\Omega} p(\omega) = 1$. p is *regular* iff $p(\omega) > 0$ for every $\omega \in \Omega$.

rProbs is the collection of all regular probability functions on Ω.

We model your belief, $\mathcal{F}$, by a set of sets of probabilities. In fact we will restrict to *regular* probabilities, so $\mathcal{F} \subseteq \wp(\text{rProbs})$. We say you *endorse* a set of probabilities P iff $P \in \mathcal{F}$.

For a description of a probability property, such as $\text{pr}(E) > 0.2$, we will use the notation $[\![\text{pr}(E) > 0.2]\!]$ for $\{p \mid p(E) > 0.2\}$.

We define what it is to be coherent as being a proper filter. That is, it is a collection of sets of probabilities closed under supersets (F4) and finite intersections (F3), and that is non-trivial: it must be non-empty (F2) and not contain $\varnothing$, so that it is not identical to $\wp(\text{rProbs})$ (F1).

Definition 2.2. Say $\mathcal{F} \subseteq \wp(\text{rProbs})$ is *coherent* iff it is a proper filter; that is:

F1. $\varnothing \notin \mathcal{F}$.

F2. $\mathcal{F} \neq \varnothing$.

F3. If $P, Q \in \mathcal{F}$ then $P \cap Q \in \mathcal{F}$.

F4. If $P \in \mathcal{F}$ and $Q \supseteq P$ then $Q \in \mathcal{F}$.

2.1 Regularity

Note that we have essentially encoded an axiom in the setup by only considering filters on *regular* probabilities. We have decided to adopt this to match the assumptions used in the desirable gambles setting that weakly dominating 0 is sufficient to be strictly desirable.[1] It would be more explicit to simply have $\mathcal{F}$ a collection of sets of possibly non-regular probabilities and then have an axiom:

[1]It also aids proofs as we obtain closed sets. We conjecture that all the results will carry through with appropriate modifications in the setting without regularity.

- For each $\omega \in \Omega$, $[\![\mathrm{pr}(\omega) > 0]\!] \in \mathcal{F}$

However this makes the notion of natural extension more complex and it is easier to simply impose it as part of the setup. This restriction to regular probabilities is not as plausible as the other axioms, but we adopt it for this paper for simplicity. Throughout the paper, we will always restrict attention to regular probabilities.

2.2 Principal filters and sets of probabilities

Special kinds of filters are those which are *principal*; that is, there is some set of probabilities, $\mathbb{P} \subseteq \mathrm{rProbs}$, with

$$P \in \mathcal{F} \text{ iff } P \supseteq \mathbb{P} \tag{1}$$

Principal filters are equivalent to the model of belief given by arbitrary sets of probabilities (restricted to regular probability measures). The restriction to principal filters, and thus the model of belief of arbitrary sets of probabilities, is given by strengthening the axiom of finite intersections (axiom F3) to arbitrary intersections:

$F_{\mathrm{inf}\cap}$. If $P_i \in \mathcal{F}$ for each $i \in I$ then $\bigcap_{i \in I} P_i \in \mathcal{F}$.
(Where I can be infinite.)

We will not adopt this strengthened axiom. Non-principal filters are important to capture the expressive power of the desirable gambles framework as it allows for non-Archimadean behaviour when we consider free filters, where $\bigcap \mathcal{F} = \varnothing$.

2.3 Totality / Ultrafilters

Another interesting class of filters are *ultrafilters*. These are filters that cannot coherently be enlarged. These filters are obtained by adding the axiom of totality:

F_{tot}. $P \in \mathcal{F}$ or $\overline{P} \in \mathcal{F}$, where $\overline{P} := \mathrm{rProbs} \setminus P$.

This says that you either endorse something or its complement.

Principal ultrafilters give us precise probabilities. Non-principal ultrafilters roughly give us hyperreal probabilities.[2]

2.4 Natural Extensions

If we obtain some endorsements of sets of probabilities but not a complete description of the agents judgements, in what cases that this be extended coherently?

We define what the resultant extension will be, if possible. This just closes a set under finite intersections and supersets.

Definition 2.3. Let $\mathcal{E} \subseteq \wp(\mathrm{rProbs})$. $\mathrm{ext}(\mathcal{E})$, is defined by $Q \in \mathrm{ext}(\mathcal{E})$ iff there are some $P_1, \ldots, P_n \in \mathcal{E}$ and $Q \supseteq P_1 \cap \ldots \cap P_n$.

This is coherent whenever $\mathcal{E}$ satisfies the finite intersection property: whenever $P_1, \ldots, P_n \in \mathcal{E}$, $P_1 \cap \ldots \cap P_n \neq \varnothing$. Such $\mathcal{E}$ are called filter subbases, and $\mathrm{ext}(\mathcal{E})$ is the filter generated by it.

[2] A formal equivalence to hyperreal probabilities still remains to be checked.

Proposition 2.4. *There is a filter $\mathcal{F} \supseteq \mathcal{E}$ iff $\varnothing \notin \mathrm{ext}(\mathcal{E})$; and the minimal such filter is* $\mathrm{ext}(\mathcal{E})$.

Another useful case is when $\mathcal{E}$ is already closed under finite intersections, then $\mathrm{ext}(\mathcal{E})$ just takes supersets. For example, if we have a descending chain $P_1 \supseteq P_2 \supseteq P_3 \supseteq \ldots$, then $Q \in \mathrm{ext}(\{P_1, P_2, \ldots\})$ iff $Q \supseteq P_i$ for some i.

3 Probability filters and desirable gambles

One of the most prominent models of belief in the imprecise probability literature is to model one's belief by a set of desirable gambles (Walley, 2000). The probability filters model contains all the representational power of such sets of desirable gambles.

Setup 3.1. $\mathcal{G}$ is the set of all gambles, which are the bounded functions from Ω to $\mathbb{R}$.[3]

$\mathcal{G}_{\gtrsim 0}$ is the set of gambles which weakly dominate 0. I.e. all $g \in \mathcal{G}$ with $g(\omega) \geqslant 0$ for all $\omega \in \Omega$, and > 0 for some $\omega \in \Omega$.[4]

For p probabilistic and g a gamble, we use $p \cdot g$ for $\sum_{\omega\in\Omega} p(\omega)g(\omega)$, which is just probabilistic expectation, $\mathrm{Exp}_p[g]$.

Definition 3.2. $\mathcal{D}$ is *coherent* if:

D1. $0 \notin \mathcal{D}$

D2. If $g \in \mathcal{G}_{\gtrsim 0}$, then $g \in \mathcal{D}$

D3. If $g \in \mathcal{D}$ and $\lambda > 0$, then $\lambda g \in \mathcal{D}$

D4. If $f, g \in \mathcal{D}$, then $f + g \in \mathcal{D}$

Given a probability filter, we can extract a set of desirable gambles using:

$$g \in \mathcal{D}_{\mathcal{F}} \text{ iff } [\![\mathrm{pr} \cdot g > 0]\!] \in \mathcal{F} \tag{2}$$

where $[\![\mathrm{pr} \cdot g > 0]\!]$ was notation for $\{p \mid p \cdot g > 0\}$.

Any coherent probability filter gives a coherent set of desirable gambles.

Theorem 3.3. *If $\mathcal{F}$ is coherent, then $\mathcal{D}_{\mathcal{F}}$ is a coherent set of desirable gambles.*

Proof. Axiom D1 follows from axiom F1 by observing that $[\![\mathrm{pr} \cdot 0 > 0]\!] = \varnothing$. Axiom D3 holds because when $\lambda > 0$, $[\![\mathrm{pr} \cdot g > 0]\!] = [\![\mathrm{pr} \cdot \lambda g > 0]\!]$.

For axiom D2 it was important that we restricted to regular probabilities. If $g \in \mathcal{G}_{\gtrsim 0}$, any $p \in \mathrm{rProbs}$ has $p \cdot g > 0$; so this follows from $\mathrm{rProbs} \in \mathcal{F}$ (using our choice to restrict to regular probabilities, and axioms F2 and F4).

Axiom D4: If $g \in \mathcal{D}_{\mathcal{F}}$ and $f \in \mathcal{D}_{\mathcal{F}}$, then $[\![\mathrm{pr} \cdot g > 0]\!] \in \mathcal{F}$ and $[\![\mathrm{pr} \cdot f > 0]\!] \in \mathcal{F}$. So $[\![\mathrm{pr} \cdot g > 0 \text{ and } \mathrm{pr} \cdot f > 0]\!] \in \mathcal{F}$ by axiom F3. And this is $\subseteq [\![\mathrm{pr} \cdot (g + f) > 0]\!]$, so this is also $\in \mathcal{F}$ by axiom F4 and thus $g + f \in \mathcal{D}_{\mathcal{F}}$. □

[3] Since Ω is finite, they are automatically bounded.

[4] These are usually simply denoted with $\mathcal{G}_{>0}$, but I keep the $\gtrsim$ to highlight the *weak* dominance component, rather than that $g(\omega) > 0$ for all $\omega \in \Omega$.

Given a set of desirable gambles, $\mathcal{D}$, we construct a filter $\mathcal{F}_\mathcal{D}$ which is the minimal filter corresponding to it. Each gamble, g, which is desirable gives us an endorsed set of probabilities: $[\![\mathrm{pr} \cdot g > 0]\!]$, which is just $\{p \mid p \cdot g > 0\}$. We then define $\mathcal{F}_\mathcal{D}$ to be the minimal filter endorsing each of these sets, i.e., it is the filter generated by them. When doing this, we do not go beyond the constraints of coherence in the desirable gamble model; and we result in a filter, $\mathcal{F}_\mathcal{D}$, with

$$g \in \mathcal{D} \text{ iff } [\![\mathrm{pr} \cdot g > 0]\!] \in \mathcal{F}_\mathcal{D} \tag{3}$$

Theorem 3.4. *If $\mathcal{D}$ is coherent, then $\mathcal{F}_\mathcal{D} := \mathrm{ext}(\{[\![\mathrm{pr} \cdot g > 0]\!] \mid g \in \mathcal{D}\})$ is coherent and $g \in \mathcal{D}$ iff $[\![\mathrm{pr} \cdot g > 0]\!] \in \mathcal{F}_\mathcal{D}$. Thus, for distinct coherent $\mathcal{D}$ and $\mathcal{D}'$, $\mathcal{F}_\mathcal{D}$ and $\mathcal{F}_{\mathcal{D}'}$ are distinct. And $\mathcal{D}_{\mathcal{F}_\mathcal{D}} = \mathcal{D}$.*

Proof. By construction $[\![\mathrm{pr} \cdot g > 0]\!] \in \mathcal{F}_\mathcal{D}$ for $g \in \mathcal{D}$. We need to show that $[\![\mathrm{pr} \cdot f > 0]\!] \notin \mathcal{F}_\mathcal{D}$ for $f \notin \mathcal{D}$. Suppose $g_1, \ldots, g_n \in \mathcal{D}$ and $[\![\mathrm{pr} \cdot f > 0]\!] \supseteq [\![\mathrm{pr} \cdot g_1 > 0]\!] \cap \ldots \cap [\![\mathrm{pr} \cdot g_n > 0]\!]$. We need to show that $f \in \mathcal{D}$. We will show that then $f \in \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$; and thus $f \in \mathcal{D}$ by coherence.

Suppose for contradiction that $f \notin \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0}) = \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \{I_{\{\omega\}} \mid \omega \in \Omega\})$; where $\mathrm{Posi}(S)$ is the positive linear combinations of S, i.e., $\mathrm{Posi}(S) = \{\sum_{i=1}^n \lambda_i s_i \mid n \in \mathbb{N}, \lambda_i > 0, s_i \in S\}$; and $I_{\{\omega\}}$ is the indicator gamble, taking value 1 at ω and 0 everywhere else. Then, by a separating hyperplane theorem (see Klee, 1955), we can find a linear functional separating them. (See fig. 1.) By normalising this, we have p with $p \cdot g_i > 0$ for each i and $p \cdot f \leqslant 0$. Since each $I_{\{\omega\}} \in \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$, each $p(\omega)$ is positive, so this gives our regular probabilistic p. So $p \cdot g_i > 0$ for each i, but $p \cdot f \leqslant 0$; and thus $[\![\mathrm{pr} \cdot f > 0]\!] \not\supseteq [\![\mathrm{pr} \cdot g_1 > 0]\!] \ldots [\![\mathrm{pr} \cdot g_n > 0]\!]$, contradicting the hypothesis.

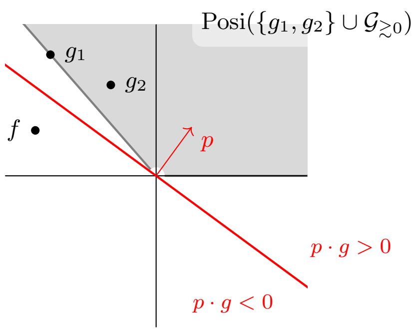

Figure 1: When $f \notin \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$, we can separate them by a hyperplane.

Thus, $f \in \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$, and so by the coherence of $\mathcal{D}$, $f \in \mathcal{D}$. Since $\varnothing = [\![\mathrm{pr} \cdot -1 > 0]\!]$, we have also shown that $\varnothing \notin \mathcal{F}_\mathcal{D}$; so it is coherent. ☐

One can also observe using this argument that for a set of gambles, G, $f \in \mathcal{D}_{\mathcal{F}_G} \iff f \in \mathrm{Posi}(G \cup \mathcal{G}_{\gtrsim 0})$, giving us just the natural extension notion on desirable gambles.

Consider $\Omega = \{\omega_t, \omega_f\}$ and a coherent set of desirable gambles with $\langle .4, -.6 \rangle \in \mathcal{D}$, but also $\langle -.4 + \epsilon, .6 + \epsilon \rangle \in \mathcal{D}$ for each $\epsilon > 0$ ($\epsilon \in \mathbb{R}$); as in fig. 2. This set of desirable gambles cannot be captured in the set of probabilities model, but it

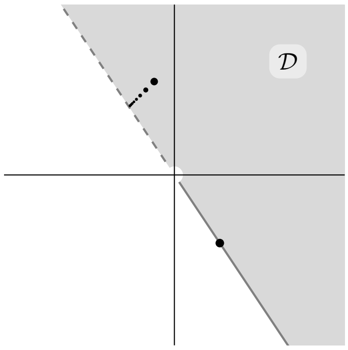

Figure 2: $\mathcal{D}$ with $\langle .4, -.6 \rangle \in \mathcal{D}$ and $\langle -.4 + \epsilon, .6 + \epsilon \rangle \in \mathcal{D}$ for each $\epsilon > 0$.

can be captured by a probability filter. Since $\langle .4, -.6 \rangle \in \mathcal{D}$, $[\![\mathrm{pr}(\omega_t) > .6]\!] \in \mathcal{F}_{\mathcal{D}}$. Since $\langle -.4 + \epsilon, .6 + \epsilon \rangle \in \mathcal{D}$, $[\![\mathrm{pr}(\omega_t) < .6 + \epsilon]\!] \in \mathcal{F}_{\mathcal{D}}$. This can be given a very natural gloss. You think that the probability exceeds .6 but not by any particular amount. This filter is non-principal (indeed it is free: $\bigcap \mathcal{F}_{\mathcal{D}} = \varnothing$). This gives it non-Archimadean behaviour. If we were to ask what one thinks the probability of ω_t is, we would have to give a hyperreal value. The model based on probability filters does not have such hyperreals as values, but they are captured by the filter structure.

Probability filters model goes beyond desirable gambles. It can, for example, capture arbitrary sets of probabilities rather than just convex ones.

4 Probability filters and choice functions

4.1 Choice functions

Choice functions provides a powerful model of belief (Seidenfeld et al., 2010). When one's beliefs are represented by a set of probabilities, a choice function will be given a set of options and select those options which have optimal expected utility according to some probability in the set (we hold a utility function as fixed). Seidenfeld et al. (2010) showed how to directly axiomatise this notion of choice function and showed its equivalence to the model of sets of probabilities.

Instead of axiomatising the choice function directly, De Bock and de Cooman (2018) give a desirability based axiomatisation. A set of gambles is said to be a desirable gamble set you think it contains at least one desirable gamble. This can be used to extract a choice function as we can then say that $o \notin C(O)$ iff $\{u(o') - u(o) \mid o' \in O\}$ is a desirable gamble set, where $u(o) : \Omega \to \mathbb{R}$ describing the utility profile of the option. We will follow this and adopt the framework of encoding desirable gamble sets.

De Bock and de Cooman (2018) make a restriction not made in Seidenfeld et al. (2010): they just consider sets which are finite, or equivalently, choices between finitely many options. We follow Seidenfeld et al. (2010) and allow sets to be infinite.

There are two further differences between the setting of De Bock and de Cooman (2018) and that of Seidenfeld et al. (2010) (see also De Bock and de Cooman, 2019): Firstly, De Bock and de Cooman do not include a mixing axiom adopted by Seidenfeld et al. Different choice rules are different ways of associating judgements with the opinion state given by a probability set, or probability filter. We choose follow E-admissibility and extract when a set is desirable using:

$$K \in \mathcal{K} \text{ iff } [\![\text{there is some } g \in A \text{ with } \mathrm{pr} \cdot g > 0]\!] \in \mathcal{F} \tag{4}$$

There doesn't need to be a particular g that is desirable; there might be disagreement about which gamble is preferable to the status quo. If we wanted to consider Walley-Sen Maximisation, we would need to consider a different way to extract the judgement from the filter:

$$A \in \mathcal{K} \text{ iff there is some } g \in A \text{ with } [\![\mathrm{pr} \cdot g > 0]\!] \in \mathcal{F}. \tag{5}$$

We use the association of eq. (4); and thus adopt a mixing axiom (our K5).

Secondly, De Bock and de Cooman do not have an Archimadean axiom. We will similarly have no Archimadean axiom as our framework is more general than the sets of probabilities framework.

We now give our notion of coherence for sets of desirable gamble sets. First, there are two notions used in the axioms, which we should state:

- $\mathrm{Posi}(S)$ is the set of positive linear combinations of members of S. That is,

$$\mathrm{Posi}(S) = \left\{ \sum_{i=1}^{n} \lambda_i s_i \;\middle|\; n \in \mathbb{N}, \lambda_i > 0, s_i \in S \right\}$$

  This is the smallest convex cone extending S.

- $\mathrm{ClPosi}(S)$ is the closure of $\mathrm{Posi}(S)$. This is the smallest closed and convex cone extending S.

Definition 4.1. $\mathcal{K} \subseteq \wp(\mathcal{G})$ is *coherent* if it satisfies

K0. $\varnothing \notin \mathcal{K}$

K1. If $A \in \mathcal{K}$ then $A \setminus \{0\} \in \mathcal{K}$.

K2. If $g \in \mathcal{G}_{\gtrsim 0}$, then $\{g\} \in \mathcal{K}$

K3. If $A_1, A_2 \in \mathcal{K}$ and for each $g_1 \in A_1$, $g_2 \in \mathcal{K}_2$, $f_{\langle g_1, g_2 \rangle}$ is some member of $\mathrm{Posi}(\{g_1, g_2\})$, then $\{f_{\langle g_1, g_2 \rangle} \mid g_1 \in A_1,\, g_2 \in A_2\} \in \mathcal{K}$

K4. If $A \in \mathcal{K}$ and $B \supseteq A$, then $B \in \mathcal{K}$

K5. If $A \in \mathcal{K}$ and $\mathrm{ClPosi}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

K6. If $A \in \mathcal{K}$ and for each $g \in A$, $f_g = g$ or f_g weakly dominates g (i.e., $f_g - g \in \mathcal{G}_{\gtrsim 0}$) then $\{f_g \mid g \in A\} \in \mathcal{K}$.

Axioms K0 to K4 follow De Bock and de Cooman (2018). We include the addition of the mixing axiom (axiom K5). It is close to the formulation of formulation of De Bock and de Cooman (2019, Sec. 8), but it follows Seidenfeld et al. (2010, axiom 2b) in taking the *closure*. This is not present in De Bock and de Cooman as they are restricting attention to finite sets of gambles.

We have added a further axiom, axiom K6 which says that if A is a desirable gamble set, and you replace some of the members of A by weak dominators, then the set remains desirable.

Whilst further discussion on these mixing and dominating axioms is deserved, we leave that to section 4.3 and move to considering its relationship to probability filters.

4.2 Probability filters and choice functions

Given a probability filter, we can extract a set of gamble sets. The intended interpretation is that $A \in \mathcal{K}$ iff you think that at least one gamble in the set is desirable, but you needn't have any particular one which is desirable, so given our understanding of desirability as that $p \cdot g > 0$, this is thus naturally given by:

$$K \in \mathcal{K}_{\mathcal{F}} \text{ iff } \{p \mid p \cdot g > 0 \text{ for some } g \in A\} \in \mathcal{F} \tag{6}$$

Note that in this, different probabilities may evaluate different gambles as being the one which are desirable.

The axioms we have imposed on $\mathcal{F}$ being coherent ensures that this set of desirable gamble sets is coherent. This tells us either than one's filter axioms are strong enough or that one's axioms on desirable gamble sets are not too strong, depending on one's perspective.

Theorem 4.2. *If $\mathcal{F}$ is coherent, then $\mathcal{K}_{\mathcal{F}}$ is coherent; where $\mathcal{K}_{\mathcal{F}}$ is the collection of finite sets of gambles satisfying eq.* (4).

Proof. Axiom K0 follows from axiom F1.

For axiom K1, since $p \cdot 0 = 0$, we have $p \cdot g > 0$ for some $g \in A$, then $p \cdot g > 0$ for some $g \in A \setminus \{0\}$; thus $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F} \implies [\![\exists g \in A \setminus \{0\} \text{ pr} \cdot g > 0]\!] \in \mathcal{F}$.

For axiom K2, if $g \in \mathcal{G}_{\gtrsim 0}$, then every $p \in$ rProbs has $p \cdot g > 0$, so $[\![\exists g \in \{g\} \text{ pr} \cdot g > 0]\!] = \text{rProbs} \in \mathcal{F}$.

Axiom K3: If $A, B \in \mathcal{K}_{\mathcal{F}}$, then $[\![\exists g \in A \text{ pr} \cdot g > 0]\!], [\![\exists g \in B \text{ pr} \cdot g > 0]\!] \in \mathcal{F}$ so by axiom F3, $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \cap [\![\exists g \in B \text{ pr} \cdot g > 0]\!] \in \mathcal{F}$. For any $p \in [\![\exists g \in A \text{ pr} \cdot g > 0]\!] \cap [\![\exists g \in B \text{ pr} \cdot g > 0]\!]$, there is some $g \in A$ and $f \in B$ with $p \cdot g > 0$ and $p \cdot f > 0$, and thus $\lambda_{g,f} g + \mu_{g,f} F > 0$, as required.

Axiom K4 follows from axiom F4 as $[\![\exists g \in B \text{ pr} \cdot g > 0]\!] \supseteq [\![\exists g \in A \text{ pr} \cdot g > 0]\!]$.

Now consider axiom K5. We consider the closure and positive hull parts separately, observing that they can be combined to obtain the closed convex. See proposition 4.5

Let $\text{Posi}(B) \supseteq A$ and $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F}$. We need to show that $[\![\exists g \in B \text{ pr} \cdot g > 0]\!] \in \mathcal{F}$. It suffices to show that $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \subseteq [\![\exists g \in B \text{ pr} \cdot g > 0]\!]$ and then use axiom F4. So we need to show that if $p \cdot g > 0$ for some $g \in A$ then $p \cdot f > 0$ for some $f \in B$. If $g \in A$ then $g = \lambda_1 f_1 + \ldots + \lambda_n f_n$ for some $f_1, \ldots, f_n \in A$ with λ_i positive. Thus, if $p \cdot g > 0$, also $p \cdot f_i > 0$ for some f_i, as required.

If $g \in A \subseteq \mathsf{closure}(B)$, then there is a sequence $f_n \in B$ with $f_n \longrightarrow g$. If $p \cdot g > 0$, then there is some f_m (in fact a tail) with $p \cdot f_m > 0$, as required.

Finally we look at axiom K6 and observe that if $p \cdot g > 0$ then $p \cdot f_g > 0$, so it holds by axiom F4. □

We can also give a filter that captures each coherent $\mathcal{K}$. This tells us that our axioms on desirable gamble sets are strong enough; or that our axioms on filters do not conflict with anything said about choice functions.

Theorem 4.3. *For any coherent $\mathcal{K}$, $\mathcal{F}_{\mathcal{K}} := \mathrm{ext}(\{p \mid p \cdot g > 0$ for some $g \in A\} \mid A \in \mathcal{K})$ is coherent with*

$$K \in \mathcal{K} \text{ iff } \{p \mid p \cdot g > 0 \text{ for some } g \in A\} \in \mathcal{F}_{\mathcal{K}} \tag{7}$$

Thus, for distinct coherent $\mathcal{K}$ and $\mathcal{K}'$, $\mathcal{F}_{\mathcal{K}}$ and $\mathcal{F}_{\mathcal{K}'}$ are distinct. And $\mathcal{K}_{\mathcal{F}_{\mathcal{K}}} = \mathcal{K}$.

The proof extends the proof used for theorem 3.4; but we will first state a lemma:

Lemma 4.4. *Suppose $\mathcal{K}$ is coherent. If $A_1, \ldots, A_n \in \mathcal{K}$ and for each sequence of $g_1 \in A_1, \ldots, g_n \in A_n$, we let $f_{\langle g_1,\ldots,g_n\rangle}$ be some member of $\mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$, then $\{f_{\langle g_1,\ldots,g_n\rangle} \mid g_1 \in A_1, \ldots, g_n \in A_n\} \in \mathcal{K}$.*

Proof. If $f_{\langle g_i\rangle} \in \mathcal{G}_{\gtrsim 0}$ for some $g_1, \ldots, g_n$, then by axiom K2, its singleton is in $\mathcal{K}$ and then by axiom K4, $\{f_{\langle g_i\rangle} \mid g_1 \in A_1, \ldots, g_n \in A_n\} \in \mathcal{K}$.

If $f_{\langle g_i\rangle} \notin \mathcal{G}_{\gtrsim 0}$, then since it is $\in \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$, there is some $h_{\langle g_i\rangle} \in \mathrm{Posi}(\{g_1, \ldots, g_n\})$ with $f_{\langle g_i\rangle} = h_{\langle g_i\rangle}$ or weakly dominating it.

By axiom K5, $\{h_{\langle g_i\rangle} \mid g_1 \in A_1, \ldots, g_n \in A_n\}$; and then by axiom K6, $\{f_{\langle g_i\rangle} \mid g_1 \in A_1, \ldots, g_n \in A_n\} \in \mathcal{K}$. □

Proof of theorem 4.3. By construction $[\![\exists g \in B \ \mathrm{pr} \cdot g > 0]\!] \in \mathcal{F}$ for $B \in \mathcal{F}$. We need to show $[\![\exists g \in B \ \mathrm{pr} \cdot g > 0]\!] \notin \mathcal{F}$ for $B \notin \mathcal{F}$. So suppose $A_1, \ldots, A_n \in \mathcal{K}$ and $[\![\exists g \in B \ \mathrm{pr} \cdot g > 0]\!] \supseteq [\![\exists g \in A_1 \ \mathrm{pr} \cdot g > 0]\!] \cap \ldots \cap [\![\exists g \in A_n \ \mathrm{pr} \cdot g > 0]\!]$. We need to show that $B \in \mathcal{K}$.

Take some $g_1 \in A_1, \ldots, g_n \in A_n$. Suppose for contradiction that there is no $f \in \mathrm{ClPosi}(B)$ with $f \in \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0}) = \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \{I_{\{\omega\}} \mid \omega \in \Omega\})$. These are both closed cones (when extended to include 0) and $\mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$ is locally compact; thus, by a separating hyperplane theorem (see Klee, 1955, Theorem 2.7), we can find a linear functional separating them. (See fig. 3.) By normalising this, we obtain our p, which we can see is regular probability as $I_{\{\omega\}} \in \mathcal{G}_{\gtrsim 0}$. So $p \cdot g_i > 0$ for each i, and thus $p \in [\![\exists g \in A_1 \ \mathrm{pr} \cdot g > 0]\!] \cap \ldots \cap [\![\exists g \in A_n \ \mathrm{pr} \cdot g > 0]\!]$ but $p \notin [\![\exists g \in B \ \mathrm{pr} \cdot g > 0]\!]$ as $p \cdot f \leqslant 0$ for all $f \in B$. Thus $[\![\exists g \in A_1 \ \mathrm{pr} \cdot g > 0]\!] \cap \ldots \cap [\![\exists g \in A_n \ \mathrm{pr} \cdot g > 0]\!] \not\subseteq [\![\exists g \in B \ \mathrm{pr} \cdot g > 0]\!]$, contradicting the hypothesis.

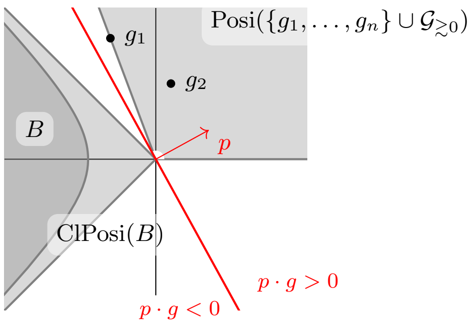

Figure 3: When $\mathrm{ClPosi}(B)$ and $\mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$ are disjoint they can be separated.

Thus, for any $g_1 \in A_1, \ldots, g_n \in A_n$ there is some $h_{\langle g_i\rangle} \in \mathrm{ClPosi}(B)$ and $h_{\langle g_i\rangle} \in \mathrm{Posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$. By lemma 4.4, $\{h_{\langle g_i\rangle} \mid g_1 \in A_1, \ldots, g_n \in A_n\} \in \mathcal{K}$. This is $\subseteq \mathrm{ClPosi}(B)$ so by axiom K5, $B \in \mathcal{K}$, as required.

Since $[\![\exists g \in \{-1\}\ \mathrm{pr} \cdot g > 0]\!] = \varnothing \notin \mathcal{F}_{\mathcal{K}}$, we also have that it is coherent. $\square$

4.3 Axiomatisation with infinite sets

4.3.1 Mixing axiom

We have adopted the mixing axiom in the form:

K5 If $A \in \mathcal{K}$ and $\mathrm{ClPosi}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

We will discuss versions of this axiom that just differ in which operation they use in place of ClPosi. For O an operator such as Conv, Posi, ClConv, let K_O be the axiom:

K_O. If $A \in \mathcal{K}$ and $O(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

Given axiom K4, we could equivalently state any of these as

K_O. If $O(B) \in \mathcal{K}$ then $B \in \mathcal{K}$.

De Bock and de Cooman (2019), working in the finite setting, choose to use the axiom $\mathrm{K}_{\mathrm{Posi}}$ instead of $\mathrm{K}_{\mathrm{Conv}}$. Similarly, we have chosen to use $\mathrm{K}_{\mathrm{ClPosi}}$ instead of Seidenfeld et al. (2010)'s use of $\mathrm{K}_{\mathrm{ClConv}}$. But similarly, here they are equivalent:

Proposition 4.5. *Suppose $\mathcal{K}$ satisfies axioms* K3 *and* K4*. Then the following are equivalent:*

1. $\mathrm{K}_{\mathrm{ClPosi}}$
2. $\mathrm{K}_{\mathsf{closure}}$ *and* $\mathrm{K}_{\mathrm{Posi}}$
3. $\mathrm{K}_{\mathrm{ClConv}}$
4. $\mathrm{K}_{\mathsf{closure}}$ *and* $\mathrm{K}_{\mathrm{Conv}}$

Proof. Item 2 $\Longrightarrow$ item 1: If $\mathrm{ClPosi}(B) = \mathsf{closure}(\mathrm{Posi}(B)) \in \mathcal{K}$ then by $\mathrm{K}_{\mathsf{closure}}$, $\mathrm{Posi}(B) \in \mathcal{K}$ and by $\mathrm{K}_{\mathrm{Posi}}$, $B \in \mathcal{K}$.

Item 1 $\Longrightarrow$ item 2: If $\mathsf{closure}(B) \in \mathcal{K}$, then $\mathrm{ClPosi}(B) \in \mathcal{K}$ by axiom K4, so $B \in \mathcal{K}$. Similarly, if $\mathrm{Posi}(B) \in \mathcal{K}$ then $\mathrm{ClPosi}(B) \in \mathcal{K}$ so $B \in \mathcal{K}$.

For item 4 $\Longleftrightarrow$ item 2, we show that $\mathrm{Conv}(B) \in \mathcal{K}$ iff $\mathrm{Posi}(B) \in \mathcal{K}$: For $\Longrightarrow$ we use axiom K4. For $\Longleftarrow$ we axiom K3, choosing appropriate scalars.

The argument for item 2 $\Longleftrightarrow$ item 1 is exactly analogous. $\square$

It is worth noting that for $\mathrm{Scalar}(S) = \{\lambda s \mid \lambda > 0,\ s \in S\}$, $\mathrm{K}_{\mathrm{Scalar}}$ follows directly from axiom K3.

4.3.2 Taking dominators

Our axiom K6 is used in theorem 4.3 to allow us to replace members of a set by weak dominators and remain desirable. This has a clear intuitive motivation: this can only improve the situation.

De Bock and de Cooman (2019, Lemma 34) show that we can use axioms K2 and K3 to see that replacing a single member of a desirable set with a dominator will remain a desirable set. We can iterate this any finite number of times, but it is not clear to me how to derive axiom K6, as it applies to infinite sets, from the remaining axioms. Since it is clearly an axiom we want, I have simply added it.

An alternative would be to add this power in to axiom K3 directly. This could be done by replacing $\mathrm{Posi}(\{g_1, g_2\})$ with $\mathrm{Posi}(\{g_1, g_2\} \cup \mathcal{G}_{\gtrsim 0})$,

- If $A_1, A_2 \in \mathcal{K}$ and for each $g_1 \in A_1$, $g_2 \in \mathcal{K}_2$, $f_{\langle g_1, g_2\rangle}$ is some member of $\mathrm{Posi}(\{g_1, g_2\} \cup \mathcal{G}_{\gtrsim 0})$, then $\{f_{\langle g_1, g_2\rangle} \mid g_1 \in A_1,\ g_2 \in A_2\} \in \mathcal{K}$

This would then give us exactly what is needed for our result (when extending to finitely many rather than just two). An alternative, which would still suffice would be to use $\mathrm{Posi}(\{g_1, g_2\}) + \mathcal{G}_{\geqslant 0} = \{h + f \mid h \in \mathrm{Posi}(\{g_1, g_2\}),\ f \in \mathcal{G}_{\gtrsim 0} \cup \{0\}\}$.

- If $A_1, A_2 \in \mathcal{K}$ and for each $g_1 \in A_1$, $g_2 \in \mathcal{K}_2$, $f_{\langle g_1, g_2\rangle}$ is some member of $\mathrm{Posi}(\{g_1, g_2\}) + \mathcal{G}_{\geqslant 0}$, then $\{f_{\langle g_1, g_2\rangle} \mid g_1 \in A_1,\ g_2 \in A_2\} \in \mathcal{K}$.

We would also be able to derive it from an infinite extension of axiom K3. That is:

✗ Suppose $\mathcal{A} \subseteq \mathcal{K}$. If for each sequence $\langle g_A\rangle_{A \in \mathcal{A}}$ with $g_A \in A$, $f_{\langle g_A\rangle}$ is a member of $\mathrm{Posi}(\{g_A\} \mid A \in \mathcal{A})$, then $\{f_{\langle g_A\rangle} \mid \langle g_A\rangle \in \prod \mathcal{A}\} \in \mathcal{K}$

But this is not valid using non-principal filters and the E-admissibility interpretation eq. (4).

Proof. Consider $\mathcal{F}$ with each $g_n \in \mathcal{D}_{\mathcal{F}}$ and $h \in \mathcal{D}_{\mathcal{F}}$, as in fig. 4, and as discussed in section 3. This will violate the infinite extension of axiom K3.

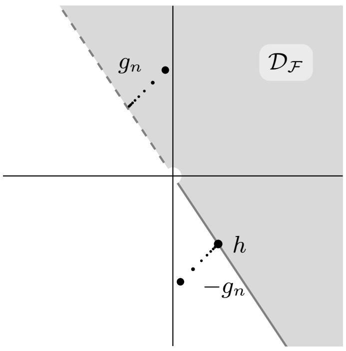

Figure 4: $h = \langle .4, -.6\rangle$; $g_n = -h + \langle 1/n, 1/n\rangle$.

Observe that $\{-g_n \mid n \in \mathbb{N}\} \in \mathcal{K}_{\mathcal{F}}$: if $p \cdot h > 0$ then for some n, $p \cdot g_n > 0$; thus $[\![\exists n \in \mathbb{N}\ \mathrm{pr} \cdot g_n > 0]\!] \supseteq [\![\mathrm{pr} \cdot h > 0]\!] \in \mathcal{F}$.

Also each singleton $\{g_n\} \in \mathcal{K}$. Consider the infinite collection of gambles given by these singletons and $\{-g_n \mid n \in \mathbb{N}\}$; each of which is a member of $\mathcal{K}$.

For each n observe that $0 \in \mathrm{Posi}(\{g_n, -g_n\}) \subseteq \mathrm{Posi}(\{g_1, g_2, \ldots\} \cup \{-g_n\})$.

So, for each sequence, 0 is a member of the relevant posi, and thus for the infinite version of the axiom to be true, it would need to be that $\{0\} \in \mathcal{K}_{\mathcal{F}}$. $\square$

5 Beyond

An encoding of which gamble sets are desirable, using eq. (4), does not suffice to tell us everything about the opinion state, as given by the filter. The expressive power of probability filters goes strictly beyond that of sets of desirable gamble sets.

We will now describe an example where two distinct filters do not differ in their evaluations of which gambles are desirable, or on their evaluations of which sets of gambles contain at least one desirable gamble. This will show us that the expressive power of probability filters goes strictly beyond that of desirable gamble sets.

The example shows an expressive limitation of the sets of desirable gamble sets framework as these filters differ in their judgements of weak preferences between gambles, i.e., when it is evaluated to be weakly preferable to the status quo; or when it is either strictly preferable or equivalent to the status quo.

Let $\Omega = \{\omega_t, \omega_f\}$. For $x \in (0,1)$, let $p_x(\omega_t) = x, p_x(\omega_f) = 1 - x$. Consider the following three belief states:

$$
\begin{aligned}
&P \in \mathcal{F}_= \text{ iff } p_{.6} \in P. && (8)\\
&P \in \mathcal{F}_> \text{ iff there is } \epsilon > 0 \text{ with } P(\omega_t) \supseteq (.6,\ .6+\epsilon). && (9)\\
&P \in \mathcal{F}_\geqslant \text{ iff there is } \epsilon > 0 \text{ with } P(\omega_t) \supseteq (.6,\ .6+\epsilon) && (10)\\
&\qquad\qquad \text{and } p_{.6} \in P.\\
&\qquad\quad \text{iff there is } \epsilon > 0 \text{ with } P(\omega_t) \supseteq [.6,\ .6+\epsilon).
\end{aligned}
$$

$\mathcal{F}_>$ is the belief state we used in section 3 to capture the half-opens set of gambles. $\mathcal{F}_=$ thinks the probability of ω_t is identical to .6. $\mathcal{F}_\geqslant$ only endorses a set of probabilities when it is endorsed by both $\mathcal{F}_>$ and $\mathcal{F}_=$. It leaves it open whether the probability of ω_t is identical to .6 or not, but commits to it being at least .6 and not exceeding .6 by any particular amount.

The set of desirable gamble sets corresponding to $\mathcal{F}_\geqslant$ and to $\mathcal{F}_=$ are identical. So the expressive power of probability filters goes strictly beyond that of desirable gamble sets. We state this as a theorem:

Theorem 5.1. *There are coherent distinct $\mathcal{F}$ and $\mathcal{F}'$ with $\mathcal{K}_\mathcal{F} = \mathcal{K}_{\mathcal{F}'}$.*

Proof. We show that $\mathcal{K}_{\mathcal{F}_=} = \mathcal{K}_{\mathcal{F}_\geqslant}$, where $\mathcal{F}_=$ and $\mathcal{F}_\geqslant$ are as in eqs. (8) and (10). We need to show $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F}_\geqslant$ iff $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F}_=$.

Suppose $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F}_\geqslant$. Then $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \supseteq (.6,\ .6+\epsilon]$ for some $\epsilon > 0$ and $p_{.6} \in [\![\exists g \in A \text{ pr} \cdot g > 0]\!]$; and thus $p_{.6} \in [\![\exists g \in A \text{ pr} \cdot g > 0]\!]$ so $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F}_=$.

Suppose $[\![\exists g \in A \text{ pr} \cdot g > 0]\!] \in \mathcal{F}_=$. So $p_{.6} \cdot g > 0$ for some $g \in A$. Since probabilistic expectation is continuous, there is some $\epsilon > 0$ with $p_\alpha \cdot g > 0$ for every $.6 \leqslant \alpha < .6 + \epsilon$. Thus $[\![\exists g \in A \text{ pr} \cdot g > 0]\!](\omega_t) \supseteq (.6,\ .6+\epsilon)$. □

However, $\mathcal{F}_\geqslant$ and $\mathcal{F}_=$ differ on whether they endorse $g = \langle -.4,\ .6\rangle$ as weakly preferable to 0. $\mathcal{F}_=$ thinks that g is equivalent in estimation to the status quo. But $\mathcal{F}_\geqslant$ leaves open whether g is equivalent to the status quo or is strictly undesirable. In both cases, g is the limit of strictly desirable gambles, but this does not entail that g is evaluated as weakly desirable: $\mathcal{F}_\geqslant$ leaves it open.

We propose that this is a limitation of the desirable gamble sets framework.

In the desirable gambles setting, we directly considered whether a set does contain a desirable gamble; that is we encoded when

$$\{p \mid \exists g \in A \quad p \cdot g > 0\} \in \mathcal{F} \tag{11}$$

This does not tell us whether a set definitely does not contain any desirable gamble, or equivalently, when it only contains gambles that are weakly desirable. That is, any one of the equivalent:

$$\begin{align} \{p \mid \neg\, \exists g \in A \quad p \cdot g > 0\} \in \mathcal{F} \tag{12}\\ \{p \mid \quad \forall g \in A \quad p \cdot g \not> 0\} \in \mathcal{F} \tag{13}\\ \{p \mid \quad \forall g \in A \quad p \cdot g \leqslant 0\} \in \mathcal{F} \tag{14}\end{align}$$

Note also that this is just the question of whether

$$\overline{\{p \mid \exists g \in A\, p \cdot g > 0\}} = \text{rProbs} \setminus \{p \mid \exists g \in A\, p \cdot g > 0\} \in \mathcal{F}.$$

Knowing that $A \notin \mathcal{F}$ does not tell us about whether $\overline{A} \in \mathcal{F}$ unless the filter is total; it might be rejected, or it may be left open.

We can also helpfully consider these questions in terms of choice functions. Seidenfeld et al. (2010)'s choice function, C, uses the judgement that the option o is optimal according to some probability in one's probability set, i.e., it maximises (possibly non-uniquely) expected utility according to that probability. When associating our filter framework to the sets of probability framework (section 2.2) we encoded judgements of when something holds according to all probabilities in the set. To phrase the choice-function notion in this framework, then, what we are encoding is the fact that you do not endorse the option to be non-optimal:

$$\begin{align} o \in C(O) &\text{ iff } \exists p \in \mathbb{P} \quad o \text{ is optimal according to } p \tag{15}\\ &\text{ iff } \exists p \in \mathbb{P}\, \forall o' \in O \quad p \cdot o \geqslant p \cdot o' \tag{16}\\ &\text{ iff } \neg\, \forall p \in \mathbb{P}\, \exists o' \in O \quad p \cdot o < p \cdot o' \tag{17}\\ &\text{ iff } \{p \mid \exists o' \in O \quad p \cdot o < p \cdot o'\} \notin \mathcal{F}_{\mathbb{P}} \tag{18}\\ &\text{ iff } \{p \mid o \text{ is not optimal according to } p\} \notin \mathcal{F}_{\mathbb{P}} \tag{19}\\ &\text{ iff you } \textit{do not} \text{ endorse that } o \text{ is } \textit{not} \text{ optimal} \tag{20}\end{align}$$

We might also wish to consider an endorsement that an option *is* optimal. In the sets of probability model, this means encoding when o is optimal according to *all* probabilities in the set. This needs to separately be encoded.

We may further need to include judgements of whether a set contains at least one weakly desirable gamble, or when a set only contain weakly desirable gambles.

$\mathcal{F}_{>}$ is a natural belief state. There are many distinct refinements of it. We need to use the axiom of choice to give ultrafilters expanding $\mathcal{F}_{>}$. We conjecture that $\mathcal{F}_{>}$ determines all judgements on whether sets contain some or only contain strictly or weakly desirable gambles, and further refinements cannot be distinguished using these.

But for any distinct filters we may be able to distinguish them on some questions of whether a set of gambles contains at least one gamble that's equal to the status quo. But full investigation of these questions remains future work.

6 Conclusion

We have considered representing beliefs by probability filters and shown it is very general. It can capture the expressive power of a whole range of frameworks, such as comparative judgements, desirable gambles, or choice functions.

It is a very natural model and is easy to work with. It provides a unifier between the model of belief given by sets of probabilities, and those of the desirable gambles kind.

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