Results about Sets of Desirable Gamble Sets

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1 Setup - Definition of Coherence

1.1 Gambles

Setup 1.1. $\mathcal{G}$ is the set of all gambles, which are the bounded functions from Ω to $\mathbb{R}$.

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When $f(\omega) \geqslant g(\omega)$ for all $\omega \in \Omega$, we will say $f \geqslant g$.

When $f(\omega) > g(\omega)$ for all $\omega \in \Omega$, we will say $f > g$, or that f *strictly dominates* g.

When $f \geqslant g$ and $g \ngeqslant f$ we say $f \gneqq g$, or that f *weakly dominates* g.. That is, when $f(\omega) \geqslant g(\omega)$ for all $\omega \in \Omega$ and $f(\omega) > g(\omega)$ for some $\omega \in \Omega$.

$\mathcal{G}_{\geqslant 0}$ is the set of gambles where $f \geqslant 0$.

$\mathcal{G}_{>0}$ is the set of gambles which strictly dominate 0. I.e., where $f > 0$.

$\mathcal{G}_{\gneqq 0}$ is the set of gambles which weakly dominate 0. I.e., where $f \gneqq 0$.

We will also make use of the positive linear hull of a set: $\mathrm{posi}(B) := \{\sum_{i=1}^{n} \lambda_i g_i \mid n \in \mathbb{N}, \lambda_i > 0, g_i \in B\}$.

1.2 Desirable gambles - usual results

Definition 1.2. $D \subseteq \mathcal{G}$ is coherent if:

(D_0) $0 \notin D$

($\mathrm{D}_{\gneqq 0}$) If $g \in \mathcal{G}_{\gneqq 0}$, then $g \in D$

(D_λ) If $g \in D$ and $\lambda > 0$, then $\lambda g \in D$

(D_+) If $f, g \in D$, then $f + g \in D$

Proposition 1.3. *For $E \subseteq \mathcal{G}$,*

(i) *If $0 \notin \mathrm{posi}(E \cup \mathcal{G}_{\gneqq 0})$ then there is some coherent D extending E; and the minimal such coherent extension is $\mathrm{posi}(E \cup \mathcal{G}_{\gneqq 0})$.*

(ii) *If $0 \in \mathrm{posi}(E \cup \mathcal{G}_{\gneqq 0})$ then there is no coherent D extending E.*

Proposition 1.4. *D is coherent iff $D = \mathrm{posi}(E \cup \mathcal{G}_{\gneqq 0})$ for some $E \subseteq \mathcal{G}$, and $0 \notin D$.*

1.3 Sets of desirable gamble sets

I will give definitions of coherence as in Campbell-Moore (forthcoming), which are based on those of De Bock and de Cooman (2018) but diverge from them in order to accommodate the fact that I am not restricting to finite sets of gambles. In section 3 we'll show they're equivalent in the case where gamble sets must be finite.

Definition 1.5. $\mathcal{K} \subseteq \wp(\mathcal{G})$ is *coherent* if it satisfies

($\mathrm{K}_\varnothing$) $\varnothing \notin \mathcal{K}$

(K_0) If $A \in \mathcal{K}$ then $A \setminus \{0\} \in \mathcal{K}$.

($\mathrm{K}_{\gneqq 0}$) If $g \in \mathcal{G}_{\gneqq 0}$, then $\{g\} \in \mathcal{K}$

($\mathrm{K}_\supseteq$) If $A \in \mathcal{K}$ and $B \supseteq A$, then $B \in \mathcal{K}$

($\mathrm{K}_{\mathrm{Dom}}$) If $A \in \mathcal{K}$ and for each $g \in A$, f_g is some gamble where $f_g \geqslant g$, then $\{f_g \mid g \in A\} \in \mathcal{K}$.

($\mathrm{K}_{\mathrm{Add}}$) If $A_1, \ldots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \ldots, g_n\rangle \in A_1 \times \ldots \times A_n$, $f_{\langle g_i \rangle}$ is some member of $\mathrm{posi}(\{g_1, \ldots, g_n\})$, then

$$\{f_{\langle g_i \rangle} \mid \langle g_1, \ldots, g_n\rangle \in A_1 \times \ldots \times A_n\} \in \mathcal{K}$$

2 Natural Extension

2.1 The key result about coherence

Our central result tells us the key important property of coherent sets of desirable gamble sets. We will afterwards show that this is is a characteristic result of coherence, i.e., it gives us the natural extension.

Theorem 2.1. *Suppose $\mathcal{K}$ is coherent and $A_1, \dots, A_n \in \mathcal{K}$.*

If B is such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \ldots \times A_n$, either $0 \in \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$, or there is some $f_{\langle g_1,\dots,g_n \rangle} \in B$ with $f_{\langle g_1,\dots,g_n \rangle} \in \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$;
Then $B \in \mathcal{K}$.

Proof. We start with a (not terribly interesting) lemma:

Sublemma 2.1.1. *For $f \in \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$, whenever $f \notin \mathcal{G}_{\gneq 0}$, there is some $h \in \operatorname{posi}(\{g_1, \dots, g_n\})$ with $f \geqslant h$.*

Proof. $f \in \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$ so $f = \sum_i \lambda_i g_i + \sum_j \mu_j p_j$ for some $p_j \in \mathcal{G}_{\gneq 0}$ and $\lambda_i, \mu_j \geqslant 0$, with at least one > 0.

Suppose $f \notin \mathcal{G}_{\gneq 0}$. Then some $\lambda_i > 0$ because otherwise $f = \sum_j \mu_j p_j$ with each $p_j \in \mathcal{G}_{\gneq 0}$, and so we'd also have that $f \in \mathcal{G}_{\gneq 0}$.

So, let $h = \sum_i \lambda_i g_i$, and we then know that $h \in \operatorname{posi}(\{g_1, \dots, g_n\})$. Since $f = h + \sum_j \mu_j p_j$ with each $p_j \in \mathcal{G}_{\gneq 0}$, we know that $f \geqslant h$. □

Assume we have coherent $\mathcal{K}$ and B satisfying the assumptions of the theorem. So we have some $A_1, \dots, A_n$ where for any sequence $\langle g_i \rangle \in \bigtimes_i A_i$, whenever $0 \notin \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_i \rangle} \in \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$ with $f_{\langle g_i \rangle} \in B$. We need to show that $B \in \mathcal{K}$.

When $0 \in \operatorname{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$, we can let $f_{\langle g_i \rangle}$ denote 0.

If there is some $f \in \mathcal{G}_{\gneq 0}$ with $f \in B$ then by axioms items $\mathrm{K}_{\gneq 0}$ and $\mathrm{K}_{\supseteq}$, $B \in \mathcal{K}$.

So we can assume that each $f_{\langle g_i \rangle} \notin \mathcal{G}_{\gneq 0}$ (noting that $0 \notin \mathcal{G}_{\gneq 0}$), so by sublemma 2.1.1, we can find $h_{\langle g_i \rangle} \in \operatorname{posi}(\{g_1, \dots, g_n\})$ with $f_{\langle g_i \rangle} \geqslant h_{\langle g_i \rangle}$. By axiom $\mathrm{K}_{\mathrm{Add}}$,

$$\{h_{\langle g_i \rangle} \mid \langle g_i \rangle \in \bigtimes_i A_i\} \in \mathcal{K}. \tag{1}$$

and then by axiom $\mathrm{K}_{\mathrm{Dom}}$,

$$\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \bigtimes_i A_i\} \in \mathcal{K}. \tag{2}$$

Since each $f_{\langle g_i \rangle} \in B \cup \{0\}$, by axiom $\mathrm{K}_{\supseteq}$, $B \cup \{0\} \in \mathcal{K}$. And thus $B \in \mathcal{K}$ by axiom K_0. □

Note that we have restricted this to finitely many $A_1, \dots, A_n$ in $\mathcal{K}$. This is because axiom $\mathrm{K}_{\mathrm{Add}}$ is restricted to finitely many members.

2.2 Defining the Natural Extension

If we start with a given set of desirable gamble sets, $\mathcal{E}$, we define $\mathrm{Kext}(\mathcal{E})$ by closing it under the construction of theorem 2.1 (except when we start with $\varnothing$ in which case we give a separate definition because that construction then doesn't add anything).

Definition 2.2. For $\mathcal{E} \subseteq \wp(\mathcal{G})$, let $\mathrm{Kext}(\mathcal{E})$ be given by:
When $\mathcal{E} \neq \varnothing$:

- $B \in \mathrm{Kext}(\mathcal{E})$ iff there are some $A_1, \ldots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, whenever $0 \notin \mathrm{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B$ where also $f_{\langle g_1, \ldots, g_n \rangle} \in \mathrm{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$.

When $\mathcal{E} = \varnothing$,

- $B \in \mathrm{Kext}(\varnothing)$ iff there is some $f \in B$ where also $f \in \mathcal{G}_{\gneq 0}$.

This will give us the natural extension notion: the minimal coherent set of desirable gamble sets extending the given set, $\mathcal{E}$. The formulation is a bit different to that of De Bock and de Cooman (2018), but they're ultimately equivalent. We'll discuss this in section 2.4.

We now see that this gives us the natural extension:

Theorem 2.3.

(i) *If* $\varnothing \notin \mathrm{Kext}(\mathcal{E})$ *then* $\mathrm{Kext}(\mathcal{E})$ *is coherent and it is the minimal coherent* $\mathcal{K}$ *extending* $\mathcal{E}$.

(ii) *If* $\varnothing \in \mathrm{Kext}(\mathcal{E})$ *then* $\mathrm{Kext}(\mathcal{E})$ *is incoherent and there is no coherent* $\mathcal{K}$ *extending* $\mathcal{E}$.

Proof. We first deal with the easy case, when $\mathcal{E} = \varnothing$: If $\mathcal{K}$ is coherent then by $\mathrm{K}_{\gneq 0}$ and $\mathrm{K}_{\supseteq}$, any $B \in \mathrm{Kext}(\varnothing)$ has $B \in \mathcal{K}$. Also observe that $\mathrm{Kext}(\varnothing)$ is coherent. So it is the minimal coherent extension of $\varnothing$.

So we now just consider when $\mathcal{E} \neq \varnothing$. As a corollary of theorem 2.1, when $\mathcal{K}$ is coherent extending $\mathcal{E}$, then $\mathcal{K} \supseteq \mathrm{Kext}(\mathcal{E})$. This is because $\mathrm{Kext}(\mathcal{E})$ was just defined to take the closure in accordance with theorem 2.1. We then need to check that $\mathrm{Kext}(\mathcal{E})$ is coherent (so long as it doesn't contain $\varnothing$).

Sublemma 2.3.1. *Suppose* $\mathcal{E} \neq \varnothing$. *If* $\varnothing \notin \mathrm{Kext}(\mathcal{E})$, *then* $\mathrm{Kext}(\mathcal{E})$ *is coherent.*

Proof. We check each of the axioms.

- $\mathrm{K}_{\varnothing}$: $\varnothing \notin \mathrm{Kext}(\mathcal{E})$ by assumption.

- K_0: Suppose $A \in \mathrm{Kext}(\mathcal{E})$. We need to show that $A \setminus \{0\} \in \mathrm{Kext}(\mathcal{E})$. Since $A \in \mathrm{Kext}(\mathcal{E})$, there are $B_1, \ldots, B_n \in \mathcal{E}$ where for each $\langle e_i \rangle \in \bigtimes B_i$, whenever $0 \notin \mathrm{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$ there is some $h_{\langle e_i \rangle} \in A$ with $h_{\langle e_i \rangle} \in \mathrm{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$. For such $\langle e_i \rangle$, $h_{\langle e_i \rangle} \neq 0$ so also $h_{\langle e_i \rangle} \in (A \setminus \{0\})$. Thus, $A \setminus \{0\} \in \mathrm{Kext}(\mathcal{E})$.

- $\mathrm{K}_{\gneq 0}$: Let $g \in \mathcal{G}_{\gneq 0}$. Since $\mathcal{E} \neq \varnothing$ there is some $B \in \mathcal{E}$. For every $e \in B$, $g \in \mathrm{posi}(\{e\} \cup \mathcal{G}_{\gneq 0})$. So by definition of Kext, $\{g\} \in \mathrm{Kext}(\mathcal{E})$.

- $K_\supseteq$: Suppose $A \in \text{Kext}(\mathcal{E})$ and $C \supseteq A$. We need to show that $C \in \text{Kext}(\mathcal{E})$. There are some $B_i \in \mathcal{E}$ where for each $\langle e_i \rangle \in \bigtimes B_i$, whenever $0 \notin \text{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$, we have some $h_{\langle e_i \rangle} \in A$ with $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$. Since $C \supseteq A$, also $h_{\langle e_i \rangle} \in C$. So $C \in \text{Kext}(\mathcal{E})$.

- K_{Dom}: Suppose $A \in \text{Kext}(\mathcal{E})$ and for each $g \in A$, f_g is a gamble with $f_g \geqslant g$. We need to show that $C := \{f_g \,|\, g \in A\} \in \text{Kext}(\mathcal{E})$. There are some $B_i \in \mathcal{E}$ where for each $\langle e_i \rangle \in \bigtimes B_i$, whenever $0 \notin \text{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $h_{\langle e_i \rangle} \in A$ with $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$. For each such $\langle e_i \rangle$, let $f_{\langle e_i \rangle}$ be the relevant $f_{h_{\langle e_i \rangle}} \geqslant h_{\langle e_i \rangle}$. So since $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$ with $f_{\langle e_i \rangle} \geqslant h_{\langle e_i \rangle}$, also $f_{\langle e_i \rangle} \in \text{posi}(\{e_1, \ldots, e_n\} \cup \mathcal{G}_{\gneq 0})$. And $f_{\langle e_i \rangle} \in C$; as required.

- K_{Add}: Suppose $A_1, \ldots, A_n \in \text{Kext}(\mathcal{E})$, and $C := \{f_{\langle g_i \rangle} \,|\, \langle g_i \rangle \in \bigtimes A_i\}$, with $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \ldots, g_n\})$ (as in K_{Add}). We need to show that $C \in \text{Kext}(\mathcal{E})$.

  For each i, there are $B^i_1, \ldots, B^i_{m_i}$ where for any $\langle e_j \rangle \in \bigtimes_j B^i_j$, whenever $0 \notin \text{posi}(\{e_1, \ldots, e_{m_i}\} \cup \mathcal{G}_{\gneq 0})$, there is some $g_{\langle e_j \rangle} \in A_i$ and $g_{\langle e_j \rangle} \in \text{posi}(\{e_1, \ldots, e_{m_i}\} \cup \mathcal{G}_{\gneq 0})$.

  Now, consider all of B^i_j for i, j. This is a finite collection. Any sequence of members of it has the form $\langle e^i_j \rangle_{i,j} \in \bigtimes_{i,j} B^i_j$. Fix such a sequence with $0 \notin \text{posi}(\{e^i_j \mid i, j\} \cup \mathcal{G}_{\gneq 0})$. We need to show there is some $f \in C$ with $f \in \text{posi}(\{e^i_j \mid i, j\} \cup \mathcal{G}_{\gneq 0})$.

  We also have $0 \notin \text{posi}(\{e^i_j \mid j\} \cup \mathcal{G}_{\gneq 0})$ for each i; so we have some $g_i := g_{\langle e^i_j \rangle_j} \in A_i$ with $g_i \in \text{posi}(\{e^i_j \mid j\} \cup \mathcal{G}_{\gneq 0})$. Since $\langle g_i \rangle \in \bigtimes A_i$, by our choice of C there is some $f \in C$ with $f \in \text{posi}(\{g_1, \ldots, g_n\})$.

  We need to show that $f \in \text{posi}(\{e^i_j \mid i, j\} \cup \mathcal{G}_{\gneq 0})$. This follows from $f \in \text{posi}(\{g_1, \ldots, g_n\})$ and each $g_i \in \text{posi}(\{e^i_j \mid i, j\} \cup \mathcal{G}_{\gneq 0})$ (since it is in $\text{posi}(\{e^i_j \mid j\} \cup \mathcal{G}_{\gneq 0})$).

  So $C \in \text{Kext}(\mathcal{E})$. □

So we have shown that when $\varnothing \notin \text{Kext}(\mathcal{E})$, $\text{Kext}(\mathcal{E})$ is the minimal coherent extension of $\mathcal{E}$.

We now just need to check that when $\varnothing \in \text{Kext}(\mathcal{E})$, there is no coherent $\mathcal{K}$ extending $\mathcal{E}$. This follows from theorem 2.1 because then $\varnothing$ is in any coherent $\mathcal{K}$ extending $\mathcal{E}$, contradicting the supposed coherence of $\mathcal{K}$. □

2.3 Comments on the Formulation of it

2.3.1 Quick notes

One component of our natural extension formulation is:

> whenever $0 \notin \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B$ where also $f_{\langle g_1, \ldots, g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$.

This can equivalently be phrased:

> there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B \cup \{0\}$ where also $f_{\langle g_1, \ldots, g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$.

We can also phrase things with intersections. I.e.,

there is some $f_{\langle g_1,\dots,g_n\rangle} \in (B \cup \{0\}) \cap \mathrm{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0})$.

Or without the reference to $f_{\langle g_1,\dots,g_n\rangle}$ at all:

$(B \cup \{0\}) \cap \mathrm{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0}) \neq \varnothing$.

Just to see what this would look like, we'd then have the definition of Kext for $\mathcal{E} \neq \varnothing$ as:

- $B \in \mathrm{Kext}(\mathcal{E})$ iff there are some $A_1,\dots,A_n \in \mathcal{E}$ such that for each sequence $\langle g_1,\dots,g_n\rangle \in A_1 \times \ldots \times A_n$, such that $(B \cup \{0\}) \cap \mathrm{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0}) \neq \varnothing$.

And we might then want to write the case for $\mathcal{E} = \varnothing$ as:

- $B \in \mathrm{Kext}(\varnothing)$ iff $B \cap \mathcal{G}_{\gneq 0} \neq \varnothing$.

I quite like the reference to the particular member, rather than talking about the intersection being non-empty, because I think it helps thinking about it and is useful to have for proofs, which is why I've opted for that.

2.3.2 On the case when $\mathcal{E} = \varnothing$

I have chosen to define the case of $\mathcal{E} \neq \varnothing$ differently from when $\mathcal{E} = \varnothing$.

To give a single definition we could follow Arne Decadt REFERENCE and include an additional separate clause for the case where $B \cap \mathcal{G}_{\gneq 0} \neq \varnothing$. So, we would offer:

For any $\mathcal{E}$ (including $\varnothing$), $B \in \mathrm{Kext}(\mathcal{E})$ iff either:

- There is some $f \in B$ where also $f \in \mathcal{G}_{\gneq 0}$. Or
- There are some $A_1,\dots,A_n \in \mathcal{E}$ such that for each sequence $\langle g_1,\dots,g_n\rangle \in A_1 \times \ldots \times A_n$, whenever $0 \notin \mathrm{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_1,\dots,g_n\rangle} \in B$ where also $f_{\langle g_1,\dots,g_n\rangle} \in \mathrm{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0})$.

An alternative would be to first add something that we know to be in all coherent $\mathcal{K}$ so that we essentially ensure we're working with non-empty $\mathcal{E}$, for example we could add $\mathcal{G}_{\gneq 0}$ itself or each singleton $\{g\}$ for $g \in \mathcal{G}_{\gneq 0}$. This would then allow us to use the theorem 2.1 characterisation. But I want to keep the non-empty case as simple as possible as it's really the interesting one, so I keep them separate.

Both of these have the disadvantage of introducing additional steps/clauses to be checked. I have thus opted to keep the definition simpler and simply deal with $\mathcal{E} = \varnothing$ manually.

2.3.3 Removing the use of $\mathcal{G}_{\gneq 0}$ in the posi:

Decant doesn't work with $\mathrm{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0})$ but instead simply with posi and $\geqslant$. His definition is (with different formulation):

For any $\mathcal{E}$ (including $\varnothing$), $B \in \mathrm{Kext}(\mathcal{E})$ iff either:

- There is some there are some $f \in B$ where also $f \in \mathcal{G}_{\gneq 0}$. Or
- There are some $A_1, \ldots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_1,\ldots,g_n \rangle} \in B$ and $h_{\langle g_1,\ldots,g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\})$ where $f_{\langle g_1,\ldots,g_n \rangle} \geqslant h_{\langle g_1,\ldots,g_n \rangle}$.

Proposition 2.4. *This is equivalent to definition 2.2*

Proof.

Sublemma 2.4.1. *$f \in \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$ iff $f \in \mathcal{G}_{\gneq 0}$ or there is some $h \in \text{posi}(\{g_1, \ldots, g_n\})$ with $f \geqslant h$.*

Proof. The left-to-right direction is sublemma 2.1.1. The right-to-left can be easily checked todo . □

Thus, definition 2.2 is equivalent to:

When $\mathcal{E} \neq \varnothing$:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_1, \ldots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_1,\ldots,g_n \rangle} \in B$ where
  - either $f_{\langle g_1,\ldots,g_n \rangle} \in \mathcal{G}_{\gneq 0}$
  - or there is some $h_{\langle g_1,\ldots,g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\})$ with $f_{\langle g_1,\ldots,g_n \rangle} \geqslant h_{\langle g_1,\ldots,g_n \rangle}$.

When $\mathcal{E} = \varnothing$,

- $B \in \text{Kext}(\varnothing)$ iff there is some $f \in B$ where also $f \in \mathcal{G}_{\gneq 0}$.

The first condition of the $\mathcal{E} \neq \varnothing$ case and the $\mathcal{E} = \varnothing$ case can be combined into the single condition that $B \cap \mathcal{G}_{\gneq 0} \neq \varnothing$. We are then left with the alternative condition that

> There are some $A_1, \ldots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, there is some $f_{\langle g_1,\ldots,g_n \rangle} \in B$ and $h_{\langle g_1,\ldots,g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\})$ where $f_{\langle g_1,\ldots,g_n \rangle} \geqslant h_{\langle g_1,\ldots,g_n \rangle}$.

So we see that the two formulations are equivalent. □

Note that it is important in doing this that the additional clause about $f \in \mathcal{G}_{\gneq 0}$ is added. That is, we cannot define $\text{Kext}(\mathcal{E})$ by simply taking our definition (definition 2.2) and replacing the existence of $f \in B \cap \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$ with the existence of $f \in B$ and $h \in \text{posi}(\{g_1, \ldots, g_n\})$ with $f \geqslant h$. [However, interestingly if we move to strict dominance in our criteria that would be possible.]

2.4 Relationship to De Bock and de Cooman (2018)

Their formulation looks a bit different. Since they both characterise natural extensions, they'll be equivalent, but in this section we consider the difference in more detail.

They:

(i) Add the singletons from $\mathcal{G}_{\gneq 0}$ to $\mathcal{E}$. This gives us $\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\gneq 0}\}$.

(ii) We add sets using posi from these sets. More carefully: For any $C_1, \ldots, C_n$ in $\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\gneq 0}\}$ (from (i)), if for each sequence $\langle g_i \rangle \in \bigtimes C_i$, $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \ldots, g_n\})$, then we add $\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \bigtimes C_i\}$. Our resultant set is called $\text{Posi}(\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\gneq 0}\})$.

(iii) Add any sets obtained by removing some gambles which are $\leqslant 0$.

(iv) Add any supersets thereof.

This gives us the final set, called: $\text{Rs}(\text{Posi}(\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\gneq 0}\}))$. They show that this is the natural extension in the finite setting. So in this setting, we know that it will be equivalent to ours, however it's open whether it gives the natural extension in the infinite setting.

It's got the same sort of moving parts as our definition: there's addition of some $\mathcal{G}_{\gneq 0}$ (in (i)), there's taking posi's (in (ii)), there's removing some obviously-bad stuff (in (iii)) and taking supersets (in (iv)). These are the same sorts of moving parts that I have.

Let's try to rework their formulation to present it in a way that is more similar to our presentation so we can evaluate how/if they are different. Firstly, we can summarise the construction to:

- $B \in \text{Kext}_{\text{DBdC}}(\mathcal{E})$ iff there are some $C_1, \ldots, C_n \in \mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\gneq 0}\}$ with $\langle g_1, \ldots, g_n \rangle \in C_1 \times \ldots \times C_n$, there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B \cup \mathcal{G}_{\leqslant 0}$ where also $f_{\langle g_1, \ldots, g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\})$.

We can then split the choice of $C_1, \ldots, C_n$ from $\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\gneq 0}\}$ into a choice separately of $A_1, \ldots, A_n \in \mathcal{E}$, and $p_1, \ldots, p_m \in \mathcal{G}_{\gneq 0}$. Sometimes no members of $\mathcal{E}$ are chosen, which should be presented as a separate clause. So we see that:

- $B \in \text{Kext}_{\text{DBdC}}(\mathcal{E})$ iff either:
  - There are some $A_1, \ldots, A_n \in \mathcal{E}$ and $p_1, \ldots, p_m \in \mathcal{G}_{\gneq 0}$ such that for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B \cup \mathcal{G}_{\leqslant 0}$ where also $f_{\langle g_1, \ldots, g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n, p_1, \ldots, p_m\})$.
  - There are some $p_1, \ldots, p_m \in \mathcal{G}_{\gneq 0}$ such that there is some $f \in B \cup \mathcal{G}_{\leqslant 0}$ where also $f \in \text{posi}(\{p_1, \ldots, p_m\})$.

Note that this is different to our:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_1, \ldots, A_n \in \mathcal{E}$ and for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B \cup \{0\}$ where also $f_{\langle g_1, \ldots, g_n \rangle} \in \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$.

Some of the differences between these are simply choices of presentation.
However, substantial differences are:

- DBdC allow the choice of the sequence $C_1, \dots, C_n$ to contain no members of $\mathcal{E}$ whatsoever. This allows their definition to apply properly to the case of $\mathcal{E} = \varnothing$, whereas we need a separate definition for that case.
- Their Posi is defined working with finitely many $C_1, \dots, C_n$, which means in any use of it we can only use finitely many members of $\mathcal{G}_{\gneq 0}$. Is that enough? It's closely related to my addition of item $\mathrm{K}_{\mathrm{Dom}}$.

  For example, suppose $A \in \mathcal{E}$ and we have for each g in A a distinct $p_g \in \mathcal{G}_{\gneq 0}$, we want to ensure that $\{g + p_g \mid g \in A\} \in \mathrm{Kext}(\mathcal{E})$.

  This doesn't immediately follow from the construction of Posi because it involves infinitely many sets: A plus the infinitely many singletons $\{p_g\}$.

  In fact, like the dominance axiom, when Ω is finite it'll be fine because $g + p_g \in \mathrm{posi}(\{g, I_{\omega_1}, \dots, I_{\omega_n}\})$ so we can just work with the finitely many sets $A, \{I_{\omega_1}\}, \dots, \{I_{\omega_n}\}$. But perhaps when Ω is infinite it'll be different.

2.5 Does DBdC get infinite addition axiom?

Should DBdC extend the construction to define Posi that it directly works with infinitely many sets?

2.6 Extending

2.6.1 Regularity

We can replace $\mathcal{G}_{\gneq 0}$ by $\mathcal{G}_{>0}$ and everything still works. Theorem 2.1 then becomes simpler because any $f_{\langle g_i \rangle} \in \mathrm{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{>0})$, has some $h_{\langle g_i \rangle} \in \mathrm{posi}(\{g_1, \dots, g_n\})$ with $f_{\langle g_i \rangle} \geqslant h_{\langle g_i \rangle}$ so we don't need a separate clause.

In fact this shows us that in the strict-dominance setting, axiom $\mathrm{K}_{\gneq 0}$ is not required as a separate axiom, though one should then include a non-trivial axiom that $\mathcal{K}$ is non-empty.

2.6.2 Extending to infinite addition

If we extend to:

($\mathrm{K}_{\mathrm{AddInf}}$) If $A_i \in \mathcal{K}$ for each $i \in I$ (possibly infinite) and for each sequence $\langle g_i \rangle_{i \in I}$ with each $g_i \in A_i$, $f_{\langle g_i \rangle}$ is some member of $\mathrm{posi}(\{g_i \mid i \in I\})$, then

$$\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \bigtimes_{i \in I} A_i\} \in \mathcal{K}$$

we obtain a strictly stronger system. Campbell-Moore (forthcoming, §4.4.1). Then the results immediately carry through now working with:

- $B \in \mathrm{Kext}(\mathcal{E})$ iff there are some $A_i \in \mathcal{K}$ (a possible infinite collection) where for each sequence $\langle g_i \rangle \in \bigtimes_i A_i$, there is some $f_{\langle g_i \rangle} \in B \cup \{0\}$ with $f_{\langle g_i \rangle} \in \mathrm{posi}(\{g_i \mid i \in I\} \cup \mathcal{G}_{\gneq 0})$.

3 Axioms when gamble sets must be finite

This section shows that in the special case when gamble sets are finite, we can use the axioms from De Bock and de Cooman (2018)

Definition 3.1. $\mathcal{K} \subseteq \wp^{\text{finite}}(\mathcal{G})$ is *coherent* if it satisfies $\mathrm{K}_\varnothing$ K_0 $\mathrm{K}_{\gneq 0}$ $\mathrm{K}_\supseteq$ and:

($\mathrm{K}_{\mathrm{AddPair}}$) If $A_1, A_2 \in \mathcal{K}$ and for each pair $g_1 \in A_1$ and $g_2 \in A_2$, $f_{\langle g_1, g_2 \rangle}$ is some member of $\mathrm{posi}(\{g_1, g_2\})$, then

$$\{f_{\langle g_1, g_2 \rangle} \mid g_1 \in A_1, g_2 \in A_2\} \in \mathcal{K}$$

These are equivalent to our earlier axioms when we are restricted to finite sets. The proof of this is contained in the next two sections.

3.1 Finite addition axiom - axiom $\mathrm{K}_{\mathrm{Add}}$

Proposition 3.2. [1]*Assume we are restricted to finite sets. Axiom* $\mathrm{K}_{\mathrm{Add}}$ *follows from axiom* $\mathrm{K}_{\mathrm{AddPair}}$ *and axiom* $\mathrm{K}_\supseteq$.

Proof. We in fact prove:

- If $A_1, \ldots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \ldots, g_n \rangle \in A_1 \times \ldots \times A_n$, there is some $f_{\langle g_1, \ldots, g_n \rangle} \in B \cap \mathrm{posi}(\{g_1, \ldots, g_n\}$;
  Then $B \in \mathcal{K}$.

which entails axiom $\mathrm{K}_{\mathrm{Add}}$. (It essentially just adds any supersets. This allows us to avoid some fiddliness in the proof.)

We work by induction on n.

Base case: For $n = 1$ use axiom $\mathrm{K}_{\mathrm{AddPair}}$ and axiom $\mathrm{K}_\supseteq$, letting $A_1 = A_2$.

Inductive step: Assume it holds for n and consider $n + 1$.

So, suppose $A_1, \ldots, A_n, A_{n+1} \in \mathcal{K}$ and for each sequence $\langle g_i \rangle \in \bigtimes_i A_i$ and $a \in A_{n+1}$, there is some $f_{\langle g_1, \ldots, g_n, a \rangle} \in B \cap \mathrm{posi}(\{g_1, \ldots, g_n, a\})$. We need to show that $B \in \mathcal{K}$.

We will abuse notation and write $f_{\langle g_i, a \rangle}$ instead of $f_{\langle g_1, \ldots, g_n, a \rangle}$.

A_{n+1} is finite, so enumerate it, $A_{n+1} = \{a_1, \ldots, a_K\}$. Define:

$$B_k := \left\{ f_{\langle g_i, a \rangle} \,\middle|\, \langle g_i \rangle \in \bigtimes_{i \leqslant n} A_i,\ a \in \{a_1, \ldots, a_k\} \right\} \cup \{a_{k+1}, \ldots, a_K\} \tag{3}$$

Where we have already replaced k-many of A_{n+1} with the relevant members of B. We show by a induction on k that $B_k \in \mathcal{K}$. Since $B_K \subseteq B$, this will suffice.[2]

Base case: $B_0 = A_{n+1} \in \mathcal{K}$.

Inductive step: we can assume that $B_k \in \mathcal{K}$ and need to show $B_{k+1} \in \mathcal{K}$.

We will construct a set C which we know to be in $\mathcal{K}$, which we can then combine with B_k using axiom $\mathrm{K}_{\mathrm{AddPair}}$ (and axiom $\mathrm{K}_\supseteq$) to get $B_{k+1} \in \mathcal{K}$.

[1]Thanks to Arthur van Camp for a key insight, and Jasper De Bock and Arne Decadt for discussion.

[2]Arthur van Camp's insight that this can be done in such a step-by-step way to just use axiom $\mathrm{K}_{\mathrm{AddPair}}$ when the sets are finite, i.e., it can be shown by a sub-induction.

By assumption, $f_{\langle g_i, a_{k+1}\rangle} \in \text{posi}(\{g_1, \ldots, g_n, a_{k+1}\})$. So there is some $h_{\langle g_i, a_{k+1}\rangle}$ where:

$$h_{\langle g_i, a_{k+1}\rangle} \in \text{posi}(\{g_1, \ldots, g_n\}), \text{ and} \tag{4}$$

$$f_{\langle g_i, a_{k+1}\rangle} \in \text{posi}(\{a_{k+1}, h_{\langle g_i, a_{k+1}\rangle}\}). \tag{5}$$

To see this: $f_{\langle g_i, a_{k+1}\rangle}$ has the form $\sum_{i \leqslant n} \lambda_i g_i + \mu a_{k+1}$. If $\lambda_i > 0$ for some i, then we put $h_{\langle g_i, a_{k+1}\rangle} := \sum_{i \leqslant n} \lambda_i g_i$. If not, then we simply let it be, for example, g_1. This is as required.

We let

$$C := \{h_{\langle g_i, a_{k+1}\rangle} \mid \langle g_i\rangle \in \bigtimes_{i \leqslant n} A_i\} \tag{6}$$

$C \in \mathcal{K}$ by the induction hypothesis (on n), using eq. (4).

We will show that for every $\langle c, b\rangle \in B_k \times C$, there is some $d \in \text{posi}(\{c, b\}) \cap B_{k+1}$; which will then allow us to use axiom $\mathrm{K}_{\mathrm{AddPair}}$ (and $\mathrm{K}_{\supseteq}$) to get that $B_{k+1} \in \mathcal{K}$.

If $c \in B_k \cap B_{k+1}$, we just put $d = c$. The only remaining member of B_k to consider is $c = a_{k+1}$. For any $b \in C$, $b = h_{\langle g_i, a_{k+1}\rangle}$ for some $\langle g_i\rangle \in \bigtimes_i A_i$. By definition of this (eq. (5)), $f_{\langle g_i, a_{k+1}\rangle} \in \text{posi}(\{a_{k+1}, b\})$. And by looking at the definition of B_{k+1} (eq. (3)), also $f_{\langle g_i, a_{k+1}\rangle} \in B_{k+1}$. So $f_{\langle g_i, a_{k+1}\rangle}$ is the required $d \in \text{posi}(\{c, b\}) \cap B_{k+1}$.

So, by axiom $\mathrm{K}_{\mathrm{AddPair}}$ and axiom $\mathrm{K}_{\supseteq}$, $B_{k+1} \in \mathcal{K}$, completing the inductive step.

This has shown by induction that $B_K \in \mathcal{K}$. $B_K \subseteq B$, so $B \in \mathcal{K}$ (using $\mathrm{K}_{\supseteq}$).

This completes the inductive step for our initial induction (on n). Thus we have shown that our initial statement holds for all n. And since it entails axiom $\mathrm{K}_{\mathrm{Add}}$, we have shown that this follows. □

In fact, one can do it without axiom $\mathrm{K}_{\supseteq}$; it just becomes even more fiddly. See appendix A for the proof.

3.2 Dominators axiom - axiom $\mathrm{K}_{\mathrm{Dom}}$

Proposition 3.3. *Assume we are restricted to finite sets. Axiom* $\mathrm{K}_{\mathrm{Dom}}$ *follows from axiom* $\mathrm{K}_{\mathrm{Add}}$ *and* $\mathrm{K}_{\gneq 0}$.

The proof is closely related to De Bock and de Cooman (2019, Lemma 34). There they show that when one replaces a single member of a desirable gamble set by a dominator it remains desirable. We could directly use this and iterate it (finitely many times) to show our result (to write this down carefully, we'd have to do a proof by induction). But it is slightly cleaner to directly rely on our already proved axiom $\mathrm{K}_{\mathrm{Add}}$ and do the replacements simultaneously.

Proof. Suppose $B = \{f_g \mid g \in A\}$, where $f_g \geqslant g$.

Now, $h_g := f_g - g \in \mathcal{G}_{\geqslant 0}$.

If $h_g \neq 0$, the singleton $\{h_g\} \in \mathcal{K}$.

So we have $A \in \mathcal{K}$ and each singleton $\{h_g\} \in \mathcal{K}$ whenever $f_g \gneq g$.

Since A is finite, this is finitely many sets.

For any $g^* \in A$, $f_{g^*} \in \text{posi}(\{g^*\} \cup \{h_g \mid g \in A\})$.

So by axiom $\mathrm{K}_{\mathrm{Add}}$, $B \in \mathcal{K}$. □

We can similarly see:

Proposition 3.4. *Axiom* $\mathrm{K}_{\mathrm{Dom}}$ *follows from axiom* $\mathrm{K}_{\mathrm{AddInf}}$ *and axiom* $\mathrm{K}_{\gneq 0}$ *even when sets can be infinite.*

4 Representation with desirable gambles

There is a representation result in the finite setting which says that every coherent $\mathcal{K}$ can be captured by a set of coherent desirable gambles. We have to generalise this result in the infinite setting if we want to avoid item $\mathrm{K}_{\mathrm{AddInf}}$ and merely impose item $\mathrm{K}_{\mathrm{Add}}$.

4.1 Sets of coherent desirable gamble sets

If we have a set of desirable gambles, D, we can extract a set of desirable gamble sets: D evaluates a gamble set B as desirable when it thinks that some member of the set is desirable

$$B \in \mathcal{K}_D \text{ iff there is some } g \in B \text{ with } g \in D \tag{7}$$

$$\text{iff } B \cap D \neq \varnothing \tag{8}$$

This results in very special kinds of sets of desirable gamble sets. We can get more if we instead look at generating a set of desirable gamble sets by using a *set* ($\mathbb{D}$) of sets of desirable gambles (Ds).

If we have a set of coherent D, $\mathbb{D}$, we can consider evaluations of B as a desirable gamble set if every $D \in \mathbb{D}$ evaluates it as desirable. That is:

$$B \in \mathcal{K}_{\mathbb{D}} \text{ iff } B \in \mathcal{K}_D \text{ for each } D \in \mathbb{D} \tag{9}$$

$$\text{iff for each } D \in \mathbb{D} \text{ there is some } g \in B \text{ with } g \in D \tag{10}$$

$$\text{iff for each } D \in \mathbb{D} \text{ there is some } g \in B \cap D \tag{11}$$

More concisely, $\mathcal{K}_{\mathbb{D}} := \bigcap_{D \in \mathbb{D}} \mathcal{K}_D$.

When $\mathbb{D}$ is a nonempty set of coherent D, $\mathcal{K}_{\mathbb{D}}$ is coherent. But it also satisfies axiom $\mathrm{K}_{\mathrm{AddInf}}$.

Proposition 4.1. *Let* $\mathbb{D}$ *be a nonempty set of coherent* D*.* $\mathcal{K}_{\mathbb{D}}$ *satisfies* K_{infadd}*.*

Proof. Let $A_i \in \mathcal{K}_{\mathbb{D}}$ with $f_{\langle g_i \rangle} \in \mathrm{posi}(\{g_i \mid i \in I\})$ with $B = \{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \bigtimes A_i\}$.

So for each $D \in \mathbb{D}$, $D \cap A_i \neq \varnothing$.

Fix any $D \in \mathbb{D}$. Take g_i to be some member of $A_i \cap D$. Now, $f_{\langle g_i \rangle} \in \mathrm{posi}(\{g_i \mid i \in I\})$, so since each $g_i \in D$, also $f_{\langle g_i \rangle} \in D$ by coherence of D. Thus $B \cap D \neq \varnothing$.

So, for any $D \in \mathbb{D}$, $B \cap D \neq \varnothing$. Thus, $B \in \mathcal{K}_{\mathbb{D}}$. ☐

Note that, as we show in Campbell-Moore (forthcoming, §4.4.1), some coherent $\mathcal{K}$ fail axiom $\mathrm{K}_{\mathrm{AddInf}}$ So these are still special kinds of coherent sets of desirable gamble sets.

In **??** we'll show that such $\mathcal{K}_{\mathbb{D}}$ are exactly the $\mathcal{K}$ that satisfy axiom item $\mathrm{K}_{\mathrm{AddInf}}$, but for now we move to consider the more general question:

What about when we allow for mere finite addition? Can we generate *all* coherent $\mathcal{K}$ in an analogous way? Answer: yes!...

4.2 Representation of all coherent $\mathcal{K}$ using Ds

Consider $\mathfrak{D}$ a set of $\mathbb{D}$'s. We will in fact impose a further constraint: that it is closed under finite intersection. We might more generally require it to be a filter.

We say that $\mathfrak{D}$ evaluates B as a desirable gamble set when there is at least one $\mathbb{D}$ in $\mathfrak{D}$ that evaluates it as desirable.

That is, we put:

$$\begin{aligned} B \in \mathcal{K}_{\mathfrak{D}} &\text{ iff there is some } \mathbb{D} \in \mathfrak{D} \text{ with } B \in \mathcal{K}_{\mathbb{D}} && (12)\\ &\text{iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D},\ B \in \mathcal{K}_D && (13)\\ &\text{iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D} \text{ there is some } g \in B \cap D && (14)\end{aligned}$$

Or, more concisely,

$$\mathcal{K}_{\mathfrak{D}} := \bigcup_{\mathbb{D}\in\mathfrak{D}} \bigcap_{D\in\mathbb{D}} \mathcal{K}_D \tag{15}$$

Just to explicitly say: we don't need to find a single $g \in B$ which is in all $D \in \mathbb{D}$, but each $D \in \mathbb{D}$ can have a different $g \in B \cap D$.

Whereas we went from D to $\mathbb{D}$ using universal quantification, we go from $\mathbb{D}$ to $\mathfrak{D}$ using existential quantification.

$\mathfrak{D}$ contains the $\mathbb{D}$ that one is happy to evaluate gamble sets with respect to. This is different from a $\mathbb{D}$ which contains the D that one thinks are still open. One is not sure about which $D \in \mathbb{D}$ is "right" so will only make judgements when agreed on by all $D \in \mathbb{D}$. But each $\mathbb{D} \in \mathfrak{D}$ is in some sense good and can be trusted to make decisions.

We require that $\mathfrak{D}$ be a (proper) filter of coherent Ds.

Definition 4.2. More precisely: $\mathfrak{D} \subseteq \wp(\{D \subseteq \mathcal{G} \mid D \text{ is coherent}\})$ st:

- $\mathfrak{D} \neq \varnothing$
- $\varnothing \notin \mathfrak{D}$
- Each $D \in \mathbb{D} \in \mathfrak{D}$ is coherent (definition 1.2)
- If $\mathbb{D}_1, \mathbb{D}_2 \in \mathfrak{D}$, then $\mathbb{D}_1 \cap \mathbb{D}_2 \in \mathfrak{D}$.
- If $\mathbb{D} \in \mathfrak{D}$ and $\mathbb{D}' \supseteq \mathbb{D}$ then $\mathbb{D}' \in \mathfrak{D}$.

We can think of $\mathfrak{D}$ as containing all one's judgements about what the "coherent" D is like. We can more simply put:

$$\begin{aligned} B \in \mathcal{K}_{\mathfrak{D}} &\text{ iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D} \text{ there is some } g \in B \cap D && (16)\\ &\text{iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st } \mathbb{D} \subseteq \{D \mid \text{there is some } g \in B \cap D\} && (17)\\ &\text{iff } \{D \mid \text{there is some } g \in B \cap D\} \in \mathfrak{D} && (18)\end{aligned}$$

Remark. In fact, our requirement that it be closed under supersets is needless for representing the $\mathcal{K}$. I do so anyway because ???

Note, if we didn't close under supersets, we do have to use eq. (14) rather than eq. (18)

Remark. Another way of going is to not require that it is closed under finite intersection, but instead to bake that into the definition, defining:

$$A \in \mathcal{K}_{\mathfrak{D}} \text{ iff there are some } \mathbb{D}_1, \ldots, \mathbb{D}_n \in \mathfrak{D} \text{ with } A \in \mathcal{K}_{\mathbb{D}_i} \text{ for each } i \tag{19}$$

$$\text{iff there is some } \mathbb{D}_1, \ldots, \mathbb{D}_n \in \mathfrak{D} \text{ st for all } D \in \mathbb{D}_1 \cap \ldots \cap \mathbb{D}_n \text{ there is some } g \in A \cap D \tag{20}$$

For this, we need to impose the "coherence" requirements on this that it has the finite intersection property: that we never have $\mathbb{D}_1 \cap \ldots \cap \mathbb{D}_n = \varnothing$. It is then equivalent. But this makes the proof a bit more cumbersome, so we stick with the first way.

We can now give our key representation result:

Theorem 4.3. *$\mathcal{K}$ is coherent iff there is some $\mathfrak{D}$ a proper filter of coherent Ds (definition 4.2) where $\mathcal{K} = \mathcal{K}_{\mathfrak{D}}$.*

Warning, the proof is sketchy! I think it works, though.

Proof. Suppose $\mathcal{K}$ is coherent.

Let $\mathfrak{D}$ be $\mathbb{D} \in \mathfrak{D}_{\mathcal{K}}$ iff $\mathbb{D} \supseteq \mathbb{D}_{\{A_1,\ldots,A_n\}}$ for some $A_1, \ldots, A_n \in \mathcal{K}$; where

$$\mathbb{D}_{\{A_1,\ldots,A_n\}} := \left\{ D \;\middle|\; \begin{array}{l} \text{there is } \langle g_i \rangle \in \bigtimes A_i \text{ with :} \\ 0 \notin \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0}) \\ D \supseteq \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0}) \\ D \text{ coherent} \end{array} \right\} \tag{21}$$

Sublemma 4.3.1. *$\mathfrak{D}$ is coherent.*

Proof. First note that each $\mathbb{D}_{\{A_1,\ldots,A_n\}}$ is non-empty. If it is empty, then this is because we have $A_1, \ldots, A_n \in \mathcal{K}$ where for each $\langle g_i \rangle \in \bigtimes A_i$, $0 \in \text{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$. But then by theorem 2.1, $\{0\} \in \mathcal{K}$, contradicting the coherence of $\mathcal{K}$.

It then suffices to show that

$$\mathbb{D}_{\{A_i^j | i,j\}} \subseteq \mathbb{D}_{\{A_i^1 | i\}} \cap \ldots \cap \mathbb{D}_{\{A_i^m | i\}} \tag{22}$$

Suppose $D \in \mathbb{D}_{\{A_i^j | i,j\}}$.

So there is $\{g_i^j \mid i,j\} \in \bigtimes_j \bigtimes_i A_i^j$ with

(i) $0 \notin \text{posi}(\{g_i^j \mid i,j\} \cup \mathcal{G}_{\gneq 0})$,

(ii) $D \supseteq \text{posi}(\{g_i^j \mid i,j\} \cup \mathcal{G}_{\gneq 0})$,

(iii) D coherent.

Consider any $k = 1, \ldots, m$

Observe that $\text{posi}(\{g_i^j \mid i,j\} \cup \mathcal{G}_{\gneq 0}) \supseteq \text{posi}(\{g_i^k \mid i\} \cup \mathcal{G}_{\gneq 0})$

So, we have $\{g_i^k \mid i,j\} \in \bigtimes_i A_i^k$ with:

(i) $0 \notin \text{posi}(\{g_i^k \mid i\} \cup \mathcal{G}_{\gneq 0})$, and

(ii) $D \supseteq \text{posi}(\{g_i^k \mid i\} \cup \mathcal{G}_{\gtrsim 0})$

(iii) D coherent

Thus $D \in \mathbb{D}_{\{A_i^k|i\}}$.

As required.

□

Sublemma 4.3.2. $\mathcal{K} = \mathcal{K}_{\mathfrak{D}}$.

Proof.

$$\begin{aligned}
&B \in \mathcal{K}_{\mathfrak{D}} && (23)\\
&\text{iff there is some } \mathbb{D} \in \mathfrak{D}_{\mathcal{K}} \text{ st for all } D \in \mathbb{D},\ B \cap D \neq \varnothing && (24)\\
&\text{iff there is some } A_1, \dots, A_n \in \mathcal{K} \text{ st for } \mathbb{D} \supseteq \mathbb{D}_{\{A_1,\dots,A_n\}}, \text{ and for all } D \in \mathbb{D},\ B \cap D \neq \varnothing && (25)\\
&\text{iff there is some } A_1, \dots, A_n \in \mathcal{K} \text{ st for all } D \in \mathbb{D}_{\{A_1,\dots,A_n\}},\ B \cap D \neq \varnothing && (26)\\
&\text{iff } \begin{array}{l}\text{there is some } A_1, \dots, A_n \in \mathcal{K} \text{ st for all } \langle g_i \rangle \in \bigtimes A_i,\\ \quad B \cap \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0}) \neq \varnothing \text{ or } 0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0})\end{array} && (27)
\end{aligned}$$

By theorem 2.1, we know that any B satisfying eq. (27) is in $\mathcal{K}$. It is also easy to see that any $B \in \mathcal{K}$ satisfies eq. (27). So we have that $\mathcal{K}_{\mathfrak{D}} = \mathcal{K}$. □

Sublemma 4.3.3. *When $\mathfrak{D}$ is a filter of coherent D, then $\mathcal{K}_{\mathfrak{D}}$ is coherent*

Proof. Check each of the axioms.

Axiom $\text{K}_{\varnothing}$... by non-empty stuff.

Axiom K_0, $\text{K}_{\gtrsim 0}$, $\text{K}_{\supseteq}$ and K_{Dom} should follow because each $\mathcal{K}_D$ satisfy them. If $A \in \mathcal{K}_{\mathfrak{D}}$, then there is some $\mathbb{D}$ with $A \in \mathcal{K}_{\mathbb{D}}$. And thus $A \in \mathcal{K}_D$ for each $D \in \mathbb{D}$. That is, there is some $g \in A \cap D$. But therefore $f_g \in D$ by properties of coherent D, thus $B \in \mathcal{K}_D$ for each $D \in \mathbb{D}$. Thus $B \in \mathcal{K}_{\mathbb{D}}$, and so $B \in \mathcal{K}_{\mathfrak{D}}$.

For axiom K_{Add}: Let $A_1, \dots, A_n \in \mathcal{K}_{\mathfrak{D}}$. So we have some $\mathbb{D}_1, \dots, \mathbb{D}_n \in \mathfrak{D}$ st for all $D \in \mathbb{D}_i$, $A_i \cap D \neq \varnothing$. Consider $\mathbb{D} = \bigcap_{i \leqslant n} \mathbb{D}_i$. So for all $D \in \mathbb{D}$, $A_i \cap D \neq \varnothing$. So one can then check that for B the relevant cross-thing, $B \cap D \neq \varnothing$. □

This proves our result. □

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A Finite addition proof without supersets

Proposition A.1. *Assume we are restricted to finite sets. Axiom* $\mathrm{K}_{\mathrm{Add}}$ *follows from axiom* $\mathrm{K}_{\mathrm{AddPair}}$.

Proof. The argument gets slightly fiddly because members may coincide and repetitions get "deleted" by sets. To avoid this issue, we need to include constants throughout to ensure distinctness.

We work by induction on n.

Base case: $n = 1$ follows directly from axiom $\mathrm{K}_{\mathrm{AddPair}}$ by putting $A_1 = A_2$.

Inductive step: Assume it holds for n and consider $n + 1$.

So, suppose $A_1, \ldots, A_n, A_{n+1} \in \mathcal{K}$ and for each sequence $\langle g_i \rangle \in \bigtimes_i A_i$ and $a \in A_{n+1}$, we have $f_{\langle g_1,\ldots,g_n,a\rangle} \in \mathrm{posi}(\{g_1, \ldots, g_n, a\})$ with

$$B = \{f_{\langle g_1,\ldots,g_n,a\rangle} \mid \langle g_i \rangle \in \bigtimes_{i \leqslant n} A_i,\ a \in A_{n+1}\} \tag{28}$$

We need to show that $B \in \mathcal{K}$.

We will abuse notation and write $f_{\langle g_i,a\rangle}$ instead of $f_{\langle g_1,\ldots,g_n,a\rangle}$.

Consider adding some scalar constants: let

$$f^*_{\langle g_i,a\rangle} := \delta_{\langle g_i,a\rangle} \times f_{\langle g_i,a\rangle} \tag{29}$$

where $\delta_{\langle g_i,a\rangle}$ are chosen to ensure that these are all distinct, and distinct from any a'. We will show that

$$B^* = \{f^*_{\langle g_i,a\rangle} \mid \langle g_i \rangle \in \bigtimes_i A_i\} \tag{30}$$

A_{n+1} is finite, so enumerate it, $A_{n+1} = \{a_1, \ldots, a_K\}$. Define:

$$B^*_k := \{f^*_{\langle g_i,a_j\rangle} \mid \langle g_i \rangle \in \bigtimes_{i \leqslant n} A_i,\ a_j \in \{a_1, \ldots, a_k\}\} \cup \{a_{k+1}, \ldots, a_K\} \tag{31}$$

Where we have already replaced k-many of A_{n+1} with the relevant members of B^*. We show by a induction on k that $B^*_k \in \mathcal{K}$.[3]

Base case: $B_0 = A_{n+1} \in \mathcal{K}$.

Inductive step: we can assume that $B_k \in \mathcal{K}$ and need to show $B_{k+1} \in \mathcal{K}$.

We will construct a set C which we know to be in $\mathcal{K}$, which we can then combine with B_k using axiom $\mathrm{K}_{\mathrm{AddPair}}$ (and axiom $\mathrm{K}_{\supseteq}$) to get $B^*_{k+1} \in \mathcal{K}$.

Sublemma A.1.1. *We can find distinct* $h^*_{\langle g_i,a_{k+1}\rangle}$ *where:*

$$h^*_{\langle g_i,a_{k+1}\rangle} \in \mathrm{posi}(\{g_1, \ldots, g_n\}),\ \textit{and} \tag{32}$$

$$f^*_{\langle g_i,a_{k+1}\rangle} \in \mathrm{posi}(\{h^*_{\langle g_i,a_{k+1}\rangle}, a_{k+1}\}). \tag{33}$$

[3] Arthur van Camp's insight that this can be done in such a step-by-step way to just use axiom $\mathrm{K}_{\mathrm{AddPair}}$, i.e., it can be shown by a sub-induction.

Proof. $f_{\langle g_i, a_{k+1}\rangle}$ has the form $\sum_{i \leqslant n} \lambda_i g_i + \mu a_{k+1}$. If $\lambda_i > 0$ for some i, then we put $h_{\langle g_i, a_{k+1}\rangle} := \sum_{i \leqslant n} \lambda_i g_i$. If not, then we simply let it be, for example, g_1.

To ensure they are distinct, we simply multiply the h by appropriate constants to find h^*. □

We let

$$C := \{h^*_{\langle g_i, a_{k+1}\rangle} \mid \langle g_i\rangle \in \bigtimes_i A_i\} \tag{34}$$

$C \in \mathcal{K}$ by the induction hypothesis (on n), using eq. (32).

Now, for each $\langle c, b\rangle$, let

$$e_{\langle c,b\rangle} := \begin{cases} c & b \in B^*_k \cap B^*_{k+1} \\ f^*_{\langle g_i, a_{k+1}\rangle} & c = a_{k+1} \text{ and } b = h^*_{\langle g_i, a_{k+1}\rangle} \end{cases} \tag{35}$$

Since each $e_{\langle c,b\rangle} \in \mathrm{posi}(\{c, b\})$, by axiom $\mathrm{K}_{\mathrm{AddPair}}$,

$$\{e_{\langle c,b\rangle} \mid c \in B_k,\, b \in C\} \in \mathcal{K} \tag{36}$$

Sublemma A.1.2.

$$B^*_{k+1} = \{e_{\langle c,b\rangle} \mid c \in B_k,\, b \in C\} \tag{37}$$

Proof. Can easily be checked. □

So, by axiom $\mathrm{K}_{\mathrm{AddPair}}$ and axiom $\mathrm{K}_{\supseteq}$, $B^*_{k+1} \in \mathcal{K}$, completing the inductive step.

This has shown by induction that $B^*_K \in \mathcal{K}$. $B^*_K = B^*$, so $B^* \in \mathcal{K}$.

We need to show that $B \in \mathcal{K}$. This works because it is obtained from B^* simply by multiplying everything by a scalar, so follows from the $n = 1$ version of axiom $\mathrm{K}_{\mathrm{Add}}$, which itself directly follows from axiom $\mathrm{K}_{\mathrm{AddPair}}$.

This shows that the statement holds for $n + 1$, completing the inductive step. Thus we have shown that it holds for all n; as required. □