Accuracy and Immodesty for the Imprecise Probabilist

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Abstract

In the imprecise probabilities framework we represent one's belief state as a set of precise probabilities. This paper considers the following rather natural principle to tell us about what imprecise options an imprecise probability recommends:

IMPRECISE RECOMMENDATION.
What a set recommends is the set of what the individuals recommend.

We will show that this principle has a number of desirable consequences for the imprecise probabilist once we tie it with accuracy-theoretic considerations. It allows for a picture of accuracy for the imprecise where any imprecise probability recommends itself, and we can obtain arguments for various principles of rationality such as Imprecise Probabilism and Imprecise Conditionalization. It can also allow for some rationally permissible options in certain scenarios which have formed a challenge case for accuracy-theory, or more generally for rationality considerations. In such scenarios, which bear a close relationship to the liar paradox, any precise probability undermines itself, but by using Imprecise Recommendation, we see that imprecise probabilities can be self-recommending, and thus candidates for rationally permissible options. We also give some kind of justification to Imprecise Recommendation by providing a story thinking about the imprecise credence as a group, or credal-committee, where each committee member (precise probability) has opinions about what is better or worse, and they come together to make a decision on which groups are better and worse.

1 Accuracy and rationality for the imprecise

In the imprecise probabilities framework we represent one's belief state as a *set* of precise probabilities.[1] This framework has proved very powerful extension of the Bayesian framework, but there are various key principles which are used by the imprecise probabilist which could do with more support. For example:

*Thanks to lots of people!

[1]See, e.g. Mahtani (2019) for an introduction to imprecise probabilities.

Imprecise-Probabilism.
One's credal set, $\mathbb{B}$, should only contains probability functions.

I.e. every member of my credal committee should satisfy the axioms of probability. And:

Imprecise-Conditionalization.
After learning E, one's updated belief state should be the set of conditionalized precise probabilities:

$$\mathbb{B}_E = \{b(-|E) \mid b \in \mathbb{B}\}.$$

These principles are often assumed by the imprecise probability theorist.

An influential method of justifying rationality constraints is the accuracy framework.[2] This justifies various rationality principles by their success at pursuing matters of epistemic value, which, specifically, is taken as their success at giving credences which are "close to the truth". This framework has been very powerful in the precise setting, for example there are dominance arguments for an agent's (precise) degrees of beliefs satisfying the axioms of probability: non-probability functions are irrational because there will be some (probabilistic) opinion which is more epistemically valuable, i.e. closer to the truth, whatever the truth turns out to be; similarly there are various arguments surrounding conditionalization as the epistemically optimal update procedure.

Can the imprecise probabilist reap the fruits of this accuracy project? I suggest yes. I try to argue that the imprecise probabilist can directly help herself to the results from the precise framework and get (pretty-much immediate) arguments for her principles such as Imprecise-Probabilism and Imprecise-Conditionalization.

The suggestion I make is that we should consider the opinions of the imprecise probabilist by looking at the opinions of the individual members of the credal-committee. We measure accuracy simply of the precise opinions, then an imprecise probability has opinions about what does best accuracy-wise by supervening on the opinions of the precise members of her credal committee. In particular, the following principle allows us to directly obtain arguments for these normative requirements such as Imprecise-Probabilism and Imprecise-Conditionalization.

Imprecise Recommendation.
What a set recommends is the set of what the individuals recommend.

For our accuracy considerations we are taking 'recommend' understood as doing best at the pursuit of accuracy. This allows accuracy considerations to apply to the imprecise, and allows us to obtain desirable consequences.

This goes against some recent papers providing various impossibility results regarding the possibility of applying accuracy considerations for the imprecise probabilist.[3] Such papers have centred around the difficulty of applying accuracy

[2] Joyce (1998, 2009); Pettigrew (2016).

[3] Seidenfeld et al. (2012); Mayo-Wilson and Wheeler (2016); Schoenfield (2017); Chambers (2008). See also Konek (2019).

considerations for the imprecise probabilist in a way that obtains immodesty for the imprecise:

IMPRECISE-IMMODESTY.
Every set of probability functions $\mathbb{B}$ should take itself to be doing best accuracy-wise.

These papers have provided impossibility results for providing a way of measuring accuracy that obtain Imprecise-Immodesty. Immodesty has been an immensely important component in the accuracy literature. Formally, it is spelled out as saying that our way of measuring accuracy should be strictly proper. The accuracy literature motivates this as a requirement for an appropriate account of accuracy, and it is technically important for their results. However, the impossibility results assume substantial matters about how accuracy considerations should be developed in the imprecise setting. The way I propose to apply accuracy considerations goes against the settings of the impossibility results primarily by not providing a specific value (precise or imprecise) that measures the accuracy of an imprecise probability at a world. Instead, we have precise accuracy values for the precise members of one's credal committee, and we use Imprecise Recommendation to obtain rationality constraints for the imprecise.

1.1 The contents of the paper to come

Section 2 spells out Imprecise Recommendation in a bit more detail then shows that when we link it with accuracy-theoretic considerations, we can obtain Imprecise-Immodesty (Section 2.1), Imprecise-Probabilism (Section 2.4) and Imprecise-Conditionalization (Section 2.2). In Section 2.3 we show that Imprecise Recommendation allows for some rationally permissible options in certain scenarios which have formed a challenge case for accuracy-theory, or more generally for rationality considerations. In such scenarios, which bear a close relationship to the liar paradox, any precise probability undermines itself, but by using Imprecise Recommendation, we see that imprecise probabilities can be self-recommending, and thus candidates for rationally permissible options.

In Section 3 we try to give more detail for why the idea in Imprecise Recommendation is plausible. We provide a story thinking about the imprecise credence as a group, or credal-committee, where each committee member has a precise degree of belief and opinions about what is better or worse, and they come together to make a decision on which groups are better and worse. This section needs more work. Firstly, there's work to be done on the formal side to define how the individual's ordering of credence function is jacked up to the committee's ordering of sets in a way that is satisfactory and does the work we need it to do (Section 3.2). Secondly, more needs to be done to think about the interpretation of the imprecise probabilities that is being presupposed and which can ground the kind of proposal that is being made here (Section 3.3). But hopefully it'll give you a sense for how this might go.

In Section 4 we finish by tying some loose ends. In Section 4.1 we talk specifically about how this avoids the impossibility results in the literature. In

Section 4.2 we present some considerations regarding obtaining a dominance result for Imprecise Probabilism.

2 Imprecise Recommendation and its Accuracy-Theoretic Consequences

Imprecise Recommendation can be spelled out a bit more precisely:

IMPRECISE RECOMMENDATION.
If each precise b has a uniquely recommended object, $\mathsf{rec}(b)$, then the imprecise object that $\mathbb{B}$ recommends is $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$.

This principle allows us to move from a notion of individual-recommendation to a notion of set-recommendation.

To determine its consequences, then, we need to provide it with a notion of individual-recommendation. The principle of Imprecise Recommendation is quite general, but we'll specifically be applying it in the case where recommendation on the precise side is given by accuracy-theoretic considerations. Each of our different applications of the accuracy argument for the imprecise will involve a slightly different consideration here, so we move directly to considering the applications.

2.1 Imprecise Immodesty

In order to obtain its results, accuracy-theorists need to say more about how to measure accuracy. One of the most common measures of accuracy for precise credences is the Brier score, which gives us:[4]

$$\mathsf{Acc}(c, w) = -\sum_{A \in \mathcal{A}} (c(A) - w(A))^2;$$

We can consider how a probability function b thinks it is doing at the pursuit of accuracy compared to other functions, c. To do this we look at the expected accuracy of adopting credence c:

$$\mathrm{Exp}_b\mathsf{Acc}(c) = \sum_{w} b(w) \times \mathsf{Acc}(c, w)$$

The Brier Score is strictly-proper, meaning:

STRICT PROPRIETY.
For b probabilistic, if $c \neq b$, $\mathrm{Exp}_b\mathsf{Acc}(c) < \mathrm{Exp}_b\mathsf{Acc}(b)$.

This says that any probabilistic credence function evaluates itself to be doing better, accuracy-wise, than any particular alternative credence function. This is

[4] *In*accuracy is typically considered, measuring distance from truth. We stick to accuracy to have that optimal relates to highest value.

taken to be fundamentally important for the accuracy project. The majority of the accuracy-theoretic arguments for any strictly proper accuracy measure.[5] The strict propriety of the accuracy measure is often justified by rational immodesty: rational agents should evaluate themselves to be doing best accuracy-wise. And if we suppose that any probability function could be rational given the right circumstances; and evaluation is to be done by consideration of expected accuracy in those circumstances; then a measure of accuracy for the precise that obtains immodesty has to be strictly proper.[6]

So, if we apply our 'recommendation' considerations using expected accuracy, we can phrase this as: for each probabilistic b, $\mathsf{rec}(b) = b$, i.e. it takes itself to be uniquely recommended. We can then use Imprecise Recommendation to determine what a set of probabilities $\mathbb{B}$ takes to be optimal. Imprecise Recommendation tells us that this is $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$. But since $\mathsf{rec}(b) = b$, this simply $= \mathbb{B}$. So $\mathbb{B}$ takes itself to be optimal, so long as each $b \in \mathbb{B}$ is probabilistic. This allows us to immediately obtain Imprecise-Immodesty:

Imprecise-Immodesty.
Every set of probability functions $\mathbb{B}$ takes itself to be doing best accuracy-wise.

at least if accuracy is measured (on the precise-side) by a strictly proper scoring rule, and use Imprecise Recommendation to jack that up to how imprecise probabilities consider accuracy.

2.2 Imprecise Conditionalization

We now consider conditionalization.

There are different ways of giving accuracy arguments for this. Are we judging what credal state is best now that I know E but I haven't yet updated my credences, or what update strategy is best? Either way, the idea is: conditionalization is recommended by each probabilistic b; so an imprecise $\mathbb{B}$ will recommend conditionalizing each of its members. And this is basically what Imprecise-Conditionalization says.

2.2.1 Restricting Possibilities

The move to the imprecise setting is most natural using the approach of Leitgeb and Pettigrew (2010) for the precise notion of recommendation. A way to view their proposal maintains that the immediate, arational effect of learning E is that all $\neg E$ possibilities drop out from the agent's credal state. It is only once this has happened that the rational aspect of belief-revision can take place. The agent is left with her previous credences in the various possibilities compatible

[5] Satisfying some basic additional principles, in particular, continuity.

[6] In the presence of extensionality, we can formulate this as: for every probability space $\langle \Omega, \mathcal{F}, \Pr\rangle$, there is an isomorphic space $\langle \Omega', \mathcal{F}', \Pr'\rangle$ such that $\Pr'$ is rationally permissible (in the right circumstances). This is then compatible with further restrictions to probabilism. See Joyce (2009, Footnote 17). This is also then compatible with not using expectations to do evaluations in certain cases, such as those considered in Section 2.3 where some deference considerations are also used.

with E, which (remaining – as yet – unmodified) fail to add up to 1 and hence fail to constitute a probability distribution. She then uses these (non-probabilistic) credences to decide what credences to adopt, by maximizing expected accuracy. This means that she will adopt the credences that maximize expected accuracy, as calculated using her prior credences over the worlds consistent with E.

$$\mathrm{Exp}_b \mathsf{Acc}(c) \restriction_E = \sum_{w \in E} b(w) \times \mathsf{Acc}(c, w)$$

And, they show that what maximises this is the conditionalized credence.

So, each b recommends $b(\cdot \mid E)$. So, Imprecise Recommendation tells us that $\mathbb{B}$ will recommend $\{b(\cdot \mid E) \mid b \in \mathbb{B}\}$. This was exactly our desired Imprecise-Conditionalisation principle.

2.2.2 Update Strategies

This argument has been criticised for the picture of updating that it encodes and how these arational credences are used to evaluate options. An alternative is from Greaves and Wallace (2006) where our prior credences b evaluate various update strategies for what credence to adopt after learning one proposition from a partition. And they show that the strategy of updating by conditionalizing on the learned member of the partition is the optimal update strategy.

To apply this to the imprecise we have to think about what the right notion of an update strategy is in the imprecise setting. Imprecise-Recommendation can directly apply to tell us that an imprecise probability $\mathbb{B}$ recommends the set of precise update strategies: $\{\langle b(\cdot \mid E), b(\cdot \mid \neg E)\rangle \mid b \in \mathbb{B}\}$. But, it is not clear how to interpret this and whether that tells us anything about how an imprecise probability $\mathbb{B}$ should be updated. Instead, then we want strategies that say: if you learn E adopt thus-and-so imprecise probability, and if you learn $\neg E$ adopt thus-and-so imprecise probability. If we turn the above set into such a strategy by just taking its projections, we will obtain: $\langle \{b(\cdot \mid E) \mid b \in \mathbb{B}\}, \{b(\cdot \mid \neg E) \mid b \in \mathbb{B}\}\rangle$. There's certainly more to be said here. But I am just going to move on now.

Before moving to Imprecise Probabilism, which is a little more subtle, I want to mention one other, non-standard, application, which shows something new arising from Imprecise Recommendation.

2.3 Undermining Probabilities

There are some particular cases in the literature that form a challenge for rationality, but where moving to imprecise probabilities and using Imprecise Recommendation results in a solution that is intuitive. This is interesting because it shows something new that Imprecise Recommendation can bring to the table, in addition to its allowing for justification of otherwise accepted principles of rationality.

Typically, accuracy considerations are applied in cases where the opinions that one adopts in a proposition has no direct impact on the world and doesn't itself create evidence. But once that assumption is dropped some challenges arise.

Consider *Passport*:

If I have degree of belief $\geqslant 1/2$ that I'll forget my passport then I won't.
If I have degree of belief $< 1/2$ that I'll forget it, then I will.

Suppose our agent knows this about herself. If she starts off with degree of belief $1/4$ that she'll forget it, then by reflecting on her opinion, she knows that she'll actually forget it, so should adopt degree of belief 1. But adopting 1 then recommends adopting 0. In this case every credal state is undermining in this way: recommending some other credal state.

However, we can find an imprecise credal state that is self-recommending: $\{0, 1\}$. 0 recommends 1; 1 recommends 0. So, using our idea encoded in Imprecise Recommendation, $\{0, 1\}$ recommends $\{\mathsf{rec}(0), \mathsf{rec}(1)\} = \{1, 0\} = \{0, 1\}$. So $\{0, 1\}$ is self-recommending.

This provides an interesting account of such cases.[7]

The way we have implemented the notion of recommendation for the precise side most closely matches that of Joyce (2018), as against implementations by Caie (2013); Greaves (2013) which are defended by Pettigrew (2018). Those authors implement accuracy-theoretic considerations in a consequentialist manner, they say a credal state is good at the pursuit of accuracy in so far as makes itself accurate; or, otherwise put, we should care about how accurate the function *would be were* it to be adopted as the agent's credences. They thus do not say that 0 recommends 1 and 1 recommends 0, as we did, but they say that recommendation isn't relative to the starting point, either credal state recommends $1/2$; it's as close as possible to the truth value that would obtain were it to be adopted (see also Carr (2017)). Konek and Levinstein (2017); Joyce (2018) both argue against such consequentialist implementations of accuracy-considerations for epistemic norms. In particular, Joyce (2018) argues that in order to count as a candidate for being one's credences, the function has to be self-recommending, using the notion of recommendation as we have done in our considerations of *Passport*, rather than in a consequentialist way. Otherwise they do not play their constitutive role in guiding one's estimations or actions and thus cannot legitimately be termed "one's credences". However, in the case of *Passport*, there are no possible functions that could count as the agent's credences. There are, however, imprecise credences by using Imprecise Recommendation. In such a case, I suggest that $\{0, 1\}$ can be a self-recommending opinion state, using Joyce's picture of recommendation, and something Joyce could allow as a candidate for being "one's credences" in such a case (indeed, in this case, the only such candidate).

Note, though, that one isn't forced to reject consequentialism to use Imprecise Recommendation. One can also use it with a consequentialist manner of applying

[7] One can also formally connect such scenarios to the liar paradox and such an account can be seen as an analogue of a supervaluational logic based Kripkean account for truth (Campbell-Moore, msa).

accuracy considerations on the precise side, it's just that in that case it's not as interesting (or plausible), as then there are already self-recommending credences, albeit non-probabilistic ones ($^1/_2$ in *Passport*, and 0 in $\neg$*Passport*).

So, we have allowed for some self-recommending *imprecise* credal state in the case of *Passport*, when no precise credal state was self-recommending. One might further question whether we can always find a self-recommending imprecise credal state. Unfortunately not. There are some recommendation functions where there will be no set that is self-recommending, for example a recommendation function which would lead to a sequence of recommendations: $0 \rightsquigarrow 1 \rightsquigarrow {}^1/_2 \rightsquigarrow {}^1/_4 \rightsquigarrow \ldots$. In this situation, there is also no set that is self-recommending in the sense of Imprecise-Recommendation. For such cases, I think we should move to an even more general setting where we allow not only imprecise probabilities but also infinitesimal probabilities. A very expressively powerful framework is that of Moss (2018) where we can be thought of as adopting all-or-nothing beliefs towards sets of probability functions. This framework naturally allows for imprecise probabilities, but also can be used to model *infinitesimal* probabilities: she can believe the probability is $< {}^1/_2$, $< {}^1/_4$, $\ldots$, but also believe it is > 0. We might then extend our notion of Imprecise-Recommendation to such probabilistic beliefs: believing probabilistic contents $\mathbb{B}$ recommends believing $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$. For this case, such an infinitesimal opinion allows for a self-recommending Belief in Probabilistic Contents. Further investigation of this is left to future work.[8]

There is a further sort of case where moving to imprecise credences doesn't always permit self-recommendation. That is when the setup of the case is extended to involve one's imprecise credences. Of can amend the situation to have, e.g. if you adopt $\{0, 1\}$ then she knows she'll forget it, so should adopt 0. As for the liar paradox, revenge will always be a challenge and its nonetheless interesting that we've made some progress. Moreover, perhaps something can be said for why it is interesting to account for cases like the original Passport.

For further discussion of the application of Imprecise Recommendation to such cases, see Campbell-Moore (msb), we now return to the main theme, how Imprecise Recommendation can be used to justify usual epistemic requirements for the imprecise.

2.4 Imprecise Probabilism

A very desirable principle is:

Imprecise-Probabilism.
One should have a credal set $\mathbb{B}$ that only contains probability functions.

The argument for probabilism on the precise side is a dominance argument: If b is non-probabilistic, there is some probabilistic c that is more accurate whatever the world is like.

[8] The alternative for such cases, which was adopted in Campbell-Moore (msa), is to allow that a set recommends the *(topological) closure* of the set of individually recommended credences.

This, then, doesn't directly talk about what these (possibly non-probabilistic) credences recommend. So it's not immediately obvious how we use it with Imprecise Recommendation which tells us what the imprecise credences recommends once we plug in a notion of what the precise credences recommend.

We could, of course, encode such recommendation talk in. We could say:

- If b is non-probabilistic and c accuracy-dominates b, then b recommends (possibly non-uniquely) c.

Which would be added to the usual recommendation for the precise probabilities

- If b is probabilistic and then it recommends the c that maximises $\mathrm{Exp}_b\mathsf{Acc}(c)$ (namely, itself).

Imprecise Recommendation tells us that $\mathbb{B}$ recommends $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$. For each probabilistic b, it will recommend itself; but non-probabilistic b's will recommend some alternative credence c. So if $\mathbb{B}$ contains some non-probabilistic b; then $\mathbb{B}$ will not recommend itself, but will recommend some alternative set where b is replaced with c.

There's a slight subtlety that we're passing over: the non-probabilistic members needn't uniquely recommend someone, but Imprecise Recommendation only applied when precise credences had unique recommendations. But that was just to keep it simple. We will extend Imprecise Recommendation to:

Imprecise Recommendation (Extended).
(Suppose each $\succ_b$ is chain-complete.[9]) If

- Every $b \in \mathbb{B}$ has some $c \in \mathbb{C}$ which is amongst its recommendations.
- Every $c \in \mathbb{C}$ is a recommendation of some $b \in \mathbb{B}$

Then $\mathbb{C}$ is amongst $\mathbb{B}$'s recommendations.

Using Imprecise Recommendation (Extended), our argument here will still work: the non-probabilistic $b \in \mathbb{B}$ will not be contained in any set that's recommended by $\mathbb{B}$ because it's not recommended by anyone.

We might, however, hope to get a *dominance* argument for Imprecise-Probabilism where we don't have to talk about what the non-probabilistic b recommend. One reason we might want this is that it's not clear that a non-probabilistic member will recommend its accuracy-dominator; it might just dig in its heels and say it's good. In Section 4.2 we return to explaining how the dominance argument might go and the challenges it faces.

[9]That is, every c is either 'amongst b's recommendations' or can be improved on by some such c', that is, a c' which cannot itself be improved on. Or, equivalently, that there is no continual sequence of improvements with no maximal member.

3 Extending and justifying Imprecise Recommendation

3.1 The idea

Imprecise Recommendation focused on the question of which set is evaluated as *most* accurate. But in general, accuracy considerations allow us to see which belief-objects are better or worse than which others. We want to develop an account which provides us with an ordering of sets, which will be grounded in the individual's orderings over individuals.

To give this story, we will use the credal-committee analogy. If you have an imprecise credence we can think of you as a group of credal-committee members. Each comittee member has an opinion about which object is best, or, more generally, a ranking of certain objects; typically these objects will be other precise credences. The committee comes together and discusses whether they think that this set is better than this other set, based on the opinions of the committee members.

We will focus on the question of how the imprecise probability ranks other imprecise probabilities, i.e. whether $\mathbb{C}' \succ_{\mathbb{B}} \mathbb{C}$. To extend this to ranking of other imprecise credal objects is straightforward. So the individuals start with orderings over the precise objects: $c' \succ_b c$. This might come from expected accuracy considerations, $c' \succ_b c$ iff $\mathrm{Exp}_b\mathsf{Acc}(c') > \mathrm{Exp}_b\mathsf{Acc}(c)$, or if b is non-probabilistic, at least her ranking will follow dominance considerations, if c' dominates c then $c' \succ_b c$. The group of $b \in \mathbb{B}$ then come together and discuss whether $\mathbb{C}' \succ_{\mathbb{B}} \mathbb{C}$ based on their rankings of the individuals in $\mathbb{C}'$ and $\mathbb{C}$.

4th Mar: slightly modified this intro

But we need to be careful about how to do this: Consider b_1 and b_2, and suppose they think themselves are optimal. b_1 will evaluate b_1 to be better than b_2 ($b_1 \succ_{b_1} b_2$), and she will thus have the opinion that $\{b_1\} \succ_{b_1} \{b_1, b_2\}$. But b_2 disagrees; she thinks that $\{b_1\} \prec_{b_2} \{b_1, b_2\}$, after all, $b_2 \succ_{b_2} b_1$. So in what sense is $\{b_1, b_2\}$ the right compromise for the group rather than $\{b_1\}$? In applying accuracy for the imprecise, Levinstein (2019) allows that either $\{b_1\}$ or $\{b_1, b_2\}$ is recommended by the set, thus allowing that imprecise probabilities think that they are among the optimal. I'm aiming for the stronger conclusion that $\{b_1, b_2\}$ uniquely evaluates itself to be optimal. This can get us stronger conclusions such as that updating by conditionalization is rationally required rather than merely permitted.

Consider again:

$$\{b_1\} \succ_{b_1} \{b_1, b_2\}$$
$$\{b_1\} \prec_{b_2} \{b_1, b_2\}$$

But we want to overall say that $\{b_1, b_2\}$ is preferred by the group, $\{b_1, b_2\}$, to $\{b_1\}$? We think that there is a different strength of opinion encoded in the two preferences above. b_2's opinion is a strong, unmovable demand, whereas b_1's opinion is weaker. Our committee member's first priority is to get their favourite person in the group. The set $\{b_1\}$ fails this from b_2's perspective. There is no way he can get on board with supporting $\{b_1\}$. Whereas $\{b_1, b_2\}$ at least has b_1's favourite member. Sure she thinks that $\{b_1\}$ is better, but she's happy to compromise on this in order to come to an agreement with her committee members.

We will encode these different strengths of opinions: $\gg$ is a strong preference that the individuals are unwilling to compromise on. These are unmovable demands of our members. $\rhd$, on the other hand, is the member's preferences that they are open to compromising on. We then have the preferences:

$$\{b_1\} \rhd_{b_1} \{b_1, b_2\}$$
$$\{b_1\} \ll_{b_2} \{b_1, b_2\}$$

Since the preference $\gg$ takes priority, we thus get that the committee's joint preference has $\{b_1, b_2\} \succ_{\{b_1,b_2\}} \{b_1\}$.

We need to describe how to construct such $\gg_{\mathbb{B}}$ and $\rhd_{\mathbb{B}}$. Then the overall $\succ_{\mathbb{B}}$ first checks for any $\gg_{\mathbb{B}}$ preferences, then breaks any ties using $\rhd_{\mathbb{B}}$.

We want to do all this in a way that obtains Imprecise Recommendation as a consequence. We will also want to obtain:

Imprecise Recommendation (Extended).
If

- Every $b \in \mathbb{B}$ has some $c \in \mathbb{C}$ which is amongst its recommendations.
- Every $c \in \mathbb{C}$ is a recommendation of some $b \in \mathbb{B}$

Then $\mathbb{C}$ is amongst $\mathbb{B}$'s recommendations.[10]

which is important to derive Imprecise Probabilism, when not all b have unique recommendations as the non-probabilistic b only have ordering-opinions in accordance with dominance.

We want to argue for Imprecise Recommendation (Extended) based on the story of the group coming together and voting. The argument will follow the following lines:

We start by considering all $b \in \mathbb{B}$'s unmovable demands, which will be encoded by $\gg_b$. If b has none of her recommended members in the $\mathbb{C}$, then she is going to refuse to accept $\mathbb{C}$; she will demand that one of her recommended members be included in $\mathbb{C}$. The other b' think that this will make the set worse, however, it will not destroy their primary goal: to get their recommended members in the set. So they will accept b's demand to include one of her recommended members.

Now, the sets that are acceptable according to all $b \in \mathbb{B}$'s unmovable demands will be those that include some recommended member for each $b \in \mathbb{B}$. But this can include some additional members who are not recommended by anyone. Everyone will agree that the set can be made better by removing this c who is not recommended by anyone. So the sets that are in the end recommended are those that only include c who are recommended by some b.

We need to develop a nice definition of the orderings that allow us to make this argument.

[10] It's still not clear to me whether we want this to hold in general or only when $\prec_b$ is chain complete, i.e. that there's no continual sequence of improvements with no maximal member.

3.2 Defining the ordering

This section has had alterations since I shared the paper on Tuesday. They're highlighted with changebars, also the original can be found in Appendix B

So, how to define these orderings in a way that allows us to obtain Imprecise Recommendation (Extended)? We should also be able to ensure that these are strict partial orders (i.e. irreflexive and transitive), and a desirable property is that by taking $\succ_b$ equivalences, we shouldn't break any $\gg_b$ or $\rhd_b$ orderings.

The following definition will allow us to obtain Imprecise Recommendation, but not Imprecise Recommendation (Extended): [NB: this whole thing is still sketchy. I've got various attempts at trying to get Imprecise Recommendation (Extended), something should work, but this was a quick fall-back, that I think (/hope) does work at least for Imprecise Recommendation.]

Definition 3.1.

- If

  $$\exists c'_* \in \mathbb{C}' \text{ such that } c'_* \succ_b c \text{ for all } c \in \mathbb{C}$$

  then $\mathbb{C}' \gg_b \mathbb{C}$. (And if $\succeq$ then $\ggeq$.)

- If $\mathbb{C}' \gg_b \mathbb{C}$ for some $b \in \mathbb{B}$ and $\mathbb{C}' \ggeq_b \mathbb{C}$ for all $b \in \mathbb{B}$, then $\mathbb{C}' \gg_{\mathbb{B}} \mathbb{C}$.

- If

  $\mathbb{C}' \subset \mathbb{C}$ such that all $c \in \mathbb{C} \setminus \mathbb{C}'$ have some $c' \in \mathbb{C}'$ with $c' \succ_b c$.

  Or, by closing this under taking equivalents (which is thus equivalent to the previous when $c \sim_b c' \implies c = c'$):

  * every $c' \in \mathbb{C}'$ has some $c \in \mathbb{C}$ with $c \sim_b c'$
  * there is some $c \in \mathbb{C}$ with no $c' \in \mathbb{C}'$ with $c \sim_b c'$
  * every $c \in \mathbb{C}$ has some $c' \in \mathbb{C}'$ with $c' \succeq_b c$

  then $\mathbb{C}' \rhd_b \mathbb{C}$.

- If $\mathbb{C}' \rhd_b \mathbb{C}$ for all $b \in \mathbb{B}$ then $\mathbb{C}' \rhd_{\mathbb{B}} \mathbb{C}$.

- If $\mathbb{C}' \gg_{\mathbb{B}} \mathbb{C}$, or $\mathbb{C}' \approx_{\mathbb{B}} \mathbb{C}$ but $\mathbb{C}' \rhd_{\mathbb{B}} \mathbb{C}$, then $\mathbb{C}' \succ_{\mathbb{B}} \mathbb{C}$.

this needs improving to get $\mathbb{C}' \gg_b \mathbb{C} \implies \mathbb{C} \rhd_b \mathbb{C}$ and to be more natural.

Proposition 3.2. *$\gg_b$ and $\rhd_b$ are irreflexive and transitive. So are $\gg_{\mathbb{B}}$ and $\rhd_{\mathbb{B}}$. I don't think $\succ_{\mathbb{B}}$ is necessarily transitive (which should of course be fixed)*

Remark. $\gg_b$: The restricted choice of quantifier order $\exists c' \forall c$ in $\gg_b$ allows it to be irreflexive. $\forall c \exists c'$ would not be irreflexive. For example $\{c_1, c_2, \ldots\}$.

Also note that this definition of $\gg_b$ is much too strong for what we'll want in the end, there's no hope to use it to get Imprecise Recommendation (Extended).[11]

[11] What we should try to use:

$$\forall c \in \mathbb{C}\ \exists c' \in \mathbb{C}'\ c' \succeq_b c \ \text{ and }\ \forall c' \in \mathbb{C}'\ \exists c \in \mathbb{C}\ c' \succeq_b c$$

$$\text{and } \exists c'_* \in \mathbb{C}'\ \exists c_* \in \mathbb{C} \begin{pmatrix} & c'_* \succ_b c_* \\ \wedge & \neg \exists c \in \mathbb{C}\ c \succeq_b c'_* \end{pmatrix}$$

$\gg_{\mathbb{B}}$: The choice of $\mathbb{C}' \underline{\gg}_b \mathbb{C}$ rather than $\mathbb{C}' \not\ll_b \mathbb{C}$ allows $\gg_{\mathbb{B}}$ to be transitive. The difference this makes: improvement in one component and incomparability in another isn't enough for an improvement. We can see this in the notion of weak dominance. Suppose G and H are incomparable. Consider:

	A	B	C
w_1	G	G-\$1	H
w_2	G	H+\$1	H

Suppose we defined weak dominance with $\not<$, then we would have: A weakly dominates B (it's better in w_1 and incomparable in w_2), and B weakly dominates C, but A is simply incomparable with C.

$\succ_{\mathbb{B}}$: To ensure that $\succ_{\mathbb{B}}$ is transitive we should extend $\rhd_b$ to a transitive relation such that $\mathbb{C}' \gg_b \mathbb{C} \implies \mathbb{C}' \rhd_b \mathbb{C}$, which is something we'd independently want anyway then we could drop the "$\rhd$" and just call this "$\succ$", with $\gg$ simply being the strong preferences.

$\rhd_b$: Some desiderata/notes for ideas to extend it to get $\gg \implies \rhd$ (think of the numbers as utility values):

- Removing not-top improves: $\{3,2,1\} \prec \{3,1\}$.
- Improving the top improves ($\gg$): $\{3,1\} \succ \{2,1\}$. Also $\{\text{Integers}\} \cup \{\infty\} > \{\text{Integers}\}$.
- A definition like $\exists c'_*$ which lies $\geqslant$ everything, and $>$ something would allow that $\{2,1\} \not\succ \{2,1\}$: 2 is better than 1.
- Simply requiring that what it improves on is not in $\mathbb{C}$ won't work:

$$\{\text{Even integers}\} \cup \{\infty\} \not\succ \{\text{Odd integers}\} \cup \{\infty\}$$

Proposition 3.3. *Suppose each b has a unique $\mathsf{rec}(b)$ ($\succ_b$ is complete). Then $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$ is uniquely $\succ_{\mathbb{B}}$ maximal.*

Proof. One can check:

1. If $\mathsf{rec}(b) \notin \mathbb{C}$ and $\mathsf{rec}(b) \in \mathbb{C}'$ then $\mathbb{C}' \gg_b \mathbb{C}$.
   If $\mathsf{rec}(b) \in \mathbb{C}$ and $\mathsf{rec}(b) \in \mathbb{C}'$ then $\mathbb{C}' \approx_b \mathbb{C}$.
2. If $\mathbb{C} \not\supseteq \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$, then $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \gg_{\mathbb{B}} \mathbb{C}$.
   If $\mathbb{C} \supseteq \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$, then $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \approx_{\mathbb{B}} \mathbb{C}$.
3. If $\mathbb{C} \supset \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$ then $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \rhd_b \mathbb{C}$ for all b,
   so $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \rhd_b \mathbb{C}$.
4. If $\mathbb{C} \neq \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$ then $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \succ_{\mathbb{B}} \mathbb{C}$. $\square$

This all becomes a little more fiddly and delicate to obtain Imprecise Recommendation (Extended). This will be done asap.

One final note: we might have hoped for our orderings is that by improving all members, the overall set cannot be made worse. But this is incompatible with our criterion of $\rhd_b$: Consider $\{c_0, c_1, \ldots\}$ with $c_0 \succ_b c_1 \succ_b c_2 \ldots$. Then using our definition of $\rhd_b$, $\{c_0, c_2, c_3, \ldots\} \rhd_b \{c_0, c_1, c_2, c_3, \ldots\}$. But by mapping $c_0 \mapsto c_0$ and $c_n \mapsto c_{n-1}$ for $n \geqslant 2$, which is a (weak-)improvement to each member, we get from $\{c_0, c_2, c_3, \ldots\}$ to $\{c_0, c_1, c_2, c_3, \ldots\}$. So it shouldn't be that this latter set is worse than the former.

3.3 Interpreting our Imprecise Probabilities

So, we have given some definitions which try to spell out a picture on which Imprecise Recommendation is plausible. We used an analogy with a group making a decision. It is still unclear whether the use of Imprecise Recommendation assumes that we have a particular interpretation of the imprecise probabilities. A simply supervaluational picture, there's nothing more than what the individuals agree on, doesn't have the complexity to obtain Imprecise Recommendation, it merely obtains a weak form of it as is used in Levinstein (2019). A picture where they're simply a different kind of opinion that should be treated on a parallel with precise probabilities wouldn't motivate the idea that ultimately accuracy theoretic considerations only directly apply to the precise, with the imprecise probabilities' opinion supervening on these. We have used a kind of mixed picture of what imprecise probabilities are to justify this. Thus, we are perhaps assuming a slightly complicated picture of what imprecise probabilities are in order to justify it. It is still open what exactly this interpretation is, and how important it is to our adoption of Imprecise Recommendation.

4 Further discussion

4.1 Connections to impossibility results

There are a few papers which have argued that it won't be possible to apply accuracy theoretic considerations to the imprecise in a way that obtains Imprecise Immodesty (Seidenfeld et al., 2012; Mayo-Wilson and Wheeler, 2016; Schoenfield, 2017; Chambers, 2008). These papers provide impossibility results showing there is no way of measuring closeness to truth in a way that every imprecise probability function evaluates itself to be best. What they try to do is to assign a specific value of accuracy to the imprecise probability function at each world.[12] We, however, don't directly assign accuracy values to the imprecise credences at the worlds. Instead, what accuracy considerations primarily give us is the measure of accuracy of a precise credence at a world, and this gives us results about accuracy of the imprecise by using Imprecise Recommendation.

To understand the difference it makes, it is worth going through how far we might go assigning accuracy to the imprecise credences at a world. We could define the notion of the accuracy of a set $\mathbb{C}$ at a world w as the imprecise value

$$\mathsf{Acc}(\mathbb{C}, w) = \{\mathsf{Acc}(c, w) \mid c \in \mathbb{C}\}.$$

Schoenfield (2017) considers a possibility like this. She criticises this by considering the following case: consider how $1/2$ evaluates $[0.4, 0.6]$. According to Immodesty, it should evaluate it to be worse that $1/2$ itself. But in each world, w, $\mathsf{Inacc}(\{1/2\}, w) = -0.25$, $\mathsf{Inacc}([0.4, 0.6], w) = [0.16, 0.36]$, so in each w they are incomparable. Thus, since $[0.4, 0.6]$ is not worse in either world, Schoenfield

[12] **?** gives an alternative response to the results where different measures are appropriate for different imprecise probabilities.

thinks that $1/2$ cannot evaluate $[0.4, 0.6]$ to be worse than $1/2$ itself, violating Immodesty. However, this will be rejected on the picture that I give. Moreover, it seems like a principle that is very strong. I think there are examples in the literature where two options are incomparable in every situation, but nonetheless one is evaluated as better than the other.

Even if the set of accuracy values at each world was identical, we take b's estimated goodness not by simply doing an expectation of the goodness of $\mathbb{C}$ in each world, but asking b what it thinks of each member $c \in \mathbb{C}$. We need the additional information of which c is which across worlds: we need[13]

$$\{\langle \mathsf{Acc}(c, w)\rangle_{w\in W} \mid c \in \mathbb{C}\}$$

rather than

$$\langle\{\mathsf{Acc}(c, w) \mid c \in \mathbb{C}\}\rangle_{w\in W}.$$

We could assign set-values for b's estimated accuracy of a set $\mathbb{C}$:

$$\mathrm{Est}_b\mathsf{Acc}(\mathbb{C}) = \{\mathrm{Est}_b\mathsf{Acc}(c) \mid c \in \mathbb{C}\}.$$

and then describe how to compare these set-values by paralleling the (a) clauses from Section 3, and thus obtaining that b evaluates $\{b\}$ to be most accurate. If we want to give an accuracy value for $\mathbb{B}$'s estimation of $\mathbb{C}$, we have to move to sets of sets. And we say how to compare these by matching our definition in Section 3, obtaining Imprecise Recommendation.

4.2 Dominance argument for Imprecise-Probabilisim?

Our argument for probabilism used the idea that non-probabilities recommend probabilities, and then looking at the set-recommendation using Imprecise Recommendation. It is desirable to directly obtain a dominance argument for Imprecise Probabilism. In this section we merely describe how one might go but note the challenge for it.

Well, we consider the accuracy value of $\mathbb{C}$ at w as:

$$\mathsf{Acc}(\mathbb{C}, w) = \{\mathsf{Acc}(c, w) \mid c \in \mathbb{C}\}.$$

We know from the usual precise result that: If $\mathbb{B}$ contains some non-probabilistic b, then there is some dominator, $\mathsf{dom}(b)$ which is more accurate in each world. If b is probabilistic, so has no such dominator, then we let $\mathsf{dom}(b) = b$.

Consider $\{\mathsf{dom}(b) \mid b \in \mathbb{B}\}$. Compare:

$$\begin{array}{c} \{\quad \mathsf{Acc}(b, w) \quad \mid b \in \mathbb{B}\} \\ \{\mathsf{Acc}(\mathsf{dom}(b), w) \mid b \in \mathbb{B}\} \end{array}$$

[13] This is related to Konek's picture in ??? and was probably motivated by our discussions on that.

We know $\mathsf{Acc}(b, w) \leqslant \mathsf{Acc}(\mathsf{dom}(b), w)$; and sometimes $<$. But the challenge is that these are supposed to be unstructured sets of numerical values. The explanation for why one is better than the other involves this pairing: knowing that $\mathsf{dom}(b)$ should be paired with b. But these should be thought of simply as sets and this fact of how they are to be paired up is not one that we can use.

5 Conclusions and Further Thoughts

There are still gaps and further work to develop and justify the proposal. The imprecise ranking we've given is still quite sparse and it should be filled out with more conditions. And at the end of Section 3 we mentioned that we are perhaps assuming a slightly complicated picture of what imprecise probabilities are in order to justify it. But the proposal is more powerful than one might have thought, and seems intuitively quite plausible, at least in its Imprecise Recommendation form. We thus think it is interesting and worth further investigation.

Of course, there are objections one might make to Imprecise Recommendation. One such objection would argue that there will be certain desirable/undesirable features of imprecise probabilities that aren't reducible to features of the precise ones (Moss). That seems right. So then what is recommended similarly might not be reducible to the precise recommendations. I'm not quite sure what to say about this. There are certain global features that could be added to Imprecise Recommendation: we might allow a set to recommend the *closure* of what the individuals recommend (see Footnote 8). Perhaps there is some systematic split where some global constraints are OK, so long as they supervene on the set of individual recommendations. But we can't have anything-goes, because that would directly result in revenge problems in the undermining scenarios applications. Will that capture the examples that Moss is thinking about?

There is also more to do on investigating the consequences and applications of Imprecise Recommendation. For example, in aggregating imprecise probabilities. Konek (2019) has recently proposed using accuracy in the imprecise setting for justifying aggregation methods. This would be similarly motivated but a very different kind of instantiation of Konek's ideas. What would it result in? On the precise side we get accuracy arguments for linear pools. On the imprecise side, then, perhaps we get an argument that it should be contained in the convex hull of the union of the $\mathbb{B}_i$s. Can we get more than that?

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A Aggregation

A la Jason, aggregate with accuracy.

Suppose you have $\mathbb{B}_i$.

Usual result, if c is not a linear pool of b_i, then there is some c' which is a linear pool which each b_i expect to be accuracy improvement.

Now, each $\mathbb{B}_i$ has its own orderings of the IPs $\mathbb{C} \succ_{\mathbb{B}_i} \mathbb{C}'$ according to expected accuracy considerations as above. What can we say about which $\mathbb{C}$ are non-dominated from a pooling perspective, i.e. which $\mathbb{C}$ are such that there is no $\mathbb{C}'$ where $\mathbb{C} \prec_{\mathbb{B}_i} \mathbb{C}'$ for all i?

Conjecture A.1. *If $\mathbb{C}$ is a subset of the convex hull of the union of the $\mathbb{B}_i$s, then it is dominated.*

Proof Attempt. Suppose $c \in \mathbb{C}$ is not in the convex hull of the union of the $\mathbb{B}_i$s.

Then it is not in the convex hull of any $\langle b_1, \ldots, b_n\rangle \in \mathbb{B}_1 \times \ldots \times \mathbb{B}_n$. Thus for each $\langle b_1, \ldots, b_n\rangle \in \mathbb{B}_1 \times \ldots \times \mathbb{B}_n$ there is some $c_{\langle b_1,\ldots,b_n\rangle}$ such that $\mathrm{Exp}_{b_i}\mathsf{Acc}(c_{\langle b_1,\ldots,b_n\rangle}) > \mathrm{Exp}_{b_i}\mathsf{Acc}(c)$.

Consider $\mathbb{C}'$ which replaces c by the collection of $c_{\langle b_1,\ldots,b_n\rangle}$. Noone thinks that c is better than all $c_{\langle b_1,\ldots,b_n\rangle}$, after all, b_i thinks that $c_{\langle b_1,\ldots,b_n\rangle}$ is better.

So $\mathbb{C} \not\succ^*_{\mathbb{B}_i} \mathbb{C}'$.

Now consider their weak preferences:

Consider $\mathbb{C} \cup \{c_{\langle b_1,\ldots,b_n\rangle} \mid b_i \in \mathbb{B}_i\} \cup \{c^*\}$ and $\mathbb{C} \cup \{c_{\langle b_1,\ldots,b_n\rangle} \mid b_i \in \mathbb{B}_i\}$. There is no $\succ^*_{\mathbb{B}}$ preferences between these two sets. But: each b^*_i has a weak preference for $\mathbb{C} \cup \{c_{\langle b_1,\ldots,b_n\rangle} \mid b_i \in \mathbb{B}_i\}$, after all, there's something there, namely any $c_{b_1,\ldots,b^*_i,\ldots b_n}$.

BLAH

B Record of defining the ordering

So, how to define these orderings in a way that allows us to obtain Imprecise Recommendation (Extended)? We should also be able to ensure that these are strict partial orders (i.e. irreflexive and transitive), and a desirable property is that by taking $\succ_b$ equivalences, we shouldn't break any $\gg_b$ or $\rhd_b$ orderings.

The following definition will allow us to obtain Imprecise Recommendation, but not Imprecise Recommendation (Extended): [NB: this whole thing is still sketchy. I've got various attempts at trying to get Imprecise Recommendation (Extended), something should work, but this was a quick fall-back, that I think (/hope) does work at least for Imprecise Recommendation.]

Definition B.1.

- If

  $$\exists c'_* \in \mathbb{C}' \text{ such that } c'_* \succ_b c \text{ for all } c \in \mathbb{C}$$

  then $\mathbb{C}' \gg_b \mathbb{C}$.

- If $\mathbb{C}' \gg_b \mathbb{C}$ for some $b \in \mathbb{B}$ and $\mathbb{C}' \not\gg_b \mathbb{C}$ for any $b \in \mathbb{B}$, then $\mathbb{C}' \gg_{\mathbb{B}} \mathbb{C}$.

- If

  $\mathbb{C}' \subset \mathbb{C}$ such that all $c \in \mathbb{C} \setminus \mathbb{C}'$ have some $c' \in \mathbb{C}'$ with $c \succ_b c'$.

  Or, by closing this under taking equivalents (which is thus equivalent to the previous when $c \sim_b c' \implies c = c'$):

  * every $c' \in \mathbb{C}'$ has some $c \in \mathbb{C}$ with $c \sim_b c'$
  * there is some $c \in \mathbb{C}$ with no $c' \in \mathbb{C}'$ with $c \sim_b c'$
  * every $c \in \mathbb{C}$ has some $c' \in \mathbb{C}'$ with $c' \succeq_b c$

  then $\mathbb{C}' \rhd_b \mathbb{C}$.

- If $\mathbb{C}' \rhd_b \mathbb{C}$ for some $b \in \mathbb{B}$ and $\mathbb{C}' \not\rhd_b \mathbb{C}$ for any $b \in \mathbb{B}$, then $\mathbb{C}' \rhd_{\mathbb{B}} \mathbb{C}$.

- If $\mathbb{C}' \gg_{\mathbb{B}} \mathbb{C}$, or $\mathbb{C}' \not\gg_{\mathbb{B}} \mathbb{C}$ but $\mathbb{C}' \rhd_{\mathbb{B}} \mathbb{C}$, then $\mathbb{C}' \succ_{\mathbb{B}} \mathbb{C}$.

The clause of $\rhd_b$ saying 'where there is no $c \succ_b c'_*$' is perhaps not immediately intuitive. The reason it is needed is to ensure that $\rhd_b$ is irreflexive.

Proposition B.2. *$\gg_b$ and $\rhd_b$ are irreflexive and transitive.*

$\forall c\ \exists c' \succ_b c$ would not be irreflexive, but the choice of quantifier order in the definition of $\gg_b$ allows it to be irreflexive.

Proof. TBC □

Proposition B.3. *Suppose each b has a unique $\mathsf{rec}(b)$ ($\succ_b$ is complete). Then $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$ is uniquely $\succ_{\mathbb{B}}$ maximal.*

Proof. One can check:

- If $\mathsf{rec}(b) \in \mathbb{C}$ then no $\mathbb{C}'$ has $\mathbb{C}' \gg_b \mathbb{C}$.
- Thus, if $\mathbb{C} \supseteq \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$, there is no $\mathbb{C}' \gg_{\mathbb{B}} \mathbb{C}$.
- If $\mathsf{rec}(b) \notin \mathbb{C}$ and $\mathsf{rec}(b) \in \mathbb{C}'$ then $\mathbb{C}' \gg_b \mathbb{C}$.
- Thus, if $\mathbb{C} \not\supseteq \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$, $\mathbb{C} \cup \{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \gg_{\mathbb{B}} \mathbb{C}$.

- So the only candidates for being maximal are those that are $\supseteq \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$.
- If $\mathbb{C} \supset \{\mathsf{rec}(b) \mid b \in \mathbb{B}\}$, then for all b, $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \rhd_b \mathbb{C}$, and so $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \rhd_{\mathbb{B}} \mathbb{C}$, and therefore $\{\mathsf{rec}(b) \mid b \in \mathbb{B}\} \succ_{\mathbb{B}} \mathbb{C}$. $\square$

This all becomes a little more fiddly and delicate to obtain Imprecise Recommendation (Extended). This will be done asap.

One final note: we might have hoped for our orderings is that by improving all members, the overall set cannot be made worse. But this is incompatible with our criterion of $\rhd_b$: Consider $\{c_0, c_1, \ldots\}$ with $c_0 \succ_b c_1 \succ_b c_2 \ldots$. Then using our definition of $\rhd_b$, $\{c_0, c_2, c_3, \ldots\} \rhd_b \{c_0, c_1, c_2, c_3, \ldots\}$. But by mapping $c_0 \mapsto c_0$ and $c_n \mapsto c_{n-1}$ for $n \geqslant 2$, which is a (weak-)improvement to each member, we get from $\{c_0, c_2, c_3, \ldots\}$ to $\{c_0, c_1, c_2, c_3, \ldots\}$. So it shouldn't be that this latter set is worse than the former.