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Beliefs as Probability Filters

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January 14, 2025

Abstract

In this paper we propose a mathematical model for imprecise probability, used to represent an agents uncertain beliefs. In this model, we focus on probability constraints and ask whether the agent believes the constraint or not. For example, she might believe that the probability that it'll rain tomorrow is at least 0.2. To formalise the model, we represent an agents uncertain belief state by a collection of probability constraints — those which she believes. We will impose a coherence constraint that the believed probability constraints should form the mathematical structure of a filter, that is, they should be closed under finite intersection and superset. In this paper we compare this model to desirability based models of uncertainty, showing that the model encompass the framework of representing uncertainty with a set of desirable gambles.

1 Introduction

In work on imprecise probabilities, various mathematical models are provided to capture an agents uncertain belief state [33]. In this paper we develop and discuss an alternative mathematical model. A key tool in our representation is to consider constraints on probability functions, for example, the constraint that the probability that it rains tomorrow is at least 0.2. We will represent an agent with a collection of probability constraints, which we gloss as those which she believes. For example, she might believe that the probability that it will rain tomorrow is at least 0.2.

We will impose a coherence constraint on the agent which requires that the believed probability constraints be closed under finite intersection and supersets, that is, they should form the mathematical structure of a filter. We thus call this model the probability filter model of belief.

In this paper, we will compare this model to the very prominent model of desirable gambles [33, 32, 24, 8]. If she believes that the probability that it rains tomorrow is at least 0.2, she will find desirable a gamble which pays out 9 if it rains and costs 1 if it does not.[1] We will show that the probability filter model can encompass the desirable gambles model. This is not possible if we were to just consider a single set of probabilities representing the agent's beliefs, often called a credal set, as is often done in philosophy [20, 21, 15]. We obtain this additional power because of our coherence requirement only requires that the probability constraints be closed under merely *finite* intersection.

[1]This makes use of our Proposition 3.5.

We will also consider the framework of choice functions, which extend the desirable gambles model by considering not only binary choice [28, 10, 11, 30]. We will show that the probability filter framework can encompass many choice functions. In particular, we will show that if we associate probability filters with choice functions using a generalisations of Levi's E-admissibility rule [20, 21], then we obtain all so-called mixing choice functions.

This paper is a development of Campbell-Moore [4], which itself builds on a joint paper with Jason Konek [7]. Key results of this paper were stated there without proof. In that paper, we interpreted the model as beliefs with probabilistic contents, as outlined in Moss [23]. There are some close relationships between the proposal here and the framework in Moss [23], however we are not committed to her interpretation and our notion of coherence is weaker than that hers. This is important for its ability to capture the expressive power of the desirable gambles framework.

This paper proceeds as follows: We introduce the proposed model in Section 2; giving the notion of coherence as being a filter on probability functions. In Section 3 we discuss the relationship between the model of probability filters and that of desirable gambles, and in Section 4 we discuss the relationship with the more general model of desirable gamble sets, or choice functions.

The key results of this paper are the facts that this new framework of probability filters extends the desirable gambles framework (Theorems 3.7 and 3.9) and that it extends the framework of sets of desirable gamble sets with a mixing axiom (Theorems 4.8 and 4.11). This demonstrates the power of the framework.

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2 Probability constraints and the model of belief

In this section we introduce the new mathematical framework for modelling uncertain beliefs. We first need to introduce the background setup and notions.

Representations are always relative to an underlying possibility space, Ω. Formally, Ω is simply a non-empty set. It represents the possible ways that the world might be. If we are just focusing on her opinions about the outcomes of a specific experiment, Ω would consist of the different possible experimental outcomes, where it is guaranteed that exactly one of these outcomes obtains. We will assume that Ω is finite; extending these results is future work.

Since we are assuming Ω is finite, we can simply work directly with a probability mass function.

Setup 2.1. A **probability function** $p : \wp(\Omega) \to \mathbb{R}$ is characterised by a probability mass function, $p : \Omega \to \mathbb{R}$, with $p(\omega) \geqslant 0$ and $\sum_{\omega\in\Omega} p(\omega) = 1$. The set of all probability functions is Probs.[2]

p is **regular** iff $p(\omega) > 0$ for every $\omega \in \Omega$. rProbs is the collection of all regular probability functions on Ω. For the purposes of the paper, we will always restrict attention to regular probability functions.[3]

For a set of regular probability functions, $Q \subseteq$ rProbs, $\overline{Q}$ is the complement of Q, relative to rProbs, i.e., $\overline{Q} =$ rProbs $\setminus Q$.

[2]Note our abuse of notation. We write $p(\omega)$ instead of $p(\{\omega\})$.

[3]This is important to show axiom ($D_{\gneq 0}$) in Theorem 3.7.

The model that we propose focuses on probability constraints. Formally, a **probability constraint** is any set of regular probability functions, $Q \subseteq \text{rProbs}$. We typically express such constraints using sentences that talk about probability. For example, the constraint that the probability that it'll rain is at least 0.2 is the set of probabilities:

$$\{p \mid p(\textsc{Rain}) \geqslant 0.2\}.$$

The constraint that it is twice as likely that John passes the exam than that Billy passes the exam is given by:

$$\{p \mid p(\textsc{John passes}) = 2 \times p(\textsc{Billy passes})\}.$$

We allow *any* set of regular probability functions to be a probability constraint. For example, we also allow non-convex sets of probabilities, for example $\{p \mid p(\textsc{Red}) \in \{0.2, 0.8\}\}$ (see further comment on non-convex sets after Proposition 2.5).

Our proposed mathematical model of uncertainty can be introduced by supposing that the agent has attitudes towards these probability constraints: she might believe, disbelieve, or suspend judgement on them. The agent might, for example, believe that the probability that it'll rain is at least 0.2. The idea behind this representation is from Moss [22, ch.1]. We also see a very closely related idea in discussions of the credal set model of probabilities in Van Fraassen [31, p.345-347], Joyce [15, p.288]. These authors treat as fundamental certain assessments about probability such as someone thinking that X is more likely than Y, or that X is twice as likely as Y. Unlike these authors, we take these probabilistic judgements be the basis for our mathematical model.[4] We provide our mathematical model by simply collecting together all those probability constraints which she believes. That is, our representation is given by a set of probability constraints, or a set of sets of probabilities: $\mathcal{F} \subseteq \wp(\text{rProbs})$.

We will say that $\mathcal{F}$ is coherent iff it is a (proper) filter on rProbs.

Definition 2.2. A set of probability constraints, i.e., a set of subsets of rProbs, $\mathcal{F} \subseteq \wp(\text{rProbs})$, is **coherent** iff it satisfies the following axioms:

($\mathrm{F}_{\cap}$) For any $P, Q \subseteq \text{rProbs}$, if $P, Q \in \mathcal{F}$ then $P \cap Q \in \mathcal{F}$.

($\mathrm{F}_{\supseteq}$) For any $P, Q \subseteq \text{rProbs}$, if $P \in \mathcal{F}$ and $Q \supseteq P$ then $Q \in \mathcal{F}$.

($\mathrm{F}_{\neq\varnothing}$) $\mathcal{F} \neq \varnothing$.

($\mathrm{F}_{\text{Proper}}$) $\varnothing \notin \mathcal{F}$.

The notion of being a (proper) filter is standard from topology, see for example [14, ch.7]. Since we are representing uncertainty with a (proper) filter on the (regular) probabilities, we will call these **probability filters**.

Axioms ($\mathrm{F}_{\neq\varnothing}$) and ($\mathrm{F}_{\text{Proper}}$) ensure that a coherent $\mathcal{F}$ is non-trivial. Axiom ($\mathrm{F}_{\neq\varnothing}$) says that there has to be some probability constraint that she believes. Axiom ($\mathrm{F}_{\text{Proper}}$) says that she does not believe $\varnothing$. The substantive axioms are axioms ($\mathrm{F}_{\cap}$) and ($\mathrm{F}_{\supseteq}$) saying that her believed probability constraints should

[4] While these authors represent the agent through the set of probabilities consistent with all the assessments, we adopt probabilistic assessments themselves as our primary representational framework. A corresponding set of probabilities can only be derived in certain cases; see Proposition 2.5.

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be closed under finite intersections and supersets. Some illustrative examples of the consequences of these are presented in 2.2.

The axioms on coherence ensures that we have the following result.

Proposition 2.3 (No Confusion). [5] *If $\mathcal{F}$ is coherent, then there is no $Q \subseteq$ rProbs where $Q \in \mathcal{F}$ and $\overline{Q} \in \mathcal{F}$.*

Proof. If both $Q \in \mathcal{F}$ and $\overline{Q} \in \mathcal{F}$, then by axiom ($\text{F}_{\cap}$), we also have $Q \cap \overline{Q} = \varnothing \in \mathcal{F}$, contradicting axiom ($\text{F}_{\text{Proper}}$). $\square$

When $Q \in \mathcal{F}$, the agent represented by $\mathcal{F}$ believes Q. Our representation only explicitly considers belief, but we can also capture disbelief by assuming she disbelieves a constraint iff she believes its complement. So she disbelieves Q when $\overline{Q} \in \mathcal{F}$. We can then capture suspension of judgement when she neither believes nor disbelieves the constraint, i.e., when $Q \notin \mathcal{F}$ and $\overline{Q} \notin \mathcal{F}$. Proposition 2.3 ensures that exactly one of these holds. and suspension of judgement We find it convenient to drop reference to the agent and just talk about $\mathcal{F}$ itself encoding beliefs to probability constraints; or sometimes, simply, that $\mathcal{F}$ believes Q.

The final notion we introduce is that of one filter being at least as committal as another. If $\mathcal{F} \subseteq \mathcal{F}'$, then we say that $\mathcal{F}'$ is **at least as committal** as $\mathcal{F}$. When $\mathcal{F} \subseteq \mathcal{F}'$, any probability constraint which is believed according to $\mathcal{F}$ is also believed according to $\mathcal{F}'$; and similarly for disbeliefs. But $\mathcal{F}'$ might believe or disbelieve some constraints which are suspended on according to $\mathcal{F}$.

2.1 Examples of probability filters

In this section, we give some important examples of coherent probability filters, $\mathcal{F}$.

Proposition 2.4. *For any non-empty set of regular probabilities, $P \subseteq$ rProbs, we can associate a set of probability constraints, $\mathcal{F}_P \subseteq \wp(\text{rProbs})$, defined by:*

$$\mathcal{F}_P = \{Q \subseteq \text{rProbs} \mid Q \supseteq P\}. \tag{1}$$

Then:

(i) *$\mathcal{F}_P$ is coherent.*

(ii) *$\mathcal{F}_P$ is the least committal coherent probability filter where the probability constraint P is believed. That is, for any coherent $\mathcal{F}'$ with $P \in \mathcal{F}'$, we have $\mathcal{F}' \supseteq \mathcal{F}_P$.*

(iii) *For $P \neq P'$, $\mathcal{F}_P \neq \mathcal{F}_{P'}$.*

Proof. It is immediate to check that $\mathcal{F}_P$ satisfies all the axioms for coherence, as given in Definition 2.2.

To observe that $\mathcal{F}_P$ is the least committal coherent opinion set where P is believed, note that any coherent $\mathcal{F}'$ with $P \in \mathcal{F}$ also has any superset of P, any $Q \supseteq P$, also in $\mathcal{F}'$, by axiom ($\text{F}_{\supseteq}$). And thus $\mathcal{F}' \supseteq \mathcal{F}_P$.

If $P \neq P'$, then either $P \not\supseteq P'$ or $P' \not\supseteq P$, so it is trivial to observe that $\mathcal{F}_P \neq \mathcal{F}_{P'}$. $\square$

[5] The name follows Quaeghebeur et al. [25, p.73]

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This result provides us with a clear connection to another popular imprecise probability model of belief: the **credal set** model, which captures an agent's belief state with a single set of probabilities, P. This can be seen as a special case of the probability filter model obtained by restricting just to those probability filters of the form $\mathcal{F}_P$ for a fixed P. In much of the literature [e.g., 21, 13], it is also assumed that P is closed and convex, but some authors argue against this restriction, for example Kyburg Jr and Pittarelli [18, 19], Joyce [15] and in much of the philosophy community the restriction is not imposed [17, 3, 27]. Our model does not impose any convexity requirement as we allow all subsets of rProbs to be probability constraints in $\mathcal{F}$. Proposition 2.4 tells us that every non-empty set of probabilities generates a unique $\mathcal{F}_P$, even when P is not convex. Our model could be restricted to only represent convex sets by inclusion of additional axioms to ensure that $\mathcal{F}$ is determined by its treatment of convex probability constraints (or by restricting the probability constraints, the sorts of things that are in $\mathcal{F}$, to just be convex sets of probabilities). We do not pursue this restriction further in this paper.

Filters that have the form $\mathcal{F}_P$ for some $P \subseteq \text{rProbs}$ are, as standard for filters, called **principal filters**.[6] Not all coherent sets of probability constraints have this form. There are some non-principal filers, and thus some coherent probability filters that do not correspond to any credal set.

Proposition 2.5. *There are coherent $\mathcal{F}$ where $\mathcal{F}$ does not have the form $\mathcal{F}_P$ for any $P \subseteq \text{rProbs}$. (Where $\mathcal{F}_P$ is specified as in Eq. (1)).*

Moreover, there are coherent $\mathcal{F}$ where $\bigcap \mathcal{F} = \varnothing$; that is, there is no single $p \in \text{rProbs}$ which is a member of every $P \in \mathcal{F}$.

We couch the proof of this proposition as an example. It is a probability filter that we will return to throughout the paper.

Example 2.6. Fix $\Omega = \{\mathrm{H}, \mathrm{T}\}$, representing the outcome of a coin toss. We will describe a probability filter $\mathcal{F}_{\text{InfBiased}}$, which does not correspond to any single set of probabilities. Instead, to describe it, we need to specify beliefs about infinitely many probability constraints.

$\mathcal{F}_{\text{InfBiased}}$ believes that the coin is biased towards heads, i.e.,

$$\{p \mid p(H) > 0.5\} \in \mathcal{F}_{\text{InfBiased}}.$$

But also believes that its probability does not exceed 0.5 by any particular amount. For example,

$$\begin{aligned}
&\{p \mid p(H) < 0.51\} \in \mathcal{F}_{\text{InfBiased}}, \\
&\{p \mid p(H) < 0.501\} \in \mathcal{F}_{\text{InfBiased}}, \\
&\{p \mid p(H) < 0.5001\} \in \mathcal{F}_{\text{InfBiased}}, \\
&\text{etc.}
\end{aligned}$$

By using the upcoming Proposition 2.12, one can observe that there is a coherent such probability filter, $\mathcal{F}_{\text{InfBiased}}$ and that the smallest such filter is generated

[6] In fact, principal filters are exactly those that are also closed under *infinite* intersections. We thus have an equivalence between the credal set framework and probability filters which satisfy this infinite intersection property.

by simply closing these specified beliefs under finite intersection and superset. This means that we have, for example, that:

$$\{p \mid 0.5 < p(H) < 0.5001\}.$$

$\mathcal{F}_{\text{InfBiased}}$ believes that the bias of the coin is strictly between 0.5 and any given amount above 0.5.

In fact, in general, $\mathcal{F}_{\text{InfBiased}}$ is given by: $P \in \mathcal{F}_{\text{InfBiased}}$ iff there is some positive real $\epsilon > 0$ with $P \supseteq \{p \mid 0.5 < p(H) < 0.5 + \epsilon\}$.[7]

Note that $\bigcap \mathcal{F}_{\text{InfBiased}} = \varnothing$. That is, there is no single p^* which satisfies every probability constraint in $\mathcal{F}_{\text{InfBiased}}$. This is because any probability function p^* with $p^*(\text{H}) > 0.5$ has $p^*(\text{H}) \geqslant 0.5 + \epsilon$ for some ϵ. In the language of filters, $\mathcal{F}_{\text{InfBiased}}$ is a *free* filter.

2.2 Natural Extensions

Probability filters are easy to work with. In particular, it is very easy to start with knowing some probability constraints as believed and draw out some consequences in virtue of coherence.

Proposition 2.7. *For any coherent $\mathcal{F}$, if $Q_1, \ldots, Q_n \in \mathcal{F}$ and $P \supseteq Q_1 \cap \ldots \cap Q_n$ then also $P \in \mathcal{F}$.*

Proof. Suppose $Q_1, \ldots, Q_n \in \mathcal{F}$. By iterated uses of axiom ($\text{F}_\cap$), we can see that $Q_1 \cap \ldots \cap Q_n \in \mathcal{F}$. Then by axiom ($\text{F}_\supseteq$), $P \in \mathcal{F}$. □

We give two illustrative instances of Proposition 2.7

Example Fact 2.8. *For coherent $\mathcal{F}$, if $\{p \mid p(E) > 0.6\} \in \mathcal{F}$ then $\{p \mid p(E) > 0.2\} \in \mathcal{F}$.*

Proof. This follows from axiom ($\text{F}_\supseteq$), since

$$\{p \mid p(E) > 0.6\} \subseteq \{p \mid p(E) > 0.2\}. \qquad \square$$

It can also be seen as a special case of Proposition 2.7 with $n = 1$.

Our examples so far have typically made use of probability constraints which have a very simple form. However any sets of probabilities count as a probability constraint, including, for example, non-convex sets. We give another example simply to highlight this fact:

Example Fact 2.9. *For coherent $\mathcal{F}$, if $\{p \mid p(E) = 0.3$ or $p(E) = 0.2\} \in \mathcal{F}$ then $\{p \mid p(E) \neq 0.5\} \in \mathcal{F}$.*

Proof. This again follows from axiom ($\text{F}_\supseteq$), since

$$\{p \mid p(E) = 0.3 \text{ or } p(E) = 0.2\} \subseteq \{p \mid p(E) \neq 0.5\}. \qquad \square$$

We give a third example where the fact that we're restricting to sets of regular probabilities is made use of.

[7]Note that there is no extension of the underlying space of reals to a space of hyperreals. The probabilities considered in the probability constraints are real-valued functions.

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Example Fact 2.10. *Suppose $E \cap F \neq \varnothing$. For coherent $\mathcal{F}$, if $\{p \,|\, p(E) \leqslant 0.3\} \in \mathcal{F}$ and $\{p \mid p(F) \leqslant p(E)\} \in \mathcal{F}$ then $\{p \mid p(E \vee F) < 0.6\} \in \mathcal{F}$.*

Proof. Since $E \cap F \neq \varnothing$, any regular probability $p \in \text{rProbs}$ has $p(E \cap F) \neq 0$; and thus $p(E \cup F) = p(E) + p(F) - p(E \cap F) < p(E) + p(F)$. So for $p \in \text{rProbs}$, if $p(E) \leqslant 0.3$ and $p(F) \leqslant p(E)$ then also $p(F) \leqslant 0.3$ and so $p(E \cup F) < 0.3 + 0.3 = 0.6$. I.e., as sets of regular probabilities,

$$\{p \mid p(E) \leqslant 0.3\} \cap \{p \mid p(F) \leqslant p(E)\} \subseteq \{p \mid p(E \vee F) < 0.5\}.$$

So, this immediately follows from Proposition 2.7. $\square$

In fact Proposition 2.7 fully characterises coherence, and determines the so-called natural extension: the smallest coherent probability filter extending a given collection of judgements of probability constraints as believed.

Definition 2.11. For non-empty $\mathcal{E} \subseteq \wp(\text{rProbs})$, $\text{ext}(\mathcal{E})$ is defined by: $P \in \text{ext}(\mathcal{E})$ iff there are some finitely many members of $\mathcal{E}$, $Q_1, \ldots, Q_n$, such that $P \supseteq Q_1 \cap \ldots \cap Q_n$. That is:

$$\text{ext}(\mathcal{E}) := \left\{ P \subseteq \text{rProbs} \;\middle|\; \begin{array}{l} \text{there are some } Q_1, \ldots, Q_n \in \mathcal{E} \\ \qquad \text{(where } n \text{ is finite and non-zero),} \\ \text{with } P \supseteq Q_1 \cap \ldots \cap Q_n. \end{array} \right\}.$$

Proposition 2.12. *Suppose $\mathcal{E} \subseteq \wp(\text{rProbs})$ is non-empty. Then:*

- *There is some coherent $\mathcal{F} \supseteq \mathcal{E}$ iff $\varnothing \notin \text{ext}(\mathcal{E})$; i.e., iff whenever $Q_1, \ldots, Q_n \in \mathcal{E}$, $Q_1 \cap \ldots \cap Q_n \neq \varnothing$ (this is usually called the 'finite intersection property').*
- *If $\varnothing \notin \text{ext}(\mathcal{E})$, then $\text{ext}(\mathcal{E})$ is the least committal coherent filter $\mathcal{F} \supseteq \mathcal{E}$.*

This is a standard result regarding filters [see, e.g., 2, p.58]. We include the proof because it is important in the results that follow.

Proof. As we noted in Proposition 2.7, if $Q_1, \ldots, Q_n \in \mathcal{F}$ with $P \supseteq Q_1 \cap \ldots \cap Q_n$ then $P \in \mathcal{F}$. So, for any coherent $\mathcal{F} \supseteq \mathcal{E}$, we must have $\mathcal{F} \supseteq \text{ext}(\mathcal{E})$.

So it suffices to check that $\text{ext}(\mathcal{E})$ is coherent iff $\varnothing \notin \text{ext}(\mathcal{E})$. Axioms ($\text{F}_\cap$) and ($\text{F}_\supseteq$) hold for any $\text{ext}(\mathcal{E})$ because we have forced $\text{ext}(\mathcal{E})$ to be closed under intersection and supersets by definition. (For the supersets, use $n = 1$.) It is non-empty by assumption on $\mathcal{E}$, since any $P \in \mathcal{E}$ is also in $\text{ext}(\mathcal{E})$. So the only remaining axiom for coherence is (F_{Proper}). Thus, $\text{ext}(\mathcal{E})$ is coherent iff (F_{Proper}) holds of it, i.e., iff $\varnothing \notin \text{ext}(\mathcal{E})$. $\square$

Another useful case is when $\mathcal{E}$ is already closed under finite intersections. In this case, $\text{ext}(\mathcal{E})$ just takes supersets. For example, if we have a descending chain $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \ldots$, then $P \in \text{ext}(\{Q_1, Q_2, \ldots\})$ iff $P \supseteq Q_i$ for some i.

Finally, we present the following result, which shows an important aspect of the model of belief. If $\mathcal{F}$ suspends judgement on Q, i.e., if $Q \notin \mathcal{F}$ and $\overline{Q} \notin \mathcal{F}$, then $\mathcal{F}$ could become more committal either way.

Proposition 2.13. *Suppose $\mathcal{F}$ is coherent. If $\mathcal{F}$ suspends judgement on a probability constraint $Q^* \subseteq \text{rProbs}$, i.e., $Q^* \notin \mathcal{F}$ and $\overline{Q^*} \notin \mathcal{F}$, then there is a coherent probability filter $\mathcal{F}_{\text{bel}}$ which is at least as committal as $\mathcal{F}$, i.e., $\mathcal{F}_{\text{bel}} \supseteq \mathcal{F}$, and which believes Q^*, i.e., $Q^* \in \mathcal{F}_{\text{bel}}$; and there is another coherent probability filter $\mathcal{F}_{\text{disbel}}$ which is at least as committal as $\mathcal{F}$, i.e., $\mathcal{F}_{\text{disbel}} \supseteq \mathcal{F}$, and which disbelieves Q^*, i.e., $\overline{Q^*} \in \mathcal{F}_{\text{disbel}}$*

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Proof. Put $\mathcal{F}_{\text{bel}} := \text{ext}(\mathcal{F} \cup \{Q^*\})$. We just need to check that $\mathcal{F}_{\text{bel}}$ is coherent. By Proposition 2.12, it suffices to check that for any $Q_1, \ldots, Q_n \in \mathcal{F}$ we have that $Q^* \cap Q_1 \cap \ldots \cap Q_n \neq \varnothing$. If $Q^* \cap Q_1 \cap \ldots \cap Q_n = \varnothing$ then $\overline{Q^*} \supseteq Q_1 \cap \ldots \cap Q_n$; and thus by Proposition 2.7, already $\overline{Q^*} \in \mathcal{F}$ by coherence of $\mathcal{F}$. So, since we have assumed that $\overline{Q^*} \notin \mathcal{F}$, we know that $\mathcal{F}_{\text{bel}}$ is coherent. By construction, it is more committal than $\mathcal{F}$ and contains Q^*.

Similarly, we can see that $\mathcal{F}_{\text{disbel}} := \text{ext}(\mathcal{F} \cup \{\overline{Q^*}\})$ is coherent since $Q^* \notin \mathcal{F}$. □

3 Probability filters and desirable gambles

One of the most prominent models of belief in the imprecise probability literature is to model one's belief by a set of desirable gambles [33, 32, 24, 8]. It is a model that encompasses many other models, such as comparative previsions.

In this section we will compare the probability filter model of belief to the set of desirable gambles model.

3.1 The desirable gambles model of belief

Setup 3.1. A **gamble** is a bounded function from Ω to $\mathbb{R}$, i.e., $g : \Omega \to \mathbb{R}$.[8] $\mathcal{G}$ is the collection of all gambles.

0 is a gamble that takes value 0 at every world.

For gambles f and g, when $f(\omega) \geqslant g(\omega)$ for all $\omega \in \Omega$, we will say $f \geqslant g$.

$\mathcal{G}_{\gneq 0}$ is the set of gambles where $f \geqslant 0$ and, for some $\omega \in \Omega$, $f(\omega) > 0$. [9]

For a probability function p and gamble g, we use $\mathbb{E}_p(g)$ for probabilistic expectation, i.e., $\mathbb{E}_p(g) = \sum_{\omega \in \Omega} p(\omega) g(\omega)$.

The positive linear hull of a set is denoted $\mathsf{posi}(B) := \{\sum_{i=1}^n \lambda_i g_i \mid n \in \mathbb{N} \setminus \{0\}, \lambda_i > 0, g_i \in B\}$.

Finally, I_E is the indicator gamble for $E \subseteq \Omega$, given by $I_E := \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E \end{cases}$.

In the desirable gambles framework for modelling uncertainty, we model the subject's beliefs by a set of desirable gambles D, i.e., the set of gambles that she evaluates as preferable to the status quo.

The standard axioms on coherence for a set of desirable gambles are as follows [33]:

Definition 3.2. A set of gambles, $D \subseteq \mathcal{G}$, is **coherent** if it satisfies the four axioms:

(D_0) $0 \notin D$.

($\mathrm{D}_{\gneq 0}$) If $g \in \mathcal{G}_{\gneq 0}$, then $g \in D$.

(D_λ) If $g \in D$ and $\lambda > 0$, then $\lambda g \in D$.

(D_+) If $f, g \in D$, then $f + g \in D$.

[8] Since Ω is finite, functions from Ω to $\mathbb{R}$ are automatically bounded.

[9] These are usually simply denoted with $\mathcal{G}_{>0}$, but I keep the $\gneq$ to highlight the *weak* dominance component, rather than that $g(\omega) > 0$ for all $\omega \in \Omega$.

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We will make use of the following standard results regarding coherent sets of desirable gambles [see, e.g., 1, Theorem 1.1]:

Proposition 3.3. *Suppose D is coherent.*
If $g_1, \dots, g_n \in D$ and $f \in \mathsf{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gneq 0})$, then $f \in D$.

3.2 Associating the frameworks

We associate probability filters with judgements regarding the desirability of gambles by using probabilistic expectations. We will say that $\mathcal{F}$ **judges gamble** g **to be desirable** iff $\{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}$; that is, if $\mathcal{F}$ believes that g has positive expected value. $D_{\mathcal{F}}$ collects all those gambles which $\mathcal{F}$ judges to be desirable.

Definition 3.4. For a probability filter, $\mathcal{F}$, a set of gambles $D_{\mathcal{F}}$ is specified by:

$$D_{\mathcal{F}} = \{g \in \mathcal{G} \mid \{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}\}.$$

We first note an equivalent formulation:

Proposition 3.5. *$g \in D_{\mathcal{F}}$ iff there is some $P \in \mathcal{F}$ with $\mathbb{E}_p(g) > 0$ for all $p \in P$*

Proof. If P is such that $\mathbb{E}_p(g) > 0$ for all $p \in P$, then $P \subseteq \{p \mid \mathbb{E}_p(g) > 0\}$. So by axiom ($\mathrm{F}_{\supseteq}$), $P \in \mathcal{F}$ implies $\{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}$, which implies that $g \in D_{\mathcal{F}}$ by Definition 3.4.

If $\{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}$, then we can put P as $\{p \mid \mathbb{E}_p(g) > 0\}$, observing that $\mathbb{E}_p(g) > 0$ for all $p \in P$. □

It is interesting to then note a special case of this result stated for principal filters (those of the form in Proposition 2.4):

Corollary 3.6. *If $\mathcal{F}$ has the form $\mathcal{F}_{P^*}$ for some $P^* \subseteq \mathrm{rProbs}$, then $g \in D_{\mathcal{F}}$ iff $\mathbb{E}_p(g) > 0$ for all $p \in P^*$.*

Having seen how we are relating the framework of probability filters to that of sets of desirable gambles framework, we will now turn to comparing the two frameworks. We will ask the following questions:

(i) Is every $D_{\mathcal{F}}$ coherent?

(ii) Does the probability filter framework encompass that of coherent sets of desirable gambles? Can every coherent D be obtained as $D_{\mathcal{F}}$ for some $\mathcal{F}$?

(iii) Does the probability filter framework go beyond that of coherent sets of desirable gambles? I.e., are there distinct $\mathcal{F}$ and $\mathcal{F}'$ where $D_{\mathcal{F}} = D_{\mathcal{F}'}$?

We will answer 'yes' to all three of these questions (Theorems 3.7, 3.9 and 3.12).

3.3 Question (i). Every $D_{\mathcal{F}}$ is coherent.

The axioms we imposed for coherence of $\mathcal{F}$, axioms ($\mathrm{F}_{\cap}$), ($\mathrm{F}_{\supseteq}$), ($\mathrm{F}_{\neq\varnothing}$) and ($\mathrm{F}_{\mathrm{Proper}}$), ensure that the resultant desirability judgements are coherent; that is, for a coherent $\mathcal{F}$, the corresponding set of desirable gambles, $D_{\mathcal{F}}$, will satisfy the axioms for coherence of a set of desirable gambles, axioms (D_0), ($\mathrm{D}_{\gneq 0}$), (D_{λ}) and (D_+).

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Theorem 3.7. *If $\mathcal{F}$ is coherent, then $D_{\mathcal{F}}$ is a coherent set of desirable gambles.*

Proof. Axiom (D_0) follows from axiom (F_{Proper}) by observing that $\{p \mid \mathbb{E}_p(0) > 0\} = \varnothing$.

Axiom ($D_{\gtrsim 0}$): If $g \in \mathcal{G}_{\gtrsim 0}$, any $p \in$ rProbs has $\mathbb{E}_p(g) > 0$; so this follows from the fact that rProbs $\in \mathcal{F}$ (using our choice to restrict to regular probabilities, and axioms ($F_{\neq\varnothing}$) and ($F_{\supseteq}$)).

Axiom (D_λ) holds because when $\lambda > 0$, $\{p \mid \mathbb{E}_p(g) > 0\} = \{p \mid \mathbb{E}_p(\lambda g) > 0\}$.

Axiom (D_+): If $g \in D_{\mathcal{F}}$ and $f \in D_{\mathcal{F}}$, then $\{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}$ and $\{p \mid \mathbb{E}_p(f) > 0\} \in \mathcal{F}$. So $\{p \mid \mathbb{E}_p(g) > 0 \text{ and } \mathbb{E}_p(f) > 0\} \in \mathcal{F}$ by axiom ($F_\cap$). If $\mathbb{E}_p(g) > 0$ and $\mathbb{E}_p(f) > 0$ then also $\mathbb{E}_p(g + f) > 0$ by the linearity of probabilistic expectation. So $g + f \in D_{\mathcal{F}}$, using Proposition 3.5. $\square$

3.4 Question (ii). Every coherent D can be obtained as $D_{\mathcal{F}}$ for some $\mathcal{F}$.

We can also reverse the process: starting with a set of desirable gambles, D, we can determine a probability filter $\mathcal{F}_D$ which is the least committal filter evaluating each $g \in D$ as desirable, i.e., which believes that the probabilistic expectation of g is positive for any $g \in D$.

Definition 3.8. For a set of gambles $D \subseteq \mathcal{G}$, define $\mathcal{F}_D \subseteq \wp(\text{rProbs})$ by:

$$\mathcal{F}_D := \text{ext}(\{\{p \mid \mathbb{E}_p(g) > 0\} \mid g \in D\}).$$

Recalling the definition of ext (Definition 2.11), we see that $P \in \mathcal{F}_D$ iff there are finitely many gambles in D, some $g_1, \ldots, g_n \in D$, with

$$P \supseteq \{p \mid \mathbb{E}_p(g_1) > 0\} \cap \ldots \cap \{p \mid \mathbb{E}_p(g_n) > 0\}.$$

Theorem 3.9. *If D is coherent, then $\mathcal{F}_D$ is coherent and $D = D_{\mathcal{F}_D}$, i.e.:*

$$f \in D \textit{ iff } \{p \mid \mathbb{E}_p(f) > 0\} \in \mathcal{F}_D. \tag{2}$$

Proof. Assume D is coherent. We will first show that $D = D_{\mathcal{F}_D}$, i.e, Eq. (2), before checking that $\mathcal{F}_D$ is coherent.

For any $f \in D$ we have $\{p \mid \mathbb{E}_p(f) > 0\} \in \mathcal{F}_D$ by construction. We need to show the converse, that is, if $\{p \mid \mathbb{E}_p(f) > 0\} \in \mathcal{F}_D$ then $f \in D$.

By definition of $\mathcal{F}_D$, if $\{p \mid \mathbb{E}_p(f) > 0\} \in \mathcal{F}_D$ then there are some $g_1, \ldots, g_n \in D$ where $\{p \mid \mathbb{E}_p(f) > 0\} \supseteq \{p \mid \mathbb{E}_p(g_1) > 0\} \cap \ldots \cap \{p \mid \mathbb{E}_p(g_n) > 0\}$. We are aiming to show that this entails that $f \in D$. We will show that it entails that $f \in \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$ and can then use Proposition 3.5 to get that $f \in D$.

We work by showing the contrapositive. That is, if $f \notin \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$, then $\{p \mid \mathbb{E}_p(f) > 0\} \not\supseteq \{p \mid \mathbb{E}_p(g_1) > 0\} \cap \ldots \cap \{p \mid \mathbb{E}_p(g_n) > 0\}$. This requires us to find some regular probability p^* with $\mathbb{E}_{p^*}(g_i) > 0$ for each $i = 1, \ldots, n$, but $\mathbb{E}_{p^*}(f) \leqslant 0$. We will do this by a separating hyperplane result (see Fig. 1).

Let $C := \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$. This is a convex cone which is closed when 0 is included (since it is finitely generated); moreover it is pointed: for any $g^* \in C$, $-g^* \notin C$; otherwise also $-g^* \in D$, so $g^* + -g^* = 0 \in D$, using Proposition 3.3.

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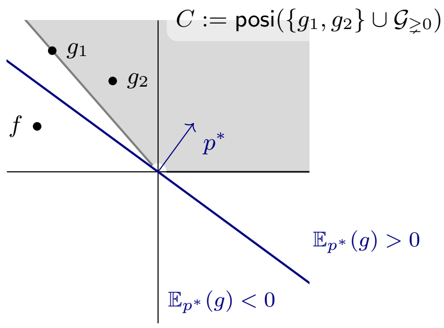

Figure 1: When $f \notin \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gnsim 0})$, we can separate them by a hyperplane.

C does not contain f, and thus, by a separating hyperplane theorem [16, Theorem 2.5],[10] we can find a linear functional T such that $T(g) > 0$ for all $g \in C$ and $T(f) \leqslant 0$. (The pointedness of C suffices for the inequality to be strict for $g \in C$.[11])

We will use this linear functional T to generate our required regular probability p^*. For any $\omega \in \Omega$, the indicator gamble of ω, $I_{\{\omega\}}$, is a member of $\mathcal{G}_{\gnsim 0}$, and thus is also a member of $C = \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gnsim 0})$. So $T(I_\omega) > 0$ for all ω. Since Ω is finite, we can also normalise T and obtain a regular probability $p^*(\omega) := \frac{T(I_{\{\omega\}})}{T(I_\Omega)}$ with $\mathbb{E}_{p^*}(g) > 0$ for all $g \in C$ and $\mathbb{E}_{p^*}(f) \leqslant 0$. So, in particular, $\mathbb{E}_{p^*}(g) > 0$ for each $g \in C$, so in particular, $\mathbb{E}_{p^*}(g_i) > 0$ for each $i = 1, \ldots, n$, and $\mathbb{E}_{p^*}(f) \leqslant 0$; as desired.

This suffices to show that $\{p \mid \mathbb{E}_p(f) > 0\} \in \mathcal{F}_D$ then $f \in D$. We have thus shown Eq. (2), i.e., that $D = D_{\mathcal{F}_D}$.

It remains to show that $\mathcal{F}_D$ is coherent. By Proposition 2.12, we just need to check that $\varnothing \notin \mathcal{F}_D$. For this, we can simply use Eq. (2) and the coherence of D to observe that $\{p \mid \mathbb{E}_p(-1) > 0\} = \varnothing \notin \mathcal{F}_D$. □

An immediate corollary of Theorem 3.9 is:

Corollary 3.10. *For distinct coherent D and D', $\mathcal{F}_D$ and $\mathcal{F}_{D'}$ are distinct.*

Proof. This follows immediately from Eq. (2). □

This tells us that the representational power of the probability filter framework is at least that of the set of desirable gambles framework. In fact it will go beyond this, which we will discuss in the next section. However first it is illustrative to consider an example of a coherent set of gambles which is not representable by a single set of probabilities. By Theorem 3.9, however, it is representable by a probability filter.

Example 3.11. Fix $\Omega = \{\mathrm{H}, \mathrm{T}\}$, representing the outcomes of a coin toss. Consider D^* as illustrated in Fig. 2.

[10] We use this form of the separating hyperplane result because it is what we need for the more general case in Theorem 4.11.

[11] Typically also $T(f) < 0$, except for in the special case where $f = 0$.

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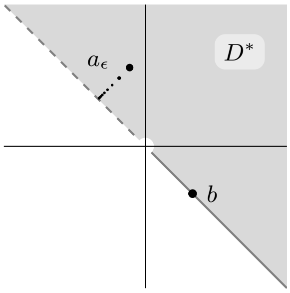

Figure 2: A half-open set of desirable gambles.

In particular, we have:

$$b := \langle 0.5, -0.5 \rangle \in D^*.$$
$$a_\epsilon := \langle -0.5, 0.5 \rangle + \langle \epsilon, \epsilon \rangle = \langle -0.5 + \epsilon, 0.5 + \epsilon \rangle \in D^* \text{ for } \epsilon \text{ any positive real.}$$

So, for example $a_{0.001} = \langle -0.499, 0.501 \rangle \in D^*$.

Observe that $\mathbb{E}_p(b) > 0$ iff $p(\mathrm{H}) > 0.5$. And that $\mathbb{E}_p(a_\epsilon) > 0$ iff $p(\mathrm{H}) < 0.5+\epsilon$. We thus see that there is no single p^* where $\mathbb{E}_p(g) > 0$ for all $g \in D^*$. (See also [1, p.20].)

D^* is, however, a coherent set of desirable gambles, and it can be represented in the probability filter framework due to Theorem 3.9. $\mathcal{F}_{D^*}$ will have that $\{p \mid \mathbb{E}_p(b) > 0\} = \{p \mid p(\mathrm{H}) > 0.5\} \in \mathcal{F}_{D^*}$ and $\{p \mid \mathbb{E}_p(a_\epsilon) > 0\} = \{p \mid p(\mathrm{H}) < 0.5 + \epsilon\} \in \mathcal{F}_{D^*}$ for each positive real ϵ. It can in fact then be observed that $\mathcal{F}_{D^*}$ is identical to $\mathcal{F}_{\mathrm{InfBiased}}$, as in Example 2.6.

3.5 Question (iii). Probability filters go beyond desirability of gambles.

The probability filter model goes beyond the model of desirable gambles. A probability filter judges a gamble to be desirable when $\{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}$. But $\mathcal{F}$ contains other kinds of probability constraints, and how $\mathcal{F}$ treats all these probability constraints is not determined by its treatment of the constraints of the form $\{p \mid \mathbb{E}_p(g) > 0\}$.

This is a consequence of the usual fact that taking a convex closure of a set of probabilities does not affect which gambles, g, are such that $\mathbb{E}_p(g) > 0$ for all p in the set.

Theorem 3.12. *There are distinct coherent probability filters $\mathcal{F}$ and $\mathcal{F}'$ where $D_{\mathcal{F}} = D_{\mathcal{F}'}$.*

Proof. Proposition 2.4 gave us that for any non-empty $P \subseteq$ rProbs, $\mathcal{F}_P$ is coherent and, by Corollary 3.6, $g \in D_{\mathcal{F}_P}$ iff $\mathbb{E}_p(g) > 0$ for all $p \in P$.

So take any non-convex P, and let P' be its convex closure. Then $\mathbb{E}_p(g) > 0$ for all $p \in P$ implies also $\mathbb{E}_p(g) > 0$ for any $p \in P'$. And so, $D_{\mathcal{F}_P} = D_{\mathcal{F}_{P'}}$. However, $\mathcal{F}_P \neq \mathcal{F}_{P'}$, for example, $P' \notin \mathcal{F}_P$. □

If we wished to restrict the expressive power of the probability filter framework to make it equivalent to that of desirable gambles, we could add an additional axiom which ensures that $\mathcal{F}$ is characterised by its judgements on probability constraints of the form $\{p \mid \mathbb{E}_p(g) > 0\}$. This is given by the following axiom:

(F_D) If $P \in \mathcal{F}$ then there are some finitely many gambles $g_1, \ldots, g_n$ with $\{p \mid \mathbb{E}_p(g_i) > 0\} \in \mathcal{F}$ for each g_i and

$$P \supseteq \{p \mid \mathbb{E}_p(g_1) > 0\} \cap \ldots \cap \{p \mid \mathbb{E}_p(g_n) > 0\}.$$

This axiom ensures that $\mathcal{F}$ is determined by its treatment of the probability constraints of the form $\{p \mid \mathbb{E}_p(g) > 0\}$. By Theorem 4.13 we know that this won't hold for all coherent $\mathcal{F}$, but we can guarantee this if we restrict to the probability filters that satisfy axiom (F_D).

Proposition 3.13. *For every coherent D, $\mathcal{F}_D$ satisfies axiom* (F_D)*. Also, if $\mathcal{F}$ is a probability filter satisfying axiom* (F_D) *then $\mathcal{F} = \mathcal{F}_{D_\mathcal{F}}$.*

Proof. To show that every $\mathcal{F}_D$ satisfies axiom (F_D), we see by the definition of $\mathcal{F}_D$ (Definition 3.8) that $P \in \mathcal{F}_D$ iff there are finitely many $g_1, \ldots, g_n \in D$ with

$$P \supseteq \{p \mid \mathbb{E}_p(g_1) > 0\} \cap \ldots \cap \{p \mid \mathbb{E}_p(g_n) > 0\}.$$

And observe that $\{p \mid \mathbb{E}_p(g_i) > 0\} \in \mathcal{F}_D$, so we have axiom ($\mathrm{F}_D$).

To show that $\mathcal{F} = \mathcal{F}_{D_\mathcal{F}}$, we also consult the definition of $D_\mathcal{F}$ (Definition 3.4) and see that

$$P \in \mathcal{F}_{D_\mathcal{F}} \text{ iff } \quad \begin{array}{l} \text{there are some } g_1, \ldots, g_n \\ \text{with } \{p \mid \mathbb{E}_p(g_i) > 0\} \in \mathcal{F} \text{ for each } i \\ \text{and } P \supseteq \{p \mid \mathbb{E}_p(g_1) > 0\} \cap \ldots \cap \{p \mid \mathbb{E}_p(g_n) > 0\}. \end{array}$$

So axiom (F_D) exactly guarantees that $\mathcal{F} \subseteq \mathcal{F}_{D_\mathcal{F}}$. We also have $\mathcal{F}_{D_\mathcal{F}} \subseteq \mathcal{F}$ just by Proposition 2.7. □

A more independent characterisation of axiom (F_D) would be desirable, but we leave this for future work. However, we do not propose it as an axiom which should be adopted when modelling uncertain belief. There are important differences of opinion which it rules out. We now turn to a more general model of uncertainty.

4 Probability filters and choice functions

4.1 Choice functions

Choice functions provide a more general model of uncertainty than sets of desirable gambles, which can be seen as restricting attention to binary choice [28]. We adopt the framework for choice functions of De Bock and De Cooman [10], De Bock and de Cooman [11], van Camp [30] where they are given a desirability based characterisation, because this then easily extends the model of desirable gambles that we have just considered. (The connection to choice is spelled out in De Bock and De Cooman [10].)

Whereas the desirable gambles framework just considers whether an individual gamble is desirable or not, in this more general framework we consider whether a set of gambles contains at least one desirable gamble. This thus provides a set

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of sets of gambles, $\mathcal{K} \subseteq \wp(\mathcal{G})$, where a set of gambles (also called a gamble set), B, is in $\mathcal{K}$ if you think that at least one member of B is desirable.

We will consider how the probability filters model links to this desirable gamble sets model. Using a probability filter, $\mathcal{F}$, we can determine judgements of when a set of gambles, $B \subseteq \mathcal{G}$ contains at least one member which is desirable. We can then collect those gamble sets which are judged as desirable into a set $\mathcal{K}_{\mathcal{F}}$. So, what is it for $\mathcal{F}$ to judge that B contains at least one member which is desirable. Does there need to be some particular member of the set which is judged as desirable, or does it suffice that you think that something is desirable without being able to identify any particular one? In the special case when filters are principal filters, this is equivalent to the question of whether a choice function is related to a credal set according to Maximality [32, 29] or E-admissibility [20, 21]; see Seidenfeld et al. [28] on how this makes a difference to the choice function associated with a non-convex set of probabilities.[12]

To apply the analogue of E-admissibility in this setting, there doesn't need to be a particular gamble that you think is desirable, so long as you think that at least one of them is.[13] That is, we can have $B \in \mathcal{K}_{\mathcal{F}}$ when there is no $g \in B$ where $\{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}$, instead we look at whether the probability constraint expressing the judgement that some member of B is something that $\mathcal{F}$ believes. That is:

$$B \in \mathcal{K}_{\mathcal{F}} \text{ iff } \{p \mid \text{there is some } g{\in}B \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F} \tag{3}$$

This is the relationship between the models which we accept as our definition.

Definition 4.1. For each probability filter, $\mathcal{F}$, define a set of gamble sets $\mathcal{K}_{\mathcal{F}}$ by:

$$\mathcal{K}_{\mathcal{F}} := \{B \subseteq \mathcal{G} \mid \{p \mid \text{there is some } g{\in}B \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}\}$$

There is an alternative association which could be given which parallels the Maximality rule:

$$B \in \mathcal{K}^{\text{Max}}_{\mathcal{F}} \text{ iff there is some } g \in B \text{ with } \{p \mid \mathbb{E}_p(g) > 0\} \in \mathcal{F}.$$

We will not further consider this latter association, but will work with $\mathcal{K}_{\mathcal{F}}$, based on E-admissibility. This is closely related to what is done by Seidenfeld et al. [28] who are axiomatise choice functions associated with credal sets by E-admissibility. It is in contrast to De Bock and De Cooman [10] who, following van Camp [30], want their axioms to allow for both choice rules, so they don't have a unique way to associate a set of desirable gamble sets with a credal set, or in our case, a probability filter.

It is useful to present an alternative characterisation:

Proposition 4.2. *$B \in \mathcal{K}_{\mathcal{F}}$ iff there is some $P \in \mathcal{F}$ where for every $p \in P$ there is some $g \in B$ where $\mathbb{E}_p(g) > 0$.*

[12] For more discussion in relation to choice functions or sets of desirable gamble sets, see [26, 30, 11]

[13] Linking to choice functions, this says you reject an option, o, from a set, O, if you think it is non-optimal, in the sense that $\{p \mid \text{there is some } o' \in O \text{ with } \mathbb{E}_pU(o') > \mathbb{E}_pU(o)\} \in \mathcal{F}$. If the filter is principal, given by credal set P, this is just when every $p \in P$ has some $o' \in O$ with $\mathbb{E}_pU(o') > \mathbb{E}_pU(o)$.

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Proof. If $B \in \mathcal{K}_\mathcal{F}$, then $P := \{p \mid \text{there is some } g \in B \text{ with } \mathbb{E}_p(g) > 0\}$ is such a P. For the converse, if there is some such $P \in \mathcal{F}$, then observe that $P \subseteq \{p \mid \text{there is some } g \in B \text{ with } \mathbb{E}_p(g) > 0\}$, so then by axiom ($\mathrm{F}_\supseteq$), $\{p \mid \text{there is some } g \in B \text{ with } \mathbb{E}_p(g) > 0\}$ is itself in $\mathcal{F}$, and thus $B \in \mathcal{K}_\mathcal{F}$. □

We also have an even simpler characterisation in the case where the filter is principal:

Proposition 4.3. *If $\mathcal{F}$ is principal, and has the form $\mathcal{F}_{P^*}$ for some $P^* \subseteq \mathrm{rProbs}$, then $B \in \mathcal{K}_\mathcal{F}$ iff for every $p \in P^*$ there is some $g \in B$ where $\mathbb{E}_p(g) > 0$.*

Proof. Proposition 4.2 gives us the right-to-left direction since $P^* \in \mathcal{F}_{P^*}$. For the left-to-right direction, we know by Proposition 4.2 that there is some $Q \in \mathcal{F}$ where every $p \in Q$ has some $g \in B$ with $\mathbb{E}_p(g) > 0$. Since $Q \supseteq P^*$, also every $p \in P^*$ has some $g \in B$ with $\mathbb{E}_p(g) > 0$, as required. □

We have now specified how we determine a set of desirable gamble sets from a given probability filter: we have adopted Eq. (3), generalising E-admissibility to the probability filter framework.

We now present our key questions, paralleling those we asked in the desirable gambles framework.

(i) Is every $\mathcal{K}_\mathcal{F}$ is coherent?

(ii) Does the probability framework filter encompass the choice functions framework? Can every coherent $\mathcal{K}$ be obtained as $\mathcal{K}_\mathcal{F}$ for some coherent probability filter, $\mathcal{F}$?

(iii) Does the probability filter framework go beyond that of coherent sets of desirable gamble sets? I.e., are there distinct $\mathcal{F}$ and $\mathcal{F}'$ where $\mathcal{K}_\mathcal{F} = \mathcal{K}_{\mathcal{F}'}$?

To give answers to these questions we will need to specify a notion of coherence for sets of desirable gamble sets, which is what we turn to in the next section.

4.2 The axioms on sets of desirable gamble sets

There are two axiomatisations in the literature for sets of desirable gamble sets: Seidenfeld et al. [28], De Bock and De Cooman [10]. De Bock and De Cooman give their axiomatisation in the framework of sets of desirable gamble sets, rather than Seidenfeld et al.'s axiomatisation of choice functions selecting admissible options from option sets. We thus primarily focus on the axiomatisation of De Bock and De Cooman.

The answer to the first question is 'yes' when we use the axiomatisation of coherence given by De Bock and De Cooman [10], i.e., any $\mathcal{K}_\mathcal{F}$ is coherent in their sense. For this to be true, it is important that De Bock and De Cooman do not impose any Archimedeanicity axiom.[14] This is in contrast to Seidenfeld et al. [28], who are axiomatising choice functions obtained from credal sets. Some $\mathcal{K}_\mathcal{F}$ will be non-Archimadean. This is because of our model of probability filters going beyond choice functions, and examples like Example 2.6.

However, simply using the axiomatisation of De Bock and De Cooman [10] we would get a negative answer to the second question: some coherent $\mathcal{K}$ cannot

[14] See also De Bock and de Cooman [11, §9].

be represented as $\mathcal{K}_{\mathcal{F}}$. This is because of our choice to associate a filter with a choice function using the analogue of E-admissibility. We do, however, capture all those coherent $\mathcal{K}$ which satisfy a so-called mixing axiom, as is adopted in the axiomatisation of Seidenfeld et al. [28]. de Cooman et al. [12] give related results representing *all* coherent $\mathcal{K}$ with filters, but these filters are not based on probabilities.

We will also need to extend the axiomatisation of De Bock and De Cooman [10] in additional ways to accommodate the fact that we allow the gamble sets $A \in \mathcal{K}$ to be infinite whereas they restrict to gamble sets being finite. This is our addition of axiom ($\mathrm{K}_{\mathrm{Improve}}$) and a slight strengthening of ($\mathrm{K}_{\mathrm{Add}}$) as compared to the analogous axiom in De Bock and De Cooman [10]. We will first present the axioms before discussing these choices.

In stating the mixing axiom, we will make use of the notion of $\mathsf{clposi}(B)$. This is the topological closure of $\mathsf{posi}(B)$ in the topology of pointwise convergence [see, e.g., 34, §42.2]. (See Setup 3.1 for the definition of posi.)

Definition 4.4. $\mathcal{K} \subseteq \wp(\mathcal{G})$ is **coherent** if it satisfies

($\mathrm{K}_{\varnothing}$) $\varnothing \notin \mathcal{K}$

(K_0) If $A \in \mathcal{K}$ then $A \setminus \{0\} \in \mathcal{K}$.

($\mathrm{K}_{\gneq 0}$) If $g \in \mathcal{G}_{\gneq 0}$, then $\{g\} \in \mathcal{K}$

($\mathrm{K}_{\supseteq}$) If $A \in \mathcal{K}$ and $B \supseteq A$, then $B \in \mathcal{K}$

($\mathrm{K}_{\mathrm{Improve}}$) If $A \in \mathcal{K}$ and for each $g \in A$, f_g is some gamble where $f_g \geqslant g$, then $\{f_g \mid g \in A\} \in \mathcal{K}$.

($\mathrm{K}_{\mathrm{Add}}$) If $A_1, \ldots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \ldots, g_n\rangle \in A_1 \times \ldots \times A_n$, $f_{\langle g_1,\ldots,g_n\rangle}$ is some member of $\mathsf{posi}(\{g_1, \ldots, g_n\})$, then $\{f_{\langle g_1,\ldots,g_n\rangle} | \langle g_1, \ldots, g_n\rangle \in A_1 \times \ldots \times A_n\} \in \mathcal{K}$

$\mathcal{K}$ is **mixing coherent** if it also satisfies:

($\mathrm{K}_{\mathrm{Mix}}$) If $A \in \mathcal{K}$ and $\mathsf{clposi}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

We immediately present an important consequence of the axiomatisation, which will play a key role in Theorem 4.11.[15]

Proposition 4.5. *Suppose $\mathcal{K}$ is coherent. If $A_1, \ldots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \ldots, g_n\rangle$ with $g_1 \in A_1, \ldots, g_n \in A_n$, we have*

$$B \cap \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0}) \neq \varnothing,$$

then $B \in \mathcal{K}$.

Proof. Assume that there are some $A_1, \ldots, A_n \in \mathcal{K}$ such that for each sequence $g_1 \in A_1, \ldots, g_n \in A_n$, we have some $f_{\langle g_1,\ldots,g_n\rangle} \in B \cap \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$.

If there is some sequence $\langle g_1^*, \ldots, g_n^*\rangle$ with $f_{\langle g_1^*,\ldots,g_n^*\rangle} \in \mathcal{G}_{\gneq 0}$, then we can immediately see that $B \in \mathcal{K}$. This is because: by axiom ($\mathrm{K}_{\gneq 0}$), the singleton $\{f_{\langle g_1^*,\ldots,g_n^*\rangle}\} \in \mathcal{K}$, and thus by axiom ($\mathrm{K}_{\supseteq}$), $B \in \mathcal{K}$.

[15] Campbell-Moore [5, Theorem 2.1] provides a slight extension of this which is sufficient to characterise the natural extension for $\mathcal{K}$.

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So we can assume that each $f_{\langle g_1,\dots,g_n\rangle} \notin \mathcal{G}_{\gneq 0}$. Since $f_{\langle g_1,\dots,g_n\rangle} \in \mathsf{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0})$ and $f_{\langle g_1,\dots,g_n\rangle} \notin \mathcal{G}_{\gneq 0}$, one can find some $h_{\langle g_1,\dots,g_n\rangle} \in \mathsf{posi}(\{g_1,\dots,g_n\})$ with $f_{\langle g_1,\dots,g_n\rangle} \geqslant h_{\langle g_1,\dots,g_n\rangle}$.

By axiom ($\mathrm{K_{Add}}$), $\{h_{\langle g_1,\dots,g_n\rangle} \mid g_1 \in A_1, \dots, g_n \in A_n\} \in \mathcal{K}$; and then by axiom ($\mathrm{K_{Improve}}$), $\{f_{\langle g_1,\dots,g_n\rangle} \mid g_1 \in A_1, \dots, g_n \in A_n\} \in \mathcal{K}$. And thus $B \in \mathcal{K}$ by axiom ($\mathrm{K}_\supseteq$). □

Corollary 4.6. *Suppose $\mathcal{K}$ is mixing coherent. If $A_1,\dots,A_n \in \mathcal{K}$ and for each sequence $\langle g_1,\dots,g_n\rangle$ with $g_1 \in A_1,\dots,g_n \in A_n$, we have $\mathsf{clposi}(B) \cap \mathsf{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0}) \neq \varnothing$, then $B \in \mathcal{K}$.*

Proof. Take $A_1,\dots,A_n \in \mathcal{K}$ and assume that for each sequence $\langle g_1,\dots,g_n\rangle$ where $g_1 \in A_1,\dots,g_n \in A_n$, $\mathsf{clposi}(B) \cap \mathsf{posi}(\{g_1,\dots,g_n\} \cup \mathcal{G}_{\gneq 0}) \neq \varnothing$. By Proposition 4.5, we know that for $B' := \mathsf{clposi}(B)$, $B' \in \mathcal{K}$. It then follows immediately from axiom ($\mathrm{K_{Mix}}$), that $B \in \mathcal{K}$. □

Proposition 4.5 and Corollary 4.6 do not in fact depend on the assumption that Ω is finite. It is for this reason that we have opted to include axiom ($\mathrm{K_{Improve}}$) in our axiomatisation. It is an open question whether it follows from the other axioms when Ω is infinite, but it follows from them when Ω is finite.

Proposition 4.7. *Suppose $\mathcal{K}$ satisfies axioms* ($\mathrm{K}_{\gneq 0}$) *and* ($\mathrm{K_{Add}}$)*. Then, given our assumption that Ω is finite, it satisfies axiom* ($\mathrm{K_{Improve}}$)*.*

Proof. Let $\Omega = \{\omega_1,\dots,\omega_n\}$. If $f_g \geqslant g$ then $f_g \in \mathsf{posi}(\{g, I_{\omega_1},\dots,I_{\omega_n}\})$. So since $I_{\omega_i} \in \mathcal{G}_{\gneq 0}$ for each $i \in \{1,\dots,n\}$, by ($\mathrm{K}_{\gneq 0}$), each singleton $\{I_{\omega_i}\} \in \mathcal{K}$; and thus $\{f_g \mid g \in A\} \in \mathcal{K}$ by axiom ($\mathrm{K_{Add}}$). □

This tells us that at least when we restrict to finite Ω, as we in fact do in this paper, axiom ($\mathrm{K_{Improve}}$) can be omitted. Adding this axiom was one of the changes that we made to the axiomatisation given by De Bock and De Cooman [10] due to our change of setting to allow the gamble sets, $A \in \mathcal{K}$, to be infinite.

The other change we have made to their axiomatisation is to strengthen axiom ($\mathrm{K_{Add}}$) so that it applies immediately to finitely many $A_1,\dots,A_n \in \mathcal{K}$; whereas the axiom of De Bock and De Cooman [10] applies just to a pair A_1 and A_2. When the gamble sets must be finite, it can be iterated to obtain axiom ($\mathrm{K_{Add}}$). This is a consequence of their representation result [10, Theorem 7]. It can also be proved directly as in Campbell-Moore [6, §3.1]. But since we are working in the setting where the gamble sets may be infinite. It is an open question whether it is derivable from the pair-version when the gamble sets are infinite (as they are in this paper); we conjecture that it does not.

The final comment we should make on the exact specification of axiom ($\mathrm{K_{Mix}}$). Our formulation of this axiom is similar to the formulation of De Bock and de Cooman [11, Sec. 8] except that it follows Seidenfeld et al. [28, axiom 2b] in taking the *closure*. This is not present in De Bock and de Cooman as they are restricting attention to finite sets of gambles. Further discussion of the formulation of this axiom is left to Appendix A, and we move to our answers to our two main questions.

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4.3 Question (i). Every $\mathcal{K}_{\mathcal{F}}$ is (mixing) coherent.

Our first result shows us that these axioms follow from our probability filter axioms. Since the axioms we give extend those of De Bock and De Cooman [10], this also shows that the De Bock and De Cooman axioms are satisfied by every $\mathcal{K}_{\mathcal{F}}$.

Theorem 4.8. *If $\mathcal{F}$ is coherent, then $\mathcal{K}_{\mathcal{F}}$ is mixing coherent.*

Proof. In the proof, we use the notation

$$[\![B]\!] := \{p \mid \text{there is some } g \in B \text{ with } \mathbb{E}_p(g) > 0\}$$

So, we can rewrite the definition of $\mathcal{K}_{\mathcal{F}}$ as $B \in \mathcal{K}_{\mathcal{F}}$ iff $[\![B]\!] \in \mathcal{F}$. We check each of the axioms.

Axiom ($K_\varnothing$) follows from axiom (F_{Proper}) as $[\![\varnothing]\!] = \varnothing \notin \mathcal{F}$.

Axiom (K_0): Note that $\mathbb{E}_p(0) = 0$. So for every p, if $g \in A$ with $\mathbb{E}_p(g) > 0$, then $g \in A \setminus \{0\}$; thus $[\![A]\!] \subseteq [\![A \setminus \{0\}]\!]$; so $[\![A]\!] \in \mathcal{F}$ implies $[\![A \setminus \{0\}]\!] \in \mathcal{F}$ by axiom ($F_\supseteq$), as required.

Axiom ($K_{\gneq 0}$): If $g \in \mathcal{G}_{\gneq 0}$, then every $p \in \text{rProbs}$ has $\mathbb{E}_p(g) > 0$, so $[\![\{g\}]\!] = \text{rProbs} \in \mathcal{F}$ (axiom ($F_{\neq\varnothing}$) and ($F_\supseteq$)).

Axiom ($K_\supseteq$) follows from axiom ($F_\supseteq$) because $B \supseteq A$ implies $[\![B]\!] \supseteq [\![A]\!]$.

Axiom (K_{Improve}): observe that if $\mathbb{E}_p(g) > 0$ then $\mathbb{E}_p(f_g) > 0$, so it holds by axiom ($F_\supseteq$).

Axiom (K_{Add}): Suppose $A_1, \ldots, A_n \in \mathcal{K}_{\mathcal{F}}$, i.e., $[\![A_1]\!], \ldots, [\![A_n]\!] \in \mathcal{F}$. Let $B = \{f_{\langle g_1,\ldots,g_n\rangle} \mid \langle g_1, \ldots, g_n\rangle \in A_1, \ldots, A_n\}$ with $f_{\langle g_1,\ldots,g_n\rangle} \in \mathsf{posi}(\{g_1, \ldots, g_n\})$. We need to show that $B \in \mathcal{K}_{\mathcal{F}}$; i.e., that $[\![B]\!] \in \mathcal{F}$. We will show that $[\![B]\!] \supseteq [\![A_1]\!] \cap \ldots \cap [\![A_n]\!]$, and can then use Proposition 2.7 to conclude that $[\![B]\!] \in \mathcal{F}$, as required. For any $p^* \in [\![A_1]\!] \cap \ldots \cap [\![A_n]\!]$, there are some $g_1^* \in A_1$, $\ldots$, $g_n^* \in A_n$ with $\mathbb{E}_{p^*}(g_i^*) > 0$ for each i. For $f_{\langle g_1^*,\ldots,g_n^*\rangle} \in \mathsf{posi}(\{g_1^*, \ldots, g_n^*\})$, also $\mathbb{E}_{p^*}(f_{\langle g_1^*,\ldots,g_n^*\rangle}) > 0$. This uses the linearity of probabilistic expectation, $\mathbb{E}_{p^*}(f_{\langle g_1^*,\ldots,g_n^*\rangle}) = \mathbb{E}_{p^*}(\sum_i \lambda_i g_i^*) = \sum_i \lambda_i \mathbb{E}_{p^*}(g_i^*) > 0$. We have thus shown that any $p \in [\![A_1]\!] \cap \ldots \cap [\![A_n]\!]$, has some $g_1 \in A_1$, $\ldots$, $g_n \in A_n$ such that $\mathbb{E}_p(f_{\langle g_1,\ldots,g_n\rangle}) > 0$; and therefore also $p \in [\![B]\!]$. So we have shown that $[\![B]\!] \supseteq [\![A_1]\!] \cap \ldots \cap [\![A_n]\!]$, as required.

We have thus shown that $\mathcal{K}_{\mathcal{F}}$ is coherent. We now need to show that it also satisfies axiom (K_{Mix}). We consider the closure and positive hull parts separately, observing that they can be combined to obtain axiom (K_{Mix}) (see proposition A.1).

Let $\mathsf{posi}(B) \supseteq A$ and $[\![A]\!] \in \mathcal{F}$. We need to show that $[\![B]\!] \in \mathcal{F}$. By axiom ($K_\supseteq$), it suffices to show that $[\![A]\!] \subseteq [\![B]\!]$. So we need to show that if $\mathbb{E}_p(f) > 0$ for some $f \in A$ then $\mathbb{E}_p(g) > 0$ for some $g \in B$. Since we have assumed that $A \subseteq \mathsf{posi}(B)$, if $f \in A$ then there are some $g_1, \ldots, g_n \in B$ and λ_i positive with $f = \lambda_1 g_1 + \ldots + \lambda_n g_n$. If $\mathbb{E}_p(f) > 0$, then also $\mathbb{E}_p(g_i) > 0$ for some i. And thus we have found some $g \in B$ with $\mathbb{E}_p(g) > 0$, as required.

If $f \in A \subseteq \mathsf{closure}(B)$, then there is a sequence, $\langle g_n\rangle$, of members of B with g_n converging to f. If $\mathbb{E}_p(f) > 0$, then there is some g_m (in fact a tail) with $\mathbb{E}_p(g_m) > 0$. So again $[\![A]\!] \subseteq [\![B]\!]$, as required. □

In obtaining this result it is important that we have not extended axiom (K_{Add}) to apply to infinitely many sets. Consider the following possible axiom for sets of desirable gambles.

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($\mathrm{K}_{\mathrm{InfAdd}}$) If $A_i \in \mathcal{K}$ for each $i \in I$ (where I is possibly infinite) and for each sequence $\langle g_i \rangle_{i \in I}$ with $g_i \in A_i$ for each $i \in I$, $f_{\langle g_i \rangle_i}$ is some member of $\mathsf{posi}(\{g_i \mid i \in I\})$, then $\{f_{\langle g_i \rangle_i} \mid \langle g_i \rangle_i \in \bigtimes_{i \in I} A_i\} \in \mathcal{K}$

This is adopted in De Bock [9]. But it is not satisfied in every $\mathcal{K}_{\mathcal{F}}$.

Proposition 4.9. *Some coherent probability filters, $\mathcal{F}$, result in a set of desirable gamble sets, $\mathcal{K}_{\mathcal{F}}$, which violate axiom* ($\mathrm{K}_{\mathrm{InfAdd}}$).

Proof. Fix $\Omega = \{\mathrm{H}, \mathrm{T}\}$ and consider $\mathcal{F}_{\mathrm{InfBiased}}$ as in Example 2.6 (see also Example 3.11); that is $\{p \mid p(\mathrm{H}) > 0.5\} \in \mathcal{F}_{>}$ and also $\{p \mid p(H) < 0.5 + \epsilon\}$ for every positive real number ϵ. Consider the gambles as in Example 3.11

$$a_\epsilon := \langle -0.5, 0.5 \rangle + \langle \epsilon, \epsilon \rangle = \langle -0.5 + \epsilon, 0.5 + \epsilon \rangle \text{ for } \epsilon \text{ any positive real.}$$

So, for example $a_{0.001} = \langle -0.499, 0.501 \rangle$.

Observe that $\mathbb{E}_p(a_\epsilon) > 0$ iff $p(\mathrm{H}) < 0.5 + \epsilon$. Thus, since $\{p \mid p(H) < 0.5 + \epsilon\} \in \mathcal{F}_{\mathrm{InfBiased}}$ for each positive real ϵ, each singleton $\{a_\epsilon\}$ is in $\mathcal{K}_{\mathcal{F}_{\mathrm{InfBiased}}}$.

Consider also

$$A^* := \{-a_\epsilon \mid \epsilon \text{ a positive real}\}$$

Consider any probability p with $p(\mathrm{H}) > 0.5$. Since probabilities are real-valued, there must be some positive real ϵ with $p(\mathrm{H}) > 0.5 + \epsilon$. For this ϵ, then, $\mathbb{E}_p(-a_\epsilon) > 0$. Thus, since $\{p \mid p(\mathrm{H}) > 0.5\} \in \mathcal{F}_{>}$, using Proposition 4.2, we have that $A^* \in \mathcal{K}_{\mathcal{F}_{\mathrm{InfBiased}}}$.

We can now show that axiom ($\mathrm{K}_{\mathrm{InfAdd}}$) fails for $\mathcal{K}_{\mathcal{F}_{\mathrm{InfBiased}}}$. Consider the infinitely many sets $A_1 = A^*$, $A_2 = \{a_{1/2}\}$, $A_3 = \{a_{1/3}\}$, and so on. We have observed that each is a member of $\mathcal{K}_{\mathcal{F}_{\mathrm{InfBiased}}}$.

To evaluate axiom ($\mathrm{K}_{\mathrm{InfAdd}}$), we need to work with sequences $\langle g_i \rangle$ where $g_i \in A_i$ for each i. Such sequences must have the form $\langle -a_\epsilon, a_{1/2}, a_{1/3}, \ldots \rangle$ where ϵ is a positive real. For each ϵ, there is some n where $\epsilon > 1/n$, so then $-a_\epsilon + a_{1/n} \leqslant 0$, and so $0 \in \mathsf{posi}(\{-a_\epsilon, a_{1/2}, a_{1/3}, \ldots\})$. Axiom ($\mathrm{K}_{\mathrm{InfAdd}}$) would then require that $\{0\} \in \mathcal{K}_{\mathcal{F}_{\mathrm{InfBiased}}}$ (by putting $f_{\langle g_i \rangle_i} = 0$ for every $\langle g_i \rangle$). But $\{0\} \notin \mathcal{K}_{\mathcal{F}_{\mathrm{InfBiased}}}$, so we violate axiom ($\mathrm{K}_{\mathrm{InfAdd}}$). □

More generally, it is important that we have not added any further axioms which are not derivable from the axioms here. This is a consequence of our main result in the next section, Theorem 4.11, which shows that every set of desirable gamble sets satisfying these axioms is obtained from some probability filter. So any axiom that goes beyond the specified axioms must fail for some $\mathcal{K}_{\mathcal{F}}$.

4.4 Question (ii). Every mixing coherent $\mathcal{K}$ is obtained from some $\mathcal{F}$.

We will now show that the probability filter model captures all the representational power of mixing coherent sets of desirable gamble sets. For any coherent mixing set of gamble sets, we can find a filter which evaluates exactly those gamble sets as having a member which is desirable.

Definition 4.10. For a set of sets of gambles, $\mathcal{K} \subseteq \wp(\mathcal{G})$, define $\mathcal{F}_{\mathcal{K}}$ by:

$$\mathcal{F}_{\mathcal{K}} := \mathrm{ext}(\{p \mid \{\text{there is some } g \in A \text{ with } \mathbb{E}_p(g) > 0\} \mid A \in \mathcal{K}\})$$

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That is, $P \in \mathcal{F}_\mathcal{K}$ iff there some finitely many sets of gambles in $\mathcal{K}$, $A_1 \in \mathcal{K}, \dots, A_n \in \mathcal{K}$, with

$$P \supseteq \left\{ p \;\middle|\; \begin{array}{l} \text{there is some } g \in A_1 \text{ with } \mathbb{E}_p(g) > 0, \text{ and} \\ \dots, \text{ and} \\ \text{there is some } g \in A_n \text{ with } \mathbb{E}_p(g) > 0 \end{array} \right\}.$$

Theorem 4.11. *For any coherent set of desirable gamble sets, $\mathcal{K}$, $\mathcal{F}_\mathcal{K}$ is coherent and $\mathcal{K} = \mathcal{K}_{\mathcal{F}_\mathcal{K}}$, i.e.,*

$$B \in \mathcal{K} \textit{ iff } \{p \mid \textit{there is some } g \in B \textit{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}_\mathcal{K} \tag{4}$$

We now present our main result. The proof strategy extends that of Theorem 3.9.

Proof. We continue to use the notation

$$[\![B]\!] := \{p \mid \text{there is some } g \in B \text{ with } \mathbb{E}_p(g) > 0\}.$$

Assume $\mathcal{K}$ is coherent. We will first show that $\mathcal{K} = \mathcal{K}_{\mathcal{F}_\mathcal{K}}$, i.e, Eq. (4), before checking that $\mathcal{F}_\mathcal{K}$ is coherent.

For any $B \in \mathcal{K}$, it is immediate from the definition (4.10) that we have $[\![B]\!] \in \mathcal{F}_\mathcal{K}$. We need to show the converse, that is, if $[\![B]\!] \in \mathcal{F}_\mathcal{K}$ then $B \in \mathcal{K}$.

By definition of $\mathcal{F}_\mathcal{K}$, if $[\![B]\!] \in \mathcal{F}_\mathcal{K}$ then there are some $A_1, \dots, A_n \in \mathcal{K}$ where $[\![B]\!] \supseteq [\![A_1]\!] \cap \dots \cap [\![A_n]\!]$. We are aiming to show that this entails that $B \in \mathcal{K}$. We will show that this entails that for each sequence $\langle g_1, \dots, g_n \rangle$ with $g_1 \in A_1, \dots, g_n \in A_n$, we have $\mathsf{clposi}(B) \cap \mathsf{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0}) \neq \varnothing$, and then use Corollary 4.6 to get that $B \in \mathcal{K}$.

We work by showing the contrapositive. That is, if $\mathsf{clposi}(B)$ and $\mathsf{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$ are disjoint, then $\mathsf{clposi}(B) \cap \mathsf{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0}) \neq \varnothing$. This requires us to find some regular probability p^* with $\mathbb{E}_{p^*}(g_i) > 0$ for each $i = 1, \dots, n$, but $\mathbb{E}_{p^*}(f) \leqslant 0$ for all $f \in B$. We will do this by using a separating hyperplane result (Fig. 3).

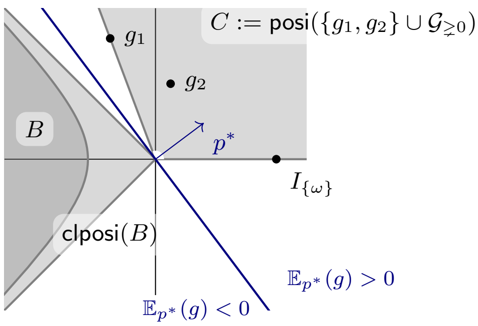

Figure 3: When $\mathsf{clposi}(B)$ and $C := \mathsf{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$ are disjoint they can be separated.

Put $C := \mathsf{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\gtrsim 0})$. This is a convex cone which is closed when 0 is included (since it is finitely generated) and disjoint from $\mathsf{clposi}(B)$. If there

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is some $g^* \in C$ with also $-g^* \in C$, then also $g^* + -g^* = 0 \in C$; and thus, since $0 \in \mathsf{clposi}(B)$, we have $0 \in \mathsf{clposi}(B) \cap \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, contradicting our assumption of disjointness.[16] Thus, we know that for any $g^* \in C$, $-g^* \notin C$, i.e., C is pointed.

We can thus use a separating hyperplane theorem [16, Theorem 2.5] to find a linear functional T such that $T(g) > 0$ for all $g \in C$ and $T(f) \leqslant 0$ for all $f \in B$. (The pointedness of C suffices for the inequality to be strict for $g \in C$.)

We will use this linear functional T to generate our required regular probability p^*. For any $\omega \in \Omega$, the indicator gamble of ω, $I_{\{\omega\}}$, is a member of $C = \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$. So $T(I_\omega) > 0$ for all ω. Since Ω is finite, we can also normalise T and obtain a regular probability $p^*(\omega) := \frac{T(I_{\{\omega\}})}{T(I_\Omega)}$ with $\mathbb{E}_{p^*}(g_i) > 0$ for each $i = 1, \ldots, n$ (as these are in C) and $\mathbb{E}_{p^*}(f) \leqslant 0$ for all $f \in B$. So $p^* \in [\![A_i]\!]$ for each i, but $p^* \notin [\![B]\!]$.

We have thus shown that if $\mathsf{clposi}(B)$ and $\mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$ are disjoint then $[\![B]\!] \not\supseteq [\![A_1]\!] \cap \ldots \cap [\![A_n]\!]$. So we have that $\mathcal{K} = \mathcal{K}_{\mathcal{F}_\mathcal{K}}$, as required.

It remains to show that $\mathcal{F}_\mathcal{K}$ is coherent. By Proposition 2.12, we just need to check that $\varnothing \notin \mathcal{F}_\mathcal{K}$. By coherence of $\mathcal{K}$, $\varnothing \notin \mathcal{K}$, so, since we have already shown that $\mathcal{K} = \mathcal{K}_{\mathcal{F}_\mathcal{K}}$., i.e., $B \in \mathcal{K}$ iff $[\![B]\!] \in \mathcal{F}_\mathcal{K}$, we have $[\![\varnothing]\!] = \{p \mid$ there is some $g \in \varnothing$ with $\mathbb{E}_p(g) > 0\} = \varnothing \notin \mathcal{F}_\mathcal{K}$, as required. □

Corollary 4.12. *For distinct coherent $\mathcal{K}$ and $\mathcal{K}'$, $\mathcal{F}_\mathcal{K}$ and $\mathcal{F}_{\mathcal{K}'}$ are distinct.*

Proof. This follows immediately from Theorem 4.11. □

4.5 Question (iii). Probability filters go beyond desirability of gamble sets.

An encoding of which gamble sets are desirable, using Eq. (3), does not suffice to tell us everything about the opinion state, as given by a probability filter. This contrasts to the case of credal sets, where Archimedean mixing coherent choice functions are expressively equivalent to sets of probabilities [28]. The expressive power of probability filters goes strictly beyond that of sets of desirable gamble sets.

Theorem 4.13. *There are distinct coherent probability filters, $\mathcal{F}$ and $\mathcal{F}'$ which result in the same sets of desirable gamble sets, i.e., $\mathcal{K}_\mathcal{F} = \mathcal{K}_{\mathcal{F}'}$.*

We couch the proof of this by setting up an example, which bears a close relationship to $\mathcal{F}_{\text{InfBiased}}$ used in Examples 2.6 and 3.11, and then proving lemma 4.15.

Example 4.14. We will describe two filters, $\mathcal{F}_{\text{Fair}}$ and $\mathcal{F}_{\text{FairOrInfBiased}}$.

Consider $\Omega = \{\text{H}, \text{T}\}$, the outcomes of a coin toss. Let p_β be the unique probability function where $p_\beta(\text{H}) = \beta$.

$\mathcal{F}_{\text{Fair}}$ is given by: $P \in \mathcal{F}_{\text{Fair}}$ iff $p_{0.5} \in P$. $\mathcal{F}_{\text{Fair}}$ believes that the coin is fair:

$$\{p \mid p(\text{H}) = 0.5\} \in \mathcal{F}_{\text{Fair}}.$$

[16] This importantly relies on the mixing axiom including the 0; without this axiom we will need to extend Corollary 4.6 so the antecedent says that whenever $0 \notin \mathsf{posi}(\{g_1, \ldots, g_n\} \cup \mathcal{G}_{\gneq 0})$, they are not disjoint; see Campbell-Moore [5].

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$\mathcal{F}_{\text{FairOrInfBiased}}$ is given by: $P \in \mathcal{F}_{\text{FairOrInfBiased}}$ iff there is some positive real $\epsilon > 0$ with $P \supseteq \{p_{0.5+\alpha} \mid 0 \leqslant \alpha < \epsilon\}$.

$\mathcal{F}_{\text{FairOrInfBiased}}$ believes that the coin is not biased against heads

$$\{p \mid p(\mathrm{H}) \geqslant 0.5\} \in \mathcal{F}_{\text{FairOrInfBiased}}$$

but also believes that it is not biased towards heads by any particular amount.

$$\{p \mid p(\mathrm{H}) < 0.5 + \epsilon\} \in \mathcal{F}_{\text{FairOrInfBiased}} \text{ for } \epsilon > 0.$$

Unlike $\mathcal{F}_{\text{Fair}}$, $\mathcal{F}_{\text{FairOrInfBiased}}$ suspends judgement on whether the coin is fair

$$\{p \mid p(\mathrm{H}) = 0.5\} \notin \mathcal{F}_{\text{FairOrInfBiased}}$$

So clearly $\mathcal{F}_{\text{Fair}}$ and $\mathcal{F}_{\text{FairOrInfBiased}}$ are distinct.

However, this difference between $\mathcal{F}_{\text{Fair}}$ and $\mathcal{F}_{\text{FairOrInfBiased}}$ does not affect any judgements of the (strict) desirability of gambles, or of whether gamble sets contain a desirable gamble. That is what we show in the next result.

Lemma 4.15. $\mathcal{K}_{\mathcal{F}_{\text{Fair}}} = \mathcal{K}_{\mathcal{F}_{\text{FairOrInfBiased}}}$

Proof. Recall $A \in \mathcal{K}_{\mathcal{F}}$ iff $\{p \mid \text{there is some } g \in A \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}$.

Since $\mathcal{F}_{\text{FairOrInfBiased}} \subseteq \mathcal{F}_{\text{Fair}}$, also $\mathcal{K}_{\mathcal{F}_{\text{FairOrInfBiased}}} \subseteq \mathcal{K}_{\mathcal{F}_{\text{Fair}}}$. We need to show the converse.

Suppose $A \in \mathcal{K}_{\mathcal{F}_{\text{Fair}}}$, so there is some gamble g^* in A with $\mathbb{E}_{p_{0.5}}(g^*) > 0$. We will find some $\epsilon > 0$ such that for all $0 \leqslant \alpha < \epsilon$, $\mathbb{E}_{p_{0.5+\alpha}}(g^*) > 0$. And thus also $\{p | \text{there is some } g \in A \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}_{\text{FairOrInfBiased}}$, so $A \in \mathcal{K}_{\mathcal{F}_{\text{FairOrInfBiased}}}$. This works because of the continuity of probabilistic expectation. We can prove it directly as follows.

As $\Omega = \{\mathrm{H}, \mathrm{T}\}$, the gamble g^* has the form $g^* = \langle g^*_{\mathrm{H}}, g^*_{\mathrm{T}} \rangle$. If $g^*_{\mathrm{H}} \geqslant g^*_{\mathrm{T}}$, then $\mathbb{E}_{p_{0.5+\alpha}}(g^*) \geqslant \mathbb{E}_{p_{0.5}}(g^*) > 0$ for all $0 \leqslant \alpha$, so any $\epsilon > 0$ is as required.

If $g^*_{\mathrm{H}} < g^*_{\mathrm{T}}$, then put

$$\epsilon = \frac{0.5g^*_{\mathrm{H}} + 0.5g^*_{\mathrm{T}}}{g^*_{\mathrm{T}} - g^*_{\mathrm{H}}}$$

and observe that for $0 \leqslant \alpha < \epsilon$,

$$\begin{aligned}
\mathbb{E}_{p_{0.5+\alpha}}(g^*) &= (0.5 + \alpha)g^*_{\mathrm{H}} + (0.5 - \alpha)g^*_{\mathrm{T}} \\
&= (0.5g^*_{\mathrm{H}} + 0.5g^*_{\mathrm{T}}) - \alpha(g^*_{\mathrm{T}} - g^*_{\mathrm{H}}) \\
&> (0.5g^*_{\mathrm{H}} + 0.5g^*_{\mathrm{T}}) - \epsilon(g^*_{\mathrm{T}} - g^*_{\mathrm{H}}) && \text{since } g^*_{\mathrm{T}} > g^*_{\mathrm{H}} \text{ and } \epsilon > \alpha \\
&= 0 && \text{by the choice of } \epsilon
\end{aligned}$$ □

This matches the case of desirable gambles framework (Section 3.5), and as in that framework, one could obtain equivalent frameworks if we restrict the probability filter framework to special kinds of filters, those satisfying axiom ($\mathrm{F}_{\mathcal{K}}$):

($\mathrm{F}_{\mathcal{K}}$) If $P \in \mathcal{F}$ then there are some finitely many sets of gambles $A_1, \ldots, A_n$ with $\{p \mid \text{there is some } g \in A_i \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}$ for each A_i and

$$P \supseteq \left\{ p \;\middle|\; \begin{array}{l} \text{there is some } g \in A_1 \text{ with } \mathbb{E}_p(g) > 0, \text{ and} \\ \ldots, \text{ and} \\ \text{there is some } g \in A_n \text{ with } \mathbb{E}_p(g) > 0 \end{array} \right\}$$

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This axiom ensures that $\mathcal{F}$ is determined by its treatment of the probability constraints of the form $\{p \mid \text{there is some } g \in A \text{ with } \mathbb{E}_p(g) > 0\}$.

An axiom very similar to this is adopted in de Cooman et al. [12].[17]

Proposition 4.16. *For every coherent mixing $\mathcal{K}$, $\mathcal{F}_{\mathcal{K}}$ satisfies axiom* ($\mathrm{F}_{\mathcal{K}}$)*. Also, if $\mathcal{F}$ is a probability filter satisfying axiom* ($\mathrm{F}_{\mathcal{K}}$) *then $\mathcal{F} = \mathcal{F}_{\mathcal{K}_{\mathcal{F}}}$.*

Proof. To show that every $\mathcal{F}_{\mathcal{K}}$ satisfies axiom ($\mathrm{F}_{\mathcal{K}}$), we see by the definition of $\mathcal{F}_{\mathcal{K}}$ (Definition 4.10) that $P \in \mathcal{F}_{\mathcal{K}}$ iff there are finitely many $A_1, \ldots, A_n \in \mathcal{K}$ with

$$P \supseteq \left\{ p \;\middle|\; \begin{array}{l} \text{there is some } g \in A_1 \text{ with } \mathbb{E}_p(g) > 0 \\ \text{and} \ldots \\ \text{and there is some } g \in A_n \text{ with } \mathbb{E}_p(g) > 0 \end{array} \right\}$$

And observe that also $\{p \mid \text{there is some } g \in A_i \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}_{\mathcal{K}}$ for each i.

To show that $\mathcal{F} = \mathcal{F}_{\mathcal{K}_{\mathcal{F}}}$, we also consult the definition of $\mathcal{K}_{\mathcal{F}}$ (Definition 4.1) and see that $P \in \mathcal{F}_{\mathcal{K}_{\mathcal{F}}}$ iff there are some finitely many sets of gambles $A_1, \ldots, A_n$ with $\{p \mid \text{there is some } g \in A_i \text{ with } \mathbb{E}_p(g) > 0\} \in \mathcal{F}$ for each A_i and

$$P \supseteq \left\{ p \;\middle|\; \begin{array}{l} \text{there is some } g \in A_1 \text{ with } \mathbb{E}_p(g) > 0, \text{ and} \\ \ldots, \text{ and} \\ \text{there is some } g \in A_n \text{ with } \mathbb{E}_p(g) > 0 \end{array} \right\}.$$

So axiom ($\mathrm{F}_{\mathcal{K}}$) exactly guarantees that $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{K}_{\mathcal{F}}}$. We also have $\mathcal{F}_{\mathcal{K}_{\mathcal{F}}} \subseteq \mathcal{F}$ just by Proposition 2.7. □

A more independent characterisation of axiom ($\mathrm{F}_{\mathcal{K}}$) would be desirable, but we leave this for future work. We do not, however, propose axiom ($\mathrm{F}_{\mathcal{K}}$) as a axiom which should be adopted when capturing an agent's uncertain belief state as it rules out important differences of opinion. For example, it rules out $\mathcal{F}_{\mathrm{Fair}}$. The view that the coin is in fact fair is not something which matters for judgements of (strict) desirability of any gambles (due to lemma 4.15; it is only $\mathcal{F}_{\mathrm{FairOrInfBiased}}$ which satisfies axiom ($\mathrm{F}_{\mathcal{K}}$)). However, I propose that this is a legitimate opinion which should be accommodated in a model of uncertain belief. Thus, adopting axiom ($\mathrm{F}_{\mathcal{K}}$) is, I propose, too strong, and our representation of uncertainty should go beyond sets of desirable gamble sets.

5 Conclusion

We have proposed representing an agent's uncertain belief state by probability filters. This captures probabilistic judgements directly by representing an agents belief state with a collection of probability constraints, which we gloss as those which she believes. The axioms we gave for coherence required that the believed probability constraints be closed under finite intersection and supersets, i.e., that they should form the mathematical structure of a filter. This provides an easy notion of natural extension.

The model is closely related to the set-of-probabilities, or credal set, model of belief, with special kinds of filters, namely principal filters, being equivalent to the set-of-probability model. By allowing also for non-principal filters as coherent, in this paper, we have shown that the model can also accommodate the power of the desirable gambles model of belief.

[17] They impose this by restricting the kinds of constraints which are considered.

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A The Formulation of the Mixing Axiom - Axiom ($\mathrm{K}_{\mathrm{Mix}}$)

We have adopted the mixing axiom in the form:

($\mathrm{K}_{\mathrm{Mix}}$) If $A \in \mathcal{K}$ and $\mathsf{clposi}(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

There are various closely related axioms that differ just in which operation

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they use in place of clposi. For O an operator such as conv, posi, clconv, let $\mathrm{K}^{O}_{\mathrm{Mix}}$ be the axiom:

($\mathrm{K}_{\mathrm{Mix}^O}$) If $A \in \mathcal{K}$ and $O(B) \supseteq A \supseteq B$ then $B \in \mathcal{K}$.

Given axiom ($\mathrm{K}_{\supseteq}$), we could equivalently state any of these as

($\mathrm{K}_{\mathrm{Mix}^O}$) If $O(B) \in \mathcal{K}$ then $B \in \mathcal{K}$.

De Bock and de Cooman [11] choose to use the axiom $\mathrm{K}^{\mathsf{posi}}_{\mathrm{Mix}}$ instead of $\mathrm{K}^{\mathsf{conv}}_{\mathrm{Mix}}$. Similarly, we have chosen to use $\mathrm{K}^{\mathsf{clposi}}_{\mathrm{Mix}}$ instead of Seidenfeld et al. [28]'s use of $\mathrm{K}^{\mathsf{clconv}}_{\mathrm{Mix}}$. (De Bock and de Cooman can omit the closure component because they restrict to finite sets of gambles.) The choice of posi or conv doesn't make a difference.

Proposition A.1. *Given axioms* ($\mathrm{K}_{\supseteq}$) *and* ($\mathrm{K}_{\mathrm{Add}}$)*, the following are equivalent:*

(i) $\mathrm{K}^{\mathsf{clposi}}_{\mathrm{Mix}}$ $=$ *axiom* ($\mathrm{K}_{\mathrm{Mix}}$)

(ii) $\mathrm{K}^{\mathsf{closure}}_{\mathrm{Mix}}$ *and* $\mathrm{K}^{\mathsf{posi}}_{\mathrm{Mix}}$

(iii) $\mathrm{K}^{\mathsf{clconv}}_{\mathrm{Mix}}$

(iv) $\mathrm{K}^{\mathsf{closure}}_{\mathrm{Mix}}$ *and* $\mathrm{K}^{\mathsf{conv}}_{\mathrm{Mix}}$

Proof. Item (ii) $\Longrightarrow$ Item (i): $\mathsf{clposi}(B) = \mathsf{closure}(\mathsf{posi}(B))$. If $\mathsf{clposi}(B) \in \mathcal{K}$, then by $\mathrm{K}^{\mathsf{closure}}_{\mathrm{Mix}}$, $\mathsf{posi}(B) \in \mathcal{K}$ and by $\mathrm{K}^{\mathsf{posi}}_{\mathrm{Mix}}$, $B \in \mathcal{K}$.

Item (i) $\Longrightarrow$ Item (ii): If $\mathsf{closure}(B) \in \mathcal{K}$, then $\mathsf{clposi}(B) \in \mathcal{K}$ by axiom ($\mathrm{K}_{\supseteq}$), so $B \in \mathcal{K}$. Similarly, if $\mathsf{posi}(B) \in \mathcal{K}$ then $\mathsf{clposi}(B) \in \mathcal{K}$ so $B \in \mathcal{K}$.

The argument for Item (iii) $\Longleftrightarrow$ Item (iv) is exactly analogous.

For Item (iv) $\Longleftrightarrow$ Item (ii), we need to show that $\mathsf{conv}(B) \in \mathcal{K}$ iff $\mathsf{posi}(B) \in \mathcal{K}$. Since $\mathsf{posi}(B) \supseteq \mathsf{conv}(B)$, one direction follows from axiom ($\mathrm{K}_{\supseteq}$). For the other direction, assume $\mathsf{posi}(B) \in \mathcal{K}$. For each $g \in \mathsf{posi}(B)$ there is some $f_g \in \mathsf{conv}(B)$ where $f_g \in \mathsf{posi}(\{g\})$ (it can simply be normalised). So by axiom ($\mathrm{K}_{\mathrm{Add}}$), $\mathsf{conv}(B) \in \mathcal{K}$. □

It is worth noting that for $\mathsf{scalar}(B) = \{\lambda_g g \mid \lambda_g > 0,\, g \in B\}$, $\mathrm{K}^{\mathsf{scalar}}_{\mathrm{Mix}}$ follows directly from axiom ($\mathrm{K}_{\mathrm{Add}}$).

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