Indeterminate Truth and Credences

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1 Introduction

Consider the following unfortunate scenario:

> PASSPORT.
> If you have credence $\geqslant 0.5$ that you'll remember your passport, then when the time comes you'll end up forgetting it (you'll get on with other things).
> And if not, then you *will* end up remembering it (you'll spend your time worrying about it).
> And you know this about yourself. What should your credence be?
> (Archer case from Joyce, 2018)

Suppose you adopt credence 0.2 that you'll remember your passport. Then you'll get on with other things and will forget your passport. And you know this. So you should be certain that you'll forget your passport, i.e. adopt credence 1. But were you to adopt credence 1 that you'll forget your passport, you would spend all your time worrying about it, and thus would remember it. And since you know this about yourself, this would recommend adopting credence value 0. More generally, any credence value you assign will undermine itself in this sort of way.

Such scenarios have recently been discussed as a challenge for rationality arguments (Caie, 2013; Greaves, 2013).[1] A rational opinion state should not undermine its own adoption. If it does, it cannot be relied on for decision making and evaluation. And in a case like PASSPORT, every credence value is undermining so there seems to be no rational options.

We parallel the kind of underminingness found in PASSPORT to that due to the liar sentence:

> `Liar`: `Liar` is not true.

We might naturally reason about the liar sentence as follows: Is it true or not? Suppose it were true. Then since it says "`Liar` is not true", and `Liar` is true, we can conclude that it is false. Suppose it were not true. Then since it says "`Liar` is not true", we can conclude it is true. Reflecting on its truth value always results in a contradictory truth value. We might thus say that it is undermining.

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[1]See also Konek and Levinstein (2017); Carr (2017); Joyce (2018); Pettigrew (2018) for discussion of such challenges.

However, following Kripke (1975) and McGee (1989) we might consider truth to be a vague notion, and consider whether sentences are definitely true or definitely not true. `Liar` will be neither. One can then describe how a proposed collection of definite truth value verdicts should be revised: a so-called Kripkean jump. And show there will be fixed points. We might describe such fixed points as collections of definite verdicts which do not undermine themselves. We suggest applying similar considerations to credences.

We allow the notion of credence to be vague and it may be that no definite credence value is assigned to PASSPORT. Our first question is exactly how to model this. We could just focus on questions like whether one's credence in φ is definitely equal to r or definitely not equal to r, or neither. But we will suggest that it's more natural to directly model vague credences with a set of precise credence functions: its set of precisifications. So instead of focusing on definite judgements and considering the collection of precisifications as derivative, we directly work with the set of precisifications. In fact, this is a model of belief which is of independent interest in formal epistemology

We then consider how to reflect on one's indeterminate credences, i.e. describe a supervaluational Kripkean jump. We provide two suggestions, both of which can be seen as being inspired by the usual supervaluational Kripkean jump for truth. The most natural supervaluational jump is to revise a set of precisifications simply by revising each of the individual precisifications (we call this Θ). But this can sometimes result in triviality. To understand this triviality and how we might avoid it, we then return to considering the supervaluational Kripkean jump for truth.

For truth, we typically focus simply on whether sentences are definitely true, definitely not true, or indefinite instead of focusing on the sets of precisifications themselves, even though it's this resultant set of precisifications that's important for defining the supervaluational Kripkean jump. We might also ask what happens when we focus on the set of precisifications themselves in the case of truth, and apply Θ. It too leads to triviality due to the McGee sentence. But the usual supervaluational Kripkean jump for truth (thought of as applying to a definite verdicts account of truth) does not correspond to Θ, but instead additional precisifications are added: those that agree on any definite truth value verdicts. And this allows the account to avoid triviality.

We propose to apply this idea when we focus on sets of precisifcations themselves, as we did in the case of credences. We define an alternative jump, Γ, which explicitly adds additional precisifications that are 'limits' of the precisifications obtained from applying Θ. Formally, we take a topological closure. And this will allow triviality to be avoided.

We will use these two jumps to propose when an imprecise credal state is undermining (if $\mathbb{C} \not\subseteq \Gamma(\mathbb{C})$ or $\mathbb{C} \not\supseteq \Theta(\mathbb{C})$). Since Γ has non-trivial fixed points, there is always some credal state which is non-undermining in this sense, and we thus propose that such credal states are candidates for being rational attitudes to adopt.

The account we provide is in fact very general and could apply to a whole range of target notions, one just needs to spell out a range of objects that play the role of the precisifications, and to describe how to revise each of these. Then we can move to considering sets of precisifications and describe a revision of a set by revising each of its members. This might lead to triviality. But by modifying the revision of the set of precisifications to also add some additional precise members, by taking the topological closure of the set, we can typically show there are non-trivial fixed points (this does involve also having a notion of closure defined on the space of precisifications). At least, this holds whenever the underlying space of precisifications is topologically compact.

1.1 Summary of the paper

We start by briefly presenting the supervaluational Kripkean account of truth in Section 2. In Section 3 we present the classical notion of a credence function. In Section 4 we then move to considering how to think about indefinite credences and suggest we should directly think about sets of precisifications. In Section 5 we consider the supervaluational jump given by just revising each member of the set, and show that this might result in triviality. Section 6 shows that the McGee sentence leads to an analogous challenge for indefinite truth, and that triviality is avoided because additional precisifications are added when applying the Kripkean jump due to the focus on definite truth value verdicts. Section 7 shows that if we define a jump on the sets of precisifcations which allows for some additional precisifications, then triviality can be avoided. Section 8 describes the generality of our construction and how it could be applied to any other notions of interest once one spells out how to revise the various precisifications. Section 9 concludes.

2 Kripkean supervaluational account of truth

We first briefly present the supervaluational account of truth.[2]

2.1 Classical truth

Setup 2.1. Let $\mathcal{L}$ be a base language in which we have the ability to code sentences, for ease we might take this to be the language of Peano Arithmetic. Let $\mathcal{L}_T$ extend this with the addition of a unary predicate, T. Sent_T denotes the sentences of this language. We will assume we have a fixed model of our base language, which we assume is the standard model of arithmetic, denoted $\mathbb{N}$.

Definition 2.2. A *precise interpretation of truth*, Q, is given by a collection of sentences, i.e. $Q \subseteq \mathsf{Sent}_T$. We denote the collection of all interpretations by $\mathcal{P}\mathsf{recs}$. $(\mathbb{N}, Q)$ refers to the classical model of $\mathcal{L}_T$ resulting from expanding the standard model of arithmetic for the base language, $\mathbb{N}$, with Q providing the sentences whose codes are in the extension of the truth predicate. So we have $(\mathbb{N}, Q) \models T\ulcorner\varphi\urcorner$ iff $\varphi \in Q$.

We then define a way to 'revise' a precise interpretation, Q, to a new precise interpretation, $\tau(Q)$.

Definition 2.3. $\tau(Q) \subseteq \mathsf{Sent}_T$ is defined by: $\varphi \in \tau(Q)$ iff $(\mathbb{N}, Q) \models \varphi$.

We thus have:

[2]For a more in depth introduction to the Kripkean account with comments on the supervaluational variant of it see, e.g., McGee (1990). We are presenting it in the way outlined in McGee (1989) rather than of Kripke (1975), in particular focusing definite verdicts rather than partial truth.

$$\varphi \in Q \text{ iff } T\ulcorner\varphi\urcorner \in \tau(Q).$$

τ can be understood as a stage of reflecting on the supposed truth values.To be materially adequate, an interpretation of truth should be a fixed point of τ, i.e. $Q = \tau(Q)$. But, the liar paradox shows us that this is not possible because for the liar sentence, `Liar`, which is equivalent to $\neg T\ulcorner\texttt{Liar}\urcorner$, we have that $\texttt{Liar} \in Q$ iff $\texttt{Liar} \notin \tau(Q)$. For an analogy with credences, we will say that every precise interpretation of truth is thus *undermining*.

2.2 Definite Truth and Supervaluationism

We now consider imprecise truth. We focus on whether sentences are definitely true, definitely not true, or indefinite. An assignment of such definite truth-value verdicts is given by:

Definition 2.4. A definite verdicts assignment, S, is given by two sets of sentences, S^+ and S^-.

S^+ contains the sentences that are definitely true, and S^- the sentences that are definitely not true. Some sentences may be neither definitely true nor definitely not true. One might also present this as providing an account of partial truth, then S^+ would provide the sentences that are true, S^- the sentences that are false, and some sentences may be neither true nor false. Our formal work can equally well be seen to apply to such an interpretation, but we follow McGee (1985) in treating it as instead a picture of definite truth.

Associated with any definite verdicts assignment is a collection of precisifcations:

Definition 2.5. $Q \subseteq \mathsf{Sent}_T$ is a *precisification* of $S = (S^+, S^-)$, $Q \in \mathsf{Precs}(S)$, iff

- If $\varphi \in S^+$ then $\varphi \in Q$.
- If $\varphi \in S^-$ then $\varphi \notin Q$.

That is, Q is a precisification if it coheres with the definite verdicts of S. One might also consider adding "admissibility conditions" which restrict the precisifications to, for example, those that are maximally consistent. We will return to this in Section 8.2.

We can also consider the definite verdicts assignment that a set of precisifications gives rise to:

Definition 2.6. Given a set of precisifications, $\mathbb{Q} \subseteq \mathcal{P}\mathsf{recs}$,

- $\varphi \in \mathsf{Def}(\mathbb{Q})^+$ iff $\varphi \in Q$ for all $Q \in \mathbb{Q}$
- $\varphi \in \mathsf{Def}(\mathbb{Q})^-$ iff $\varphi \notin Q$ for all $Q \in \mathbb{Q}$

That is, it is the collection of precise interpretations which agree with the definite verdicts as spelled out by S.

The supervaluational Kripke jump revises a definite verdicts assignment as follows:

Definition 2.7. $\mathcal{J}(S) := (\mathcal{J}(S)^+, \mathcal{J}(S)^-)$ where

$$\varphi \in \mathcal{J}(S)^+ \text{ iff } (\mathbb{N}, Q) \models \varphi \text{ for all } Q \in \mathsf{Precs}(S)$$
$$\varphi \in \mathcal{J}(S)^- \text{ iff } (\mathbb{N}, Q) \not\models \varphi \text{ for all } Q \in \mathsf{Precs}(S)$$

Given our definitions of τ and Def, we can immediately see this is equivalent to:

Proposition 2.8.

$$\mathcal{J}(S) = \mathsf{Def}(\{\tau(Q) \mid Q \in \mathsf{Precs}(S)\})$$

We can further simplify this by making another definition:

Definition 2.9. For $\mathbb{Q} \subseteq \mathcal{P}\mathsf{recs}$,

$$\Theta_\tau(\mathbb{Q}) := \{\tau(Q) \mid Q \in \mathbb{Q}\}$$

We then immediately have:

Proposition 2.10.

$$\mathcal{J}(S) = \mathsf{Def}(\Theta_\tau(\mathsf{Precs}(S)))$$

We can describe this jump as follows: (i) starting with a definite verdicts assignment, use Precs to move to the corresponding set of precisifications, (ii) use τ to revise each of these, and (iii) use Def to move from the resultant collection of revised precise interpretations to the definite verdicts.

One can then show that there are fixed points of $\mathcal{J}$: accounts of definite truth that are non-undermining.

3 Precise credences

We now move to developing the analogous tools for the case of credences. We first start just by specifying the notion of credence function that we're working with in the classical setting.

Setup (Credence functions)**.** We start with a non-empty set of sentences, $\mathcal{A}$, which we call our *agenda*.[3] This could be all sentences of a given language, but it can also be more restrictive, for example we might consider cases where we are only looking at your credence in a single sentence, so where $\mathcal{A}$ is a singleton. For example we might just be interested in the credence that you'll forget your passport, so $\mathcal{A}$ might just contain the sentence saying that you'll forget your passport.

A *credence function* on an agenda $\mathcal{A}$ is a function, c, from $\mathcal{A}$ to $[0, 1]$; i.e. it associates with each sentence in $\mathcal{A}$ a degree of belief, which is a real number between 0 and 1 inclusive. $\mathsf{Creds}_\mathcal{A}$ is the set of all credence functions, i.e. all functions from $\mathcal{A}$ to $[0, 1]$. If $\mathcal{A}$ just contains a single sentence, a credence function can be thought of simply as a value in $[0, 1]$ and we will call this a *credence value.*

It wouldn't affect our account if we restrict $\mathsf{Creds}_\mathcal{A}$ to just those functions that are finitely-additive probabilities (or, more carefully, which are extendable to functions on a Boolean algebra which satisfy the axioms of finitely-additive probability theory[4]), as this is still a compact space. However, we could not restrict attention to *countably*-additive probabilities.[5]

[3]It would not affect our account if we took them to be propositions understood in a different way, e.g. they could be sets of possible worlds.

[4]See, e.g., Pettigrew (2016, Def 1.0.1).

[5]To see that this is not compact, observe that the limit of a convergent sequence of countably additive probabilities might be merely finitely additive. This is not the case for finite additivity.

In the case of truth, we gave a way of revising the precise interpretations of truth: τ. For credences, we will just take a revision function as given, and our supervaluational account will be a general one that can apply to any given revision function.[6] Formally:

Definition 3.1. A *revision function* is a function, ρ, from $\mathsf{Creds}_{\mathcal{A}}$ to $\mathsf{Creds}_{\mathcal{A}}$.

Passport is a story which directly describes how one should revise one's credence in the target-proposition, that you'll remember your passport, under a step of reflection on the proposed credence value. If we let $\mathcal{A}$ simply contain this one proposition, then credence functions are just values $x \in [0, 1]$, and the revision function that Passport gives rise to is:

$$\rho_{\textsc{Passport}}(x) = \begin{cases} 1 & x < 0.5 \\ 0 & x \geqslant 0.5 \end{cases}$$

One can see that there are no fixed points of $\rho_{\textsc{Passport}}$. That is, every precise credence function undermines its own adoption. In this sense Passport can be related to the liar sentence: in both cases, all precise values are undermining.

There are other cases that also give rise to the same revision function as Passport, for example:

> Bad Navigator.
> You've come to a crossroads and are wondering whether you need to turn left or right to get to your hotel. You know you're a really bad navigator. In particular, you believe that if you have credence $\geqslant 0.5$ that left is the way to your hotel, then it's actually right; and if not, then it's actually to the left. What should your credence be that it's actually right?
> (Extremal version of an example in Egan and Elga, 2005)

In this case, the change in one's credence isn't due to the causal structure, but instead simply that the credence that one adopts affects the evidence that one has about the situation. But the same revision function describes this case. The same revision function would also arise when considering the following self-referential sentence:

`PrLiar`: Your credence in `PrLiar` is not $\geqslant 0.5$.

In this case, the revision is due to semantic features of the sentence.

You might also be uncertain about whether you are is in a Passport-like case:

> Basketball.
> You think there's a small chance, 1% chance that whether you'll be able to shoot this free-throw is dependent on the credence you adopt in it in a Passport style way, i.e., where if you have credence $\geqslant 0.5$ then you'll fail, and if not then you'll succeed. But you're 99% sure that it's just a normal case and you have a 50% chance of success.

[6] In fact, many situations will not give rise to a recommendation *function*. Instead, maybe there are ties. Our whole account can be expanded to deal with ties, see Footnote 22. However, this would complicate the presentation and the parallel to the truth case, so I assume that we have a revision function.

This leads to the revision function

$$\rho_{\textsc{Basketball}}(x) = \begin{cases} 0.01 \times 1 + 0.99 \times 0.5 = 0.505 & x \geqslant 0.5 \\ 0.01 \times 0 + 0.99 \times 0.5 = 0.495 & x < 0.5 \end{cases}$$

It also might be that one's credence doesn't directly provide evidence about the truth of the sentence, but instead it affects the chances. Consider, for example, the following scenario discussed by Greaves (2013):

> PROMOTION.
> "Alice is up for promotion. Her boss, however, is a deeply insecure type: he is more likely to promote Alice if she comes across as lacking in confidence. Furthermore, Alice is useless at play-acting, so she will come across that way iff she really does have a low degree of belief that shes going to get the promotion. Specifically, the chance of her getting the promotion will be $1 - x$, where x is whatever degree of belief she chooses to have in the proposition P that she will be promoted. What credence in P is it epistemically rational for Alice to have?" (Greaves, 2013, pp.1–2)

(Moreover Greaves assumes "that the agent is aware of the specification of [...] her case".) If Alice considers adopting credence 0.2 in P; then the chance of P would be 0.8, and she knows that, so that would recommend adopting credence 0.8. More generally, the description of this case directly provides us with the revision funtion

$$\rho_{\textsc{Promotion}}(x) = 1 - x.$$

Unlike for the PASSPORT revision function, this function does have a fixed point: 0.5.

All the cases we've seen so far are unusual cases. In normal cases, the credence one adopts provides no additional evidence about the situation at hand.

> RAIN.
> The credence that you adopt that it is going to rain tomorrow provides no additional evidence about the likelihood of rain.

In this case, $\rho(x) = x$. More generally, in normal cases $\rho(c) = c$ for all c, or at least all c which are probabilistic. And most rationality theorising has focused on these "safe" cases.[7]

This same revision function might also arise in a case where the credence one adopts does provide additional information:

> LEAP.
> The chance you'll successfully leap across this chasm is identical to your credence. (Greaves, 2013, see also James, 1897)

We do not assume any further modelling of this revision function, we simply assume that any scenario gives rise to such a revision function. Any further modelling of ρ would have allow for the range of cases mentioned so far. It has to allow for logical, causal and evidential impact (as in `PrLiar`, PASSPORT, and BADNAVIGATOR), and this might go via chance (like PROMOTION) or be

[7] In fact, one might only have $\rho(c) = c$ if c satisfies further principles of rationality, for example the Principal Principle.

directly about the proposition (as in PASSPORT), or be associated with further uncertainty. Our account does not depend on any further specifics of the revision function, we can simply take it as an input to our account. The revision function should encode the idea of reflecting on one's credences, and we are adopting the idea that to be rational, a precise credence function should be a fixed point of this revision function.[8]

There are two suggestions for how one could include further modelling or explanation of the revision function.

Firstly, one might simply take $\rho(c)$ to be c conditionalized on "c is my credence function". If we assume that one's initial credence function satisfies certain other constraints of rationality such as deferring to chances (by satisfying the so-called Principal Principle) this should lead to the revision functions proposed. This would be the implementation that is similar to that of Joyce (2018).[9]

Alternatively, one might want to explicitly include a possible worlds structure in the modelling and then define ρ using this. This is particularly natural for accounting for a language with sentences that can talk about the credence on has in that very sentence. This kind of picture has commonly been used in when developing accounts for languages with modal predicates (Stern, 2015; Nicolai, 2018; Campbell-Moore, 2015; Halbach and Welch, 2009; Halbach et al., 2003). All the considerations in this paper can also directly be applied to such a setting (see also Section 8). But it is not obvious how to apply this kind of analysis to cases like PROMOTION where the impact goes via chances. And insofar as it deals with PASSPORT or BADNAVIGATOR it just treats them like `PrLiar`, for example, we wouldn't represent 'the hotel is to the left' as an atomic sentence, as would be most natural, but instead as a sentence that refers to itself. Whilst this leads to the right revision function, it does not seem to be the right analysis of the sentence itself. We thus find it valuable to not encode further modelling such as this, but to simply provide the account for any specified revision function.

4 Supervaluational Accounts of Credences

In the case of truth we focused on definite truth value verdicts. When considering indeterminate credences, what should we think about? We suggest that we directly work with a set of precise credence functions, that is, it is given by some $\mathbb{C} \subseteq \mathsf{Creds}_{\mathcal{A}}$. The precise credence functions in the set will be called the 'precisifications' of the indeterminate credal state. So, for example, if our agenda contains a single proposition, a precise credence function is some real number between 0 and 1, whereas a vague credal state is given by a set of numbers, e.g., $\{0.2, 0.3\}$, or $[0.2, .0.3]$. It remains indefinite which of the credence values in the set it is, but it is, for example, definitely not 0.9.

This model of belief is closely related to one that is familiar in formal epistemology under the term "imprecise probabilities", "indeterminate probabilities"

[8] For arguments for this in our "unusual" cases see, primarily, Joyce (2018). For this argument in "safe" cases, see discussions of immodesty, e.g., Joyce (2009); Lewis (1971). Someone like Pettigrew (2018) who disagrees with Joyce on these "unusual" cases might be though of as agreeing with the idea that one's credence should be a fixed point of ρ, but instead works with an implementation of ρ which is consequentialist. It turns out then that ρ does not depend on the input value at all, and thus it always has a fixed point.

[9] Joyce in fact proposes that $\rho(c)$ is the function that minimizes expected inaccuracy, given that c is chosen (Joyce, 2018, p.257), but this will typically simply be the c thus conditionalized.

or "mushy credences".[10] It has been proposed for a range of reasons, including being able to represent incomparability as distinct from indifference, distinguishing between lack of evidence and symmetric evidence, allowing for suspension of judgement, and rationalising intuitively rational responses to certain decision problems (Joyce, 2010; Bradley, 2015; Levi, 1978; Jeffrey, 1983). Though its interpretation is debated.

To more closely match the application to truth, we might identify some particular judgements and ask whether they definitely hold, definitely don't hold, or hold indeterminately. For example, we might only care about whether one's credence in φ is definitely equal to r, definitely not equal to r or neither.[11] But the account will be then be very weak. For example, a case like Basketball would not get assigned a credence value at all. This thus doesn't respect the fact that your credence should definitely be $\geqslant 0.3$, which we would get out of the more expressive framework when we look directly at the set of precisifications. Furthermore, this is a difference that might be used in decision making. One might instead then try to be more expansive about the kinds of definite verdicts that are being considered. We might consider whether your credence is definitely $\geqslant r$ or definitely not $\geqslant r$. Again, this can be criticised for leaving out potentially definite judgements such as that φ is more likely than ψ, or that φ is evidence for ψ, or that I'm certain in at least one of $\varphi_1, \varphi_2, \ldots$.

By focusing on sets of precisifications themselves, however, every definite judgement is encoded. Any $\mathbb{B} \subseteq \mathsf{Creds}_{\mathcal{A}}$ can be thought of as a property of one's credences, for example, $\mathbb{B} = \{c \mid c(\varphi) > c(\varphi)\}$ is the property that φ is more likely than ψ; $\mathbb{B} = \{c \mid c(\varphi \mid \psi) > c(\varphi)\}$ is the property that ψ is evidence for φ; and $\mathbb{B} = \{c \mid c(\varphi_k) = 1 \text{ for some } k)\}$ is the property that you are certain of at least one of $\varphi_1, \varphi_2, \ldots$. For any set of precisifications $\mathbb{C}$, we can say whether it definitely satisfies that property, definitely doesn't, or neither, by considering whether $\mathbb{C} \supseteq \mathbb{B}$, $\mathbb{C} \cap \mathbb{B} = \varnothing$, or neither. Focusing on the set of precisifications, $\mathbb{C}$, itself is equivalent to focusing on definite judgements on *all* properties, at least when we ignore any differences in definite judgement assignments that don't correspond to differences in resultant precisifications.[12] And since differences that don't constitute differences in precisifications will not affect supervaluational considerations, considering sets of precisifications themselves is the most general model available. Focusing on any particular definite judgements can then be considered as special cases of our general account. Unlike for truth, though, the

[10]There is in fact a whole range of models of belief that are discussed. Many of these are weaker than arbitrary sets of probabilities, e.g., upper-lower probability models. However, especially under the term "imprecise probabilities", some models are stronger as they can encode opinions that would only be captured by non-Archimedean probability functions. Campbell-Moore and Konek (2019) presents a very model of belief that is more general than all those considered in the imprecise probability literature, and where considerations such as those in this paper might also be able to be applied. Moreover, some of the issues we find of ensuring fixed points may be more easily avoided.

[11]Follow the idea of Kripke (1975) that some sentences don't express propositions, and those that express propositions get normal credence values, and those that don't don't get assigned values at all. This would correspond to only caring about whether one's credences is definitely equal to r without caring about whether it's definitely not equal to r. But in a case like BadNavigator, the sentence 'the hotel is to the left' seems to express a proposition as normal. What's weird in this case is not what the sentence expresses, but instead, what rationality says about such a case.

[12]For example if two definite judgement assignments both have no precisifications, they are treated as identical.

kinds of triviality issues we face when working with sets of precise credences will often arise for these other models, at least whenever one is interested both in whether a property is definitely satisfied and whether it is definitely not satisfied. (See Section 8.2 for further discussion.)

5 Revision of a set of credences

Recall our presentation of the supervaluational Kripkean jump for truth in Proposition 2.10:

$$\mathcal{J}(S) = \mathsf{Def}(\Theta_\tau(\mathsf{Precs}(S)))$$

We described that with the following procedure: (i) use Precs to move from a definite verdicts assignment to the corresponding set of precisifications, (ii) use τ to revise each of these, and (iii) use Def to move from the resultant set of revised precise interpretations back to the define verdicts.

Perhaps what motivates this definition of $\mathcal{J}$ is just Θ_τ, which revises a set of precisifications, but since $\mathcal{J}$ is defined on the definite verdicts model rather than sets of precisifications, we have to also introduce stages (i) and (iii) to find Θ_τ's treatment of definite verdicts.

In the case of credences, however, we have suggested working directly with a set of precisifications, and want to know how to revise that. So stages (i) and (iii) aren't needed and we might suggest that the analogous way to revise a set of precisifications is just to revise each of the precisifications, i.e., just apply stage (ii). We defined Θ_τ for the case of truth as $\Theta_\tau(\mathbb{Q}) := \{\tau(Q) \mid Q \in \mathbb{Q}\}$. We now simply present this as a more general definition that can apply to any revision function:

Definition 5.1. For a given (fixed) revision function ρ,

$$\Theta_\rho(\mathbb{C}) = \{\rho(c) \mid c \in \mathbb{C}\}.$$

We will generally drop the subscript as it's typically clear which revision function is used.

This simply takes the collection of revised individuals. It is a very intuitive notion of revision applied to a indeterminate credence, understood as a set of precisifications, and seems to follow naturally from the supervaluationist idea that what happens on the supervaluational-side supervenes on what happens on the precise side.

Consider Passport, BadNavigator or `PrLiar`. We simply focus on an agenda consisting of the single sentence at stake in each of these scenarios, so credence functions are given by real numbers between 0 and 1. Supervaluational credences are given by sets of precise credences, so this will be a set of real numbers between 0 and 1. Consider adopting the set consisting just of the two extremal credences, $\{0, 1\}$. Revision of precise credences is spelled out by $\rho_{\textsc{Passport}}$. In particular, credence 0 recommends adopting credence 1; and 1 recommends 0. So $\Theta(\{0, 1\}) = \{\rho(0), \rho(1)\} = \{1, 0\} = \{0, 1\}$. I.e. the indeterminate credence $\{0, 1\}$ is a fixed point of Θ. Whilst each precisification is undermining, the set, as a whole, then seems to be a non-undermining attitude to adopt in these cases.

However, there is a formal issue facing this proposal. Θ does not always have a non-trivial fixed point. Thus, if our notion of recommendation is spelled out with Θ, we still might end up with a situation where every credal state, precise or imprecise, recommends another one, so is undermining, and thus not a candidate for the rational response to the situation.

5.1 Θ doesn't always have a fixed point — Spring

Consider the following kind of scenario:[13]

> Spring.
> You know that you're always overconfident in this type of situation. Except you also know that a credence value of 0 would be wrong.

What revision function does this lead to? It will have that $\rho(0) > 0$, and for all $x > 0$, $\rho(x) < x$. To say more about it, though, we need further details about this case: 'how overconfident?' 'how wrong?'. In fact, I think natural ways of adding to this story will not guarantee a notion of a particular credence value being recommended, instead it might allow for ties. But for simplicity, this paper focuses on the case where we have a fully specified revision function.[14] In fact it doesn't matter how we spell it out, any revision function with these properties leads to a Θ which has no fixed points. To work with a concrete example, we suppose that additional details are added to the case so that we obtain the following revision function:[15]

$$\rho_{\text{Spring}}(x) = \begin{cases} 1 & x = 0 \\ \frac{x}{2} & x > 0 \end{cases}$$

See Fig. 1 for an illustration.

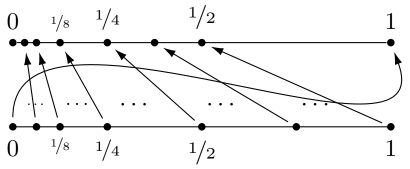

Figure 1: Illustration of ρ_{Spring}.

As in the cases like Passport, every credence value is undermining. But, unlike in Passport, there is also no set of precise credence values which is a fixed point of Θ.

Proposition 5.2. *There is no (non-empty) fixed point of Θ_{Spring}.*

Proof. We first observe that for any $\mathbb{C}$ and $n \geqslant 1$, any $x \in \Theta^n(\mathbb{C})$ has $0 < x \leqslant 1/2^{n-1}$.

Base case: For any $x > 0$, $\rho(x) = x/2 > 0$ (and $\rho(x) \leqslant 1$). Also $\rho(0) = 1 > 0$. Thus, for any $x \in \mathbb{C} \subseteq [0, 1]$ has $0 < \rho(x) \leqslant 1$.

[13] Thanks to Adam Coulton for highlighting that my argument that Θ has a fixed point didn't guarantee it is non-trivial.

[14] See also Footnote 22.

[15] Our choice of $\rho(0) = 1$ is an extreme way to spell out the details of the story: if you assign credence 0, you think it's definitely true. We have made this choice as it is then parallel the McGee sentence which we will discuss in Section 6.1.

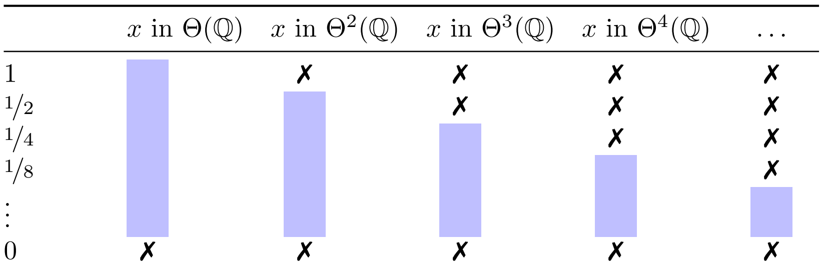

Figure 2: Illustration of Θ with SPRING. x can lie only in the gap between the crosses.

Inductive step: for any $x \in \Theta^n(\mathbb{C})$, $0 < x \leqslant 1/2^{n-1}$. So $\rho(x)$, which $= \frac{x}{2}$ has $0 < \rho(x) \leqslant \frac{1/2^{n-1}}{2}$. Therefore, for any $y \in \Theta^{n+1}(\mathbb{C})$, $y = \rho(x)$ for some $x \in \Theta^n(\mathbb{C})$, $0 < y \leqslant 1/2^n$, as required.

Now, suppose $\mathbb{C} = \Theta(\mathbb{C})$. Then $\mathbb{C} = \Theta^n(\mathbb{C})$ for all n. So any $x \in \mathbb{C}$ has $x \leqslant 1/2^{n-1}$ for all n. But the only such x is 0, and we also require that $x > 0$. So $\mathbb{C} = \varnothing$. □

So, this supervaluational jump does not guarantee that undermining credal states can be avoided.

We will propose an alternative supervaluational jump which will always allow for fixed points. In order to introduce and motivate this, we will reconsider the supervaluational Kripkean jump for truth.

6 Supervaluational Kripkean jump for truth as it applies to sets-of-precisifications

Some revision functions, such as that for PASSPORT, do lead to fixed points of Θ, whereas the revision function for SPRING rules them out. What about the revision function for truth, τ? Θ_τ does not have any fixed points either.[16] To show this, we note that the McGee sentence leads to SPRING-style phenomena.

However, the usual revision jump for truth, $\mathcal{J}$, does not in fact correspond to Θ but instead to $\mathsf{Precs} \circ \mathsf{Def} \circ \Theta$. This is because the choice of focusing only on definite truth value verdicts is less expressively powerful than sets of precisifications. Θ does not determine a set of precisifications that corresponds to a definite verdicts assignment. This means that when we look at the effect of $\mathcal{J}$, additional precisifications are added: those that the definite verdicts assignment does not have the resources to exclude. This allows $\mathcal{J}$ to have fixed points, where Θ does not.

We will use this insight to define an alternative jump on sets of precisifications which we can apply to credences in cases like SPRING in order to obtain fixed points.

[16] When we conceive of it as apply to the whole language, which includes, for example, the McGee sentence. It does have fixed points when applied, e.g., just to `Liar`.

6.1 There are no non-trivial fixed points of Θ_τ

Truth has its own case which leads to Spring-like phenomena, the McGee sentence, `McGee`,

`McGee`: Some truth iteration of `McGee` is not true.

Or, more formally, where

$$\texttt{McGee} \text{ is equivalent to } \neg\forall k{>}0\ T^k\ulcorner\texttt{McGee}\urcorner.$$

We can use this to show:

Proposition 6.1 (See also Halbach, 2014, Thereom 14.11)**.** *There is no (non-empty) fixed point of Θ_τ.*

Proof. We first observe that for any $\mathbb{Q}$ and $n > 0$, any $Q \in \Theta^n(\mathbb{Q})$ has $T^i\ulcorner\texttt{McGee}\urcorner \in Q$ for all $i \leqslant n-1$, but also has some $T^k\ulcorner\texttt{McGee}\urcorner \notin Q$. (See Fig. 3.)

	Q in $\Theta(\mathbb{Q})$	Q in $\Theta^2(\mathbb{Q})$	Q in $\Theta^3(\mathbb{Q})$	Q in $\Theta^4(\mathbb{Q})$	$\cdots$
`McGee`		true	true	true	
$T\ulcorner\texttt{McGee}\urcorner$	some not-true		true	true	
$T^2\ulcorner\texttt{McGee}\urcorner$		some not-true		true	
$T^3\ulcorner\texttt{McGee}\urcorner$			some not-true	some nt	
$\vdots$					

Figure 3: Illustration of Θ with the McGee sentence

Base case: We need to show that any $Q \in \Theta(\mathbb{Q})$, some $T^k\ulcorner\texttt{McGee}\urcorner \notin Q$. Since we always have $\varphi \in Q$ iff $T\ulcorner\varphi\urcorner \in \tau(Q)$, if $T^k\ulcorner\texttt{McGee}\urcorner \notin Q$, then $T^{k+1}\ulcorner\texttt{McGee}\urcorner \notin \tau(Q)$. So we just need to consider Q where there is no such k. For such Q, $(\mathbb{N}, Q) \not\models \texttt{McGee}$, so $\texttt{McGee} \notin \tau(Q)$.

Inductive step: by our inductive hypothesis we have that for any $Q \in \Theta^n(\mathbb{Q})$ has $T^i\ulcorner\texttt{McGee}\urcorner \in Q$ for all $i \leqslant n-1$, but also has some $T^k\ulcorner\texttt{McGee}\urcorner \notin Q$. So, since $T\ulcorner\varphi\urcorner \in \tau(Q)$ iff $\varphi \in Q$, we have $T^{i+1}\ulcorner\texttt{McGee}\urcorner \in Q$ for all $i \leqslant n-1$ and $T^{k+1}\ulcorner\texttt{McGee}\urcorner \notin Q$. Thus $T^i\ulcorner\texttt{McGee}\urcorner \in \rho(Q)$ for all $0 < i \leqslant n$ and some $T^k\ulcorner\texttt{McGee}\urcorner \notin \rho(Q)$. Also $(\mathbb{N}, Q) \models \texttt{McGee}$, so $\texttt{McGee} \in \tau(Q)$ as required.

Now, suppose $\mathbb{Q} = \Theta(\mathbb{Q})$. Then also $\mathbb{Q} = \Theta^n(\mathbb{Q})$ for all n. So any $Q \in \mathbb{Q}$ has $T^n\ulcorner\texttt{McGee}\urcorner \in Q$ for all n. But also $T^k\ulcorner\texttt{McGee}\urcorner \notin Q$ for some k. So $\mathbb{Q} = \varnothing$. $\square$

We've thus seen that like for our Spring case, Θ_τ can result in triviality.

6.2 How $\mathcal{J}$ acts on sets of precisifications

$\mathcal{J}$ does not act on sets-of-precisifications in accordance with Θ. Suppose we start with a definite verdicts assignment with $\mathsf{Precs}(S) = \mathbb{Q}$. $\mathcal{J}(S)$ is found by applying Θ to $\mathbb{Q}$ and then recovering its definite verdicts. If we now look at the set of precisifications corresponding to the resultant definite verdicts assignment it does not return $\Theta(\mathbb{Q})$, but instead, $\Delta(\mathbb{Q})$:

Definition 6.2. For a set of precisifications, $\mathbb{Q}$,

$$\Delta(\mathbb{Q}) := \mathsf{Precs}(\mathsf{Def}(\Theta(\mathbb{Q})))$$

Proposition 6.3. *If* $\mathsf{Precs}(S) = \mathbb{Q}$*, then* $\mathsf{Precs}(\mathcal{J}(S)) = \Delta(\mathbb{Q})$*.*

Proof. Suppose $\mathsf{Precs}(S) = \mathbb{Q}$. Then

$$\begin{aligned}\mathsf{Precs}(\mathcal{J}(S)) &= \mathsf{Precs}(\mathsf{Def}(\Theta(\mathsf{Precs}(S)))) && \text{Proposition 2.10}\\ &= \mathsf{Precs}(\mathsf{Def}(\Theta(\mathbb{Q}))) && \text{as } \mathsf{Precs}(S) = \mathbb{Q}\\ &= \Delta(\mathbb{Q}) && \text{definition of } \Delta\end{aligned}$$ □

How does $\Delta(\mathbb{Q})$ relate to $\Theta(\mathbb{Q})$?

Proposition 6.4. $Q^* \in \mathsf{Precs}(\mathsf{Def}(\mathbb{Q}))$ *iff for any* φ*,*

- *if* $\varphi \in Q$ *for all* $Q \in \mathbb{Q}$ *then* $\varphi \in Q^*$*, and*
- *if* $\varphi \notin Q$ *for all* $Q \in \mathbb{Q}$ *then* $\varphi \notin Q^*$*.*

Proof. This follows immediately from the definitions of Precs and Def. □

Corollary 6.5. $\mathsf{Precs}(\mathsf{Def}(\mathbb{Q})) \supseteq \mathbb{Q}$ *for any* $\mathbb{Q}$*. Thus, also,* $\Delta(\mathbb{Q}) \supseteq \Theta(\mathbb{Q})$*.*

Proof. Follows immediately from Proposition 6.4. □

However, Δ adds additional precisifications. And this is what allows $\mathcal{J}$, and thus Δ, to have fixed points, whereas Θ has none.

Proposition 6.6. *There is some* $Q_{\textsf{all-true}}$ *in* $\Delta(\mathcal{P}\mathsf{recs})$ *with* $T^k\ulcorner\mathtt{McGee}\urcorner \in Q_{\textsf{all-true}}$ *for all* k*. There is no such* $Q_{\textsf{all-true}}$ *in* $\Theta(\mathcal{P}\mathsf{recs})$*.*

Proof. We argued in Proposition 6.1 that there is no such $Q_{\textsf{all-true}}$ in $\Theta(\mathcal{P}\mathsf{recs})$. We need to show there is some such in $\Delta(\mathcal{P}\mathsf{recs})$.

Consider any $Q_0 \in \mathcal{P}\mathsf{recs}$. Set $Q_n = \tau^n(Q_0)$. Note that $Q_n \in \Theta^n(\mathcal{P}\mathsf{recs})$, and also that $Q_n \in \Theta(\mathcal{P}\mathsf{recs})$.

Define $Q_{\textsf{all-true}}$ by $\varphi \in Q_{\textsf{all-true}}$ iff φ is stably true in $\langle Q_n \rangle$, that is there is some k with $\varphi \in Q_n$ for all $n > k$. Since every $T^n\ulcorner\mathtt{McGee}\urcorner$ is stable true, it is in $Q_{\textsf{all-true}}$.[17]

We can then show that $Q_{\textsf{all-true}} \in \mathsf{Precs}(\mathsf{Def}(\Theta(\mathcal{P}\mathsf{recs})))$ using Proposition 6.4 and that $\{Q_1, Q_2, \ldots\} \subseteq \Theta(\mathcal{P}\mathsf{recs})$. If $\varphi \in Q$ for all $Q \in \Theta(\mathcal{P}\mathsf{recs})$, then $\varphi \in Q_n$ for all $n > 0$, so $\varphi \in Q_{\textsf{all-true}}$. If $\varphi \notin Q$ for all $Q \in \Theta(\mathcal{P}\mathsf{recs})$, then $\varphi \notin Q_n$ for any $n > 0$, so $\varphi \notin Q_{\textsf{all-true}}$. □

We have thus seen that $\Delta(\mathbb{Q})$ differs from the set of individually revised interpretations, $\Theta(\mathbb{Q})$, in that it will add some additional precise interpretations, such as $Q_{\textsf{all-true}}$, which are not individually revised to, but which cohere with any of the revised precisifications' unanimous verdicts on truth values. (See Fig. 4.)

[17] This style of argument can directly be used to show that Γ has non-trivial fixed points, see Footnote 21.

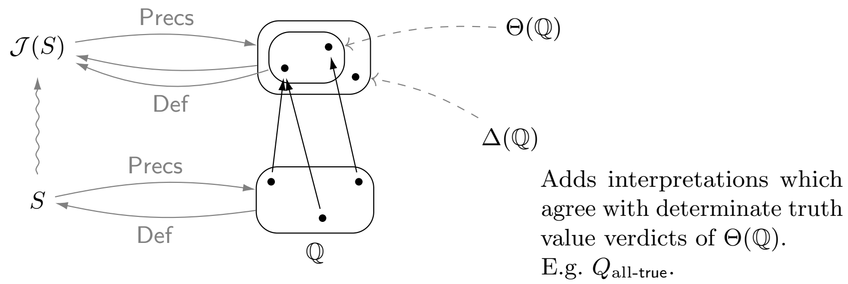

Figure 4: Revising definite verdicts assignment and revising the set of precisifications

7 Reconsidering the jump for indeterminate credence

How can we take these insights and apply them to credence, where we are working directly with the set-of-precisifications model? When viewed through the lenses of sets-of-precisifications, the usual supervaluational jump for truth, $\mathcal{J}$, adds additional precisifications: it corresponds to Δ rather than Θ. We similarly propose an alternative jump for imprecise credences, Γ, which also adds additional precisifications. Which ones? For truth, Δ adds to Θ any precise interpretations which agree on any truth value verdicts that are unanimously agreed on by all $Q \in \Theta(\mathbb{Q})$. For example, in the case of `McGee` it adds some $Q_{\textsf{all-true}}$. For credences, we will instead directly use underlying structure of the real numbers, and take a topological closure.

That is, we will define:

Definition 7.1. For $\mathbb{C} \subseteq \mathsf{Creds}_{\mathcal{A}}$,

$$\Gamma(\mathbb{C}) := \mathsf{closure}(\Theta(\mathbb{C})).$$

where:

Definition 7.2. $c^* \in \mathsf{closure}(\mathbb{C})$ iff there is a sequence, $\langle c_\alpha \rangle$, (not necessarily following the revision function) with each $c_\alpha \in \mathbb{C}$ and where $\langle c_\alpha \rangle$ converges to c^*, i.e., where for all $\varphi \in \mathcal{A}$ and for all $\epsilon > 0$, there is some β such that for all $\alpha > \beta$, $|c_\alpha(\varphi) - c^*(\varphi)| < \epsilon$.[18]

Recall that in the case of Spring,

$$\Theta(\{0, 1, 1/2, 1/4, \ldots\}) = \{1, 1/2, 1/4, 1/8, \ldots\}.$$

so

$$\begin{aligned}\Gamma(\{0, 1, 1/2, 1/4, \ldots\}) &= \mathsf{closure}(\{1, 1/2, 1/4, 1/8, \ldots\})\\ &= \{0, 1, 1/2, 1/4, 1/8, \ldots\},\end{aligned}$$

that is, it adds the additional credence value of 0, which is the limit of the sequence $\langle 1, 1/2, 1/4, \ldots \rangle$ This set is a fixed point of Γ in the Spring case. (See Fig. 5.)

[18]If $\mathcal{A}$ is countable, we just need to look at ω-length sequences.

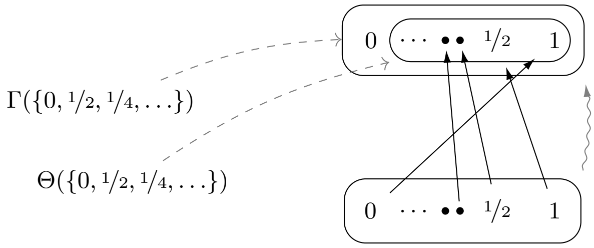

Figure 5: By including the additional credence, 0, we find a fixed point in the case of Spring.

This notion of closure is obtained from the so-called topology of pointwise convergence (with the natural topology on $[0, 1]$). We also define:

Definition 7.3. $\mathbb{C}$ is *closed* if $\mathbb{C} = \mathsf{closure}(\mathbb{C})$.

and there is an important property of this: it is (topologically) *compact*, that is:

Proposition 7.4. *If $\mathcal{E}$ is a collection of* closed *sets which has the finite intersection property,*

i.e. for any finite collection $E_1, \ldots, E_k \in \mathcal{E}$, $E_1 \cap \ldots \cap E_k \neq \varnothing$,

then $\bigcap \mathcal{E} \neq \varnothing$.

This is a consequence of Tychonoff's theorem and the compactness of $[0, 1]$.[19] There some other properties that are equivalent to compactness: that every convergent sequence has a limit point, or that every sequence whatsoever has a cluster point.[20] If we call a collection of sets consistent if it has some common member, i.e. has non-empty intersection, then we can describe this as: any collection of *closed* sets which is finitely consistent is consistent. Since we are looking for fixed points of Γ rather than Θ we can focus just on closed properties in the notion of compactness, as is done in the topological definition of compactness.

This allows us to show:

Theorem 7.5. *For any ρ, there is a non-empty fixed point of Γ.*

Proof. Define a sequence $\mathbb{C}_0 := \mathsf{Creds}_{\mathcal{A}}$, $\mathbb{C}_{\alpha+1} := \Gamma(\mathbb{C}_\alpha)$, $\mathbb{C}_\mu := \bigcap_{\alpha<\mu} \mathbb{C}_\alpha$.

Γ is monotone, that is:

Sublemma 7.5.1. *If $\mathbb{C} \supseteq \mathbb{C}'$ then $\Gamma(\mathbb{C}) \supseteq \Gamma(\mathbb{C}')$.*

Proof. It is easy to observe that Θ is monotone, that is, if $\mathbb{C} \supseteq \mathbb{C}'$ then $\Theta(\mathbb{C}) \supseteq \Theta(\mathbb{C}')$.

[19] See, e.g., Willard (1970, Thm. 42.3). See Campbell-Moore (2019) for more details on this as applied both to credence and truth.

[20] See, e.g., Willard (1970, Thm. 17.4). See also Footnotes 21 and 24 for use of this alternative picture and a comment that it then offers a close relationship to the revision theory.

Also, $\mathsf{closure}$ is monotone: Suppose $\mathbb{C} \supseteq \mathbb{C}'$. For any, $c^* \in \mathsf{closure}(\mathbb{C}')$, there is a sequence $\langle c_\alpha \rangle$ in $\mathbb{C}'$ which convergs to c^*. This sequence is also a sequence in any $\mathbb{C} \supseteq \mathbb{C}'$. So $c^* \in \mathsf{closure}(\mathbb{C})$.

And thus, Γ, which is the result of composing these, is also monotone. $\square$

So, by starting with $\mathbb{C}_0 = \mathsf{Creds}_{\mathcal{A}}$, where we have $\mathbb{C}_0 \supseteq \Gamma(\mathbb{C}_0)$, we have that for $\alpha < \beta$, $\mathbb{C}_\alpha \supseteq \mathbb{C}_\beta$; and there must be a (possibly empty) fixed point of Γ.

We need to check that this fixed point is non-empty, which we do by induction.[21]

- Base case: $\mathbb{C}_0 = \mathsf{Creds}_{\mathcal{A}} \neq \varnothing$.
- Successor case: For any $c \in \mathbb{C}_\alpha$, $\rho(c) \in \Theta(\mathbb{C}_\alpha)$, and since $\mathsf{closure}(\mathbb{C}') \supseteq \mathbb{C}'$ for any $\mathbb{C}'$ (as the constant sequence $\langle c', c', \ldots \rangle$ converges to c), also $\rho(c) \in \mathsf{closure}(\Theta(\mathbb{C}_\alpha)) = \mathbb{C}_{\alpha+1}$.
- Limit case: Suppose each $\mathbb{C}_\alpha \neq \varnothing$ for $\alpha < \mu$. $\{\mathbb{C}_\alpha \mid \alpha < \mu\}$ is a collection of closed subsets of Creds. For $\alpha < \beta$, $\mathbb{C}_\alpha \supseteq \mathbb{C}_\beta$, so any finite subcollection has a non-empty intersection. Thus, by Proposition 7.4, $\mathbb{C}_\mu \neq \varnothing$. $\square$

We will now use these revision jumps to spell out a notion of underminingness applied to imprecise credences, and use this result to show that there are always some non-undermining credal states.

7.1 Underminingness

We had originally considered suggesting that an imprecise credal state is undermining if $\Theta(\mathbb{C}) \neq \mathbb{C}$. If this is the right notion of underminingness, then in the case of SPRING, not only are all precise credence functions undermining, also all the imprecise credences are too. However, we have now defined a new jump, Γ and we instead propose to extend the notion of underminingness to the imprecise setting by saying:[22]

Definition 7.6. $\mathbb{C}$ is *non-undermining* iff $\Theta(\mathbb{C}) \subseteq \mathbb{C} \subseteq \Gamma(\mathbb{C})$.

This says that every $c \in \mathbb{C}$ should have its recommended credence in the set, i.e. $\Theta(\mathbb{C}) \subseteq \mathbb{C}$, and that every credence function in the set should either be recommended by someone in the set or be the limit of a sequence of such recommended functions, i.e., $\mathbb{C} \subseteq \mathsf{closure}(\Theta(\mathbb{C})) = \Gamma(\mathbb{C})$.

This definition allows that any $\mathbb{C}$ which is a fixed point of Θ is non-undermining.[23] So are any fixed points of Γ, and thus by Theorem 7.5 there is al-

[21] Alternative arguments are possible: Firstly, we could observe that what compactness shows us is that the non-empty closed subsets of $\mathbb{C}$ forms a ccpo in the sense of Visser (1984). And since we can restrict attention to the closed subsets for the purposes of Γ, there must be a fixed point. Secondly, we can define a revision sequence: $c_0 \in \mathsf{Creds}_{\mathcal{A}}$, $c_{\alpha+1} = \rho(c_\alpha)$, and let c_μ be a cluster point of the preceding sequence (in Campbell-Moore (2019) we proposed using this as the limit criterion in the revision theory), and observe that $c_\alpha \in \mathbb{C}_\alpha$. This relies on the ability to always find a cluster point, which is equivalent to compactness.

[22] To extend this to the case where ρ doesn't pick out a unique credence function but can allow for ties, we will say that $\mathbb{C}$ is non-undermining iff each $c \in \mathbb{C}$ has at least one of their maximally recommended credences in $\mathbb{C}$, and everything in $\mathbb{C}$ is in the closure of recommended credences.

[23] This is a key reason to give our definition rather than saying that it has to be a fixed point of Γ. I would like to say the further thing that they are preferable: if they exist then they are required, but the criterion does not do this. To account for this intuition, we might also define a notion of recommendation for imprecise credences as: $\mathbb{C}$ recommends $\Theta(\mathbb{C})$ and say that ideally one's credal state should be self-recommending, but if it cannot be, it should at least be non-undermining.

ways some non-undermining opinion state. A state can also be non-undermining without being a fixed point of either of these if it contain some but not all members of the closure.[24]

What credal states are non-undermining in the cases mentioned? (Most of these were introduced in Section 3.)

- For the PASSPORT case, $\{0, 1\}$ is the only credal state which is non-undermining. The same revision function is used for BADNAVIGATOR and `PrLiar`, so the same holds for these cases too.
- BASKETBALL is similar, and the only non-undermining credal state is $\{0.495, 0.505\}$.
- For normal cases, like RAIN, where $\rho(c) = c$ for all c, every imprecise credal state, $\mathbb{C}$, is a fixed point of Θ, and thus is non-undermining.

  Note that if we *required* states to be fixed points of Γ, then a set which is not closed, such as $(0.2, 0.8) = \{x \mid 0.2 < x < 0.8\}$ would be undermining as it doesn't contain its limit points of 0.2 and 0.8. But we would like to say that it is non-undermining, and we can do so with our definition.
- Since the revision function of LEAP is identical to that of RAIN, it too says that any $\mathbb{C}$ is non-undermining.
- For an extremal version of LEAP where $\rho(x) = 1$ if $x \geqslant 0.5$ and $\rho(x) = 0$ if $x < 0.5$, the non-undermining options are 0, 1 and $\{0, 1\}$.[25]
- For PROMOTION, the precise credence 0.5 is non-undermining. But so is any imprecise credence with $x \in \mathbb{C}$ iff $1 - x \in \mathbb{C}$, e.g., $\{0.2, 0.8\}$, or $(0.2, 0.8)$.
- For SPRING, $\{0, 1, 1/2, 1/4, \ldots\}$ is a fixed point of Γ, it is the only non-undermining state.

8 Other applications

We have here considered the notions of truth and rational credence. But the considerations and construction we give here is very general. It could fruitfully apply to any target domain, for example: reference or satisfaction; membership or exemplification; necessity or knowledge; or decision theoretic or game theoretic rationality.[26]

All one needs to be able to apply it to a target domain is to specify a collection of all potential precisifciations, $\mathcal{P}$, and how to revise each of them, i.e. specify a revision function, $\rho : \mathcal{P} \to \mathcal{P}$.

[24] A further advantage of this definition is it leads to a nice relationship with the revision theory of Gupta and Belnap (1993). If one has a revision sequence, following ρ as the revision step and using the limit criterion that the limit be a cluster point of the preceding sequence, then the collection of members of the looping part of the sequence is non-undermining. So are any unions of such revision loops. (However, there are non-undermining states which are not unions of such revision loops.)

[25] This could be described as a "truth-teller" variant of `PrLiar`.

[26] I have mainly here selected examples that are discussed in applications of the revision theory of Gupta and Belnap (1993), where there is a revision function one would ideally like to find fixed points of.

For example, for truth as presented here, $\mathcal{P} = \mathcal{P}\mathsf{recs} = \wp(\mathsf{Sent}_T)$ and ρ was spelled out with τ, and for credences, $\mathcal{P} = \mathsf{Creds}_{\mathcal{A}}$ and ρ was taken as given.

We might also use this general framework to see how it applies to credences as specified over possible world structures. This is particularly useful for developing a semantics for a language with sentences saying something about their credence values. Given a fixed possible world structure, with various worlds and probabilistic accessibility relations between them, we consider a member of $\mathcal{P}$ to be given by an assignment of a credence function at each world: a credal-evaluation-function. We can then directly define the revision of a credal-evaluation-function by taking the weighted proportion of the accessible worlds where the sentence is evaluated as true when the initial credal-evaluation-function provides the interpretation of the credence function symbol at the various worlds.[27]

8.1 Working directly with sets of precisifications

Once we specify these two inputs, all the considerations of the paper can immediately apply.

We firstly consider what a supervaluational picture of these notions are. In the case of truth we considered an independently given account: we were only interested in the definite truth-value verdicts. However, in general, we might think that the indeterminacy of the notion should just be spelled out by providing a set of precisifications, as we did for credences. And this offers the most general model compatible with supervaluationism.

Working with a set of precisifications, we then have a proposal for a jump:

$$\Theta(\mathbb{P}) := \{\rho(p) \mid p \in \mathbb{P}\} \quad \text{ for } \mathbb{P} \subseteq \mathcal{P}$$

However, this does not in general allow one to find fixed points. We thus instead suggested allowing for some additional precisifications to be added when taking the jump: those that are close to being recommended. We spelled this out with a notion of $\mathsf{closure}$ and defined:

$$\Gamma(\mathbb{P}) := \mathsf{closure}(\Theta(\mathbb{P})).$$

The features that we need of $\mathsf{closure} : \wp(\mathcal{P}) \to \wp(\mathcal{P})$ are:

Assumption 1.

- $\mathsf{closure}(\mathbb{P}) \supseteq \mathbb{P}$ for all $\mathbb{P}$.
- If $\mathbb{P} \supseteq \mathbb{P}'$ then $\mathsf{closure}(\mathbb{P}) \supseteq \mathsf{closure}(\mathbb{P}')$.

In the case of probability, this was naturally given by the closure notion from the topology of pointwise convergence built over the natural topology on $[0, 1]$.

The key property we need for the claim that there are non-trivial fixed points of Γ is compactness:

Definition 8.1. $\mathcal{P}$ (with $\mathsf{closure}$) is *compact* iff Proposition 7.4 holds of it.

We can then ensure there are non-empty fixed points.

Proposition 8.2. *If $\mathcal{P}$ is compact (and $\mathcal{P} \neq \varnothing$), then there is some non-empty fixed point of Γ.*

[27] See Campbell-Moore (2016, secs. 5.3, 2.4.2).

Proof. Exactly as in Theorem 7.5. Assumption 1 describes the properties of $\mathsf{closure}$ that were used in that argument. $\square$

We might apply this directly to truth if we are interested not only in the definite truth value verdicts but the more general structure as encoded in a set of precisifications. This would, for example, also care about whether it's definite that at least one of φ or ψ is true, or if it's definite that truth is maximally consistent, or even if it's definite that truth is ω-consistent. To do this, we need to define an appropriate notion of $\mathsf{closure}$. One might simply take $\mathsf{closure}(\mathbb{Q}) := \mathsf{Precs} \circ \mathsf{Def}\,(\mathbb{Q})$, in which case $\Gamma(\mathbb{Q}) = \Delta(\mathbb{Q})$, and then one can check that this is compact. However, we might instead think about the precise interpretations of truth as functions from Sent_T to $\{\mathsf{true}, \mathsf{not\text{-}true}\}$ and use the convergence topology, which can be specified as: $Q^* \in \mathsf{closure}(\mathbb{Q})$ iff for every finite set of sentences Σ there is some $Q \in \mathbb{Q}$ with $Q\restriction_\Sigma = Q^*\restriction_\Sigma$ (see Campbell-Moore, 2019, for further details). By Tychonoff's theorem, this is compact so our results will apply to it. Note, though, that generally $\mathsf{closure}(\mathbb{Q}) \subset \mathsf{Precs} \circ \mathsf{Def}\,(\mathbb{Q})$, and so $\Gamma(\mathbb{Q}) \subset \Delta(\mathbb{Q})$. For example, every $Q \in \Gamma(\mathbb{Q})$ is maximally consistent, whereas members of $\Delta(\mathbb{Q})$ needn't be.[28]

8.2 Working with alternative models

One might be interested in working with an independently specified supervaluational model of one's target notions rather than working directly with sets of precisifications. This is, for example, what was done for truth when we considered definite truth value assignments.

Call the collection of all one's independently specified models to be $\mathcal{D}$, with d being the members of it. For example d is a definite verdicts assignment and $\mathcal{D}$ the collection of all definite verdicts assignments.

One can also consider definite judgements assignments for other target notions, for example credence. In general, given a range of properties one is interested in, such as, 'φ is true', or 'my credence in φ is $\geqslant r$', a definite judgements assignment would provide judgements as to which of these definitely hold and which definitely don't hold.

Since we are assuming supervaluationism, what's important for describing a supervaluational jump is a set of precisifications. We thus need to have a "translation procedure" between these supervaluational models and sets of precisifications. So we assume we have functions:

$$\mathsf{P} : \mathcal{D} \to \mathcal{P}$$
$$\mathsf{D} : \mathcal{P} \to \mathcal{D}.$$

For example, if one considers a definite judgements assignment, d, we will define $\mathsf{P}(\mathrm{d})$ as the collection of precisifications that cohere with these judgements as analogous to Definition 2.5, and $\mathsf{D}(\mathbb{P})$ to give the determinate judgements of $\mathbb{P}$, as analogous to Definition 2.6.

[28]It is interesting to note that if one considers restricting $\mathcal{P}\mathsf{recs}$ to $\mathcal{P}\mathsf{recs}_{\mathrm{MaxCons}}$, then $\mathsf{closure}(\mathbb{Q}) = \mathsf{Precs} \circ \mathsf{Def}\,(\mathbb{Q})$, so then $\Gamma(\mathbb{Q}) = \Delta(\mathbb{Q})$. This is because the treatment of a finite collection of sentences by a maximal consistent interpretation is characterised by a single sentence.

One might also think of the admissibility conditions as modifying P, for example considering

$$\mathsf{Precs}_{\mathrm{MaxCons}}(S) = \{Q \in \mathsf{Precs}(S) \mid Q \text{ is maximal and consistent}\}$$
$$\mathsf{Precs}_{\omega\mathrm{Cons}}(S) = \{Q \in \mathsf{Precs}(S) \mid Q \text{ is } \omega\text{-consistent}\}$$

$\mathcal{P}$ and $\mathcal{D}$ typically each have their own expressive power which is not captured in the other, so these won't be one-to-one functions.[29] However, focusing on the sets of precisifications specified by a definite judgements assignment is equivalent to focusing on those definite judgements assignments that are specified by a set of precisifications. That is, once we ignore any of the additional expressive power of the two frameworks, and just restrict attention to the d or $\mathbb{P}$ that are in the ranges of D and P, they are isomorphic under P and D. We in general take this as an assumption:

Assumption 2. $\mathsf{P} \circ \mathsf{D} \circ \mathsf{P}\,(\mathrm{d}) = \mathsf{P}(\mathrm{d})$, and $\mathsf{D} \circ \mathsf{P} \circ \mathsf{D}\,(\mathbb{P}) = \mathsf{D}(\mathbb{P})$.

We also have to assume:

Assumption 3. $\mathsf{P} \circ \mathsf{D}$ is monotone, that is: If $\mathbb{P} \supseteq \mathbb{P}'$, then $\mathsf{P}(\mathsf{D}(\mathbb{P})) \supseteq \mathsf{P}(\mathsf{D}(\mathbb{P}'))$.

This is equivalent to there being an ordering $\preceq$ on $\mathcal{D}$ where both P and D are order-preserving,[30] that is, that

$$\text{If } \mathrm{d} \preceq \mathrm{d}', \text{ then } \mathsf{P}(\mathrm{d}) \supseteq \mathsf{P}(\mathrm{d}'),$$
$$\text{If } \mathbb{P} \supseteq \mathbb{P}' \text{ then } \mathsf{D}(\mathbb{P}) \preceq \mathsf{D}(\mathbb{P}').$$

This will hold for the definite judgement assignments model by defining $\mathrm{d} \preceq \mathrm{d}'$ if every judgement that definitely holds according to d also definitely holds according to d′, and every judgement that definitely doesn't hold according to d also definitely doesn't hold according to d′.

We also need a further assumption to be able to apply Theorem 7.5 to the general setting:

Assumption 4. If $\mathbb{P}$ is non-empty, then so is $\mathsf{P} \circ \mathsf{D}\,(\mathbb{P})$.

To ensure this is satisfied when working with definite judgements assignments where admissibility conditions are added to explicitly modify P, we need to restrict our space of all possible precisifications, $\mathcal{P}$, to just those that are admissible, e.g.:

$$\mathcal{P}\mathsf{recs}_{\mathrm{MaxCons}} = \{Q \mid Q \text{ is maximal and consistent}\}$$
$$\mathcal{P}\mathsf{recs}_{\omega\mathrm{Cons}}(S) = \{Q \mid Q \text{ is } \omega\text{-consistent}\}$$

Since $\mathcal{P}\mathsf{recs}_{\omega\mathrm{Cons}}$ is not compact, our result will then not apply to it. And indeed, the McGee phenomena shows that there are no fixed points of $\mathsf{Def} \circ \Gamma \circ \mathsf{P}$ (when $\mathcal{P}$ is thus restricted) because the required $Q_{\textsf{all-true}}$, which was added by

[29]If we focus on definite judgements assignments where we consider *all* properties, i.e. all $\mathbb{B} \subseteq \mathcal{P}$, rather than just, e.g., definite truth value assignments, then the definite judgements model is at least as expressive as the sets of precisifications model and P is one-to-one.

[30]To show this suffices, note that the composition of monotone functions is monotone. To see it's necessary, define $\mathrm{d} \preceq \mathrm{d}'$ iff $\mathsf{P}(\mathrm{d}) \supseteq \mathsf{P}(\mathrm{d}')$.

taking closures, is not a member of $\mathcal{P}_{\omega\text{Cons}}$, so is no longer added.[31] $\mathcal{P}\mathsf{recs}_{\text{MaxCons}}$, however, is compact.[32]

We can then state our main result:

Proposition 8.3. *If $\mathcal{P}$ is compact, then there is a non-trivial fixed point of $\mathcal{K} := \mathsf{D} \circ \Gamma \circ \mathsf{P}$.*

Proof. Define recursively $\mathbb{P}_0 = \mathcal{P}$, $\mathbb{P}_{\alpha+1} = \Gamma \circ \mathsf{P} \circ \mathsf{D}(\mathbb{P}_\alpha)$, and $\mathbb{P}_\mu = \bigcap_{\alpha<\mu} \mathbb{P}_\alpha$. Also define $\mathrm{d}_\alpha = \mathsf{D}(\mathbb{P}_\alpha)$ and $\mathbb{P}'_\alpha = \mathsf{P}(\mathrm{d}_\alpha)$. See Fig. 6.

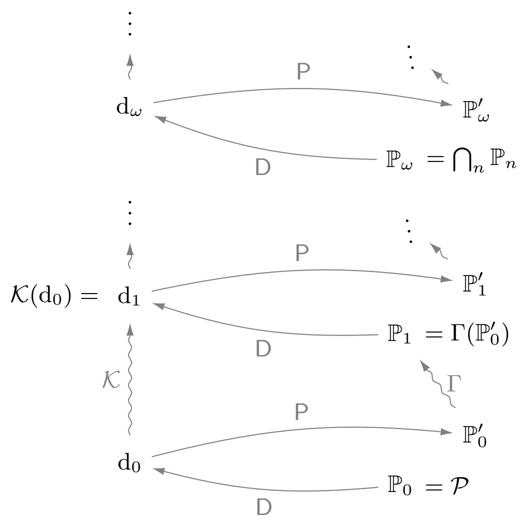

Figure 6: To show there is a fixed point

Since we have assumed $\mathsf{P} \circ \mathsf{D}$ is monotone, and so are Θ and $\mathsf{closure}$, we get that $\Gamma \circ \mathsf{P} \circ \mathsf{D} = \mathsf{closure} \circ \Theta \circ \mathsf{P} \circ \mathsf{D}$ is monotone. So it has a fixed point.

To show it is non-trivial we show by (simultaneous) induction that each $\mathbb{P}_\alpha$ and $\mathbb{P}'_\alpha$ are non-empty. The argument for $\mathbb{P}_\alpha$ is exactly as in Theorem 7.5. Since $\mathbb{P}'_\alpha = \mathsf{P} \circ \mathsf{D}(\mathbb{P}_\alpha)$, we also get that $\mathbb{P}'_\alpha$ is non-empty by assumption 4. ☐

We can now state certain cases where we can work directly with $\mathcal{J} = \mathsf{D} \circ \Theta \circ \mathsf{P}$ and avoid triviality.

Corollary 8.4. *If $\mathsf{P} \circ \mathsf{D}(\mathbb{P}) \supseteq \mathsf{closure}(\mathbb{P})$ for all $\mathbb{P}$, then $\mathsf{D} \circ \Theta \circ \mathsf{P} = \mathsf{D} \circ \Gamma \circ \mathsf{P}$. So if $\mathcal{P}$ is compact, $\mathcal{J} := \mathsf{D} \circ \Theta \circ \mathsf{P}$ also has non-trivial fixed points.*

Proof. By assumption 1 and the assumption in the result,

$$\mathsf{P} \circ \mathsf{D}(\Theta(\mathbb{P})) \supseteq \mathsf{closure}(\Theta(\mathbb{P})) = \Gamma(\mathbb{P}) \supseteq \Theta(\mathbb{P})$$

so by assumption 3,

$$\mathsf{P} \circ \mathsf{D} \circ \mathsf{P} \circ \mathsf{D}(\Theta(\mathbb{P})) \supseteq \mathsf{P} \circ \mathsf{D} \circ \Gamma(\mathbb{P}) \supseteq \mathsf{P} \circ \mathsf{D} \circ \Theta(\mathbb{P})$$

[31] For all I know, $\mathsf{Def} \circ \Gamma \circ \mathsf{Precs}_{\omega\text{Cons}}$ might have fixed points when we keep $\mathcal{P} = \wp(\mathsf{Sent}_T)$, in particular, containing such ω-inconsistent $Q_{\textsf{all-true}}$; but our argument doesn't guarantee this.

[32] As it is a closed subset of the compact space $\wp(\mathsf{Sent}_T)$.

And the left-hand side $= \mathsf{P} \circ \mathsf{D} \circ \Theta(\mathbb{P})$ by assumption 2. Thus, $\mathsf{P} \circ \mathsf{D} \circ \Theta(\mathbb{P}) = \mathsf{P} \circ \mathsf{D} \circ \Gamma(\mathbb{P})$. So, by applying D to both sides, and replacing $\mathbb{P}$ with $\mathsf{P}(\mathrm{d})$, we have (using assumption 2) $\mathsf{D} \circ \Theta \circ \mathsf{P}(\mathrm{d}) = \mathsf{D} \circ \Gamma \circ \mathsf{P}(\mathrm{d})$ as required. □

This holds for the case of truth even when $\mathcal{P}$ is restricted with an admissibility condition.[33]

However, it will not hold when considering definite judgement assignments in the case of credences. Consider just focusing on definite judgements as to whether one's credence in φ is $> r$ or not. However, for $\mathbb{C} = \{x \mid x > 0\}$, $\mathsf{closure}(\mathbb{C}) = \{x \mid x \geqslant 0\}$, but $x^* = 0 \notin \mathsf{P} \circ \mathsf{D}(\mathbb{C})$ as $x^* \not> 0$. The Spring case will show there's no non-trivial fixed points of $\mathsf{D} \circ \Theta \circ \mathsf{P}$, but there are non-trivial fixed points of $\mathsf{D} \circ \Gamma \circ \mathsf{P}$.

We would find analogous examples for any property one focuses on, e.g., whether one's credence is $\geqslant r$, at least if one is interested both in whether it definitely holds and whether it definitely doesn't hold. Formally, this is because, for the case of credences, a set and its complement cannot both be closed (unless it's $\varnothing$ or $\mathsf{Creds}_{\mathcal{A}}$). To avoid such worries, and have that $\mathsf{D} \circ \Theta \circ \mathsf{P}$ always has non-trivial fixed points, one would need to only focus just on the closed properties, e.g., whether one's credence is definitely $\geqslant r$ without considering whether it's definitely not $\geqslant r$. This would certainly be possible. But, whenever one is also interested in some non-closed property, such as whether one's credences is definitely not $\geqslant r$, then to avoid triviality, one needs to add closures explicitly when defining the jump. It was a special feature of truth-values (they're discrete) that meant that focussing on definite truth value verdicts allowed triviality to be avoided.[34]

9 Conclusion

One might consider modelling indeterminate truth with sets of precisifications, but it is more normal when working with the Kripke account to just focus on the definite verdicts themselves. We have seen that this choice makes a substantial technical contribution: it allows the account to avoid triviality.

However, one might want to directly work with the sets of precifications themselves, and we focused on this when considering rational credences. Acknowledging that the natural supervaluational jump defined on sets of precisifications, Θ, may lead to triviality, we considered what allows the choice to focus on the determinate verdicts in the case of truth to avoid triviality. The reason was that precisifications are only ruled out when they disagree on definite verdicts, and this allows that additional precisifications, such as $Q_{\textsf{all-true}}$, which are not in $\Theta(\mathbb{Q})$, are re-included when applying the supervaluational Kripkean jump.

We can still work with sets of precisifications but nonetheless avoid triviality if we replicate this by defining an alternative jump on sets of precisifications which additionally takes the closure of the revised set. This will allow us to recover fixed points. The property that ensures this is that the underlying space of precisifications is topologically compact.

[33]Though if introduce admissibility conditions by restricting Precs without changing $\mathcal{P}$, it only holds if the admissibility condition is closed. For example, it does not then hold for $\mathcal{P}_{\omega\mathrm{Cons}}$, as some $Q_{\textsf{all-true}}$ is in the closure but is ω-inconsistent.

[34]As both the property of being true and being not-true are closed properties.

In fact, we furthermore see that the complications that are found in the case of credences when focusing on sets of precisifications cannot be avoided by moving to focusing only on determinate judgements unless one imposes seemingly ad-hoc restrictions on the judgements that are considered.

We used these considerations to propose a notion of when sets of credence functions are undermining. And we showed that there is always some non-undermining response to a situation. Thus, our seeming epistemic dilemma is avoided by allowing imprecise attitudes.

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