Further technical results for 'Strict Propriety is Weak'

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Abstract

This document gives careful arguments for the claims made in the paper 'Strict Propriety is Weak'. In particular, we are careful about infinity considerations.

1 Infinity considerations

Remark 1.1 (Schervish et al., 2009, p.210)**.** To even make sense of additivity, there might be problems if we allow both $+\infty$ and $-\infty$. Suppose we have $\mathsf{acc}(0, \mathsf{t}) = -\infty$, $\mathsf{acc}(0, \mathsf{f}) = -\infty$. Consider $p(\psi_1) = p(\psi_2) = 0$, with ψ_1 true at w, and ψ_2 false at w. $\mathsf{Acc}(p, w) = \mathsf{acc}(p(\psi_1), \mathsf{t}) + \mathsf{acc}(p(\psi_2), \mathsf{f}) + \ldots = \infty - \infty + \ldots =$???.

So: generally when dealing with global scores, we rule out having both $\pm\infty$.

Note, also that if both $\pm\infty$ are allowed, Schervish et al. (2009, Example 5) show that incoherent need not be weakly dominated even with strictly proper scoring rule.

Definition 1.2. Definitions made to deal with infinities:

Following Schervish (1989), we define acc can take values $\pm\infty$.

$$\mathrm{Exp}_x \mathsf{acc}(y) = x\mathsf{acc}(y, \mathsf{t}) + (1 - x)\mathsf{acc}(y, \mathsf{f})$$

$0 \times \pm\infty$ is taken to be 0 if it appears as one of the constituents of the equation.

This is then only ill-defined if we have $\infty - \infty$. In that case, we consider $\lim_{t \longrightarrow z} \mathrm{Exp}_x \mathsf{acc}(t)$. We assume this converges (possibly to $\pm\infty$)[1]. (qu: exactly what assuptions would this converges assumption correspond to?[2] Note, though, this is only successful in the single-proposition case, see Remark 1.1, so we don't worry about it too much.)

[1] Just checking that: what does it even mean to be continuous at a point where it $= -\infty$? Means it $\longrightarrow -\infty$. Define open sets as $(a, \infty]$ to make this definition work.

[2] NB: if it's $\infty - \infty$ all over the place, e.g. $g_0 = +\infty$, $g_1 = -\infty$, this is problematic. I wonder if it follows from: finite on $(0, 1)$ and continuous at endpoints. Not sure.

More generally,

$$\mathrm{Exp}_p\mathsf{Acc}(q) := \sum_w p(w)\mathsf{Acc}(q,w)$$

As above, if we have $0 \times (-\infty)$ somewhere inside this sum, we define that to be $= 0$. NB, this definition needs to ensure **??** holds.

Remark 1.3. Just allowing one-sided infinities, let's say $-\infty$:

General problems with infinities come from:

- Weak facts about Acc entailing weak facts about acc. Because:
  - $A + B \geqslant A' + B$ need not entail $A \geqslant A'$ when $B = \pm\infty$. It might be that both sides are infinite, and finite wrong-way orders in A are engulfed by infinites from B.
- Strong facts about acc entailing strong facts about Acc.
  - Because:$A > A'$ and $B \geqslant B'$ need not entail $A + B > A' + B$ when $B = \pm\infty$.

We're safe in:

- Strong facts about Acc entailing strong facts about acc. Because:
  - $A + B > A' + B$ entails $A + B$ is finite, so both A and B are finite. A' might still be $= -\infty$, but we nonetheless have $A > A'$.
- Weak facts about acc entailing weak facts about Acc.
  - Because:$A \geqslant A'$ and $B \geqslant B'$ always entails $A + B \geqslant A' + B$ even when we're dealing with infinities.

We need to check the cases where these sorts of things are used and show that there's special reasons why we don't encounter the worrying instances of these. One result that will help us with this:

Proposition 1.4. *Suppose that we have assumed that we only have $-\infty$, and it'll only appear at endpoints of $\mathsf{acc}(0, \mathsf{t})$ or $\mathsf{acc}(1, \mathsf{f})$. Then $\mathrm{Exp}_p\mathsf{Acc}(p)$ is finite.*

Proof. Note that $\mathrm{Exp}_x\mathsf{acc}(x)$ is always finite: there's always 0-weight on the infinite values.

Our stipulation then allows us to push that to Acc. □

In fact, this is a consequence of strict truth directedness and only allowing $-\infty$:

Lemma 1.5. *If acc is strictly truth directed, then the only possible infinite values are at the endpoints. If we've only allowed $-\infty$, this'll be at $\mathsf{acc}(0, \mathsf{t})$ or $\mathsf{acc}(1, \mathsf{f})$. this'll appear at*

This does not follow from weak truth directedness or weak propriety: consider 4.1 of Schervish (1989).

One final result:

Lemma 1.6 (Schervish et al. (2009)). *If* acc *is weakly proper, and* $\mathsf{acc}(x, v) = -\infty$ *for some* $x \in (0, 1)$ *and* v*, then for all* y *there is* v *with* $\mathsf{acc}(y, v) = -\infty$*.*

Proof Attempt. Suppose $\mathsf{acc}(x_0, \mathsf{t}) = -\infty$. But $\mathrm{Exp}_{x_0}\mathsf{acc}(x_0) = x_0 \times \mathsf{acc}(x_0, \mathsf{t}) + (1 - x_0)\mathsf{acc}(x_0, \mathsf{f}) = -\infty$. Since by weak propriety, this is optimal, we must have $\mathrm{Exp}_{x_0}\mathsf{acc}(y) = -\infty$ for all y.

2 Results

2.1 Lemmas for main result

Remark 2.1 (Thought about additivity). There are two ways to think about the algebra and additivity.

1. There is a fixed $\mathcal{A}$ and a fixed Acc defined on credences over this agenda, which our, e.g., strict propriety assumptions work on. We then need to derive facts about strict propriety of acc
2. Acc is thought of as a function which when given an agenda spits out a measure of accuracy. Saying "Acc is strictly proper" then means it is strictly proper on every agenda. Then claims like "Acc is strictly proper implies acc is strictly proper" is trivial by letting the agenda be a singleton.

It's nice if we can think about the former, it's weaker. Pettigrew in **?** does the latter.

Proposition 2.2. *Suppose* Acc *is additive, with* acc *as the component function.* Acc *is weakly proper iff* acc *is weakly proper. And similarly for strict propriety.*

Assumes:

- *Not both* $\pm\infty$ *(to make sense of additive: see Remark 1.1).*
- Acc *weakly proper implies* acc *weakly proper, and* acc *strictly proper implies* Acc *strictly proper both assume: only* $-\infty$ *allowed and this'll appear at the endpoints of* $\mathsf{acc}(0, \mathsf{t})$ *or* $\mathsf{acc}(1, \mathsf{f})$*.*

Before proving it, we need a further result:

Proposition 2.3. *If* Acc *is additive,*

$$\mathrm{Exp}_p\mathsf{Acc}(q) := \sum_{\varphi} \mathrm{Exp}_{p(\varphi)}\mathsf{acc}(q(\varphi)). \tag{1}$$

Proof.

$$\begin{aligned}
\mathrm{Exp}_p \mathsf{Acc}(q) &= \sum_w p(w)\mathsf{Acc}(q,w) && (2)\\
&= \sum_w p(w) \sum_\varphi \mathsf{acc}(q(\varphi), w(\varphi)) && (3)\\
&= \sum_\varphi \sum_w p(w)\mathsf{acc}(q(\varphi), w(\varphi)) && (4)\\
&= \sum_\varphi \left(\sum_{w(\varphi)=\mathsf{t}} p(w)\mathsf{acc}(q(\varphi),\mathsf{t}) + \sum_{w(\varphi)=\mathsf{f}} p(w)\mathsf{acc}(q(\varphi),\mathsf{f}) \right) && (5)\\
&= \sum_\varphi p(\varphi)\mathsf{acc}(q(\varphi),\mathsf{t}) + (1-p(\varphi))\mathsf{acc}(q(\varphi),\mathsf{f}) && (6)\\
&= \sum_\varphi \mathrm{Exp}_{p(\varphi)}\mathsf{acc}(q(\varphi)) && (7)
\end{aligned}$$

Now, does this all work if there are infinities around? The only stipulation we made involved $0 \times -\infty$; and we made this stipulation both on the Acc and acc side. So we can ignore (in both cases) any terms where $p(w)/p(\varphi) = 0$. If we're dealing with more than finitely many worlds we might have to be careful. ☐

I haven't done this super carefully.

Now, return to proving Proposition 2.2:

Proof. Suppose Acc is strictly proper. $\mathrm{Exp}_p\mathsf{Acc}(q) < \mathrm{Exp}_p\mathsf{Acc}(p)$ for all probabilistic p.

Consider $x \in [0,1]$. Take any p probabilistic with $p(\psi) = x$ for some ψ. Now for $y \neq x$ consider $q(\psi) = \begin{cases} y & \varphi = \psi \\ p(\varphi) & \varphi \neq \psi \end{cases}$.

$$\begin{aligned}
0 &> \mathrm{Exp}_p\mathsf{Acc}(p) - \mathrm{Exp}_p\mathsf{Acc}(q) && (8)\\
&= \sum_\varphi \mathrm{Exp}_{p(\varphi)}\mathsf{acc}(p(\varphi)) - \mathrm{Exp}_{p(\varphi)}\mathsf{acc}(q(\varphi)) && (9)\\
&= \mathrm{Exp}_x\mathsf{acc}(x) - \mathrm{Exp}_x\mathsf{acc}(y) && (10)
\end{aligned}$$

Similarly, Acc is weakly proper entails acc is weakly proper.

We now consider the acc to Acc direction. Suppose acc is strictly proper.

$$\begin{aligned}
&\mathrm{Exp}_p\mathsf{Acc}(p) - \mathrm{Exp}_p\mathsf{Acc}(q) && (11)\\
&= \mathrm{Exp}_p \sum_\varphi \mathsf{acc}(p(\varphi)) - \mathrm{Exp}_p \sum_\varphi \mathsf{acc}(q(\varphi)) && (12)\\
&= \sum_\varphi \underbrace{\left(\mathrm{Exp}_{p(\varphi)}\mathsf{acc}(p(\varphi)) - \mathrm{Exp}_{p(\varphi)}\mathsf{acc}(q(\varphi))\right)}_{\geqslant 0 \text{ for all } \varphi, \text{ and } > 0 \text{ for some } \varphi} && (13)\\
&> 0 && (14)
\end{aligned}$$

Similarly for weak propriety.

Now consider infinity. In lieu of Remark 1.3 the argument involving strict propriety in $\mathsf{Acc} \implies \mathsf{acc}$, and weak propriety in $\mathsf{acc} \implies \mathsf{Acc}$ are fine (although note that they shouldn't be formulated with $-$ but directly using the addition and ordering facts as discussed in that remark).

Think about weakly proper Acc entails weakly proper acc. This might be a problem: Consider $x = 0$. Choose a probability p with $p(\psi) = 0$. It might be that p has to assign extremal values to other propositions. This might lead to a problem. However, by Proposition 1.4, we can make assumptions so that we're not ultimately having to deal with a case of infinities. Similarly for strictly proper acc entails strictly proper Acc. With the additional assumptions, we can use Proposition 1.4 to avoid worries with infinities.

Note, if we drop the idea that it is additive with the same score for each proposition, we [to be checked] have that Acc is strictly proper iff each constituatnt acc is weakly proper and at least one of them is strictly proper. □

Proposition 2.4. Acc *is strictly truth directed, i.e.:*

- *if $c(\varphi) \leqslant c(\varphi)$ for all φ true at w, and $c(\varphi) \geqslant c'(\varphi)$ for all φ false at w, with at least one strict inequality, then $\mathsf{Acc}(c, w) < \mathsf{Acc}(c', w)$.*

iff acc *is, i.e.*

- *If $0 \leqslant x < y \leqslant 1$ then $\mathsf{acc}(x, \mathsf{t}) < \mathsf{acc}(y, \mathsf{t})$ and $\mathsf{acc}(x, \mathsf{f}) > \mathsf{acc}(y, \mathsf{f})$.*

And similarly for weak truth directedness.
Assumes: Not both $\pm\infty$ (see Remark 1.1).

Proof. Suppose $0 \leqslant x < y \leqslant 1$. Construct some global c_x and c_y which only differ on φ and take values x and y there. They then satisfy the antecedent of the global truth condition clause so $\mathsf{Acc}(c_x, \mathsf{t}) > \mathsf{Acc}(c_y, \mathsf{t})$; but by additivity this can only come from the $\mathsf{acc}(x, \mathsf{t})$ component.

Suppose acc is strictly truth directed. If we have c, c' satisfying the antecedent of the global strict truth directendess, then we have $\geqslant$ for each component, and $>$ for some (by local truth directedness); so adding these together we get $>$.

As in Remark 1.3, we need to consider:

Weak truth directedness of Acc implies weak truth directedness of acc: Suppose $\mathsf{acc}(x, \mathsf{t}) < \mathsf{acc}(y, \mathsf{t})$ (failure of weak-truth-directedness of acc). So $\mathsf{acc}(y, \mathsf{t})$ is finite. Then construct c_x and c_y taking value y on every other proposition. Note that we didn't need to choose it probabilistic so this is fine. Then we aren't in a problematic case.

Strict truth directedness of acc implies strict truth directedness of Acc:

not needed for our argument so I leave it for now.
I haven't checked the infinity assumptions there

□

Note that I had tried to weaken the global clause to: if c assignes higher credence to *all* truths, it has higher accuracy. But I don't think this suffices. Now I forget why.

Proposition 2.5. *Suppose* acc *is weakly proper and strictly truth directed. Then it is strictly proper.*

Assumes: only $-\infty$ *allowed (it'll then be at the endpoint).*

Actually, this argument still works if both $\pm\infty$ *allowed and we can define* $\mathrm{Exp}_x\mathsf{acc}(1) = \lim_{t\longrightarrow 1}\mathrm{Exp}_x\mathsf{acc}(t)$.

Proof. Consider $0 \leqslant x < y < z \leqslant 1$ with acc finite:

$$\mathrm{Exp}_x\mathsf{acc}(y) - \mathrm{Exp}_x\mathsf{acc}(z) \tag{15}$$

$$= x\underbrace{(\mathsf{acc}(y,\mathsf{t}) - \mathsf{acc}(z,\mathsf{t})))}_{<0} + (1-x)\underbrace{(\mathsf{acc}(y,\mathsf{f}) - \mathsf{acc}(z,\mathsf{f}))}_{>0} \quad \text{(by Truth-Directedness)} \tag{16}$$

$$> y(\mathsf{acc}(y,\mathsf{t}) - \mathsf{acc}(z,\mathsf{t}))) + (1-y)(\mathsf{acc}(y,\mathsf{f}) - \mathsf{acc}(z,\mathsf{f})) \quad \text{(since } x<y\text{)} \tag{17}$$

$$= \mathrm{Exp}_y\mathsf{acc}(y) - \mathrm{Exp}_y\mathsf{acc}(z) \tag{18}$$

$$\geqslant 0 \quad \text{(weak propriety at } y\text{)} \tag{19}$$

So, for all $x < y < z$, $\mathrm{Exp}_x\mathsf{acc}(y) < \mathrm{Exp}_x\mathsf{acc}(z)$. I.e., $\mathrm{Exp}_x\mathsf{acc}(y)$ is strictly decreasing on $y > x$. We can similarly show that $\mathrm{Exp}_x\mathsf{acc}(y)$ is strictly increasing on $y < x$. Thus, $x = y$ is a unique maximum.

Now, suppose we are dealing with infinities. Suppose we just have the one-sided infinity, $-\infty$. We then can only possibly have infinity values at $\mathsf{acc}(1,\mathsf{f})$ or $\mathsf{acc}(0,\mathsf{t})$. When considering accuracy values of y and z with $x < y < z \leqslant 1$, we are only possibly dealing with accuracy value of $-\infty$ at $z = 1$. But, then $\mathrm{Exp}_x\mathsf{acc}(1) = -\infty$, whereas accuracy values at y are finite so $\mathrm{Exp}_x\mathsf{acc}(y)$ is finite and thus $> \mathrm{Exp}_x\mathsf{acc}(z)$. So, even when we are dealing with $-\infty$ we can still show $\mathrm{Exp}_x\mathsf{acc}(y) > \mathrm{Exp}_x\mathsf{acc}(z)$ for $x < y < z \leqslant 1$.

Similarly, we can still show $\mathrm{Exp}_x\mathsf{acc}(y) < \mathrm{Exp}_x\mathsf{acc}(z)$ for $0 \leqslant z < y < x \leqslant 1$ even when $y = 0$ where we might have $\mathsf{acc}(y,\mathsf{t}) = -\infty$.

What about when ∞ *and* $-\infty$ are allowed?

Then $\mathrm{Exp}_x\mathsf{acc}(z) = x \times \infty + (1-x) \times -\infty = ?$. As SSK/Schervish do, we define this by $\lim_{t\longrightarrow z}\mathrm{Exp}_x\mathsf{acc}(t)$, which we assume to exist. What makes it exist? Here's a way to ensure it exists: only allow infty at endpoints (follows from strict truth directedness anyway) and have that acc is cts at endpoints. Will that then allow the argument to go through?

In that case, we already have that $\mathrm{Exp}_x\mathsf{acc}(y)$ is strictly decreasing as y converges up to 1; thus, at its limit of $z = 1$, this is $< \mathrm{Exp}_x\mathsf{acc}(x)$. □

Corollary 2.6. *Suppose* Acc *is weakly proper, additive and strictly truth directed. Then it is strictly proper.*

Assumes: Only $-\infty$ *allowed.*

Proof. Weak propriety of Acc entails weak propriety of the associated acc Proposition 2.2. Similarly for strict truth directedness (Proposition 2.4). So, by Proposition 2.5, acc is strictly proper. And thus Acc is also strictly proper (Proposition 2.2)

We have only allowed $-\infty$. Since acc is strictly truth directed, we use **??** to see that we only have $-\infty$ at the endpoints; so the condition for Proposition 2.2 is satisfied. (Note that this does assume our Definition 1.2) □

2.2 Result needed specifically for the stronger claim

Proposition 2.7. *Schervish (1989, Lemma A1) If* acc *is weakly proper, it is weakly truth directed on* $[0, 1]$. *Similarly for strict.*

Assumes: only have $\pm\infty$ *at endpoints.*

Proof. Consider $0 \leqslant y < x \leqslant 1$ and where acc is finite.

$$\mathrm{Exp}_x(\mathsf{acc}(x) - \mathsf{acc}(y)) \geqslant 0 \geqslant \mathrm{Exp}_y(\mathsf{acc}(x) - \mathsf{acc}(y)) \quad \text{weak prop} \tag{20}$$

$$\text{So } \begin{array}{l} x(\mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t})) + (1-x)(\mathsf{acc}(x,\mathsf{f}) - \mathsf{acc}(y,\mathsf{f})) \\ \geqslant 0 \geqslant y(\mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t})) + (1-y)(\mathsf{acc}(x,\mathsf{f}) - \mathsf{acc}(y,\mathsf{f})) \end{array} \quad \text{expanding} \tag{21}$$

$$\text{So } (x-y)(\mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t})) \geqslant (x-y)(\mathsf{acc}(x,\mathsf{f}) - \mathsf{acc}(y,\mathsf{f})) \quad \text{rearranging (ignoring 0)} \tag{22}$$

$$\text{So } \mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t}) \geqslant \mathsf{acc}(x,\mathsf{f}) - \mathsf{acc}(y,\mathsf{f}) \tag{23}$$

$$\text{Now } x(\mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t})) + (1-x)(\mathsf{acc}(x,\mathsf{f}) - \mathsf{acc}(y,\mathsf{f})) \geqslant 0 \quad \text{From 21} \tag{24}$$

$$\text{So } x(\mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t})) + (1-x)(\mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t})) \geqslant 0 \quad \text{Using 23} \tag{25}$$

$$\text{I.e. } \mathsf{acc}(x,\mathsf{t}) - \mathsf{acc}(y,\mathsf{t}) \geqslant 0 \tag{26}$$

$$\text{I.e. } \mathsf{acc}(x,\mathsf{t}) \geqslant \mathsf{acc}(y,\mathsf{t}) \tag{27}$$

Similarly, using the $0 \geqslant$ component of 21 we get $\mathsf{acc}(y,\mathsf{f}) \geqslant \mathsf{acc}(x,\mathsf{f})$.

Now, consider, the possibility of $\pm\infty$. By assumption, this can only be at 0 or 1. Consider $x = 1$. Note, $\mathsf{acc}(1,\mathsf{t}) = \mathrm{Exp}_1\mathsf{acc}(1) \leqslant \mathrm{Exp}_1\mathsf{acc}(y) = \mathsf{acc}(y,\mathsf{t})$; and similarly for $\mathsf{acc}(x,\mathsf{f}) \leqslant \mathsf{acc}(0,\mathsf{f})$. This is because we stipulate $0 \times \pm\infty = 0$ in the expectation equation.. □

Corollary 2.8. *If* Acc *is merely weakly proper and additive, it is constant on some interval.*

Assumes: Only $-\infty$ *allowed. And that this appears only at endpoints.*

Proof. Merely weakly proper additive Acc have merely weakly proper associated acc (Proposition 2.2) which are thus weakly truth directed on $[0,1]$ (Proposition 2.7). Since acc being strongly truth directed entails strict propriety (Proposition 2.5); acc is thus *merely* weakly truth directed, i.e. constant on some interval, $[a,b] \subseteq [0,1]$. Thus Acc is contant on some region (given by $[a,b]^n$).

We need this further assumption about endpoints for Proposition 2.2. □

But in light of Lemma 1.6, we need simply assume the weaker: For at least some x, both $\mathsf{acc}(x,\mathsf{t})$ and $\mathsf{acc}(x,\mathsf{f})$ are finite.

3 Interesting notes

SSK show: if sc rules are discts (even if strictly proper) an incoherent need not be dominated!

References

Mark J Schervish. A general method for comparing probability assessors. *The Annals of Statistics*, 17.4:1856–1879, 1989.

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