

Results about Sets of Desirable Gamble Sets

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1 Setup - Definition of Coherence

1.1 Gambles

Setup 1.1. \mathcal{G} is the set of all gambles, which are the bounded functions from Ω to \mathbb{R} .

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When $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, we will say $f \geq g$.

When $f(\omega) > g(\omega)$ for all $\omega \in \Omega$, we will say $f > g$, or that f *strictly dominates* g .

When $f \geq g$ and $g \not\geq f$ we say $f \succ g$, or that f *weakly dominates* g . That is, when $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$ and $f(\omega) > g(\omega)$ for some $\omega \in \Omega$.

$\mathcal{G}_{\geq 0}$ is the set of gambles where $f \geq 0$.

$\mathcal{G}_{> 0}$ is the set of gambles which strictly dominate 0. I.e., where $f > 0$.

$\mathcal{G}_{\succeq 0}$ is the set of gambles which weakly dominate 0. I.e., where $f \succeq 0$.

We will also make use of the positive linear hull of a set: $\text{posi}(B) := \{\sum_{i=1}^n \lambda_i g_i \mid n \in \mathbb{N}, \lambda_i > 0, g_i \in B\}$.

1.2 Desirable gambles - usual results

Definition 1.2. $D \subseteq \mathcal{G}$ is coherent if:

(D₀) $0 \notin D$

(D _{≥ 0}) If $g \in \mathcal{G}_{\geq 0}$, then $g \in D$

(D _{λ}) If $g \in D$ and $\lambda > 0$, then $\lambda g \in D$

(D₊) If $f, g \in D$, then $f + g \in D$

Proposition 1.3. For $E \subseteq \mathcal{G}$,

(i) If $0 \notin \text{posi}(E \cup \mathcal{G}_{\succeq 0})$ then there is some coherent D extending E ; and the minimal such coherent extension is $\text{posi}(E \cup \mathcal{G}_{\succeq 0})$.

(ii) If $0 \in \text{posi}(E \cup \mathcal{G}_{\succeq 0})$ then there is no coherent D extending E .

Proposition 1.4. D is coherent iff $D = \text{posi}(E \cup \mathcal{G}_{\succeq 0})$ for some $E \subseteq \mathcal{G}$, and $0 \notin D$.

1.3 Sets of desirable gamble sets

I will give definitions of coherence as in Campbell-Moore (2021), which are based on those of De Bock and de Cooman (2018) but diverge from them in order to accommodate the fact that I am not restricting to finite sets of gambles. In section 3 we'll show they're equivalent in the case where gamble sets must be finite.

Definition 1.5. $\mathcal{K} \subseteq \wp(\mathcal{G})$ is *coherent* if it satisfies

(K _{\emptyset}) $\emptyset \notin \mathcal{K}$

(K₀) If $A \in \mathcal{K}$ then $A \setminus \{0\} \in \mathcal{K}$.

(K _{≥ 0}) If $g \in \mathcal{G}_{\geq 0}$, then $\{g\} \in \mathcal{K}$

(K _{\supseteq}) If $A \in \mathcal{K}$ and $B \supseteq A$, then $B \in \mathcal{K}$

(K_{Dom}) If $A \in \mathcal{K}$ and for each $g \in A$, f_g is some gamble where $f_g \geq g$, then $\{f_g \mid g \in A\} \in \mathcal{K}$.

(K_{Add}) If $A_1, \dots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, $f_{\langle g_i \rangle}$ is some member of $\text{posi}(\{g_1, \dots, g_n\})$, then

$$\{f_{\langle g_i \rangle} \mid \langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n\} \in \mathcal{K}$$

2 Natural Extension

2.1 The key result about coherence

Our central result tells us the key important property of coherent sets of desirable gamble sets. We will afterwards show that this is a characteristic result of coherence, i.e., it gives us the natural extension.

Theorem 2.1. *Suppose \mathcal{K} is coherent and $A_1, \dots, A_n \in \mathcal{K}$.*

If B is such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, either $0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, or there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$ with $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$; Then $B \in \mathcal{K}$.

Proof. We start with a (not terribly interesting) lemma:

Sublemma 2.1.1. *For $f \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, whenever $f \notin \mathcal{G}_{\geq 0}$, there is some $h \in \text{posi}(\{g_1, \dots, g_n\})$ with $f \geq h$.*

Proof. $f \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ so $f = \sum_i \lambda_i g_i + \sum_j \mu_j p_j$ for some $p_j \in \mathcal{G}_{\geq 0}$ and $\lambda_i, \mu_j \geq 0$, with at least one > 0 .

Suppose $f \notin \mathcal{G}_{\geq 0}$. Then some $\lambda_i > 0$ because otherwise $f = \sum_j \mu_j p_j$ with each $p_j \in \mathcal{G}_{\geq 0}$, and so we'd also have that $f \in \mathcal{G}_{\geq 0}$.

So, let $h = \sum_i \lambda_i g_i$, and we then know that $h \in \text{posi}(\{g_1, \dots, g_n\})$. Since $f = h + \sum_j \mu_j p_j$ with each $p_j \in \mathcal{G}_{\geq 0}$, we know that $f \geq h$. \square

Assume we have coherent \mathcal{K} and B satisfying the assumptions of the theorem. So we have some A_1, \dots, A_n where for any sequence $\langle g_i \rangle \in \times_i A_i$, whenever $0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ with $f_{\langle g_i \rangle} \in B$. We need to show that $B \in \mathcal{K}$.

When $0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, we can let $f_{\langle g_i \rangle}$ denote 0.

If there is some $f \in \mathcal{G}_{\geq 0}$ with $f \in B$ then by axioms items $K_{\geq 0}$ and K_{\supseteq} , $B \in \mathcal{K}$.

So we can assume that each $f_{\langle g_i \rangle} \notin \mathcal{G}_{\geq 0}$ (noting that $0 \notin \mathcal{G}_{\geq 0}$), so by sublemma 2.1.1, we can find $h_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$ with $f_{\langle g_i \rangle} \geq h_{\langle g_i \rangle}$. By axiom K_{Add} ,

$$\{h_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times_i A_i\} \in \mathcal{K}. \quad (1)$$

and then by axiom K_{Dom} ,

$$\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times_i A_i\} \in \mathcal{K}. \quad (2)$$

Since each $f_{\langle g_i \rangle} \in B \cup \{0\}$, by axiom K_{\supseteq} , $B \cup \{0\} \in \mathcal{K}$. And thus $B \in \mathcal{K}$ by axiom K_0 . \square

Note that we have restricted this to finitely many A_1, \dots, A_n in \mathcal{K} . This is because axiom K_{Add} is restricted to finitely many members.

2.2 Understanding this

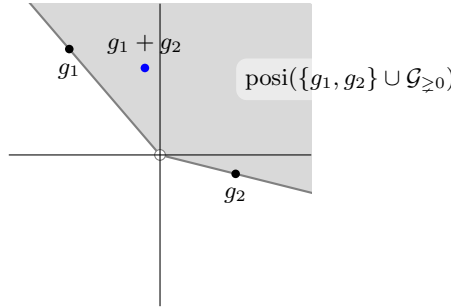
This will in fact give us maximal information about coherence.

But first we give an idea of why this might be the right thing to do and to help understand the criterion.

Suppose $A_1, A_2 \in \mathcal{K}$. We can then consider what else should be in \mathcal{K} .

You think that A_1 contains some desirable gamble, and so does A_2 . But you leave it open which member of A_1 is desirable and which member of A_2 .

We can go through each member of A_1 and consider what are the consequences of that gamble being the one in virtue of which you think A_1 contains a desirable gamble. Similarly for A_2 . Consider the possibility that A_1 is desirable in virtue of g_1 , and A_2 is desirable in virtue of g_2 . In that case, also $g_1 + g_2$ is desirable, and more generally any $f \in \text{posi}(\{g_1, g_2\} \cup \mathcal{G}_{\geq 0})$ is desirable. So any C with some $f \in \text{posi}(\{g_1, g_2\} \cup \mathcal{G}_{\geq 0}) \cap C$ contains some desirable gamble.



But this is all conditional on g_1 and g_2 being the desirable members. If B is a set such that *whichever* members of A_1 and A_2 are considered to be the relevant ones in virtue of which A_i desirable, we can have that some member of B is desirable, then we can outright conclude that B contains a desirable gamble. We needn't be able to point to any particular member of B which we evaluate as desirable. But whenever a member of A_1 and a member of A_2 are desirable, then so is a member of B .

Suppose

$$A_1 = \{a_1, a_2\} \quad (3)$$

$$A_2 = \{c_1, c_2\} \quad (4)$$

$$B = \{f^1, f^2\} \quad (5)$$

Suppose these values are as described by fig. 1 and $A_1, A_2 \in \mathcal{K}$, also $B \in \mathcal{K}$.

So we have described a general idea motivating:

If for each pair $\langle g_1, g_2 \rangle \in A_1 \times A_2$, there is some $f_{\langle g_1, g_2 \rangle} \in \text{posi}(\{g_1, g_2\} \cup \mathcal{G}_{\geq 0})$ with $f_{\langle g_1, g_2 \rangle} \in B$. Then $B \in \mathcal{K}$.

In our particular case, put $f_{\langle a_1, c_1 \rangle} = f_{\langle a_1, c_2 \rangle} = f^1$ with $f_{\langle a_2, c_1 \rangle} = f_{\langle a_2, c_2 \rangle} = f^2$.

A proof of the legitimacy of this, given the axioms on coherence, is in theorem 2.1 since it is a weaker condition than that used in that result.

This story can easily be extended to multiple sets, if A_1, \dots, A_n are all taken to contain at least one desirable gamble, and whenever we pick some particular members of them and consider those to be desirable, a member of B is also desirable, then we conclude that B contains a desirable gamble. That is:

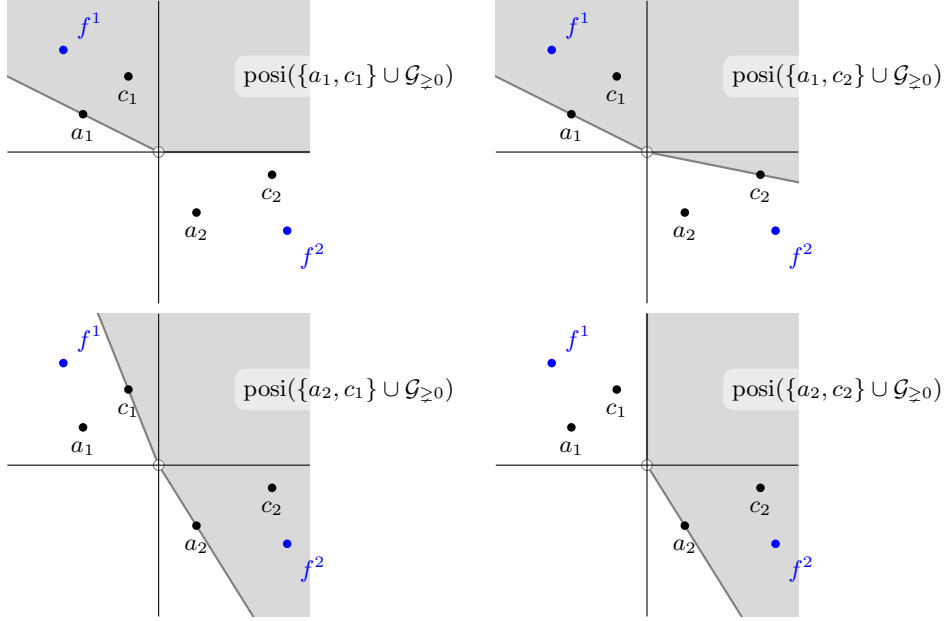


Figure 1:

If for each sequence $\langle g_i \rangle \in \times A_i$, there is some $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ with $f_{\langle g_i \rangle} \in B$. Then $B \in \mathcal{K}$.

Can it be extended to infinitely many sets? Well, this will go beyond the axioms that I imposed on coherence. A thought behind this: I am unable to perform infinite reasoning with desirability. Or alternatively, maybe I am unable to simultaneously consider the infinitely many sets and pick the relevant members. Or perhaps one can only make finitely many conditional assumptions.

This gives us something quite close to that of theorem 2.1.

However, it misses the clause that says that when $0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ we don't need to find a $f_{\langle g_i \rangle} \in B$. This is because when $0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, g_1, \dots, g_n are mutually incompatible. Suppose we just consider $n = 2$ and we have $g_1 \in A_1$ and $g_2 \in A_2$ with $0 \in \text{posi}(\{g_1, g_2\} \cup \mathcal{G}_{\geq 0})$. It cannot be that, simultaneously, A_1 is desirable in virtue of g_1 and A_2 is desirable in virtue of g_2 . The desirability of g_1 rules out the desirability of g_2 . So we can ignore this choice of $\langle g_i \rangle$. We only need to find the relevant $f_{\langle g_i \rangle}$ when the choice of these g_i are mutually compatible.

Consider:

$$A_1 = \{g_1, 0\} \tag{6}$$

$$A_2 = \{g_2, 0\} \tag{7}$$

The only choice of a member of A_1 and a member of A_2 which do not result in a desirable gamble set containing 0 is the choice of g_1 from A_1 and g_2 from A_2 . So we will be able to conclude that $\{g_1 + g_2\} \in \mathcal{K}$ since for the only coherent choice of a member of each of the sets has $g_1 + g_2$ in the relevant posi, which is $\text{posi}(\{g_1, g_2\} \cup \mathcal{G}_{\geq 0})$. Without this additional clause we would only be able

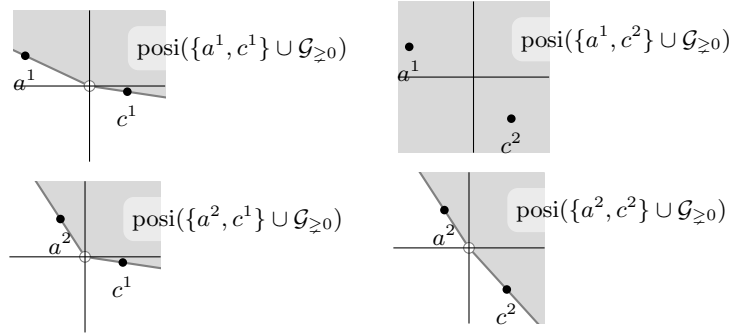
to conclude that, for example, $\{0, g_1 + g_2\} \in \mathcal{K}$. To be able to remove the 0 from this set we need our additional clause that allows us to ignore mutually incompatible choices.

Let's see a further example where the gambles are really mutually incompatible rather than simply being 0 themselves. Consider:

$$A = \{a^1, a^2\} \in \mathcal{K} \quad (8)$$

$$C = \{c^1, c^2\} \in \mathcal{K} \quad (9)$$

given as in:



Since here $0 \in \text{pos}(\{a^1, c^2\} \cup \mathcal{G}_{\geq 0})$, we do not need to consider this case. Instead we just need to ensure that there is some member of B in each of $\text{pos}(\{a^1, c^1\} \cup \mathcal{G}_{\geq 0})$, $\text{pos}(\{a^2, c^1\} \cup \mathcal{G}_{\geq 0})$ and $\text{pos}(\{a^2, c^2\} \cup \mathcal{G}_{\geq 0})$. In fact, in this case it wouldn't be an issue to ensure that we have some member of B in $\text{pos}(\{a^1, c^2\} \cup \mathcal{G}_{\geq 0})$ because this contains all gambles whatsoever. But not all cases are like this.

This then fully characterises coherence.

find a case where it actually makes a difference when it's not 0 in original set.

2.3 Defining the Natural Extension

If we start with a given set of desirable gamble sets, \mathcal{E} , we define $\text{Kext}(\mathcal{E})$ by closing it under the construction of theorem 2.1 (except when we start with \emptyset in which case we give a separate definition because that construction then doesn't add anything).

Definition 2.2. For $\mathcal{E} \subseteq \wp(\mathcal{G})$, let $\text{Kext}(\mathcal{E})$ be given by:

When $\mathcal{E} \neq \emptyset$:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_1, \dots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, whenever $0 \notin \text{pos}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$ where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{pos}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$.

When $\mathcal{E} = \emptyset$,

- $B \in \text{Kext}(\emptyset)$ iff there is some $f \in B$ where also $f \in \mathcal{G}_{\geq 0}$.

This will give us the natural extension notion: the minimal coherent set of desirable gamble sets extending the given set, \mathcal{E} . The formulation is a bit

different to that of De Bock and de Cooman (2018), but they're ultimately equivalent. We'll discuss this in section 2.5.

We now see that this gives us the natural extension:

Theorem 2.3.

- (i) *If $\emptyset \notin \text{Kext}(\mathcal{E})$ then $\text{Kext}(\mathcal{E})$ is coherent and it is the minimal coherent \mathcal{K} extending \mathcal{E} .*
- (ii) *If $\emptyset \in \text{Kext}(\mathcal{E})$ then $\text{Kext}(\mathcal{E})$ is incoherent and there is no coherent \mathcal{K} extending \mathcal{E} .*

Proof. We first deal with the easy case, when $\mathcal{E} = \emptyset$: If \mathcal{K} is coherent then by $\text{K}_{\geq 0}$ and K_{\supseteq} , any $B \in \text{Kext}(\emptyset)$ has $B \in \mathcal{K}$. Also observe that $\text{Kext}(\emptyset)$ is coherent. So it is the minimal coherent extension of \emptyset .

So we now just consider when $\mathcal{E} \neq \emptyset$. As a corollary of theorem 2.1, when \mathcal{K} is coherent extending \mathcal{E} , then $\mathcal{K} \supseteq \text{Kext}(\mathcal{E})$. This is because $\text{Kext}(\mathcal{E})$ was just defined to take the closure in accordance with theorem 2.1. We then need to check that $\text{Kext}(\mathcal{E})$ is coherent (so long as it doesn't contain \emptyset).

Sublemma 2.3.1. *Suppose $\mathcal{E} \neq \emptyset$. If $\emptyset \notin \text{Kext}(\mathcal{E})$, then $\text{Kext}(\mathcal{E})$ is coherent.*

Proof. We check each of the axioms.

- K_{\emptyset} : $\emptyset \notin \text{Kext}(\mathcal{E})$ by assumption.
- K_0 : Suppose $A \in \text{Kext}(\mathcal{E})$. We need to show that $A \setminus \{0\} \in \text{Kext}(\mathcal{E})$. Since $A \in \text{Kext}(\mathcal{E})$, there are $B_1, \dots, B_n \in \mathcal{E}$ where for each $\langle e_i \rangle \in \times B_i$, whenever $0 \notin \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$ there is some $h_{\langle e_i \rangle} \in A$ with $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$. For such $\langle e_i \rangle$, $h_{\langle e_i \rangle} \neq 0$ so also $h_{\langle e_i \rangle} \in (A \setminus \{0\})$. Thus, $A \setminus \{0\} \in \text{Kext}(\mathcal{E})$.
- $\text{K}_{\geq 0}$: Let $g \in \mathcal{G}_{\geq 0}$. Since $\mathcal{E} \neq \emptyset$ there is some $B \in \mathcal{E}$. For every $e \in B$, $e \in \text{posi}(\{e\} \cup \mathcal{G}_{\geq 0})$. So by definition of Kext , $\{g\} \in \text{Kext}(\mathcal{E})$.
- K_{\supseteq} : Suppose $A \in \text{Kext}(\mathcal{E})$ and $C \supseteq A$. We need to show that $C \in \text{Kext}(\mathcal{E})$. There are some $B_i \in \mathcal{E}$ where for each $\langle e_i \rangle \in \times B_i$, whenever $0 \notin \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$, we have some $h_{\langle e_i \rangle} \in A$ with $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$. Since $C \supseteq A$, also $h_{\langle e_i \rangle} \in C$. So $C \in \text{Kext}(\mathcal{E})$.
- K_{Dom} : Suppose $A \in \text{Kext}(\mathcal{E})$ and for each $g \in A$, f_g is a gamble with $f_g \geq g$. We need to show that $C := \{f_g \mid g \in A\} \in \text{Kext}(\mathcal{E})$. There are some $B_i \in \mathcal{E}$ where for each $\langle e_i \rangle \in \times B_i$, whenever $0 \notin \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$, there is some $h_{\langle e_i \rangle} \in A$ with $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$. For each such $\langle e_i \rangle$, let $f_{\langle e_i \rangle}$ be the relevant $f_{h_{\langle e_i \rangle}} \geq h_{\langle e_i \rangle}$. So since $h_{\langle e_i \rangle} \in \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$ with $f_{\langle e_i \rangle} \geq h_{\langle e_i \rangle}$, also $f_{\langle e_i \rangle} \in \text{posi}(\{e_1, \dots, e_n\} \cup \mathcal{G}_{\geq 0})$. And $f_{\langle e_i \rangle} \in C$; as required.
- K_{Add} : Suppose $A_1, \dots, A_n \in \text{Kext}(\mathcal{E})$, and $C := \{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times A_i\}$, with $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$ (as in K_{Add}). We need to show that $C \in \text{Kext}(\mathcal{E})$. For each i , there are $B_1^i, \dots, B_{m_i}^i$ where for any $\langle e_j \rangle \in \times_j B_j^i$, whenever $0 \notin \text{posi}(\{e_1, \dots, e_{m_i}\} \cup \mathcal{G}_{\geq 0})$, there is some $g_{\langle e_j \rangle} \in A_i$ and $g_{\langle e_j \rangle} \in \text{posi}(\{e_1, \dots, e_{m_i}\} \cup \mathcal{G}_{\geq 0})$.

Now, consider all of B_j^i for i, j . This is a finite collection. Any sequence of members of it has the form $\langle e_j^i \rangle_{i,j} \in \times_{i,j} B_j^i$. Fix such a sequence with $0 \notin \text{posi}(\{e_j^i \mid i, j\} \cup \mathcal{G}_{\geq 0})$. We need to show there is some $f \in C$ with $f \in \text{posi}(\{e_j^i \mid i, j\} \cup \mathcal{G}_{\geq 0})$.

We also have $0 \notin \text{posi}(\{e_j^i \mid j\} \cup \mathcal{G}_{\geq 0})$ for each i ; so we have some $g_i := g_{\langle e_j^i \rangle_j} \in A_i$ with $g_i \in \text{posi}(\{e_j^i \mid j\} \cup \mathcal{G}_{\geq 0})$. Since $\langle g_i \rangle \in \times A_i$, by our choice of C there is some $f \in C$ with $f \in \text{posi}(\{g_1, \dots, g_n\})$.

We need to show that $f \in \text{posi}(\{e_j^i \mid i, j\} \cup \mathcal{G}_{\geq 0})$. This follows from $f \in \text{posi}(\{g_1, \dots, g_n\})$ and each $g_i \in \text{posi}(\{e_j^i \mid i, j\} \cup \mathcal{G}_{\geq 0})$ (since it is in $\text{posi}(\{e_j^i \mid j\} \cup \mathcal{G}_{\geq 0})$).

So $C \in \text{Kext}(\mathcal{E})$. \square

So we have shown that when $\emptyset \notin \text{Kext}(\mathcal{E})$, $\text{Kext}(\mathcal{E})$ is the minimal coherent extension of \mathcal{E} .

We now just need to check that when $\emptyset \in \text{Kext}(\mathcal{E})$, there is no coherent \mathcal{K} extending \mathcal{E} . This follows from theorem 2.1 because then \emptyset is in any coherent \mathcal{K} extending \mathcal{E} , contradicting the supposed coherence of \mathcal{K} . \square

2.4 Comments on the Formulation of it

2.4.1 Quick notes

One component of our natural extension formulation is:

whenever $0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$
where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$.

This can equivalently be phrased:

there is some $f_{\langle g_1, \dots, g_n \rangle} \in B \cup \{0\}$ where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$.

We can also phrase things with intersections. I.e.,

there is some $f_{\langle g_1, \dots, g_n \rangle} \in (B \cup \{0\}) \cap \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$.

Or without the reference to $f_{\langle g_1, \dots, g_n \rangle}$ at all:

$(B \cup \{0\}) \cap \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0}) \neq \emptyset$.

Just to see what this would look like, we'd then have the definition of Kext for $\mathcal{E} \neq \emptyset$ as:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_1, \dots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, such that $(B \cup \{0\}) \cap \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0}) \neq \emptyset$.

And we might then want to write the case for $\mathcal{E} = \emptyset$ as:

- $B \in \text{Kext}(\emptyset)$ iff $B \cap \mathcal{G}_{\geq 0} \neq \emptyset$.

I quite like the reference to the particular member, rather than talking about the intersection being non-empty, because I think it helps thinking about it and is useful to have for proofs, which is why I've opted for that.

2.4.2 On the case when $\mathcal{E} = \emptyset$

I have chosen to define the case of $\mathcal{E} \neq \emptyset$ differently from when $\mathcal{E} = \emptyset$.

To give a single definition we could follow Arne Decadt [REFERENCE](#) and include an additional separate clause for the case where $B \cap \mathcal{G}_{\geq 0} \neq \emptyset$. So, we would offer:

For any \mathcal{E} (including \emptyset), $B \in \text{Kext}(\mathcal{E})$ iff either:

- There is some $f \in B$ where also $f \in \mathcal{G}_{\geq 0}$. Or
- There are some $A_1, \dots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$ where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$.

An alternative would be to first add something that we know to be in all coherent \mathcal{K} so that we essentially ensure we're working with non-empty \mathcal{E} , for example we could add $\mathcal{G}_{\geq 0}$ itself or each singleton $\{g\}$ for $g \in \mathcal{G}_{\geq 0}$. This would then allow us to use the theorem 2.1 characterisation. But I want to keep the non-empty case as simple as possible as it's really the interesting one, so I keep them separate.

Both of these have the disadvantage of introducing additional steps/clauses to be checked. I have thus opted to keep the definition simpler and simply deal with $\mathcal{E} = \emptyset$ manually.

2.4.3 Removing the use of $\mathcal{G}_{\geq 0}$ in the posi:

Decadt doesn't work with $\text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ but instead simply with posi and \geq . His definition is (with different formulation):

For any \mathcal{E} (including \emptyset), $B \in \text{Kext}(\mathcal{E})$ iff either:

- There is some there are some $f \in B$ where also $f \in \mathcal{G}_{\geq 0}$. Or
- There are some $A_1, \dots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$ and $h_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$ where $f_{\langle g_1, \dots, g_n \rangle} \geq h_{\langle g_1, \dots, g_n \rangle}$.

Proposition 2.4. *This is equivalent to definition 2.2*

Proof.

Sublemma 2.4.1. *$f \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ iff $f \in \mathcal{G}_{\geq 0}$ or there is some $h \in \text{posi}(\{g_1, \dots, g_n\})$ with $f \geq h$.*

Proof. The left-to-right direction is sublemma 2.1.1. The right-to-left can be easily checked [todo](#). \square

Thus, definition 2.2 is equivalent to:

When $\mathcal{E} \neq \emptyset$:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_1, \dots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$ where

- either $f_{\langle g_1, \dots, g_n \rangle} \in \mathcal{G}_{\geq 0}$
- or there is some $h_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$ with $f_{\langle g_1, \dots, g_n \rangle} \geq h_{\langle g_1, \dots, g_n \rangle}$.

When $\mathcal{E} = \emptyset$,

- $B \in \text{Kext}(\emptyset)$ iff there is some $f \in B$ where also $f \in \mathcal{G}_{\geq 0}$.

The first condition of the $\mathcal{E} \neq \emptyset$ case and the $\mathcal{E} = \emptyset$ case can be combined into the single condition that $B \cap \mathcal{G}_{\geq 0} \neq \emptyset$. We are then left with the alternative condition that

There are some $A_1, \dots, A_n \in \mathcal{E}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, whenever $0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B$ and $h_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$ where $f_{\langle g_1, \dots, g_n \rangle} \geq h_{\langle g_1, \dots, g_n \rangle}$.

So we see that the two formulations are equivalent. \square

Note that it is important in doing this that the additional clause about $f \in \mathcal{G}_{\geq 0}$ is added. That is, we cannot define $\text{Kext}(\mathcal{E})$ by simply taking our definition (definition 2.2) and replacing the existence of $f \in B \cap \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ with the existence of $f \in B$ and $h \in \text{posi}(\{g_1, \dots, g_n\})$ with $f \geq h$. [However, interestingly if we move to strict dominance in our criteria that would be possible.]

2.5 Relationship to De Bock and de Cooman (2018)

Their formulation looks a bit different. Since they both characterise natural extensions, they'll be equivalent, but in this section we consider the difference in more detail.

They:

- Add the singletons from $\mathcal{G}_{\geq 0}$ to \mathcal{E} . This gives us $\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\geq 0}\}$.
- We add sets using posi from these sets. More carefully: For any C_1, \dots, C_n in $\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\geq 0}\}$ (from (i)), if for each sequence $\langle g_i \rangle \in \times C_i$, $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$, then we add $\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times C_i\}$. Our resultant set is called $\text{Posi}(\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\geq 0}\})$.
- Add any sets obtained by removing some gambles which are ≤ 0 .
- Add any supersets thereof.

This gives us the final set, called: $\text{Rs}(\text{Posi}(\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\geq 0}\}))$. They show that this is the natural extension in the finite setting. So in this setting, we know that it will be equivalent to ours, however it's open whether it gives the natural extension in the infinite setting.

It's got the same sort of moving parts as our definition: there's addition of some $\mathcal{G}_{\geq 0}$ (in (i)), there's taking posi's (in (ii)), there's removing some obviously-bad stuff (in (iii)) and taking supersets (in (iv)). These are the same sorts of moving parts that I have.

Let's try to rework their formulation to present it in a way that is more similar to our presentation so we can evaluate how/if they are different. Firstly, we can summarise the construction to:

- $B \in \text{Kext}_{\text{DBdC}}(\mathcal{E})$ iff there are some $C_1, \dots, C_n \in \mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\geq 0}\}$ with $\langle g_1, \dots, g_n \rangle \in C_1 \times \dots \times C_n$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B \cup \mathcal{G}_{\leq 0}$ where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$.

We can then split the choice of C_1, \dots, C_n from $\mathcal{E} \cup \{\{g\} \mid g \in \mathcal{G}_{\geq 0}\}$ into a choice separately of $A_1, \dots, A_n \in \mathcal{E}$, and $p_1, \dots, p_m \in \mathcal{G}_{\geq 0}$. Sometimes no members of \mathcal{E} are chosen, which should be presented as a separate clause. So we see that:

- $B \in \text{Kext}_{\text{DBdC}}(\mathcal{E})$ iff either:
 - There are some $A_1, \dots, A_n \in \mathcal{E}$ and $p_1, \dots, p_m \in \mathcal{G}_{\geq 0}$ such that for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B \cup \mathcal{G}_{\leq 0}$ where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n, p_1, \dots, p_m\})$.
 - There are some $p_1, \dots, p_m \in \mathcal{G}_{\geq 0}$ such that there is some $f \in B \cup \mathcal{G}_{\leq 0}$ where also $f \in \text{posi}(\{p_1, \dots, p_m\})$.

Note that this is different to our:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_1, \dots, A_n \in \mathcal{E}$ and for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B \cup \{0\}$ where also $f_{\langle g_1, \dots, g_n \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$.

Some of the differences between these are simply choices of presentation. However, substantial differences are:

- DBdC allow the choice of the sequence C_1, \dots, C_n to contain no members of \mathcal{E} whatsoever. This allows their definition to apply properly to the case of $\mathcal{E} = \emptyset$, whereas we need a separate definition for that case.
- Their Posi is defined working with finitely many C_1, \dots, C_n , which means in any use of it we can only use finitely many members of $\mathcal{G}_{\geq 0}$. Is that enough? It's closely related to my addition of item K_{Dom} .

For example, suppose $A \in \mathcal{E}$ and we have for each g in A a distinct $p_g \in \mathcal{G}_{\geq 0}$, we want to ensure that $\{g + p_g \mid g \in A\} \in \text{Kext}(\mathcal{E})$.

This doesn't immediately follow from the construction of Posi because it involves infinitely many sets: A plus the infinitely many singletons $\{p_g\}$.

In fact, like the dominance axiom, when Ω is finite it'll be fine because $g + p_g \in \text{posi}(\{g, I_{\omega_1}, \dots, I_{\omega_n}\})$ so we can just work with the finitely many sets $A, \{I_{\omega_1}\}, \dots, \{I_{\omega_n}\}$. But perhaps when Ω is infinite it'll be different.

2.6 Does DBdC get infinite addition axiom?

Should DBdC extend the construction to define Posi that it directly works with infinitely many sets?

2.7 Extending

2.7.1 Regularity

We can replace $\mathcal{G}_{\geq 0}$ by $\mathcal{G}_{> 0}$ and everything still works. Theorem 2.1 then becomes simpler because any $f_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{> 0})$, has some $h_{\langle g_i \rangle} \in \text{posi}(\{g_1, \dots, g_n\})$ with $f_{\langle g_i \rangle} \geq h_{\langle g_i \rangle}$ so we don't need a separate clause.

In fact this shows us that in the strict-dominance setting, axiom $K_{\geq 0}$ is not required as a separate axiom, though one should then include a non-trivial axiom that \mathcal{K} is non-empty.

2.7.2 Extending to infinite addition

If we extend to:

(K_{AddInf}) If $A_i \in \mathcal{K}$ for each $i \in I$ (possibly infinite) and for each sequence $\langle g_i \rangle_{i \in I}$ with each $g_i \in A_i$, $f_{\langle g_i \rangle}$ is some member of $\text{posi}(\{g_i \mid i \in I\})$, then

$$\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \prod_{i \in I} A_i\} \in \mathcal{K}$$

we obtain a strictly stronger system. Campbell-Moore (2021, §4.4.1). Then the results immediately carry through now working with:

- $B \in \text{Kext}(\mathcal{E})$ iff there are some $A_i \in \mathcal{K}$ (a possible infinite collection) where for each sequence $\langle g_i \rangle \in \prod_i A_i$, there is some $f_{\langle g_i \rangle} \in B \cup \{0\}$ with $f_{\langle g_i \rangle} \in \text{posi}(\{g_i \mid i \in I\} \cup \mathcal{G}_{\geq 0})$.

3 Axioms when gamble sets must be finite

This section shows that in the special case when gamble sets are finite, we can use the axioms from De Bock and de Cooman (2018)

Definition 3.1. $\mathcal{K} \subseteq \wp^{\text{finite}}(\mathcal{G})$ is *coherent* if it satisfies K_{\emptyset} K_0 $K_{\geq 0}$ K_{\supseteq} and:

(K_{AddPair}) If $A_1, A_2 \in \mathcal{K}$ and for each pair $g_1 \in A_1$ and $g_2 \in A_2$, $f_{\langle g_1, g_2 \rangle}$ is some member of $\text{posi}(\{g_1, g_2\})$, then

$$\{f_{\langle g_1, g_2 \rangle} \mid g_1 \in A_1, g_2 \in A_2\} \in \mathcal{K}$$

These are equivalent to our earlier axioms when we are restricted to finite sets. The proof of this is contained in the next two sections.

3.1 Finite addition axiom - axiom K_{Add}

Proposition 3.2. ¹Assume we are restricted to finite sets. Axiom K_{Add} follows from axiom K_{AddPair} and axiom K_{\supseteq} .

¹Thanks to Arthur van Camp for a key insight, and Jasper De Bock and Arne Decadt for discussion.

Proof. We in fact prove:

- If $A_1, \dots, A_n \in \mathcal{K}$ and for each sequence $\langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n$, there is some $f_{\langle g_1, \dots, g_n \rangle} \in B \cap \text{posi}(\{g_1, \dots, g_n\})$;
Then $B \in \mathcal{K}$.

which entails axiom K_{Add} . (It essentially just adds any supersets. This allows us to avoid some fiddliness in the proof.)

We work by induction on n .

Base case: For $n = 1$ use axiom K_{AddPair} and axiom K_{\supseteq} , letting $A_1 = A_2$.

Inductive step: Assume it holds for n and consider $n + 1$.

So, suppose $A_1, \dots, A_n, A_{n+1} \in \mathcal{K}$ and for each sequence $\langle g_i \rangle \in \times_i A_i$ and $a \in A_{n+1}$, there is some $f_{\langle g_1, \dots, g_n, a \rangle} \in B \cap \text{posi}(\{g_1, \dots, g_n, a\})$. We need to show that $B \in \mathcal{K}$.

We will abuse notation and write $f_{\langle g_i, a \rangle}$ instead of $f_{\langle g_1, \dots, g_n, a \rangle}$.

A_{n+1} is finite, so enumerate it, $A_{n+1} = \{a_1, \dots, a_K\}$. Define:

$$B_k := \left\{ f_{\langle g_i, a \rangle} \mid \langle g_i \rangle \in \times_{i \leq n} A_i, a \in \{a_1, \dots, a_k\} \right\} \cup \{a_{k+1}, \dots, a_K\} \quad (10)$$

Where we have already replaced k -many of A_{n+1} with the relevant members of B . We show by a induction on k that $B_k \in \mathcal{K}$. Since $B_K \subseteq B$, this will suffice.²

Base case: $B_0 = A_{n+1} \in \mathcal{K}$.

Inductive step: we can assume that $B_k \in \mathcal{K}$ and need to show $B_{k+1} \in \mathcal{K}$.

We will construct a set C which we know to be in \mathcal{K} , which we can then combine with B_k using axiom K_{AddPair} (and axiom K_{\supseteq}) to get $B_{k+1} \in \mathcal{K}$.

By assumption, $f_{\langle g_i, a_{k+1} \rangle} \in \text{posi}(\{g_1, \dots, g_n, a_{k+1}\})$. So there is some $h_{\langle g_i, a_{k+1} \rangle}$ where:

$$h_{\langle g_i, a_{k+1} \rangle} \in \text{posi}(\{g_1, \dots, g_n\}), \text{ and} \quad (11)$$

$$f_{\langle g_i, a_{k+1} \rangle} \in \text{posi}(\{a_{k+1}, h_{\langle g_i, a_{k+1} \rangle}\}). \quad (12)$$

To see this: $f_{\langle g_i, a_{k+1} \rangle}$ has the form $\sum_{i \leq n} \lambda_i g_i + \mu a_{k+1}$. If $\lambda_i > 0$ for some i , then we put $h_{\langle g_i, a_{k+1} \rangle} := \sum_{i \leq n} \lambda_i g_i$. If not, then we simply let it be, for example, g_1 . This is as required.

We let

$$C := \{h_{\langle g_i, a_{k+1} \rangle} \mid \langle g_i \rangle \in \times_{i \leq n} A_i\} \quad (13)$$

$C \in \mathcal{K}$ by the induction hypothesis (on n), using eq. (11).

We will show that for every $\langle c, b \rangle \in B_k \times C$, there is some $d \in \text{posi}(\{c, b\}) \cap B_{k+1}$; which will then allow us to use axiom K_{AddPair} (and K_{\supseteq}) to get that $B_{k+1} \in \mathcal{K}$.

If $c \in B_k \cap B_{k+1}$, we just put $d = c$. The only remaining member of B_k to consider is $c = a_{k+1}$. For any $b \in C$, $b = h_{\langle g_i, a_{k+1} \rangle}$ for some $\langle g_i \rangle \in \times_i A_i$. By definition of this (eq. (12)), $f_{\langle g_i, a_{k+1} \rangle} \in \text{posi}(\{a_{k+1}, b\})$. And by looking at the definition of B_{k+1} (eq. (10)), also $f_{\langle g_i, a_{k+1} \rangle} \in B_{k+1}$. So $f_{\langle g_i, a_{k+1} \rangle}$ is the required $d \in \text{posi}(\{c, b\}) \cap B_{k+1}$.

²Arthur van Camp's insight that this can be done in such a step-by-step way to just use axiom K_{AddPair} when the sets are finite, i.e., it can be shown by a sub-induction.

So, by axiom K_{AddPair} and axiom K_{\supseteq} , $B_{k+1} \in \mathcal{K}$, completing the inductive step.

This has shown by induction that $B_K \in \mathcal{K}$. $B_K \subseteq B$, so $B \in \mathcal{K}$ (using K_{\supseteq}).

This completes the inductive step for our initial induction (on n). Thus we have shown that our initial statement holds for all n . And since it entails axiom K_{Add} , we have shown that this follows. \square

I think that one can do it without axiom K_{\supseteq} ; though it'll be fiddlier. See appendix A for an attempted proof.

3.2 Dominators axiom - axiom K_{Dom}

Proposition 3.3. *Assume we are restricted to finite sets. Axiom K_{Dom} follows from axiom K_{Add} and $K_{\geq 0}$.*

The proof is closely related to De Bock and de Cooman (2019, Lemma 34). There they show that when one replaces a single member of a desirable gamble set by a dominator it remains desirable. We could directly use this and iterate it (finitely many times) to show our result (to write this down carefully, we'd have to do a proof by induction). But it is slightly cleaner to directly rely on our already proved axiom K_{Add} and do the replacements simultaneously.

Proof. Suppose $B = \{f_g \mid g \in A\}$, where $f_g \geq g$.

Now, $h_g := f_g - g \in \mathcal{G}_{\geq 0}$.

If $h_g \neq 0$, the singleton $\{h_g\} \in \mathcal{K}$.

So we have $A \in \mathcal{K}$ and each singleton $\{h_g\} \in \mathcal{K}$ whenever $f_g \geq g$.

Since A is finite, this is finitely many sets.

For any $g^* \in A$, $f_{g^*} \in \text{posi}(\{g^*\} \cup \{h_g \mid g \in A\})$.

So by axiom K_{Add} , $B \in \mathcal{K}$. \square

We can similarly see:

Proposition 3.4. *Axiom K_{Dom} follows from axiom K_{AddInf} and axiom $K_{\geq 0}$ even when sets can be infinite.*

4 Representation with desirable gambles

There is a representation result in the finite setting which says that every coherent \mathcal{K} can be captured by a set of coherent desirable gambles. We have to generalise this result in the infinite setting if we want to avoid item K_{AddInf} and merely impose item K_{Add} .

4.1 Sets of coherent desirable gamble sets

If we have a set of desirable gambles, D , we can extract a set of desirable gamble sets: D evaluates a gamble set B as desirable when it thinks that some member of the set is desirable

$$B \in \mathcal{K}_D \text{ iff there is some } g \in B \text{ with } g \in D \quad (14)$$

$$\text{iff } B \cap D \neq \emptyset \quad (15)$$

This results in very special kinds of sets of desirable gamble sets. We can get more if we instead look at generating a set of desirable gamble sets by using a *set* (\mathbb{D}) of sets of desirable gambles (D s).

If we have a set of coherent D , \mathbb{D} , we can consider evaluations of B as a desirable gamble set if every $D \in \mathbb{D}$ evaluates it as desirable. That is:

$$B \in \mathcal{K}_{\mathbb{D}} \text{ iff } B \in \mathcal{K}_D \text{ for each } D \in \mathbb{D} \quad (16)$$

$$\text{iff for each } D \in \mathbb{D} \text{ there is some } g \in B \text{ with } g \in D \quad (17)$$

$$\text{iff for each } D \in \mathbb{D} \text{ there is some } g \in B \cap D \quad (18)$$

More concisely, $\mathcal{K}_{\mathbb{D}} := \bigcap_{D \in \mathbb{D}} \mathcal{K}_D$.

When \mathbb{D} is a nonempty set of coherent D , $\mathcal{K}_{\mathbb{D}}$ is coherent. But it also satisfies axiom K_{AddInf} .

Proposition 4.1. *Let \mathbb{D} be a nonempty set of coherent D s. $\mathcal{K}_{\mathbb{D}}$ satisfies K_{infadd} .*

Proof. Let $A_i \in \mathcal{K}_{\mathbb{D}}$ with $f_{\langle g_i \rangle} \in \text{posi}(\{g_i \mid i \in I\})$ with $B = \{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times A_i\}$.

So for each $D \in \mathbb{D}$, $D \cap A_i \neq \emptyset$.

Fix any $D \in \mathbb{D}$. Take g_i to be some member of $A_i \cap D$. Now, $f_{\langle g_i \rangle} \in \text{posi}(\{g_i \mid i \in I\})$, so since each $g_i \in D$, also $f_{\langle g_i \rangle} \in D$ by coherence of D . Thus $B \cap D \neq \emptyset$.

So, for any $D \in \mathbb{D}$, $B \cap D \neq \emptyset$. Thus, $B \in \mathcal{K}_{\mathbb{D}}$. \square

Note that, as we show in Campbell-Moore (2021, §4.4.1), some coherent \mathcal{K} fail axiom K_{AddInf} . So these are still special kinds of coherent sets of desirable gamble sets.

In ?? we'll show that such $\mathcal{K}_{\mathbb{D}}$ are exactly the \mathcal{K} that satisfy axiom item K_{AddInf} , but for now we move to consider the more general question:

What about when we allow for mere finite addition? Can we generate *all* coherent \mathcal{K} in an analogous way? Answer: yes!...

4.2 Representation of all coherent \mathcal{K} using D s

A gamble set B was evaluated by \mathbb{D} by seeing if every D in \mathbb{D} evaluated B as desirable. Now, we consider different possible evaluations, as given by a collection of \mathbb{D} , and simply ask that one of them evaluates B as desirable.

That is, we put:

$$B \in \mathcal{K}_{\mathfrak{D}} \text{ iff there is some } \mathbb{D} \in \mathfrak{D} \text{ with } B \in \mathcal{K}_{\mathbb{D}} \quad (19)$$

$$\text{iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D}, B \in \mathcal{K}_D \quad (20)$$

$$\text{iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D} \text{ there is some } g \in B \cap D \quad (21)$$

Or, more concisely,

$$\mathcal{K}_{\mathfrak{D}} := \bigcup_{\mathbb{D} \in \mathfrak{D}} \bigcap_{D \in \mathbb{D}} \mathcal{K}_D \quad (22)$$

Whereas we went from D to \mathbb{D} using universal quantification, we go from \mathbb{D} to \mathfrak{D} using existential quantification.

\mathfrak{D} contains the \mathbb{D} that one is happy to evaluate gamble sets with respect to. This is different from a \mathbb{D} which contains the D that one thinks are still open. One is not sure about which $D \in \mathbb{D}$ is “right” so will only make judgements

when agreed on by all $D \in \mathbb{D}$. But each $\mathbb{D} \in \mathfrak{D}$ is in some sense good and can be trusted to make decisions.

In general, this will not generate something coherent. There is no guarantee that K_{Add} will hold, as we can select A_i s evaluated as desirable by different \mathbb{D}_i , and there's no guarantee that the relevant $\{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times A_i\}$ is evaluated as desirable by any \mathbb{D} in \mathfrak{D} . We will impose requirement on \mathfrak{D} which will ensure this. The property that we will impose is that it should be downwards closed:³ when $\mathbb{D}_1, \mathbb{D}_2 \in \mathfrak{D}$, there is some $\mathbb{D} \in \mathfrak{D}$ with $\mathbb{D} \subseteq \mathbb{D}_1 \cap \mathbb{D}_2$. This will ensure that by taking some A_1 evaluated as desirable by \mathbb{D}_1 and A_2 by \mathbb{D}_2 , then \mathbb{D} will evaluate the required $\{f_{\langle g_1, g_2 \rangle} \mid \langle g_1, g_2 \rangle \in A_1 \times A_2\}$ as desirable; and it can be extended to finitely many members to get axiom K_{Add} .

This then allows us to represent all coherent \mathcal{K} , where we impose axiom K_{Add} but not necessarily axiom K_{AddInf} .

Theorem 4.2. \mathcal{K} is coherent iff there is some \mathfrak{D} a non-empty set of \mathbb{D} s, which are themselves non-empty sets of coherent D s (??), where \mathfrak{D} is downwards closed and $\mathcal{K} = \mathcal{K}_D$, i.e.,:

$$B \in \mathcal{K} \text{ iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D} \text{ there is some } g_D \in B \cap D \quad (23)$$

Proof. Suppose \mathcal{K} is coherent.

Let $\mathfrak{D}_{\mathcal{K}} := \{\mathbb{D}_{\{A_1, \dots, A_n\}} \mid A_1, \dots, A_n \text{ are finitely many members of } \mathcal{K}\}$ where

$$\mathbb{D}_{\{A_1, \dots, A_n\}} := \left\{ D \text{ coherent} \left| \begin{array}{l} \text{there is } \langle g_1, \dots, g_n \rangle \in A_1 \times \dots \times A_n \text{ with} \\ 0 \notin \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0}) \text{ and} \\ D \supseteq \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0}) \end{array} \right. \right\} \quad (24)$$

Sublemma 4.2.1. $\mathfrak{D}_{\mathcal{K}}$ is a non-empty set of \mathbb{D} s, which are themselves non-empty sets of coherent D s (??), and $\mathfrak{D}_{\mathcal{K}}$ is downwards closed

Proof. $\mathfrak{D}_{\mathcal{K}}$ is non-empty because \mathcal{K} is non-empty and for any $A \in \mathcal{K}$, $\mathbb{D}_{\{A\}} \in \mathfrak{D}_{\mathcal{K}}$.

Each $\mathbb{D}_{\{A_1, \dots, A_n\}}$ is a non-empty set of coherent D s. It is a set of coherent D s by definition; we just need to show that it is non-empty. Suppose that $A_1, \dots, A_n \in \mathcal{K}$ with $\mathbb{D}_{\{A_1, \dots, A_n\}} = \emptyset$, i.e., for each $\langle g_i \rangle \in \times A_i$, $0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ (because otherwise $\text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0})$ is coherent). But then by theorem 2.1, then $\{0\} \in \mathcal{K}$, contradicting the coherence of \mathcal{K} . So $\emptyset \notin \mathfrak{D}_{\mathcal{K}}$.

It remains to show that $\mathfrak{D}_{\mathcal{K}}$ is downwards closed. We will show that

$$\mathbb{D}_{A_1^1, \dots, A_{n_1}^1} \cap \mathbb{D}_{A_1^2, \dots, A_{n_2}^2} \supseteq \mathbb{D}_{A_1^1, \dots, A_{n_1}^1, A_1^2, \dots, A_{n_2}^2} \quad (25)$$

Suppose $D \in \mathbb{D}_{A_1^1, \dots, A_{n_1}^1, A_1^2, \dots, A_{n_2}^2}$. So there is $\langle g_1^1, \dots, g_{n_1}^1, g_1^2, \dots, g_{n_2}^2 \rangle \in A_1^1 \times \dots \times A_{n_1}^1 \times A_1^2 \times \dots \times A_{n_2}^2$ with $0 \notin \text{posi}(\{g_1^1, \dots, g_{n_1}^1, g_1^2, \dots, g_{n_2}^2\} \cup \mathcal{G}_{\geq 0})$ and $D \supseteq \text{posi}(\{g_1^1, \dots, g_{n_1}^1, g_1^2, \dots, g_{n_2}^2\} \cup \mathcal{G}_{\geq 0})$.

Since $\text{posi}(\{g_1^1, \dots, g_{n_1}^1, g_1^2, \dots, g_{n_2}^2\} \cup \mathcal{G}_{\geq 0}) \supseteq \text{posi}(\{g_1^1, \dots, g_{n_1}^1\} \cup \mathcal{G}_{\geq 0})$, also $D \supseteq \text{posi}(\{g_1^1, \dots, g_{n_1}^1\} \cup \mathcal{G}_{\geq 0})$ and $0 \notin \text{posi}(\{g_1^1, \dots, g_{n_1}^1\} \cup \mathcal{G}_{\geq 0})$. So $D \in \mathbb{D}_{A_1^1, \dots, A_{n_1}^1}$. Similarly $D \in \mathbb{D}_{A_1^2, \dots, A_{n_2}^2}$. Thus $D \in \mathbb{D}_{A_1^1, \dots, A_{n_1}^1} \cap \mathbb{D}_{A_1^2, \dots, A_{n_2}^2}$, as required. \square

³I had originally required that it is closed under finite intersection. Thanks to Jasper De Bock for noticing that my proof didn't then work.

Sublemma 4.2.2.

$B \in \mathcal{K}$ iff there is some $\mathbb{D} \in \mathfrak{D}$ st for all $D \in \mathbb{D}$ there is some $g \in B \cap D$ (26)

Proof.

$$B \in \mathcal{K}_{\mathfrak{D}\mathcal{K}} \quad (27)$$

$$\text{iff there is some } \mathbb{D} \in \mathfrak{D}_{\mathcal{K}} \text{ st for all } D \in \mathbb{D}, B \cap D \neq \emptyset \quad (28)$$

$$\text{iff there is some } A_1, \dots, A_n \in \mathcal{K} \text{ st } \mathbb{D} \supseteq \mathbb{D}_{\{A_1, \dots, A_n\}}, \text{ and for all } D \in \mathbb{D}, B \cap D \neq \emptyset \quad (29)$$

$$\text{iff there is some } A_1, \dots, A_n \in \mathcal{K} \text{ st for all } D \in \mathbb{D}_{\{A_1, \dots, A_n\}}, B \cap D \neq \emptyset \quad (30)$$

$$\text{iff } \begin{aligned} &\text{there is some } A_1, \dots, A_n \in \mathcal{K} \text{ st for all } \langle g_i \rangle \in \times A_i, \\ &B \cap \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0}) \neq \emptyset \text{ or } 0 \in \text{posi}(\{g_1, \dots, g_n\} \cup \mathcal{G}_{\geq 0}) \end{aligned} \quad (31)$$

By theorem 2.1, we know that any B satisfying eq. (31) is in \mathcal{K} . It is also easy to see that any $B \in \mathcal{K}$ satisfies eq. (31) (by $n = 1$, $A_1 = B$). So we have that $\mathcal{K}_{\mathfrak{D}} = \mathcal{K}$. \square

Sublemma 4.2.3. *When \mathfrak{D} is a non-empty set of \mathbb{D} s, which are themselves non-empty sets of coherent D s (??), and \mathfrak{D} is downwards closed, then $\mathcal{K}_{\mathfrak{D}}$ is coherent, where $B \in \mathcal{K}_{\mathfrak{D}}$ iff there is some $\mathbb{D} \in \mathfrak{D}$ st for all $D \in \mathbb{D}$, $B \cap D \neq \emptyset$.*

Proof. We check each of the axioms.

Axiom \mathcal{K}_{\emptyset} : Note that $\emptyset \cap D \neq \emptyset$ for any D . Also each $\mathbb{D} \neq \emptyset$, so $\emptyset \notin \mathcal{K}_{\mathfrak{D}}$ for any \mathfrak{D} . And thus $\emptyset \notin \mathcal{K}_{\mathfrak{D}}$.

Axiom \mathcal{K}_0 , $\mathcal{K}_{\geq 0}$, \mathcal{K}_{\supseteq} and \mathcal{K}_{Dom} hold because each $\mathcal{K}_{\mathbb{D}}$ satisfies each of them.

Axiom \mathcal{K}_0 : Suppose $A \in \mathcal{K}_{\mathfrak{D}}$. Then there is some \mathbb{D} with $A \in \mathcal{K}_{\mathbb{D}}$. And thus $A \in \mathcal{K}_D$ for each $D \in \mathbb{D}$. That is, there is some $g \in A \cap D$. Since $0 \notin D$, also $(A \setminus \{0\}) \cap D \neq \emptyset$. So $(A \setminus \{0\}) \in \mathcal{K}_{\mathbb{D}}$. Thus $(A \setminus \{0\}) \in \mathcal{K}_{\mathfrak{D}}$.

Axiom $\mathcal{K}_{\geq 0}$: Suppose $g \in \mathcal{G}_{\geq 0}$. Then $g \in D$ for each coherent D . Consider any $\mathbb{D} \in \mathfrak{D}$. Then $g \in D$ for each $D \in \mathbb{D}$. Thus $g \in \mathcal{K}_{\mathbb{D}}$; thus $g \in \mathcal{K}_{\mathfrak{D}}$.

Axiom \mathcal{K}_{\supseteq} : Suppose $A \in \mathcal{K}_{\mathfrak{D}}$ and $B \supseteq A$. Then there is some \mathbb{D} with $A \in \mathcal{K}_{\mathbb{D}}$. And thus $A \cap D \neq \emptyset$ for each $D \in \mathbb{D}$. So also $B \cap D \neq \emptyset$ for each $D \in \mathbb{D}$. So $B \in \mathcal{K}_{\mathbb{D}}$. So $B \in \mathcal{K}_{\mathfrak{D}}$.

Axiom \mathcal{K}_{Dom} : Suppose $A \in \mathcal{K}_{\mathfrak{D}}$ and each $f_g \geq g$. There is some \mathbb{D} with $A \in \mathcal{K}_{\mathbb{D}}$. And thus $A \in \mathcal{K}_D$ for each $D \in \mathbb{D}$. That is, there is some $g \in A \cap D$. But therefore $f_g \in D$ by properties of coherent D , thus $B \in \mathcal{K}_D$ for each $D \in \mathbb{D}$. Thus $B \in \mathcal{K}_{\mathfrak{D}}$, and so $B \in \mathcal{K}_{\mathfrak{D}}$.

For axiom \mathcal{K}_{Add} :

Let $A_1, \dots, A_n \in \mathcal{K}_{\mathfrak{D}}$ and $B = \{f_{\langle g_i \rangle} \mid \langle g_i \rangle \in \times A_i\}$ where $f_{\langle g_i \rangle} \in \text{posi}(\{g_i\})$. Since $A_i \in \mathcal{K}_{\mathfrak{D}}$, we have some $\mathbb{D}_i \in \mathfrak{D}$ with $A_i \in \mathcal{K}_{\mathbb{D}_i}$. By downwards closure, there is some $\mathbb{D}^* \in \mathfrak{D}$ with $\mathbb{D}^* \subseteq \mathbb{D}_1 \cap \dots \cap \mathbb{D}_n$. We will show that $B \in \mathcal{K}_{\mathbb{D}^*}$. For any $D \in \mathbb{D}^*$, also $D \in \mathbb{D}_i$ for each i , and thus there is some $g_i \in D \cap A_i$. Since D is coherent containing g_1, \dots, g_n and $f_{\langle g_i \rangle} \in \text{posi}\{g_1, \dots, g_n\}$, also $f_{\langle g_i \rangle} \in D$. So $D \cap B \neq \emptyset$. Since this holds for all $D \in \mathbb{D}^*$, $B \in \mathcal{K}_{\mathbb{D}^*}$. So $B \in \mathcal{K}_{\mathfrak{D}}$, as required. \square

This proves our result. \square

It might be more intuitive to think about \mathfrak{D} not simply as a downwards closed set of \mathbb{D} s, but as a filter of \mathbb{D} s. That is, we might also close it under supersets. This won't make a difference to the representation, because adding supersets doesn't add any new members to $\mathcal{K}_{\mathfrak{D}}$.

We can then think of \mathfrak{D} as containing all one's judgements about what the “coherent” D is like. This would allow us to give a different formulation.

$$B \in \mathcal{K}_{\mathfrak{D}} \text{ iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st for all } D \in \mathbb{D} \text{ there is some } g \in B \cap D \quad (32)$$

$$\text{iff there is some } \mathbb{D} \in \mathfrak{D} \text{ st } \mathbb{D} \subseteq \{D \mid \text{there is some } g \in B \cap D\} \quad (33)$$

$$\text{iff } \{D \mid \text{there is some } g \in B \cap D\} \in \mathfrak{D} \quad (34)$$

There is a further alternative we might do which removes requirements on \mathfrak{D} : we can drop the requirement that \mathfrak{D} be downwards closed, and instead bake that into the characterisation of $\mathcal{K}_{\mathfrak{D}}$. We could say:

$$A \in \mathcal{K}_{\mathfrak{D}} \text{ iff there are some } \mathbb{D}_1, \dots, \mathbb{D}_n \in \mathfrak{D} \text{ with } A \in \mathcal{K}_{\mathbb{D}_i} \text{ for each } i \quad (35)$$

$$\text{iff there is some } \mathbb{D}_1, \dots, \mathbb{D}_n \in \mathfrak{D} \text{ st for all } D \in \mathbb{D}_1 \cap \dots \cap \mathbb{D}_n \text{ there is some } g \in A \cap D \quad (36)$$

Now we get axiom K_{Add} for free, however axiom K_0 might now fail. We still need a “coherence” requirement on this. The relevant requirement is that \mathfrak{D} has the finite intersection property: that we never have $\mathbb{D}_1 \cap \dots \cap \mathbb{D}_n = \emptyset$. It is then equivalent.

Corollary 4.3. *\mathcal{K} is coherent iff there is some \mathfrak{D} a non-empty set of \mathbb{D} s, which are themselves non-empty sets of coherent D s (??), where \mathfrak{D} has the finite intersection property (i.e., no $\mathbb{D}_1 \cap \dots \cap \mathbb{D}_n = \emptyset$), and*

$$B \in \mathcal{K} \text{ iff there is some } \mathbb{D}_1, \dots, \mathbb{D}_n \in \mathfrak{D} \text{ st for all } D \in \mathbb{D}_1 \cap \dots \cap \mathbb{D}_n, \emptyset \neq A \cap D \quad (37)$$

Proof. If \mathfrak{D} is such a set which has the finite intersection property, consider \mathfrak{D}' which closes it under finite intersections, i.e., $\mathfrak{D}' := \{\mathbb{D}_1 \cap \dots \cap \mathbb{D}_n \mid \mathbb{D}_1, \dots, \mathbb{D}_n \text{ are finitely many members of } \mathfrak{D}\}$. By assumption that \mathfrak{D} has the finite intersection property, \mathfrak{D}' doesn't contain the emptyset. It immediately satisfies all the other assumptions of theorem 4.2.

Also observe that:

$$\text{There is some } \mathbb{D}_1, \dots, \mathbb{D}_n \in \mathfrak{D} \text{ st for all } D \in \mathbb{D}_1 \cap \dots \cap \mathbb{D}_n, \emptyset \neq A \cap D \quad (38)$$

$$\text{iff There is some } \mathbb{D}' \text{ st for all } D \in \mathbb{D}', \emptyset \neq A \cap D \quad (39)$$

So we see that for any such \mathfrak{D} with the finite intersection property, we can simply close it under finite intersections and use the representation of theorem 4.2 \square

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A Finite addition proof without supersets

Proposition A.1. *Assume we are restricted to finite sets. Axiom K_{Add} follows from axiom K_{AddPair} .*

Jasper: the pf doesn't work: what if multiple of the f's are all 0?

Proof Attempt. The argument gets slightly fiddly because members may coincide and repetitions get “deleted” by sets. To avoid this issue, we need to include constants throughout to ensure distinctness.

We work by induction on n .

Base case: $n = 1$ follows directly from axiom K_{AddPair} by putting $A_1 = A_2$.

Inductive step: Assume it holds for n and consider $n + 1$.

So, suppose $A_1, \dots, A_n, A_{n+1} \in \mathcal{K}$ and for each sequence $\langle g_i \rangle \in \times_i A_i$ and $a \in A_{n+1}$, we have $f_{\langle g_1, \dots, g_n, a \rangle} \in \text{posi}(\{g_1, \dots, g_n, a\})$ with

$$B = \{f_{\langle g_1, \dots, g_n, a \rangle} \mid \langle g_i \rangle \in \times_{i \leq n} A_i, a \in A_{n+1}\} \quad (40)$$

We need to show that $B \in \mathcal{K}$.

We will abuse notation and write $f_{\langle g_i, a \rangle}$ instead of $f_{\langle g_1, \dots, g_n, a \rangle}$.

Consider adding some scalar constants: let

$$f_{\langle g_i, a \rangle}^* := \delta_{\langle g_i, a \rangle} \times f_{\langle g_i, a \rangle} \quad (41)$$

where $\delta_{\langle g_i, a \rangle}$ are chosen to ensure that these are all distinct, and distinct from any a' . We will show that

$$B^* = \{f_{\langle g_i, a \rangle}^* \mid \langle g_i \rangle \in \times_i A_i\} \quad (42)$$

A_{n+1} is finite, so enumerate it, $A_{n+1} = \{a_1, \dots, a_K\}$. Define:

$$B_k^* := \{f_{\langle g_i, a_j \rangle}^* \mid \langle g_i \rangle \in \times_{i \leq n} A_i, a_j \in \{a_1, \dots, a_k\} \cup \{a_{k+1}, \dots, a_K\}\} \quad (43)$$

Where we have already replaced k -many of A_{n+1} with the relevant members of B^* . We show by a induction on k that $B_k^* \in \mathcal{K}$.⁴

⁴Arthur van Camp's insight that this can be done in such a step-by-step way to just use axiom K_{AddPair} , i.e., it can be shown by a sub-induction.

Base case: $B_0 = A_{n+1} \in \mathcal{K}$.

Inductive step: we can assume that $B_k \in \mathcal{K}$ and need to show $B_{k+1} \in \mathcal{K}$.

We will construct a set C which we know to be in \mathcal{K} , which we can then combine with B_k using axiom $\text{K}_{\text{AddPair}}$ (and axiom K_{\supseteq}) to get $B_{k+1}^* \in \mathcal{K}$.

Sublemma A.1.1. We can find distinct $h_{\langle g_i, a_{k+1} \rangle}^*$ where:

$$h_{\langle g_i, a_{k+1} \rangle}^* \in \text{posi}(\{g_1, \dots, g_n\}), \text{ and} \quad (44)$$

$$f_{\langle g_i, a_{k+1} \rangle}^* \in \text{posi}(\{h_{\langle g_i, a_{k+1} \rangle}^*, a_{k+1}\}). \quad (45)$$

Proof. $f_{\langle g_i, a_{k+1} \rangle}$ has the form $\sum_{i \leq n} \lambda_i g_i + \mu a_{k+1}$. If $\lambda_i > 0$ for some i , then we put $h_{\langle g_i, a_{k+1} \rangle} := \sum_{i \leq n} \lambda_i g_i$. If not, then we simply let it be, for example, g_1 .

To ensure they are distinct, we simply multiply the h by appropriate constants to find h^* . \square

We let

$$C := \{h_{\langle g_i, a_{k+1} \rangle}^* \mid \langle g_i \rangle \in \bigtimes_i A_i\} \quad (46)$$

$C \in \mathcal{K}$ by the induction hypothesis (on n), using eq. (44).

Now, for each $\langle c, b \rangle$, let

$$e_{\langle c, b \rangle} := \begin{cases} c & b \in B_k^* \cap B_{k+1}^* \\ f_{\langle g_i, a_{k+1} \rangle}^* & c = a_{k+1} \text{ and } b = h_{\langle g_i, a_{k+1} \rangle}^* \end{cases} \quad (47)$$

Since each $e_{\langle c, b \rangle} \in \text{posi}(\{c, b\})$, by axiom $\text{K}_{\text{AddPair}}$,

$$\{e_{\langle c, b \rangle} \mid c \in B_k, b \in C\} \in \mathcal{K} \quad (48)$$

Sublemma A.1.2.

$$B_{k+1}^* = \{e_{\langle c, b \rangle} \mid c \in B_k, b \in C\} \quad (49)$$

Proof. Can easily be checked. \square

So, by axiom $\text{K}_{\text{AddPair}}$ and axiom K_{\supseteq} , $B_{k+1}^* \in \mathcal{K}$, completing the inductive step.

This has shown by induction that $B_K^* \in \mathcal{K}$. $B_K^* = B^*$, so $B^* \in \mathcal{K}$.

We need to show that $B \in \mathcal{K}$. This works because it is obtained from B^* simply by multiplying everything by a scalar, so follows from the $n = 1$ version of axiom K_{Add} , which itself directly follows from axiom $\text{K}_{\text{AddPair}}$.

This shows that the statement holds for $n + 1$, completing the inductive step. Thus we have shown that it holds for all n ; as required.