Additivity and the opinion-set

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WARNING: Updates coming soon!

1 Agenda and probabilities

Definition 1.1 (Worlds, propositions and Boolean algebras)**.** There are different possible setups:

1. - Start with a collection, W.
   - Propositions, $A \in \mathcal{P}$ are conceived of as $A \subseteq W$.
   - $\mathcal{P}$ forms a Boolean algebra if
     - $W \in \mathcal{P}$,
     - $\varnothing \in \mathcal{P}$,
     - $A \in \mathcal{P} \implies W \setminus A \in \mathcal{P}$,
     - $A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$,
   - To link to the sentential framework, we define $w(A) = \mathsf{t}$ if $w \in A$, and $w(A) = \mathsf{f}$ if $w \notin A$.
2. - We start with $\mathcal{P}$ a collection of sentences in a language.
   - $\mathcal{P}$ forms a Boolean algebra if
     - $\top \in \mathcal{P}$,
     - $\bot \in \mathcal{P}$,
     - $A \in \mathcal{P} \implies \neg A \in \mathcal{P}$,
     - $A, B \in \mathcal{P} \implies A \wedge B \in \mathcal{P}$.
   - W is collection of classically consistent functions $w : \mathcal{P} \to \{\mathsf{t}, \mathsf{f}\}$ (it might be a subset of the collection of all classical logic models).
3. [Stating the previous one slightly more generally]
   - Start with $\mathcal{P}$ simply as a non-empty set.
   - W is some collection of functions $w : \mathcal{P} \to \{\mathsf{t}, \mathsf{f}\}$.

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- $\mathcal{P}$ forms a Boolean algebra if
  - There is some $A \in \mathcal{P}$ such that $w(A) = \mathsf{t}$ for all $w \in W$. Let $\top$ denote some such A.
  - There is some $A \in \mathcal{P}$ such that $w(A) = \mathsf{f}$ for all $w \in W$. Let $\perp$ denote some such A.
  - If $A \in \mathcal{P}$ then there is some $B \in \mathcal{P}$ such that $w(A) = \mathsf{t}$ iff $w(B) = \mathsf{f}$. Let $\neg A$ denote some such B.
  - If $A, B \in \mathcal{P}$ then there is some $C \in \mathcal{P}$ such that $w(C) = \mathsf{t}$ iff $w(A) = \mathsf{t}$ and $w(B) = \mathsf{t}$. Let $A \wedge B$ denote some such C.

I think we might also need to require that W is closed (using the powerset topology on $\{\mathsf{t}, \mathsf{f}\}^{\mathcal{P}}$).

I typically like to think in the sentential framework, but it shouldn't matter for this discussion. We can move back and forth between the frameworks. I think in some sense they're equivalent. There will be different once we consider, e.g., weakening the logic. But that's not a job for now.

We need to assume that there's a Boolean structure around in the background to make use of the usual definition of probabilities (see below comment). If we want to use alternative definitions of probability we can completely delete that requirement that $\mathcal{P}$ forms a Boolean algebra.

We then define:

Definition 1.2 (Opinion set)**.** An opinion set, $\mathcal{O}$, is a collection of propositions, i.e., $\mathcal{A} \subseteq \mathcal{P}$.

This is the collection of propositions that you in fact have opinions about. We then have:

Definition 1.3 (Agenda)**.** An agenda, $\mathcal{A}$, is any collection of propositions that you have opinions about, i.e., $\mathcal{A} \subseteq \mathcal{O}$.

Formally, $\mathcal{A}$ and $\mathcal{O}$ are just the same sorts of objects: collections of propositions. So all our definitions, e.g., of credences, are defined for either of these. I don't know anyone else distinguishing them this sort of way, I'm playing with doing it. The idea of why I've kept them separate is that they seem to go with different pictures: An agenda is simply the propositions that I happen to be considering right now. Whereas an "opinion set" is the collection of all propositions that I have an opinion about. An accuracy measure that measures the accuracy of my (total) opinions, should rather work with $\mathcal{O}$. Pettigrew (2016) works with what he calls an "opinion set", but I think he rather uses the picture behind the agenda as I pictured it here. Fitelson uses "agenda". A number of authors call (either of) these an "algebra". But this really implies that they're Boolean algebras, which I don't want to imply.

We here allow them to be infinite. But we can't actually do accuracy stuff with infinite agendas yet.

Definition 1.4 (Credences)**.** A credence function on $\mathcal{A}$ is a function $c : \mathcal{A} \to [0, 1]$. $\mathsf{Creds}_{\mathcal{A}}$ is the collection of all credence functions on $\mathcal{A}$.

Note I've put $[0, 1]$ into the definition of credence function. One needn't do that. But perhaps I like Richard's idea that the numbers are arbitrary bounds, so it'd be meaningless to be outside them. He talks about this in a blog. I'm not sure if there's a better reference.

I'm not sure this is the best definition to take as the primary one, but here's what it is to be probabilistic (eg Pettigrew, 2016)

Definition 1.5 (Probability)**.** Assume that $\mathcal{P}$ forms a Boolean algebra (thus any $\mathcal{A}$ can be extended to a Boolean algebra)...

$c \in \mathsf{Creds}_{\mathcal{A}}$ is probabilistic iff there is Boolean algebra, $\mathcal{B} \subseteq \mathcal{P}$ which extends $\mathcal{A}$ and with $p : \mathcal{B} \to [0, 1]$ extending c, with p a finitely additive probability function which extends c. I.e.:

- $p(W) = 1$.
- $p(A) \geqslant 0$ for all A.
- $p(A \cup B) = p(A) + p(B)$ if $A \cap B = \varnothing$.

1.1 Other characterisations of being probabilistic

We also know we can characterise it as:

- $c \in \mathsf{Probs}_{\mathcal{A}}$ iff c is in the convex hull of the $w : \mathcal{A} \to \{0, 1\}$. If $\mathcal{A}$ is finite, this is just that there are $\lambda_1, \ldots, \lambda_n$ positive summing to 1 and $w_1, \ldots, w_n \in W$ such that $c(\varphi_i) = \sum_i \lambda_i w_i(A)$. If $\mathcal{A}$ is infinite, then ...

To do. Remember we ensured W closed.

- $c \in \mathsf{Probs}_{\mathcal{A}}$ iff c is not Dutch bookable, i.e. there is no finite $A_1, \ldots, A_n \in \mathcal{A}$ and $s_1, \ldots, s_n \in \mathbb{R}$ such that $\sum_i s_i(w(A_i) - c(A_i)) < 0$ for all w.

The advantage of these characterisations is we didn't then need to initially assume a $\mathcal{P}$ forming a Boolean algebra. We might just start with a fixed $\mathcal{P}$ which is the collection of propositions I have opinions about, call this $\mathcal{O}$. We don't need to assume any closure principles. And we define probabilistic directly using this structure. I actually think that's a nicer way to go.

But some people will be unhappy with that as the primitive definition of probabilistic. We'd need to justify it more. Anyway, since I already wrote it up with the Boolean extensions, I'll leave it for now.

As an interesting sidenote, perhaps there are more general axiomatisations.For example:

Proposition 1.6. *If $\mathcal{A}$ is closed under conjunctions and containing $\varnothing$ and W, then p is extendable to a probability function iff:*

- *Normalization:* $p(W) = 1$
- *Emptyset:* $p(\varnothing) = 0$

- *∞-valuation (via-entailments): For C in $\mathcal{A}$ and K finite $\subseteq \mathcal{A}$,*

$$\text{If } C \supseteq \bigcup K, \text{ then } p(C) \geqslant \sum_{\varnothing \neq J \subseteq K} (-1)^{\#J+1} p(\bigcap J)$$
$$\text{If } C \subseteq \bigcup K, \text{ then } p(C) \leqslant \sum_{\varnothing \neq J \subseteq K} (-1)^{\#J+1} p(\bigcap J)$$

What about when $\mathcal{A}$ is not closed under conjunctions? I think this axiomatisation won't be enough, but I haven't actually got a counterexample.

2 Accuracy

2.1 Definition and some comments

Definition 2.1. An $\mathcal{A}$ accuracy measure is a function $\mathsf{Acc} : \mathsf{Creds}_{\mathcal{A}} \times W_{\mathcal{A}} \to \mathbb{R} \cup \{-\infty\}$.[1]

(Some people might think that accuracy is only something we apply to coherent agents, so it's $\mathsf{Acc} : \mathsf{Probs}_{\mathcal{A}} \times W \to \mathbb{R} \cup \{-\infty\}$. Of course such accuracy measures won't be useful for dominance arguments for probabilisim, basically by definition.)

Note that I've also put it into the definition of an accuracy measure that it is "extensional": it only depends on one's credences in $\mathcal{A}$ propositions, and the truth values of those propositions. One might want to include that as a separate axiom of accuracy instead.

Because of this assumption of extensionality we can see:

Remark. For $\mathcal{A}$ a singleton, $\mathcal{A} = \{B\}$, where B is contingent, $\mathsf{Creds}_{\{B\}} = [0, 1]$, and $W_{\{B\}} = \{\mathsf{t}, \mathsf{f}\}$, so a $\{B\}$ accuracy measure is a function from $[0, 1] \times \{\mathsf{t}, \mathsf{f}\}$ to $\mathbb{R} \cup \{-\infty\}$.

We can define axioms, or properties, of such $\mathcal{A}$ accuracy measures, for example:

Definition 2.2 (Strict propriety)**.** Let Acc be an $\mathcal{A}$ accuracy measure. Acc is strictly proper if for all $p \in \mathsf{Probs}_{\mathcal{A}}$ and $c \in \mathsf{Creds}_{\mathcal{A}} \setminus \{p\}$,

$$\mathrm{Exp}_p \mathsf{Acc}(p) > \mathrm{Exp}_p \mathsf{Acc}(c).$$

Actually for $\mathcal{A}$ non-Boolean, we need to be a little careful about defining expectation and how this works. We should probably actually define it as Richard does in Pettigrew (2016, Def.2.2.1). Though note that for additive Acc we don't need to be careful (**?**, ftnte.5).

[1] $W_{\mathcal{A}}$ is $\{w{\restriction_{\mathcal{A}}} \mid w \in W\}$.

2.2 Pictures of accuracy

There are two different pictures of accuracy.

One might think that really we should just measure and be interested in one's total accuracy, i.e., work with a $\mathcal{O}$ accuracy measure for $\mathcal{O}$ the fixed collection of all propositions that I have opinions about. So this measures the accuracy of my actual (total) credence function on the domain on which it's actually defined.

For $\mathcal{A} \subset \mathcal{O}$ we have to ask whether there is a philosophically interesting notion of $\mathcal{A}$ accuracy? Or is it just a formal tool to be able to prove things about $\mathcal{O}$ accuracy (in the case, e.g., where $\mathcal{O}$ accuracy is additive we can prove things about the local accuracy and get stuff about global accuracy.)

Even if we think that there is an independent philosophically interesting notion of $\mathcal{A}$ accuracy, I wonder whether that's a domain where our normal accuracy theoretic justifications work. For example, suppose on $\mathcal{A} \subset \mathcal{O}$ I think that some other $\mathcal{A}$ credences would be accuracy-theoretic better, but nonetheless when we take all propositions into account ($\mathcal{O}$), I think my credences are the best. Then that doesn't seem like such a bad epistemic failing. So this only motivates rational immodesty wrt $\mathcal{O}$ not $\mathcal{A}$ accuracy more generally. Similarly, I think we may only be able to draw normative conclusions from $\mathcal{O}$ accuracy considerations. For example, if we knew that my opinions on a $\mathcal{A} \subset \mathcal{O}$ were accuracy dominated, it might (prima facie) nonetheless be that on $\mathcal{O}$ my opinions are non-dominated. (I guess that can't actually happen, but that's something to be checked.)

We now go through how things look on the two pictures of accuracy.

2.3 Only $\mathcal{O}$

Now suppose we think that the only meaningful notion of accuracy is an $\mathcal{O}$ notion for fixed $\mathcal{O}$. We can still consider "local accuracy measures" as formal tools.

We would define accuracy:

Definition 2.3. An $\mathcal{O}$ accuracy measure is (weightedly-)additive if there are functions $\mathsf{acc}_B : [0,1] \times \{\mathsf{t}, \mathsf{f}\} \to \mathbb{R} \cup \{-\infty\}$ for every $B \in \mathcal{P}$ such that

$$\mathsf{Acc}(c, w) = \sum_{B \in \mathcal{A}} \mathsf{acc}_B(c(B), w(B)).$$

Such acc are called local accuracy measures.

A feature of this picture is that acc is merely a formal tool: it's just the function that generates Acc. We directly justify propriety of Acc. Any properties of acc are formal tools. We don't get to justify propriety of acc by thinking about goodness of individual credence functions. Instead we justify strict propriety of Acc, and if we want to use any properties of acc we should show they follow from properties of Acc. So we are then interested in proving results like

Proposition 2.4. *Suppose* Acc *is additive, and is generated by* acc_B*. and* Acc *is strictly proper, then so is* acc_B*, i.e.:*

For any $x \in [0,1]$ and $y \neq x$,

$$x \times \mathsf{acc}(x, \mathsf{t}) + (1-x) \times \mathsf{acc}(x, \mathsf{f}) > x \times \mathsf{acc}(y, \mathsf{t}) + (1-x) \times \mathsf{acc}(y, \mathsf{f})$$

2.4 $\mathcal{O}$ and some $\mathcal{A}$ accuracy are meaningful

We might alternatively think that there are actual $\mathcal{A}$ accuracy measures for various $\mathcal{A}$. So here we're not really working with a single accuracy measure but a family of them for various $\mathcal{A}$.

Definition 2.5. For $\mathfrak{A}$ a collection of agendas, an $\mathfrak{A}$ accuracy notion defines an $\mathcal{A}$ accuracy measure for any $\mathcal{A} \in \mathfrak{A}$.

Which range? Perhaps: any $\mathcal{A} \subseteq \mathcal{O}$. Or perhaps just the $\mathcal{A}$ which form Boolean algebras, or perhaps just the $\mathcal{A}$ which are partitions. Or perhaps just $\mathcal{O}$ and any singletons.

If we have a more general picture of accuracy, an $\mathfrak{A}$ accuracy notion, where the singletons are in $\mathfrak{A}$, we probably want to define additivity as:

Definition 2.6. If $\mathfrak{A}$ contains all the singletons $\{B\}$ for $B \in \mathcal{O}$. For an $\mathfrak{A}$ accuracy notion, we can let $\mathsf{acc}_B : [0,1] \times \{\mathsf{t}, \mathsf{f}\} \to \mathbb{R} \cup \{-\infty\}$ be the $\{B\}$ accuracy measure given by the accuracy notion. Then we can say that this accuracy notion is additive if for every $\mathcal{A} \in \mathfrak{A}$, the $\mathcal{A}$ accuracy measure is given by:

$$\mathsf{Acc}(c, w) = \sum_{B \in \mathcal{A}} \mathsf{acc}_B(c(B), w(B)).$$

Here we might think we get to directly justify properties of accuracy measures by justifying properties of the local measures. I think Pettigrew (2016) does it like this.

Alternatively, we still think that the justification is directly of the $\mathcal{O}$ accuracy measure, and any properties of the local measures are derivative. So we have to justify strict propriety of acc by going via the strict propriety of the $\mathcal{O}$ accuracy measure.

2.5 Examples

Whenever we define a function $\mathsf{acc} : [0,1] \times \{\mathsf{t}, \mathsf{f}\} \to \mathbb{R} \cup \{-\infty\}$, we can generate accuracy measures for every $\mathcal{A} \subseteq \mathcal{P}$ just using additivity. So, for example:

Example 2.7. The Brier score is $\mathsf{BS}(x, \mathsf{t}) = 1 - (1-x)^2$, $\mathsf{BS}(x, \mathsf{f}) = 1 - x^2$.[2]

For any $\mathcal{A}$, we have an $\mathcal{A}$ accuracy measure: $\mathsf{BS}(c, w) = \sum_{A \in \mathcal{A}} \mathsf{BS}(c(A), w(A))$.

Example 2.8. The global log score could also be though of this way: $\mathsf{GlLog}(x, \mathsf{t}) = \ln(x)$, $\mathsf{GlLog}(x, \mathsf{f}) = 0$. For any $\mathcal{A}$, we have an $\mathcal{A}$ accuracy measure: $\mathsf{GlLog}(c, w) = \sum_{A \in \mathcal{A}} \mathsf{GlLog}(c(A), w(A))$.

[2] I added $1-$ to make it an accuracy measure rather than inaccuracy.

But this isn't how people usually conceive of it. I think people only think: the global log score is a score that's only defined when $\mathcal{A}$ is a partition. And in that case, $\mathsf{GlLog}(c, w) = \ln(c(A_w))$ for the $A_w \in \mathcal{A}$ which is true at w.

In fact, I think the Global Log score is usually thought of not as an $\mathcal{A}$ accuracy measure in my sense at all, they think: it's only defined over *probability* functions.

References

Richard Pettigrew. *Accuracy and the Laws of Credence*. Oxford University Press, 2016.