$$\int f(ax+b)\,dx = \frac{1}{a}\int f(ax+b)\,d(ax+b)$$

eg: $$\int e^{\boxed{2x+3}}\,dx = \frac{1}{2}\int e^{2x+3}\,d(2x+3) = \frac{1}{2}e^{2x+3}+C$$

$$\int a^x\,dx = \frac{a^x}{\ln a}+C$$

$$\int \frac{1}{x} f(\ln x)\,dx = \int f(\ln x)\cdot(\ln x)'\,dx = \int f(\ln x)\,d\ln x = F(\ln x)+C$$

eg: $$\int \frac{2}{x}\cdot 3^{(\ln x)}\,dx = 2\int 3^{\ln x}\cdot(\ln x)'\,dx = 2\int \underbrace{3^{\ln x}\,d\ln x} = 2\cdot\frac{3^{\ln x}}{\ln 3}+C$$

$$\int \frac{1}{x}\sqrt{\ln x}\,dx = \int \sqrt{\ln x}\cdot(\ln x)'\,dx = \int \sqrt{\ln x}\,d(\ln x) = \frac{2}{3}(\ln x)^{\frac{3}{2}}+C$$

$$\int \frac{a}{\sqrt{x}} f(\sqrt{x})\,dx = 2a\int f(\sqrt{x})\,(\sqrt{x})'\,dx = 2a\int f(\sqrt{x})\,d\sqrt{x} = 2a F(\sqrt{x})+C$$

eg: $$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin\sqrt{x}\cdot(\sqrt{x})'\,dx = -2\cos\sqrt{x}+C$$

$$\int \frac{1}{\sqrt{x}} e^{3\sqrt{x}+2}\,dx = \frac{2}{3}\int e^{3\sqrt{x}+2}\cdot(3\sqrt{x}+2)'\,dx = \frac{2}{3}e^{3\sqrt{x}+2}+C$$

$$\int \frac{a}{x^2} f\left(\frac{1}{x}\right) dx = -a\int f\left(\frac{1}{x}\right)\cdot\left(\frac{1}{x}\right)' dx = -a F\left(\frac{1}{x}\right) + C$$

eg. $$\int \frac{1}{x^2} e^{\frac{1}{x}} dx = -\int e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)' dx = -\int e^{\frac{1}{x}} d\frac{1}{x} = -e^{\frac{1}{x}} + C$$

$$\int e^{\frac{3}{x}-1}\cdot\frac{1}{x^2} dx = -\frac{1}{3}\int e^{\frac{3}{x}-1}\cdot\left(\frac{3}{x}-1\right)' dx = -\frac{1}{3} e^{\frac{3}{x}-1} + C$$

$$\int x\sqrt{1+x^2}\, dx$$

$$= \frac{1}{2}\int \sqrt{1+x^2}\cdot(1+x^2)' dx$$

$$= \frac{1}{2}\int \sqrt{1+x^2}\, d(1+x^2)$$

$$= \frac{1}{2}\cdot\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

$$\int \frac{x}{\sqrt{1+x^2}} dx$$

$$\frac{1}{2}\int \frac{1}{\sqrt{1+x^2}}\cdot(1+x^2)' dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}} d(1+x^2)$$

$$= \sqrt{1+x^2} + C$$

$$\int \cos^2 x\cdot\sin x\, dx$$

$$= -\int \cos^2 x\cdot(\cos x)' dx$$

$$= -\int \cos^2 x\, d\cos x$$

$$= -\frac{1}{3}\cos^3 x + C$$

$$\int \frac{x}{1+x^4}dx$$

解 原式 $= \frac{1}{2}\int \frac{1}{1+(x^2)^2}\cdot (x^2)' dx$

$$= \frac{1}{2}\int \frac{1}{1+(x^2)^2} dx^2$$

$$= \frac{1}{2}\arctan x^2 + C$$

$$\int \frac{1}{x(1+\ln^2 x)}dx$$

$$= \int \frac{1}{1+\ln^2 x}\cdot (\ln x)' dx$$

$$= \arctan(\ln x) + C$$

$$\int \frac{1}{1+e^x}dx$$

$$= \int \frac{(1+e^x)-e^x}{1+e^x}dx$$

$$= \int 1\, dx - \int \frac{1}{1+e^x} d(e^x+1)$$

$$= x - \ln(e^x+1) + C$$

$$\int \frac{1}{1-\sin x}dx$$

$$\int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}dx$$

$$= \int \frac{1+\sin x}{\cos^2 x}dx$$

$$= \tan x - \int \frac{1}{\cos^2 x} d\cos x$$

$$= \tan x + \frac{1}{\cos x} + C$$

$$\int \frac{1}{x^2}dx$$

$$-\frac{1}{x}$$

$$\int \frac{1}{x^2+x-2}\,dx \qquad \frac{1}{a\times b}=\frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) \qquad \int \frac{1}{x^2+2x+1}\,dx$$

$$=\int \frac{1}{(x-1)(x+2)}\,dx \qquad \int \frac{1}{(x+1)^2}\,dx$$

$$=\frac{1}{3}\int \frac{1}{x-1}\,dx-\frac{1}{3}\int \frac{1}{x+2}\,dx \qquad =\int \frac{1}{(x+1)^2}\,d(x+1)$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\ln|x+2|+C \qquad =-\frac{1}{x+1}+C$$

$$=\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$$

$$\int \frac{1}{(x-1)(x+2)(x-3)}\,dx$$

$$=\int \frac{a}{x-1}+\frac{b}{x+2}+\frac{c}{x-3}\,dx=\int \frac{a(x^2-x-5)+b(x^2-4x+3)+c(x^2+x-2)}{(x-1)(x+2)(x-3)}\,dx$$

$$\begin{cases} a+b+c=0 \\ -a-4b+c=0 \\ -5a+3b-2c=1 \end{cases}$$

$$\int \frac{x+1}{x^2+2x+6}\,dx$$

$$\frac{1}{2}\int \frac{1}{x^2+2x+6}\,(x^2+2x+6)'\,dx$$

$$=\frac{1}{2}\int \frac{1}{x^2+2x+6}\,d(x^2+2x+6)$$

$$=\frac{1}{2}\ln(x^2+2x+6)+C$$

$$\int \frac{1}{x^2+x+1}\,dx$$

$$=\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,d(x+\frac{1}{2})$$

$$=\int \frac{1}{(\frac{\sqrt{3}}{2})^2+(x+\frac{1}{2})^2}\,d(x+\frac{1}{2})=\frac{2\sqrt{3}}{3}\arctan\frac{2\sqrt{3}(x+\frac{1}{2})}{3}+C$$

$$\int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\arctan\frac{x}{a}+C$$

设 $\int f(x)\,dx=F(x)+C$. $\int f(2+3e^x)\cdot e^x\,dx=$ $\underline{\frac{1}{3}F(2+3e^x)+C}$

分析 $\frac{1}{3}\int f(2+3e^x)\cdot(2+3e^x)'\,dx$

3. 根式代换（直 $\xrightarrow{反}$ 凑 $\xrightarrow{否}$ 根

1) 简单无理根式代换（根号下为一次函数）

▲ ① 形如 $\sqrt[n]{ax+b}$ 型，令 $\sqrt[n]{ax+b}=t$，解出 x、dx

② 形如 $\sqrt[n]{\frac{ax+b}{cx+d}}$ 型，令 $\sqrt[n]{\frac{ax+b}{cx+d}}=t$ 解出 x、dx

③ 含 $\sqrt[m]{ax+b}$ $\sqrt[n]{ax+b}$ 型，令 $\sqrt[p]{ax+b}=t$，解出 x、dx

p是m, n的最小公倍数.

eg1) $\int\sqrt{\frac{x}{1-x}}\,dx$

$t^2=\frac{x}{1-x}$

$t^2-t^2x=x$

解：令 $\sqrt{\frac{x}{1-x}}=t$，$x=\frac{t^2}{1+t^2}$，

$dx=\frac{2t}{(1+t^2)^2}dt$

原式 $=\int\frac{t\cdot 2t}{(1+t^2)^2}dt=2\int\frac{t^2}{(1+t^2)^2}dt$

$=2\int\frac{(t^2+1)-1}{(1+t^2)^2}dt$

$=2\int\frac{1}{1+t^2}dt-\int\frac{2}{(1+t^2)^2}dt$

令 $t=\tan u$

$\int(\cos^2 u)^2\,du$

eg2) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx \Rightarrow 2\int \frac{\arctan\sqrt{x}}{1+(\sqrt{x})^2}d\sqrt{x} = 2\int \arctan\sqrt{x}\, d\arctan\sqrt{x}$

解：令 $\sqrt{x}=t$，$x=t^2$　$dx=dt^2=2t\,dt$

原式 $=\int \frac{\arctan t\cdot 2t}{t\cdot(1+t^2)}dt$

$=\int \frac{2\arctan t}{1+t^2}dt$

$=2\int \arctan t\cdot(\arctan t)'\,dt$

$=2\int \arctan t\, d\arctan t$

$=\arctan^2 t+C$

$=\arctan^2\sqrt{x}+C$