五. 微分形式

1. 微分形式

l次形式(l形式)　　$\omega_{a_1\cdots a_l} = \omega_{[a_1\cdots a_l]}$　(写作 ω)

L-form

1) 分量形式　$\omega_{a_1\cdots a_l} = \omega_{[a_1\cdots a_l]} \Rightarrow \forall$基底, $\omega_{\mu_1\cdots\mu_l} = \omega_{[\mu_1\cdots\mu_l]}$　基底: $\omega_{a_1\cdots a_l} = \sum_C \omega_{\mu_1\cdots\mu_l}(e^{\mu_1})_{a_1}\wedge(e^{\mu_2})_{a_2}\cdots$

$\exists$基底, $\omega_{\mu_1\cdots\mu_l} = \omega_{[\mu_1\cdots\mu_l]} \Rightarrow \omega_{a_1\cdots a_l} = \omega_{[a_1\cdots a_l]}$

2) 赤维塔符号　$\omega_{a_1\cdots a_l} = \delta_\pi \omega_{\pi(1)\cdots\pi(l)}$,　$\forall$基底 $\omega_{\mu_1\cdots\mu_l} = \delta_\pi \omega_{\pi(1)\cdots\pi(l)}$

└ 重复指标为0

V上全体l形式记为 $\Lambda(l)$,　$\Lambda(1) = V^*$ (对偶矢)　$\Lambda(0) = R$

$\dim \Lambda(l) = C_n^l$　$\Lambda(l) \subset V_n$　在V_n上选择l个分量作形式

楔形积: $(\omega\wedge\mu)_{a_1\cdots a_l b_1\cdots b_m} := \frac{(l+m)!}{l!m!}\omega_{[a_1\cdots a_l}\mu_{b_1\cdots b_m]}$, $\omega\in\Lambda(l)$, $\mu\in\Lambda(m)$,

wedge product　$\wedge: \Lambda(l)\times\Lambda(m)\to\Lambda(l+m)$

1) 不交换律: $\omega\wedge\mu = (-1)^{lm}\mu\wedge\omega$

2) 结合律: $(\omega\wedge\mu)\wedge\nu = \omega\wedge(\mu\wedge\nu)$

3) 分配律(线性): $(\alpha\omega+\beta\mu)\wedge\nu = \alpha(\omega\wedge\nu)+\beta(\mu\wedge\nu)$

外微分算符：

exterior differentiation operator

$$(d\omega)_{ba_1\cdots a_l} := (l+1)\nabla_{[b}\omega_{a_1\cdots a_l]}$$

1) 对Λ(0)用导数 $(df)_a = \nabla_a f$

2) 对 $\tilde{\nabla}_a$ 不限形式 $\nabla_{[b}\omega_{a_1\cdots a_l]} = \tilde{\nabla}_{[b}\omega_{a_1\cdots a_l]}$ ($C^c{}_{ab} = C^c{}_{ba} \to 0$)

3) 对分量 $(d\omega)_{ba_1\cdots a_l} = \sum_c (d\omega_{\mu_1\cdots\mu_l})_b \wedge (e^{\mu_1})_{a_1}\cdots\cdots$ (用 ∇_a 为 ∂_a 去证

4) $d\circ d = 0$ Pf： 用 $\partial_{[a}\partial_{b]}T^{\cdots}\cdots = 0$

闭的： M上 $\omega \in \Lambda(l)$， $d\omega = 0$

恰当的： $\exists \mu \in \Lambda(l-1)$， $d\mu = \omega$

当M局部平凡，则：ω恰当→ω闭

流形满足

$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \rightleftharpoons df = Xdx + Ydy$

2、流形上积分

流形可定向： n维流形M上存在 C^0 且处处非0的n维形式场 ε

orientable

实质：单号

莫比乌斯带不可定向

对 $\varepsilon_1, \varepsilon_2$，若∃函数 $h>0$，使 $\varepsilon_1 = h\varepsilon_2$ ($\varepsilon_1, \varepsilon_2$ 为 C^0，处处非0)

右手的： $\exists h>0$， $\varepsilon = h(e^1)_{a_1}\wedge\cdots(e^n)_{a_n}$ → 对偶基矢展开 $\omega = \omega_{1\cdots n}(x^1\cdots x^n) \cdot dx^1\wedge dx^2\cdots dx^n$

左手： $h<0$

积分： $\int_G \omega := \underbrace{\int_{\psi[G]}}_{\text{映射到}R^n\text{的坐标点}} \omega_{1\cdots n}(x^1\cdots x^n)dx^1\cdots dx^n$

右手

$\int_G: \Lambda(n) \to \int_n R^n \to R$

坐标系选择无关性： $\omega = \omega_{a_1a_2\cdots}\, dx^1\wedge dx^2\cdots = \omega_{b_1b_2\cdots}\, dx'^1 dx'^2\cdots$ —与形式场特点有关

$\omega_{b_1b_2\cdots} = \omega_{a_1a_2\cdots}\frac{\partial(x^1, x^2, x\cdots)}{\partial(x'^1, x'^2, \cdots)}$ 满足积分换元 (雅克比)

└ 左手系：$\int_G \omega := -\int_{\psi[G]} \omega_{1\cdots n}(x^1\cdots x^n)\,dx^1\cdots dx^n$

1) 积分的正负取决于流形的定向，定向变则变号

限制 $\tilde{\mu}_{a_1\cdots a_l}$：　$\mu_{a_1\cdots a_l} \in \Lambda(l)$，在 $\phi[S]$ 上，对 $\forall q \in \phi[S]$，$w_1{}^a \cdots w_l{}^a \in W_q$，$\tilde{\mu}$ 是 $\phi[S]$ 上形式场

restriction

$$\tilde{\mu}_{a_1\cdots a_l}|_q (w_1)^{a_1}\cdots(w_l)^{a_l} = \mu_{a_1\cdots a_l}|_q (w_1)^{a_1}\cdots(w_l)^{a_l}$$

μ 虽是 l 维，但并不一定全在 $\phi[S]$ 上，这里可以看作是 μ 在 $\phi[S]$ 上的投影

3. Stokes定理

带边流形　　边界记为 ∂N，为 $n-1$ 维流形

manifold with boundary　　内部　为 $i(N) = N - \partial N$

Stokes定理：n 维定向流形 M 的紧致子集 n 维带边流形 N，M 上 $n-1$ 形式场，至少 C^1 ω

$$\int_{i(N)} d\omega = \int_{\partial N} \omega$$

证明被略

4. 体元

体元： n维可定向流形上任一 C^0，处处非0的形式场 ε

volume element

对于可定向连通流形，定向只有2个，体元无限

+度规 g^{ab} ： $\varepsilon^{b_1\cdots b_L} \equiv \varepsilon_{a_1\cdots a_L}\cdot g^{a_1b_1}\cdots g^{a_Lb_L} \rightarrow \varepsilon^{a_1\cdots a_L}\varepsilon_{a_1\cdots a_L} = n!(\varepsilon_{12\cdots L})^2\cdot(-1)^s$

（g^{ab}特征值为负的个数）

$\int_N \varepsilon$ = 体积

对 g^{ab} 的正交归一基底 $\{(e^\mu)_a\}$ 若 $\varepsilon_{1\cdots n}=\pm1$ （ $\varepsilon_{a_1\cdots a_n}=\pm(e^1)_{a_1}\wedge(e^2)_{a_2}\cdots$ ）

$\varepsilon^{a_1\cdots}\varepsilon_{a_1\cdots}=(-1)^s n!$

与度规适配的体元

对基底 $\{(e^\mu)_a\}$ $\varepsilon_{a_1\cdots a_n}=\pm\sqrt{|g|}\,(e^1)_{a_1}\wedge\cdots\wedge(e^n)_{a_n}$ ，g 为 g_{ab} 在 $\{(e^\mu)_a\}$ 分量的行列式

$|g|\rightarrow$ 绝对值

pf: $(-1)^s n! = \varepsilon^{a_1\cdots a_n}\varepsilon_{a_1\cdots a_n} = g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\cdot\varepsilon_{\nu_1\cdots\nu_n}\varepsilon_{\mu_1\cdots\mu_n}$

$\left(\begin{matrix}0 & \text{重复}\\ 1 & \text{偶排列}\\ -1 & \text{奇排列}\end{matrix}\right) = \hat{\varepsilon}_{\mu_1\cdots\mu_n}$ 为 Levi-Civita 记号，

对适配体元： $\varepsilon_{\mu_1\cdots\mu_n}=\varepsilon_{123\cdots n}\cdot\hat{\varepsilon}_{\mu_1\cdots\mu_n}$

$= \sum_{\pi(\mu_i)}\sum_{\pi(\nu_i)} g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\cdot\varepsilon_{\nu_1\cdots\nu_n}\varepsilon_{\mu_1\cdots\mu_n}$

$= \sum_{\pi(\nu_i)} g^{1\nu_1}\cdots g^{n\nu_n}\cdot\varepsilon_{123\cdots n}\varepsilon_{\nu_1\cdots\nu_n} + \cdots\cdots$

$= (\varepsilon_{12\cdots n})^2\cdot\sum_\pi\sum_\pi\hat{\varepsilon}_\mu\prod g^{\mu\nu}\cdot \underbrace{n!}$ (n!个式子)

$= (\varepsilon_{12\cdots n})^2\det(g^{\mu\nu})\cdot n!$

$g^{\mu\nu}$ 与 $g_{\mu\nu}$ 互逆 $\rightarrow \det(g^{\mu\nu}) = \dfrac{1}{\det(g_{\mu\nu})} = \dfrac{1}{g}$

$(-1)^s = \dfrac{1}{g}(\varepsilon_{123\cdots n})^2 \rightarrow \varepsilon_{a_1\cdots a_n}=\pm\sqrt{|g|}\,(e^1)_{a_1}\wedge(e^2)_{a_2}\cdots$

1) 导数算符 $\nabla_a g^{ab}=0 \rightarrow \nabla_a\varepsilon_{a_1\cdots a_L}=0$

引理：对克氏符 $\delta^i{}_j = \begin{cases}1\\0\end{cases}$，　$\delta^{[a_1}{}_{a_1} \cdots \delta^{a_j}{}_{a_j}\, \delta^{a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n} = \frac{(n-j)!\,j!}{n!}\, \delta^{[a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n}$

pf：$\delta^{[a_1}{}_{a_1} \delta^{a_2}{}_{b_2} \cdots \delta^{a_n]}{}_{b_n} = \frac{1}{n}\, \delta^{[a_2}{}_{b_2} \cdots \delta^{a_n]}{}_{b_n}$　（$\delta^{a_1}{}_{a_1} = 1$，$\delta^{a_i}{}_{a_1} = 0$）

$\downarrow$

$\delta^{[a_2}{}_{a_2} \delta^{a_3}{}_{b_3} \cdots \delta^{a_n]}{}_{b_n} = \frac{2}{n-1}\, \delta^{[a_3}{}_{b_3} \cdots \delta^{a_n]}{}_{b_n}$　（a_2 可取2值　）

$\downarrow$

$\delta^{[a_j}{}_{a_j}\, \delta^{a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n} = \frac{j}{n-j+1}\, \delta^{[a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n} \to$ 原式

$$\varepsilon^{a_1 \cdots a_j a_{j+1} \cdots a_n}\, \varepsilon_{a_1 \cdots a_j b_{j+1} \cdots b_j} = (-1)^s (n-j)!\; \delta^{[a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n}$$

Pf：$\varepsilon^{a_1 \cdots a_n}\, \varepsilon_{b_1 \cdots b_n} = K\, \delta^{[a_1}{}_{b_1} \delta \cdots \delta^{a_n]}{}_{b_n}$　（同为一维矢量）

$\downarrow$

$\varepsilon^{a_1 \cdots a_n} \varepsilon_{b_1 \cdots b_n}\, \varepsilon_{a_1 \cdots a_n} \varepsilon^{b_1 \cdots b_n} = K \cdot (-1)^s n! \to K = (-1)^s n! \to \varepsilon^{a_1 \cdots a_n} \varepsilon_{b_1 \cdots b_n} = (-1)^s n!\, \delta^{[a_1}{}_{b_1} \delta^{a_2}{}_{b_2} \cdots \delta^{a_n]}{}_{b_n}$

$\downarrow$

加缩并：$\varepsilon^{a_1 \cdots a_j a_{j+1} \cdots a_n}\, \varepsilon_{a_1 \cdots a_j b_{j+1} \cdots b_j} = (-1)^s (n-j)!\; \delta^{[a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n}$

5. 函数积分：$\int_M f := \int_M f\varepsilon$

Guess定理：$\int_{i(N)} (\nabla_b v^b)\, \varepsilon = \int_{\partial N} v^b \varepsilon_{b a_1 \cdots a_{n-1}}$

pf：$\omega \equiv v^b \varepsilon_{b a_1 \cdots a_{n-1}}$ 的外微分 $d\omega = n \nabla_{[c}\, v^b \varepsilon_{|b| a_1 \cdots a_{n-1}]} = h\, \varepsilon_{c a_1 \cdots a_{n-1}}$ 与 $\varepsilon^{c a_1 \cdots a_{n-1}}$ 缩并

$= h \cdot (-1)^s n!$

$\downarrow$

$\varepsilon^{c a_1 \cdots a_{n-1}} \cdot d\omega = n\, \varepsilon^{[c a_1 \cdots a_{n-1}]} \nabla_c v^b \varepsilon_{b a_1 \cdots a_{n-1}}$

$= n\, \varepsilon^{[c a_1 \cdots a_{n-1}]} \varepsilon_{b a_1 \cdots a_{n-1}} \nabla_c v^b = (-1)^s \cdot n \cdot (n-1)! \cdot \delta_b{}^c \nabla_c v^b = (-1)^s \cdot n! \cdot \nabla_b v^b \to h = \nabla_b v^b$

见上框

$\downarrow$

$\int_{i(N)} (\nabla_b v^b)\, \varepsilon = \int_{\partial N} v^b \varepsilon_{b a_1 \cdots a_{n-1}}$

∂N 非类光超曲面时：$n^a n_a = \pm 1$

∂N 的诱导体元 $\hat{\varepsilon}_{a_1 \cdots a_{n-1}} = n^b \varepsilon_{b a_1 \cdots a_{n-1}}$

$$\downarrow$$

$$\hat{\varepsilon}^{a_1 \cdots a_{n-1}} \hat{\varepsilon}_{a_1 \cdots a_{n-1}} = (-1)^{\hat{s}} (n-1)!$$

∂N 为类光超曲面：$n^a n_a = 0$

?

另一形式 Guess：$\int_{i(N)} (\nabla_a v^a)\, \varepsilon = \pm \int_{\partial N} v^a n_a \hat{\varepsilon}$

pf：只需证：$\int_{\partial N} v^b \varepsilon_{b a_1 \cdots a_{n-1}} = \int_{\partial N} v^b n_a \hat{\varepsilon}$

↓限制积分：

6. 对偶微分形式

对偶微分形式 $^{*}\omega \in \Lambda_M(n-l)$, $\omega \in \Lambda_M(l)$

dual form $^{*}\omega_{a_1\cdots a_{n-l}} := \frac{1}{l!}\omega^{b_1\cdots b_l}\varepsilon_{b_1\cdots b_l a_1\cdots a_{n-l}}$, $\omega^{b_1\cdots b_l} = g^{a_1b_1}g^{a_2b_2}\cdots\cdots\omega_{a_1\cdots a_l}$

Hodge star $*: \Lambda_M(l) \to \Lambda_M(n-l)$

对标量: $^{*}f_{a_1\cdots a_n} = \frac{1}{0!}f\varepsilon_{a_1\cdots a_n} = f\varepsilon_{a_1\cdots a_n}$

1) $^{*}(^{*}f) = (-1)^{s}f$ easy to prove

$^{**}\omega = (-1)^{s+l(n-l)}\omega$

R^3的向量积: $\vec{A}\cdot\vec{B} = A_aB^a$

$\vec{A}\times\vec{B} = {}^{*}\omega_c = \varepsilon_{abc}A^aB^b$

$\vec{\nabla}f$: $\partial_a f$

$\nabla^2 A = \partial_a\partial^a A^b$

$\vec{\nabla}\cdot\vec{A} = \partial_a A^a$

$\vec{\nabla}\times\vec{A} = \varepsilon^{abc}\partial_a A_b$

$\vec{\nabla}\cdot(\vec{A}\vec{B}) = \partial_a(A^aB^b)$

$\vec{\nabla}\vec{A} = \partial_a A^b$

$\nabla^2 f = \partial_a\partial^a f$

7. 标架:

联络系数 $\gamma^{\sigma}{}_{\mu\tau}$: 基矢场移动 $(e_\tau)^b\nabla_b(e_\mu)^a = \gamma^{\sigma}{}_{\mu\tau}(e_\sigma)^a$ $\Rightarrow$ $\gamma^{\sigma}{}_{\mu\tau} = (e^\sigma)_a(e_\tau)^b\nabla_b(e_\mu)^a$

connection coefficients

(克氏符 $\Gamma^{\sigma}{}_{\mu\tau}$ 定义于坐标基底: $(\partial/\partial x^\tau)^b\nabla_b(\partial/\partial x^\mu)^a = \Gamma^{\sigma}{}_{\mu\tau}(\partial/\partial x^\sigma)^a$

联络1形式 $\omega_\mu{}^\nu{}_a := -\gamma^{\nu}{}_{\mu\tau}(e^\tau)_a$ $-\gamma^{\nu}{}_{\mu i}$: 为其第i个分量

connection-1-form

$$\omega_\mu{}^\nu{}_a = -(e^\nu)_c\underbrace{(e_\tau)^b\nabla_b(e_\mu)^c}\cdot\underbrace{(e^\tau)_a}_{1\text{形式}} \quad (\delta^b{}_a\nabla_b = \nabla_a) \quad = -(e^\nu)_c\nabla_a(e_\mu)^c = -\overbrace{\nabla_a[(e^\nu)_c(e_\mu)^c]}^{\nabla_a\delta^\nu{}_\mu} + (e_\mu)^c\nabla_a(e^\nu)_c$$

$= (e_\mu)^c \nabla_a (e^\nu)_c$

基底又称为标架 (有时专指非坐标基底)

frame, 4维专称 tetrad

嘉当第一结构方程 1) $\vec{d}e^\nu = -\vec{e}^\mu \wedge \vec{\omega}_\mu{}^\nu$ (此处为形式场)

Cartan

pf: $-(e^\mu)_a \wedge \omega_\mu{}^\nu{}_a = -(e^\mu)_a \wedge [(e_\mu)^c \nabla_b (e^\nu)_c] = -2(e^\mu)_{[a}(e_\mu)^c \nabla_{b]}(e^\nu)_c = -2\delta^c{}_{[a}\nabla_{b]}(e^\nu)_c$
$= -2\nabla_{[b}(e^\nu)_{a]} = de^\nu$

计算黎曼曲率张量: $R_{ab\mu}{}^\nu = R_{abc}{}^d (e_\mu)^c (e^\nu)_d$ → 记作形式场 $\vec{R}_\mu{}^\nu$ 的分量 $(\vec{R}_\mu{}^\nu)_{ab}$ $\quad \vec{R}_\mu{}^\nu = \frac{1}{2} R_{\rho\sigma\mu}{}^\nu e^\rho \wedge e^\sigma$

2) $R_{ab\mu}{}^\nu = -R_{ba\mu}{}^\nu$

嘉当第二结构方程 2) $R_\mu{}^\nu = d\omega_\mu{}^\nu + \omega_\mu{}^\lambda \wedge \omega_\lambda{}^\nu$

pf: $R_{ab\mu}{}^\nu = 2(e_\mu)^c \nabla_{[a}\nabla_{b]}(e^\nu)_c$, $\quad (e_\mu)^c \nabla_a \nabla_b (e_\nu)_c = \nabla_a[(e_\mu)^c \nabla_b (e^\nu)_c] - \nabla_b (e^\nu)_c \cdot \nabla_a (e_\mu)^c$
$= \nabla_a \omega_\mu{}^\nu{}_b - \nabla_b (e^\nu)_c \cdot \underset{\sim}{\delta^c_d} \nabla_a (e_\mu)^d$
$= \nabla_a \omega_\mu{}^\nu{}_b - (e_\lambda)^c \nabla_b (e^\nu)_c \cdot (e^\lambda)_d \nabla_a e_\mu{}^d$
$= \nabla_a \omega_\mu{}^\nu{}_b - \omega_\mu{}^\lambda{}_a \cdot \omega_\lambda{}^\nu{}_b$

$R_{ab\mu}{}^\nu = 2(\nabla_{[a}\omega_\mu{}^\nu{}_{b]} - \omega_\mu{}^\lambda{}_{[a}\omega_\lambda{}^\nu{}_{b]})$
$= (d\vec{\omega}_\mu{}^\nu - \vec{\omega}_\mu{}^\lambda \wedge \vec{\omega}_\lambda{}^\nu)_{ab}$

此时须有 ∇_a 无挠

∇_a 与度规适配: $\nabla_a g_{bc} = 0$: $\quad g_{\mu\nu} = g_{bc} \cdot (e_\mu)^b (e_\nu)^c$

$\xrightarrow{\text{记号}} (e_\mu)_a \equiv g_{ab}(e_\mu)^b$

$(e^\mu)_a = g^{\mu\nu}(e_\nu)_a$, $(e_\mu)^a = g_{\mu\nu}(e^\nu)^a$

$\omega_{\mu\nu a} := g_{\lambda\nu} \cdot \omega_\mu{}^\lambda{}_a$

刚性标架: $\nabla_a g_{\mu\nu} = 0$ ($g_{\mu\nu}$ 为常数, 不同于 $\nabla_a g_{bc} = 0$, 如 $\eta^\mu{}_\nu$)

rigid frame

1) 使 $\omega_{\mu\nu a} = (e_\mu)_b \nabla_a (e_\nu)^b$

2) $\omega_{\mu\nu a} = -\omega_{\nu\mu a}$ $\quad (\nabla_a((e_\mu)_b (e_\nu)^b) = 0)$

里奇旋转系数:

Ricci rotation coefficients

用标架算曲率: 1) 选标架 2) 算 $\vec{\omega}_\mu{}^\nu$ 3) 嘉当2方程得 $R_\mu{}^\nu$

设 $(e_\nu)_\lambda$ 为 $(e_\nu)_a$ 在 $\{x^\mu\}$ 上的 λ 分量 $(e_\nu)_\lambda = (e_\nu)_a \cdot (\partial/\partial x^\lambda)^a$

得 $\Lambda_{\mu\nu\rho} \equiv [(e_\nu)_{\lambda,\tau} - (e_\nu)_{\tau,\lambda}](e_\mu)^\lambda (e_\rho)^\tau$ $\quad (\Lambda_{\mu\nu\rho} = -\Lambda_{\rho\nu\mu})$

↳ $\omega_{\mu\nu\rho} = \frac{1}{2}(\Lambda_{\mu\nu\rho} + \Lambda_{\rho\mu\nu} - \Lambda_{\nu\rho\mu})$

pf: $\Lambda_{\mu\nu\rho} = [\nabla_a (e_\nu)_b - \nabla_b (e_\nu)_a] \underbrace{(dx^\tau)^a (\partial/\partial x^\lambda)^b (e_\mu)^\tau (e_\rho)^\lambda}_{(e_\mu)^a (e_\rho)^b}$

$= \omega_{\mu\nu\rho} - \omega_{\rho\nu\mu}$