四. 旋量场

1. Weyl 方程

考虑一阶偏微分的场方程

$\partial_0\psi = b^i\partial_i\psi + C\psi \xrightarrow{\partial_0} \partial_0^2\psi = (b^i\partial_i + C)\partial_0\psi = (b^i\partial_i + c)^2\psi = \left[\frac{1}{2}(b^ib^j + b^jb^i)\partial_i\partial_j + 2cb^i\partial_i + c^2\right]\psi$

协变性要求：$[b_i, b_j] = -2g_{ij}$，$C=0$

$C=0$，$\{b^i, b^j\} = -2g^{ij}$（反对易）

有协变性 → $\gamma^\mu\partial_\mu\phi = 0$ d'Alembert方程

$b_i = \pm\sigma_i$，

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$(\sigma^i)^2 = 1$，$[\sigma^i, \sigma^j] = 0$

每个中为2分量波函数$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

Weyl方程：$\partial_0\psi = \pm\sigma^i\partial_i\psi$（2个方程），光速无质量粒子

角动量算符 $s^i = \frac{1}{2}\sigma^i$，$[s^i, s^j] = i\varepsilon^{ij}{}_k s^k$，$s^2 = \frac{3}{4} = \frac{1}{2}(1+\frac{1}{2}) \to$ 自旋为$\frac{1}{2}$

→ 描述$m=0$，自旋为$\frac{1}{2}$的光速粒子

方程的解：$\psi = e^{ik^ix_i} \longrightarrow -k^0 = \pm\vec{\sigma}\cdot\vec{k}$，$k^0 = \pm\omega = \pm|\vec{k}|$ → $\xi = \frac{\vec{\sigma}\cdot\vec{k}}{|\vec{k}|} = \pm 1$ 螺旋性

$\xi = +1 \to$ 自旋与动量同向 → 右旋态

$\xi = -1 \to$ 反向 → 左旋

空间变换：

空间转动：$x^{i\prime} = a^i{}_j x^j$，$(a^i{}_j a^i{}_k = g^i{}_k)$ 场变换 $\psi' = \Lambda\psi$，$\partial_i = a^i{}_i\partial_j'$

这里注意，逆变与协变的转动

$\partial_i x^i = \partial_i' x^{i\prime} = a^k{}_i a^i{}_j \partial_k x^j = \partial_k x^k$

↓

方程：$\Lambda\partial_0\psi = \Lambda\sigma^i\partial_i\Lambda^{-1}\psi' \to \partial_0\psi' = \pm\Lambda\sigma^i\Lambda^{-1}a^j{}_i\partial_j'\psi' \to a^j{}_i\Lambda\sigma^i\Lambda^{-1} = \sigma^j \to \Lambda^{-1}\sigma^i\Lambda = a^i{}_j\sigma^j$

这里保证转动后方程不变

↓ 转动变化

无穷小转动：$a^i{}_j = g^i{}_j + \varepsilon^i{}_{jk}\theta^k \longrightarrow$ 无限小转动矩阵：$\Lambda = 1 + i\varepsilon_i\sigma^i$，$\varepsilon_i$为小量

$\Lambda^{-1}\sigma^i\Lambda = \sigma^i + i\varepsilon_j(\sigma^i\sigma^j - \sigma^j\sigma^i) + o(\varepsilon_i) = \sigma^i - 2\varepsilon_j\epsilon^{ij}{}_k\sigma^k = \sigma^i + \epsilon^{ij}{}_k\sigma^j\theta^k$

$\varepsilon_i = \frac{\theta^i}{2} \to \Lambda = 1 - \frac{i}{2}\vec{\theta}\cdot\vec{\sigma}$

↓ 有限角度 $\vec{\theta} = \vec{n}\theta$；$\sigma_n = \vec{n}\cdot\vec{\sigma}$

$\Lambda = e^{-\frac{i\vec{\theta}\cdot\vec{\sigma}}{2}} = e^{-\frac{i\theta\sigma_n}{2}}$ $\left(\frac{d\Lambda}{d\theta^i} = -\frac{i}{2}\theta^i\sigma_i\right) \to \Lambda = e^{-\frac{i\vec{\theta}\cdot\vec{\sigma}}{2}}$

$= D^{\frac{1}{2}}(\vec{n}, \theta)$ → 三维转动下，中内部的二维空间同转（$\dim SU(2) = 3$）

$\vec{S}=\frac{\vec{\sigma}}{2}$ 为自旋角动量算符 . 中内部转 → 旋量场

时空反演

空间反演P $a^\mu{}_\nu=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix} \xrightarrow{\text{代入方程:}} \partial_0\psi'=\pm\Lambda\sigma^i\Lambda^{-1}\underbrace{a^j{}_i}_{-1}\partial_j\psi'$ $\quad \partial_0\psi=\mp\Lambda\sigma^i\Lambda^{-1}\partial_i\psi$

反射不变条件 ↓

$\Lambda\sigma^i\Lambda^{-1}=-\sigma^i \to [\Lambda,\sigma^i]=0 \to$ 不存在这的Λ

时间 $a^\mu{}_\nu=\begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} \longrightarrow \Lambda\sigma^i\Lambda^{-1}=-\sigma^i \to$ 同上 宇称不守恒

时空同时反演 → 守恒

正反粒子变换C Weyl方程共轭 $\cdot\sigma^2$: $\partial_0\psi_c=\mp\sigma^i\partial_i\psi_c$, $\psi_c=C\psi\equiv\eta_c\sigma^2\psi^*$ → ψ_c与ψ为螺旋性相反的正反粒子

$\sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\quad \sigma^2(\sigma^i)^*=-\sigma^i\sigma^2$

C+P → $\Lambda^{-1}\sigma^i\Lambda=\sigma^i$ → CP后Weyl方程不变 → CP不变性

2. Dirac方程 : 自旋为$\frac{1}{2}$, 有质量 $\xrightarrow{m=0}$ 简化为Weyl

耦合: 1. 左旋态、右旋态可叠加 (相对观看 $\overset{\frac{v}{2}}{\circlearrowleft}\leftarrow v$ → $\overset{右}{\circlearrowleft}$)

$$\begin{cases}(\partial_0+\sigma^i\partial_i)\psi_R=-im\psi_L\\(\partial_0-\sigma^i\partial_i)\psi_L=-im\psi_R\end{cases}\longrightarrow(\partial_0+\sigma^i\partial_i)(\partial_0-\sigma^i\partial_i)=(\partial_0^2+\underbrace{\sigma^i\sigma^j\partial_i\partial_j}_{\text{计j项反对易}})=\partial_\mu\partial^\mu$$

↓ 耦合

$$\partial_\mu\partial^\mu\psi_i=-m^2\psi_i\quad,\ i=R,L$$

Dirac方程 $(\partial_0+\alpha^i\partial_i+im\beta)\psi=0$, $\alpha=\begin{pmatrix}\sigma&0\\0&-\sigma\end{pmatrix}$, $\beta=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. 可有: $\psi=\begin{pmatrix}\psi_R\\\psi_L\end{pmatrix}$ → 内部有一自由度

表象:

$p\gg mc$ → 用Weyl表象 ψ_R,ψ_L

低速/质量大 → Dirac表象 $\psi=\begin{pmatrix}\varphi\\\chi\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\psi_R+\psi_L\\\psi_R-\psi_L\end{pmatrix}$ $\quad u=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ 变换

$\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\alpha=\begin{pmatrix}0&\sigma\\\sigma&0\end{pmatrix}$, $\begin{cases}\partial_0\varphi+\sigma\cdot\nabla\chi+im\varphi=0\\\partial_0\chi+\underbrace{\sigma\cdot\nabla}_{k\to0\text{可略}}\varphi-im\chi=0\end{cases}$ → $k\to0$时, φ正频项, χ负频项

?各种表象出现的原因

态矢有多少性质?

Majorana表象

粒子态 ψ_P，反粒子态 ψ_A，变换：$\psi_A = \hat{C}\psi = \eta_c \sigma^2 \psi_P^*$

4元数与σ矩阵

（未发现有趣性质）

扯上为 $\frac{1}{\sqrt{2}}$?

$\psi = \begin{pmatrix} \psi_A \\ \psi_P \end{pmatrix} = \begin{pmatrix} \eta_c \sigma^2 \psi_P^* \\ \psi_P \end{pmatrix}$，取 $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sigma^2 \\ \sigma^2 & -1 \end{pmatrix}$，$\psi_M = U\psi$，$\psi_M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sigma^2 \\ \sigma^2 & -1 \end{pmatrix}\begin{pmatrix} \eta_c \sigma^2 \psi_P^* \\ \psi_P \end{pmatrix} = \sqrt{2}\begin{pmatrix} i\sigma^2 \mathrm{Im}\,\psi_P \\ -\mathrm{Re}\,\psi_P \end{pmatrix}$

实结果 ψ_M，有：$\alpha_M^i = U\alpha_w^i U^\dagger$，$\beta_M^i = U\beta_w U^\dagger$ → α_M^i 为实，β_M 为虚 → 全方程 $(\partial_0 + \alpha^i\partial_i + im\beta)\psi_M = 0$ 全实数

$\alpha_M^1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}$，$\alpha_M^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$，$\alpha_M^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$，$\beta_M = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$

手征投影算符：

?联系

定义 $P_L = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$，$P_R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ⟶ Dirac 表象 $P_L\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_L \\ -\psi_L \end{pmatrix}$，$P_R\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_R \\ \psi_R \end{pmatrix}$

$P_L^2 = P_L$，$P_R^2 = P_R$；$P_R P_L = P_L P_R = 0$，$P_R + P_L = 1$ → $P_L\psi$ 与 $P_R\psi$ 正交

协变形式：（Dirac表象）

1. Dirac 矩阵为厄米矩阵：$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$，$\{\alpha_i, \beta\} = 0$，$\beta^2 = 1$

γ^μ 是起何作用 ?

2. γ 矩阵：$\gamma^0 = \beta$，$\gamma^i = \beta\alpha^i$，→ $(\gamma^0)^\dagger = \gamma^0$；$(\gamma^i)^\dagger = -\gamma^i$ （$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$，$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$）

1. 形式上 使方程保持协变性

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ 相当于一个"体元" ?

2. 每一个 γ^μ 对应着 ∂_μ

3. 定义 $\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$，$\{\gamma^5, \gamma^\mu\} = 0$，$(\gamma^5)^2 = 1$

而 $\gamma^\mu A_\mu$ 的实际效果

$P_L = \frac{1}{2}(1 - \gamma^5)$，$P_R = \frac{1}{2}(1 + \gamma^5)$ $P_L\psi$ 与 $P_R\psi$ 为 γ^5 本征态 → $\gamma^5 P_L\psi = -P_L\psi$，$\gamma^5 P_R\psi = P_R\psi$ → γ^5 称手征算符

$(i\gamma^\mu\partial_\mu - m)\psi = 0$，共轭：$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$，$\bar{\psi} = \psi^\dagger\gamma^0$ （Dirac 共轭）

Feynman 符号：γ 与 4-矢量相乘：$\gamma^\mu\partial_\mu = \not{\partial}$，$\gamma^\mu k_\mu = \not{k}$，$\gamma^\mu p_\mu = \not{p}$

$(i\not{\partial} - m)\psi = 0$

平面波解：Dirac 表象的 φ 与 χ 形式同 K-G 方程 ⟶ 设 $\psi(x) = W \cdot e^{-ik^\mu x_\mu}$，$k^0 = \sqrt{\vec{k}^2 + m^2} = \omega$，$W = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

代入：$(\not{k} - m)\psi = (\gamma^0 k_0 + \gamma^i k_i - m)W = 0$ → $\begin{pmatrix} m - k_0 & \vec{\sigma}\cdot\vec{k} \\ -\vec{\sigma}\cdot\vec{k} & m + k_0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0$

当 W 为螺旋度 s 本征态 → $\vec{\sigma}\cdot\vec{k} \Rightarrow s|\vec{k}|$

正频 $u(k, s) = W(\omega, k, s)$

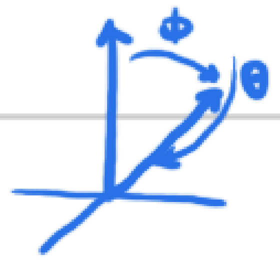
负频 $v(k, s) = W(-\omega, -k, s)$

共轭

$(\gamma^i)^\dagger = -\gamma^i \longrightarrow \quad w^\dagger(\gamma^0 k_0 - \gamma^i k_i - m) = 0$

$\downarrow$

$w^\dagger(式)w + w^\dagger(式)w \longrightarrow w^\dagger(\omega, k, s)\gamma^0 w(\omega, k, s') = \frac{m}{k^0} w^\dagger(\omega, k, s)\, w(\omega, k, s')$ $(\bar{w} = w^\dagger \gamma^0)$



空间转动：

$D^{\frac{1}{2}}(n_z \to n_k) = D^{\frac{1}{2}}(n_z, \phi)\, D^{\frac{1}{2}}(n_y, \theta) = e^{-i\phi\sigma^3/2} \cdot e^{-i\theta\sigma^2/2}$

$= \begin{pmatrix} e^{-\frac{i\phi}{2}}\cos\frac{\theta}{2} & -e^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ e^{\frac{i\phi}{2}}\sin\frac{\theta}{2} & e^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}$

ξ 2个本征态：$\xi_+ = D^{\frac{1}{2}}(n_z \to n_k)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\frac{i\theta}{2}}\cos\frac{\theta}{2} \\ e^{\frac{i\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}$

$\xi_- = D^{\frac{1}{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-\frac{i\phi}{2}}\sin\frac{\theta}{2} \\ e^{\frac{i\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}$

分离正负频：$\bar{u}u = \frac{m}{k^0} u^\dagger u$; $\bar{v}v = -\frac{m}{k^0} v^\dagger v \rightarrow u(k, s) = N\begin{pmatrix} \zeta \\ \frac{\vec{\sigma}\cdot\vec{k}}{\omega+m}\zeta \end{pmatrix}$; $v(k, s) = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{\omega+m}\eta \\ \eta \end{pmatrix}$

归一化：$u^\dagger(k, s)u(k, s') = 2\omega \cdot \delta_{ss'}$; $v^\dagger(k, s)v(k, s') = 2\omega\delta_{ss'}$

$\downarrow$

$N = \sqrt{\omega + m}$ $(\xi^\dagger\xi = \eta^\dagger\eta = 1$ (归一的值)$)$

$\downarrow$

正交归一关系

$$\begin{cases} \bar{u}(k, s)u(k, s') = (\omega+m)\left(1 - \frac{\omega^2 - m^2}{(\omega+m)^2}\right)\delta_{ss'} = 2m\delta_{ss'} \\ \bar{v}(k, s)v(k, s') = (\omega+m)\left(\frac{\omega^2 - m^2}{(\omega+m)^2} - 1\right)\delta_{ss'} = -2m\delta_{ss'} \\ \bar{u}v = \bar{v}u = 0 \end{cases}$$

$\downarrow$

投影算符：$P_+ = \frac{1}{2m}\sum_s u(k, s)\bar{u}(k, s)$, $P_- = -\frac{1}{2m}\sum_s v(k, s)\bar{v}(k, s)$ $\Longrightarrow$ 代入u,v表达式：$P_+ = \frac{\omega+m}{2m}\sum \begin{pmatrix} \zeta \\ \frac{\vec{\sigma}\cdot\vec{k}}{\omega+m}\zeta \end{pmatrix}\left(\zeta^\dagger, -\frac{\vec{\sigma}\cdot\vec{k}}{\omega+m}\zeta^\dagger\right)$

$= \frac{1}{2m}\sum\begin{pmatrix} \omega+m\,\xi\xi^\dagger & -\xi\xi^\dagger(\sigma\cdot k) \\ \sigma\cdot k\,\xi\xi^\dagger & -\sigma\cdot k\,\xi\xi^\dagger\frac{\sigma\cdot k}{\omega+m} \end{pmatrix} = \frac{\gamma^\mu k_\mu + m}{2m} = \frac{\not{k} + m}{2m}$

$\uparrow$

$\sum\xi\xi^\dagger = \xi_+\xi_+^\dagger + \xi_-\xi_-^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$P_- = \frac{-\not{k} + m}{2m}$

$\downarrow$

$P_+ u(s) = u(s)$, $P_- v(s) = v(s)$, else $= 0$

$(P_+)^2 = P_+$, $(P_-)^2 = P_-$; $P_+P_- = P_-P_+ = 0$; $P_+ + P_- = 1$

u,v完备性：$\frac{1}{2m}\sum_s u\bar{u} - v\bar{v} = 1$

?门内空间？

与 $\mathbb{C}^n$ 关，实数内积空间

3. Dirac方程变换性质

1. 相对论协变：

Lorentz变换：$x'^{\mu} = a^{\mu}{}_{\nu} x^{\nu}$，$\psi' = \Lambda \psi \Rightarrow \partial_{\mu} = a^{\nu}{}_{\mu} \partial'_{\nu}$

$(i \Lambda \gamma^{\mu} \Lambda^{-1} a^{\nu}{}_{\mu} \partial'_{\nu} - m)\psi' = 0 \xrightarrow{\text{形式不变}} \underline{\underline{\Lambda \gamma^{\mu} \Lambda^{-1} a^{\nu}{}_{\mu} = \gamma^{\nu}}}$

无限小正规变换：$a^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}$，正规：$\langle av, av\rangle = \langle v, v\rangle$ $a_{\lambda}{}^{\mu} a^{\lambda}{}_{\nu} = g^{\mu}{}_{\nu} \rightarrow \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$

设 $\Lambda = 1 + a\,\varepsilon_{\mu\nu}\gamma^{\mu}\gamma^{\nu} \rightarrow$ 代入：$a\,\varepsilon_{\nu\lambda}(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda} - \gamma^{\nu}\gamma^{\lambda}\gamma^{\mu}) = \varepsilon^{\mu}{}_{\nu}\gamma^{\nu}$

$\varepsilon_{\mu\nu}$反对称 + γ反对易：$a\,\varepsilon_{\nu\lambda}(\underbrace{-\gamma^{\nu}\gamma^{\mu}\gamma^{\lambda} - \gamma^{\nu}\gamma^{\lambda}\gamma^{\mu}}_{-\gamma^{\nu}\{\gamma^{\mu},\gamma^{\lambda}\}} + \{\gamma^{\mu},\gamma^{\nu}\}\gamma^{\lambda}) = -2a\,\varepsilon_{\nu}{}^{\mu}\gamma^{\nu} + 2a\,\varepsilon^{\mu}{}_{\lambda}\gamma^{\lambda}$

$= 4a\,\varepsilon^{\mu}{}_{\nu}\gamma^{\nu} = \varepsilon^{\mu}{}_{\nu}\gamma^{\nu}$

$a = \frac{1}{4}$

$a = \frac{1}{4}$

$\Lambda = 1 + \frac{1}{8}\varepsilon_{\mu\nu}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$

双线性协变： Dirac共轭 $\bar{\psi} = \psi^{\dagger}\gamma^{0}$，$\Lambda^{\dagger} = \gamma^{0}\Lambda^{-1}\gamma^{0} \rightarrow \bar{\psi}' = \psi'^{\dagger}\gamma^{0} = (\Lambda\psi)^{\dagger}\gamma^{0} = \psi^{\dagger}\Lambda^{\dagger}\gamma^{0} = \bar{\psi}\Lambda^{-1}$

γ线性 + ψ线性组合

$\bar{\psi}\psi$	$\bar{\psi}\gamma^{\mu}\psi$	$\bar{\psi}\gamma^{\mu}\gamma^{\nu}\psi$	$\bar{\psi}\gamma^{5}\psi$	$\bar{\psi}\gamma^{5}\gamma^{\mu}\psi$
标量	矢量	反对称张量	赝标量	赝矢量

Pf: 1) $\bar{\psi}'\psi' = \bar{\psi}\Lambda^{-1}\Lambda\psi = \bar{\psi}\psi$　2) $\bar{\psi}'\gamma^{\mu}\psi' = \bar{\psi}\Lambda^{-1}\gamma^{\mu}\Lambda\psi = a^{\mu}{}_{\nu}\bar{\psi}\gamma^{\nu}\psi$　3) $\bar{\psi}'\gamma^{\mu}\gamma^{\nu}\psi' = \bar{\psi}\Lambda^{-1}\gamma^{\mu}\Lambda\Lambda^{-1}\gamma^{\nu}\Lambda\psi = a^{\mu}{}_{\lambda}a^{\nu}{}_{\rho}\bar{\psi}\gamma^{\lambda}\gamma^{\rho}\psi$

4) $\bar{\psi}'\gamma^{5}\psi' = \bar{\psi}\Lambda^{-1}\gamma^{5}\Lambda\psi = \bar{\psi}\gamma^{5}\psi$，5) 同 2)

流矢量：$j^{\mu} = q\cdot\bar{\psi}\gamma^{\mu}\psi \rightarrow \partial_{\mu}j^{\mu} = 0$ 为标量，L变换不变

$j^{0} = q\bar{\psi}\gamma^{0}\psi = q\psi^{\dagger}\psi = q$；$j^{i} = q\bar{\psi}\gamma^{i}\psi = q\psi^{\dagger}\gamma^{0}\gamma^{i}\psi = q\psi^{\dagger}\beta\beta\alpha^{i}\psi = q\psi^{\dagger}\alpha^{i}\psi$

2. 时空对称性：

空间转动：$\varepsilon_{ij} = \varepsilon_{ijk}\theta^{k}$，同前 $\rightarrow \Lambda = 1 - \frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}$，$\Sigma^{i} = -\frac{i}{2}\varepsilon^{i}{}_{jk}\gamma^{j}\gamma^{k} = \begin{pmatrix}\sigma^{i} & 0\\ 0 & \sigma^{i}\end{pmatrix}$

$[\Sigma^{i}, \Sigma^{j}] = -2g^{ij}$，$[S^{i}, S^{j}] = i\varepsilon^{ij}{}_{k}S^{k}$

空间反射：$a^{\mu}{}_{\nu} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$，$\det(a^{\mu}{}_{\nu}) = -1 \rightarrow$ 非正规L变换

$\Lambda^{-1}\gamma^{0}\Lambda = \gamma^{0}$；$\Lambda^{-1}\gamma^{i}\Lambda = -\gamma^{i} \rightarrow P = \Lambda = \eta_{P}\gamma^{0}$，$\eta_{P}^{*}\eta_{P} = 1 \rightarrow$ 空间反射宇称守恒

?コ内部空间与外部空间耦合？

$\bar\psi\psi$	$\bar\psi\gamma^\mu\psi$	$\bar\psi\gamma^\mu\gamma^\nu\psi$	$\bar\psi\gamma^5\psi$	$\bar\psi\gamma^5\gamma^\mu\psi$
标量	矢量	反对称张量	赝标量	赝矢量

1) 略． 2) $\bar\psi\Lambda^{-1}\gamma^\mu\Lambda\psi = \begin{cases} \mu=0 \to \text{同上} \\ \mu=i \to -\bar\psi\gamma^i\psi \end{cases}$ 3) 同2)

4) $\bar\psi'\gamma^5\psi' = \bar\psi\Lambda^{-1}\gamma^5\Lambda\psi = -\bar\psi\gamma^5\psi$

5) $\bar\psi'\gamma^5\gamma^\mu\psi' = \bar\psi\Lambda^{-1}\gamma^5\Lambda\Lambda^{-1}\gamma^\mu\Lambda\psi = \begin{cases} \mu=0 & -\bar\psi\gamma^5\gamma^0\psi \\ \mu=i & \bar\psi\gamma^5\gamma^0\psi \end{cases}$

时间反演：$a^\mu{}_\nu = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$，Dirac方程复共轭：$(-i\gamma^{\mu *}\partial_\mu - m)\psi^* = 0$

$$[i(\gamma^0\partial_0 - \gamma^1\partial_1 + \gamma^2\partial_2 - \gamma^3\partial_3) - m]\psi^* = 0 \to [i(\Lambda\gamma^0\Lambda^{-1}\partial_0' - \Lambda\gamma^1\Lambda^{-1}\partial_1' + \Lambda\gamma^2\Lambda^{-1}\partial_2' - \Lambda\gamma^3\Lambda^{-1}\partial_3') - m]\Lambda\psi^* = 0$$

η_c的含义？コ

$$\begin{matrix} \Lambda\gamma^0\Lambda^{-1} = \gamma^0 & \Lambda\gamma^1\Lambda^{-1} = -\gamma^1 \\ \Lambda\gamma^2\Lambda^{-1} = \gamma^2 & \Lambda\gamma^3\Lambda^{-1} = -\gamma^3 \end{matrix} \Rightarrow \Lambda = \eta_T\gamma^1\gamma^3,\quad \eta_T^*\eta_T = 1. \quad T\psi = \Lambda\psi^*$$

↳反幺正

3.正反粒子变换：

$\psi_c = C\psi = \eta_c\gamma^2\psi^*$，若$\eta_c = i \to \eta_c\gamma^2$幺正　　$\gamma^{2*} = -\gamma^2$, else $\gamma^{\nu *} = \gamma^\nu$

引入规范场$A_\mu \to$ Dirac方程：$[\gamma^\mu(i\partial_\mu - qA_\mu) - m] = 0$；复共轭：$[\gamma^{\mu *}(-i\partial_\mu - qA_\mu) - m]\psi = 0$

故$\eta_c\gamma^2$作用：$[\gamma^\mu(i\partial_\mu + qA_\mu) - m] = 0 \Rightarrow$ 守恒荷反号（电荷）

⇓

Dirac方程正反粒子变换 = 电荷共轭变换

Majorana表象中ψ为实场，无规范变换⇒无规范场作用

↓

为中性粒子．$q = -q = 0$

4. 旋量场量子化

1.拉氏密度与观测量密度

为何 ?コ　拉氏密度 $\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ → $\frac{\partial\mathcal{L}}{\partial\bar\psi} = 0 \to (i\partial\!\!\!/ - m)\psi = 0$

代入：$\mathcal{L} = \bar\psi \cdot 0 = 0$　　$\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi} = 0 \to -m\bar\psi - i\partial_\mu\bar\psi\gamma^\mu = 0$?コ

旋量场的观测量密度：　　★故以前所有$\langle\psi^+, \psi^+\rangle = -1 \to \langle i\psi^+, i\psi^+\rangle = 1$

正则动量：$\pi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = i\bar\psi\gamma_0 = i\psi^+$ → ψ与$i\psi^+$构成相空间（当∂_μ为1阶出现该现象）

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\partial_0\psi = \psi^\dagger(-i\alpha_0\cdot\nabla + m\beta)\psi$$

$$P^\mu = \pi\partial_\mu\psi - \mathcal{L} = \psi^\dagger i\partial^\mu\psi$$

$$j^\mu = q\bar{\psi}\gamma^\mu\psi \to Q = q\psi^\dagger\psi$$

猜想：Dirac方程-年有自由度隐入内部空间？⊐

2、Jordan-Wigner 量子化

?⊐ 正则与反正则 — P与q是否独立？

问题：正则量子化得到的对易关系为 Bose 子

Fermi子 → 应为反对易

动量空间：
$$\psi(x,t) = \sum_s \int \frac{d^3k}{(2\pi)^3 2w}\left[u(k,s)\, c_{ks}\, e^{-i(wt-\vec{k}\cdot\vec{x})} + v(k,\eta)\, d^\dagger_{k\eta}\, e^{i(wt-kx)}\right]$$
$$\psi^\dagger(x,t) = \sum_s \int \frac{d^3k}{(2\pi)^3 2w}\left[u^\dagger(k,\eta)\, c^\dagger_{k\eta}\, e^{i(wt-\vec{k}\cdot\vec{x})} + v^\dagger(k,\eta)\, d_{k\eta}\, e^{-i(wt-kx)}\right]$$

$$H = \int d^3x\, \psi^\dagger i\partial_0\psi = \sum_s \int d^3k\; w\,(c^\dagger_{ks}c_{ks} - d_{ks}d^\dagger_{ks})$$

Jordan-wigner 量子化：$\{c_{ks}, c^\dagger_{k's'}\} = \delta_{ss'}\delta(k-k')$ else $=0$

$\{d_{ks}, d^\dagger_{k's'}\} = \delta_{ss'}\delta(k-k')$

$$H = \sum_s \int d^3k\, w\left[c^\dagger_{ks}c_{ks} + d^\dagger_{ks}d_{ks} - \delta(0)\right]$$

零点能

?] ，$\sigma_x, \sigma_y, \sigma_z$ 的本征矢

$\psi, \psi^\dagger$ 的反对易：

$$\{\psi_\alpha(x), \psi^\dagger_\beta(x')\}_{t=t'} = \sum_{ss'}\int d^3k\, d^3k' \left\{u_\alpha(k,s)c_{ks}\varphi_k(x) + v_\alpha(k,s)d^\dagger_{ks}\varphi^\dagger_k(x),\; u^\dagger_\beta(k',s')c^\dagger_{k's'}\varphi^\dagger_{k'}(x') + v^\dagger_\beta(k',s')d_{k's'}\varphi_{k'}(x')\right\}_{t=t'}$$

$$= \sum_{ss'}\int d^3k\; \delta_{ss'}\, u_\alpha(k,s)u^\dagger_\beta(k,s')\varphi_k(x)\varphi^\dagger_k(x') - v_\alpha v^\dagger_\beta\, \varphi^\dagger_k(x)\varphi_k(x')$$

?⊐ 计算

$$= \int \frac{d^3k}{(2\pi)^3 2w}\, \delta_{\alpha\beta}\gamma^0\left[(\not{k}+m)e^{-ik(x-x')} - (\not{k}-m)e^{ik(x-x')}\right] = \int \frac{d^3k}{(2\pi)^3 2w}\, 2w\, e^{ik(x-x')}\delta_{\alpha\beta} = \delta_{\alpha\beta}\,\delta(x-x')$$

协变反对易：

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')] = \int\frac{d^3k}{(2\pi)^3 2\omega}\left[(\not{k}+m)e^{-i(x-x')}+(\not{k}-m)e^{i(x-x')}\right]_{\alpha\beta} = (i\not{\partial}+m)_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3 2\omega}\left(e^{-ik(x-x')}-e^{ik(x-x')}\right)$$

$$= i\,\underbrace{(i\not{\partial}+m)_{\alpha\beta}\Delta(x-x')}_{S_{\alpha\beta}(x-x')}$$

else $=0$

箱归一化 $\int d^3k \to \sum_k \frac{(2\pi)^3}{V}$; $k\to\frac{2\pi}{L}(n,m,l)$; $\delta(k-k')\to\frac{V}{(2\pi)^3}\delta_{kk'}$

则: $c_k \to \sqrt{\frac{V}{(2\pi)^3}}\,c_k$, $d_k\to\sqrt{\frac{V}{(2\pi)^3}}\,d_k$

$$\psi(x,t) = \sum_{k,s}\frac{1}{\sqrt{2\omega V}}\left[u(k,s)c_{ks}e^{-i(\omega t-\vec{k}\vec{x})}+v(k,s)d^\dagger_{ks}e^{i(\omega t-\vec{k}\vec{x})}\right]$$

粒子数表象 $N_{ks} = c^\dagger_{ks}c_{ks} \longrightarrow N_{ks}|n_{ks}\rangle = n_{ks}|n_{ks}\rangle$

$c^\dagger_{ks}|n_{ks}\rangle = C|n_{ks}+1\rangle \to C^*C = 1-n_{ks}$

$C=\sqrt{1-n_{ks}}$

(对 d_{ks}, $d^\dagger_{ks}$ 同样)

$\hookrightarrow (c_{ks})^2 = \frac{1}{2}\{c_{ks},c_{ks}\}=0$ ，同 $c^\dagger_{ks}$

$$(N_{ks})^2 = c^\dagger_{ks}c_{ks}c^\dagger_{ks}c_{ks} = c^\dagger_{ks}(1-c^\dagger c)c_{ks} = N_{ks} \longrightarrow n_{ks}=0,1$$

$[N_{ks}, c^\dagger_{ks}] = c^\dagger_{ks}(1-2c^\dagger_{ks}c_{ks}) = c^\dagger_{ks}$ $c^\dagger_{ks}|0\rangle = |1\rangle$, $c^\dagger_{ks}|1\rangle = \underbrace{0}_{\text{真空}}$

↓取共

$[N_{ks}, c_{ks}] = -c_{ks}$ $c_{ks}|1\rangle = |0\rangle$, $c_{ks}|0\rangle = 0$

$$[N_{ks}, N_{k's'}] = c^\dagger_k c_k c^\dagger_{k'}c_{k'} - c^\dagger_{k'}c_{k'}c^\dagger_k c_k = 0$$

Pauli不相容：反对易：$|1_{ks},1_{k's'}\rangle = c^\dagger_{ks}c^\dagger_{k's'}|0\rangle \to |1_{ks},1_{k's'}\rangle = -|1_{k's'},1_{ks}\rangle \to$ 同态不能有2个粒子

反粒子

$P = \sum_s\int d^3k\,\vec{k}\,(c^\dagger_{ks}c_{ks} - d_{ks}d^\dagger_{ks})$ J-W量子化使得动量相反，电性相同

$Q = \sum_s\int d^3k\,q\,(c^\dagger_{ks}c_{ks} + d_{ks}d^\dagger_{ks})$ (正则 → 动同荷反)

正则量子化 ← Heisenberg 不确定原理

J-W量子化 ← 微观因果性原理

5. 微观因果性原理（对所有场的要求）

→ 类空间隔的两点无关（即算子对易）

对于Dirac旋量，仅反

对易满足微观因果

场的观测量为双线性 → $O = \int d^3x\, O(x,t)$

$O(x) = \psi_r(x)\psi_s(x)$ → 由 ψ 与 $\psi^\dagger$（或 ψ 与 $\psi^\dagger$ 的分量）线性组合而成

$[O(x), O(x')] = 0$ 对于 $(x-x')^2 < 0$（类空）

↓ 一般化

$[\psi_r(x), \psi_s(x')] = 0$ 或 $\{\psi_r(x), \psi_s(x')\} = 0$ 对 $(x-x')^2 < 0$

实标量：$\phi^\dagger = \phi$，则 $[\phi(x), \phi(x')] = 0$，$(x-x')^2 < 0$

复标量：创仍同上

Dirac场：$\{\psi_\alpha(x), \psi_\beta(x')\} = \{\bar\psi_\alpha(x), \bar\psi_\beta(x')\} = 0$，$(x-x')^2 < 0$

$\{\psi_\alpha(x), \bar\psi_\beta(x')\} = 0$ $(x-x')^2 < 0$ → $S_{\alpha\beta}(x-x') = (i\partial\!\!\!/ + m)\underbrace{\Delta(x-x')}_{(x-x')^2<0\text{时全}0} = 0$ 对 $(x-x')^2 < 0$

标量场：$[\phi(x), \phi(x')] = 0$，$[\phi(x), \phi^\dagger(x')] = i\Delta(x-x')$，$\{\phi(x), \phi^\dagger(x')\} = \Delta_1(x-x')$

Dirac：$[\psi_\alpha(x), \psi_\beta(x')] = 0$，$\{\psi_\alpha(x), \bar\psi_\beta(x')\} = iS_{\alpha\beta}(x-x')$，$[\psi_\alpha(x), \bar\psi_\beta(x')] = S_{1\alpha\beta}(x-x')$

$S_{1\alpha\beta}(x-x') = (i\partial\!\!\!/ + m)\Delta_1(x-x')$，类空时

$\Delta_1(x-x') \neq 0$ 满足 K-G方程

解一下?□ → $|x-x'| > \lambda = \frac{1}{m}$ 时，$\Delta_1(x-x') \sim -\frac{1}{(x-x')^2} e^{-\sqrt{(x-x')^2}/m}$ ——→ 破坏微观因果

定域性：$\phi(x)$ 完全定域，算符构成的观测量不一定定域

eg：实标的 $N_k = \int d^3k\, a_k^\dagger a_k$ → $\phi(x,t) = \phi^{(+)}(x,t) + \phi^{(-)}(x,t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}} e^{-ikx} a_k + \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}} e^{ikx} a_k^\dagger$

?□待分析 ——→ $\mathcal{N}_k = \phi^{(-)} i\overleftrightarrow{\partial_0}\phi^{(+)}$ → $[\mathcal{N}_k(x), \mathcal{N}_k(x')] = \begin{cases} \infty & |x-x'| \to 0 \\ 0, & |x-x'| > \lambda \end{cases}$

?□ 二次量子化（schrödinger场）