Assignment_4

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1 Assignment 4

Realfagslektormaster, programing course
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1.1 1

1.1.1 (*i*)

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        import math


        def zeta(m, s):
            """Zeta function for a singluar m and s."""
            result = 0
            for n in range(int(m)):
                result += 1/np.power(float(n+1), s)

            return result


        def zeta_function_m(s=4):
            """Iterative Zeta function for a range of m's."""
            start = 2
            stop = 40

            x = np.linspace(start, stop, stop - start + 1)
            y = np.zeros(len(x))
            for i in range(len(x)):
                y[i] = zeta(x[i], s)

            return x, y


        def zeta_function_s(m):
```

```
    """Iterative Zeta function for a range of s's."""
    start = -5
    stop = 5

    x = np.linspace(start, stop, stop - start + 1)
    y = np.zeros(len(x))
    for i in range(len(x)):
        y[i] = zeta(m, x[i])

    return x, y


#fig = plt.figure()
x, y = zeta_function_s(5)
#print(x)
plt.plot(x, y)
plt.show()

# zeta_function_m(s)
#    m controls smoothness of curve.
#    s controls bend.
#    s = 0 results in a straight line, from about (0,0) to (m,m)
#    As s goes towards -inf, the curve will bend to the bottom right.
#    As s goes towards +inf, the curve will bend to the top left.

# zeta_function_s(m)
#    s range (stop - start) controls the smoothness of the curve.
#    m < 2 results in a straight horizontal line at 1.
#    As m->+inf, the curve will bend to the bottom left.
#    This will also happen if the s range becomes larger.
```

```
<Figure size 640x480 with 1 Axes>
```

1.1.2 *(ii)*

$$\zeta(2) = \lim_{m \to \infty} \zeta(2)_m = \frac{\pi^2}{6}$$

In [2]:
```
print("\n\nEstimation, zeta(2) = pi^2/6")
estimate = zeta(1000, 2)
print(estimate)

actual_value = math.pi**2/6
print(actual_value)

print("Absolutt error: " + str(abs(estimate - actual_value)))
```

```
Estimation, zeta(2) = pi^2/6
1.6439345666815615
1.6449340668482264
Absolutt error: 0.0009995001666649461
```

$$\zeta(4) = \lim_{m\to\infty} \zeta(4)_m = \frac{\pi^4}{90}$$

```
In [3]: print("\n\nEstimation, zeta(4) = pi^4/90")
        estimate = zeta(1000, 4)
        print(estimate)

        actual_value = math.pi**4/90
        print(actual_value)

        print("Absolutt error: " + str(abs(estimate - actual_value)))
```

```
Estimation, zeta(4) = pi^4/90
1.082323233378306
1.082323233711138
Absolutt error: 3.328319841955363e-10
```

1.1.3 (*iii*)

For any $r \gg 0$, $|\zeta(2)_m - \frac{\pi^2}{6}| < 10^{-1}$, for any $m \geq N_r = 10^r$
I.e. if $r = 3.5$, m must be at least $10^{3.5} \approx 3162.27766....$ In this case, $m \geq 3162$.

```
In [4]: def check_r(r):
            m = 0
            while abs(zeta(m, 2) - np.pi**2/6) >= 10**-r:
                m += 1
                if m > 4000:
                    return m
            return m

        print(check_r(2))
```

```
100
```

1.2 2

1.2.1 (*i*)

```
In [5]: import matplotlib.pyplot as plt
        import numpy as np
```

```
def plot_ell(c, a, b, min_x=0, max_x=2*np.pi, d=100):
    t = np.linspace(min_x, max_x, d)
    #x = f(t, a)
    #y = g(t, b)
    x, y = c(t, a, b)
    plt.plot(x, y)
    plt.show()


def plot(f, g, min_x=0, max_x=2*np.pi, d=100):
    t = np.linspace(min_x, max_x, d)
    x = f(t)
    y = g(t)
    plt.plot(x, y)
    plt.show()
```

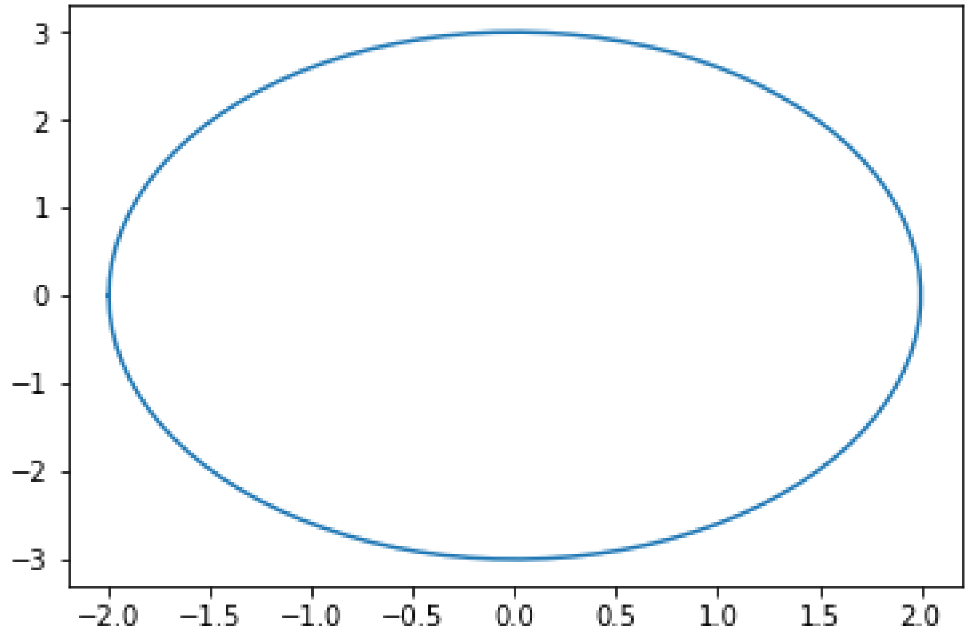
1.2.2 *(ii)*

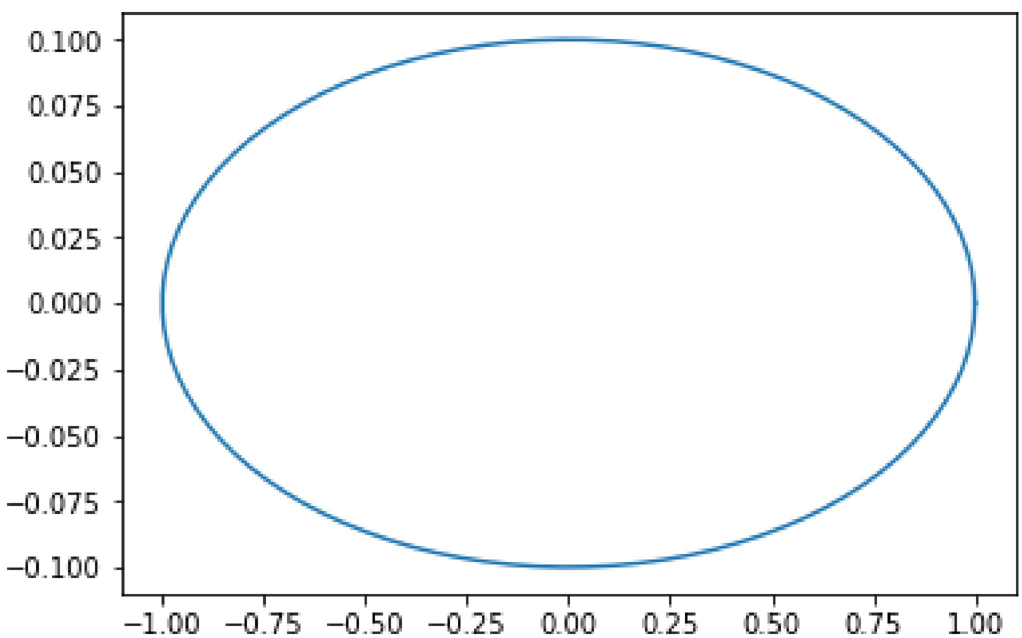
Any non-zero values of a and b will make the following formula describe an ellipse:

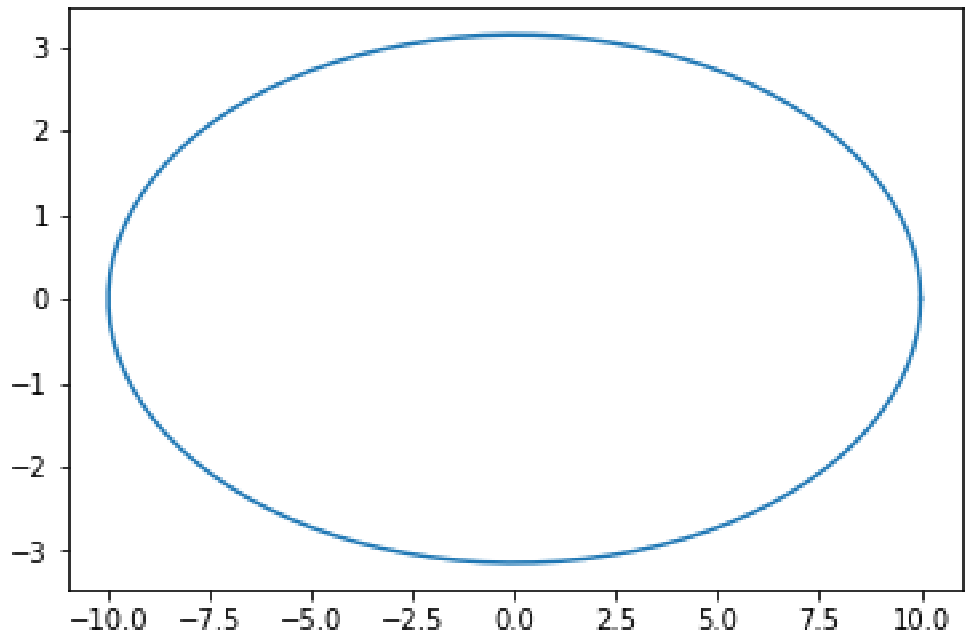
$$c(\theta) = \big(a\,cos(\theta), b\,sin(\theta)\big), \quad 0 \leq \theta \leq 2\pi$$

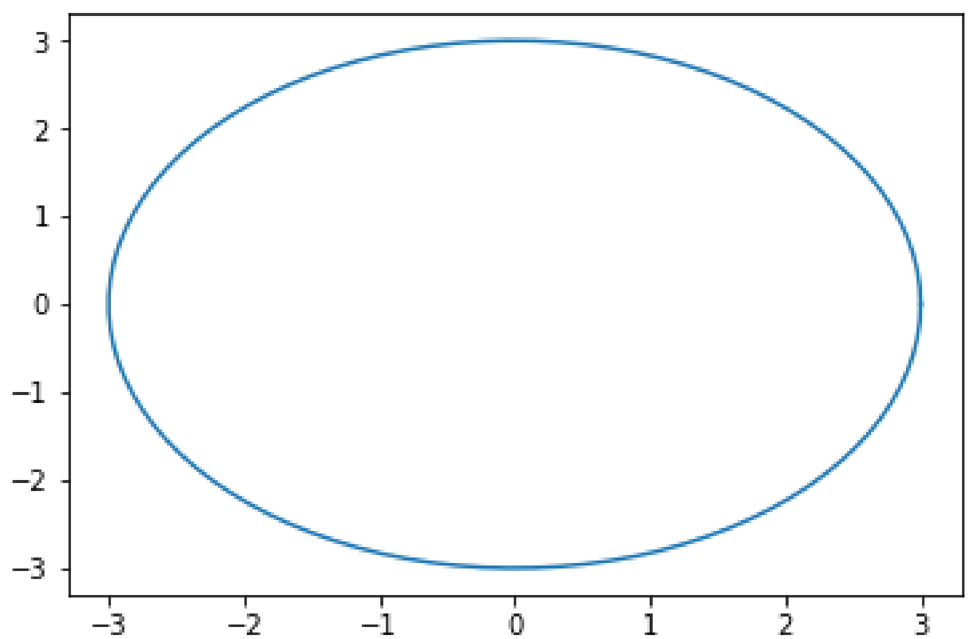
In [6]:
```
import matplotlib.pyplot as plt
import numpy as np


def f(t, a):
    return a * np.cos(t)


def g(t, b):
    return b * np.sin(t)


def c(t, a, b):
    x = f(t, a)
    y = g(t, b)
    return x, y


plot_ell(c, -2, -3)
plot_ell(c, 1, 0.1)
plot_ell(c, 10, np.pi)
plot_ell(c, 3, 3)
```

```
# a describes the x-scale. b describes the y-scale.
# The ellipse will have an r in the y-axis of 2*b, from -b to b.
# The ellipse will have an r in the x-axis of 2*a, from -a to a.
# Whether a and/or b are positive or negative does not affect the result.
```









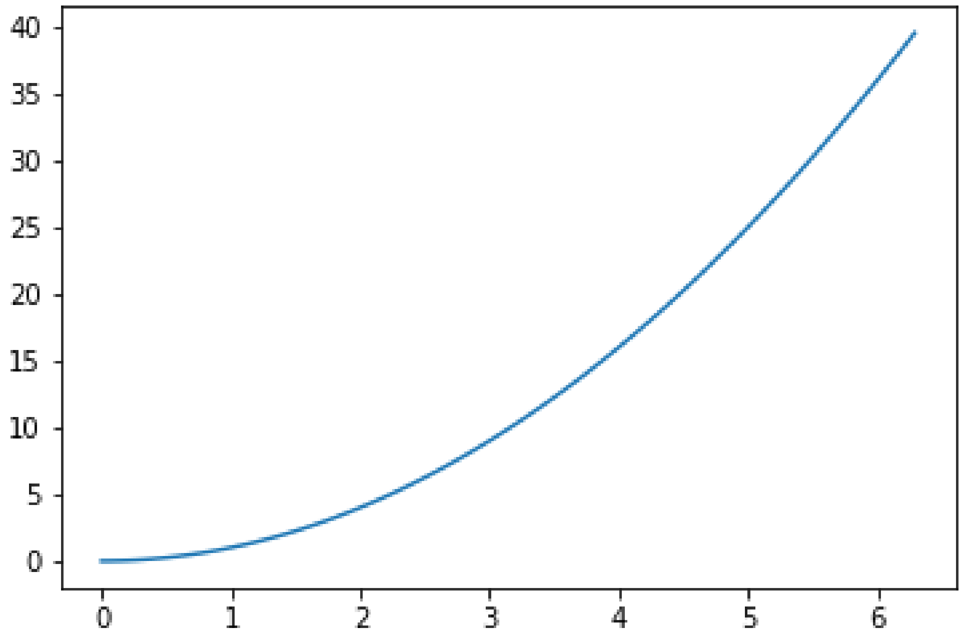
1.2.3 (iii) (pt. 1)

Numerical derivation and ellipse arc length finder, based on numerical integration.

In [7]:
```
import matplotlib.pyplot as plt
import numpy as np


def derr(f, x):
    h = 1/10000
    dy = (f(x+h) - f(x))/h
    return dy


def arc_length(f, g, a, b):
    d = 10000
    dt = 1/d
    integral = 0
    for i in range(d-1):
        t = (i*dt)*(b-a) + a
        func = np.sqrt(derr(f, t)**2 + derr(g, t)**2)
        integral += func*dt*(b-a)

    return integral
```

In [8]:
```
def f(t):
    return t


def g(t):
    return t**2


print(arc_length(f, g, 0, 1))


plot(f, g)
```

```
1.478719258945574
```



In [9]:
```
import numpy as np

def df(t):
    """ g(t) = t"""
    return 1


def dg(t):
    """f(t) = t^2"""
    return 2*t


def f(t):
    return np.sqrt(df(t)**2 + dg(t)**2)


def arc_length(a, b):
    d = 10000
    dt = 1/d
    integral = 0
    for i in range(d-1):
        t = (i*dt)*(b-a) + a
        integral += f(t)*dt*(b-a)

    return integral
```

```
print("x:\t", "1.4789428575445974 (W|A)")
print("x_est:\t", arc_length(0, 1))
```

```
x:          1.4789428575445974 (W|A)
x_est:          1.4786574667270478
```

1.2.4 $(iv) + (v)$

Arc length using numerical elliptical integration. Solution for both (iv) and (v).

```
In [10]: import numpy as np


         def e_sq(a, b):
             return np.sqrt(1 - b**2/a**2)


         def f(a, b, theta):
             return np.sqrt(1 - e_sq(a, b)**2 * np.sin(theta)**2)


         def arc_length_ell(a, b):
             d = 10000
             integral = 0
             for i in range(d):
                 theta = i/d * np.pi/2
                 integral += f(a, b, theta)

             return 4*a*integral/d*np.pi/2


         print(arc_length_ell(10, 5))
         # 10,5 = 48.44224110273838 (W|A)

         #x = np.linspace(0, 2*np.pi, 100)
         #plt.plot(x, f(10, 5, x))
         #plt.show()
```

```
48.44381189906502
```

1.2.5 (vi)

The defined range for the function E can be decided by its root.

$$E(e) = 4a \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 sin^2(\theta)}\, d\theta$$

If $1 - e^2 sin^2(\theta) < 0$, the root becomes imaginary. We must therefore find for which values of e, $e^2 sin^2(\theta)$ will always be lower than or equal to one.

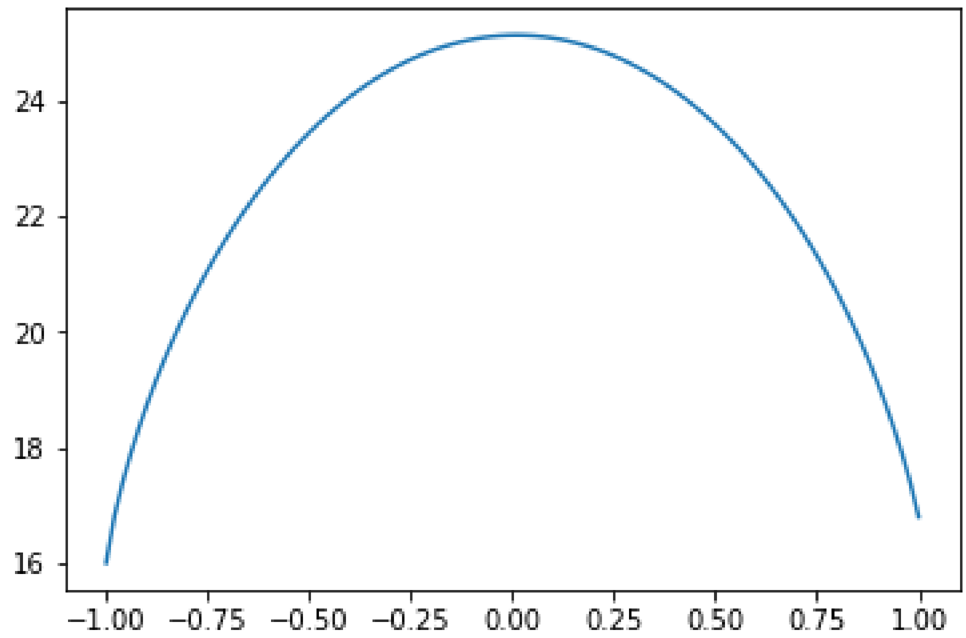
$$e^2 sin^2(\theta) \leq 1 \implies e^2 \leq \frac{1}{sin^2(\theta)} \implies e \leq \frac{1}{sin(\theta)}$$

Since $sin(\theta)$ can never exceed 1, we know that $sin(\theta) \leq 1 \implies \frac{1}{sin(\theta} \geq 1$. Therefore $e^2 \leq 1 \implies |e| \leq 1$

1.2.6 (vii)

Using $|e| \leq 1$, we can quite easily change the code from (iv) and (v).

```
In [11]: import matplotlib.pyplot as plt
         import numpy as np


         def f(e, theta):
             return np.sqrt(1 - e**2 * np.sin(theta)**2)


         def arc_length_ell(e, a, b, A):
             d = 1000
             dt = 1/d
             integral = 0
             for i in range(d):
                 theta = (i*dt) * (b - a) + a
                 integral += f(e, theta)

             return 4*A*integral*dt*(b - a)


         a = 0
         b = 2*np.pi
         A = 1
         e_min = -1
         e_max = 1

         x = np.linspace(e_min, e_max, 100)
         y = np.zeros(len(x))
         for i in range(len(x)):
             e = i*(e_max - e_min)/len(x) + e_min
             y[i] = arc_length_ell(e, a, b, A)


         plt.plot(x, y)
         plt.show()
```



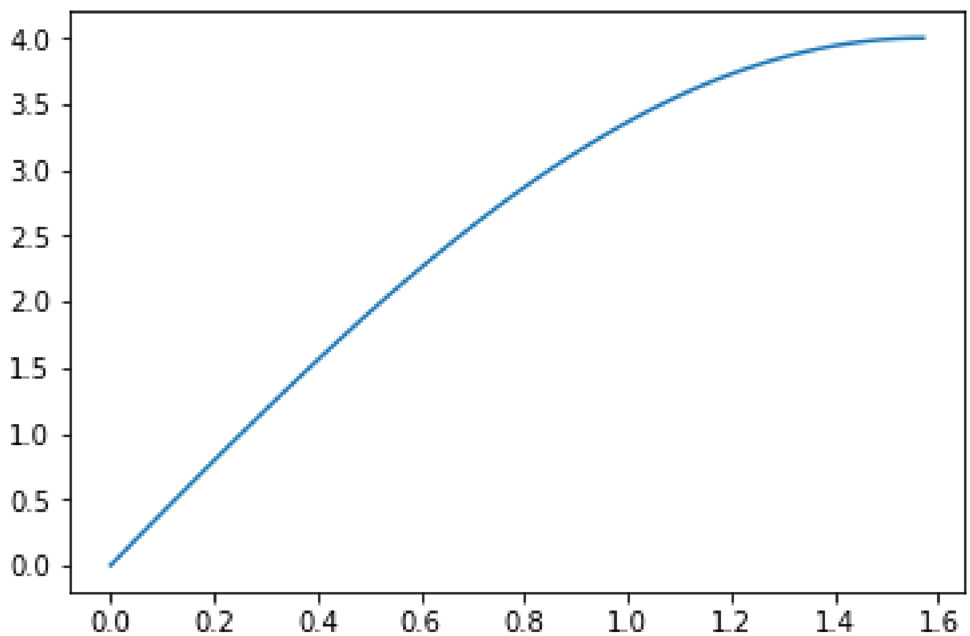
1.2.7 $(viii)$

Here, we will do the same as (vii), but with a constant e and a variable x as upper limit for the integral, and 0 as its bottom limit. With e close to 0, you get an almost linear function. When e moves closer to 1, it will zigzag more. (This is true when b is 2π instead of $\frac{pi}{2}$, as I found out when mistakenly using the wrong limit.

When using an $|e| > |$, it will be linear, with a fall off as $x \to \infty$. (It will also give you error messages, as you are going outside its range, so do it at your own risk.)

```
In [12]: import matplotlib.pyplot as plt
         import numpy as np


         def f(e, theta):
             return np.sqrt(1 - e**2 * np.sin(theta)**2)


         def arc_length_ell(e, a, b, A):
             d = 1000
             dt = 1/d
             integral = 0
             for i in range(d):
                 theta = (i*dt) * (b - a) + a
                 integral += f(e, theta)
```

```
    return 4*A*integral*dt*(b - a)


a = 0
b = np.pi/2
A = 1

e = 1

x = np.linspace(a, b, 100)
y = np.zeros(len(x))

for i in range(len(x)):
    y[i] = arc_length_ell(e, a, x[i], A)

plt.plot(x, y)
plt.show()
```

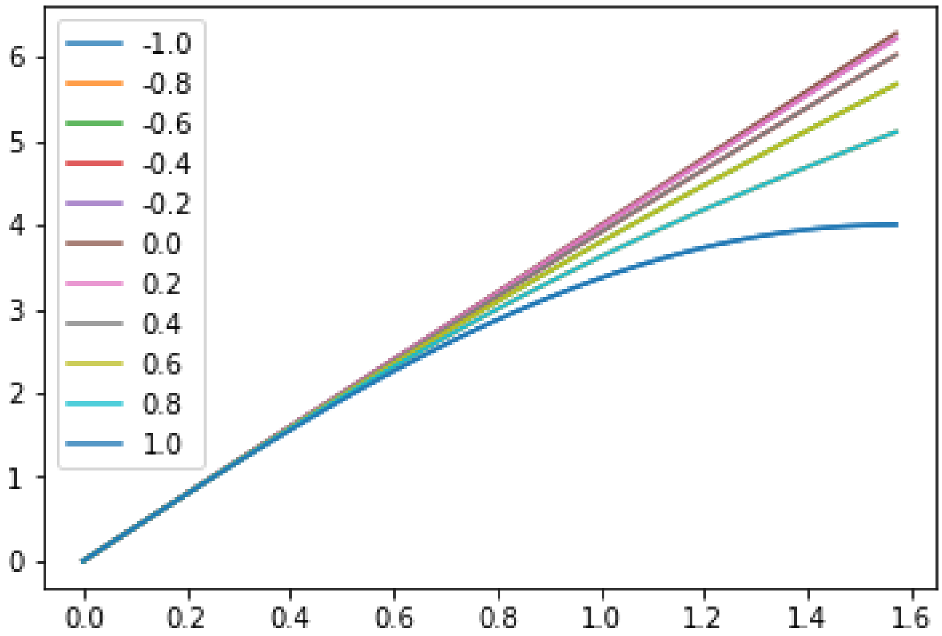


1.2.8 (ix)

It takes a few seconds to calculate.

```
In [13]: import matplotlib.pyplot as plt
         import numpy as np
```

```python
def f(e, theta):
    return np.sqrt(1 - e**2 * np.sin(theta)**2)


def arc_length_ell(e, a, b, A):
    d = 1000
    dt = 1/d
    integral = 0
    for i in range(d):
        theta = (i*dt) * (b - a) + a
        integral += f(e, theta)

    return 4*A*integral*dt*(b - a)



a = 0
b = np.pi/2
A = 1

e = np.linspace(-1, 1, 11)
for j in range(len(e)):
    x = np.linspace(a, b, 100)
    y = np.zeros(len(x))

    for i in range(len(x)):
        y[i] = arc_length_ell(e[j], a, x[i], A)

    plt.plot(x, y)


for i in range(len(e)):
    e[i] = round(e[i], 1)

plt.legend(e)
plt.show()
```



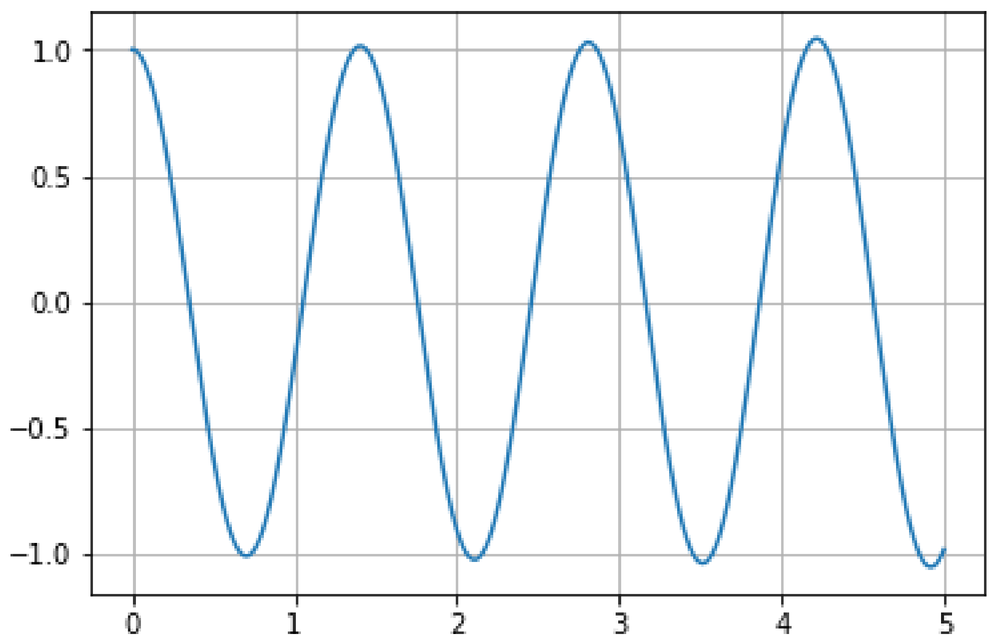
1.3 3 - Differential Equations and Euler's Method

In [40]:
```
import numpy as np
import matplotlib.pylab as plt

h = 0.001

t_0 = 0
t_max = 5


def A(t, y):
    return [y[1], F(t,y[0],y[1])]

def _next_y(t, y):
    return y + np.dot(A(t, y),h)

def _euler(t_0, t_max, y_0):
    t_lst = [t_0]
    z_lst = [y_0]
    y_lst = [y_0[0]]
    lst = np.linspace(t_0, t_max, int((t_max-t_0)/h))

    for i in range(1,len(lst)):
        t_lst.append(t_0 + i*h)
```

```
            z_lst.append(_next_y(t_0+(i-1)*h, z_lst[i-1]))
            y_lst.append(z_lst[i][0])
        return t_lst, y_lst
```

(i)
Using the Euler's Method Python code.
Graphs show meters of displacement/movement m and seconds s.

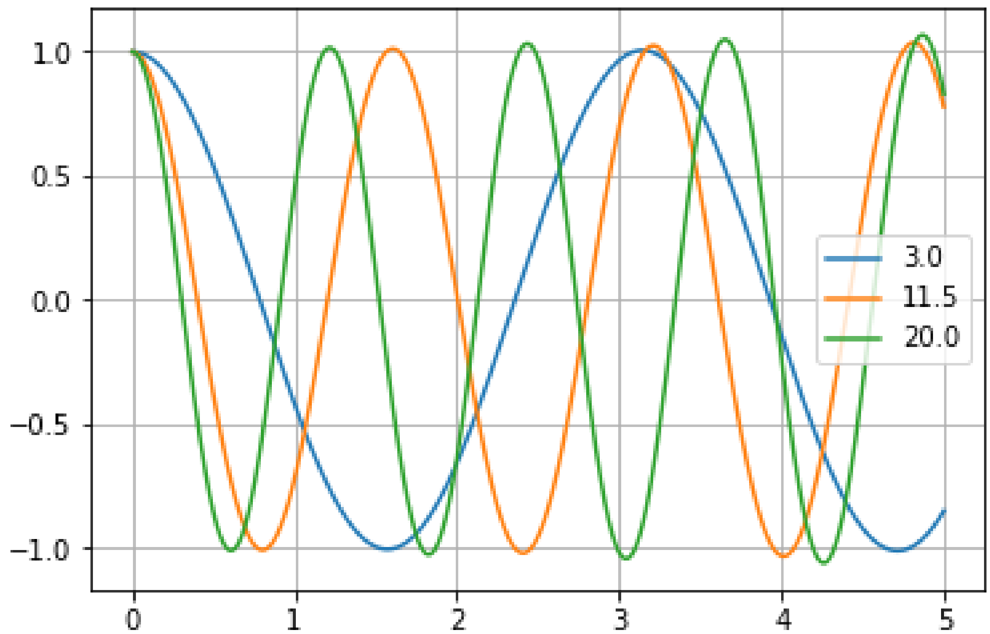
```
In [41]: def F(t,y,yprime):          # y'' = F(t,y,y')
             return -k/m * y


         k = 15
         m = 0.75

         fig = plt.figure()
         ax1 = fig.add_subplot(111)
         ax1.grid(True)

         y_0 = [1,0]   # startverdier [y(0),y'(0)]

         x, y = _euler(t_0, t_max, y_0)
         ax1.plot(x, y)

         fig.show()
```



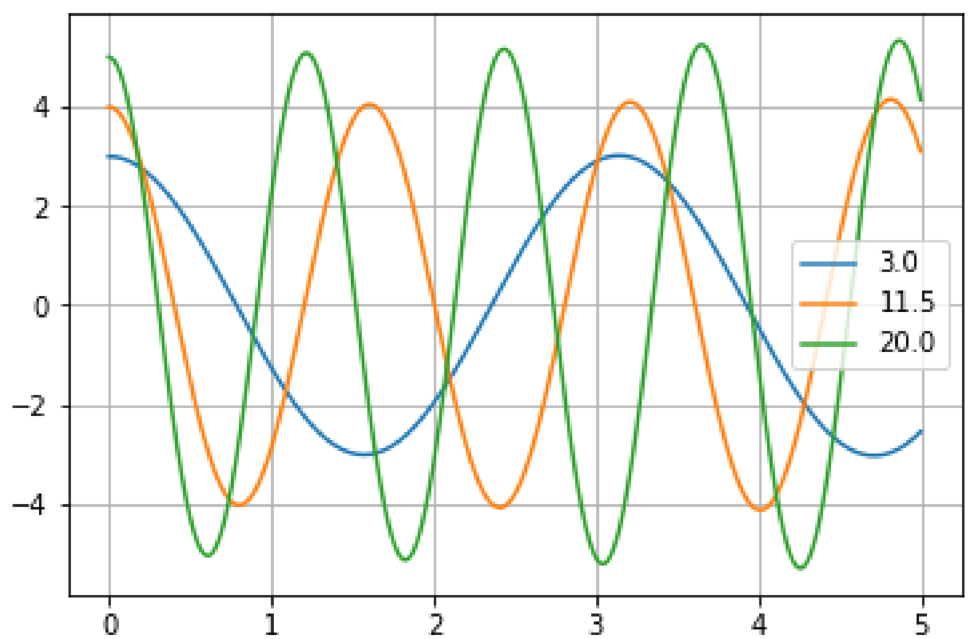
1.3.1 (*ii*)

This shows how the spring system will act with three different forces applied by the spring. $3.0\frac{N}{m}$, $11.5\frac{N}{m}$ and $20\frac{N}{m}$. The pattern is clear: the stronger the force, the faster the mass is moved/accelerated, and the faster initial speed, the higher amplitudes you get. This seems reasonable by physical standards.

```
In [16]: def F(t,y,yprime):            # y'' = F(t,y,y')
             return -k/m * y


         k = 0
         m = 0.75

         fig = plt.figure()
         ax1 = fig.add_subplot(111)
         ax1.grid(True)

         x_base = np.linspace(3, 20, 3)

         y_0 = [1, 0]

         for i in range(len(x_base)):
             k = x_base[i]
             x, y = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)

         plt.legend(x_base)
         fig = plt.figure()
         ax1 = fig.add_subplot(111)
         ax1.grid(True)

         for i in range(len(x_base)):
             y_0 = [i+3, 0]
             k = x_base[i]
             x, y = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)

         plt.legend(x_base)
         fig.show()
```



1.0
0.5
0.0
−0.5
−1.0
0
1
2
3
4
5
3.0
11.5
20.0



4
2
0
−2
−4
0
1
2
3
4
5
3.0
11.5
20.0

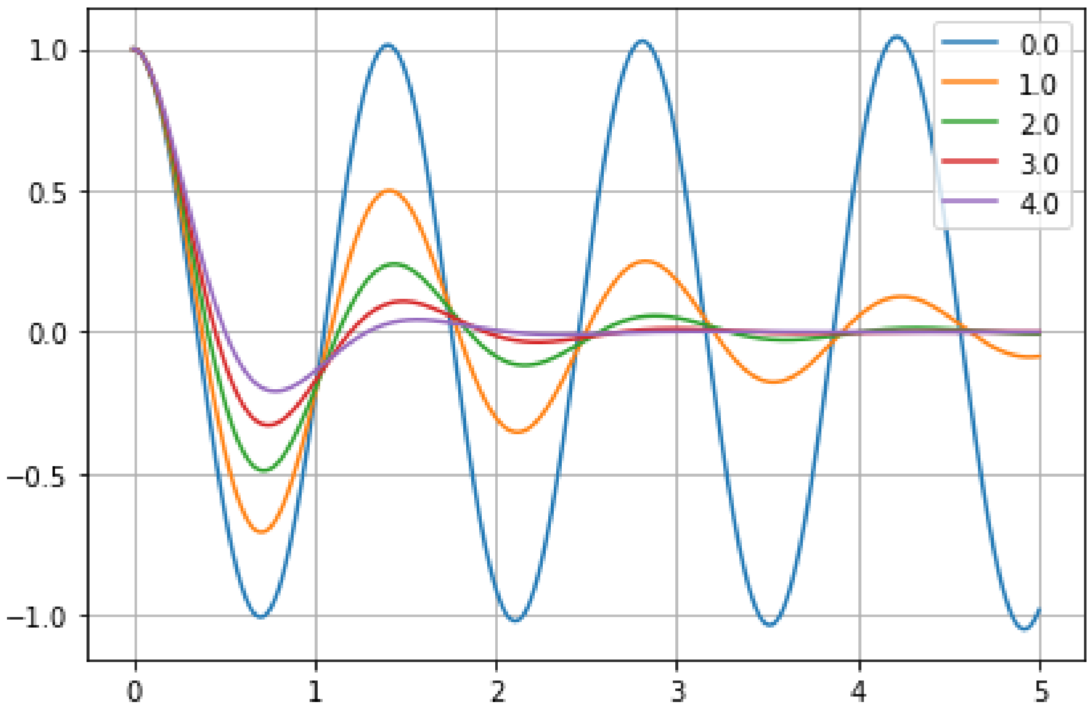
1.3.2 (iii)

Here, the friction α (alpha), which is now part of the differential equation

$$-\alpha y_p - \frac{k}{m} y$$

(where y is the displacement in meters) has the values of $\alpha \in [0, 4]$ with a step size of 1. The higher the friction, the faster the mass slows down.

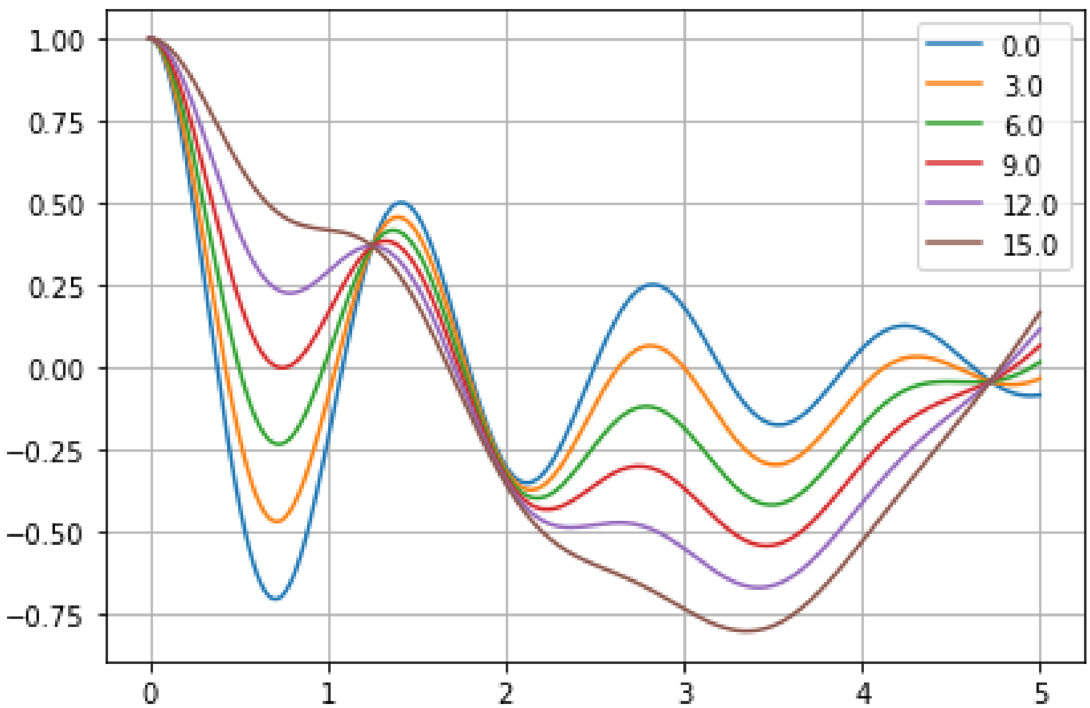
```
In [17]: def F(t,y,yprime):           # y'' = F(t,y,y')
             return - k/m * y - alpha*yprime


         k = 15
         m = 0.75
         alpha = 0

         fig = plt.figure()
         ax1 = fig.add_subplot(111)
         ax1.grid(True)

         base = np.linspace(0, 4, 5)

         for i in range(len(base)):
             y_0 = [1, 0]
             alpha = base[i]
             x, y = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)
             base[i] = round(base[i], 2)

         fig.tight_layout()
         plt.legend(base)
         fig.show()
```



1.3.3 (*iv*)

```
In [18]: def F(t,y,yprime):           # y'' = F(t,y,y')
             return -alpha*yprime - k/m*y + AA*np.cos(t)


         k = 15
         m = 0.75
         alpha = 1
         AA = 0

         fig = plt.figure()
         ax1 = fig.add_subplot(111)
         ax1.grid(True)

         base = np.linspace(0, 15, 6)

         for i in range(len(base)):
             y_0 = [1, 0]
             AA = base[i]
             x, y = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)
             base[i] = round(base[i], 2)
```

```
fig.tight_layout()
plt.legend(base)
fig.show()
```



1.3.4 (v)

Use any of the following functions by first running them, and then running the plotting below.

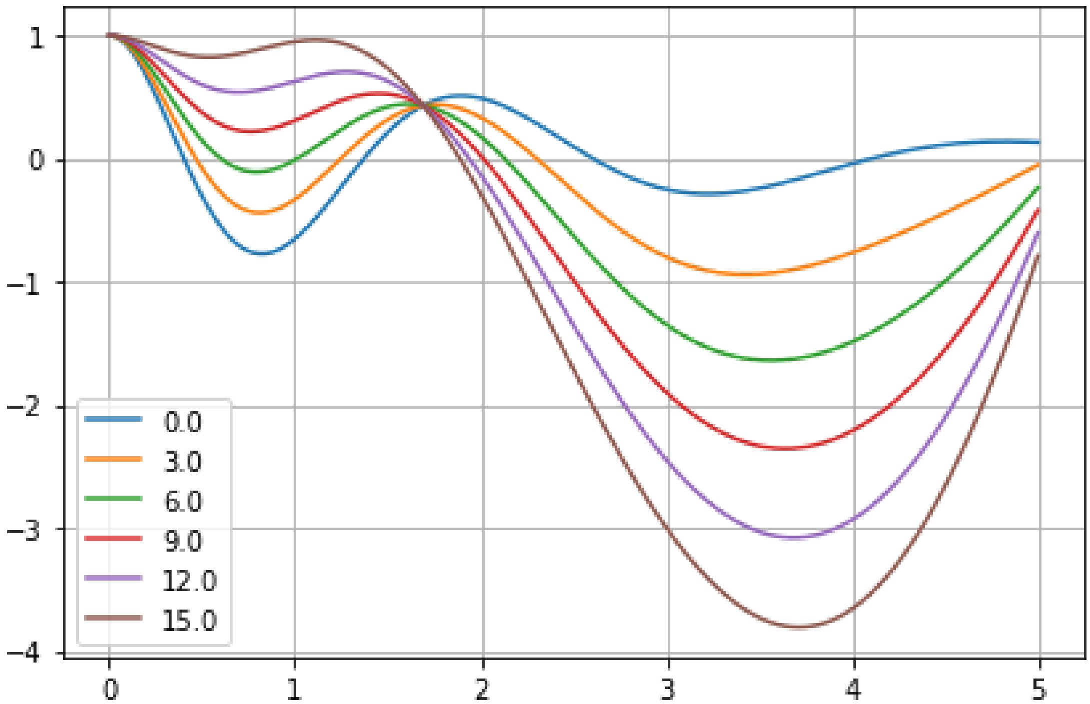
```
In [19]: def F(t,y,yprime):
             return -alpha*yprime - t*k/m*y + AA*np.cos(t)
```

```
In [20]: def F(t,y,yprime):
             return -alpha*yprime - t**2*k/m*y + AA*np.cos(t)
```

```
In [21]: def F(t,y,yprime):
             return -alpha*yprime - 1/(t+1)*k/m*y + AA*np.cos(t)
```

```
In [22]: k = 15
         m = 0.75
         alpha = 1
         AA = 0

         fig = plt.figure()
```

```
ax1 = fig.add_subplot(111)
ax1.grid(True)

base = np.linspace(0, 15, 6)

for i in range(len(base)):
    y_0 = [1, 0]
    AA = base[i]
    x, y = _euler(t_0, t_max, y_0)
    ax1.plot(x, y)
    base[i] = round(base[i], 2)

fig.tight_layout()
plt.legend(base)
fig.show()
```



1.3.5 (*vi*)

Using the Euler's Method + phase code.

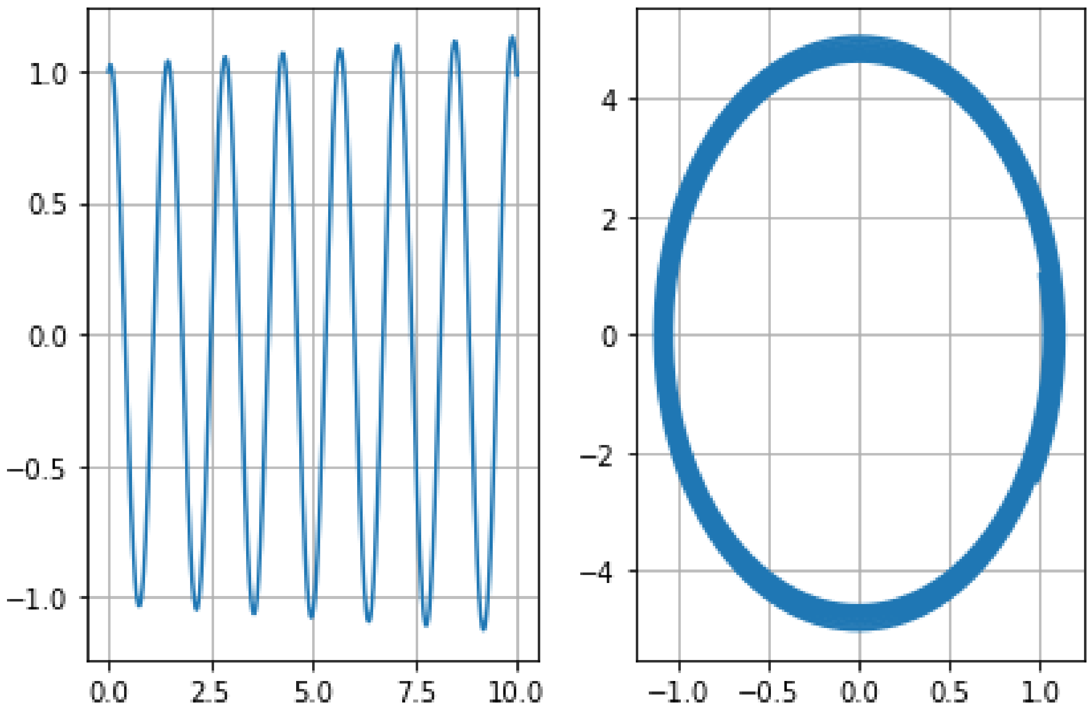
```
In [23]: import numpy as np
         import matplotlib.pylab as plt

         h = 0.001
```

```
t_0 = 0
t_max = 10


def A(t, y):
    return [y[1], F(t,y[0],y[1])]

def _next_y(t, y):
    return y + np.dot(A(t, y),h)

def _euler(t_0, t_max, y_0):
    t_lst = [t_0]
    z_lst = [y_0]
    y_lst = [y_0[0]]
    yprime_lst = [y_0[1]]
    lst = np.linspace(t_0, t_max, int((t_max-t_0)/h))

    for i in range(1,len(lst)):
        t_lst.append(t_0 + i*h)
        z_lst.append(_next_y(t_0+(i-1)*h, z_lst[i-1]))
        y_lst.append(z_lst[i][0])
        yprime_lst.append(z_lst[i][1])
    return t_lst, y_lst, yprime_lst
```

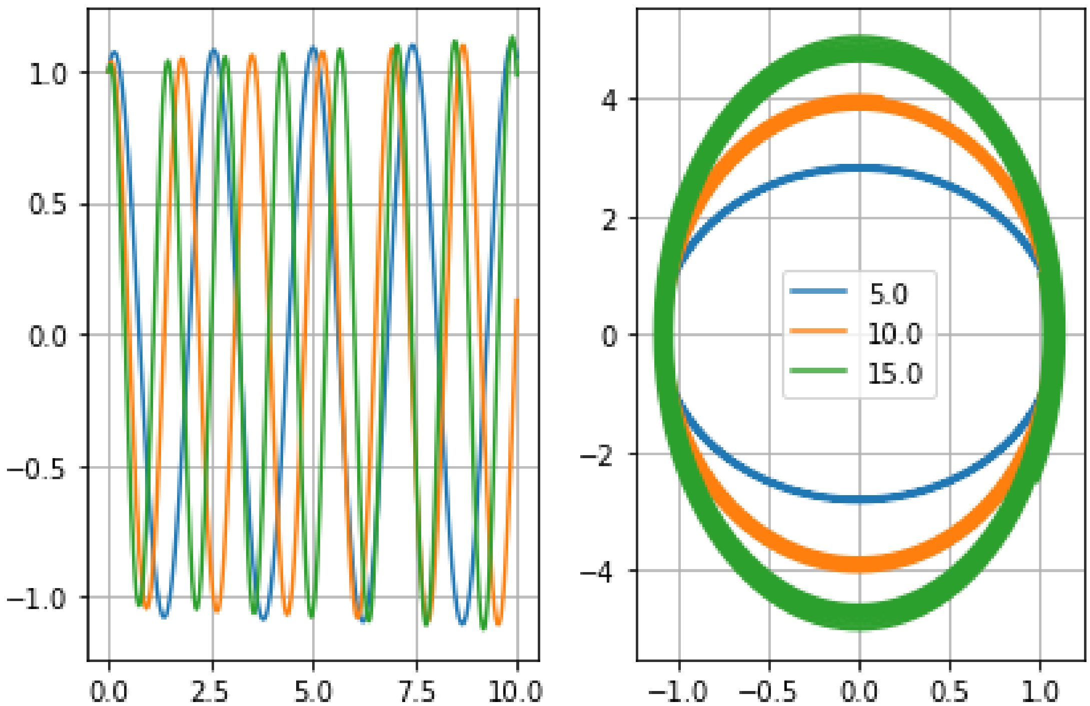
In [24]:
```
def F(t,y,yprime):
    return -alpha*yprime - k/m*y + AA*np.cos(t)
```

In [25]:
```
k = 15
m = 0.75

alpha = 0
AA = 0

y_0 = [1,1]      # startverdier y_0 = [y(0), y'(0)]
x, y, yprime = _euler(t_0, t_max, y_0)
```

In [26]:
```
fig = plt.figure()
ax1 = fig.add_subplot(121)
ax1.grid(True)
ax2 = fig.add_subplot(122)
ax2.grid(True)

ax1.plot(x, y)
ax2.plot(y,yprime)

fig.tight_layout()
fig.show()
```

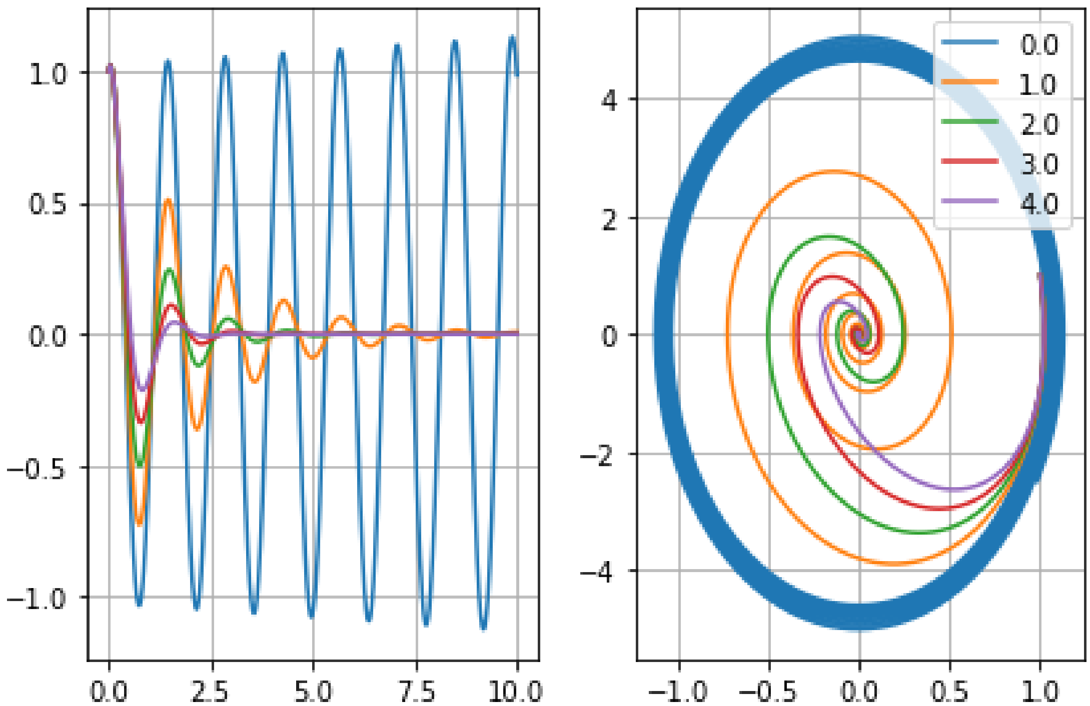



```
In [27]: fig = plt.figure()
         ax1 = fig.add_subplot(121)
         ax1.grid(True)
         ax2 = fig.add_subplot(122)
         ax2.grid(True)

         base = np.linspace(5, 15, 3)

         for i in range(len(base)):
             k = base[i]
             x, y, yprime = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)
             ax2.plot(y,yprime)
             base[i] = round(base[i], 2)

         fig.tight_layout()
         plt.legend(base)
         fig.show()
```



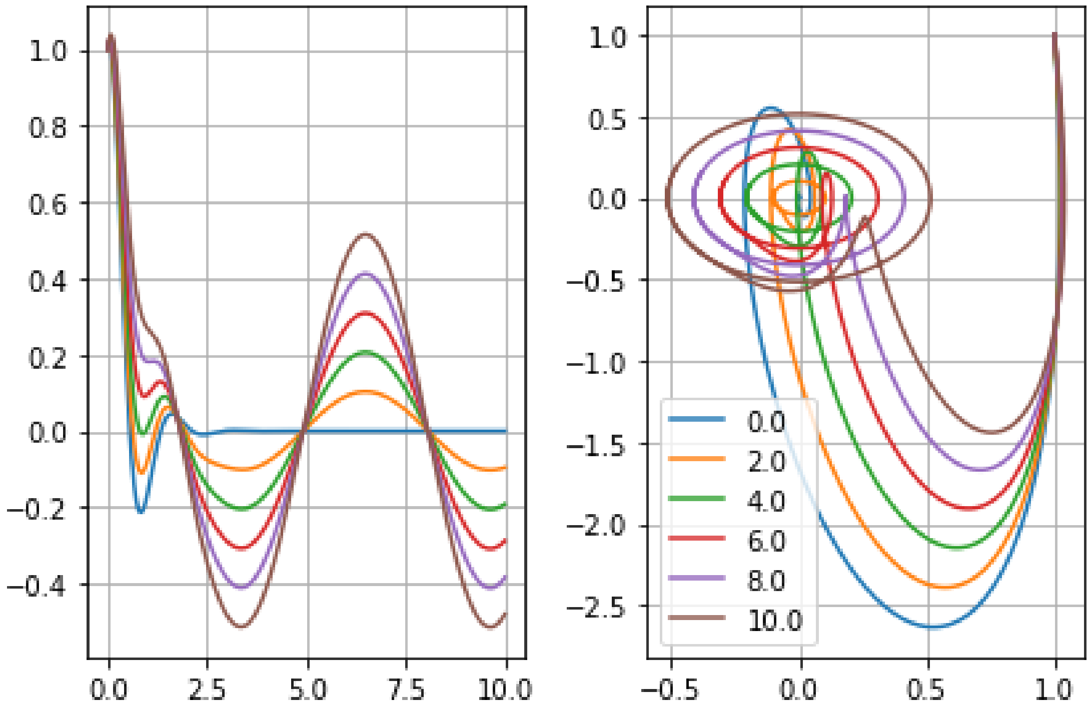
Higher friction α makes the mass reduce its speed, and makes the (y, y_p) diagram spiral inwards.

```
In [28]: fig = plt.figure()
         ax1 = fig.add_subplot(121)
         ax1.grid(True)
         ax2 = fig.add_subplot(122)
         ax2.grid(True)

         base = np.linspace(0, 4, 5)

         for i in range(len(base)):
             alpha = base[i]
             x, y, yprime = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)
             ax2.plot(y,yprime)
             base[i] = round(base[i], 2)

         fig.tight_layout()
         plt.legend(base)
         fig.show()
```



Changing the AA (A) value.

```
In [29]: def F(t,y,yprime):
             return -alpha*yprime - k/m*y + AA*np.cos(t)


         fig = plt.figure()
         ax1 = fig.add_subplot(121)
         ax1.grid(True)
         ax2 = fig.add_subplot(122)
         ax2.grid(True)

         base = np.linspace(0, 10, 6)

         for i in range(len(base)):
             AA = base[i]
             x, y, yprime = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)
             ax2.plot(y,yprime)
             base[i] = round(base[i], 2)

         fig.tight_layout()
         plt.legend(base)
         fig.show()
```



Experimentation with changes in $\alpha(t)$ and $\omega^2(t)$.

Making A (from $A\,cos(t)$) more affecting will make the mass start swinging more like a sine wave, while making A smaller will make it less affecting.

$A\,cos(t)$ will be the base line for the displacement, where the system will converge, assuming friction. $\omega^2(t)$ will force movement back and forth as long as it is not overpowered by the friction force. $\alpha(t)$ will reduce the movement over time, and make the displacement converge.

Changing $\alpha(t)$ will affect how the

Making the friction force or the spring force positive will make the system quickly spiral out of control as it becomes a positive feedback loop.

```
In [30]: k = 15
         m = 0.75
         AA = 2
         alpha = 1
```

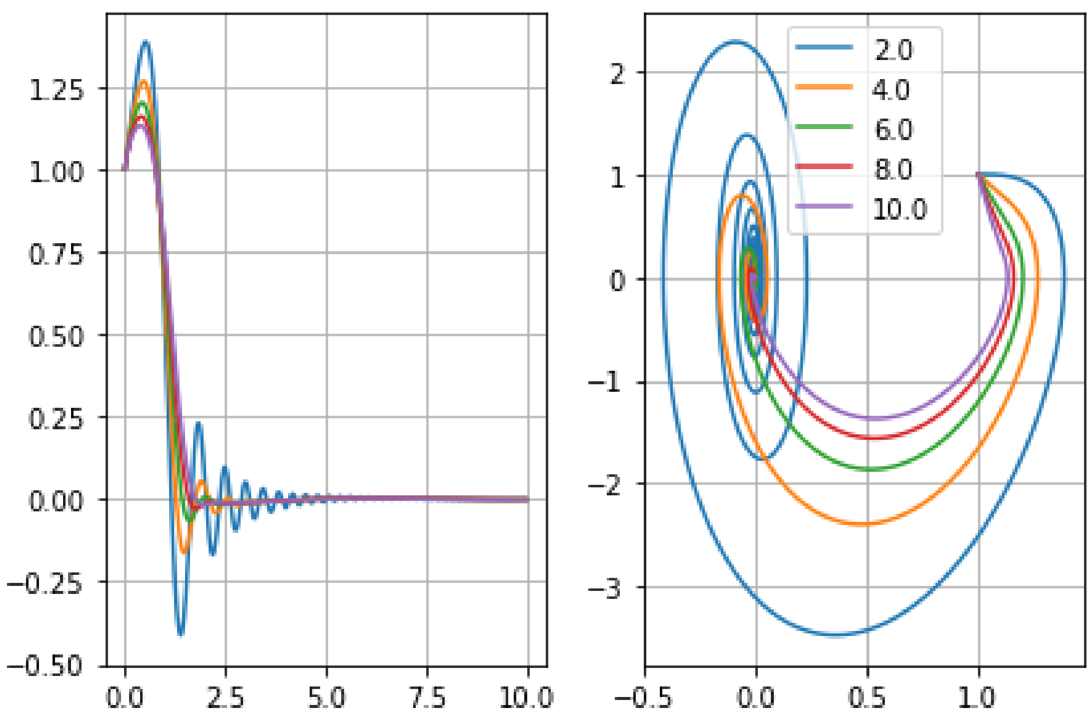
```
In [31]: def F(t,y,yprime):
             return -alpha*yprime - k/m*y + AA*np.cos(t)
```

```
In [32]: def F(t,y,yprime):
             return -alpha*yprime - t*k/m*y - AA*np.cos(t)
```

```
In [33]: def F(t,y,yprime):
             return -alpha*yprime - k/m*y + 10*AA*np.cos(t)
```

```
In [34]: def F(t,y,yprime):
             return -alpha*yprime - t**2*k/m*y + AA*np.cos(t)
```

```
In [35]: fig = plt.figure()
         ax1 = fig.add_subplot(121)
         ax1.grid(True)
         ax2 = fig.add_subplot(122)
         ax2.grid(True)

         base = np.linspace(2, 10, 5)

         for i in range(len(base)):
             alpha = base[i]
             x, y, yprime = _euler(t_0, t_max, y_0)
             ax1.plot(x, y)
             ax2.plot(y,yprime)
             base[i] = round(base[i], 2)

         fig.tight_layout()
         plt.legend(base)
         fig.show()
```



1.3.6 (*vii*)

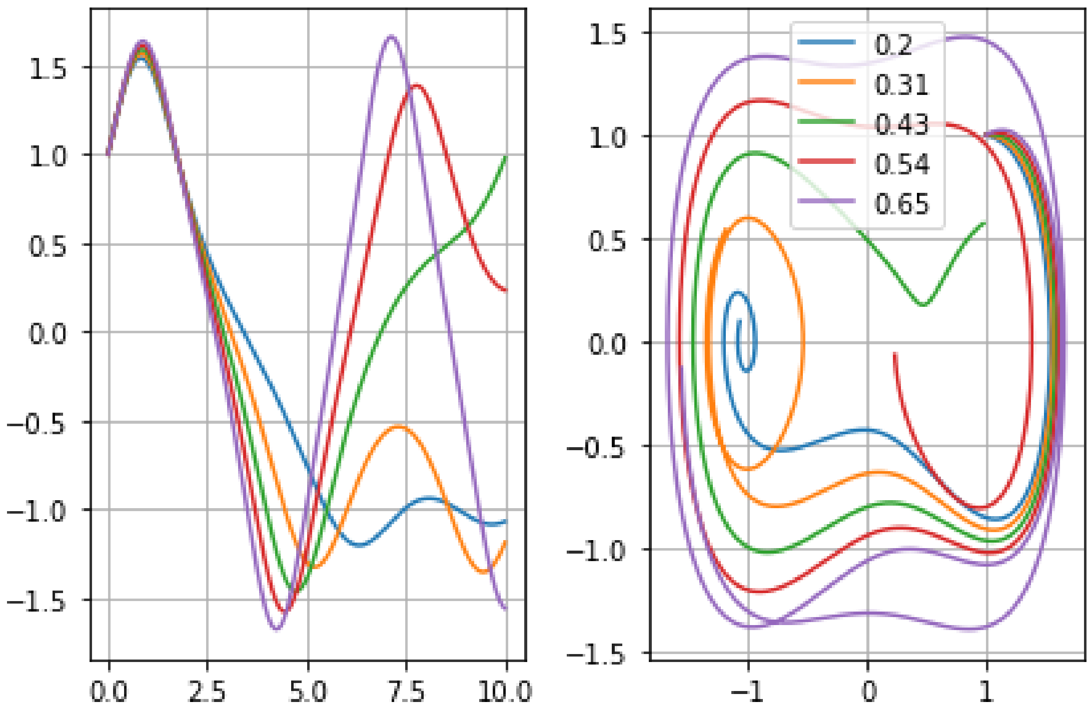
The Duffing equation, shown below, describe an oscillator using Hooke's law.

$$\frac{d^2s}{dt^2} + \alpha(t)\frac{ds}{dt} + \omega^2(t)\,s(t) + \gamma\,s(t)^3 = \beta\,cos(ut), \quad \beta, u \in \mathbb{R}$$

Conclusion: Reminds me of a Picasso.

In [36]:
```
k = 15
m = 0.75
AA = 2
alpha = 0.3
beta = 1
gamma = 1
omega2 = -1
u = 1.2
y = [1, 0]
```

In [37]:
```
def F(t,y,yprime):
    return -alpha*yprime - omega2*y - gamma*y**3 + beta*np.cos(u*t)


fig = plt.figure()
ax1 = fig.add_subplot(121)
ax1.grid(True)
ax2 = fig.add_subplot(122)
ax2.grid(True)

base = np.linspace(0.2, 0.65, 5)

for i in range(len(base)):
    beta = base[i]
    x, y, yprime = _euler(t_0, t_max, y_0)
    ax1.plot(x, y)
    ax2.plot(y,yprime)
    base[i] = round(base[i], 2)

fig.tight_layout()
plt.legend(base)
fig.show()
```

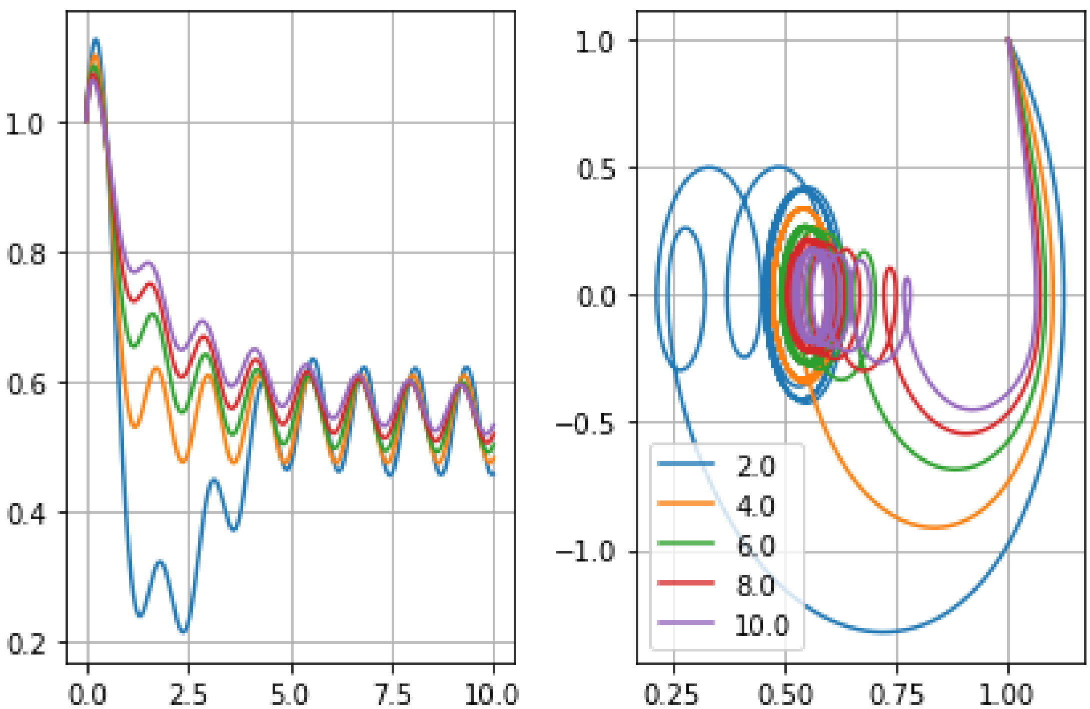


1.3.7 *(viii)*

In [38]:
```
k = 15
m = 0.75
alpha = 0.7
beta = 2
gamma = 5
omega2 = -1.5
u = 5
y = [-1, -1]
```

In [39]:
```
def F(t,y,yprime):
    return -alpha*yprime - omega2*y - gamma*y**3 + beta*np.cos(u*t)


fig = plt.figure()
ax1 = fig.add_subplot(121)
ax1.grid(True)
ax2 = fig.add_subplot(122)
ax2.grid(True)

base = np.linspace(2, 10, 5)
```

```
for i in range(len(base)):
    alpha = base[i]
    x, y, yprime = _euler(t_0, t_max, y_0)
    ax1.plot(x, y)
    ax2.plot(y,yprime)
    base[i] = round(base[i], 2)

fig.tight_layout()
plt.legend(base)
fig.show()
```




```
In [ ]:
```