1 Vector Spaces

1.1 Sols of simultaneous linear equations

Theorem 1.1.4 *Solution sets of inhomogeneous systems*
If the solution set of a linear system of equations is non-empty, then we obtain all solutions by adding component-wise an arbitrary solution of the associated homogenised system to a fixed solution of the system.

1.2 Fields & vector spaces

Definition 1.2.1.1 *Fields*
A *field* F is a set with functions

$$\begin{aligned} \text{addition } &= + : F \times F \to F\ ;\ (\lambda,\mu) \mapsto \lambda+\mu \\ \text{multiplication } &= . : F \times F \to F; (\lambda,\mu) \mapsto \lambda\mu \end{aligned}$$

such that $(F,+)$ and $(F \setminus \{0\}, .)$ are abelian groups, with

$$\lambda(\mu+\nu) = \lambda\mu + \lambda\nu \in F, \quad \forall \lambda\nu \in F$$

The neutral elements are called $0_F, 1_F$. In particular

$$\lambda+\mu = \mu+\lambda,\ \lambda.\mu = \mu.\lambda,\ \lambda+0_F = \lambda,\ \lambda.1_F = \lambda \in F, \quad \forall \lambda,\mu \in F$$

For every $\lambda \in F$ there exists $-\lambda \in F$ such that

$$\lambda + (-) = 0_F \in F$$

For every $\lambda \neq 0 \in F$ there exists $\lambda^{-1} \neq 0 \in F$ such that

$$\lambda(\lambda^{-1}) = 1_F \in F$$

Definition 1.2.1.2 *Vector space*
A *vector space* V over a *field* F is a pair consisting of an abelian group $V = (V,+)$ and a mapping

$$F \times V \to V : (\lambda, \vec{v}) \mapsto \lambda\vec{v}$$

such that for all $\lambda,\mu \in F$ and $\vec{v},\vec{w} \in V$ the following identities hold:

$$\begin{aligned} \lambda(\vec{v}+\vec{w}) &= (\lambda\vec{v}) + (\lambda\vec{w}) && \text{(distributivity)} \\ (\lambda+\mu)\vec{v} &= (\lambda\vec{v}) + (\mu\vec{v}) && \text{(distributivity)} \\ \lambda(\mu\vec{v}) &= (\lambda\mu)\vec{v} && \text{(associativity)} \\ 1_F\vec{v} &= \vec{v} \end{aligned}$$

A vector space V over a field F is called an *F-vector space.*

Lemma 1.2.2 Product with the scalar zero
If V is a vector space and $\vec{v} \in V$, then $0\vec{v} = \vec{0}$

Lemma 1.2.3 Product with the scalar (-1)
If V is a vector space and $\vec{v} \in V$, then $(-1)\vec{v} = -\vec{v}$.

Lemma 1.2.4 Product with the zero vector
If V is a vector space over a field F, then $\lambda\vec{0} = \vec{0}$ for all $\lambda \in F$. Furthermore, if $\lambda\vec{v} = \vec{0}$, then either $\lambda = 0$ or $\vec{v} = \vec{0}$.

1.3 Products of sets and of vector spaces

1.4 Vector subspaces

Definition 1.4.1 *Vector subspaces*
A subset U of a vector space V is called a *vector subspace* or *subspace* if U contains $\vec{0}$ and

$$\vec{u},\vec{v} \in U \text{ and } \lambda \in F \implies \vec{u}+\vec{v} \in U \text{ and } \lambda\vec{u} \in U$$

Proposition 1.4.5 Generating a vector subspace from a subset
Let T be a subset of a vector space V over a field F. Then amongst all vector subspace of V that include T, there is a smallest vector subspace

$$\langle T \rangle = \langle T \rangle_F \subseteq V$$

It can be described as the set of all vectors $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r$ with $\alpha_1,\ldots,\alpha_r \in F$ and $\vec{v}_1,\ldots,\vec{v}_r \in T$, together with $\vec{0}$ in the case $T = \emptyset$.

Definition 1.4.7 *Generating set*
A subset of a vector space is called a *generating set* of our vector space if its span is all of the vector space. A vector space that has a finite generating set is said to be *finitely generated.*

Definition 1.4.9 *Power Set & System of Subsets*
The set of all subsets $\mathcal{P}(X) = \{U : U \subseteq X\}$ of X is the *power set* of X.
A subset of $\mathcal{P}(X)$ is a *system of subsets of X*.
Given such a system $\mathcal{U} \subseteq \mathcal{P}(X)$ we can create two new subsets of X, the *union* and the *intersection* of the sets of our system $\mathcal{U}$:

$$\bigcup_{U \in \mathcal{U}} U = \{x \in X : \exists U \in \mathcal{U}.\, x \in U\}$$

$$\bigcap_{U \in \mathcal{U}} U = \{x \in X : x \in U\ \forall\ U \in \mathcal{U}\}$$

In particular the intersection of the empty system of subsets of X is X, and the union of the empty system of subsets X is the empty set.

1.5 Linear independence and bases

Definition 1.5.1 *Linear independence*
A subset L of a vector space V is *linearly independent* if for all pairwise different vectors $\vec{v}_1,\ldots,\vec{v}_r \in L$ and arbitrary vectors $\alpha_1,\ldots,\vec{v}_r \in F$,

$$\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0} \implies \alpha_1 = \cdots = \alpha_r = 0$$

Definition 1.5.2 *Linear dependence*
A subset L of a vector space V is called *linearly dependent* if it is not linearly independent.

Definition 1.5.8 *Basis*
A *basis* of a vector space V is a linearly independent generating set in V.

Theorem 1.5.11 Linear combinations of basis elements
Let F be a field, V be a vector space over F, and $\vec{v}_1,\ldots,\vec{v}_r \in V$ vectors. The family $(\vec{v}_i)_{1\le i\le r}$ is a basis of V if and only if the following "evaluation" mapping

$$\begin{aligned} \Phi : F^r &\to V \\ (\alpha_1,\ldots,\alpha_r) &\mapsto \alpha\vec{v}_1 + \cdots + \alpha_r\vec{v}_r \end{aligned}$$

is a bijection.

Theorem 1.5.12 Characterisation of bases
The following are equivalent for a subset E of a vector space V:

1. E is a basis, i.e. a linearly independent generating set;
2. E is minimal among all generating sets, meaning that $E \setminus \{\vec{v}\}$ does not generate V, $\forall \vec{v} \in E$;
3. E is maximal among all linearly independent subsets, meaning that $E \cup \{\vec{v}\}$ is not linearly independent $\forall \vec{v} \in V$.

Corollary 1.5.13 The existence of a basis
Let V be a finitely generated vector space over a field F. The V has a basis.

Theorem 1.5.14 (Useful variant on the Characterisation of bases)
Let V be a vector space.

1. If $L \subset V$ is a linearly independent subset and E is minimal amongst all generating sets of our vector space with the property that $L \subseteq E$, then E is a basis.
2. If $E \subseteq V$ is a generating set and if L is maximal amongst all linearly independent subsets of our vector space with the property $L \subseteq E$, then L is basis.

Definition 1.5.15 *Free vector space*
Let X be a set and F a field. The set $\text{Maps}(X, F)$ of all mappings $f : X \to F$ becomes an F-vector space with the operations of point-wise addition and multiplication by a scalar. The subset of all mappings which send almost all elements of X to zero is a vector subspace

$$F\langle X\rangle \subseteq \text{Maps}(X, F)$$

This vector subspace is called the *free vector space on the set* X.

Theorem 1.5.16 (Useful variant on Linear combinations of basis elements)
Let F be a field, V an F-vector space, and $(\vec{v}_i)_{i\in I}$ a family of vectors from the vector space V. The following are equivalent:

1. The family $(\vec{v}_i)_{i\in I}$ is a basis for V;
2. For each vector $\vec{v} \in V$ there is precisely one family $(a_i)_{i\in I}$ of elements of our field F, almost all of which are zero and such that

$$\vec{v} = \sum_{i\in I} a_i\vec{v}_i$$

1.6 Dimension of a vector space

Theorem 1.6.1 Fundamental estimate of linear algebra
No linearly independent subset of a given vector space has more elements than a generating set. Thus if V is a vector space, $L \subset V$ a linearly independent subset, and $E \subseteq V$ a generating set, then:

$$|L| \leq |E|$$

Theorem 1.6.2 Steinitz exchange theorem
Let V be a vector space, $L \subset V$ and finite linearly independent subset, and $E \subseteq V$ and generating set. Then there is an injection $\Phi : L \to E$ such that $(E \setminus \Phi(L)) \cup L$ is also a generating set for V.

We can swap out some elements of a generating set by the elements of our linearly independent set, and still keep a generating set.

Lemma 1.6.3 Exchange lemma
Let V be a vector space, $M \subseteq V$ a linearly independent subset, and $E \subseteq V$ a generating subset, such that $M \subseteq E$. If $\vec{w} \in V \setminus M$ is a vector set not belonging to M such that $M \cup \{\vec{w}\}$ is linearly independent, then there exists $\vec{e} \in E \setminus M$ such that $\{E \setminus \{\vec{e}\}\} \cup \{\vec{w}\}$ is a generating set for V.

Corollary 1.6.4 Cardinality of bases
Let V be a finitely generated vector space.

1. V has a finite basis;
2. V cannot have an infinite basis;
3. Any two bases of V have the same number of elements.

Definition 1.6.5 *Dimension*
The cardinality of one (and each) basis of a finitely generated vector space V is called the *dimension* of V and is denoted $\dim V$. If the vector space is not finitely generated, then $\dim V = \infty$ and V is *infinite dimensional.*

Corollary 1.6.8 Cardinality criterion for bases
Let V be a finitely generated vector space.

1. Each linearly independent subset $L \subset V$ has at most $\dim V$ elements, and if $|L| = \dim V$, then L is actually a basis;
2. Each generating set $E \subseteq V$ has at least $\dim V$ elements, and if $|E| = \dim V$ then E is actually a basis.

Corollary 1.6.9 Dimension estimate for vector subspaces
A proper vector subspace of a finite dimensional vector space has itself a strictly smaller dimension.

Remark 1.6.10 If $U \subseteq V$ is a vector subspace of an arbitrary vector space, then we have $\dim U \leq \dim V$ and if we have $\dim U = \dim V < \infty$ then it follows that $U = V$.

Notation If V is a vector space, and U, W are subspaces of V, then we define $U + W$ to be the subspace $\langle U \cup W\rangle$ of V generated by U and W together.

Theorem 1.6.11 The dimension theorem
Let V be a vector space containing vector subspaces $U, W \subseteq V$. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

1.7 Linear mappings

Definition 1.7.1 *Linear mapping*
Let V, W be vector spaces over a field F. A mapping $f : V \to W$ is called *linear* if for all $\vec{v}_1, \vec{v}_2 \in V$ and $\lambda \in F$ we have

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$
$$f(\lambda\vec{v}_1) = \lambda f(\vec{v}_1)$$

A bijective linear mapping is called an *isomorphism* of vector spaces. If there is an isomorphism of vector spaces, we call them *isomorphic.* A homomorphism from one vector space to itself is called an *endomorphism.* An isomorphism of a vector space to itself is called an *automorphism.*

Definition 1.7.5 *Fixed point*
A point that is sent to itself by a mapping is called a *fixed point* of the mapping. Given a mapping $f : X \to X$, we denote the set of fixed points by

$$X^f = \{x \in X : f(x) = x\}$$

Definition 1.7.6 *Complementary*
Two vector subspace V_1, V_2 of a vector space V are *complementary* if addition defines a bijection $V_1 \times V_2 \to V$

Theorem 1.7.7 Classification of vector spaces by their dimension
Let $n \in \mathbb{N}$. Then a vector space over a field F is isomorphic to F^n if and only if it has dimension n.

Lemma 1.7.8 Linear mappings and bases
Let V, W be vector spaces over F and let $B \subset V$ be a basis. Then restriction of a mapping gives a bijection

$$\text{Hom}_F(V, W) = \text{Hom}(V, W) \subseteq \text{Maps}(V, W)$$
$$f \mapsto f|_B$$

In other words, each linear mapping determines and is completely determined by the values it takes on a basis.

Proposition 1.7.9

1. Every injective linear mapping $f : V \to W$ has a *left inverse*, in other words a linear mapping $g : W \to V$ such that $g \circ f = \text{id}_V$
2. Every surjective linear mapping $f : V \to W$ has a *right inverse*, in other words a linear mapping $g : W \to V$ such that $f \circ g = \text{id}_W$

1.8 Rank-Nullity theorem

Definition 1.8.1 *Image, Kernel*
The *image* of a linear mapping $f : V \to W$ is the subset $\mathrm{im}(f) = f(V) \subseteq W$. It is a vector subspace of W. The preimage of the zero vector of a linear mapping $f : V \to W$ is denoted by

$$\ker(f) \equiv f^{-1}(0) = \{v \in V : f(v) = 0\}$$

and is called the *kernel* of the linear mapping f. The kernel is a vector subspace of V.

Lemma 1.8.2 A linear mapping $f : V \to W$ is injective if and only if $\ker_f = 0$.

Theorem 1.8.4 Rank-Nullity theorem
Let $f : V \to W$ be a linear mapping between vector spaces. Then

$$\begin{aligned} \mathrm{dim}V &= \dim(\mathrm{ker}f) + \dim(\mathrm{im}f) \\ &= \text{nullity } + \text{ rank} \end{aligned}$$

Corollary 1.8.5 (Dimension theorem, again)
Let V be a vector space, and $U, W \subseteq V$ vector subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \mathrm{dim}U + \mathrm{dim}W$$

Definition *Idempotent*
An element f of a set with composition or product is called *idempotent* if $f^2 = f$.

2 Linear Mappings and Matrices

2.1 Linear mappings $F^m \to F^n$ and matrices

Theorem 2.1.1 Linear mappings $F^m \to F^n$ and matrices
Let F be a field and let $m, n \in \mathbb{N}$. There is a bijection between the space of linear mappings $F^m \to F^n$ and the set of matrices with n rows and m columns and entries in F

$$\begin{aligned} \mathrm{M} : \mathrm{Hom}_F(F^m, F^n) &\to \ \mathrm{Mat}(n \times m; F) \\ f &\mapsto [f] \end{aligned}$$

This attaches to each linear mapping f its *representing matrix* $\mathrm{M}(f) \equiv [f]$. The columns of this matrix are the images under f of the standard basis elements of F^m

$$[f] \equiv (f(\mathbf{e}_1)|f(\mathbf{e}_2)|\cdots|f(\mathbf{e}_m))$$

Definition 2.1.6 *Product*
Let $n, m, l \in \mathbb{N}$, F and field, and let $A \in \mathrm{Mat}(n \times m; F)$ and $B \in \mathrm{Mat}(m \times l; F)$ be matrices. The *product* $A \circ B = AB \in \mathrm{Mat}(n \times l; F)$ is the matrix defined by

$$(AB)_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk}$$

Matrix multiplication produces a mapping

$$\begin{aligned} \mathrm{Mat}(n \times m; F) \times \mathrm{Mat}(m \times l; F) &\to \mathrm{Mat}(m \times l; F) \\ (A, B) &\mapsto AB \end{aligned}$$

Theorem 2.1.8 Composition of linear mappings and products of matrices
Let $g : F^l \to F^m$ and $f : F^m \to F^n$ be linear mappings. The representing matrix of their composition is the product of their representing matrices

$$[f \circ g] = [f] \circ [g]$$

Proposition 2.1.9 Calculating with matrices
Let $k, l, m, n \in \mathbb{N}, A, A' \in \mathrm{Mat}(n \times m; F), B, B' \in \mathrm{Mat}(m \times l; F), C \in \mathrm{Mat}(l \times k; F)$ and $I = I_m$. Then the following hold for matrix multiplication

$$\begin{aligned} (A + A')B &= AB + A'B \\ A(B + B') &= AB + AB' \\ IB &= B \\ AI &= A \\ (AB)C &= A(BC) \end{aligned}$$

Definition 2.2.1 *Invertible*
A matrix A is called *invertible* if there exist matrices B and C such that $BA = I$ and $AC = I$.

Definition 2.2.2 *Elementary matrix*
An *elementary matrix* is any square matrix that differs from the identity matrix in at most one entry.

Theorem 2.2.3 Every square matrix can be written as a product of elementary matrices.

Definition 2.2.4 *Smith Normal Form*
Any matrix whose only non-zero entries lie on the diagonal, and which has first 1s on along the diagonal followed by 0s is in *Smith Normal Form.*

Theorem 2.2.5 Transformation of a matrix into Smith-Normal form
For each matrix $A \in \mathrm{Mat}(n \times m; F)$ there exist invertible matrices P and Q such that PAQ is a matrix in Smith Normal Form.

Definition 2.2.6 *Rank*
The *column rank* of a matrix $A \in \mathrm{Mat}(n \times m; F)$ is the dimension of the subspace of F^n generated by the columns of A. Similarly, the *row rank* of A is the dimension of the subspace of F^m generated by the rows of A.

Theorem 2.2.7 The column rank and the row rank of any matrix are equal.

Definition 2.2.8 *Full rank*
Whenever the rank of a matrix is equal to the number of rows (or columns — whichever is smaller), it has *full rank.*

2.2 Abstract linear mappings and matrices

Theorem 2.3.1 Abstract linear mappings and matrices
Let F be a field, V and W vector spaces over F with ordered bases $\mathcal{A} = (\vec{v}_1, \ldots, \vec{v}_m)$ and $\mathcal{B} = (\vec{w}_1, \ldots, \vec{w}_n)$. Then to each linear mapping $f : V \to W$ we associated a *representing matrix* ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ whose entries a_{ij} are defined by the identity

$$f(\vec{v}_j) = a_{1j}\vec{w}_1 + \cdots + a_{nj}\vec{w}_n \in W$$

This produces a bijection, which is even an isomorphism of vector spaces

$$\begin{aligned} \mathrm{M}_{\mathcal{B}}^{\mathcal{A}} : \mathrm{Hom}_F(V, W) &\tilde{\to} \mathrm{Mat}(n \times m; F) \\ f &\mapsto_{\mathcal{B}} [f]_{\mathcal{A}} \end{aligned}$$

Theorem 2.3.2 The representing matrix of a composition of linear mappings
Let F be a field and U, V, W finite-dimensional vector spaces over F with ordered bases $\mathcal{A}, \mathcal{B}, \mathcal{C}$. If $f : U \to V$ and $g : V \to W$ are linear mappings, then the representing matrix of the composition $g \circ f : U \to W$ is the matrix product of the representing matrices of f and g

$${}_{\mathcal{C}}[g \circ f]_{\mathcal{A}} = {}_{\mathcal{C}} [g]_{\mathcal{B}} \circ_{\mathcal{B}} [f]_{\mathcal{A}}$$

Definition 2.3.3 *Representation of a vector with respect to a basis*
Let V be a finite-dimensional vector spaces with an ordered

basis $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_m)$ We denote the inverse to the bijection $\Phi_{\mathcal{A}} : F^m \to V, (\alpha_1, \dots, \alpha_m)^T \mapsto \alpha_1\vec{v}_1 + \cdots + \alpha_m\vec{v}_m$ by

$$\vec{v} \mapsto_{\mathcal{A}} [\vec{v}]$$

The column vector ${}_{\mathcal{A}}[\vec{v}]$ is called the *representation of the vector $\vec{v}$ with respect to the basis $\mathcal{A}$*.

Theorem 2.3.4 Representation of the image of a vector
Let V, W be finite-dimensional vector-spaces over F with ordered bases $\mathcal{A}, \mathcal{B}$ and let $f : V \to W$ be a linear mapping. The following holds for $\vec{v} \in V$:

$${}_{\mathcal{B}}[f(\vec{v})] = {}_{\mathcal{B}}[f]_{\mathcal{A}} \circ_{\mathcal{A}} [\vec{v}]$$

2.3 Change of a matrix by change of basis

Definition 2.4.1 *Change of basis matrix*
Let $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_n)$ and $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$ be ordered bases of the same F-vector space V. Then the matrix representing the identity mapping with respect to these bases ${}_{\mathcal{B}}[\mathrm{id}_V]_{\mathcal{A}}$ is called a *change of basis matrix*. By definition, its entries are given by the equalities $\vec{v}_j = \sum_{i=1}^{n} a_{ij}\vec{w}_i$.

Theorem 2.4.3 Change of basis
Let V and W be finite-dimensional vector-spaces over F and let $f : V \to W$ be a linear mapping. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V and $\mathcal{B}, \mathcal{B}'$ are ordered bases of W. Then

$${}_{\mathcal{B}'}[f]_{\mathcal{A}'} = {}_{\mathcal{B}'}[\mathrm{id}_W]_{\mathcal{B}} \circ_{\mathcal{B}} [\mathrm{f}]_{\mathcal{A}} \circ_{\mathcal{A}} [\mathrm{id}_V]_{\mathcal{A}'}$$

Corollary 2.4.4 Let V be a finite-dimensional vector-space and let $f : V \to V$ be an endomorphism of V. Suppose that $\mathcal{A}, \mathcal{A}'$ are ordered bases of V. Then

$${}_{\mathcal{A}'}[f]_{\mathcal{A}'} = {}_{\mathcal{A}'}[\mathrm{id}_V]_{\mathcal{A}'}^{-1} \circ_{\mathcal{A}} [\mathrm{f}]_{\mathcal{A}} \circ_{\mathcal{A}} [\mathrm{id}_V]_{\mathcal{A}'}$$

Theorem 2.4.5 Smith Normal Form
Let $f : V \to W$ be a linear mapping between finite-dimensional F-vector spaces. There exist an ordered basis $\mathcal{A}$ of V and an ordered basis $\mathcal{B}$ of W such that the representing matrix ${}_{\mathcal{B}}[f]_{\mathcal{A}}$ has zero entries everywhere except possibly on the diagonal, and along the diagonal there are 1s first, followed by 0s.

Definition 2.4.6 *Trace*
The *trace* of a square matrix is defined to be the sum of its diagonal entries. We denote this by $\mathrm{tr}(A)$

Definition *Nilpotent*
An endomorphism $f : V \to V$ of an F-vector space is called *nilpotent* if and only if there exists $d \in \mathbb{N}$ such that $f^d = 0$.

3 Rings and Modules

3.1 Rings

Definition 3.3.1 *Ring*
A *ring* is a set with two operations $(R, +, .)$ that satisfy

1. $(R, +)$ is an abelian group;
2. $(R, \cdot)$ is a *monoid*; this means that the second operation $\cdot : R \cdot R \to R$ is associative and that there is an *identity element* $1 = 1_R \in R$.
3. The distributive laws hold.

The two operations are called *addition* and *multiplication* in our ring.
A ring in which multiplication is commutative is a *commutative ring.*

Proposition 3.1.7 Divisibility by sum
A natural number is divisible by 3 (respectively 9) precisely when the sum of its digits is divisible by 3 (respectively 9).

Definition 3.1.8 *Field*
A *field* F is a non-zero commutative ring in which every non-zero element $a \in F$ has an inverse $a^{-1} \in F$.

Proposition 3.1.11
Let $m \in \mathbb{Z}^+$. The commutative ring $\mathbb{Z}/m\mathbb{Z}$ is a field if and only if m is prime.

3.2 Properties of rings

Lemma 3.2.1 Additive inverses
Let R be a ring and let $a, b \in R$. Then

1. $0a = 0 = a0$
2. $(-a)b = -(ab) = a(-b)$
3. $(-a)(-b) = ab$

Definition 3.2.3 *Multiple of an element*
Let $m \in \mathbb{Z}$. The *m-th multiple ma of an element* a in abelian group R is

$$ma = \underbrace{a + a + \cdots + a}_{m \text{ terms}} \quad \text{if } m > 0$$

$0a = 0$, and negative multiples are defined by $(-m)a = -(ma)$.

Lemma 3.2.4 Rules for multiples
Let R be a ring, let $a, b \in R$ and let $m, n \in \mathbb{Z}$. Then

1. $m(a + b) = ma + mb$;
2. $(m + n)a = ma + na$;
3. $m(na) = (mn)a$;
4. $m(ab) = (ma)b = a(mb)$;
5. $(ma)(nb) = (mn)(ab)$;

Definition 3.2.6 *Unit*
Let R be a ring. An element $a \in R$ is called a *unit* if it is invertible in R or (in other words) has a multiplicative inverse in R.

Proposition 3.2.10 The set $R^\times$ of units in a ring R forms a group under multiplication.

Definition 3.2.13 *Integral domains*
An *integral domain* is a non-zero commutative ring that has no zero-divisors.

Proposition 3.2.16 Cancellation law for integral domains
Let R be an integral domain and let $a, b, c \in R$.

$$ab = ac \text{ and } a \neq 0 \implies b = c$$

Proposition 3.2.17 Let $m \in \mathbb{N}$. Then $\mathbb{Z}/m\mathbb{Z}$ is an integral domain if and only if m is prime.

Theorem 3.2.18 Every *finite* integral domain is a field.

3.3 Polynomials

Definition 3.3.1 *Polynomials over rings*
Let R be a ring. A *polynomial over R* is an expression of the form

$$P = a_0 + a_1X + a_2X^2 + \cdots + a_mX^m$$

for some $m \in \mathbb{N}$ and elements $a_i \in R$ for $i \in [0, m]$.
The set of all polynomials over R is denoted by $R[X]$.
In case a_m is non-zero, the polynomial P has *degree m*, written $\deg(P)$, and a_m is its *leading coefficient*.
When the leading coefficient is 1, the polynomial is a *monic polynomial.*
A polynomial of degree one is called *linear*, a polynomial of degree two is called *quadratic*, and a polynomial of degree three is called *cubic*.

Definition 3.3.2 *Ring of polynomials*
The set $R[X]$ is a ring called the *ring of polynomials over R*. The zero and the identity of $R[X]$ are the zero and identity of R, respectively.

Lemma 3.3.3

1. If R is ring with no zero-divisors, then $R[X]$ has no zero-divisors and $\deg(PQ) = \deg(P) + \deg(Q)$ for non-zero $P, Q \in R[X]$.
2. If R is an integral domain, then so is $R[X]$

Theorem 3.3.4 Division and remainder
Let R be an integral domain, and let $P, Q \in R[X]$ with Q monic. Then there exists unique $A, B \in R[X]$ such that $P = AQ + B$ and $\deg(B) < \deg(Q)$ or $B = 0$.

Definition 3.3.6 *Evaluated & Root*
Let R be a commutative ring and $P \in R[X]$ a polynomial. Then the polynomial P can be *evaluated* at $\lambda \in R$ to produce $P(\lambda)$ by replacing the powers of X in the polynomial P by the corresponding powers of λ. This gives a mapping

$$R[X] \to \text{Maps}(R, R)$$

An element $\lambda \in R$ is a *root* of P if $P(\lambda) = 0$.

Proposition 3.3.9 Let R be a commutative ring, let $\lambda \in R$ and $P(X) \in R[X]$. Then λ is a root of $P(X)$ if and only if $(X - \lambda)$ divides $P(X)$.

Theorem 3.3.10 Let R a ring, or more generally, an integral domain. Then an non-zero polynomial $P \in R[X] \setminus \{0\}$ has at most $\deg(P)$ roots in R.

Definition 3.3.11 *Algebraically closed*
A field F is *algebraically closed* if each non-constant polynomial $P \in F[X] \setminus F$ with coefficients F has a root in F.

Theorem 3.3.13 *Fundamental theorem of algebra*
If F is an algebraically closed field, then every non-zero polynomial $P \in F[X] \setminus \{0\}$ *decomposes into linear factors*

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with $n \geq 0, c \in F^\times$ and $\lambda_1, \ldots, \lambda_n \in F$. This decomposition is unique up to reordering of the factors.

Theorem 3.3.14 If F is an algebraically closed field, then every non-zero polynomial $P \in F[X] \setminus \{0\}$ *decomposes into linear factors*

$$P = c(X - \lambda_1) \cdots (X - \lambda_n)$$

with $n \geq 0, c \in F^\times$ and $\lambda_1, \ldots, \lambda_n \in F$. This decomposition in unique up to reordering the factors.

3.4 Homomorphisms, Ideals, and Subrings

Definition 3.4.1 *Ring homomorphism*
Let R and S be rings. A mapping $f : R \to S$ is a *ring homomorphism* if the following hold $\forall x, y \in R$

$$f(x + y) = f(x) + f(y)$$
$$f(xy) = f(x)f(y)$$

Prelude to ideals
Let $f : R \to S$ be a ring homomorphism with $\ker f = \{r \in R : f(r) = 0_S\}$. Then $\ker f$ is:

- a subgroup of R under addition
- $0_R \in \ker f$
- closed under multiplication
- closed under left and right multiplication by arbitrary elements of R
  i.e. $x \in \ker f \implies rx, xr \in \ker f \ \forall r \in R$

Lemma 3.4.5 Let R and S be rings and $f : R \to S$ a ring homomorphism. Then $\forall x, y, \in R$ and $m \in \mathbb{Z}$

1. $f(0_R) = 0_S$
2. $f(-x) = -f(x)$
3. $f(x - y) = f(x) - f(y)$
4. $f(m \cdot x) = m \cdot f(x)$

Where mx denotes the m-th multiple of x.

Definition 3.4.7 *Ideal*
A subset I of a ring R is an *ideal*, written $I \trianglelefteq R$, if the following hold:

1. $I \neq \emptyset$
2. I is closed under subtraction (it's a subgroup)
3. $\forall i \in I$ and $\forall r \in R$ we have $ri, ir \in I$ (I is closed under multiplication by elements of R)

Ideals satisfy the properties of rings, except possibly the existence of a multiplicative identity.

Ideals are subrings which are closed under multiplication with elements from the *ring* — not just elements from within the ideal!

Definition 3.4.11 *Generated ideal*
Let R be a commutative ring and let $T \subset R$. Then the *ideal of R generated by T* is the set

$${}_R\langle T \rangle = \{r_1 t_1 + \cdots + r_m t_m : t_1, \ldots, t_m \in T, r_1, \ldots, r_m \in R\}$$

together with the zero element in the case $T = \emptyset$.

Proposition 3.4.14 Let R be a commutative ring and let $T \subseteq R$. Then ${}_R\langle T \rangle$ is the smallest ideal of R that contains T.

Definition 3.4.15 *Principal ideal*
Let R be a commutative ring. An ideal $I \trianglelefteq R$ is called a *principal ideal* if $I = \langle t \rangle$ for some $t \in R$.

Definition 3.4.17 *Kernel*
Let R and S be rings, and let $f : R \to S$ be a ring homomorphism. Since F is in particular a group homomorphism from $(R, +)$ to $(S, +)$, the *kernel* of f already has a meaning:

$$\ker f = \{r \in R : f(r) = 0_S\}$$

Proposition 3.4.18 Let R and S be rings and $f : R \to S$ a ring homomorphism. Then $\ker f$ is an ideal of R.

Lemma 3.4.20 f is injective if and only if $\ker f = \{0\}$

Lemma 3.4.21 The intersection of any collection of ideals of a ring R is an ideal of R.

Lemma 3.4.22 Let I and J be ideals of a ring R. Then

$$I + J = \{a + b : a \in I, b \in J\}$$

is an ideal of R.

Definition 3.4.23 *Subring*
Let R be a ring. A subset $R' \subseteq R$ is a *subring* of R if R' is itself a ring under the operations of addition and multiplication defined in R.

Proposition 3.4.26 Test for a subring
Let R be a ring, and $R' \subseteq R$. Then R' is a subring if and only if

1. R' has a multiplicative identity, and
2. R' is closed under subtraction, and
3. R' is closed under multiplication.

Proposition 3.4.29 Let R and S be rings and $f : R \to S$ a ring homomorphism.

1. If R' is a subring of R then $f(R')$ is a subring of S. In particular, f is a subring of S.

2. Assume that $f(1_R) = 1_S$. Then if x is a unit in R, $f(x)$ is a unit is in S and $(f(x))^{-1} = f(x^{-1})$. In this case f restricts to a group homomorphism $f|_{R^\times} : R^\times \to S^\times$.

3.5 Equivalence Relations

Definition 3.5.1 *Equivalence relation*
A *relation* R on a set X is a subset $R \subseteq X \times X$. R is an *equivalence relation on* X when $\forall x, y, z \in X$ the following hold:

1. *Reflexivity*: xRx
2. *Symmetry*: $xRy \iff yRx$
3. *Transitivity*: xRy and $yRz \implies xRz$

Definition 3.5.3
Suppose that $\sim$ is an equivalence relation on a set X. For $x \in X$ the set $E(x) \equiv \{z \in X : z \sim x\}$ is called the *equivalence class* of x.

A subset $E \subseteq X$ is called an *equivalence class* for $\sim$ if $\exists x \in X \ni E = E(x)$.

An element of an equivalence class is called a *representative* of the class.

A subset $Z \subseteq X$ containing precisely one element from each equivalence class is called a *system of representatives* for the equivalence relation.

Definition 3.5.5 *Set of equivalence classes*
Given an equivalence relation $\sim$ on the set X, the *set of equivalence classes*, which is a subset of $\mathcal{P}(X)$, is

$$(X/\sim) \equiv \{E(x) : x \in X\}$$

There is a canonical mapping can : $X \to (X/\sim)$, $x \mapsto E(x)$. It is obviously a surjection.

(I think it is also a homomorphism, which would then force $\overline{f}$ to also be a homomorphism, and thus facilitate the proof of the First Isomorphism Theorem.)

Remark
Suppose that $\sim$ is an equivalence relation on X. If $f : X \to Z$ is a mapping with the property that $x \sim y \implies f(x) = f(y)$, then there is a unique mapping $\overline{f} : (X \setminus \sim) \to Z$ with $f = \overline{f} \circ \text{can}$. Its definition is easy: $f(E(x)) = f(x)$. This property is called the *universal property of the set of equivalence classes.*

$$\begin{tikzcd} X \arrow[r, "\text{can}"] \arrow[dr, "f"'] & (X/\sim) \arrow[d, dashed, "\overline{f}"] \\ & Z \end{tikzcd}$$

Definition 3.5.7 *Well-defined*
A mapping $g : (X/\sim) \to Z$ is *well-defined* if there is a mapping $f : X \to Z$ such that f has the property $x \sim y \implies f(x) = f(y)$ and $g = \overline{f}$.

3.6 Factor Rings

Prelude
Let $f : R \to S$ be a ring homomorphism, such that

$$x \sim y \iff f(x) = f(y) \iff f(x-y) = 0 \iff x-y \in \ker f$$

Then:
$$E(x) = x + \ker f \equiv \{x + k : k \in \ker f\}$$

So we have that:

- the rule $x \sim y \iff x - y \in \ker f$ is an equivalence relation;
- the equivalence classes are the sets $x + \ker f$ for $x \in R$;
- the set of equivalence classes $(R / \sim)$ is a ring, isomorphic to a subring of S.

Definition 3.6.1 *Cosets*
Let $I \trianglelefteq R$ be an ideal in a ring R. The set

$$x + I \equiv \{x + i : i \in I\} \subseteq R$$

is a *coset of I in R*, or *the coset of x with respect to I in R.*

Definition 3.6.3 *Factor ring*
Let R be a ring, $I \trianglelefteq R$ be an ideal, and $\sim$ the equivalence relation defined by $x \sim y \iff x - y \in I$. Then R/I, the *factor ring of R by I* or the *quotient of R by I*, is the set $(R / \sim)$ of cosets of I in R.

$$R/I = \{r + I : r \in R\}$$

Theorem 3.6.4
Let R be a ring, and $I \trianglelefteq R$ an ideal. Then R/I is a ring, where the operation of addition is defined by

$$(x + I)\dot{+}(y + I) = (x + y) + I \quad \forall x, y \in R$$

and multiplication is defined by

$$(x + I) \cdot (y + I) = xy + I \quad \forall x, y \in R$$

Theorem 3.6.7 Universal Property of Factor Rings
Let R be a ring, and $I \trianglelefteq R$.

1. The mapping can : $R \to R/I$ with $\text{can}(r) = r + I$ is a surjective ring homomorphism with kernel I.
2. If $f : R \to S$ is a ring homomorphism with $f(I) = \{0_S\}$, so that $I \subseteq \ker f$, then there is a unique ring homomorphism $\overline{f} : R/I \to S$ such that $f = \overline{f} \circ \text{can}$.

Theorem 3.6.9 First Isomorphic Theorem for Rings
Let R and S be rings. Then every ring homomorphism $f : R \to S$ induces a ring isomorphism

$$\overline{f} : R/\ker f \tilde{\to} \text{im} f$$

3.7 Modules

Definition 3.7.1 A *(left) module M over a ring R* is a pair consisting of an abelian group $M = (M, \dot{+})$ and a mapping

$$R \times M \to M$$
$$(r, a) \mapsto ra$$

such that $\forall r, s \in R$ and $a, b \in M$ the following identities hold:

$$\begin{aligned} r(a\dot{+}b) &= (ra)\dot{+}(rb) && \text{(distributivity)} \\ (r + s)a &= (ra)\dot{+}(sa) && \text{(distributivity)} \\ r(sa) &= (rs)a && \text{(associativity)} \\ 1_R a &= a \end{aligned}$$

i.e. a vector space, but with a *ring* instead of a *field.*

Lemma 3.7.8 Let R be a ring, and M an R-module.

1. $0_R a = 0_M \ \forall a \in M$
2. $r0_M = 0_M \ \forall r \in R$
3. $(-r)a = r(-a) = -(ra), \quad \forall r \in R, a \in M$. (Here, the first negative is in R, and the last two negatives are in M.)

Definition 3.7.11 *R-homomorphism*
Let R be a ring, and let M, N be R-modules. A mapping $f : M \to N$ is an *R-homomorphism* if the following hold $\forall a, b \in M$ and $r \in R$:

$$f(a + b) = f(a) + f(b)$$
$$f(ra) = rf(a)$$

The *kernel* of f is $\ker f = \{a \in M : f(a) = 0_N\} \subseteq M$ and the *image* of f is $\text{im} f = \{f(a) : a \in M\} \subseteq N$.
If f is a bijection then it is an *isomorphism.*

Definition 3.7.15 *Submodule*
A non-empty subset M' of an R-module M is a *submodule* if M' is an R-module with respect to the operations of the R-module M *restricted* to M'.

Proposition 3.7.20 Test for a submodule
Let R be a ring and let M be an R-module. A subset $M' \subseteq M$ is a submodule if and only if

1. $0_M \in M'$
2. $a, b \in M' \implies a - b \in M'$
3. $r \in R, a \in M' \implies ra \in M'$

Lemma 3.7.21
Let $f : M \to N$ be an R-homomorphism. Then $\ker f$ is a submodule of M and $\mathrm{im} f$ is a submodule of N.

Lemma 3.7.22
Let R be a ring, let M and N be R-modules and let $f : M \to N$ be an R-homomorphism. Then f is injective if and only if $\ker f = \{0_M\}$.

Definition 3.7.23 *Generated submodule*
Let R be a ring, M an R-module, and let $T \subseteq M$. Then the *submodule of* M *generated by* T is the set

$$_R\langle T\rangle = \{r_1t_1 + \cdots + r_mt_m : t_1, \ldots, t_m \in T, r_1, \ldots, r_m \in R\},$$

together with the zero element in case $T = \emptyset$.
The module M is *finitely generated* if it is generated by a finite set: $M =_r \langle\{t_1, \ldots, t_n\}$.
It is *cyclic* f it is generated by a singleton: $M =_R \langle t\rangle$.

Lemma 3.7.28 Let $T \subseteq M$. Then $_r\langle T\rangle$ is the smallest submodule of M that contains T.

Lemma 3.7.29 The intersection of any collection of submodules of M is a submodule of M.

Lemma 3.7.30 Let M_1 and M_2 be submodules of M. Then

$$M_1 + M_2 = \{a + b : a \in M_1, b \in M_2\}$$

is a submodule of M.

Definition 3.7.31.1 *Coset*
Let R be a ring, M an R-module, and N a submodule of M. For each $a \in M$, the *coset of* a *with respect to* N *in* M is

$$a + N = \{a + b : b \in N\}.$$

It is a coset of N in the abelian group M and is is an equivalence class for the equivalence relation $a \sim b \iff a - b \in N$.

Definition 3.7.31.2 *Factor*
M/N, the *factor of* M *by* N or the *quotient of* M *by* N, is the set $(M / \sim)$ of all cosets of N in M.

$$M/N = \{a + N : a \in M\}$$

This becomes an R-module by introducing the operations of addition and multiplication as follows:

$$(a + N)\dot{+}(b + N) = (a + b) + N$$
$$r(a + N) = ra + N$$

for all $a, b \in M, r \in R$.

Theorem 3.7.31.3 *Factor module*

- The zero of M/N is the coset $0_{M/N} = 0_M + N$.
- The negative of $a + N \in M/N$ is the coset $-(a + N) = (-a) + N$.
- The R-module M/N is the *factor module* of M by the submodule N.

Theorem 3.7.32 The Universal Property of Factor Modules
Let R be a ring, and let L and M be R-modules, and N a sub-module of M.

1. The mapping $\mathrm{can} : M \to M/N$ sending a to $a+N$, $\forall a \in M$ is a surjective R-homomorphism with kernel N.
2. If $f : M \to L$ is an R-homomorphism with $f(N) = \{0_L\}$, so that $N \subseteq \ker f$, then there is a unique homomorphism $\overline{f} : M/N \to L$ such that $f = \overline{f} \circ \mathrm{can}$.

Theorem 3.7.33 First Isomorphism Theorem for Modules
Let R be a ring and let M and N be R-modules. Then every R-homomorphism $f : M \to N$ induces a R-isomorphism

$$\overline{f} : M/\ker f \to \mathrm{im} f$$

4 Determinants & Eigenvalues

4.1 The sign of a permutation

Definition 4.1.1 *Transposition*
The group of all permutations of the set $\{1, 2, \ldots, n\}$, also known as bijections from $\{1, 2, \ldots, n\}$ to itself, is denoted by $\mathfrak{S}_n$ and called the *n-th symmetric group*. It is a group under composition and has $n!$ elements.

A *transposition* is a permutation that swaps two elements of the set and leaves all the others unchanged.

Definition 4.1.2 *Inversion & Sign*
An *inversion* of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i, j) such that $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$. The number of inversions of the permutation σ is called the *length of* σ and written $\ell(\sigma)$. In formulas:

$$\ell(\sigma) = |\{(i, j) : i < j \text{ but } \sigma(i) > \sigma(j)\}|$$

The *sign of* σ is defined to be the parity of the number of inversions of σ. In formulas:

$$\mathrm{sgn}(\sigma) = (-1)^{\ell(\sigma)}$$

A permutation whose sign is $+1$, in other words which has even length, is called an *even permutation*, while a permutation whose sign is -1, in other words which has odd length, is called an *odd permutation.*

Lemma 4.1.5 (Multiplicativity of the sign)
For each $n \in \mathbb{N}$ the sign of a permutation produces a group homomorphism $\mathrm{sgn} : \mathfrak{S}_n \to \{+1, -1\}$ from the symmetric group to the two-element group of signs. In formulas:

$$\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) \quad \forall \sigma, \tau \in \mathfrak{S}_n$$

Definition 4.1.7 *Alternating group*
For $n \in \mathbb{N}$, the set of even permutations in $\mathfrak{S}_n$ forms a subgroup of $\mathfrak{S}_n$ because it is the kernel of the group homomorphism $\mathrm{sgn} : \mathfrak{S}_n \to \{+1, -1\}$. This group is the *alternating group* and is denoted A_n.

4.2 Determinants & what they mean

Definition 4.2.1 Let R be a commutative ring and $n \in \mathbb{N}$. The *determinant* is a mapping $\det : \mathrm{Mat}(n; R) \to R$ from square matrices with coefficients in R to the ring R that is given by the following formula:

$$A \mapsto \det(A) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)}$$

This formula is called the *Leibniz formula.*
The degenerate case $n = 0$ assigns the value 1 as the determinant of the "empty matrix".

The connection between determinants and volumes
The determinant of a matrix is equal to the scaling factor it performs.

The connection between determinants and orientation
The sign of the determinant determines the orientation: $\det = +1$ preserves the orientation; $\det = -1$ reverses the orientation.

4.3 Characterising the determininant

Definition 4.3.1 *Bi-linear forms*
Let U, V, W be F-vector spaces.
A *bi-linear form on $U \times V$ with values in W* is a mapping $H : U \times V \to W$ which is a linear mapping in both of its entries.
This means that it must satisfy the following properties for all $u_1, u_2 \in U;\ v_1, v_2 \in V;\ \lambda \in F$:

$$\begin{aligned} H(u_1 + u_2, v_1) &= H(u_1, v_1) + H(u_2, v_1) \\ H(u_1, v_1 + v_2) &= H(u_1, v_1) + H(u_1, v_2) \\ H(u_1, \lambda v_1) &= \lambda H(u_1, v_1) \\ H(\lambda u_1, v_1) &= \lambda H(u_1, v_1) \end{aligned}$$

The first two conditions state that for any fixed $v \in V$ the mapping $H(-, v) : U \to W$ is linear. H is a *bi-linear form.*
A bi-linear form H is *symmetric* if $U = V$ and

$$H(u, v) = H(v, u) \quad \forall u, v \in U$$

while it is *alternating* or *antisymmetric* if $U = V$ and

$$H(u, u) = 0 \quad \forall u \in U$$

Definition 4.3.3 *Multi-linear forms*
Let $V_1, \dots, V_n, W$ be F-vector spaces. A mapping $H : V_1 \times V_2 \times \cdots \times V_n \to W$ is a *multi-linear form* or *multi-linear* if for each j, the mapping $V_j \to W$ defined by $v_j \mapsto H(v_1, \dots, v_j, \dots, v_n)$, with $v_i \in V_i$ arbitrary fixed vectors of V_i for $i \neq j$, is linear. In the case $n = 2$, this is exactly the definition of a bi-linear mapping.

Definition 4.3.4 *Alternating*
Let V and W be F-vector spaces. A multi-linear form $H : V \times \cdots \times V \to W$ is *alternating* if it vanishes on every n-tuple of elements of V that has at least two entries equal, in other words if:

$$(\exists i \neq j \text{ with } v_i = v_j) \implies H(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = 0$$

In the case $n = 2$, this is exactly the definition of an alternating or anti-symmetric bi-linear mapping.

Theorem 4.3.6 Characterisation of the determinant
Let F be a field. The mapping

$$\det : \mathrm{Mat}(n; F) \to F$$

is the unique, alternating, multi-linear form on n-tuples of column vectors with values in F that takes the value 1_F on the identity matrix.

1. Is it a multi-linear form?
2. Does it go from $F^n \times \cdots \times F^n \to F$?
3. Is it alternating?
4. Does it take the value 1 on the identity?

If (and only if) answered *yes* to all, then we have a determinant.

4.4 Rules for calculating with determinants

Theorem 4.4.1 Multiplicativity of the determinant
Let R be a commutative ring and let $A, B \in \mathrm{Mat}(n; R)$. Then

$$\det(AB) = \det(A)\det(B)$$

Theorem 4.4.2 Determinantal criterion for invertibility
The determinant of a square matrix with entries in a field F is non-zero if and only if the matrix is invertible.

Lemma 4.4.4 The determinant of a square matrix and the transpose of the square matrix are equal, that is, for all $A \in \mathrm{Mat}(n; R)$ with R a commutative ring

$$\det(A^T) = \det(A)$$

Definition 4.4.6 *Cofactor*
Let $A \in \mathrm{Mat}(n; R)$ for some commutative ring R and $n \in \mathbb{N}$. Let $i, j \in (1, n) \subset \mathbb{N}$. Then the *$(i, j)$ cofactor of A* is $C_{ij} = (-1)^{i+j} \det(A\langle i, j\rangle)$ where $A\langle i, j\rangle$ is the matrix obtained by deleting the i-th row and the j-th column.

Theorem 4.4.7 Laplace's expansion of the determinant
Let $A = (a_{ij})$ be an $(n \times n)$ matrix with entries from a commutative ring R.

For a fixed i, the *i-th row expansion of the determinant* is

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

and for a fixed j, the *j-th column expansion of the determinant* is

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Definition 4.4.8 *Adjugate matrix*
Let A be an $(n \times n)$ matrix whose entries are $\mathrm{adj}(A)_{ij} = C_{ji}$ where C_{ji} is the (j, i) cofactor.

Theorem 4.4.9 Cramer's rule
Let A be an $(n \times n)$ matrix with entries in a commutative ring R. Then

$$A \cdot \mathrm{adj}(A) = (\det A) I_n$$

Corollary 4.4.11 Invertibility of matrices
A square matrix with entries in a commutative ring R is invertible if and only if its determinant is a unit in R. That is, $A \in \mathrm{Mat}(n; R)$ is invertible if and only if $\det(A) \in R^\times$.

4.5 Eigenvalues & Eigenvectors

Definition 4.5.1 *Eigenvalue*
Let $f : V \to V$ be an endomorphism of an F-vector space V. A scalar $\lambda \in F$ is an *eigenvalue* of f if and only if there exists a non-zero vector $\vec{v} \in V$ such that $f(\vec{v}) = \lambda \vec{v}$.

Each such vector is called an *eigenvector of f with eigenvalue λ*.

For any $\lambda \in F$, the *eigenspace of f with eigenvalue λ* is

$$E(\lambda, f) = \{\vec{v} \in V \ : \ f(\vec{v}) = \lambda \vec{v}\}$$

When $\lambda = 1$, this is equivalent to having a *fixed-point mapping*.

When $\lambda = 0$, this is equivalent to the *kernel* of the mapping.

The corresponding *eigenvectors* are the null-space of $(A - \lambda I_n)$

Theorem 4.5.4 Existence of Eigenvalues
Each endomorphism of a non-zero finite-dimensional vector space over an algebraically closed field has an eigenvalue.

Definition 4.5.6 *Characteristic polynomial*
Let R be a commutative ring and let $A \in \mathrm{Mat}(n; R)$ be a square matrix with entries in R. The polynomial $\det(A - xI_n) \in R[x]$ is called the *characteristic polynomial of the matrix A*. It is denoted by

$$\chi_A(x) \equiv \det(A - xI_n)$$

where χ stands for χaracteristic.

Theorem 4.5.8 Eigenvalues and characteristic polynomials
Let F be a field and $A \in \mathrm{Mat}(n; F)$ a square matrix with entries in F. The eigenvalues of the linear mapping $A : F^n \to F^n$ are exactly the roots of the characteristic polynomial χ_A.

Remark 4.5.9

1. Recall from *Example 3.5.2* that square matrices $A, B \in \mathrm{Mat}(n; R)$ of the same size are *conjugate* if
$$B = P^{-1}AP \in \mathrm{Mat}(n; R)$$
for an invertible $P \in \mathrm{GL}(n; R)$. Conjugacy is an equivalence relation on $\mathrm{Mat}(n; R)$.

2. The motivation for conjugacy comes from the various matrix representations of an endomorphism $f : V \to V$ of an n-dimensional vector space V over a field F. Let $A = (a_{ij}) =_{\mathcal{A}} [f]_{\mathcal{A}}, B = (b_{ij}) =_{\mathcal{B}} [f]_{\mathcal{B}} \in \mathrm{Mat}(n; F)$ be the matrices of f with respect to bases $\mathcal{A} = (\vec{v}_1, \dots, \vec{v}_n)$, $\mathcal{B} = (\vec{w}_1, \dots, \vec{w}_n)$ for V
$$f(\vec{v}_j) = \sum_{i=1}^{n} a_{ij}\vec{v}_i, \; f(\vec{w}_j) = \sum_{i=1}^{n} b_{ij}\vec{w}_i \in V.$$
The change of basis matrix $P = (p_{ij}) =_{\mathcal{A}} [id_V]_{\mathcal{B}} \in \mathrm{Mat}(n; F)$ is invertible, with
$$\vec{w}_j = \sum_{i=1}^{n} p_{ij}\vec{v}_i \in V.$$
We have the identity
$$B = P^{-1}AP \in \mathrm{Mat}(n; F)$$
so A, B are conjugate.

3. *Key observation:* the characteristic polynomials of conjugate $A, B \in \mathrm{Mat}(n; R)$ are the same
$$\begin{aligned}\chi_B(x) &= \det(B - xI_n) = \det(P^{-1}AP - xI_n) \\ &= \det(P^{-1}(A - xI_n)P) \\ &= \det(P)^{-1}\det(A - xI_n)\det(P) \\ &= \det(A - xI_n) = \chi_A(x) \in R[x]\end{aligned}$$

4. In view of (2) and (3) we can define the characteristic polynomial of an endomorphism $f : V \to V$ of an n-dimensional vector space over a field F to be
$$\chi_f(x) = \chi_A(x) \in F[x]$$
with $A =_{\mathcal{A}} [f]_{\mathcal{A}} \in \mathrm{Mat}(n; R)$ the matrix of f with respect to *any* basis $\mathcal{A}$ of V. Thanks to *Theorem 4.5.8* the eigenvalues of f are exactly the roots of χ_f, the characteristic polynomial of f.

4.6 Triangularisable & Diagonalisable

Proposition 4.6.1 Triangularisability
Let $f : V \to V$ be an endomorphism of a finite-dimensional F-vector space V. The following two statements are equivalent:

1. The vector space V has an ordered basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ such that
$$\begin{aligned}f(\vec{v}_1) &= a_{11}\vec{v}_1 \\ f(\vec{v}_2) &= a_{12}\vec{v}_1 + a_{22}\vec{v}_2 \\ &\vdots \\ f(\vec{v}_n) &= a_{1n}\vec{v}_1 + a_{2n}\vec{v}_2 + \cdots + a_{nn}\vec{v}_n \in V\end{aligned}$$
(so that the first basis vector $\vec{v}_1$ is an eigenvector, with eigenvalue a_{11}) or equivalently such that the $n \times n$ matrix ${}_{\mathcal{B}}[f]_{\mathcal{B}} = (a_{ij})$ representing f with respect to $\mathcal{B}$ is upper triangular.
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{1n} \\ 0 & 0 & a_{33} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$
When this happens, f is *triangularisable*.

2. The characteristic polynomial $\chi_{f(x)}$ of f decomposes into linear factors in $F[x]$.

Remark 4.6.2

1. An endomorphism $A : F^n \to F^n$ is triangularisable if and only if $A = (a_{ij})$ is conjugate to an upper triangular matrix $B = (b_{ij})$ ($b_{ij} = 0$ for $i > j$), with $P^{-1}AP = B$ for an invertible matrix P.

2. Any endomorphism of a finite dimensional $\mathbb{C}$-vector space (or any algebraically closed vector space) is triangularisable.

3. An endomorphism $f : V \to V$ of a n-dimensional F-vector space is triangularisable if and only if there is a sequence of subspaces
$$V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$
such that V_i is i-dimensional and $f(V_i) \subseteq V_i$.

Remark 4.6.4
A matrix $A \in \mathrm{Mat}(n\,; F)$ is nilpotent if and only if $\chi_A(x) = (-x)^n$.

Definition 4.6.5 *Diagonalisable*
An endomorphism $f : V \to V$ of an F-vector space V is *diagonalisable* if and only if there exists a basis of V consisting of eigenvectors of f.

If V is finite-dimensional, then this is the same as saying that there exists an ordered basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ such that the corresponding matrix representing f is diagonal, that is ${}_{\mathcal{B}}[f]_{\mathcal{B}} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. In this case, of course, $f(\vec{v}_i) = \lambda_i v_i$.

A square matrix $A \in \mathrm{Mat}(n; F)$ is *diagonalisable* if and only if the corresponding linear mapping $F^n \to F^n$ given by the left multiplication of A is diagonalisable. This just means that A is conjugate to a diagonal matrix: there exists an invertible matrix $P \in \mathrm{GL}(n; F)$ such that $P^{-1}AP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. In this case, the columns of P are the vectors of a basis of F^n consisting of eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$.

Lemma 4.6.8 Linear independence of Eigenvectors
Let $f : V \to V$ be an endomorphism of a vector space V and let $\vec{v}_1, \dots, \vec{v}_n$ be eigenvectors of f with pairwise different eigenvalues $\lambda_1, \dots, \lambda_n$.
Then the vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

Theorem 4.6.9 Cayley-Hamilton Theorem
Let $A \in \mathrm{Mat}(n; R)$ be a square matrix with entries in a commutative ring R. Then evaluating its characteristic polynomial $\chi_A(x) \in R[x]$ at the matrix A gives zero.

4.7 Google's PageRank Algorithm

Definition 4.7.5 *Markov matrix*
A matrix M whose entries are non-negative and such that the sum of the entries of each column equals 1 is a *Markov matrix* or a *stochastic matrix.*

Lemma 4.7.6
Suppose that $M \in \mathrm{Mat}(n; R)$ is a Markov matrix. Then $\lambda = 1$ is an eigenvalue of M.

Theorem 4.7.10 *Perron-Frobenius Theorem*
If $M \in \mathrm{Mat}(n; \mathbb{R})$ is a Markov matrix all of whose entries are positive, then the eigenspace $\mathrm{E}(1, M)$ is one dimensional. There there exists a unique basis vector $\vec{v} \in \mathrm{E}(1, M)$ all of whose entries are positive real numbers, $v_i > 0 \; \forall i$, and such that the sum of its entries is 1, $\sum_{i=1}^{n} v_i = 1$.

5 Inner Product Spaces

5.1 Inner Product Spaces: Definitions

Definition 5.1.1 *Real inner product space*
Let V be a vector space over R. An *inner product* on V is a mapping

$$(-,-) : V \times V \to \mathbb{R}$$

that satisfies the following for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda, \mu \in \mathbb{R}$:

1. $(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z})$ (bi-linear)
2. $(\vec{x}, \vec{y}) = (\vec{y}, \vec{x})$ (symmetric)
3. $(\vec{x}, \vec{x}) \geq 0$, with equality if and only if $\vec{x} = \vec{0}$. (positive definite)

A *real inner product space* is a real vector space endowed with an inner product.

Definition 5.1.3 *Complex inner product space*
Let V be a vector space over $\mathbb{C}$. An *inner product* on V is a mapping

$$(-,-) : V \times V \to \mathbb{C}$$

that satisfies the following for all $\vec{x}, \vec{y}, \vec{z} \in V$ and $\lambda, \mu \in \mathbb{C}$:

1. $(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda(\vec{x}, \vec{z}) + \mu(\vec{y}, \vec{z})$ (bi-linear)
2. $(\vec{x}, \vec{y}) = \overline{(\vec{y}, \vec{x})}$ (symmetric)
3. $(\vec{x}, \vec{x}) \geq 0$, with equality if and only if $\vec{x} = \vec{0}$. (positive definite)

Here $\overline{z}$ denotes the complex conjugate of z. A *complex inner product space* is a complex vector space endowed with an inner product.

Definition *Skew-linear*
A mapping $f : V \to W$ between complex vector spaces is *skew-linear* if $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ and $f(\lambda\vec{v}_1) = \overline{\lambda}f(\vec{v}_1)$ for all $\vec{v}_1, \vec{v}_2 \in V$ and all $\lambda \in \mathbb{C}$.

Definition *Sesquilinear*
A complex form that is *skew-linear* in its second variable. When such a form is commutative, it is *hermitian.*

Terminology

- A finite-dimensional real inner product space is a *Euclidean vector space.*
- A complex inner product space is a *unitary space* or *pre-Hilbert space.*
- A finite-dimensional inner product space is a *finite-dimensional Hilbert space.*

Definition 5.1.5 *Length or Inner Product Norm*
In a real or complex inner product space the *length* or *inner product norm* or *norm* $\|\vec{v}\| \in \mathbb{R}$ of a vector $\mathbf{v}$ is defined as the non-negative square root

$$\|\vec{v}\| = \sqrt{(\vec{v}, \vec{v})}$$

Vectors whose length is 1 are called *units.* Two vectors $\vec{v}, \vec{w}$ are *orthogonal* and we write

$$\vec{v} \perp \vec{w}$$

if and only if $(\vec{v}, \vec{w}) = 0$.

Definition 5.1.7 *Orthonormal family*
A family $(\vec{v}_i)_{i \in I}$ for vectors from an inner product space is an *orthogonal family* if all the vectors v_i have length 1 and if they are pairwise orthogonal to each other, which, using the Kronecker delta, means

$$(\vec{v}_i, \vec{v}_j) = \delta_{ij}$$

An orthonormal family that is a basis is an *orthonormal basis.*

Theorem 5.1.10
Every finite dimensional inner product space has an orthonormal basis.

5.2 Orthogonal Complements & Projections

Definition 5.2.1 *Orthogonal*
let V be an inner product space and let $T \subseteq V$ be an arbitrary subset. Define

$$T^{\perp} = \{\vec{v} \in V : \vec{v} \perp \vec{t},\ \forall \vec{t} \in T\},$$

calling this set the *orthogonal* to T.

Proposition 5.2.2
Let V be an inner product space and let U be a finite dimensional subspace of V. Then U and $U^{\perp}$ are complementary (*Definition 1.7.6*). In other words

$$V = U \oplus U^{\perp}$$

Definition 5.2.3 *Orthogonal complement*
Let U be a finite dimensional subspace of an inner product space V. The space $U^{\perp}$ is the *orthogonal complement* to U. The *orthogonal projection from V onto U* is the mapping

$$\pi_U : V \to V$$

that sends $\vec{v} = \vec{p} + \vec{r}$ to $\vec{p}$.
(With $\vec{v} \in U \oplus U^{\perp},\ p \in U,\ r \in U^{\perp}$.)

Proposition 5.2.4 Let U be a finite-dimensional subspace of an inner product space V and let π_U be the orthogonal projection from V to U.

1. π_U is a linear mapping with $\mathrm{im}(\pi_U) = U$ and $\ker(\pi_U) = U^{\perp}$.
2. If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis of U, then π_U is given by the following formula for all $\vec{v} \in V$

$$\pi_U(\vec{v}) = \sum_{i=1}^{n} (\vec{v}, \vec{v}_i)\vec{v}_i$$

3. $\pi_U^2 = \pi_U$, that is π_U is an idempotent.

Theorem 5.2.5 Cauchy-Schwarz Inequality
Let $\vec{v}, \vec{w}$ be vectors in an inner product space. Then

$$|(\vec{v}, \vec{w})| \leq \|\vec{v}\|\|\vec{w}\|$$

with equality if and only if $\vec{v}$ and $\vec{w}$ are linearly dependent.

Corollary 5.2.6
The norm $\|\cdot\|$ on an inner product space V satisfies, for any $\vec{v}, \vec{w} \in V$ and scalar λ:

1. $\|\vec{v}\| \geq 0$ with equality if and only if $\vec{v} = \vec{0}$
2. $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$
3. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$, the *triangle inequality.*

Theorem 5.2.7
Let $\vec{v}_1, \ldots, \vec{v}_k$ be linearly independent vectors in an inner product space V. Then there exists an orthonormal family $\vec{w}_1, \ldots, \vec{w}_k$ with the property that for all $1 \leq i \leq k$

$$\vec{w}_i \in \mathbb{R}_{<0}\ \vec{v}_i + \langle \vec{v}_{i-1}, \ldots, \vec{v}_1 \rangle$$

Gram-Schmidt process

$$\vec{u}_1 = \vec{v}_1, \qquad \vec{e}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|}$$
$$\vec{u}_2 = \vec{v}_2 - \pi_{\vec{u}_1}(\vec{v}_2), \qquad \vec{e}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|}$$
$$\vec{u}_3 = \vec{v}_3 - \pi_{\vec{u}_1}(\vec{v}_3) - \pi_{\vec{u}_2}(\vec{v}_3), \qquad \vec{e}_3 = \frac{\vec{u}_3}{\|\vec{u}_3\|}$$
$$\vdots \qquad \vdots$$
$$\vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \pi_{\vec{u}_j}(\vec{v}_k), \qquad \vec{e}_k = \frac{\vec{u}_k}{\|\vec{u}_k\|}$$

5.3 Adjoints & Self-Adjoints

Definition 5.3.1 *Adjoint*
Let V be an inner product space. Then two endomorphisms $T, S : V \to V$ are called *adjoint* to one another if the following holds for all $\vec{v}, \vec{w} \in V$:

$$(T\vec{v}, \vec{w}) = (\vec{v}, S\vec{w})$$

In this case, $S = T^*$, and S is the *adjoint* of T.

Theorem 5.3.4 Existence of the adjoint
Let V be a finite dimensional inner product space. Let $T : V \to V$ be an endomorphism. Then T^* exists. That is, there exists a unique linear mapping $T^* : V \to V$ such that for all $\vec{v}, \vec{w} \in V$

$$(T\vec{v}, \vec{w}) = (\vec{v}, T^*\vec{w})$$

Definition 5.3.5 *Self-adjoint*
An endomorphism of an inner product space $T : V \to V$ is *self-adjoint* if it is equal to its own adjoint, that is if $T^* = T$.

Theorem 5.3.7 Let $T : V \to V$ be a self-adjoint linear mapping of an inner product space V.

1. Every eigenvalue of T is real.
2. If λ and μ are distinct Eigenvalues of T with corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then $\vec{v}, \vec{w} = 0$.
3. T has an eigenvalue.

Theorem 5.3.9 The Spectral Theorem for Self-Adjoint Endomorphisms
Let V be a finite dimensional inner product space and let $T : V \to V$ be a self-adjoint linear mapping. Then V has an orthogonal basis consisting of eigenvectors of T.

Definition 5.3.11 *Orthogonal matrix*
An *orthogonal matrix* is an $n \times n$ matrix P with real entries such that $P^T P = I_n$. In other words, and orthogonal matrix is a square matrix P with real entries such that $P^{-1} = P^T$.

Corollary 5.3.12 The Spectral Theorem for Real Symmetric Matrices
Let A be a real $(n \times n)$-symmetric matrix. Then there is an $(n \times n)$-orthogonal matrix P such that

$$P^T AP = P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

where $\lambda_1, \dots, \lambda_n$ are the (necessarily real) eigenvalues of A, repeated according to their multiplicity as roots of the characteristic polynomial of A.

Definition 5.3.14 *Unitary matrix*
A *unitary matrix* is an $(n \times n)$-matrix P with complex entries such that $\overline{P}^T P = I_n$. In other words, a unitary matrix is a square matrix P with complex entries such that $P^{-1} = \overline{P}^T$.

Corollary 5.3.15 The Spectral Theorem for Hermitian Matrices
Let A be an $(n \times n)$-hermitian matrix. Then there is an $(n \times n)$-unitary matrix P such that

$$\overline{P}^T AP = P^{-1}AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

where $\lambda_1, \dots, \lambda_n$ are the (necessarily real) eigenvalues of A, repeated according to their multiplicity as roots of the characteristic polynomial of A.

6 Jordan Normal Form

6.1 Motivation

6.2 Statement of JNF & Strategy of Proof

Definition 6.2.1 *Nilpotent Jordan block*
Given an integer $r \geq 1$ define a $(r \times r)$-matrix $J(r)$, called the *nilpotent Jordan block of size* r, by the rule $J(r)_{ij} = 1$ for $j = i + 1$ and $J(r)_{ij} = 0$ otherwise.

$$J(r) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In particular $J(1)$ is (1×1)-matrix whose only entry is zero.

Given an integer $r \geq 1$ and a scalar $\lambda \in F$ define an $(r \times r)$-matrix $J(r, \lambda)$, called the *Jordan block of size* r *and eigenvalue* λ, by the rule

$$J(r, \lambda) = \lambda I_r + J(r) = D + N$$

with $\lambda I_r = \operatorname{diag}(\lambda, \lambda, \dots, \lambda) = D$ diagonal and $J(r) = N$ nilpotent

$$J(r, \lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

such that $DN = ND$.

Theorem 6.2.2 *Jordan Normal Form*
Let F be an algebraically closed field. Let V be a finite-dimensional vector space, and let $\phi : V \to V$ be an endomorphism of V with characteristic polynomial

$$\chi_\phi(x) = (\lambda_1 - x)^{a_1}(\lambda_2 - x)^{a_2} \cdots (\lambda_s - x)^{a_s} \in F[x] (a_i \geq 1, \sum_{i=1}^{s} a_i = n)$$

for distinct $\lambda_1, \lambda_2, \dots, \lambda_s \in F$. Then there exists an ordered basis $\mathcal{B}$ of V such that the matrix of ϕ with respect to the basis $\mathcal{B}$ is block diagonal with Jordan blocks on the diagonal

$$_{\mathcal{B}}[\phi]_{\mathcal{B}} = \operatorname{diag}(J(r_{1,1}, \lambda_1), \dots, J(r_{1,m_1}, \lambda_1), J(r_{2,1}, \lambda_2), \dots, J(r_{s,m_s}, \lambda_s))$$

with $r_{2,1}, \dots, r_{1,m_1}, r_{2,1}, \dots, r_{s,m_s} \geq 1$ such that

$$a_i = r_{i,1} + r_{i,2} + \cdots + r_{i,m_i} (1 \leq i \leq s)$$

6.3 The proof of Jordan Normal Form

Lemma 6.3.1 There exist polynomials $Q_j(x) \in F[x]$ such that

$$\sum_{j=1}^{s} P_j(x)Q_j(x) = 1$$

Definition 6.3.2 *Generalised eigenspace*
The *generalised eigenspace* of A with eigenvalue λ, $E^{\text{gen}}(\lambda, A)$, is the following subspace of V

$$E^{\text{gen}}(\lambda, A) = \{\vec{v} \in V : (A - \lambda \operatorname{id}_V)^r \vec{v} = \vec{0}\}$$

Remark 6.3.3 The actual eigenspace is defined by

$$E(\lambda, A) = \{\vec{v} \in V : (A - \lambda \mathrm{id}_V)\vec{v} = \vec{0}\}.$$

- $\dim(E(\lambda, A))$ is the *geometric multiplicity* of λ.
- $\dim(E^{\mathrm{gen}}(\lambda, A))$ is the *algebraic multiplicity* of λ.

Definition 6.3.4 *Stable*
Let $f : X \to X$ be a mapping from a set X to itself. A subset $Y \subseteq X$ is *stable under* f precisely when $f(Y) \subseteq Y$, that is if $y \in Y \implies f(y) \in Y$.

Proposition 6.3.5 The direct sum decomposition.
For each $1 \leq i \leq s$, let

$$\mathcal{B}_i = \{\vec{v}_{ij} \in V : 1 \leq j \leq a_i\}$$

is a basis of $E^{\mathrm{gen}}(\lambda_i, \phi)$, where a_i is the algebraic multiplicity of ϕ with eigenvalue λ_i, such that $\sum_{i=1}^{s} a_i = n$ is the dimension of V.

1. Each $E^{\mathrm{gen}}(\lambda_i, \phi)$ is stable under ϕ.
2. For each $\vec{v} \in V$ there exist unique $\vec{v}_i \in E^{\mathrm{gen}}(\lambda_i, \phi)$ such that $\vec{v} = \sum_{i=1}^{s} \vec{v}_i$. In other words, there is a direct sum decomposition

   $$V = \bigoplus_{i=1}^{s} E^{\mathrm{gen}}(\lambda_i, \phi)$$

   with ϕ restricting to endomorphism of the summands

   $$\phi_i = \phi| : E^{\mathrm{gen}}(\lambda_i, \phi) \to E^{\mathrm{gen}}(\lambda_i, \phi)$$

3. Then

   $$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_s = \{\vec{v}_i : 1 \leq i \leq s, 1 \leq j \leq a_i\}$$

   is a basis of V. The matrix of the endomorphism ϕ with respect to this basis is given by the block diagonal matrix

   $${}_{\mathcal{B}}[\phi]_{\mathcal{B}} = \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_s \end{pmatrix} \in \mathrm{Mat}(n; F)$$

   with $B_i =_{\mathcal{B}_i} [\phi_i]_{\mathcal{B}_i} \in \mathrm{Mat}(a_i; F)$.

Lemma 6.3.6 For each i, define a linear mapping

$$\psi_i : \frac{W}{W_{i-1}} \to \frac{W_{i-1}}{W_{i-2}}$$

by $\psi(\vec{w} + W_{i-1}) = \psi(\vec{w}) + W_{i-2}$ for $\vec{w} \in W_i$. Then ψ_i is well-defined and injective.

Lemma 6.3.7 Let $f : X \to Y$ be an injective linear mapping between the F-vector spaces X and Y. If $\{\vec{x}_1, \ldots, \vec{x}_t\}$ is a linearly independent set in X, then $\{f(\vec{x}_1), \ldots, f(\vec{x}_t)\}$ is a linearly independent set in Y.

Lemma 6.3.8 The set of elements $\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$ constructed in the algorithm above is a basis for W.

Proposition 6.3.9 Let $\mathcal{B}$ be the ordered basis of W constructed above ($\{\vec{v}_{j,k} : 1 \leq j \leq m, 1 \leq k \leq d_j\}$). Then

$$\begin{aligned} {}_{\mathcal{B}}[\psi]_{\mathcal{B}} = \mathrm{diag}\, & \underbrace{J(m), \ldots, J(m)}_{d_m \text{ times}} \\ & \underbrace{J(m-1), \ldots, J(m-1)}_{d_{m-1} - d_m \text{ times}}, \ldots, \underbrace{J(1), \ldots, J(1)}_{d_1 - d_2 \text{ times}} \end{aligned}$$

where $J(r)$ denotes the *nilpotent Jordan block of size R*.

6.4 Example of a Jordan Normal Form

6.5 PageRank and Jordan Normal Form

Lemma 6.5.1
If $M \in \mathrm{Mat}(n; \mathbb{R})$ is a Markov matrix all of whose entries are positive. Consider M as a complex matrix, all of whose entries happen to be real. If $\lambda \in \mathbb{C}$ is an eigenvalue of M, then either $\lambda = 1$ or $|\lambda| < 1$.

7 Reference

7.1 Terminology of Algebraic Structures

Single-operation structures

	Closure	*Associativity*	*Identity*	*Inverses*
Group	✓	✓	✓	✓
Monoid	✓	✓	✓	-
Semi-group	✓	✓	-	-
Magma	✓	-	-	-

Double-operation structures

Structure	*Addition*	*Multiplication*
Field	Abelian Group	Abelian Group
Division Ring	Abelian Group	Non-Abelian Group
Ring	Abelian Group	Monoid

7.2 Morphisms

Linear Mapping
Where V, W are vector spaces:
A linear mapping is a mapping $f : V \to W$ where the following hold:

$$f(\lambda \vec{v}_1 + \vec{w}_1) = \lambda f(\vec{v}_1) + f(\vec{w}_1)$$

(It is a homomorphism over vector spaces.) *Bi-linear forms*
Where U, V, W are vector spaces:
A bi-linear form is a mapping $f : U \times V \to W$ where the following hold:

$$\begin{aligned} f(u_1 + u_2, v_1) &= f(u_1, v_1) + f(u_2, v_1) \\ f(\lambda u_1, v_1) &= \lambda f(u_1, v_1) \end{aligned}$$

and again for the second parameter. *Homomorphism*
Where A, B are algebraic structures, a homomorphism $f : G \to H$ preserves the structure of the algebraic properties.

- Vector space homomorphism (Linear Mapping)

  $$\begin{aligned} f(x+y) &= f(x) + f(y) && \text{Addition-preservation} \\ f(x \cdot y) &= f(x) \cdot f(y) && \text{Multiplication-preservation} \end{aligned}$$

- Group homomorphism

  $$f(x+y) = f(x) + f(y) \qquad \text{Addition-preservation}$$

  Unity and inverse preservation follow from addition-preservation.

- Ring homomorphism

  $$\begin{aligned} f(x+y) &= f(x) + f(y) && \text{Addition-preservation} \\ f(x \cdot y) &= f(x) \cdot f(y) && \text{Multiplication-preservation} \\ f(e_G) &= e_H && \text{Unity-preservation} \end{aligned}$$

  Additive unity and inverse preservation follow.

Isomorphism A bijective homomorphism.

Endomorphism A homomorphism from a set to itself.

Automorphism A isomorphism from a set to itself.