

# Honours Analysis Notes

Anthony Catterwell

April 18, 2019

## Contents

<b>1</b>	<b>The Real Number System</b>	<b>2</b>
1.1	Introduction . . . . .	2
1.2	Ordered Field Axioms . . . . .	2
1.3	Completeness Axiom . . . . .	3
1.4	Mathematical Induction . . . . .	4
1.5	Inverse Functions and Images . . . . .	4
1.6	Countable and Uncountable Sets . . . . .	6
<b>2</b>	<b>Sequences in <math>\mathbb{R}</math></b>	<b>7</b>
2.1	Limits of Sequences . . . . .	7
2.2	Limit Theorems . . . . .	7
2.3	Bolzano-Weierstrass Theorem . . . . .	9
2.4	Cauchy Sequences . . . . .	9
2.5	Limits Supremum and Infimum . . . . .	9
<b>3</b>	<b>Functions on <math>\mathbb{R}</math></b>	<b>11</b>
3.1	Two-Sided Limits . . . . .	11
3.2	One-Sided Limits and Limits at Infinity . . . . .	12
3.3	Continuity . . . . .	13
3.4	Uniform Continuity . . . . .	14
<b>4</b>	<b>Differentiability on <math>\mathbb{R}</math></b>	<b>16</b>
4.1	The Derivative . . . . .	16
4.2	Differentiability Theorems . . . . .	16
4.3	Mean Value Theorem . . . . .	17
4.4	Taylor's Theorem and L'Hopital's Rule . . . . .	18
4.5	Inverse Function Theorems . . . . .	18
<b>5</b>	<b>Riemann Integration</b>	<b>19</b>
5.1	Introduction . . . . .	19
5.2	Step functions and their integrals . . . . .	19
5.3	Riemann-integrable functions and their integrals . . . . .	19
5.4	Fundamental Theorem of Calculus, and Practical Integration . . . . .	20
5.5	Integrals and uniform limits of sequences and series of functions . . . . .	21
5.6	A couple of odds and ends . . . . .	21
<b>6</b>	<b>Infinite Series of Real Numbers</b>	<b>22</b>
6.1	Introduction . . . . .	22
6.2	Series with Nonnegative Terms . . . . .	23
6.3	Absolute Convergence . . . . .	24
6.4	Alternating Series . . . . .	25

# 1 The Real Number System

## 1.1 Introduction

## 1.2 Ordered Field Axioms

- **Postulate 1** *Field Axioms*

There are functions  $+$  and  $\cdot$  defined on  $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ , which satisfy the following properties  $\forall a, b, c \in \mathbb{R}$

- *Closure Properties*:  $a + b, a \cdot b \in \mathbb{R}$
- *Associative Properties*:  $a + (b + c) = (a + b) + c$  and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- *Commutative Properties*:  $a + b = b + a$  and  $a \cdot b = b \cdot a$
- *Distributive Law*:  $a \cdot (b + c) = a \cdot b + a \cdot c$
- *Existence of Additive Identity*: There is a unique element  $0 \in \mathbb{R}$  such that  $0 + a = a$  for all  $a \in \mathbb{R}$
- *Existence of Multiplicative Identity*: There is a unique element  $1 \in \mathbb{R}$  such that  $1 \neq 0$  and  $1 \cdot a = a$  for all  $a \in \mathbb{R}$
- *Existence of Additive Inverses*: For every  $x \in \mathbb{R}$  there is a unique element  $-x \in \mathbb{R}$  such that

$$x + (-x) = 0$$

- *Existence of Multiplicative Inverses*: For every  $x \in \mathbb{R} \setminus \{0\}$  there is a unique element  $x^{-1} \in \mathbb{R}$  such that

$$x \cdot (x^{-1}) = 1$$

- **Postulate 2** *Order Axioms*

There is a relation  $<$  on  $\mathbb{R} \times \mathbb{R}$  that has the following properties:

- *Trichotomy Property*: Given  $a, b \in \mathbb{R}$ , one and only one of the following statements hold:

$$a < b, b < a, \text{ or } a = b$$

- *Transitive Property*: For  $a, b, c \in \mathbb{R}$

$$a < b \text{ and } b < c \implies a < c$$

- *Additive Property*: For  $a, b, c \in \mathbb{R}$

$$a < b \text{ and } c \in \mathbb{R} \implies a + c < b + c$$

- *Multiplicative Properties*: For  $a, b, c \in \mathbb{R}$

$$a < b \text{ and } c > 0 \implies ac < bc$$

and

$$a < b \text{ and } c < 0 \implies bc < ac$$

- **Remark 1.1**

We will assume that the sets  $\mathbb{N}$  and  $\mathbb{Z}$  satisfy the following properties:

1. If  $n, m \in \mathbb{Z}$ , then  $n + m, n - m$  and  $mn$  belong to  $\mathbb{Z}$
2. If  $n \in \mathbb{Z}$ , then  $n \in \mathbb{N}$  if and only if  $n \geq 1$

3. There is no  $n \in \mathbb{Z}$  that satisfies  $0 < n < 1$

• **Definition 1.4** *Absolute Value*

The *absolute value* of a number  $a \in \mathbb{R}$  is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

• **Remark 1.5** The *absolute value* is multiplicative; that is,  $|ab| = |a||b| \forall a, b \in \mathbb{R}$

• **Theorem 1.6** *Fundamental Theorem of Absolute Values*

Let  $a \in \mathbb{R}$  and  $M \geq 0$ . Then  $|a| \leq M \iff -M \leq a \leq M$ .

• **Theorem 1.7** The absolute value satisfies the following three properties:

1. *Positive Definite*: For all  $a \in \mathbb{R}$ ,  $|a| > 0$  with  $|a| = 0$  if and only if  $a = 0$ .
2. *Symmetric*: For all  $a, b \in \mathbb{R}$ ,  $|a - b| = |b - a|$ ,
3. *Triangle Inequalities*: For all  $a, b \in \mathbb{R}$ ,

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

• **Theorem 1.9** Let  $x, y, a \in \mathbb{R}$

1.  $x < y + \epsilon \forall \epsilon > 0 \iff x \leq y$
2.  $x > y - \epsilon \forall \epsilon > 0 \iff x \geq y$
3.  $|a| < \epsilon \forall \epsilon > 0 \iff a = 0$

### 1.3 Completeness Axiom

• **Definition 1.10** *Upper bounds*

Let  $E \subset \mathbb{R}$  be non-empty

1. The set  $E$  is said to be *bounded above* if and only if there is an  $M \in \mathbb{R}$  such that  $a \leq M$  for all  $a \in E$ , in which case  $M$  is called an *upper bound* of  $E$ .
2. A number  $s$  is called a *supremum* of the set  $E$  if and only if  $s$  is an upper bound of  $E$  and  $s \leq M$  for all upper bounds  $M$  of  $E$ . (In this case we shall say that  $E$  has a *finite supremum*  $s$  and write  $s = \sup E$ )

• **Remark 1.12** If a set has one upper bound, it has infinitely many upper bounds.

• **Remark 1.13** If a set has a supremum, then it has only one supremum.

• **Theorem** *Approximation Property for Suprema*

If  $E$  has a finite supremum and  $\epsilon > 0$  is any positive number, then there is a point  $a \in E$  such that

$$\sup E - \epsilon < a \leq \sup E$$

• **Theorem 1.15**

If  $E \subset \mathbb{Z}$  has a supremum, then  $\sup E \in E$ . In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

• **Postulate 3** *Completeness Axiom*

If  $E$  is a nonempty subset of  $\mathbb{R}$  that is bounded above, then  $E$  has a finite supremum.

• **Theorem 1.16** *The Archimedean Principle*

Given real numbers  $a$  and  $b$ , with  $a > 0$ , there is an integer  $n \in \mathbb{N}$  such that  $b < na$ .

- **Theorem 1.18** *Density of Rationals*

If  $a, b, \in \mathbb{R}$  satisfy  $a < b$ , then there is a  $q \in \mathbb{Q}$  such that  $a < q < b$ .

- **Definition 1.19** *Upper bounds*

Let  $E \subseteq \mathbb{R}$  be nonempty

1. The set  $E$  is said to be *bounded below* if and only if there is an  $m \in \mathbb{R}$  such that  $a \geq m$ , in which case  $m$  is called a *lower bound* of the set  $E$ .
2. A number  $t$  is called an *infimum* of the set  $E$  if and only if  $t$  is a lower bound of  $E$  and  $t \geq m$  and write  $t = \inf E$ .
3.  $E$  is said to be *bounded* if and only if it is bounded both above and below.

- **Theorem 1.20** *Reflection Principle*

Let  $E \subseteq \mathbb{R}$  be nonempty

1.  $E$  has a supremum if and only if  $-E$  has an infimum, in which case

$$\inf(-E) = -\sup E$$

2.  $E$  has an infimum if and only if  $-E$  has a supremum, in which case

$$\sup(-E) = -\inf E$$

- **Theorem 1.21** *Monotone Property*

Suppose that  $A \subseteq B$  are nonempty subsets of  $\mathbb{R}$ .

1. If  $B$  has a supremum, then  $\sup A \leq \sup B$ .
2. If  $B$  has an infimum, then  $\inf A \geq \inf B$ .

## 1.4 Mathematical Induction

- **Theorem 1.22** *Well-Ordering Principle*

If  $E$  is a nonempty subset of  $\mathbb{N}$ , then  $E$  has a least element (i.e.  $E$  has a finite infimum and  $\inf E \in E$ ).

- **Theorem 1.23**

Suppose for each  $n \in \mathbb{N}$  that  $A(n)$  is a proposition which satisfies the following two properties:

1.  $A(1)$  is true.
2. For every  $n \in \mathbb{N}$  for which  $A(n)$  is true,  $A(n+1)$  is also true.

Then  $A(n)$  is true for all  $n \in \mathbb{N}$ .

- **Theorem 1.26** *Binomial Formula*

If  $a, b, \in \mathbb{R}, n \in \mathbb{N}$  and  $0^0$  is interpreted to be 1, then

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

## 1.5 Inverse Functions and Images

- **Definition 1.29** *Injection, Surjection, Bijection*

Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$

1.  $f$  is said to be *injective* if and only if

$$x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \implies x_1 = x_2$$

2.  $f$  is said to be *surjective* if and only if

$$\forall y \in Y \exists x \in X \ni y = f(x)$$

3.  $f$  is called *bijective* if and only if it is both injective and surjective

• **Theorem 1.30**

Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$ . Then the following three statements are equivalent.

1.  $f$  has an inverse;
2.  $f$  is injective from  $X$  onto  $Y$ ;
3. There is a function  $g : Y \rightarrow X$  such that

$$g(f(x)) = x \quad \forall x \in X$$

and

$$f(g(y)) = y \quad \forall y \in Y$$

Moreover, for each  $f : X \rightarrow Y$ , there is only one function  $g$  that satisfies these. It is the inverse function  $f^{-1}$ .

• **Remark 1.31**

Let  $I$  be an interval and let  $f : I \rightarrow \mathbb{R}$ . If the derivative of  $f$  is either always positive on  $I$ , or always negative on  $I$ , then  $f$  is injective on  $I$ .

• **Definition 1.33** *Image*

Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$ . The *image* of a set  $E \subseteq X$  under  $f$  is the set

$$f(E) := \{y \in Y : y = f(x) \text{ for some } x \in E\}$$

The *inverse image* of a set  $E \subseteq Y$  under  $f$  is the set

$$f^{-1}(E) := \{x \in X : f(x) = y \text{ for some } y \in E\}$$

• **Definition 1.35** *Union, Intersection*

Let  $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$  be a collection of sets.

1. The *union* of the collection  $\mathcal{E}$  is the set

$$\bigcup_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$$

2. The *intersection* of the collection  $\mathcal{E}$  is the set

$$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

• **Theorem 1.36** *DeMorgan's Laws*

Let  $X$  be a set and  $\{E_\alpha\}_{\alpha \in A}$  be a collection of subsets of  $X$ . If for each  $E \subseteq X$  the symbol  $E^c$  represents the set  $X \setminus E$ , then

$$\left( \bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

and

$$\left( \bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

• **Theorem 1.37**

Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$ .

1. If  $\{E_\alpha\}_{\alpha \in A}$  is a collection of subsets of  $X$ , then

$$f\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f(E_\alpha) \quad \text{and} \quad f\left(\bigcap_{\alpha \in A} E_\alpha\right) \subseteq \bigcap_{\alpha \in A} f(E_\alpha)$$

2. If  $B$  and  $C$  are subsets of  $X$ , then  $f(C) \setminus f(B) \subseteq f(C \setminus B)$

3. If  $\{E_\alpha\}_{\alpha \in A}$  is a collection of subsets of  $Y$ , then

$$f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha)$$

4. If  $B$  and  $C$  are subsets of  $Y$ , then  $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$ .

5. If  $E \subseteq f(X)$ , then  $f(f^{-1}(E)) = E$ , but if  $E \subseteq X$ , then  $E \subseteq f^{-1}(f(E))$ .

## 1.6 Countable and Uncountable Sets

• **Definition 1.38** *Countable Uncountable* Let  $E$  be a set.

1.  $E$  is said to be *finite* if and only if either  $E = \emptyset$  or there exists an injective function which takes  $\{1, 2, \dots, n\}$  onto  $E$ , for some  $n \in \mathbb{N}$ .
2.  $E$  is said to be *countable* if and only if there exists an injective function which takes  $\mathbb{N}$  onto  $E$ .
3.  $E$  is said to be *at most countable* if and only if  $E$  is either finite or countable.
4.  $E$  is said to be *uncountable* if and only if  $E$  is neither finite nor countable.

• **Remark 1.39** *Cantor's Diagonalisation Argument*

The open interval  $(0, 1)$  is uncountable.

• **Lemma 1.40**

A nonempty set  $E$  is at most countable if and only if there is a function  $g$  from  $\mathbb{N}$  onto  $E$ .

• **Theorem 1.41**

Suppose  $A$  and  $B$  are sets.

1. If  $A \subseteq B$  and  $B$  is at most countable, then  $A$  is at most countable.
2. If  $A \subseteq B$  and  $A$  is uncountable, then  $B$  is uncountable.
3.  $\mathbb{R}$  is uncountable.

• **Theorem 1.42**

Let  $A_1, A_2, \dots$  be at most countable sets.

1. Then  $A_1 \times A_2$  is at most countable.
2. If

$$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j \in \mathbb{N}} A_j := \{x : x \in A_j \text{ for some } j \in \mathbb{N}\},$$

then  $E$  is at most countable.

• **Remark 1.43**

The sets  $\mathbb{Z}$  and  $\mathbb{Q}$  are countable, but the set of irrationals is uncountable.

## 2 Sequences in $\mathbb{R}$

### 2.1 Limits of Sequences

- **Definition 2.1** *Convergence*

A sequence of real numbers  $\{x_n\}$  is set to *converge* to a real number  $a \in \mathbb{R}$  if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  (which in general depends on  $\epsilon$ ) such that

$$n \geq N \implies |x_n - a| < \epsilon$$

- **Remark 2.4** A sequence can have at most one limit.

- **Definition 2.5** *Subsequence*

By a *subsequence* of a sequence  $\{x_n\}_{n \in \mathbb{N}}$ , we shall mean a sequence of the form  $\{x_{n_k}\}_{k \in \mathbb{N}}$ , where each  $n_k \in \mathbb{N}$  and  $n_1 < n_2 < \dots$ .

- **Remark 2.6**

If  $\{x_n\}_{n \in \mathbb{N}}$  converges to  $a$  and  $\{x_{n_k}\}_{k \in \mathbb{N}}$  is any subsequence of  $\{x_n\}_{n \in \mathbb{N}}$ , then  $x_{n_k}$  converges to  $a$  as  $k \rightarrow \infty$ .

- **Definition 2.7** *Bounded Sequences*

Let  $\{x_n\}$  be a sequence of real numbers.

1. The sequence  $\{x_n\}$  is said to be *bounded above* if and only if the set  $\{x_n : n \in \mathbb{N}\}$  is bounded above.
2. The sequence  $\{x_n\}$  is said to be *bounded below* if and only if the set  $\{x_n : n \in \mathbb{N}\}$  is bounded below.
3.  $\{x_n\}$  is said to be *bounded* if and only if it is bounded both above and below.

- **Theorem 2.8** Every convergent sequence is bounded.

### 2.2 Limit Theorems

- **Theorem 2.9** *Squeeze Theorem*

Suppose that  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{w_n\}$  are real sequences.

1. If  $x_n \rightarrow a$  and  $y_n \rightarrow a$  as  $n \rightarrow \infty$ , and if there is an  $N_0 \in \mathbb{N}$  such that

$$x_n \leq w_n \leq y_n \text{ for } n \geq N_0$$

then  $w_n \rightarrow a$  as  $n \rightarrow \infty$ .

2. If  $x_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $\{y_n\}$  is bounded, then  $x_n y_n \rightarrow 0$  as  $n \rightarrow \infty$ .

- **Theorem 2.11**

Let  $E \subset \mathbb{R}$ . If  $E$  has a finite supremum (respectively, a finite infimum), then there is a sequence  $x_n \in E$  such that  $x_n \rightarrow \sup E$  (respectively, a sequence  $y_n \in E$  such that  $y_n \rightarrow \inf E$ ) as  $n \rightarrow \infty$ .

- **Theorem 2.12**

Suppose that  $\{x_n\}$  and  $\{y_n\}$  are real sequences and that  $\alpha \in \mathbb{R}$ . If  $\{x_n\}$  and  $\{y_n\}$  are convergent, then

- 1.

$$\lim_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n$$

- 2.

$$\lim_{n \rightarrow \infty} (\alpha x_n) = \alpha \lim_{n \rightarrow \infty} x_n$$

and

3.

$$\lim_{n \rightarrow \infty} (x_n y_n) = \left( \lim_{n \rightarrow \infty} x_n \right) \left( \lim_{n \rightarrow \infty} y_n \right)$$

If, in addition,  $y_n \neq 0$  and  $\lim_{n \rightarrow \infty} y_n \neq 0$ , then

4.

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}$$

(In particular, all these limits exist.)

• **Definition 2.14** *Divergence*

Let  $\{x_n\}$  be a sequence of real numbers.

1.  $\{x_n\}$  is said to *diverge* to  $+\infty$  if and only if for each  $M \in \mathbb{R}$  there is an  $N \in \mathbb{N}$  such that

$$n \geq N \implies x_n > M$$

2.  $\{x_n\}$  is said to *diverge* to  $-\infty$  if and only if for each  $M \in \mathbb{R}$  there is an  $N \in \mathbb{N}$  such that

$$n \geq N \implies x_n < M$$

• **Theorem 2.15**

Suppose that  $\{x_n\}$  and  $\{y_n\}$  are real sequences such that  $x_n \rightarrow +\infty$  (respectively,  $x_n \rightarrow -\infty$ ) as  $n \rightarrow \infty$ .

1. If  $y_n$  is bounded below (respectively,  $y_n$  is bounded above), then

$$\lim_{n \rightarrow \infty} (x_n + y_n) = +\infty \quad (\text{respectively, } \lim_{n \rightarrow \infty} (x_n + y_n) = -\infty)$$

2. If  $\alpha > 0$ , then

$$\lim_{n \rightarrow \infty} (\alpha x_n) = +\infty \quad (\text{respectively, } \lim_{n \rightarrow \infty} (\alpha x_n) = -\infty)$$

3. If  $y_n > M_0$  for some  $M_0 > 0$  and all  $n \in \mathbb{N}$ , then

$$\lim_{n \rightarrow \infty} (x_n y_n) = +\infty \quad (\text{respectively, } \lim_{n \rightarrow \infty} (x_n y_n) = -\infty)$$

4. If  $\{y_n\}$  is bounded and  $x_n \neq 0$ , then

$$\lim_{n \rightarrow \infty} \frac{y_n}{x_n} = 0$$

• **Corollary 2.16**

Let  $\{x_n\}, \{y_n\}$  be real sequences and  $\alpha, x, y$  be extended real numbers. If  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , as  $n \rightarrow \infty$ , then

$$\lim_{n \rightarrow \infty} (x_n + y_n) = x + y$$

provided that the right side is not of the form  $\infty - \infty$ , and

$$\lim_{n \rightarrow \infty} (\alpha x_n) = \alpha x, \quad \lim_{n \rightarrow \infty} (x_n y_n) = xy$$

provided that none of these products is of the form  $0 \cdot \pm\infty$ .

• **Theorem 2.17** *Comparison Theorem*

Suppose that  $\{x_n\}$  and  $\{y_n\}$  are convergent sequences. If there is an  $N_0 \in \mathbb{N}$  such that

$$x_n \leq y_n \text{ for } n \geq N_0$$

then

$$\lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} y_n$$

In particular, if  $x_n \in [a, b]$  converges to some point  $c$ , then  $c$  must belong to  $[a, b]$ .



## 2.3 Bolzano-Weierstrass Theorem

- **Definition 2.18** *Increasing, Decreasing* Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence of real numbers.
  1.  $\{x_n\}$  is said to be *increasing* (respectively, *strictly increasing*) if and only if  $x_1 \leq x_2 \leq \dots$  (respectively,  $x_1 < x_2 < \dots$ ).
  2.  $\{x_n\}$  is said to be *decreasing* (respectively, *strictly decreasing*) if and only if  $x_1 \geq x_2 \geq \dots$  (respectively,  $x_1 > x_2 > \dots$ ).
  3.  $\{x_n\}$  is said to be *monotone* if and only if it is either increasing or decreasing.
- **Theorem 2.19** *Monotone Convergence Theorem*  
if  $\{x_n\}$  is increasing and bounded above, or if  $\{x_n\}$  is decreasing and bounded below, then  $\{x_n\}$  converges to a finite limit.
- **Definition 2.22** *Nested*  
A sequence of sets  $\{I_n\}_{n \in \mathbb{N}}$  is said to be *nested* if and only if

$$I_1 \supseteq I_2 \supseteq \dots$$

- **Theorem 2.23** *Nested Interval Property*  
If  $\{I_n\}_{n \in \mathbb{N}}$  is a nested sequence of nonempty closed bounded intervals, then  $E := \bigcap_{n=1}^{\infty} I_n$  is nonempty. Moreover, if the lengths of these intervals satisfy  $|I_n| \rightarrow 0$  as  $n \rightarrow \infty$  then  $E$  is a single point.
- **Remark 2.24** The Nested Interval Property might not hold if “closed” is omitted.
- **Remark 2.25** The Nested Interval Property might not hold if “bounded” is omitted.
- **Theorem 2.26** *Bolzano-Weierstrass Theorem*  
Every bounded sequence of real numbers has a convergent subsequence.

## 2.4 Cauchy Sequences

- **Definition 2.27** *Cauchy*  
A sequence of points  $x_n \in \mathbb{R}$  is said to be *Cauchy* (in  $\mathbb{R}$ ) if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that
 
$$n, m \geq N \implies |x_n - x_m| < \epsilon$$
- **Remark 2.28** If  $\{x_n\}$  is convergent, then  $\{x_n\}$  is Cauchy.
- **Theorem 2.29** *Cauchy*  
Let  $\{x_n\}$  be a sequence of real numbers. Then  $\{x_n\}$  is Cauchy if and only if  $\{x_n\}$  converges (to some point  $a \in \mathbb{R}$ ).
- **Remark 2.31** A sequence that satisfies  $x_{n+1} - x_n \rightarrow 0$  is not necessarily Cauchy.

## 2.5 Limits Supremum and Infimum

- **Definition 2.32** *Limit Supremum & Infimum*  
Let  $\{x_n\}$  be a real sequence. Then the *limit supremum* of  $\{x_n\}$  is the extended real number

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} (\sup_{k \geq n} x_k)$$

and the *limit infimum* of  $\{x_n\}$  is the extended real number

$$\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} (\inf_{k \geq n} x_k)$$

- **Theorem 2.35**

Let  $\{x_n\}$  be a sequence of real numbers,  $s = \limsup_{n \rightarrow \infty} x_n$ , and  $t = \liminf_{n \rightarrow \infty} x_n$ . Then there are subsequences  $\{x_{n_k}\}_{k \in \mathbb{N}}$  and  $\{x_{\ell_j}\}_{j \in \mathbb{N}}$  such that  $x_{n_k} \rightarrow s$  as  $k \rightarrow \infty$  and  $x_{\ell_j} \rightarrow t$  as  $j \rightarrow \infty$ .

- **Theorem 2.36**

Let  $\{x_n\}$  be a real sequence and  $x$  be an extended real number. Then  $x_n \rightarrow x$  as  $n \rightarrow \infty$  if and only if

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = x$$

- **Theorem 2.37**

Let  $\{x_n\}$  be a sequence of real numbers. Then  $\limsup_{n \rightarrow \infty} x_n$  (respectively,  $\liminf_{n \rightarrow \infty} x_n$ ) is the largest value (respectively, the smallest value) to which some subsequences of  $\{x_n\}$  converges. Namely, if  $x_{n_k} \rightarrow x$  as  $k \rightarrow \infty$ , then

$$\liminf_{n \rightarrow \infty} x_n \leq x \leq \limsup_{n \rightarrow \infty} x_n$$

- **Remark 2.38** If  $\{x_n\}$  is any sequence of real numbers, then

$$\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n$$

- **Remark 2.39** A real sequence  $\{x_n\}$  is bounded above if and only if  $\limsup_{n \rightarrow \infty} x_n < \infty$ , and is bounded below if and only if  $\liminf_{n \rightarrow \infty} x_n > -\infty$ .

- **Theorem 2.40**

If  $x_n \leq y_n$  for  $n$  large, then

$$\limsup_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} y_n \quad \text{and} \quad \liminf_{n \rightarrow \infty} y_n \leq \liminf_{n \rightarrow \infty} x_n$$

### 3 Functions on $\mathbb{R}$

#### 3.1 Two-Sided Limits

- **Definition 3.1** *Limits*

Let  $a \in \mathbb{R}$ , let  $I$  be an open interval which contains  $a$ , and let  $f$  be a real function defined everywhere on  $I$  except possibly at  $a$ . Then  $f(x)$  is said to *converge to  $L$ , as  $x$  approaches  $a$* , if and only if for every  $\epsilon > 0$  there is a  $\delta > 0$  (which in general depends on  $\epsilon, f, I$ , and  $a$ ) such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

In this case we write

$$L = \lim_{x \rightarrow a} f(x) \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow a$$

and call  $L$  the *limit* of  $f(x)$  as  $x$  approaches  $a$ .

- **Remark 3.4**

Let  $a \in \mathbb{R}$ , let  $I$  be an open interval which contains  $a$ , and let  $f, g$  be real functions defined everywhere on  $I$  except possibly at  $a$ . If  $f(x) = g(x)$  for all  $x \in I \setminus \{a\}$  and  $f(x) \rightarrow L$  as  $x \rightarrow a$ , then  $g(x)$  also has a limit as  $x \rightarrow a$ , and

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x)$$

- **Theorem 3.6** *Sequential Characterisation of Limits*

Let  $a \in \mathbb{R}$ , let  $I$  be an open interval which contains  $a$ , and let  $f$  be a real function defined everywhere on  $I$  except possibly at  $a$ . Then

$$L = \lim_{x \rightarrow a} f(x)$$

exists if and only if  $f(x_n) \rightarrow L$  as  $n \rightarrow \infty$  for every sequence  $\{x_n\} \in I \setminus \{a\}$  which converges to  $a$  as  $n \rightarrow \infty$ .

- **Theorem 3.8**

Suppose that  $a \in \mathbb{R}$ , that  $I$  is an open interval which contains  $a$ , and that  $f, g$ , are real functions defined everywhere on  $I$  except possibly at  $a$ . If  $f(x)$  and  $g(x)$  converge as  $x$  approaches  $a$ , then so do  $(f + g)(x)$ ,  $(fg)(x)$ ,  $(\alpha f)(x)$ , and  $(f/g)(x)$  (when the limit of  $g(x)$  is nonzero). In fact,

$$\begin{aligned} \lim_{x \rightarrow a} (f + g)(x) &= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) \\ \lim_{x \rightarrow a} (\alpha f)(x) &= \alpha \lim_{x \rightarrow a} f(x) \\ \lim_{x \rightarrow a} (fg)(x) &= \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) \end{aligned}$$

and (when the limit of  $g(x)$  is nonzero)

$$\lim_{x \rightarrow a} \left( \frac{f}{g} \right)(x) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

- **Theorem 3.9** *Squeeze Theorem for Functions*

Suppose that  $a \in \mathbb{R}$ , that  $I$  is an open interval which contains  $a$ , and that  $f, g, h$  are real functions defined everywhere on  $I$  except possibly at  $a$ .

1. If  $g(x) \leq h(x) \leq f(x) \forall x \in I \setminus \{a\}$ , and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = L,$$

then the limit of  $h(x)$  exists, as  $x \rightarrow a$ , and

$$\lim_{x \rightarrow a} h(x) = L.$$

2. If  $|g(x)| \leq M \forall x \in I \setminus \{a\}$  and  $f(x) \rightarrow 0$  as  $x \rightarrow a$ , then

$$\lim_{x \rightarrow a} f(x)g(x) = 0$$

• **Theorem 3.10** *Comparison Theorem for Functions*

Suppose that  $a \in \mathbb{R}$ , that  $I$  is an open interval which contains  $a$ , and that  $f, g$  are real functions defined everywhere on  $I$  except possibly at  $a$ . If  $f$  and  $g$  have a limit as  $x$  approaches  $a$  and  $f(x) \leq g(x) \forall x \in I \setminus \{a\}$ , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

### 3.2 One-Sided Limits and Limits at Infinity

• **Definition 3.12** *Converge from left & right*

Let  $a \in \mathbb{R}$  and  $f$  be a real function.

1.  $f(x)$  is said to *converge to  $L$  as  $x$  approaches  $a$  from the right* if and only if  $f$  is defined on some open interval  $I$  with left endpoint  $a$  and for every  $\epsilon > 0$  there is a  $\delta > 0$  (which in general depends on  $\epsilon, f, I$ , and  $a$ ) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call  $L$  the *right-hand limit* of  $f$  at  $a$ , and denote it by

$$f(a+) := L =: \lim_{x \rightarrow a+} f(x)$$

2.  $f(x)$  is said to *converge to  $L$  as  $x$  approaches  $a$  from the left* if and only if  $f$  is defined on some open interval  $I$  with left endpoint  $a$  and for every  $\epsilon > 0$  there is a  $\delta > 0$  (which in general depends on  $\epsilon, f, I$ , and  $a$ ) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call  $L$  the *left-hand limit* of  $f$  at  $a$ , and denote it by

$$f(a-) := L =: \lim_{x \rightarrow a-} f(x)$$

• **Theorem 3.14**

Let  $f$  be a real function. Then the limit

$$\lim_{x \rightarrow a} f(x)$$

exists and equals  $L$  if and only if

$$L = \lim_{x \rightarrow a+} f(x) = \lim_{x \rightarrow a-} f(x)$$

• **Definition 3.15** *Convergence*

Let  $a, L \in \mathbb{R}$  and let  $f$  be a real function.

1.  $f(x)$  is said to *converge to  $L$  as  $x \rightarrow \infty$*  if and only if there exists a  $c > 0$  such that  $(c, \infty) \subset \text{Dom}(f)$  and given  $\epsilon > 0$  there is an  $M \in \mathbb{R}$  such that  $x > M$  implies  $|f(x) - L| < \epsilon$ , in which case we shall write

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow \infty$$

Similarly,  $f(x)$  is said to *converge to  $L$  as  $x \rightarrow -\infty$*  if and only if there exists a  $c > 0$  such that  $(-\infty, -c) \subset \text{Dom}(f)$  and given  $\epsilon > 0$  there is  $M \in \mathbb{R}$  such that  $x > M$  implies  $|f(x) - L| < \epsilon$ , in which case we shall write

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow -\infty$$

2. The function  $f(x)$  is said to converge to  $\infty$  as  $x \rightarrow a$  if and only if there is an open interval  $I$  containing  $a$  such that  $I \setminus \{a\} \subset \text{Dom}(f)$  and given  $M \in \mathbb{R}$  there is a  $\delta > 0$  such that  $0 \leq |x - a| < \delta$  implies  $f(x) > M$ , in which case we shall write

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{or} \quad f(x) \rightarrow \infty \text{ as } x \rightarrow a$$

Similarly,  $f(x)$  is said to *converge* to  $-\infty$  as  $x \rightarrow a$  if and only if there is an open interval  $I$  containing  $a$  such that  $I \setminus \{a\} \subset \text{Dom}(f)$  and given  $M \in \mathbb{R}$  there is a  $\delta > 0$  such that  $0 < |x - a| < \delta$  implies  $f(x) < M$ , in which case we shall write

$$\lim_{x \rightarrow a} f(x) = -\infty \quad \text{or} \quad f(x) \rightarrow -\infty \text{ as } x \rightarrow a$$

• **Theorem 3.17**

Let  $a$  be an extended real number, and let  $I$  be a nondegenerate open interval which either contains  $a$  or has  $a$  as one of its endpoints. Suppose further that  $f$  is a real function defined on  $I$  except possibly at  $a$ . Then

$$\lim_{x \rightarrow a; x \in I} f(x)$$

exists and equals  $L$  if and only if  $f(x_n) \rightarrow L$  for all sequences  $x_n \in I$  which satisfy  $x_n \neq a$  and  $x_n \rightarrow a$  as  $n \rightarrow \infty$ .

### 3.3 Continuity

• **Definition 3.19** *Continuous*

Let  $E$  be a nonempty subset of  $\mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ .

1.  $f$  is said to be *continuous at a point*  $a \in \mathbb{E}$  if and only if given  $\epsilon > 0$  there is a  $\delta > 0$  (which in general depends on  $\epsilon$ ,  $f$ , and  $a$ ) such that

$$|x - a| < \delta \quad \text{and} \quad x \in E \implies |f(x) - f(a)| < \epsilon$$

2.  $f$  is said to be *continuous on*  $E$  if and only if  $f$  is continuous at every  $x \in E$ .

• **Remark 3.20**

Let  $I$  be an open interval which contains a point  $a$  and  $f : I \rightarrow \mathbb{R}$ . Then  $f$  is continuous at  $a \in I$  if and only if

$$f(a) = \lim_{x \rightarrow a} f(x)$$

• **Theorem 3.21**

Suppose that  $E$  is a nonempty subset of  $\mathbb{R}$ , that  $a \in E$ , and that  $f : E \rightarrow \mathbb{R}$ . Then the following statements are equivalent:

1.  $f$  is continuous at  $a \in E$ .
2. If  $x_n$  converges to  $a$  and  $x_n \in E$ , then  $f(x_n) \rightarrow f(a)$  as  $n \rightarrow \infty$ .

• **Theorem 3.22**

Let  $E$  be a nonempty subset of  $\mathbb{R}$  and  $f, g : E \rightarrow \mathbb{R}$ . If  $f, g$  are continuous at a point  $a \in E$  (respectively continuous on the set  $E$ ), then so are  $f + g$ ,  $fg$ , and  $\alpha f$  (for any  $\alpha \in \mathbb{R}$ ). Moreover,  $f/g$  is continuous at  $a \in E$  when  $g(a) \neq 0$  (respectively, on  $E$  when  $g(x) \neq 0 \forall x \in E$ ).

• **Definition 3.23** *Composition*

Suppose that  $A$  and  $B$  are subsets of  $\mathbb{R}$ , that  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$ . If  $F(A) \subseteq B$  for every  $x \in A$ , then the *composition* of  $g$  with  $f$  is the function  $g \circ f : A \rightarrow \mathbb{R}$  defined by

$$(g \circ f)(x) := g(f(x)), \quad x \in A$$

- **Theorem 3.24**

Suppose that  $A$  and  $B$  are subsets of  $\mathbb{R}$ , that  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$ , and that  $f(x) \in B \forall x \in A$ .

1. If  $A := I \setminus \{a\}$ , where  $I$  is a nondegenerate interval which either contains  $a$  or has  $a$  as one of its endpoints, if

$$L := \lim_{x \rightarrow a; x \in I} f(x)$$

exists and belongs to  $B$ , and if  $g$  is continuous and  $L \in B$ , then

$$(g \circ f)(x) = g\left(\lim_{x \rightarrow a; x \in I} f(x)\right)$$

2. If  $f$  is continuous at  $a \in A$  and  $g$  is continuous at  $f(a) \in B$ , then  $g \circ f$  is continuous at  $a \in A$ .

- **Definition 3.25** *Bounded*

Let  $E$  be a nonempty subset of  $\mathbb{R}$ . A function  $f : E \rightarrow \mathbb{R}$  is said to be *bounded* on  $E$  if and only if there is an  $M \in \mathbb{R}$  such that  $|f(x)| \leq M$  for all  $x \in E$ , in which case we shall say that  $f$  is *dominated* by  $M$  on  $E$ .

- **Theorem 3.26** *Extreme Value Theorem*

If  $I$  is a closed, bounded interval and  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$ , then  $f$  is bounded on  $I$ . Moreover if

$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x)$$

then there exist points  $x_m, x_M \in I$  such that

$$f(x_M) = M \quad \text{and} \quad f(x_m) = m$$

- **Remark 3.27** The Existence Value Theorem is false if either “closed” or “bounded” is dropped from the hypotheses.

- **Lemma 3.28**

Suppose that  $a < b$  and that  $f : [a, b] \rightarrow \mathbb{R}$ . If  $f$  is continuous at a point  $x_0 \in [a, b]$  and  $f(x_0) > 0$ , then there exist a positive number  $\epsilon$  and a point  $x_1 \in [a, b]$  such that  $x_1 > x_0$  and  $f(x) > \epsilon \forall x \in [x_0, x_1]$ .

- **Theorem 3.29** *Intermediate Value Theorem*

Suppose that  $a < b$  and that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. If  $y_0$  lies between  $f(a)$  and  $f(b)$ , then there is an  $x_0 \in (a, b)$  such that  $f(x_0) = y_0$ .

- **Remark 3.34** The composition of two functions  $g \circ f$  can be nowhere continuous, even though  $f$  is discontinuous only on  $\mathbb{Q}$  and  $g$  is discontinuous at only one point.

### 3.4 Uniform Continuity

- **Definition 3.35** *Uniform continuity*

Let  $E$  be a nonempty subset of  $\mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ . Then  $f$  is said to be *uniformly continuous* on  $E$  if and only if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \implies |f(x) - f(a)| < \epsilon$$

- **Lemma 3.38**

Suppose that  $E \subseteq \mathbb{R}$  and that  $f : E \rightarrow \mathbb{R}$  is uniformly continuous. If  $x_n \in E$  is Cauchy, then  $f(x_n)$  is Cauchy.

- **Theorem 3.39**

Suppose that  $I$  is a closed, bounded interval. If  $f : I \rightarrow \mathbb{R}$  is continuous on  $I$ , then  $f$  is uniformly continuous on  $I$ .

- **Theorem 3.40**

Suppose that  $a < b$  and that  $f : (a, b) \rightarrow \mathbb{R}$ . Then  $f$  is uniformly continuous on  $(a, b)$  if and only if  $f$  can be continuously extended to  $[a, b]$ ; that is, if and only if there is a continuous function  $g : [a, b] \rightarrow \mathbb{R}$  which satisfies

$$f(x) = g(x), \quad x \in (a, b)$$

## 4 Differentiability on $\mathbb{R}$

### 4.1 The Derivative

- **Definition 4.1** *Differentiable*

A real function  $f$  is said to be *differentiable* at a point  $a \in \mathbb{R}$  if and only if  $f$  is defined on some open interval  $I$  containing  $a$  and

$$f'(a) := \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists. In this case  $f'(a)$  is called the *derivative* of  $f$  at  $a$ .

- **Theorem 4.2**

A real function  $f$  is differentiable at some point  $a \in \mathbb{R}$  if and only if there exist an open interval  $I$  and a function  $F : I \rightarrow \mathbb{R}$  such that  $a \in I$ ,  $f$  is defined on  $I$ ,  $F$  is continuous at  $a$ , and

$$f(x) = F(x)(x - a) + f(a)$$

holds for all  $x \in I$  in which case  $F(a) = f'(a)$ .

- **Theorem 4.3**

A real function  $f$  is differentiable at  $a$  if and only if there is a function  $T$  of the form  $T(x) := m(x)$  such that

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - t(h)}{h} = 0$$

- **Theorem 4.4**

If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

- **Definition 4.6** *Continuously differentiable*

Let  $I$  be a nondegenerate interval.

1. A function  $f : I \rightarrow \mathbb{R}$  is said to be *differentiable* on  $I$  if and only if

$$f'_i(a) := \lim_{x \rightarrow a; x \in I} \frac{f(x) - f(a)}{x - a}$$

exists and is finite for every  $a \in I$ .

2.  $f$  is said to be *continuously differentiable* on  $I$  if and only if  $f'_I$  exists and is continuous on  $I$ .

- **Remark 4.9**

$f(x) = |x|$  is differentiable on  $[0, 1]$  and on  $[-1, 0]$  but not on  $[-1, 1]$ .

### 4.2 Differentiability Theorems

- **Theorem 4.10**

Let  $f$  and  $g$  be real functions and  $a \in \mathbb{R}$ . If  $f$  and  $g$  are differentiable at  $a$ , then  $f + g$ ,  $\alpha f$ ,  $f \cdot g$ , and [when  $g(a) \neq 0$ ]  $f/g$  are all differentiable at  $a$ . In fact,

$$\begin{aligned} (f + g)'(a) &= f'(a) + g'(a) \\ (\alpha f)'(a) &= \alpha f'(a) \\ (f \cdot g)'(a) &= g(a)g'(a) + f(a)g'(a) \\ \left(\frac{f}{g}\right)'(a) &= \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)} \end{aligned}$$



- **Theorem 4.11** *Chain Rule*

Let  $f$  and  $g$  be real functions. If  $f$  is differentiable at  $a$  and  $g$  is differentiable at  $f(a)$ , then  $g \circ f$  is differentiable at  $a$  with

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

### 4.3 Mean Value Theorem

- **Lemma 4.12** *Rolle's Theorem*

Suppose that  $a, b \in \mathbb{R}$  with  $a < b$ . If  $f$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and if  $f(a) = f(b)$ , then  $f'(c) = 0$  for some  $c \in (a, b)$ .

- **Remark 4.13**

The continuity hypothesis on Rolle's Theorem cannot be relaxed at even one point in  $[a, b]$ .

- **Remark 4.14**

The differentiability hypothesis on Rolle's Theorem cannot be relaxed at even one point in  $[a, b]$ .

- **Theorem 4.15**

Suppose that  $a, b \in \mathbb{R}$  with  $a < b$ .

1. *Generalised Mean Value Theorem*: If  $f, g$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there is a  $c \in (a, b)$  such that

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

2. *Mean Value Theorem*: If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there is a  $c \in (a, b)$  such that

$$f(b) - f(a) = f'(c)(b - a)$$

- **Definition 4.16** *Increasing, Monotone, Decreasing*

Let  $E$  be a nonempty subset of  $\mathbb{R}$  and  $f : E \rightarrow \mathbb{R}$ .

1.  $f$  is said to be *increasing* (respectively, *strictly increasing*) on  $E$  if and only if  $x_1, x_2 \in E$  and  $x_1 < x_2 \implies f(x_1) \leq f(x_2)$  [respectively,  $f(x_1) < f(x_2)$ ].
2.  $f$  is said to be *decreasing* (respectively, *strictly decreasing*) on  $E$  if and only if  $x_1, x_2 \in E$  and  $x_1 < x_2 \implies f(x_1) \geq f(x_2)$  [respectively,  $f(x_1) > f(x_2)$ ].
3.  $f$  is said to be *monotone* (respectively, *strictly monotone*) on  $E$  if and only if  $f$  is either decreasing or increasing (respectively, either strictly decreasing or strictly increasing) on  $E$ .

- **Theorem 4.17**

Suppose that  $a, b \in \mathbb{R}$ , with  $a < b$ , that  $f$  is continuous on  $[a, b]$ , and that  $f$  is differentiable on  $(a, b)$ .

1. If  $f'(x) > 0$  [respectively  $f'(x) < 0$ ] for all  $x \in (a, b)$ , then  $f$  is strictly increasing (respectively, strictly decreasing) on  $[a, b]$ .
2. If  $f'(x) = 0$  for all  $x \in (a, b)$ , then  $f$  is constant on  $[a, b]$ .
3. If  $g$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and if  $f'(x) = g'(x)$  for all  $x \in (a, b)$ , then  $f - g$  is constant on  $[a, b]$ .

- **Theorem 4.18**

Suppose that  $f$  is increasing on  $[a, b]$

1. If  $c \in [a, b)$ , then  $f(c+)$  exists and  $f(c) \leq f(c+)$ .
2. If  $c \in (a, b]$ , then  $f(c-)$  exists and  $f(c-) \leq f(c)$ .

- **Theorem 4.19**

If  $f$  is monotone on an interval  $I$ , then  $f$  has at most countable many points of discontinuity on  $I$ .

- **Theorem 4.21 Bernoulli's Inequality**

Let  $\alpha$  be a positive real number. If  $0 < \alpha < 1$ , then  $(1+x)^\alpha \leq 1 + \alpha x \ \forall x \in [-1, \infty)$ , and if  $\alpha \geq 1$ , then  $(1+x)^\alpha \geq 1 + \alpha x \ \forall x \in [-1, \infty)$ .

- **Theorem 4.23 Intermediate Value Theorem for Derivatives**

Suppose that  $f$  is differentiable on  $[a, b]$  with  $f'(a) \neq f'(b)$ . If  $y_0$  is a real number which lies between  $f'(a)$  and  $f'(b)$ , then there is an  $x_0 \in (a, b)$  such that  $f'(x_0) = y_0$ .

#### 4.4 Taylor's Theorem and L'Hopital's Rule

- **Theorem 4.24 Taylor's Formula**

Let  $n \in \mathbb{N}$  and let  $a, b$  be extended real numbers with  $a < b$ . If  $f : (a, b) \rightarrow \mathbb{R}$ , and if  $f^{(n+1)}$  exists on  $(a, b)$ , then for each pair of points  $(x, x_0 \in (a, b))$  there is a number  $c$  between  $x$  and  $x_0$  such that

$$f(x) = f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

- **Theorem 4.27 L'Hopital's Rule**

Let  $a$  be an extended real number and  $I$  be an open interval which either contains  $a$  or has  $a$  as an endpoint. Suppose that  $f$  and  $g$  are differentiable on  $I \setminus \{a\}$  and that  $g(x) \neq 0 \neq g'(x) \ \forall x \in I \setminus \{a\}$ . Suppose further that

$$A := \lim_{x \rightarrow a; x \in I} f(x) = \lim_{x \rightarrow a; x \in I} g(x)$$

is either 0 or  $\infty$ . If

$$B := \lim_{x \rightarrow a; x \in I} \frac{f'(x)}{g'(x)}$$

exists as an extended real number, then

$$\lim_{x \rightarrow a; x \in I} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a; x \in I} \frac{f'(x)}{g'(x)}$$

#### 4.5 Inverse Function Theorems

- **Theorem 4.32**

Let  $I$  be a nondegenerate interval and suppose that  $f : I \rightarrow \mathbb{R}$  is injective. If  $f$  is continuous on  $I$ , then  $J := f(I)$  is an interval,  $f$  is strictly monotone on  $I$ , and  $f^{-1}$  is continuous and strictly monotone on  $J$ .

- **Theorem 4.33 Inverse Function Theorem**

Let  $I$  be an open interval and  $f : I \rightarrow \mathbb{R}$  be injective and continuous. If  $b = f(a)$  for some  $a \in I$  and if  $f'(a)$  exists and is nonzero, then  $f^{-1}$  is differentiable at  $b$  and  $(f^{-1})'(b) = \frac{1}{f'(a)}$ .

## 5 Riemann Integration

### 5.1 Introduction

### 5.2 Step functions and their integrals

- **Definition 1** *Step function*

We say that  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is a *step function* if there exist real numbers  $x_0 < x_1 < \dots < x_n$  (for some  $n \in \mathbb{N}$ ) such that

1.  $\phi(x) = 0$  for  $x < x_0$  and  $x > x_n$
2.  $\phi$  is constant on  $(x_{j-1}, x_j)$   $1 \leq j \leq n$ .

- **Definition 2**

If  $\phi$  is a step function with respect to  $\{x_0, x_1, \dots, x_n\}$  which takes the value  $c_j$  on  $(x_{j-1}, x_j)$ , then

$$\int \phi := \sum_{j=1}^n c_j (x_j - x_{j-1})$$

- **Proposition 1**

If  $\phi$  and  $\psi$  are step functions and  $\alpha$  and  $\beta \in \mathbb{R}$ , then

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi.$$

### 5.3 Riemann-integrable functions and their integrals

- **Definition 3** *Riemann-integrable*

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say that  $f$  is *Riemann-integrable* if for every  $\epsilon > 0$  there exist step functions  $\phi$  and  $\psi$  such that

$$\phi \leq f \leq \psi$$

and

$$\int \psi - \int \phi < \epsilon$$

- **Theorem 1**

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Riemann-integrable if and only if

$$\sup\left\{\int \phi : \phi \text{ is a step function and } \phi \leq f\right\} = \inf\left\{\int \psi : \psi \text{ is a step function and } \psi \geq f\right\}.$$

- **Definition 4**

If  $f$  is Riemann-integrable we define its integral  $\int f$  as the common value

$$\int f := \sup\left\{\int \phi : \phi \text{ is a step function and } \phi \leq f\right\} = \inf\left\{\int \psi : \psi \text{ is a step function and } \psi \geq f\right\}.$$

- **Theorem 2**

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Riemann-integrable if and only if there exist sequences of step functions  $\phi_n$  and  $\psi_n$  such that

$$\phi_n \leq f \leq \psi_n \quad \forall n, \quad \text{and} \quad \int \psi_n - \int \phi_n \rightarrow 0$$

If  $\phi_n$  and  $\psi_n$  are any sequences of step functions satisfying above, then

$$\int \phi_n \rightarrow \int f \quad \text{and} \quad \int \psi_n \rightarrow \int f$$

as  $n \rightarrow \infty$ .

• **Lemma 1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function with bounded support  $[a, b]$ . The following are equivalent:

1.  $f$  is Riemann-integrable.
2. for every  $\epsilon > 0$  there exist  $a = x_0 < \dots < x_n = b$  such that, if  $M_j$  and  $m_j$  denote the supremum and infimum values of  $f$  on  $[x_{j-1}, x_j]$  respectively, then

$$\sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$

3. for every  $\epsilon > 0$  there exist  $a = x_0 < \dots < x_n = b$  such that, with  $I_j = (x_{j-1}, x_j)$  for  $j \geq 1$ ,

$$\sum_{j=1}^n \sup_{x, y \in I_j} |f(x) - f(y)| |I_j| < \epsilon.$$

For  $f : \mathbb{R} \rightarrow \mathbb{R}$  a bounded function with bounded support  $[a, b]$  and for  $a = x_0 < \dots < x_n = b$ , let  $I_j = (x_{j-1}, x_j)$ ,  $m_j := \inf_{x \in I_j} f(x)$  and  $M_j := \sup_{x \in I_j} f(x)$ . Define the *lower step function of  $f$  with respect to  $\{x_0, \dots, x_n\}$*  as

$$\phi_*(x) = \sum_{j=1}^n m_j \chi_{I_j} + \sum_{j=0}^n \chi_{x_j}$$

and the *upper step function of  $f$  with respect to  $\{x_0, \dots, x_n\}$*  as

$$\phi^*(x) = \sum_{j=1}^n M_j \chi_{I_j} + \sum_{j=0}^n \chi_{x_j}$$

Note that  $\phi_*$  and  $\phi^*$  are step functions, and that  $\phi_* \leq f \leq \phi^*$ .

• **Theorem 3**

Suppose  $f$  and  $g$  are Riemann-integrable and  $\alpha$  and  $\beta$  are real numbers. Then

1.  $\alpha f + \beta g$  is Riemann-integrable and

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

2. If  $f \geq 0$  then  $\int f \geq 0$ ; if  $f \leq g$  then  $\int f \leq \int g$ .
3.  $|f|$  is Riemann-integrable and  $|\int f| \leq \int |f|$
4.  $\max\{f, g\}$  and  $\min\{f, g\}$  are Riemann-integrable.
5.  $fg$  is Riemann-integrable.

• **Theorem 4**

If  $g : [a, b] \rightarrow \mathbb{R}$  is continuous, and  $f$  defined by  $f(x) = g(x)$  for  $a \leq x \leq b$ ,  $f(x) = 0$  for  $x \notin [a, b]$  then  $f$  is Riemann-integrable.

## 5.4 Fundamental Theorem of Calculus, and Practical Integration

• **Theorem 5**

Let  $g : [a, b] \rightarrow \mathbb{R}$  be Riemann-integrable. For  $a \leq x \leq b$  let  $G(x) = \int_a^x g$ . Suppose  $g$  is continuous at  $x$  for some  $x \in [a, b]$ . [If  $x$  is an endpoint, we mean one-sided continuous.] Then  $G$  is differentiable at  $x$  and  $G'(x) = g(x)$ . [If  $x$  is an endpoint, we mean one-sided differentiable.]

• **Theorem 6**

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  has continuous derivative  $f'$  on  $[a, b]$ . Then

$$\int_a^b f' = f(b) - f(a).$$

## 5.5 Integrals and uniform limits of sequences and series of functions

- **Theorem 7**

Suppose that  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  is a sequence of Riemann-integrable functions which converges uniformly to a function  $f$ . Suppose that  $f_n$  and  $f$  are zero outside some common interval  $[a, b]$ . Then  $f$  is Riemann-integrable and

$$\int f = \lim_{n \rightarrow \infty} \int f_n.$$

## 5.6 A couple of odds and ends

- Improper integrals
- Integral test

## 6 Infinite Series of Real Numbers

### 6.1 Introduction

- **Definition 6.1** *Partial sum*

Let  $S = \sum_{k=1}^{\infty} a_k$  be an infinite series with terms  $a_k$ .

1. For each  $n \in \mathbb{N}$ , the *partial sum of  $S$  of order  $n$*  is defined by

$$s_n := \sum_{k=1}^n a_k$$

2.  $S$  is said to *converge* if and only if its sequence of partial sums  $\{s_n\}$  converges to some  $s \in \mathbb{R}$  as  $n \rightarrow \infty$ ; that is, if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that  $n \geq N \implies |s_n - s| < \epsilon$ . In this case we shall write

$$\sum_{k=1}^{\infty} a_k = s$$

and call  $s$  the *sum*, or *value*, of the series  $\sum_{k=1}^{\infty} a_k$

3.  $S$  is said to *diverge* if and only if its sequence of partial sums  $\{s_n\}$  does not converge as  $n \rightarrow \infty$ . When  $s_n$  diverges to  $+\infty$  as  $n \rightarrow \infty$ , we shall also write

$$\sum_{k=1}^{\infty} a_k = s$$

- **Theorem 6.5** *Divergence Test* Let  $\{a_k\}_{k \in \mathbb{N}}$  be a sequence of real numbers. If  $a_k$  does not converge to zero, then the series  $\sum_{k=1}^{\infty} a_k$  diverges.

- **Theorem 6.6** *Telescoping Series*

If  $\{a_k\}$  is a convergent real sequence, then

$$\text{sum}_{k=1}^{\infty} (a_k - a_{k+1}) = a_1 - \lim_{k \rightarrow \infty} a_k$$

- **Theorem 6.7** *Geometric Series*

Suppose that  $x \in \mathbb{R}$ , that  $N \in \{0, 1, \dots\}$ , and that  $0^0$  is interpreted to be 1. Then the series  $\sum_{k=N}^{\infty} x^k$  converges if and only if  $|x| < 1$ , in which case

$$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}$$

In particular,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1.$$

- **Theorem 6.8** *The Cauchy Criterion*

Let  $\{a_k\}$  be a real sequence. Then the infinite series  $\sum_{k=1}^{\infty} a_k$  converges if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$m \geq n \geq N \implies \left| \sum_{k=n}^m a_k \right| < \epsilon$$

- **Corollary 6.9**

Let  $\{a_k\}$  be a real sequence. Then the infinite series  $\sum_{k=1}^{\infty} a_k$  converges if and only if given  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$n \geq N \implies \left| \sum_{k=n}^{\infty} a_k \right| < \epsilon$$

- **Theorem 6.10**

Let  $\{a_k\}$  and  $\{b_k\}$  be real sequences. If  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  are convergent series, then

$$\sum_{i=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

and

$$\sum_{k=1}^{\infty} (\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$$

for any  $\alpha \in \mathbb{R}$ .

## 6.2 Series with Nonnegative Terms

- **Theorem 6.11**

Suppose that  $a_k \geq 0$  for large  $k$ . Then  $\sum_{k=1}^{\infty} a_k$  converges if and only if its sequence of partial sums  $\{s_n\}$  is bounded; that is, if and only if there exists a finite number  $M > 0$  such that

$$\left| \sum_{i=1}^n a_k \right| \leq M \quad \forall n \in \mathbb{N}$$

- **Theorem 6.12 Integral Test**

Suppose that  $f : [1, \infty) \rightarrow \mathbb{R}$  is positive and decreasing on  $[1, \infty)$ . Then  $\sum_{k=1}^{\infty} f(k)$  converges if and only if  $f$  is improperly integrable on  $[1, \infty)$ ; that is if and only if

$$\int_1^{\infty} f(x) \, dx < \infty$$

- **Corollary 6.13 *p-Series Test*** The series

$$\sum_{i=1}^{\infty} \frac{1}{k^p}$$

converges if and only if  $p > 1$ .

- **Theorem 6.14 Comparison Test**

Suppose that  $0 \leq a_k \leq b_k$  for large  $k$ .

1. If  $\sum_{k=1}^{\infty} b_k < \infty$ , then  $\sum_{k=1}^{\infty} a_k < \infty$ .
2. If  $\sum_{k=1}^{\infty} b_k = \infty$ , then  $\sum_{k=1}^{\infty} a_k = \infty$ .

- **Theorem 6.16 Limit Comparison Test**

Suppose that  $a_k \geq 0$ , that  $b_k > 0$  for large  $k$ , and that  $L := \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists as an extended real number.

1. If  $0 < L < \infty$ , then  $\sum_{k=1}^{\infty} a_k$  converges if and only if  $\sum_{k=1}^{\infty} b_k$  converges.
2. If  $L = 0$  and  $\sum_{k=1}^{\infty} b_k$  converges then  $\sum_{k=1}^{\infty} a_k$  converges.
3. If  $L = \infty$  and  $\sum_{k=1}^{\infty} b_k$  diverges then  $\sum_{k=1}^{\infty} a_k$  diverges.

### 6.3 Absolute Convergence

- **Definition 6.18** *Absolute & Conditional Convergence*

Let  $S = \sum_{k=1}^{\infty} a_k$  be an infinite series.

1.  $S$  is said to *converge absolutely* if and only if  $\sum_{k=1}^{\infty} |a_k| < \infty$
2.  $S$  is said to *converge conditionally* if and only if  $S$  converges but not absolutely.

- **Remark 6.19**

A series  $\sum_{k=1}^{\infty} a_k$  converges absolutely if and only if for every  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$m > n \geq N \implies \sum_{k=1}^{\infty} |a_k| < \epsilon$$

- **Remark 6.20**

If  $\sum_{k=1}^{\infty} a_k$  converges absolutely, then  $\sum_{k=1}^{\infty} a_k$  converges, but not conversely. In particular, there exist conditionally convergent series.

- **Definition 6.21** *Limit supremum*

The *limit supremum* of a sequence of real numbers  $\{x_k\}$  is defined to be

$$\limsup_{k \rightarrow \infty} x_k := \lim_{n \rightarrow \infty} \left( \sup_{k > n} x_k \right).$$

- **Remark 6.22**

Let  $x \in \mathbb{R}$  and  $\{x_k\}$  be a real sequence.

1. If  $\limsup_{k \rightarrow \infty} x_k < x$ , then  $x_k < x$  for large  $k$ .
2. If  $\limsup_{k \rightarrow \infty} x_k > x$ , then  $x_k > x$  for infinitely many  $k$ s.
3. If  $x_k \rightarrow x$  as  $k \rightarrow \infty$ , then  $\limsup_{k \rightarrow \infty} x_k = x$ .

- **Theorem 6.23** *Root Test*

Let  $a_k \in \mathbb{R}$  and  $r := \limsup_{k \rightarrow \infty} |a_k|^{\frac{1}{k}}$ .

1. If  $r < 1$ , then  $\sum_{k=1}^{\infty} a_k$  converges absolutely.
2. If  $r > 1$ , then  $\sum_{k=1}^{\infty} a_k$  diverges.

- **Theorem 6.24** *Ratio test*

Let  $a_k \in \mathbb{R}$  with  $a_k \neq 0$  for large  $k$  and suppose that

$$r = \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}$$

exists as an extended real number.

1. If  $r < 1$ , then  $\sum_{k=1}^{\infty} a_k$  converges absolutely.
2. If  $r > 1$ , then  $\sum_{k=1}^{\infty} a_k$  diverges.

- **Remark 6.25** The Root and Ratio tests are inconclusive when  $r = 1$ .

- **Definition 6.26** *Rearrangement*

A series  $\sum_{j=1}^{\infty} b_j$  is called a *rearrangement* of a series  $\sum_{k=1}^{\infty} a_k$  if and only if there is an injection  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$b_{f(k)} = a_k, \quad k \in \mathbb{N}$$



- **Theorem 6.27**

If  $\sum_{k=1}^{\infty} a_k$  converges absolutely and  $\sum_{j=1}^{\infty} b_j$  is any rearrangement of  $\sum_{k=1}^{\infty} a_k$ , then  $\sum_{j=1}^{\infty} b_j$  converges and

$$\sum_{k=1}^{\infty} a_k = \sum_{j=1}^{\infty} b_j$$

- **Lemma 6.28**

- **Theorem 6.29** *Riemann*

## 6.4 Alternating Series

- **Theorem 6.30** *Abel's Formula*

Let  $\{a_k\}_{k \in \mathbb{N}}$  and  $\{b_k\}_{k \in \mathbb{N}}$  be real sequences, and for each pair of integers  $n \geq m \geq 1$  set

$$A_{n,m} := \sum_{k=m}^n a_k$$

Then

$$\sum_{k=m}^n a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m} (b_{k+1} - b_k)$$

for all integers  $n > m \geq 1$ .

- **Theorem 6.31** *Dirichlet's Test*

Let  $a_k, b_k \in \mathbb{R}$  for  $k \in \mathbb{N}$ . If the sequence of partial sums  $s_n = \sum_{k=1}^n a_k$  is bounded and  $b_k \downarrow 0$  as  $k \rightarrow \infty$ , then  $\sum_{k=1}^{\infty} a_k b_k$  converges.

- **Corollary 6.32** *Alternating Series Test*

If  $a_k \downarrow 0$  as  $k \rightarrow \infty$ , then

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

converges.

## 7 Infinite Series of Functions