Honours Analysis Notes

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1.1 Introduction

1.2 Ordered Field Axioms

- **Postulate 1** *Field Axioms*
  There are functions $+$ and $\cdot$ defined on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, which satisfy the following properties $\forall a, b, c \in \mathbb{R}$
  - *Closure Properties*: $a + b,\ a \cdot b \in \mathbb{R}$
  - *Associative Properties*: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  - *Commutative Properties*: $a + b = b + a$ and $a \cdot b = b \cdot a$
  - *Distributive Law*: $a \cdot (b + c) = a \cdot b + a \cdot c$
  - *Existence of Additive Identity*: There is a unique element $0 \in \mathbb{R}$ such that $0 + a = a$ for all $a \in \mathbb{R}$
  - *Existence of Multiplicative Identity*: There is a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$
  - *Existence of Additive Inverses*: For every $x \in \mathbb{R}$ there is a unique element $-x \in \mathbb{R}$ such that
  $$x + (-x) = 0$$
  - *Existence of Multiplicative Inverses*: For every $x \in \mathbb{R}\backslash\{0\}$ there is a unique element $x^{-1} \in \mathbb{R}$ such that
  $$x \cdot (x^{-1}) = 1$$
- **Postulate 2** *Order Axioms*
  There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:
  - *Trichotomy Property*: Given $a, b \in \mathbb{R}$, one and only one of the following statements hold:
  $$a < b,\ b < a,\ \text{or } a = b$$
  - *Transitive Property*: For $a, b, c, \in \mathbb{R}$
  $$a < b \text{ and¡}c \implies a < c$$
  - *Additive Property*: For $a, b, c \in \mathbb{R}$
  $$a < b \text{ and } c \in \mathbb{R} \implies a + c < b + c$$
  - *Multiplicative Properties*: For $a, b, c, \in \mathbb{R}$
  $$a < b \text{ and } c > 0 \implies ac < bc$$
  and
  $$a < b \text{ and } c < 0 \implies bc < ac$$
- **Remark 1.1**
  We will assume that the sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties:
  1. If $n, m \in \mathbb{Z}$, then $n + m, n - m$ and mn belong to $\mathbb{Z}$
  2. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$

3. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$

- **Definition 1.4** *Absolute Value*
The *absolute value* of a number $a \in \mathbb{R}$ is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

- **Remark 1.5** The *absolute value* is multiplicative; that is, $|ab| = |a||b|\ \forall a, b, \in \mathbb{R}$

- **Theorem 1.6** *Fundamental Theorem of Absolute Values*
Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M \iff -M \leq a \leq M$.

- **Theorem 1.7** The absolute value satisfies the following three properties:

  1. *Positive Definite*: For all $a \in \mathbb{R}$, $|a| > 0$ with $|a| = 0$ if and only if $a = 0$.
  2. *Symmetric*: For all $a, b, \in \mathbb{R}$, $|a - b| = |b - a|$,
  3. *Triangle Inequalities*: For all $a, b \in \mathbb{R}$,

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

- **Theorem 1.9** Let $x, y, a \in \mathbb{R}$

  1. $x < y + \epsilon\ \forall \epsilon > 0 \iff x \leq y$
  2. $x > y - \epsilon\ \forall \epsilon > 0 \iff x \geq y$
  3. $|a| < \epsilon\ \forall \epsilon > 0 \iff a = 0$

1.3 Completeness Axiom

- **Definition 1.10** *Upper bounds*
Let $E \subset \mathbb{R}$ be non-empty

  1. The set E is said to be *bounded above* if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$, in which case M is called an *upper bound* of E.
  2. A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E has a *finite supremeum* s and write $s = \sup E$)

- **Remark 1.12** If a set has one upper bound, it has infinitely many upper bounds.

- **Remark 1.13** If a set has a supremum, then it has only one supremum.

- **Theorem** *Approximation Property for Suprema*
If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that

$$\sup E - \epsilon < a \leq \sup E$$

- **Theorem 1.15**
If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

- **Postulate 3** *Completeness Axiom*
If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

- **Theorem 1.16** *The Archimedean Principle*
Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

- **Theorem 1.18** *Density of Rationals*
  If $a, b, \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

- **Definition 1.19** *Upper bounds*
  Let $E \in \mathbb{R}$ be nonempty

  1. The set E is said to be *bounded below* if and only if there is an $m \in \mathbb{R}$ such that $a \geq E$, in which case m is called a *lower bound* of the set E.
  2. A number t is called an *infimum* of the set E if and only if t is a lower bound of E and $t \geq m$ and write $t = \inf E$.
  3. E is said to be *bounded* if and only if it is bounded both above and below.

- **Theorem 1.20** *Reflection Principle*
  Let $E \in \mathbb{R}$ be nonempty

  1. E has a supremum if and only if $-E$ has an infimum, in which case
  $$\inf(-E) = -\sup E$$
  2. E has an infimum if and only if $-E$ has a supremum, in which case
  $$\sup(-E) = -\inf E$$

- **Theorem 1.21** *Monotone Property*
  Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.

  1. If B has a supremum, then $\sup A \leq \sup B$.
  2. If B has an infimum, then $\inf A \geq \inf B$.

1.4 Mathematical Induction

- **Theorem 1.22** *Well-Ordering Principle*
  If E is a nonempty subset of $\mathbb{N}$, then E has a least element (i.e. E has a finite infimum and $\inf E \in E$).

- **Theorem 1.23**
  Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition which satisfies the following two properties:

  1. $A(1)$ is true.
  2. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.

  Then $A(n)$ is true for all $n \in \mathbb{N}$.

- **Theorem 1.26** *Binomial Formula*
  If $a, b, \in \mathbb{R}, n \in \mathbb{N}$ and 0^0 is interpreted to be 1, then
  $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1.5 Inverse Functions and Images

- **Definition 1.29** *Injection, Surjection, Bijection*
  Let X and Y be sets and $f : X \to Y$

  1. f is said to be *injective* if and only if
  $$x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \implies x_1 = x_2$$

2. f is said to be *surjective* if and only if

$$\forall y \in Y \ \exists x \in X \ni y = f(x)$$

3. f is called *bijective* if and only if it is both injective and surjective

- **Theorem 1.30**
  Let X and Y be sets and $f : X \to Y$. Then the following three statements are equivalent.

  1. f has an inverse;
  2. f is injective from X onto Y;
  3. There is a function $g : Y \to X$ such that

  $$g(f(x)) = x \quad \forall x \in X$$

  and

  $$f(g(y)) = y \quad \forall y \in Y$$

  Moreover, for each $f : X \to Y$, there is only one function g that satisfies these. It is the inverse function f^{-1}.

- **Remark 1.31**
  Let I be an interval and let $f : I \to \mathbb{R}$. If the derivative of f is either always positive on I, or always negative on I, then f is injective on I.

- **Definition 1.33** *Image*
  Let X and Y be sets and $f : X \to Y$. The *image* of a set $E \subseteq X$ under f is the set

  $$f(E) := \{y \in Y : y = f(x) \text{ for some } x \in E\}$$

  The *inverse image* of a set $E \subseteq Y$ under f is the set

  $$f^{-1}(E) := \{x \in X : f(x) = y \text{ for some } y \in E\}$$

- **Definition 1.35** *Union, Intersection*
  Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ be a collection of sets.

  1. The *union* of the collection $\mathcal{E}$ is the set

  $$\bigcup_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$$

  2. The *intersection* of the collection $\mathcal{E}$ is the set

  $$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

- **Theorem 1.36** *DeMorgan's Laws*
  Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. If for each $E \subseteq X$ the symbol E^c represents the set $X \setminus E$, then

  $$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

  and

  $$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

- **Theorem 1.37**
  Let X and Y be sets and $f : X \to Y$.

  1. If $\{E_\alpha\}\alpha \in A$ is a collection of subsets of X, then
  $$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha) \quad \text{and} \quad f\left(\bigcap_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(E_\alpha)$$
  2. If B and C are subsets of X, then $f(C) \setminus f(B) \subseteq f(C \setminus B)$
  3. If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of Y, then
  $$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha)$$
  4. If B and C are subsets of Y, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.
  5. If $E \subseteq f(x)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$.

1.6 Countable and Uncountable Sets

- **Definition 1.38** *Countable Uncountable* Let E be a set.
  1. E is said to be *finite* if and only if either $E = \emptyset$ or there exists an injective function which takes $\{1, 2, \ldots, n\}$ onto E, for some $n \in \mathbb{N}$.
  2. E is said to be *countable* if and only if there exists and injective function which takes $\mathbb{N}$ onto E.
  3. E is said to be *at most countable* if and only if E is either finite or countable.
  4. E is said to be *uncountable* if and only if E is neither finite nor countable.
- **Remark 1.39** *Cantor's Diagonalisation Argument*
  The open interval $(0, 1)$ is uncountable.
- **Lemma 1.40**
  A nonempty set E is at most countable if and only if there is a function g from $\mathbb{N}$ onto E.
- **Theorem 1.41**
  Suppose A and B are sets.
  1. If $A \subseteq B$ and B is at most countable, then A is at most countable.
  2. If $A \subseteq B$ and A is uncountable, then B is uncountable.
  3. $\mathbb{R}$ is uncountable.
- **Theorem 1.42**
  Let $A_1, A_2, \ldots$ be at most countable sets.
  1. Then $A_1 \times A_2$ is at most countable.
  2. If
  $$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j\in\mathbb{N}} A_j := \{x : x \in A_j \text{ for some } j \in \mathbb{N}\},$$
  then E is at most countable.
- **Remark 1.43**
  The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but the set of irrationals is uncountable.