Honours Analysis Notes

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1 The Real Number System

1.1 Introduction

1.2 Ordered Field Axioms

- **Postulate 1** *Field Axioms*
  There are functions $+$ and $\cdot$ defined on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, which satisfy the following properties $\forall a, b, c \in \mathbb{R}$
  - *Closure Properties*: $a + b,\ a \cdot b \in \mathbb{R}$
  - *Associative Properties*: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  - *Commutative Properties*: $a + b = b + a$ and $a \cdot b = b \cdot a$
  - *Distributive Law*: $a \cdot (b + c) = a \cdot b + a \cdot c$
  - *Existence of Additive Identity*: There is a unique element $0 \in \mathbb{R}$ such that $0 + a = a$ for all $a \in \mathbb{R}$
  - *Existence of Multiplicative Identity*: There is a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$
  - *Existence of Additive Inverses*: For every $x \in \mathbb{R}$ there is a unique element $-x \in \mathbb{R}$ such that
    $$x + (-x) = 0$$
  - *Existence of Multiplicative Inverses*: For every $x \in \mathbb{R} \backslash \{0\}$ there is a unique element $x^{-1} \in \mathbb{R}$ such that
    $$x \cdot (x^{-1}) = 1$$
- **Postulate 2** *Order Axioms*
  There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:
  - *Trichotomy Property*: Given $a, b \in \mathbb{R}$, one and only one of the following statements hold:
    $$a < b,\ b < a,\ \text{or } a = b$$
  - *Transitive Property*: For $a, b, c, \in \mathbb{R}$
    $$a < b \text{ and¡}c \implies a < c$$
  - *Additive Property*: For $a, b, c \in \mathbb{R}$
    $$a < b \text{ and } c \in \mathbb{R} \implies a + c < b + c$$
  - *Multiplicative Properties*: For $a, b, c, \in \mathbb{R}$
    $$a < b \text{ and } c > 0 \implies ac < bc$$
    and
    $$a < b \text{ and } c < 0 \implies bc < ac$$
- **Remark 1.1**
  We will assume that the sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties:
  1. If $n, m \in \mathbb{Z}$, then $n + m, n - m$ and mn belong to $\mathbb{Z}$
  2. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$

3. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$

- **Definition 1.4** *Absolute Value*
The *absolute value* of a number $a \in \mathbb{R}$ is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

- **Remark 1.5** The *absolute value* is multiplicative; that is, $|ab| = |a||b|\ \forall a, b, \in \mathbb{R}$

- **Theorem 1.6** *Fundamental Theorem of Absolute Values*
Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M \iff -M \leq a \leq M$.

- **Theorem 1.7** The absolute value satisfies the following three properties:

1. *Positive Definite*: For all $a \in \mathbb{R}$, $|a| > 0$ with $|a| = 0$ if and only if $a = 0$.
2. *Symmetric*: For all $a, b, \in \mathbb{R}$, $|a - b| = |b - a|$,
3. *Triangle Inequalities*: For all $a, b \in \mathbb{R}$,

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

- **Theorem 1.9** Let $x, y, a \in \mathbb{R}$

1. $x < y + \epsilon\ \forall \epsilon > 0 \iff x \leq y$
2. $x > y - \epsilon\ \forall \epsilon > 0 \iff x \geq y$
3. $|a| < \epsilon\ \forall \epsilon > 0 \iff a = 0$

1.3 Completeness Axiom

- **Definition 1.10** *Upper bounds*
Let $E \subset \mathbb{R}$ be non-empty

1. The set E is said to be *bounded above* if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$, in which case M is called an *upper bound* of E.
2. A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E has a *finite supremeum* s and write $s = \sup E$)

- **Remark 1.12** If a set has one upper bound, it has infinitely many upper bounds.

- **Remark 1.13** If a set has a supremum, then it has only one supremum.

- **Theorem** *Approximation Property for Suprema*
If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that

$$\sup E - \epsilon < a \leq \sup E$$

- **Theorem 1.15**
If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

- **Postulate 3** *Completeness Axiom*
If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

- **Theorem 1.16** *The Archimedean Principle*
Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

- **Theorem 1.18** *Density of Rationals*
  If $a, b, \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

- **Definition 1.19** *Upper bounds*
  Let $E \in \mathbb{R}$ be nonempty

  1. The set E is said to be *bounded below* if and only if there is an $m \in \mathbb{R}$ such that $a \geq E$, in which case m is called a *lower bound* of the set E.
  2. A number t is called an *infimum* of the set E if and only if t is a lower bound of E and $t \geq m$ and write $t = \inf E$.
  3. E is said to be *bounded* if and only if it is bounded both above and below.

- **Theorem 1.20** *Reflection Principle*
  Let $E \in \mathbb{R}$ be nonempty

  1. E has a supremum if and only if $-E$ has an infimum, in which case
  $$\inf(-E) = -\sup E$$
  2. E has an infimum if and only if $-E$ has a supremum, in which case
  $$\sup(-E) = -\inf E$$

- **Theorem 1.21** *Monotone Property*
  Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.

  1. If B has a supremum, then $\sup A \leq \sup B$.
  2. If B has an infimum, then $\inf A \geq \inf B$.

1.4 Mathematical Induction

- **Theorem 1.22** *Well-Ordering Principle*
  If E is a nonempty subset of $\mathbb{N}$, then E has a least element (i.e. E has a finite infimum and $\inf E \in E$).

- **Theorem 1.23**
  Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition which satisfies the following two properties:

  1. $A(1)$ is true.
  2. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.

  Then $A(n)$ is true for all $n \in \mathbb{N}$.

- **Theorem 1.26** *Binomial Formula*
  If $a, b, \in \mathbb{R}, n \in \mathbb{N}$ and 0^0 is interpreted to be 1, then
  $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1.5 Inverse Functions and Images

- **Definition 1.29** *Injection, Surjection, Bijection*
  Let X and Y be sets and $f : X \to Y$

  1. f is said to be *injective* if and only if
  $$x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \implies x_1 = x_2$$

2. f is said to be *surjective* if and only if

$$\forall y \in Y \ \exists x \in X \ni y = f(x)$$

3. f is called *bijective* if and only if it is both injective and surjective

- **Theorem 1.30**
  Let X and Y be sets and $f : X \to Y$. Then the following three statements are equivalent.

  1. f has an inverse;
  2. f is injective from X onto Y;
  3. There is a function $g : Y \to X$ such that

  $$g(f(x)) = x \quad \forall x \in X$$

  and

  $$f(g(y)) = y \quad \forall y \in Y$$

  Moreover, for each $f : X \to Y$, there is only one function g that satisfies these. It is the inverse function f^{-1}.

- **Remark 1.31**
  Let I be an interval and let $f : I \to \mathbb{R}$. If the derivative of f is either always positive on I, or always negative on I, then f is injective on I.

- **Definition 1.33** *Image*
  Let X and Y be sets and $f : X \to Y$. The *image* of a set $E \subseteq X$ under f is the set

  $$f(E) := \{y \in Y : y = f(x) \text{ for some } x \in E\}$$

  The *inverse image* of a set $E \subseteq Y$ under f is the set

  $$f^{-1}(E) := \{x \in X : f(x) = y \text{ for some } y \in E\}$$

- **Definition 1.35** *Union, Intersection*
  Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ be a collection of sets.

  1. The *union* of the collection $\mathcal{E}$ is the set

  $$\bigcup_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$$

  2. The *intersection* of the collection $\mathcal{E}$ is the set

  $$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

- **Theorem 1.36** *DeMorgan's Laws*
  Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. If for each $E \subseteq X$ the symbol E^c represents the set $X \setminus E$, then

  $$\left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

  and

  $$\left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

- **Theorem 1.37**
  Let X and Y be sets and $f : X \to Y$.
  1. If $\{E_\alpha\}\alpha \in A$ is a collection of subsets of X, then
  $$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha) \quad \text{and} \quad f\left(\bigcap_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(E_\alpha)$$
  2. If B and C are subsets of X, then $f(C) \setminus f(B) \subseteq f(C \setminus B)$
  3. If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of Y, then
  $$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha)$$
  4. If B and C are subsets of Y, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.
  5. If $E \subseteq f(x)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$.

1.6 Countable and Uncountable Sets

- **Definition 1.38** *Countable Uncountable* Let E be a set.
  1. E is said to be *finite* if and only if either $E = \emptyset$ or there exists an injective function which takes $\{1, 2, \ldots, n\}$ onto E, for some $n \in \mathbb{N}$.
  2. E is said to be *countable* if and only if there exists and injective function which takes $\mathbb{N}$ onto E.
  3. E is said to be *at most countable* if and only if E is either finite or countable.
  4. E is said to be *uncountable* if and only if E is neither finite nor countable.
- **Remark 1.39** *Cantor's Diagonalisation Argument*
  The open interval $(0, 1)$ is uncountable.
- **Lemma 1.40**
  A nonempty set E is at most countable if and only if there is a function g from $\mathbb{N}$ onto E.
- **Theorem 1.41**
  Suppose A and B are sets.
  1. If $A \subseteq B$ and B is at most countable, then A is at most countable.
  2. If $A \subseteq B$ and A is uncountable, then B is uncountable.
  3. $\mathbb{R}$ is uncountable.
- **Theorem 1.42**
  Let $A_1, A_2, \ldots$ be at most countable sets.
  1. Then $A_1 \times A_2$ is at most countable.
  2. If
  $$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j\in\mathbb{N}} A_j := \{x : x \in A_j \text{ for some } j \in \mathbb{N}\},$$
  then E is at most countable.
- **Remark 1.43**
  The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but the set of irrationals is uncountable.

2 Sequences in $\mathbb{R}$

2.1 Limits of Sequences

- **Definition 2.1** *Convergence*
A sequence of real numbers $\{x_n\}$ is set to *converge* to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

$$n \geq N \implies |x_n - a| < \epsilon$$

- **Remark 2.4** A sequence can have at most one limit.

- **Definition 2.5** *Subsequence*
By a *subsequence* of a sequence $\{x_n\}_{n\in\mathbb{N}}$, we shall mean a sequence of the form $\{x_{nk}\}_{k\in\mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$.

- **Remark 2.6**
If $\{x_n\}_{n\in\mathbb{N}}$ converges to a and $\{x_{nk}\}k \in \mathbb{N}$ is any subsequence of $\{x_n\}_{n\in\mathbb{N}}$, then x_{nk} converges to a as $k \to \infty$.

- **Definition 2.7** *Bounded Sequences*
Let $\{x_n\}$ be a sequence of real numbers.

  1. The sequence $\{x_n\}$ is said to be *bounded above* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above.

  2. The sequence $\{x_n\}$ is said to be *bounded below* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded below.

  3. $\{x_n\}$ is said to be *bounded* if and only if it is bounded both above and below.

- **Theorem 2.8** Every convergent sequence is bounded.

2.2 Limit Theorems

- **Theorem 2.9** *Squeeze Theorem*
Suppose that $\{x_n\}, \{y_n\}$, and $\{w_n\}$ are real sequences.

  1. If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if there is an $N_0 \in \mathbb{N}$ such that

  $$x_n \leq w_n \leq y_n \text{ for } n \geq N_0$$

  then $w_n \to a$ as $n \to \infty$.

  2. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

- **Theorem 2.11**
Let $E \subset \mathbb{R}$. If E has a finite supremum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ (respectively, a sequence $y_n \in E$ such that $y_n \to \inf E$) as $n \to \infty$.

- **Theorem 2.12**
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then

  1.
  $$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

  2.
  $$\lim_{n\to\infty} (\alpha x_n) = \alpha \lim_{n\to\infty} x_n$$

  and

3.
$$\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$$
If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then

4.
$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$
(In particular, all these limits exist.)

- **Definition 2.14** *Divergence*
  Let $\{x_n\}$ be a sequence of real numbers.

  1. $\{x_n\}$ is said to *diverge* to $+\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
  $$n \geq N \implies x_n > M$$
  2. $\{x_n\}$ is said to *diverge* to $-\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
  $$n \geq N \implies x_n < M$$

- **Theorem 2.15**
  Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \to +\infty$ (respectively, $x_n \to -\infty$) as $n \to \infty$.

  1. If y_n is bounded below (respectively, y_n is bounded above), then
  $$\lim_{n\to\infty}(x_n + y_n) = +\infty \quad \text{(respectively, } \lim_{n\to\infty}(x_n + y_n) = -\infty\text{)}$$
  2. If $\alpha > 0$, then
  $$\lim_{n\to\infty}(\alpha x_n) = +\infty \quad \text{(respectively, } \lim_{n\to\infty}(\alpha x_n) = -\infty\text{)}$$
  3. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$, then
  $$\lim_{n\to\infty}(x_n y_n) = +\infty \quad \text{(respectively, } \lim_{n\to\infty}(x_n y_n) = -\infty\text{)}$$
  4. If $\{y_n\}$ is bounded and $x_n \neq 0$, then
  $$\lim_{n\to\infty}\frac{y_n}{x_n} = 0$$

- **Corollary 2.16**
  Let $\{x_n\}$, $\{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$, as $n \to \infty$, then
  $$\lim_{n\to\infty}(x_n + y_n) = x + y$$
  provided that the right side is not of the form $\infty - \infty$, and
  $$\lim_{n\to\infty}(\alpha x_n) = \alpha x, \quad \lim_{n\to\infty}(x_n y_n) = xy$$
  provided that none of these products is of the form $0 \cdot \pm\infty$.

- **Theorem 2.17** *Comparison Theorem*
  Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that
  $$x_n \leq y_n \text{ for } n \geq N_0$$
  then
  $$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$
  In particular, if $x_n \in [a, b]$ converges to some point c, then c must belong to $[a, b]$.

2.3 Bolzano-Weierstrass Theorem

- **Definition 2.18** *Increasing, Decreasing* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.
  1. $\{x_n\}$ is said to be *increasing* (respectively, *strictly increasing*) if and only if $x_1 \le x_2 \le \cdots (\text{respectively}, x_1 < x_2 < \cdots)$.
  2. $\{x_n\}$ is said to be *decreasing* (respectively, *strictly decreasing*) if and only if $x_1 \ge x_2 \ge \cdots (\text{respectively}, x_1 > x_2 > \cdots)$.
  3. $\{x_n\}$ is said to be *monotone* if and only if it is either increasing or decreasing.
- **Theorem 2.19** *Monotone Convergence Theorem*
  if $\{x_n\}$ is increasing and bounded above, or if $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.
- **Definition 2.22** *Nested*
  A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be *nested* if and only if

$$I_1 \supseteq I_2 \supseteq \cdots$$

- **Theorem 2.23** *Nested Interval Property*
  If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then $E := \bigcap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the lengths of these intervals satisfy $|I_n \to 0$ as $n \to \infty$ then E is a single point.
- **Remark 2.24** The Nested Interval Property might not hold if "closed" is omitted.
- **Remark 2.25** The Nested Interval Property might not hold if "bounded" is omitted.
- **Theorem 2.26** *Bolzano-Weierstrass Theorem*
  Every bounded sequence of real numbers has a convergent subsequence.

2.4 Cauchy Sequences

- **Definition 2.27** *Cauchy*
  A sequence of points $x_n \in \mathbb{R}$ is said to be *Cauchy* (in R) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \ge N \implies |x_n - x_m| < \epsilon$$

- **Remark 2.28** If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.
- **Theorem 2.29** *Cauchy*
  Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges (to some point $a \in \mathbb{R}$).
- **Remark 2.31** A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

2.5 Limits Supremum and Infimum

- **Definition 2.32** *Limit Supremum & Infimum*
  Let $\{x_n\}$ be a real sequence. Then the *limit supremum* of $\{x_n\}$ is the extended real number

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{k\ge n} x_k)$$

and the *limit infimum* of $\{x_n\}$ is the extended real number

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{k\ge n} x_k)$$

- **Theorem 2.35**
  Let $\{x_n\}$ be a sequence of real numbers, $s = \limsup_{n\to\infty} x_n$, and $t = \liminf_{n\to\infty} x_n$. Then there are subsequences $\{x_{nk}\}_{k\in\mathbb{N}}$ and $\{x_{\ell j}\}_{j\in\mathbb{N}}$ such that $x_{nk} \to s$ as $k \to \infty$ and $x_{\ell j} \to t$ as $j \to \infty$.

- **Theorem 2.36**
  Let $\{x_n\}$ be a real sequence and x be an extended real number. Then $x_n \to x$ as $n \to \infty$ if and only if
  $$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$$

- **Theorem 2.37**
  Let $\{x_n\}$ be a sequence of real numbers. Then $\limsup_{n\to\infty} x_n$ (respectively, $\liminf_{n\to\infty}$) is the largest value (respectively, the smallest value) to which some subsequences of $\{x_n\}$ converges. Namely, if $x_{nk} \to x$ as $k \to \infty$, then
  $$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n$$

- **Remark 2.38** If $\{x_n\}$ is any sequence of real numbers, then
  $$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

- **Remark 2.39** A real sequence $\{x_n\}$ is bounded above if and only if $\limsup_{n\to\infty} x_n < \infty$, and is bounded below if and only if $\liminf_{n\to\infty} x_n > -\infty$.

- **Theorem 2.40**
  If $x_n \leq y_n$ for n large, then
  $$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n \quad \text{and} \quad \liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty} y_n$$

3 Functions on R

3.1 Two-Sided Limits

- **Definition 3.1** *Limits*
  Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

  $$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

  In this case we write

  $$L = \lim_{x \to a} f(x) \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

  and call L the *limit* of $f(x)$ as x approaches a.

- **Remark 3.4**
  Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined everywhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \setminus \{a\}$ and $f(x) \to L$ as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and

  $$\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

- **Theorem 3.6** *Sequential Characterisation of Limits*
  Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then

  $$L = \lim_{x \to a} f(x)$$

  exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $\{x_n\} \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

- **Theorem 3.8**
  Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, are real functions defined everywhere on I except possibly at a. If $f(x)$ and $g(x)$ converge as x approaches a, then so do $(f + g)(x), (fg)(x), (\alpha f)(x)$, and $(f/g)(x)$ (when the limit of $g(x)$ is nonzer). In fact,

  $$\begin{aligned} \lim_{x \to a} (f + g)(x) &= \lim_{x \to a} + \lim_{x \to a} g(x) \\ \lim_{x \to a} (\alpha f)(x) &= \lim_{x \to a} f(x) \\ \lim_{x \to a} (fg)(x) &= \lim_{x \to a} \lim_{x \to a} g(x) \end{aligned}$$

  and (when the limit of $g(x)$ is nonzero)

  $$\lim_{x \to a} \left(\frac{f}{g} \right) (x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

- **Theorem 3.9** *Squeeze Theorem for Functions*
  Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined everywhere on I except possibly at a.

  1. If $g(x) \leq h(x) \leq f(x)$ $\forall x \in I \setminus \{a\}$, and

     $$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,$$

     then the limit of $h(x)$ exists, as $x \to a$, and

     $$\lim_{x \to a} h(x) = L.$$

2. If $|g(x)| \leq M \ \forall x \in I \setminus \{a\}$ and $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} f(x)g(x) = 0$$

- **Theorem 3.10** *Comparison Theorem for Functions*
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined everywhere on I except possibly at a. If f and g have a limit as x approaches a and $f(x) \leq g(x) \ \forall x \in I \setminus \{a\}$, then
$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

3.2 One-Sided Limits and Limits at Infinity

- **Definition 3.12** *Converge from left & right*
Let $a \in \mathbb{R}$ and f be a real function.

1. $f(x)$ is said to *converge to L as x approaches a from the right* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *right-hand limit* of f at a, and denote it by

$$f(a+) := L =: \lim_{x \to a+} f(x)$$

2. $f(x)$ is said to *converge to L as x approaches a from the left* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *left-hand limit* of f at a, and denote it by

$$f(a-) := L =: \lim_{x \to a-} f(x)$$

- **Theorem 3.14**
Let f be a real function. Then the limit

$$\lim_{x \to a} f(x)$$

exists and equals L if and only if

$$L = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

- **Definition 3.15** *Convergence*
Let $a, L, \in \mathbb{R}$ and let f be a real function.

1. $f(x)$ is said to *converge* to L as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subset \text{Dom}(f)$ and given $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

Similarly, $f(x)$ is said to *converge* to L as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(infty, -c) \subset \text{Dom}(f)$ and given $\epsilon > 0$ there is $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

2. The function $f(x)$ is said to converge to ∞ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathrm{R}$ there is a $\delta > 0$ such that $0 \leq |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to a$$

Similarly, $f(x)$ is said to *converge* to $-\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to a$$

- **Theorem 3.17**
Let a be an extended real number, and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then

$$\lim_{x \to a; x \in I} f(x)$$

exists and equals L if and only if $f(x_n) \to L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \to a$ as $n \to \infty$.

3.3 Continuity

- **Definition 3.19** *Continuous*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.

1. f is said to be *continuous at a point* $a \in \mathbb{E}$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, and a) such that

$$|x - a| < delta \quad \text{and} \quad x \in E \implies |f(x) - f(a)| < \epsilon$$

2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

- **Remark 3.20**
Let I be an open interval which contains a point a and $f : I \to \mathbb{R}$. Then f is continuous at $a \in \mathbb{I}$ if and only if

$$f(a) = \lim_{x \to a} f(x)$$

- **Theorem 3.21**
Suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f : E \to \mathbb{R}$. Then the following statements are equivalent:

1. f is continuous at $a \in E$.
2. If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

- **Theorem 3.22**
Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (respectively continuous on the set E), then so are $f + g$, fg, and αf (for any $\alpha \in \mathbb{R}$). Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$ (respectively, on E when $g(x) \neq 0 \ \forall x \in E$).

- **Definition 3.23** *Composition*
Suppose that A and B are subsets of R, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $F(A) \subseteq B$ for every $x \in A$, then the *composition* of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by

$$(g \circ f)(x) := g(f(x)), \quad x \in A$$

- **Theorem 3.24**
  Suppose that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that $f(x) \in B \ \forall x \in A$.

  1. If $A := I \setminus \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if
  $$L := \lim_{x \to a; x \in I} f(x)$$
  exists and belongs to B, and if g is continuous and $L \in B$, then
  $$(g \circ f)(x) = g\left(\lim_{x \to a; x \in I} f(x)\right)$$
  2. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.

- **Definition 3.25** *Bounded*
  Let E be a nonempty subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be *bounded* on E if and only if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$, in which case we shall say that f is *dominated* by M on E.

- **Theorem 3.26** *Extreme Value Theorem*
  If I a is closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover if
  $$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x)$$
  then there exist points $x_m, x_M \in I$ such that
  $$f(x_M) = M \quad \text{and} \quad f(x_m) = m$$

- **Remark 3.27** The Existence Value Theorem is false if either "closed" or "bounded" is dropped from the hypotheses.

- **Lemma 3.28**
  Suppose that $a < B$ and that $f : [a, b) \to \mathbb{R}$. If f is continuous at a point $x_0 \in [a, b)$ and $f(x_0) > 0$, then there exist a positive number ϵ and a point $x_1 \in [a, b)$ such that $x_1 > x_0$ and $f(x) > \epsilon \ \forall x \in [x_0, x_1]$.

- **Theorem 3.29** *Intermediate Value Theorem*
  Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

- **Remark 3.34** The composition of two functions $g \circ f$ can be nowhere continuous, even though f is discontinuous only on $\mathbb{Q}$ and g is discontinuous at only one point.

3.4 Uniform Continuity

- **Definition 3.35** *Uniform continuity*
  Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then $\mho$ is said to be *uniformly continuous* on E if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that
  $$|x - a| < delta \quad \text{and} \quad x, a, \in E \implies |f(x) - f(a)| < \epsilon$$

- **Lemma 3.38**
  Suppose that $E \subseteq \mathbb{R}$ and that $f : E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, the $f(x_n)$ is Cauchy.

- **Theorem 3.39**
  Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

- **Theorem 3.40**
  Suppose that $a < b$ and that $f : (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if f can be continuously extended to $[a, b]$; that is, if and only if there is a continuous function $g : [a, b] \to \mathbb{R}$ which satisfies
  $$f(x) = g(x), \quad x \in (a, b)$$

4 Differentiability on R

4.1 The Derivative

- **Definition 4.1** *Differentiable*
  A real function f is said to be *differentiable* at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and
  $$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
  exists. In this case $f'(a)$ is called the *derivative* of f at a.

- **Theorem 4.2**
  A real function f is differentiable at some point $a \in \mathbb{R}$ if and only if there exist an open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at a, and
  $$f(x) = F(x)(x-a) + f(a)$$
  holds for all $x \in I$ in which case $F(a) = f'(a)$.

- **Theorem 4.3**
  A real function f is differentiable at a if and only if there is a function T of the form $T(x) := m(x)$ such that
  $$\lim_{h \to 0} \frac{f(a+h) - f(a) - t(h)}{h} = 0$$

- **Theorem 4.4**
  If f is differentiable at a, then f is continuous at a.

- **Definition 4.6** *Continuously differentiable*
  Let I be a nondegenerate interval.

  1. A function $f : I \to \mathbb{R}$ is said to be *differentiable* on I if and only if
     $$f_i'(a) := \lim_{x \to a; x \in I} \frac{f(x) - f(a)}{x - a}$$
     exists and is finite for every $a \in I$.
  2. f is said to be *continuously differentiable* on I if and only if f_I' exists and is continuous on I.

- **Remark 4.9**
  $f(x) = |x|$ is differentiable on $[0, 1]$ and on $[-1, 0]$ but not on $[-1, 1]$.

4.2 Differentiability Theorems

- **Theorem 4.10**
  Let f and g be real functions and $\alpha \in \mathbb{R}$. If f and g are differentiable at a, then $f + g$, αf, $f \cdot g$, and [when $g(a) \neq 0$] f/g are all differentiable at a. In fact,
  $$(f+g)'(a) = f'(a) + g'(a)$$
  $$(\alpha f)'(a) = \alpha f'(a)$$
  $$(f \cdot g)'(a) = g(a)g'(a) + f(a)g'(a)$$
  $$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$$

- **Theorem 4.11** *Chain Rule*
  Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with
  $$(g \circ f)'(a) = g'(f(a))f'(a)$$

4.3 Mean Value Theorem

- **Lemma 4.12** *Rolle's Theorem*
  Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.
- **Remark 4.13**
  The continuity hypothesis on Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.
- **Remark 4.14**
  The differentiability hypothesis on Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.
- **Theorem 4.15**
  Suppose that $a, b, \in \mathbb{R}$ with $a < b$.
  1. *Generalised Mean Value Theorem*: If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that
     $$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$
  2. *Mean Value Theorem*: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that
     $$f(b) - f(a) = f'(c)(b - A)$$
- **Definition 4.16** *Increasing, Monotone, Decreasing*
  Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.
  1. f is said to be *increasing* (respectively, *strictly increasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \leq f(x_2)$ [respectively, $f(x_1) < f(x_2)$].
  2. f is said to be *decreasing* (respectively, *strictly decreasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \geq f(x_2)$ [respectively, $f(x_1) > f(x_2)$].
  3. f is said to be *monotone* (respectively, *strictly monotone*) on E if and only if f is either decreasing or increasing (respectively, either strictly decreasing or strictly increasing) on E.
- **Theorem 4.17**
  Suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a, b]$, and that f is differentiable on (a, b).
  1. If $f'(x) > 0$ [respectively $f'(x) < 0$] for all $x \in (a, b)$, then f is strictly increasing (respectively, strictly decreasing) on $[a, b]$.
  2. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
  3. If g is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on $[a, b]$.
- **Theorem 4.18**
  Suppose that f is increasing on $[a, b]$
  1. If $c \in [a, b)$, then $f(c+)$ exists and $f(c) \leq f(c+)$.
  2. If $c \in (a, b]$, then $f(c-)$ exists and $f(c-) \leq f(c)$.

- **Theorem 4.19**
  If f is monotone on an interval I, then f has at most countable many points of discontinuity on I.

- **Theorem 4.21** *Bernoulli's Inequality*
  Let α be a positive real number. If $0 < \alpha < 1$, then $(1+x)^\alpha \leq 1 + \alpha x\ \forall x \in [-1, \infty)$, and if $\alpha \geq 1$, then $(1+x)^\alpha \geq 1 + \alpha x\ \forall x \in [-1, \infty)$.

- **Theorem 4.23** *Intermediate Value Theorem for Derivatives*
  Suppose that f is differentiable on $[a, b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a, b)$ such that $f'(x_0) = y_0$.

4.4 Taylor's Theorem and L'Hopital's Rule

- **Theorem 4.24** *Taylor's Formula*
  Let $n \in \mathbb{N}$ and let a, b be extended real numbers with $a < b$. If $f : (a, b) \to \mathbb{R}$, and if $f^{(n+1)}$ exists on (a, b), then for each pair of points $(x, x_0 \in (a, b)$ there is a number c between x and x_0 such that
  $$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

- **Theorem 4.27** *L'Hopital's Rule*
  Let a be an extended real number and I be an open interval which either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I \setminus \{a\}$ and that $g(x) \neq 0 \neq g'(x)\ \forall x \in I \setminus \{a\}$. Suppose further that
  $$A := \lim_{x \to a; x \in I} f(x) = \lim_{x \to a; x \in I} g(x)$$
  is either 0 or ∞. If
  $$B := \lim_{x \to a; x \in I} \frac{f'(x)}{g'(x)}$$
  exists as an extended real number, then
  $$\lim_{x \to a; x \in I} \frac{f(x)}{g(x)} = \lim_{x \to a; x \in I} \frac{f'(x)}{g'(x)}$$

4.5 Inverse Function Theorems

- **Theorem 4.32**
  Let I be a nondegenerate interval and suppose that $f : I \to \mathbb{R}$ is injective. If f is continuous on I, then $J := f(I)$ is an interval, f is strictly monotone on I, and f^{-1} is continuous and strictly monotone on J.

- **Theorem 4.33** *Inverse Function Theorem*
  Let I be an open interval and $f : I \to \mathbb{R}$ be injective and continuous. If $b = f(a)$ for some $a \in I$ and if $f'(a)$ exists and is nonzero, then f^{-1} is differentiable at b and $(f^{-1})'(b) = \frac{1}{f'(a)}$.

5 Riemann Integration

5.1 Introduction

5.2 Step functions and their integrals

- **Definition 1** *Step function*
  We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a *step function* if there exist real numbers $x_0 < x_1 < \cdots < x_n$ (for some $n \in \mathbb{N}$) such that

  1. $\phi(x) = 0$ for $x < x_0$ and $x > x_n$
  2. ϕ is constant on $(x_{j-1}, x_j) 1 \leq j \leq n$.

- **Definition 2**
  If ϕ is a steup function with respect to $\{x_0, x_1, \ldots, x_n\}$ which takes the value c_j on (x_{j-1}, x_j), then
  $$\int \phi := \sum_{j=1}^{n} c_j (x_j - x_{j-1})$$

- **Proposition 1**
  If ϕ and ψ are step functions and α and $\beta \in \mathbb{R}$, then
  $$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi.$$

5.3 Riemann-integrable functions and their integrals

- **Definition 3** *Riemann-integrable*
  Let $f : \mathbb{R}\ddot{0}\mathbb{R}$. We say that f is *Riemann-integrable* if for every $\epsilon > 0$ there exist step functions ϕ and ψ such that
  $$\phi \leq f \leq \psi$$
  and
  $$\int \psi - \int \phi < \epsilon$$

- **Theorem 1**
  A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if
  $$\sup\{\int \phi : \phi \text{ is a step function and } \phi \leq f\} = \inf\{\int \psi : \psi \text{ is a step function and } \psi \geq f\}.$$

- **Definition 4**
  If f is Riemann-integrable we define its integral $\int f$ as the common value
  $$\int f := \sup\{\int \phi : \phi \text{ is a step function and } \phi \leq f\} = \inf\{\int \psi : \psi \text{ is a step function and} \psi \geq f\}.$$

- **Theorem 2**
  A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if there exist sequences of step functions ϕ_n and ψ_n such that
  $$\phi_n \leq f \leq \psi_n \ \forall n, \quad \text{and} \quad \int \psi_n - \int \phi_n \to 0$$
  If ϕ_n and ψ_n are any sequences of step functions satisfying above, then
  $$\int \phi_n \to \int f \text{ and } \int \psi_n \to \int f$$
  as $n \to \infty$.

- **Lemma 1**
Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with bounded support $[a, b]$. The following are equivalent:

  1. f is Riemann-integrable.
  2. for every $\epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ such that, if M_j and m_j denote the supremum and infimum values of$f$ on $[x_{j-1}, x_j]$ respectively, then
  $$\sum_{j-1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon$$
  3. for every $\epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ such that, with $I_j = (x_{j-1}, x_j)$ for $j \geq 1$,
  $$\sum_{j=1}^{n} \sup_{x,y,\in I_j} |f(x) - f(y)||I_j| < \epsilon.$$

  For $f : \mathbb{R} \to \mathbb{R}$ a bounded function with bounded support $[a, b]$ and for $a = x_0 < \cdots < x_n = b$, let $I_j = (x_{j-1}, x_j), m_j := \inf_{x \in I_j} f(x)$ and $M_j := \sup_{x \in I_j} f(x)$. Define the *lower step function of f with respect to* $\{x_0, \ldots, x_n\}$ as
  $$\phi_*(x) = \sum_{j=1}^{n} m_j \chi_{I_j} + \sum_{j=0}^{n} \chi_{x_j}$$
  and the *upper step function of f with respect to* $\{x_0, \ldots, x_n\}$ as
  $$\phi^*(x) = \sum_{j=1}^{n} M_j \chi_{I_j} + \sum_{j=0}^{n} \chi_{x_j}$$
  Note that ϕ_* and ϕ^* are step functions, and that $\phi_* \leq f \geq \phi^*$.

- **Theorem 3**
Suppose f and g are Riemann-integrable and α and β are real numbers. Then

  1. $\alpha f + \beta g$ is Riemann-integrable and
  $$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$
  2. If $f \geq 0$ then $\int f \geq 0$; if $f \leq g$ then $\int f \leq \int g$.
  3. $|f|$ is Riemann-integrable and $|\int f| \leq \int |f|$
  4. $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann-integrable.
  5. fg is Riemann-integrable.

- **Theorem 4**
If $g : [a, b] \to \mathbb{R}$ is continuous, and f defined by $f(x) = g(x)$ for $a \leq x \leq b$, $f(x) = 0$ for $x \notin [a, b]$ then f is Riemann-integrable.

5.4 Fundamental Theorem of Calculus, and Practical Integration

- **Theorem 5**
Let $g : [a, b] \to \mathbb{R}$ be Riemann-integrable. For $a \leq x \leq b$ let $G(x) = \int_a^x g$. Suppose g is continuous at x for some $x \in [a, b]$. [If x is an endpoint, we mean one-sided continuous.] Then G is differentiable at x and $G'(x) = g(x)$. [If x is an endpoint, we mean one-sided differentiable.]

- **Theorem 6**
Suppose $f : [a, b] \to \mathbb{R}$ has continuous derivative f' on $[a, b]$. Then
$$\int_a^b f' = f(b) - f(a).$$

5.5 Integrals and uniform limits of sequences and series of functions

- **Theorem 7**
  Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence if Riemann-integrable functions which converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then f is Riemann-integrable and

$$\int f = \lim_{n\to\infty} \int f_n.$$

5.6 A couple of odds and ends

- **Improper integrals**
- **Integral test**

6 Infinite Series of Real Numbers

6.1 Introduction

- **Definition 6.1** *Partial sum*
  Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series with terms a_k.

  1. For each $n \in \mathbb{N}$, the *partial sum of S of order n* is defined by
  $$s_n := \sum_{k=1}^{n} a_k$$
  2. S is said to *converge* if and only if its sequence of partial sums $\{s_n\}$ converges to some $s \in \mathbb{R}$ as $n \to \infty$; that is, if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N \implies |s_n - s| < \epsilon$. In this case we shall write
  $$\sum_{k=1}^{\infty} a_k = s$$
  and call s the *sum*, or *value*, of the series $\sum_{k=1}^{\infty} a_k$
  3. S is said to *diverge* if and only if its sequence of partial sums $\{s_n\}$ does not converge as $n \to \infty$. When s_n diverges to $+\infty$ as $n \to \infty$, we shall also write
  $$\sum_{k=1}^{\infty} a_k = s$$

- **Theorem 6.5** *Divergence Test* Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers. If a_k does not converge to zero, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

- **Theorem 6.6** *Telescoping Series*
  If $\{a_k\}$ is a convergent real sequence, then
  $$sum_{k=1}^{\infty}(a_k - a_{k+1}) = a_1 - \lim_{k \to \infty} a_k$$

- **Theorem 6.7** *Geometric Series*
  Suppose that $x \in \mathbb{R}$, that $N \in \{0, 1, \ldots\}$, and that 0^0 is interpreted to be 1. Then the series $\sum_{k=N}^{\infty} x^k$ converges if and only if $|x| < 1$, in which case
  $$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}$$
  In particular,
  $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1.$$

- **Theorem 6.8** *The Cauchy Criterion*
  Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
  $$m \geq n \geq N \implies \left|\sum_{k=n}^{n} a_k\right| < \epsilon$$

- **Corollary 6.9**
  Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
$$n \geq N \implies \left|\sum_{k=n}^{\infty} a_k\right| < \epsilon$$

- **Theorem 6.10**
  Let $\{a_k\}$ and $\{b_k\}$ be real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then
$$\sum_{i=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$
  and
$$\sum_{k=1}^{\infty}(\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$$
  for any $\alpha \in \mathbb{R}$.

6.2 Series with Nonnegative Terms

- **Theorem 6.11**
  Suppose that $a_k \geq 0$ for large k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if its sequence of partial sums $\{s_n\}$ is bounded; that is, if and only if there exists a finite number $M > 0$ such that
$$\left|\sum_{i=1}^{n} a_k\right| \leq M \ \forall n \in \mathbb{N}$$

- **Theorem 6.12** *Integral Test*
  Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly integrable on $[1, \infty)$; that is if and only if
$$\int_1^{\infty} f(x)\,\mathrm{d}x < \infty$$

- **Corollary 6.13** *p-Series Test* The series
$$\sum_{i=1}^{\infty} \frac{1}{k^p}$$
  converges if and only if $p > 1$.

- **Theorem 6.14** *Comparison Test*
  Suppose that $0 \leq a_k \leq b_k$ for large k.
  1. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.
  2. If $\sum_{k=1}^{\infty} b_k = \infty$, then $\sum_{k=1}^{\infty} a_k = \infty$.

- **Theorem 6.16** *Limit Comparison Test*
  Suppose that $a_k \geq 0$, that $b_k > 0$ for large k, and that $L := \lim_{n\to\infty} \frac{a_n}{b_n}$ exists as an extended real number.
  1. If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.
  2. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.
  3. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

6.3 Absolute Convergence

- **Definition 6.18** *Absolute & Conditional Convergence*
  Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series.

  1. S is said to *converge absolutely* if and only if $\sum_{k=1}^{\infty} |a_k| < \infty$
  2. S is said to *converge conditionally* if and only if S converges but not absolutely.

- **Remark 6.19**
  A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \implies \sum_{k=1}^{\infty} |a_k| < \epsilon$$

- **Remark 6.20**
  If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k$ converges, but not conversely. In particular, there exist conditionally convergent series.

- **Definition 6.21** *Limit supremum*
  The *limit supremum* of a sequence of real numbers $\{x_k\}$ is defined to be

$$\limsup_{k\to\infty} x_k := \lim_{n\to\infty} \left(\sup_{k>n} x_k \right).$$

- **Remark 6.22**
  Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence.

  1. If $\limsup_{k\to\infty} x_k < x$, then $x_k < x$ for large k.
  2. If $\limsup_{k\to\infty} x_k > x$, then $x_k > x$ for infinitely many ks.
  3. If $x_k \to x$ as $x \to \infty$, then $\limsup_{k\to\infty} x_k = x$.

- **Theorem 6.23** *Root Test*
  Let $a_k \in \mathbb{R}$ and $r := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

  1. If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
  2. If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

- **Theorem 6.24** *Ratio test*
  Let $a_k \in \mathbb{R}$ with $a_k \neq 0$ for large k and suppose that

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

  exists as an extended real number.

  1. If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
  2. If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

- **Remark 6.25** The Root and Ratio tests are inconclusive when $r = 1$.

- **Definition 6.26** *Rearrangement*
  A series $\sum_{j=1}^{\infty} b_j$ is called a *rearrangement* of a series $\sum_{k=1}^{\infty} a_k$ if and only if there is an injection $f : \mathbb{N} \to \mathbb{N}$ such that

$$b_{f(k)} = a_k, \quad k \in \mathbb{N}$$

- **Theorem 6.27**
  If $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\sum_{j=1}^{\infty} b_j$ is any rearrangement of $\sum_{k=1}^{\infty} a_k$, then $\sum_{j=1}^{\infty} b_j$ converges and
  $$\sum_{k=1}^{\infty} a_k = \sum_{j=1}^{\infty} b_j$$
- **Lemma 6.28**
- **Theorem 6.29** *Riemann*

6.4 Alternating Series

- **Theorem 6.30** *Abel's Formula*
  Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be real sequences, and for each pair of integers $n \geq m \geq 1$ set
  $$A_{n,m} := \sum_{k=m}^{n} a_k$$
  Then
  $$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$$
  for all integers $n > m \geq 1$.
- **Theorem 6.31** *Dirichlet's Test*
  Let $a_k, b_k \in \mathbb{R}$ for $k \in \mathbb{N}$. If the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ is bounded an $b_k \downarrow 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} a_k b_k$ converges.
- **Corollary 6.32** *Alternating Series Test*
  If $a_k \downarrow 0$ as $k \to \infty$, then
  $$\sum_{k=1}^{\infty} (-1)^k a_k$$
  converges.

7 Infinite Series of Functions

7.1 Uniform Convergence of Sequences

- **Definition 7.1** *Pointwise Convergence*
  Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to *converge pointwise* on E if and only if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$.

- **Remark 7.2**
  Let E be a nonempty subset of $\mathbb{R}$. Then a sequence of functions f_n converges pointwise on E, as $n \to \infty$ if and only if for every $\epsilon > 0$ and $x \in E$ there is an $N \in \mathbb{N}$ (which may depend on x as well as ϵ) such that
  $$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

- **Remark 7.3**
  The pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable).

- **Remark 7.4**
  The pointwise limit of integrable functions is not necessarily integrable.

- **Remark 7.5**
  There exist differentiable functions f_n and f such that $f_n \to f$ pointwise on $[0, 1]$ but
  $$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)'$$
  for $x = 1$.

- **Remark 7.6**
  There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0, 1]$ but
  $$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x$$

- **Definition 7.7** *Uniform Convergence*
  Let E be a nonempty subset of $\mathbb{R}$. A sequence of function $f_n : E \to \mathbb{R}$ is said to *converge uniformly* on E to a function f if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
  $$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$
  for all $x \in E$.

- **Theorem 7.9**
  Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_n \to f$ uniformly on E, as $n \to \infty$. If f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \in E$.

- **Theorem 7.10**
  Suppose that $f_n \to f$ uniformly on a closed interval $[a, b]$. if each f_n is integrable on $[a, b]$, then so is f and
  $$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x$$
  In fact, $\lim_{n\to\infty} \int_a^x f_n(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t$ uniformly for $x \in [a, b]$.

- **Lemma 7.11** *Uniform Cauchy Criterion*
  Let E be a nonempty subset of $\mathbb{R}$ and let $f_n : E \to \mathbb{R}$ be a sequence of functions. Then f_n converges uniformly on E if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
  $$n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon$$
  for all $xinE$.

- **Theorem 7.12**
  Let (a, b) be a bounded interval and suppose that f_n is a sequence of funtions which converges at some $x_0 \in (a, b)$. If each f_n is differentiable on (a, b), and f_n' converges uniformly on (a, b) as $n \to \infty$, the f_n converges uniformly on (a, b) and
  $$\lim_{n\to\infty} f_n'(x) = \Big(\lim_{n\to\infty} f_n(x)\Big)'$$
  for each $x \in (a, b)$.

7.2 Uniform Convergence of Series

- **Definition 7.13** *Convergence*
  Let f_k be a sequence of real functions defined on some set E and set
  $$s_n(k) := \sum_{k=1}^{n} f_k(x), \quad x \in E, n \in \mathbb{N}$$
  1. The series $\sum_{k=1}^{n} f_k(x)$ is said to *converge pointwise* on E if and only if the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$.
  2. The series $\sum_{k=1}^{n} f_k(x)$ is said to *converge uniformly* on E if and only if the sequence $s_n(x)$ converges uniformly on E as $n \to \infty$.
  3. The series $\sum_{k=1}^{n} f_k(x)$ is said to *converge absolutely (pointwise)* on E if and only if $\sum_{k=1}^{n} |f_k(x)|$ converges for each $x \in E$.

- **Theorem 7.14**
  Let E be a nonempty subset of $\mathbb{R}$ and let $\{f_k\}$ be a sequence of real functions defined on E.
  1. Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.
  2. *Term-by-term integration.* Suppose that $E = [a, b]$ and that each f_k is integrable on $[a, b]$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on $[a, b]$, then f is integrable on $[a, b]$ and
  $$\int_a^b \sum_{k=1}^{\infty} f_k(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,\mathrm{d}x.$$
  3. *Term-by-term differentiation.* Suppose that E is a bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^{\infty} f_k$ converges at some $x_0 \in E$, and $\sum_{k=1}^{f} {}'_k$ converges uniformly on E, then $f := \sum_{k=1}^{\infty} f_k$ converges uniformly on E, f is differentiable on E, and
  $$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x)$$
  for $x \in E$.

- **Theorem 7.15** *Weierstrass M-Test*
  Let E be a nonempty subset of $\mathbb{R}$, let $f_k : E \to \mathbb{R}, k \in \mathbb{N}$, and suppose that $M_k \geq 0$ satisfies $\sum_{k=1}^{\infty} M_k < \infty$. If $|f_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in E$, then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on E.

- **Theorem 7.16*** *Dirichlet's Test for Uniform Convergence*
  Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_k, g_k : E \to \mathbb{R}, k \in \mathbb{N}$. If
  $$\left|\sum_{k=1}^{n} f_k(x)\right| \leq M < \infty$$
  for $n \in \mathbb{N}$ and $x \in E$, and if $g_k \downarrow 0$ uniformly on E as $k \to \infty$, then $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

7.3 Power Series

- **Definition** *Power Series*
  Let (a_n) be a sequence of real numbers, and $c \in \mathbb{R}$. A *power series* is a series of the form
  $$\sum_{n=1}^{\infty} a_n (x-c)^n$$
  With a_n being the *coefficients* and c its centre.
- **Definition** *Radius of Convergence*
  The *radius of convergence* R of the power series
  $$\sum_{n=1}^{\infty} a_n (x-c)^n$$
  is defined by
  $$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$$
  unless $(a_n r^n)$ is bounded for all $r \geq 0$, in which case we declare $R = \infty$.
- **Theorem 1**
  Suppose the radius of convergence R satisfies $0 < R < \infty$. If $|x-c| < R$, the power series converges absolutely. If $|x-c| > R$, the power series diverges.
- **Theorem 2**
  Assume that $R > 0$. Suppose that $0 < r < R$. Then the series converges uniformly and absolutely on $|x-c| \leq r$ to a continuous function f. hence
  $$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$
  defines a continuous function $f : (cR, c+R) \to \mathbb{R}$.
- **Lemma**
  The two power series $\sum_{n=1}^{\infty} a_n (x-c)^n$ and $\sum_{n\to\infty} n a_n (x-c)^{n-1}$ have the same radius of convergence.
- **Theorem 3**
  Suppose the radius of convergence of the power series is R. Then the function
  $$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$
  is infinitely differentiable on $|x-c| < R$, and for such x,
  $$f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$
  and the series converges absolutely, and also uniformly on $[c-r, c+r]$ $\forall r < R$. Moreover
  $$a_n = \frac{f^{(n)}(c)}{n!}$$

8 Metric Spaces

8.1 Introduction

- **Definition 10.1** *Metric Space*
  A *metric space* is a set X together with a function $\rho : X \times X \to \mathbb{R}$ (called the *metric* of ρ) which satisfies the following properties for all $x, y, z \in X$:

$$\begin{array}{rl} \text{Positive Definite} & \rho(x,y) \geq 0 \text{ with } \rho(x,y) = 0 \iff x = y \\ \text{Symmetric} & \rho(x,y) = \rho(y,x) \\ \text{Triangle Inequality} & \rho(x,y) \leq \rho(x,z) < \rho(z,y) \end{array}$$

- **Definition 10.7** *Ball*
  Let $a \in X$ and $r > 0$. Then *open ball* (in X) with *centre* a and *radius* r is the set

$$B_r(a) := \{x \in X : \rho(x,a) < r\}$$

  and the *closed ball* (in X) with *centre* a and *radius* r is the set

$$\{x \in X : \rho(x,a) \leq r\}$$

- **Definition 10.8** *Open & Closed*
  1. A set $V \subseteq X$ is said to be *open* if and only if for every $x \in V$ there is an $\epsilon > 0$ such that the open ball $B_\epsilon(x)$ is contained in V.
  2. A set $E \subseteq X$ is set to be *closed* if and only if $E^c := X \setminus E$ is open.
- **Remark 10.9** Every open ball is open, and every closed ball is closed.
- **Remark 10.10** If $a \in X$, then $X \setminus \{a\}$ is open, and $\{a\}$ is closed.
- **Remark (10.11)** In an arbitrary metric space, the empty set $\emptyset$ and the whole space X are both open and closed.
- **Definition 10.13** *Convergence, Cauch, & Boundedness*
  Let $\{x_n\}$ be a sequence in X.
  1. $\{x_n\}$ *converges* (in X) if there is a point $a \in X$ (called the *limit* of x_n) such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \rho(x_n, a) < \epsilon.$$

  2. $\{x_n\}$ is *Cauchy* if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies \rho(x_n, x_m) < \epsilon.$$

  3. $\{x_n\}$ is *bounded* if there is an $M > 0$ and a $b \in X$ such that $\rho(x_n, b) \leq M$ for all $n \in \mathbb{N}$.
- **Theorem 10.14**
  Let X be a metric space.
  1. A sequence X can have at most one limit.
  2. If $x_n \in X$ converges to a and $\{x_{nk}\}$ is any subsequence of $\{x_n\}$, then x_{nk} converges to a as $k \to \infty$.
  3. Every convergent sequence X is bounded.

4. Every convergent sequence in X is Cauchy.

- **Theorem 10.16**
Let $E \subseteq X$. Then E is closed if and only if the limit of every convergent sequence $x_k \in E$ satisfies
$$\lim_{k\to\infty} x_k \in E.$$

- **Remark 10.17** The discrete space contains bounded sequence which have no convergent subsequences.

- **Remark 10.18** The metric space $X = \mathbb{Q}$ contains Cauchy sequences which do not converge.

- **Definition 10.19** *Completeness*
A metric space X is said to be *complete* if and only if every Cauchy sequence $x_n \in X$ converges to some point in X.

- **Remark 10.20**
By *10.19*, a complete metric space X satisfies two properties:

  1. Every Cauchy sequence in X converges;
  2. the limit of every Cauchy sequence in X stay in X.

- **Theorem 10.21**
Let X be a complete metric space E be a subset of X. Then E (as a subspace) is complete if and only if E as a (subset) is closed.

8.2 Limits of Functions

- **Definition 10.22** *Cluster Point*
A point $a \in X$ is said to be a *cluster point* (of X) if and only if $B_\delta(a)$ contains infinitely many points for each $\delta > 0$.

- **Definition 10.25** *Converge*
Let a be a cluster point of X and $f : X \setminus \{a\} \to Y$. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that
$$0 < \rho(x, a) < \delta \implies \tau(f(x), L) < \epsilon.$$
In this case we write $f(x) \to L$ as $x \to a$, or
$$L = \lim_{x\to a} f(x),$$
and call L the *limit* of $f(x)$ as x approaches a.

- **Theorem 10.26**
Let a be a cluster point of X and $f, g : X \setminus \{a\} \to Y$.

  1. If $f(x) = g(x)\ \forall x \in X \setminus \{a\}$ and $f(x)$ has a limit as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and
  $$\lim_{x\to a} g(x) = \lim_{x\to a} f(x).$$
  2. *Sequential characterisation of limits.* The limit
  $$L := \lim_{x\to a} f(x)$$
  exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in X \setminus \{a\}$ which converges to a as $n \to \infty$.

3. Suppose that $Y = \mathbb{R}^n$. If $f(x)$ and $g(x)$ have a limit as x approaches a, then so do $(f+g)(x), (f \cdot g)(x), (\alpha f)(x)$, and $(f/g)(x)$ [when $Y = \mathbb{R}$ and the limit of $g(x)$ is nonzero]. In fact,

$$\lim_{x \to a}(f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x),$$
$$\lim_{x \to a}(\alpha f)(x) = \alpha \lim_{x \to a} f(x),$$
$$\lim_{x \to a}(f \cdot)(x) = \lim_{x \to a} \cdot \lim_{x \to a} g(x)$$

and [when $Y = \mathbb{R}$ and the limit of $g(x)$ is nonzero]

$$\lim_{x \to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

4. *Squeeze Theorem for Functions.* Suppose that $Y = \mathbb{R}$. If $h : X \setminus \{a\} \to \mathbb{R}$ satisfies $g(x) \leq h(x) \leq f(x)\ \forall x \in X \setminus \{a\}$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$$

then the limit of h exists, as $x \to a$, and

$$\lim_{x \to a} h(x) = L$$

5. *Comparison Theorem for Functions.* Suppose that $Y = \mathbb{R}$. If $f(x) \leq g(x)\ \forall X \setminus \{a\}$, and if f and g have a limit as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

- **Definition 10.27** *Continuity*
  Let E be a nonempty subset of X and $f : E \to Y$.

  1. f is said to be *continuous at a point* $a \in E$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that
  $$\rho(x, a) < \delta \text{ and } \in E \implies \tau(f(x), f(a)) < \epsilon.$$
  2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

- **Theorem 10.28**
  Let E be a nonempty subset of X and $f, g : E \to Y$.

  1. f is continuous at $a \in E$ if and only if $f(x_n) \to f(a)$, as $n \to \infty$, for all sequences $x_n \in E$ which converge to a.
  2. Suppose that $Y = \mathbb{R}^n$. If f, g are continuous at a point $a \in E$ (respectively continuous on a set E), then so are $f + g, f \cdot g$, and αf (for any $\alpha \in \mathbb{R}$). Moreover, in the case $Y = \mathbb{R}$, f/g is continuous at $a \in E$ when $g(a) \neq 0$ [respectively, on E when $g(x) \neq 0,\ \forall x \in E$].

- **Theorem 10.29**
  Suppose that X, Y, and Z are metric space and that a is a cluster point of X. Suppose further that $f : X \to Y$ and $g : f(X) \to Z$. If $f(x) \to L$ as a $x \to a$ and g is continuous at L, then
  $$\lim_{x \to a}(g \circ f)(x) = g\left(\lim_{x \to a} f(x)\right).$$

- **Definition 10.30** *Bolzano-Weierstrass Property*
  X is said to satisfy the *Bolzano-Weierstrass Property* if and only if every bounded sequence $x_n \in X$ has a convergent subsequence.

8.3 Interior, Closure, and Boundary

- **Theorem 10.31**
  Let X be a metric space.

  1. If $\{V_\alpha\}_{\alpha\in A}$ is any collection of open sets in X, then
  $$\bigcup_{\alpha\in A} V_\alpha$$
  is open.
  2. If $\{V_k : k = 1, 2, \ldots, n\}$ is a finite collection of open sets in X, then
  $$\bigcap_{k=1}^{n} V_k := \bigcap_{k\in\{1,2,\ldots,n\}} V_k$$
  is open.
  3. If $\{E_\alpha\}_{\alpha\in A}$ is any collection of closed sets in X, then
  $$\bigcap_{\alpha\in A} E_\alpha$$
  is closed.
  4. If $\{E_k : k = 1, 2, \ldots, n\}$ is a finite collection of closed sets in X, then
  $$\bigcup_{k=1}^{n} E_k := \bigcup_{k\in\{1,2,\ldots,n\}} E_k$$
  is closed.
  5. If V is open in X and E is closed in X, then $V \setminus E$ is open and $E \setminus V$ is closed.

- **Remark 10.32**
  Statements 2 and 4 of *Theorem 10.31* are false if arbitrary collections are used in place of finite collections.

- **Definition 10.33** *Interior & Closure*
  Let E be a subset of a metric space X.

  1. The *interior* of E is the set
  $$E^O := \bigcup\{V : V \subseteq E \text{ and } V \text{is open in } X\}.$$
  2. The *closure* of E is the set
  $$\overline{E} := \bigcup\{B : B \supseteq E \text{ and } B \text{ is closed in } X\}.$$

- **Theorem 10.34**
  Let $E \subseteq X$. Then

  1. $E^O \subseteq E \subseteq \overline{E}$,
  2. if V is open and $V \subseteq E$, then $V \subseteq E^0$, and
  3. if C is closed and $C \supseteq E$, then $C \supseteq \overline{E}$.

- **Definition 10.37** *Boundary*
  Let $E \subseteq X$. The *boundary* of E is the set
  $$\partial E := \{x \in X : \forall r > 0,\ B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \cup E^c \neq \emptyset\}.$$
  [We refer to the last two conditions in the definition of ∂E by saying $B_r(x)$ *intersects* E and E^c.]

- **Theorem 10.39**
  Let $E \subseteq X$. Then
  $$\partial E = \overline{E} \setminus E^0.$$

- **Theorem 10.40**
  Let $A, B \subseteq X$. Then
  1. $(A \cup B)^O \supseteq A^O \cup B^O, \quad (A \cap B)^O = A^O \cap B^O$,
  2. $\overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$,
  3. $(A \cup B) \subseteq A \cup B$, and $(A \cap B) \subseteq (A \cap B) \cup (B \cap A) \cup (A \cap B)$.

8.4 Compact Sets

- **Definition 10.41** *Covering*
  Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ be a collection of subsets of a metric space X and suppose that E is a subset of X.
  1. $\mathcal{V}$ is said to *cover* E if and only if
  $$E \subseteq \bigcup_{\alpha \in A} V_\alpha.$$
  2. $\mathcal{V}$ is said to be an *open covering* of E if and only if $\mathcal{V}$ covers E and each V_α is open.
  3. Let $\mathcal{V}$ be a covering of E. $\mathcal{V}$ is said to have a *finite* (respectively *countable*) *subcovering* if and only if there is a finite (respectively, countable) subset A_0 of A such that $\{V_\alpha\}_{\alpha \in A_0}$ covers E.

- **Definition 10.42** *Compact*
  A subset H of a metric space X is said to be *compact* if and only if every open covering of H has a finite subcover.

- **Remark 10.43** The empty set and all finite subsets of a metric space are compact.

- **Remark 10.44** A compact set is always closed.

- **Remark 10.45** A closed subset of a compact set is compact.

- **Theorem 10.46**
  Let H be a subset of a metric space X. If H is compact, then H is closed and bounded.

- **Remark 10.47** The converse of *Theorem 10.46* is false for arbitrary metric spaces

- **Definition 10.48** *Separable*
  A metric space X is said to be *separable* if and only if it contains a countable dense subset (i.e. if and only if there is a countable subset Z of X such that for every point $a \in X$ there is a sequence $x_k \in Z$ such that $x_k \to a$ as $k \to \infty$).

- **Theorem 10.49** *Lindelöf*
  Let E be a subset of a separable metric space X. If $\{V_\alpha\}_{\alpha \in A}$ is a collection of open sets and $E \subseteq \bigcup_{\alpha \in A} V_\alpha$, then there is a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ of A such that
  $$E \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}$$

- **Theorem 10.50** *Heine-Borel*
  Let X be a separable metric space which satisfies the *Bolzano-Weierstrass Property*, and H be a subset of X. Then H is compact if and only if it is closed and bounded.

- **Definition 10.51** *Uniform Continuity*
  Let X be a metric space, E be a nonempty subset of X, and $f : E \to Y$. Then f is said to be *uniformly continuous* on E if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x, a) < \delta \text{ and } x, a \in E \implies \tau(f(x), f(a)) < \epsilon.$$

- **Theorem 10.52**
  Suppose that E is a compact subset of X and that $f : X \to Y$. Then f is uniformly continuous on E if and only if f is continuous on E.

8.5 Connected Sets

- **Definition 10.53** *Separate & Connected*
  Let X be a metric space.

  1. A pair of nonempty open sets U, V in X is said to *separate* X if and only if $X = U \cup V$ and $U \cap V = \emptyset$.

  2. X is said to be *connected* if and only if X cannot be separated by any pair of open sets U, V.

- **Definition 10.54** *Relatively open & closed*
  Let X be a metric space and $E \subseteq X$.

  1. A set $U \subseteq E$ is said to be *relatively open* in E if and only if there is a set V open in X such that $U = E \cup V$.

  2. A set $A \subseteq E$ is said to be *relatively closed* in E if and only if there is a set C closed in X such that $A = E \cap V$.

- **Remark 10.55**
  Let $E \subseteq X$. If there exists a pair of open sets A, B in X which separate E, then E is not connected.

- **Theorem 10.56**
  A subset E of $\mathbb{R}$ is connected if and only if E is an interval.

8.6 Continuous Functions

- **Theorem 10.58**
  Suppose that $f : X \to Y$. Then f is continuous if and only if $f^{-1}(V)$ is open in X for every open V in Y.

- **Corollary 10.59**
  Let EX and $f : E \to Y$. Then f is continuous on E if and only if $f^{-1}(V) \cap E$ is relatively open in E for all open sets V in Y.

- **Theorem 10.61**
  If H is compact in X and $f : H \to Y$ is continuous on H, then $f(H)$ is compact in Y.

- **Theorem 10.62**
  If E is connected in X and $f : E \to Y$ is continuous on E, then $f(E)$ is connected in Y.

- **Theorem 10.63** *Extreme Value Theorem*
  Let H be a nonempty, compact subset of X and suppose that $f : H \to \mathbb{R}$ is continuous. Then

$$M := \sup\{f(x) : x \in H\} \quad \text{and} \quad m := \inf\{f(x) : x \in H\}$$

are finite real numbers and there exist points $x_M, x_m \in H$ such that $M = f(x_M)$ ad $m = f(x_m)$.

- **Theorem 10.64**
  if H is a compact subset of X and $f : H \to Y$ is injective and continuous, then f^{-1} is continuous on $f(H)$.

9 Contraction Mapping & ODEs

9.1 Banach's Contraction Mapping Theorem

- **Definition** *Contraction*
  Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if there exists a number α with $0 < \alpha < 1$ such that
  $$d(f(x), f(y)) \leq \alpha d(x, y)\ \forall x, y \in X.$$
  Note the target space and the domain mus be the same.

- **Remark**

  1. It is *really* important that α be *strictly less* than 1. It's also really important that we have $d(f(x), f(y)) \leq \alpha d(x, y)$ and *not just* $d(f(x), f(y)) < d(x, y)\ \forall x, y \in X$. So $f(x) = \cos(x)$ is not a contraction on $\mathbb{R}$.
  2. The constant $\alpha < 1$ is called the *contraction constant* of f.

- **Theorem**
  If (X, d) is a complete metric space and if $f : X \to X$ is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$.

- **Remarks**
  1. It's really important that X be complete.
  2. It's really important that the image of X under f is contained in X.
  3. A point x such that $f(x) = x$ is called a *fixed point* of f.

- **Remarks**

9.2 Existence and uniqueness for solutions to ODEs

- **Definition** *Lipschitz Condition*
  Suppose $A \in \mathbb{R}$, $\rho, r > 0$, and $F : [A - \rho, A + \rho] \times [-r, r] \to \mathbb{R}$ is continuous. Suppose also that for all $x, y \in [A - \rho, A + \rho]$ and all $t \in [-r, r]$ we have, for some $M > 0$
  $$|F(x, t) - F(y, t)| \leq M|x - y|$$

- **Theorem** *Picard*
  Suppose F satisfies the Lipschitz Condition. Then there exists an $s > 0$ such that the ODE
  $$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, t)$$
  $$x(0) = A$$
  has a unique solution $x(t)$ for $|t| < s$.