Honours Analysis Notes

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1 The Real Number System

1.1 Introduction

1.2 Ordered Field Axioms

- **Postulate 1** *Field Axioms*
  There are functions $+$ and $\cdot$ defined on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, which satisfy the following properties $\forall a, b, c \in \mathbb{R}$
  - *Closure Properties*: $a + b,\ a \cdot b \in \mathbb{R}$
  - *Associative Properties*: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  - *Commutative Properties*: $a + b = b + a$ and $a \cdot b = b \cdot a$
  - *Distributive Law*: $a \cdot (b + c) = a \cdot b + a \cdot c$
  - *Existence of Additive Identity*: There is a unique element $0 \in \mathbb{R}$ such that $0 + a = a$ for all $a \in \mathbb{R}$
  - *Existence of Multiplicative Identity*: There is a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$
  - *Existence of Additive Inverses*: For every $x \in \mathbb{R}$ there is a unique element $-x \in \mathbb{R}$ such that
  $$x + (-x) = 0$$
  - *Existence of Multiplicative Inverses*: For every $x \in \mathbb{R}\backslash\{0\}$ there is a unique element $x^{-1} \in \mathbb{R}$ such that
  $$x \cdot (x^{-1}) = 1$$

- **Postulate 2** *Order Axioms*
  There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:
  - *Trichotomy Property*: Given $a, b \in \mathbb{R}$, one and only one of the following statements hold:
  $$a < b,\ b < a,\ \text{or } a = b$$
  - *Transitive Property*: For $a, b, c, \in \mathbb{R}$
  $$a < b \text{ and¡}c \implies a < c$$
  - *Additive Property*: For $a, b, c \in \mathbb{R}$
  $$a < b \text{ and } c \in \mathbb{R} \implies a + c < b + c$$
  - *Multiplicative Properties*: For $a, b, c, \in \mathbb{R}$
  $$a < b \text{ and } c > 0 \implies ac < bc$$
  and
  $$a < b \text{ and } c < 0 \implies bc < ac$$

- **Remark 1.1**
  We will assume that the sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties:
  1. If $n, m \in \mathbb{Z}$, then $n + m, n - m$ and mn belong to $\mathbb{Z}$
  2. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$

3. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$

- **Definition 1.4** *Absolute Value*
The *absolute value* of a number $a \in \mathbb{R}$ is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

- **Remark 1.5** The *absolute value* is multiplicative; that is, $|ab| = |a||b| \ \forall a, b, \in \mathbb{R}$

- **Theorem 1.6** *Fundamental Theorem of Absolute Values*
Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M \iff -M \leq a \leq M$.

- **Theorem 1.7** The absolute value satisfies the following three properties:

  1. *Positive Definite*: For all $a \in \mathbb{R}$, $|a| > 0$ with $|a| = 0$ if and only if $a = 0$.
  2. *Symmetric*: For all $a, b, \in \mathbb{R}$, $|a - b| = |b - a|$,
  3. *Triangle Inequalities*: For all $a, b \in \mathbb{R}$,

$$|a + b| \leq |a| + |b| \quad \text{and} \quad ||a| - |b|| \leq |a - b|$$

- **Theorem 1.9** Let $x, y, a \in \mathbb{R}$

  1. $x < y + \epsilon \ \forall \epsilon > 0 \iff x \leq y$
  2. $x > y - \epsilon \ \forall \epsilon > 0 \iff x \geq y$
  3. $|a| < \epsilon \ \forall \epsilon > 0 \iff a = 0$

1.3 Completeness Axiom

- **Definition 1.10** *Upper bounds*
Let $E \subset \mathbb{R}$ be non-empty

  1. The set E is said to be *bounded above* if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$, in which case M is called an *upper bound* of E.
  2. A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E has a *finite supremeum* s and write $s = \sup E$)

- **Remark 1.12** If a set has one upper bound, it has infinitely many upper bounds.

- **Remark 1.13** If a set has a supremum, then it has only one supremum.

- **Theorem** *Approximation Property for Suprema*
If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that

$$\sup E - \epsilon < a \leq \sup E$$

- **Theorem 1.15**
If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

- **Postulate 3** *Completeness Axiom*
If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

- **Theorem 1.16** *The Archimedean Principle*
Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

- **Theorem 1.18** *Density of Rationals*
  If $a, b, \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

- **Definition 1.19** *Upper bounds*
  Let $E \in \mathbb{R}$ be nonempty

  1. The set E is said to be *bounded below* if and only if there is an $m \in \mathbb{R}$ such that $a \geq E$, in which case m is called a *lower bound* of the set E.
  2. A number t is called an *infimum* of the set E if and only if t is a lower bound of E and $t \geq m$ and write $t = \inf E$.
  3. E is said to be *bounded* if and only if it is bounded both above and below.

- **Theorem 1.20** *Reflection Principle*
  Let $E \in \mathbb{R}$ be nonempty

  1. E has a supremum if and only if $-E$ has an infimum, in which case
  $$\inf(-E) = -\sup E$$
  2. E has an infimum if and only if $-E$ has a supremum, in which case
  $$\sup(-E) = -\inf E$$

- **Theorem 1.21** *Monotone Property*
  Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.

  1. If B has a supremum, then $\sup A \leq \sup B$.
  2. If B has an infimum, then $\inf A \geq \inf B$.

1.4 Mathematical Induction

- **Theorem 1.22** *Well-Ordering Principle*
  If E is a nonempty subset of $\mathbb{N}$, then E has a least element (i.e. E has a finite infimum and $\inf E \in E$).

- **Theorem 1.23**
  Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition which satisfies the following two properties:

  1. $A(1)$ is true.
  2. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.

  Then $A(n)$ is true for all $n \in \mathbb{N}$.

- **Theorem 1.26** *Binomial Formula*
  If $a, b, \in \mathbb{R}, n \in \mathbb{N}$ and 0^0 is interpreted to be 1, then
  $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1.5 Inverse Functions and Images

- **Definition 1.29** *Injection, Surjection, Bijection*
  Let X and Y be sets and $f : X \to Y$

  1. f is said to be *injective* if and only if
  $$x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \implies x_1 = x_2$$

2. f is said to be *surjective* if and only if

$$\forall y \in Y \ \exists x \in X \ni y = f(x)$$

3. f is called *bijective* if and only if it is both injective and surjective

- **Theorem 1.30**
  Let X and Y be sets and $f : X \to Y$. Then the following three statements are equivalent.

  1. f has an inverse;
  2. f is injective from X onto Y;
  3. There is a function $g : Y \to X$ such that

  $$g(f(x)) = x \quad \forall x \in X$$

  and

  $$f(g(y)) = y \quad \forall y \in Y$$

  Moreover, for each $f : X \to Y$, there is only one function g that satisfies these. It is the inverse function f^{-1}.

- **Remark 1.31**
  Let I be an interval and let $f : I \to \mathbb{R}$. If the derivative of f is either always positive on I, or always negative on I, then f is injective on I.

- **Definition 1.33** *Image*
  Let X and Y be sets and $f : X \to Y$. The *image* of a set $E \subseteq X$ under f is the set

  $$f(E) := \{y \in Y : y = f(x) \text{ for some } x \in E\}$$

  The *inverse image* of a set $E \subseteq Y$ under f is the set

  $$f^{-1}(E) := \{x \in X : f(x) = y \text{ for some } y \in E\}$$

- **Definition 1.35** *Union, Intersection*
  Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ be a collection of sets.

  1. The *union* of the collection $\mathcal{E}$ is the set

  $$\bigcup_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$$

  2. The *intersection* of the collection $\mathcal{E}$ is the set

  $$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

- **Theorem 1.36** *DeMorgan's Laws*
  Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. If for each $E \subseteq X$ the symbol E^c represents the set $X \setminus E$, then

  $$\left(\bigcup_{\alpha \in A} E_\alpha \right)^c = \bigcap_{\alpha \in A} E_\alpha^c$$

  and

  $$\left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

- **Theorem 1.37**
  Let X and Y be sets and $f : X \to Y$.
  1. If $\{E_\alpha\}\alpha \in A$ is a collection of subsets of X, then
  $$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha) \quad \text{and} \quad f\left(\bigcap_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(E_\alpha)$$
  2. If B and C are subsets of X, then $f(C) \setminus f(B) \subseteq f(C \setminus B)$
  3. If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of Y, then
  $$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha)$$
  4. If B and C are subsets of Y, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.
  5. If $E \subseteq f(x)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$.

1.6 Countable and Uncountable Sets

- **Definition 1.38** *Countable Uncountable* Let E be a set.
  1. E is said to be *finite* if and only if either $E = \emptyset$ or there exists an injective function which takes $\{1, 2, \dots, n\}$ onto E, for some $n \in \mathbb{N}$.
  2. E is said to be *countable* if and only if there exists and injective function which takes $\mathbb{N}$ onto E.
  3. E is said to be *at most countable* if and only if E is either finite or countable.
  4. E is said to be *uncountable* if and only if E is neither finite nor countable.
- **Remark 1.39** *Cantor's Diagonalisation Argument*
  The open interval $(0, 1)$ is uncountable.
- **Lemma 1.40**
  A nonempty set E is at most countable if and only if there is a function g from $\mathbb{N}$ onto E.
- **Theorem 1.41**
  Suppose A and B are sets.
  1. If $A \subseteq B$ and B is at most countable, then A is at most countable.
  2. If $A \subseteq B$ and A is uncountable, then B is uncountable.
  3. $\mathbb{R}$ is uncountable.
- **Theorem 1.42**
  Let $A_1, A_2, \dots$ be at most countable sets.
  1. Then $A_1 \times A_2$ is at most countable.
  2. If
  $$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j\in\mathbb{N}} A_j := \{x : x \in A_j \text{ for some } j \in \mathbb{N}\},$$
  then E is at most countable.
- **Remark 1.43**
  The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but the set of irrationals is uncountable.

2 Sequences in $\mathbb{R}$

2.1 Limits of Sequences

- **Definition 2.1** *Convergence*
  A sequence of real numbers $\{x_n\}$ is set to *converge* to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

  $$n \geq N \implies |x_n - a| < \epsilon$$

- **Remark 2.4** A sequence can have at most one limit.
- **Definition 2.5** *Subsequence*
  By a *subsequence* of a sequence $\{x_n\}_{n\in\mathbb{N}}$, we shall mean a sequence of the form $\{x_{nk}\}_{k\in\mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$.
- **Remark 2.6**
  If $\{x_n\}_{n\in\mathbb{N}}$ converges to a and $\{x_{nk}\}k \in \mathbb{N}$ is any subsequence of $\{x_n\}_{n\in\mathbb{N}}$, then x_{nk} converges to a as $k \to \infty$.
- **Definition 2.7** *Bounded Sequences*
  Let $\{x_n\}$ be a sequence of real numbers.
  1. The sequence $\{x_n\}$ is said to be *bounded above* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above.
  2. The sequence $\{x_n\}$ is said to be *bounded below* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded below.
  3. $\{x_n\}$ is said to be *bounded* if and only if it is bounded both above and below.
- **Theorem 2.8** Every convergent sequence is bounded.

2.2 Limit Theorems

- **Theorem 2.9** *Squeeze Theorem*
  Suppose that $\{x_n\}, \{y_n\}$, and $\{w_n\}$ are real sequences.
  1. If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if there is an $N_0 \in \mathbb{N}$ such that

     $$x_n \leq w_n \leq y_n \text{ for } n \geq N_0$$

     then $w_n \to a$ as $n \to \infty$.
  2. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.
- **Theorem 2.11**
  Let $E \subset \mathbb{R}$. If E has a finite supremum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ (respectively, a sequence $y_n \in E$ such that $y_n \to \inf E$) as $n \to \infty$.
- **Theorem 2.12**
  Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then
  1. $$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$
  2. $$\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$$

     and

3.
$$\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$$
If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then

4.
$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$
(In particular, all these limits exist.)

- **Definition 2.14** *Divergence*
  Let $\{x_n\}$ be a sequence of real numbers.
  1. $\{x_n\}$ is said to *diverge* to $+\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
  $$n \geq N \implies x_n > M$$
  2. $\{x_n\}$ is said to *diverge* to $-\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
  $$n \geq N \implies x_n < M$$

- **Theorem 2.15**
  Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \to +\infty$ (respectively, $x_n \to -\infty$) as $n \to \infty$.
  1. If y_n is bounded below (respectively, y_n is bounded above), then
  $$\lim_{n\to\infty}(x_n + y_n) = +\infty \quad \text{(respectively, } \lim_{n\to\infty}(x_n + y_n) = -\infty)$$
  2. If $\alpha > 0$, then
  $$\lim_{n\to\infty}(\alpha x_n) = +\infty \quad \text{(respectively, } \lim_{n\to\infty}(\alpha x_n) = -\infty)$$
  3. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$, then
  $$\lim_{n\to\infty}(x_n y_n) = +\infty \quad \text{(respectively, } \lim_{n\to\infty}(x_n y_n) = -\infty)$$
  4. If $\{y_n\}$ is bounded and $x_n \neq 0$, then
  $$\lim_{n\to\infty}\frac{y_n}{x_n} = 0$$

- **Corollary 2.16**
  Let $\{x_n\}$, $\{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$, as $n \to \infty$, then
  $$\lim_{n\to\infty}(x_n + y_n) = x + y$$
  provided that the right side is not of the form $\infty - \infty$, and
  $$\lim_{n\to\infty}(\alpha x_n) = \alpha x, \quad \lim_{n\to\infty}(x_n y_n) = xy$$
  provided that none of these products is of the form $0 \cdot \pm\infty$.

- **Theorem 2.17** *Comparison Theorem*
  Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that
  $$x_n \leq y_n \text{ for } n \geq N_0$$
  then
  $$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$
  In particular, if $x_n \in [a, b]$ converges to some point c, then c must belong to $[a, b]$.

2.3 Bolzano-Weierstrass Theorem

- **Definition 2.18** *Increasing, Decreasing* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.
  1. $\{x_n\}$ is said to be *increasing* (respectively, *strictly increasing*) if and only if $x_1 \leq x_2 \leq \cdots (\text{respectively}, x_1 < x_2 < \cdots)$.
  2. $\{x_n\}$ is said to be *decreasing* (respectively, *strictly decreasing*) if and only if $x_1 \geq x_2 \geq \cdots (\text{respectively}, x_1 > x_2 > \cdots)$.
  3. $\{x_n\}$ is said to be *monotone* if and only if it is either increasing or decreasing.
- **Theorem 2.19** *Monotone Convergence Theorem*
  if $\{x_n\}$ is increasing and bounded above, or if $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.
- **Definition 2.22** *Nested*
  A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be *nested* if and only if

  $$I_1 \supseteq I_2 \supseteq \cdots$$

- **Theorem 2.23** *Nested Interval Property*
  If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then $E := \bigcap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the lengths of these intervals satisfy $|I_n \to 0$ as $n \to \infty$ then E is a single point.
- **Remark 2.24** The Nested Interval Property might not hold if “closed” is omitted.
- **Remark 2.25** The Nested Interval Property might not hold if “bounded” is omitted.
- **Theorem 2.26** *Bolzano-Weierstrass Theorem*
  Every bounded sequence of real numbers has a convergent subsequence.

2.4 Cauchy Sequences

- **Definition 2.27** *Cauchy*
  A sequence of points $x_n \in \mathbb{R}$ is said to be *Cauchy* (in R) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

  $$n, m \geq N \implies |x_n - x_m| < \epsilon$$

- **Remark 2.28** If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.
- **Theorem 2.29** *Cauchy*
  Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges (to some point $a \in \mathbb{R}$).
- **Remark 2.31** A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

2.5 Limits Supremum and Infimum

- **Definition 2.32** *Limit Supremum & Infimum*
  Let $\{x_n\}$ be a real sequence. Then the *limit supremum* of $\{x_n\}$ is the extended real number

  $$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{k\geq n} x_k)$$

  and the *limit infimum* of $\{x_n\}$ is the extended real number

  $$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{k\geq n} x_k)$$

- **Theorem 2.35**
  Let $\{x_n\}$ be a sequence of real numbers, $s = \limsup_{n\to\infty} x_n$, and $t = \liminf_{n\to\infty} x_n$. Then there are subsequences $\{x_{nk}\}_{k\in\mathbb{N}}$ and $\{x_{\ell j}\}_{j\in\mathbb{N}}$ such that $x_{nk} \to s$ as $k \to \infty$ and $x_{\ell j} \to t$ as $j \to \infty$.

- **Theorem 2.36**
  Let $\{x_n\}$ be a real sequence and x be an extended real number. Then $x_n \to x$ as $n \to \infty$ if and only if
  $$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$$

- **Theorem 2.37**
  Let $\{x_n\}$ be a sequence of real numbers. Then $\limsup_{n\to\infty} x_n$ (respectively, $\liminf_{n\to\infty}$) is the largest value (respectively, the smallest value) to which some subsequences of $\{x_n\}$ converges. Namely, if $x_{nk} \to x$ as $k \to \infty$, then
  $$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n$$

- **Remark 2.38** If $\{x_n\}$ is any sequence of real numbers, then
  $$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

- **Remark 2.39** A real sequence $\{x_n\}$ is bounded above if and only if $\limsup_{n\to\infty} x_n < \infty$, and is bounded below if and only if $\liminf_{n\to\infty} x_n > -\infty$.

- **Theorem 2.40**
  If $x_n \leq y_n$ for n large, then
  $$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n \quad \text{and} \quad \liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty} y_n$$

3 Functions on R

3.1 Two-Sided Limits

- **Definition 3.1** *Limits*
  Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that
  $$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$
  In this case we write
  $$L = \lim_{x \to a} f(x) \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$
  and call L the *limit* of $f(x)$ as x approaches a.

- **Remark 3.4**
  Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined everywhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \setminus \{a\}$ and $f(x) \to L$ as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and
  $$\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

- **Theorem 3.6** *Sequential Characterisation of Limits*
  Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then
  $$L = \lim_{x \to a} f(x)$$
  exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $\{x_n\} \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

- **Theorem 3.8**
  Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, are real functions defined everywhere on I except possibly at a. If $f(x)$ and $g(x)$ converge as x approaches a, then so do $(f+g)(x), (fg)(x), (\alpha f)(x)$, and $(f/g)(x)$ (when the limit of $g(x)$ is nonzer). In fact,
  $$\begin{aligned} \lim_{x \to a} (f+g)(x) &= \lim_{x \to a} + \lim_{x \to a} g(x) \\ \lim_{x \to a} (\alpha f)(x) &= \lim_{x \to a} f(x) \\ \lim_{x \to a} (fg)(x) &= \lim_{x \to a} \lim_{x \to a} g(x) \end{aligned}$$
  and (when the limit of $g(x)$ is nonzero)
  $$\lim_{x \to a} \left(\frac{f}{g} \right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

- **Theorem 3.9** *Squeeze Theorem for Functions*
  Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined everywhere on I except possibly at a.

  1. If $g(x) \le h(x) \le f(x)$ $\forall x \in I \setminus \{a\}$, and
     $$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,$$
     then the limit of $h(x)$ exists, as $x \to a$, and
     $$\lim_{x \to a} h(x) = L.$$

2. If $|g(x)| \leq M \ \forall x \in I \setminus \{a\}$ and $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} f(x)g(x) = 0$$

- **Theorem 3.10** *Comparison Theorem for Functions*
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined everywhere on I except possibly at a. If f and g have a limit as x approaches a and $f(x) \leq g(x) \ \forall x \in I \setminus \{a\}$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

3.2 One-Sided Limits and Limits at Infinity

- **Definition 3.12** *Converge from left & right*
Let $a \in \mathbb{R}$ and f be a real function.

1. $f(x)$ is said to *converge to L as x approaches a from the right* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *right-hand limit* of f at a, and denote it by

$$f(a+) := L =: \lim_{x \to a+} f(x)$$

2. $f(x)$ is said to *converge to L as x approaches a from the left* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *left-hand limit* of f at a, and denote it by

$$f(a-) := L =: \lim_{x \to a-} f(x)$$

- **Theorem 3.14**
Let f be a real function. Then the limit

$$\lim_{x \to a} f(x)$$

exists and equals L if and only if

$$L = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

- **Definition 3.15** *Convergence*
Let $a, L, \in \mathbb{R}$ and let f be a real function.

1. $f(x)$ is said to *converge* to L as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subset \mathrm{Dom}(f)$ and given $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

Similarly, $f(x)$ is said to *converge* to L as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(infty, -c) \subset \mathrm{Dom}(f)$ and given $\epsilon > 0$ there is $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

2. The function $f(x)$ is said to converge to ∞ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \text{Dom}(f)$ and given $M \in \text{R}$ there is a $\delta > 0$ such that $0 \leq |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to a$$

Similarly, $f(x)$ is said to *converge* to $-\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \text{Dom}(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to a$$

- **Theorem 3.17**
Let a be an extended real number, and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then

$$\lim_{x \to a; x \in I} f(x)$$

exists and equals L if and only if $f(x_n) \to L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \to a$ as $n \to \infty$.

3.3 Continuity

- **Definition 3.19** *Continuous*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.

1. f is said to be *continuous at a point* $a \in \mathbb{E}$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, and a) such that

$$|x - a| < delta \quad \text{and} \quad x \in E \implies |f(x) - f(a)| < \epsilon$$

2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

- **Remark 3.20**
Let I be an open interval which contains a point a and $f : I \to \mathbb{R}$. Then f is continuous at $a \in \mathbb{I}$ if and only if

$$f(a) = \lim_{x \to a} f(x)$$

- **Theorem 3.21**
Suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f : E \to \mathbb{R}$. Then the following statements are equivalent:

1. f is continuous at $a \in E$.
2. If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

- **Theorem 3.22**
Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (respectively continuous on the set E), then so are $f + g$, fg, and αf (for any $\alpha \in \mathbb{R}$). Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$ (respectively, on E when $g(x) \neq 0 \ \forall x \in E$).

- **Definition 3.23** *Composition*
Suppose that A and B are subsets of R, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $F(A) \subseteq B$ for every $x \in A$, then the *composition* of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by

$$(g \circ f)(x) := g(f(x)), \quad x \in A$$

- **Theorem 3.24**
Suppose that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that $f(x) \in B \; \forall x \in A$.

  1. If $A := I \setminus \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if
$$L := \lim_{x \to a; x \in I} f(x)$$
exists and belongs to B, and if g is continuous and $L \in B$, then
$$(g \circ f)(x) = g\left(\lim_{x \to a; x \in I} f(x)\right)$$
  2. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.

- **Definition 3.25** *Bounded*
Let E be a nonempty subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be *bounded* on E if and only if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$, in which case we shall say that f is *dominated* by M on E.

- **Theorem 3.26** *Extreme Value Theorem*
If I a is closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover if
$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x)$$
then there exist points $x_m, x_M \in I$ such that
$$f(x_M) = M \quad \text{and} \quad f(x_m) = m$$

- **Remark 3.27** The Existence Value Theorem is false if either "closed" or "bounded" is dropped from the hypotheses.

- **Lemma 3.28**
Suppose that $a < B$ and that $f : [a, b) \to \mathbb{R}$. If f is continuous at a point $x_0 \in [a, b)$ and $f(x_0) > 0$, then there exist a positive number ϵ and a point $x_1 \in [a, b)$ such that $x_1 > x_0$ and $f(x) > \epsilon \; \forall x \in [x_0, x_1]$.

- **Theorem 3.29** *Intermediate Value Theorem*
Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

- **Remark 3.34** The composition of two functions $g \circ f$ can be nowhere continuous, even though f is discontinuous only on $\mathbb{Q}$ and g is discontinuous at only one point.

3.4 Uniform Continuity

- **Definition 3.35** *Uniform continuity*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then $\mho$ is said to be *uniformly continuous* on E if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that
$$|x - a| < delta \quad \text{and} \quad x, a, \in E \implies |f(x) - f(a)| < \epsilon$$

- **Lemma 3.38**
Suppose that $E \subseteq \mathbb{R}$ and that $f : E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, the $f(x_n)$ is Cauchy.

- **Theorem 3.39**
  Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

- **Theorem 3.40**
  Suppose that $a < b$ and that $f : (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if f can be continuously extended to $[a, b]$; that is, if and only if there is a continuous function $g : [a, b] \to \mathbb{R}$ which satisfies
  $$f(x) = g(x), \quad x \in (a, b)$$