1 The Real Number System

1.1 Introduction

1.2 Ordered Field Axioms

Remark 1.1
We will assume that the sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties:

1. If $n, m \in \mathbb{Z}$, then $n+m, n-m$ and mn belong to $\mathbb{Z}$
2. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$
3. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$

Definition 1.4 *Absolute Value*
The *absolute value* of a number $a \in \mathbb{R}$ is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

Remark 1.5 The *absolute value* is multiplicative; that is, $|ab| = |a||b| \; \forall a, b, \in \mathbb{R}$

Theorem 1.6 *Fundamental Theorem of Absolute Values*
Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M \iff -M \leq a \leq M$.

Theorem 1.7 The absolute value satisfies the following three properties:

1. *Positive Definite*: For all $a \in \mathbb{R}$, $|a| > 0$ with $|a| = 0$ if and only if $a = 0$.
2. *Symmetric*: For all $a, b, \in \mathbb{R}$, $|a-b| = |b-a|$,
3. *Triangle Inequalities*: For all $a, b \in \mathbb{R}$
   $|a+b| \leq |a| + |b|$ and $||a| - |b|| \leq |a-b|$

Theorem 1.9 Let $x, y, a \in \mathbb{R}$

1. $x < y + \epsilon \; \forall \epsilon > 0 \iff x \leq y$
2. $x > y - \epsilon \; \forall \epsilon > 0 \iff x \geq y$
3. $|a| < \epsilon \; \forall \epsilon > 0 \iff a = 0$

1.3 Completeness Axiom

Definition 1.10 *Upper bounds*
Let $E \subset \mathbb{R}$ be non-empty

1. The set E is said to be *bounded above* if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$, in which case M is called an *upper bound* of E.
2. A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E has a *finite supremum* s and write $s = \sup E$.)

Remark 1.12 If a set has one upper bound, it has infinitely many upper bounds.

Remark 1.13 If a set has a supremum, then it has only one supremum.

Theorem *Approximation Property for Suprema*
If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that
$\sup E - \epsilon < a \leq \sup E$

Theorem 1.15
If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

Postulate 3 *Completeness Axiom*
If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

Theorem 1.16 *The Archimedean Principle*
Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

Theorem 1.18 *Density of Rationals*
If $a, b, \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

Definition 1.19 *Upper bounds*
Let $E \in \mathbb{R}$ be nonempty

1. The set E is said to be *bounded below* if and only if there is an $m \in \mathbb{R}$ such that $a \geq E$, in which case m is called a *lower bound* of the set E.
2. A number t is called an *infimum* of the set E if and only if t is a lower bound of E and $t \geq m$ and write $t = \inf E$.
3. E is said to be *bounded* if and only if it is bounded both above and below.

Theorem 1.20 *Reflection Principle*
Let $E \in \mathbb{R}$ be nonempty

1. E has a supremum if and only if $-E$ has an infimum, in which case $\inf(-E) = -\sup E$.
2. E has an infimum if and only if $-E$ has a supremum, in which case $\sup(-E) = -\inf E$

Theorem 1.21 *Monotone Property*
Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.

1. If B has a supremum, then $\sup A \leq \sup B$.
2. If B has an infimum, then $\inf A \geq \inf B$.

1.4 Mathematical Induction

Theorem 1.22 *Well-Ordering Principle*
If E is a nonempty subset of $\mathbb{N}$, then E has a least element (i.e. E has a finite infimum and $\inf E \in E$).

Theorem 1.23
Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition which satisfies the following two properties:

1. $A(1)$ is true.
2. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.

Then $A(n)$ is true for all $n \in \mathbb{N}$.

Theorem 1.26 *Binomial Formula*
If $a, b, \in \mathbb{R}, n \in \mathbb{N}$ and 0^0 is interpreted to be 1, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1.5 Inverse Functions and Images

Definition 1.29 *Injection, Surjection, Bijection*
Let X and Y be sets and $f : X \to Y$

1. f is said to be *injective* if and only if
   $$x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \implies x_1 = x_2$$
2. f is said to be *surjective* if and only if
   $$\forall y \in Y \; \exists x \in X \ni y = f(x)$$
3. f is called *bijective* if and only if it is both injective and surjective

Theorem 1.30
Let X and Y be sets and $f : X \to Y$. Then the following three statements are equivalent.

1. f has an inverse;
2. f is injective from X onto Y;

3. There is a function $g : Y \to X$ such that
   $g(f(x)) = x \quad \forall x \in X$ and
   $f(g(y)) = y \quad \forall y \in Y$

Moreover, for each $f : X \to Y$, there is only one function g that satisfies these. It is the inverse function f^{-1}.

Remark 1.31
Let I be an interval and let $f : I \to \mathbb{R}$. If the derivative of f is either always positive on I, or always negative on I, then f is injective on I.

Definition 1.33 *Image*
Let X and Y be sets and $f : X \to Y$. The *image* of a set $E \subseteq X$ under f is the set

$$f(E) := \{y \in Y : y = f(x) \text{ for some } x \in E\}$$

The *inverse image* of a set $E \subseteq Y$ under f is the set

$$f^{-1}(E) := \{x \in X : f(x) = y \text{ for some } y \in E\}$$

Definition 1.35 *Union, Intersection*
Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ be a collection of sets.

1. The *union* of the collection $\mathcal{E}$ is the set
$$\bigcup_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$$
2. The *intersection* of the collection $\mathcal{E}$ is the set
$$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

Theorem 1.36 *DeMorgan's Laws*
Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. If for each $E \subseteq X$ the symbol E^c represents the set $X \setminus E$, then $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$ and $\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$

Theorem 1.37
Let X and Y be sets and $f : X \to Y$.

1. If $\{E_\alpha\}\alpha \in A$ is a collection of subsets of X, then
   $f\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f(E_\alpha)$ and
   $f\left(\bigcap_{\alpha \in A} E_\alpha\right) \subseteq \bigcap_{\alpha \in A} f(E_\alpha)$
2. If B and C are subsets of X, then $f(C) \setminus f(B) \subseteq f(C \setminus B)$
3. If $\{E_\alpha\}_{\alpha \in A}$ is a collection of subsets of Y, then
   $f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha)$ and
   $f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha)$
4. If B and C are subsets of Y, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.
5. If $E \subseteq f(x)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$.

1.6 Countable and Uncountable Sets

Definition 1.38 *Countable & Uncountable*
Let E be a set.

1. E is said to be *finite* if and only if either $E = \emptyset$ or there exists an injective function which takes $\{1, 2, \dots, n\}$ onto E, for some $n \in \mathbb{N}$.
2. E is said to be *countable* if and only if there exists and injective function which takes $\mathbb{N}$ onto E.
3. E is said to be *at most countable* if and only if E is either finite or countable.
4. E is said to be *uncountable* if and only if E is neither finite nor countable.

Remark 1.39 *Cantor's Diagonalisation Argument*
The open interval $(0, 1)$ is uncountable.

Lemma 1.40
A nonempty set E is at most countable if and only if there is a function g from $\mathbb{N}$ onto E.

Theorem 1.41
Suppose A and B are sets.

1. If $A \subseteq B$ and B is at most countable, then A is at most countable.
2. If $A \subseteq B$ and A is uncountable, then B is uncountable.
3. $\mathbb{R}$ is uncountable.

Theorem 1.42
Let $A_1, A_2, \dots$ be at most countable sets.

1. Then $A_1 \times A_2$ is at most countable.
2. If
$$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j \in \mathbb{N}} A_j := \{x : x \in A_j \text{ for some } j \in \mathbb{N}\},$$
   then E is at most countable.

Remark 1.43
The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but the set of irrationals is uncountable.

2 Sequences in $\mathbb{R}$

2.1 Limits of Sequences

Definition 2.1 *Convergence*
A sequence of real numbers $\{x_n\}$ is set to *converge* to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

$$n \geq N \implies |x_n - a| < \epsilon$$

Remark 2.4 A sequence can have at most one limit.

Definition 2.5 *Subsequence*
By a *subsequence* of a sequence $\{x_n\}_{n \in \mathbb{N}}$, we shall mean a sequence of the form $\{x_{n_k}\}_{k \in \mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$.

Remark 2.6
If $\{x_n\}_{n \in \mathbb{N}}$ converges to a and $\{x_{nk}\}k \in \mathbb{N}$ is any subsequence of $\{x_n\}_{n \in \mathbb{N}}$, then x_{nk} converges to a as $k \to \infty$.

Definition 2.7 *Bounded Sequences*
Let $\{x_n\}$ be a sequence of real numbers.

1. The sequence $\{x_n\}$ is said to be *bounded above* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above.
2. The sequence $\{x_n\}$ is said to be *bounded below* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded below.
3. $\{x_n\}$ is said to be *bounded* if and only if it is bounded both above and below.

Theorem 2.8 Every convergent sequence is bounded.

2.2 Limit Theorems

Theorem 2.9 *Squeeze Theorem*
Suppose that $\{x_n\}, \{y_n\}$, and $\{w_n\}$ are real sequences.

1. If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if there is an $N_0 \in \mathbb{N}$ such that
$$x_n \leq w_n \leq y_n \text{ for } n \geq N_0$$
   then $w_n \to a$ as $n \to \infty$.
2. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Theorem 2.11
Let $E \subset \mathbb{R}$. If E has a finite supremum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ (respectively, a sequence $y_n \in E$ such that $y_n \to \inf E$) as $n \to \infty$.

Theorem 2.12
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then

1. $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$
2. $\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$ and
3. $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$
   If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then
4. $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$
   (In particular, all these limits exist.)

Definition 2.14 *Divergence*
Let $\{x_n\}$ be a sequence of real numbers.

1. $\{x_n\}$ is said to *diverge* to $+\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
   $n \geq N \implies x_n > M$
2. $\{x_n\}$ is said to *diverge* to $-\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
   $n \geq N \implies x_n < M$

Theorem 2.15
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \to +\infty$ (respectively, $x_n \to -\infty$) as $n \to \infty$.

1. If y_n is bounded below (respectively, y_n is bounded above), then $\lim_{n\to\infty}(x_n + y_n) = +\infty$
2. If $\alpha > 0$, then $\lim_{n\to\infty}(\alpha x_n) = +\infty$
3. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$, then $\lim_{n\to\infty}(x_n y_n) = +\infty$
4. If $\{y_n\}$ is bounded and $x_n \neq 0$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$

Corollary 2.16
Let $\{x_n\}$, $\{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$, as $n \to \infty$, then
$\lim_{n\to\infty}(x_n + y_n) = x + y$ provided that the right side is not of the form $\infty - \infty$, and
$\lim_{n\to\infty}(\alpha x_n) = \alpha x, \quad \lim_{n\to\infty}(x_n y_n) = xy$
provided that none of these products is of the form $0 \cdot \pm\infty$.

Theorem 2.17 *Comparison Theorem*
Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that $x_n \leq y_n$ for $n \geq N_0$ then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.
In particular, if $x_n \in [a, b]$ converges to some point c, then c must belong to $[a, b]$.

2.3 Bolzano-Weierstrass Theorem

Definition 2.18 *Increasing, Decreasing*
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

1. $\{x_n\}$ is said to be *increasing* (respectively, *strictly increasing*) if and only if $x_1 \leq x_2 \leq \cdots$ (respectively, $x_1 < x_2 < \cdots$).
2. $\{x_n\}$ is said to be *decreasing* (respectively, *strictly decreasing*) if and only if $x_1 \geq x_2 \geq \cdots$ (respectively, $x_1 > x_2 > \cdots$).
3. $\{x_n\}$ is said to be *monotone* if and only if it is either increasing or decreasing.

Theorem 2.19 *Monotone Convergence Theorem*
If $\{x_n\}$ is increasing and bounded above, or if $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

Definition 2.22 *Nested*
A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be *nested* if and only if $I_1 \supseteq I_2 \supseteq \cdots$.

Theorem 2.23 *Nested Interval Property*
If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then $E := \bigcap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$ then E is a single point.

Remark 2.24 The Nested Interval Property might not hold if "closed" is omitted.

Remark 2.25 The Nested Interval Property might not hold if "bounded" is omitted.

Theorem 2.26 *Bolzano—Weierstrass Theorem*
Every bounded sequence of real numbers has a convergent subsequence.

2.4 Cauchy Sequences

Definition 2.27 *Cauchy*
A sequence of points $x_n \in \mathbb{R}$ is said to be *Cauchy* (in R) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
$n, m \geq N \implies |x_n - x_m| < \epsilon$

Remark 2.28 If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

Theorem 2.29 *Cauchy*
Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges (to some point $a \in \mathbb{R}$).

Remark 2.31 A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

2.5 Limits Supremum and Infimum

Definition 2.32 *Limit Supremum & Infimum*
Let $\{x_n\}$ be a real sequence. Then the *limit supremum* of $\{x_n\}$ is the extended real number
$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}(\sup_{k\geq n} x_k)$
and the *limit infimum* of $\{x_n\}$ is the extended real number
$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{k\geq n} x_k)$

Theorem 2.35
Let $\{x_n\}$ be a sequence of real numbers,
$s = \limsup_{n\to\infty} x_n$, and $t = \liminf_{n\to\infty} x_n$.
Then there are subsequences $\{x_{nk}\}_{k\in\mathbb{N}}$ and $\{x_{\ell j}\}_{j\in\mathbb{N}}$ such that $x_{nk} \to s$ as $k \to \infty$ and $x_{\ell j} \to t$ as $j \to \infty$.

Theorem 2.36
Let $\{x_n\}$ be a real sequence and x be an extended real number. Then $x_n \to x$ as $n \to \infty$ if and only if
$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.

Theorem 2.37
Let $\{x_n\}$ be a sequence of real numbers. Then $\limsup_{n\to\infty} x_n$ (respectively, $\liminf_{n\to\infty}$) is the largest value (respectively, the smallest value) to which some subsequences of $\{x_n\}$ converges. Namely, if $x_{nk} \to x$ as $k \to \infty$, then
$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n$.

Remark 2.38 If $\{x_n\}$ is any sequence of real numbers, then $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Remark 2.39 A real sequence $\{x_n\}$ is bounded above if and only if $\limsup_{n\to\infty} x_n < \infty$, and is bounded below if and only if $\liminf_{n\to\infty} x_n > -\infty$.

Theorem 2.40
If $x_n \leq y_n$ for n large, then
$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$ and
$\liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty} y_n$

3 Functions on R

3.1 Two-Sided Limits

Definition 3.1 *Limits*
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let

f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

In this case we write

$$L = \lim_{x \to a} f(x) \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

and call L the *limit* of $f(x)$ as x approaches a.

Remark 3.4
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined everywhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \setminus \{a\}$ and $f(x) \to L$ as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

Theorem 3.6 *Sequential Characterisation of Limits*
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then

$$L = \lim_{x \to a} f(x)$$

exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $\{x_n\} \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

Theorem 3.8
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, are real functions defined everywhere on I except possibly at a. If $f(x)$ and $g(x)$ converge as x approaches a, then so do $(f+g)(x), (fg)(x), (\alpha f)(x)$, and $(f/g)(x)$ (when the limit of $g(x)$ is nonzer). In fact,

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} + \lim_{x \to a} g(x)$$
$$\lim_{x \to a} (\alpha f)(x) = \lim_{x \to a} f(x)$$
$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} \lim_{x \to a} g(x)$$

and (when the limit of $g(x)$ is nonzero)

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Theorem 3.9 *Squeeze Theorem for Functions*
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined everywhere on I except possibly at a.

1. If $g(x) \le h(x) \le f(x)\ \forall x \in I \setminus \{a\}$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,$$

then the limit of $h(x)$ exists, as $x \to a$, and

$$\lim_{x \to a} h(x) = L.$$

2. If $|g(x)| \le M\ \forall x \in I \setminus \{a\}$ and $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} f(x)g(x) = 0$$

Theorem 3.10 *Comparison Theorem for Functions*
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined everywhere on I except possibly at a. If f and g have a limit as x approaches a and $f(x) \le g(x)\ \forall x \in I \setminus \{a\}$, then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

3.2 One-Sided Limits and Limits at Infinity

Definition 3.12 *Converge from left & right*
Let $a \in \mathbb{R}$ and f be a real function.

1. $f(x)$ is said to *converge to L as x approaches a from the right* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *right-hand limit* of f at a, and denote it by

$$f(a+) := L =: \lim_{x \to a+} f(x)$$

2. $f(x)$ is said to *converge to L as x approaches a from the left* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *left-hand limit* of f at a, and denote it by

$$f(a-) := L =: \lim_{x \to a-} f(x)$$

Theorem 3.14
Let f be a real function. Then the limit

$$\lim_{x \to a} f(x)$$

exists and equals L if and only if

$$L = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

Definition 3.15 *Convergence*
Let $a, L, \in \mathbb{R}$ and let f be a real function.

1. $f(x)$ is said to *converge* to L as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subset \mathrm{Dom}(f)$ and given $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

Similarly, $f(x)$ is said to *converge* to L as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(infty, -c) \subset \mathrm{Dom}(f)$ and given $\epsilon > 0$ there is $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

2. The function $f(x)$ is said to converge to ∞ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathrm{R}$ there is a $\delta > 0$ such that $0 \le |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to a$$

Similarly, $f(x)$ is said to *converge* to $-\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to a$$

Theorem 3.17
Let a be an extended real number, and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then

$$\lim_{x \to a; x \in I} f(x)$$

exists and equals L if and only if $f(x_n) \to L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \to a$ as $n \to \infty$.

3.3 Continuity

Definition 3.19 *Continuous*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.

1. f is said to be *continuous at a point* $a \in \mathbb{E}$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, and a) such that
$$|x - a| < \delta \quad \text{and} \quad x \in E \implies |f(x) - f(a)| < \epsilon$$
2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

Remark 3.20
Let I be an open interval which contains a point a and $f : I \to \mathbb{R}$. Then f is continuous at $a \in I$ if and only if
$$f(a) = \lim_{x \to a} f(x)$$

Theorem 3.21
Suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f : E \to \mathbb{R}$. Then the following statements are equivalent:

1. f is continuous at $a \in E$.
2. If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

Theorem 3.22
Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (respectively continuous on the set E), then so are $f + g$, fg, and αf (for any $\alpha \in \mathbb{R}$). Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$ (respectively, on E when $g(x) \neq 0 \; \forall x \in E$).

Definition 3.23 *Composition*
Suppose that A and B are subsets of R, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $F(A) \subseteq B$ for every $x \in A$, then the *composition* of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by
$$(g \circ f)(x) := g(f(x)), \quad x \in A$$

Theorem 3.24
Suppose that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that $f(x) \in B \; \forall x \in A$.

1. If $A := I \backslash \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if
$$L := \lim_{x \to a; x \in I} f(x)$$
exists and belongs to B, and if g is continuous and $L \in B$, then
$$\lim_{x \to a; x \in I} (g \circ f)(x) = g\left(\lim_{x \to a; x \in I} f(x)\right)$$
2. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.

Definition 3.25 *Bounded*
Let E be a nonempty subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be *bounded* on E if and only if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$, in which case we shall say that f is *dominated* by M on E.

Theorem 3.26 *Extreme Value Theorem*
If I a is closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover if
$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x)$$
then there exist points $x_m, x_M \in I$ such that
$$f(x_M) = M \quad \text{and} \quad f(x_m) = m$$

Remark 3.27 The Existence Value Theorem is false if either "closed" or "bounded" is dropped from the hypotheses.

Lemma 3.28
Suppose that $a < B$ and that $f : [a, b) \to \mathbb{R}$. If f is continuous at a point $x_0 \in [a, b)$ and $f(x_0) > 0$, then there exist a positive number ϵ and a point $x_1 \in [a, b)$ such that $x_1 > x_0$ and $f(x) > \epsilon \; \forall x \in [x_0, x_1]$.

Theorem 3.29 *Intermediate Value Theorem*
Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

Remark 3.34 The composition of two functions $g \circ f$ can be nowhere continuous, even though f is discontinuous only on $\mathbb{Q}$ and g is discontinuous at only one point.

3.4 Uniform Continuity

Definition 3.35 *Uniform continuity*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be *uniformly continuous* on E if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that
$$|x - a| < \delta \quad \text{and} \quad x, a, \in E \implies |f(x) - f(a)| < \epsilon \quad \forall a \in E$$

Lemma 3.38
Suppose that $E \subseteq \mathbb{R}$ and that $f : E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, the $f(x_n)$ is Cauchy.

Theorem 3.39
Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

Theorem 3.40
Suppose that $a < b$ and that $f : (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if f can be continuously extended to $[a, b]$; that is, if and only if there is a continuous function $g : [a, b] \to \mathbb{R}$ which satisfies
$$f(x) = g(x), \quad x \in (a, b)$$

4 Differentiability on R

4.1 The Derivative

Definition 4.1 *Differentiable*
A real function f is said to be *differentiable* at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and
$$f'(a) := \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
exists. In this case $f'(a)$ is called the *derivative* of f at a.

Theorem 4.2
A real function f is differentiable at some point $a \in \mathbb{R}$ if and only if there exist an open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at a, and
$$f(x) = F(x)(x - a) + f(a)$$
holds for all $x \in I$ in which case $F(a) = f'(a)$.

Theorem 4.3
A real function f is differentiable at a if and only if there is a function T of the form $T(x) := m(x)$ such that
$$\lim_{h \to 0} \frac{f(a + h) - f(a) - t(h)}{h} = 0$$

Theorem 4.4
If f is differentiable at a, then f is continuous at a.

Definition 4.6 *Continuously differentiable*
Let I be a nondegenerate interval.

1. A function $f : I \to \mathbb{R}$ is said to be *differentiable* on I if and only if

$$f_i'(a) := \lim_{x \to a; x \in I} \frac{f(x) - f(a)}{x - a}$$

exists and is finite for every $a \in I$.

2. f is said to be *continuously differentiable* on I if and only if f_I' exists and is continuous on I.

Remark 4.9
$f(x) = |x|$ is differentiable on $[0, 1]$ and on $[-1, 0]$ but not on $[-1, 1]$.

4.2 Differentiability Theorems

Theorem 4.10
Let f and g be real functions and $\alpha \in \mathbb{R}$. If f and g are differentiable at a, then $f + g$, αf, $f \cdot g$, and [when $g(a) \neq 0$] f/g are all differentiable at a. In fact,

$$(f + g)'(a) = f'(a) + g'(a)$$
$$(\alpha f)'(a) = \alpha f'(a)$$
$$(f \cdot g)'(a) = g(a)g'(a) + f(a)g'(a)$$
$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$$

Theorem 4.11 *Chain Rule*
Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

4.3 Mean Value Theorem

Lemma 4.12 *Rolle's Theorem*
Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Remark 4.13
The continuity hypothesis on Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.

Remark 4.14
The differentiability hypothesis on Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.

Theorem 4.15
Suppose that $a, b, \in \mathbb{R}$ with $a < b$.

1. *Generalised Mean Value Theorem*: If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

2. *Mean Value Theorem*: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - A)$$

Definition 4.16 *Increasing, Monotone, Decreasing*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.

1. f is said to be *increasing* (respectively, *strictly increasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \leq f(x_2)$ [respectively, $f(x_1) < f(x_2)$].
2. f is said to be *decreasing* (respectively, *strictly decreasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \geq f(x_2)$ [respectively, $f(x_1) > f(x_2)$].
3. f is said to be *monotone* (respectively, *strictly monotone*) on E if and only if f is either decreasing or increasing (respectively, either strictly decreasing or strictly increasing) on E.

Theorem 4.17
Suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a, b]$, and that f is differentiable on (a, b).

1. If $f'(x) > 0$ [respectively $f'(x) < 0$] for all $x \in (a, b)$, then f is strictly increasing (respectively, strictly decreasing) on $[a, b]$.
2. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
3. If g is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on $[a, b]$.

Theorem 4.18
Suppose that f is increasing on $[a, b]$

1. If $c \in [a, b)$, then $f(c+)$ exists and $f(c) \leq f(c+)$.
2. If $c \in (a, b]$, then $f(c-)$ exists and $f(c-) \leq f(c)$.

Theorem 4.19
If f is monotone on an interval I, then f has at most countable many points of discontinuity on I.

Theorem 4.21 *Bernoulli's Inequality*
Let α be a positive real number. If $0 < \alpha < 1$, then $(1 + x)^\alpha \leq 1 + \alpha x \ \forall x \in [-1, \infty)$, and if $\alpha \geq 1$, then $(1 + x)^\alpha \geq 1 + \alpha x \ \forall x \in [-1, \infty)$.

Theorem 4.23 *Intermediate Value Theorem for Derivatives*
Suppose that f is differentiable on $[a, b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a, b)$ such that $f'(x_0) = y_0$.

4.4 Taylor's Theorem and L'Hopital's Rule

Theorem 4.24 *Taylor's Formula*
Let $n \in \mathbb{N}$ and let a, b be extended real numbers with $a < b$. If $f : (a, b) \to \mathbb{R}$, and if $f^{(n+1)}$ exists on (a, b), then for each pair of points $x, x_0 \in (a, b)$ there is a number c between x and x_0 such that
$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$

Theorem 4.27 *L'Hopital's Rule*
Let a be an extended real number and I be an open interval which either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I \setminus \{a\}$ and that $g(x) \neq 0 \neq g'(x) \ \forall x \in I \setminus \{a\}$. Suppose further that

$$A := \lim_{x \to a; x \in I} f(x) = \lim_{x \to a; x \in I} g(x)$$

is either 0 or ∞. If

$$B := \lim_{x \to a; x \in I} \frac{f'(x)}{g'(x)}$$

exists as an extended real number, then

$$\lim_{x \to a; x \in I} \frac{f(x)}{g(x)} = \lim_{x \to a; x \in I} \frac{f'(x)}{g'(x)}$$

4.5 Inverse Function Theorems

Theorem 4.32
Let I be a nondegenerate interval and suppose that $f : I \to \mathbb{R}$ is injective. If f is continuous on I, then $J := f(I)$ is an interval, f is strictly monotone on I, and f^{-1} is continuous and strictly monotone on J.

Theorem 4.33 *Inverse Function Theorem*
Let I be an open interval and $f : I \to \mathbb{R}$ be injective and continuous. If $b = f(a)$ for some $a \in I$ and if $f'(a)$ exists and is nonzero, then f^{-1} is differentiable at b and $(f^{-1})'(b) = \frac{1}{f'(a)}$.

5 Riemann Integration

5.1 Introduction

5.2 Step functions and their integrals

Definition 1 *Step function*
We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a *step function* if there exist real numbers $x_0 < x_1 < \cdots < x_n$ (for some $n \in \mathbb{N}$) such that

1. $\phi(x) = 0$ for $x < x_0$ and $x > x_n$
2. ϕ is constant on $(x_{j-1}, x_j) 1 \le j \le n$.

Definition 2
If ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ which takes the value c_j on (x_{j-1}, x_j), then

$$\int \phi := \sum_{j=1}^{n} c_j (x_j - x_{j-1})$$

Proposition 1
If ϕ and ψ are step functions and α and $\beta \in \mathbb{R}$, then

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi.$$

5.3 Riemann-integrable functions and their integrals

Definition 3 *Riemann-integrable*
Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is *Riemann-integrable* if for every $\epsilon > 0$ there exist step functions ϕ and ψ such that $\phi \le f \le \psi$ and $\int \psi - \int \phi < \epsilon$

Theorem 1
A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if
$\sup\{\int \phi : \phi$ is a step function and $\phi \le f\} =$
$\inf\{\int \psi : \psi$ is a step function and $\psi \ge f\}$.

Definition 4
If f is Riemann-integrable we define its integral $\int f$ as the common value
$\int f := \sup\{\int \phi : \phi$ is a step function and $\phi \le f\}$
$= \inf\{\int \psi : \psi$ is a step function and$\psi \ge f\}$.

Theorem 2
A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if there exist sequences of step functions ϕ_n and ψ_n such that $\phi_n \le f \le \psi_n \; \forall n$, and $\int \psi_n - \int \phi_n \to 0$
If ϕ_n and ψ_n are any sequences of step functions satisfying above, then $\int \phi_n \to \int f$ and $\int \psi_n \to \int f$ as $n \to \infty$.

Lemma 1
Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with bounded support $[a, b]$. The following are equivalent:

1. f is Riemann-integrable.
2. for every $\epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ such that, if M_j and m_j denote the supremum and infimum values of$f$ on $[x_{j-1}, x_j]$ respectively, then $\sum_{j-1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon$
3. for every $\epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ such that, with $I_j = (x_{j-1}, x_j)$ for $j \ge 1$, $\sum_{j=1}^{n} \sup_{x,y,\in I_j} |f(x) - f(y)||I_j| < \epsilon$.
   For $f : \mathbb{R} \to \mathbb{R}$ a bounded function with bounded support $[a, b]$ and for $a = x_0 < \cdots < x_n = b$, let $I_j = (x_{j-1}, x_j), m_j := \inf_{x \in I_j} f(x)$ and $M_j := \sup_{x \in I_j} f(x)$.
   Define the *lower step function of f with respect to* $\{x_0, \ldots, x_n\}$ as
   $\phi_*(x) = \sum_{j=1}^{n} m_j \chi_{I_j} + \sum_{j=0}^{n} \chi_{x_j}$
   and the *upper step function of f with respect to* $\{x_0, \ldots, x_n\}$ as
   $\phi^*(x) = \sum_{j=1}^{n} M_j \chi_{I_j} + \sum_{j=0}^{n} \chi_{x_j}$
   Note that ϕ_* and ϕ^* are step functions, and that $\phi_* \le f \le \phi^*$.

Theorem 3
Suppose f and g are Riemann-integrable and α and β are real numbers. Then

1. $\alpha f + \beta g$ is Riemann-integrable and
   $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$
2. If $f \ge 0$ then $\int f \ge 0$; if $f \le g$ then $\int f \le \int g$.
3. $|f|$ is Riemann-integrable and $|\int f| \le \int |f|$
4. $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann-integrable.
5. fg is Riemann-integrable.

Theorem 4
If $g : [a, b] \to \mathbb{R}$ is continuous, and f defined by $f(x) = g(x)$ for $a \le x \le b$, $f(x) = 0$ for $x \notin [a, b]$ then f is Riemann-integrable.

5.4 Fundamental Theorem of Calculus, and Practical Integration

Theorem 5
Let $g : [a, b] \to \mathbb{R}$ be Riemann-integrable. For $a \le x \le b$ let $G(x) = \int_a^x g$. Suppose g is continuous at x for some $x \in [a, b]$. [If x is an endpoint, we mean one-sided continuous.] Then G is differentiable at x and $G'(x) = g(x)$. [If x is an endpoint, we mean one-sided differentiable.]

Theorem 6
Suppose $f : [a, b] \to \mathbb{R}$ has continuous derivative f' on $[a, b]$. Then $\int_a^b f' = f(b) - f(a)$.

5.5 Integrals and uniform limits of sequences and series of functions

Theorem 7
Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence if Riemann-integrable functions which converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then f is Riemann-integrable and $\int f = \lim_{n\to\infty} \int f_n$.

6 Infinite Series of Real Numbers

6.1 Introduction

Definition 6.1 *Partial sum*
Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series with terms a_k.

1. For each $n \in \mathbb{N}$, the *partial sum of S of order n* is defined by
   $s_n := \sum_{k=1}^{n} a_k$
2. S is said to *converge* if and only if its sequence of partial sums $\{s_n\}$ converges to some $s \in \mathbb{R}$ as $n \to \infty$; that is, if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \ge N \implies |s_n - s| < \epsilon$. In this case we shall write
   $\sum_{k=1}^{\infty} a_k = s$
   and call s the *sum*, or *value*, of the series $\sum_{k=1}^{\infty} a_k$
3. S is said to *diverge* if and only if its sequence of partial sums $\{s_n\}$ does not converge as $n \to \infty$. When s_n diverges to $+\infty$ as $n \to \infty$, we shall also write
   $\sum_{k=1}^{\infty} a_k = s$

Theorem 6.5 *Divergence Test*
Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. If a_k does not converge to zero, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 6.6 *Telescoping Series*
If $\{a_k\}$ is a convergent real sequence, then

$$\sum_{k=1}^{\infty} (a_k - a_{k+1}) = a_1 - \lim_{k\to\infty} a_k$$

Theorem 6.7 *Geometric Series*
Suppose that $x \in \mathbb{R}$, that $N \in \{0, 1, \ldots\}$, and that 0^0 is in-

terpreted to be 1. Then the series $\sum_{k=N}^{\infty} x^k$ converges if and only if $|x| < 1$, in which case

$$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}$$

In particular,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1.$$

Theorem 6.8 *The Cauchy Criterion*
Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m \geq n \geq N \implies \left| \sum_{k=n}^{m} a_k \right| < \epsilon$$

Corollary 6.9
Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \left| \sum_{k=n}^{\infty} a_k \right| < \epsilon$$

Theorem 6.10
Let $\{a_k\}$ and $\{b_k\}$ be real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then
$\sum_{i=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$
and
$\sum_{k=1}^{\infty}(\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$
for any $\alpha \in \mathbb{R}$.

6.2 Series with Nonnegative Terms

Theorem 6.11
Suppose that $a_k \geq 0$ for large k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if its sequence of partial sums $\{s_n\}$ is bounded; that is, if and only if there exists a finite number $M > 0$ such that $|\sum_{i=1}^{n} a_k| \leq M \ \forall n \in \mathbb{N}$.

Theorem 6.12 *Integral Test*
Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly integrable on $[1, \infty)$; that is if and only if

$$\int_1^{\infty} f(x) \, \mathrm{d}x < \infty$$

Corollary 6.13 *p-Series Test* The series

$$\sum_{i=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Theorem 6.14 *Comparison Test*
Suppose that $0 \leq a_k \leq b_k$ for large k.

> If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$. If $\sum_{k=1}^{\infty} b_k = \infty$, then $\sum_{k=1}^{\infty} a_k = \infty$.

Theorem 6.16 *Limit Comparison Test*
Suppose that $a_k \geq 0$, that $b_k > 0$ for large k, and that $L := \lim_{n\to\infty} \frac{a_n}{b_n}$ exists as an extended real number.

1. If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.
2. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.
3. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

6.3 Absolute Convergence

Definition 6.18 *Absolute & Conditional Convergence*
Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series.

1. S is said to *converge absolutely* if and only if $\sum_{k=1}^{\infty} |a_k| < \infty$
2. S is said to *converge conditionally* if and only if S converges but not absolutely.

Remark 6.19
A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \implies \sum_{k=n}^{m} |a_k| < \epsilon$$

Remark 6.20
If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k$ converges, but not conversely. In particular, there exist conditionally convergent series.

Definition 6.21 *Limit supremum*
The *limit supremum* of a sequence of real numbers $\{x_k\}$ is defined to be

$$\limsup_{k\to\infty} x_k := \lim_{n\to\infty} \left(\sup_{k>n} x_k \right).$$

Remark 6.22
Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence.

1. If $\limsup_{k\to\infty} x_k < x$, then $x_k < x$ for large k.
2. If $\limsup_{k\to\infty} x_k > x$, then $x_k > x$ for infinitely many ks.
3. If $x_k \to x$ as $x \to \infty$, then $\limsup_{k\to\infty} x_k = x$.

Theorem 6.23 *Root Test*
Let $a_k \in \mathbb{R}$ and $r := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

1. If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
2. If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 6.24 *Ratio test*
Let $a_k \in \mathbb{R}$ with $a_k \neq 0$ for large k and suppose that

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

exists as an extended real number.

1. If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
2. If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.25 The Root and Ratio tests are inconclusive when $r = 1$.

Definition 6.26 *Rearrangement*
A series $\sum_{j=1}^{\infty} b_j$ is called a *rearrangement* of a series $\sum_{k=1}^{\infty} a_k$ if and only if there is an injection $f : \mathbb{N} \to \mathbb{N}$ such that

$$b_{f(k)} = a_k, \quad k \in \mathbb{N}$$

Theorem 6.27
If $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\sum_{j=1}^{\infty} b_j$ is any rearrangement of $\sum_{k=1}^{\infty} a_k$, then $\sum_{j=1}^{\infty} b_j$ converges and

$$\sum_{k=1}^{\infty} a_k = \sum_{j=1}^{\infty} b_j$$

6.4 Alternating Series

Theorem 6.30 *Abel's Formula*
Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be real sequences, and for each pair of integers $n \geq m \geq 1$ set
$A_{n,m} := \sum_{k=m}^{n} a_k$ Then
$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$ for all integers $n > m \geq 1$.

Theorem 6.31 *Dirichlet's Test*
Let $a_k, b_k \in \mathbb{R}$ for $k \in \mathbb{N}$. If the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ is bounded and $b_k \to 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Corollary 6.32 *Alternating Series Test*
If $a_k \to 0$ as $k \to \infty$, then
$\sum_{k=1}^{\infty} (-1)^k a_k$
converges.

7 Infinite Series of Functions

7.1 Uniform Convergence of Sequences

Definition 7.1 *Pointwise Convergence*
Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to *converge pointwise* on E if and only if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$.

Remark 7.2
Let E be a nonempty subset of $\mathbb{R}$. Then a sequence of functions f_n converges pointwise on E, as $n \to \infty$ if and only if for every $\epsilon > 0$ and $x \in E$ there is an $N \in \mathbb{N}$ (which may depend on x as well as ϵ) such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

Remark 7.3
The pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable).

Remark 7.4
The pointwise limit of integrable functions is not necessarily integrable.

Remark 7.5
There exist differentiable functions f_n and f such that $f_n \to f$ pointwise on $[0, 1]$ but

$$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)'$$

for $x = 1$.

Remark 7.6
There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0, 1]$ but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x$$

Definition 7.7 *Uniform Convergence*
Let E be a nonempty subset of $\mathbb{R}$. A sequence of function $f_n : E \to \mathbb{R}$ is said to *converge uniformly* on E to a function f if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

for all $x \in E$.

Theorem 7.9
Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_n \to f$ uniformly on E, as $n \to \infty$. If f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \in E$.

Theorem 7.10
Suppose that $f_n \to f$ uniformly on a closed interval $[a, b]$. If each f_n is integrable on $[a, b]$, then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x$$

In fact, $\lim_{n\to\infty} \int_a^x f_n(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t$ uniformly for $x \in [a, b]$.

Lemma 7.11 *Uniform Cauchy Criterion*
Let E be a nonempty subset of $\mathbb{R}$ and let $f_n : E \to \mathbb{R}$ be a sequence of functions. Then f_n converges uniformly on E if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$.

Theorem 7.12
Let (a, b) be a bounded interval and suppose that f_n is a sequence of funtions which converges at some $x_0 \in (a, b)$. If each f_n is differentiable on (a, b), and f_n' converges uniformly on (a, b) as $n \to \infty$, the f_n converges uniformly on (a, b) and $\lim_{n\to\infty} f_n'(x) = (\lim_{n\to\infty} f_n(x))'$
for each $x \in (a, b)$.

7.2 Uniform Convergence of Series

Definition 7.13 *Convergence*
Let f_k be a sequence of real functions defined on some set E and set

$$s_n(k) := \sum_{k=1}^{n} f_k(x), \quad x \in E, n \in \mathbb{N}$$

1. The series $\sum_{k=1}^{n} f_k(x)$ is said to *converge pointwise* on E if and only if the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$.
2. The series $\sum_{k=1}^{n} f_k(x)$ is said to *converge uniformly* on E if and only if the sequence $s_n(x)$ converges uniformly on E as $n \to \infty$.
3. The series $\sum_{k=1}^{n} f_k(x)$ is said to *converge absolutely (pointwise)* on E if and only if $\sum_{k=1}^{n} |f_k(x)|$ converges for each $x \in E$.

Theorem 7.14
Let E be a nonempty subset of $\mathbb{R}$ and let $\{f_k\}$ be a sequence of real functions defined on E.

1. Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.
2. *Term-by-term integration.* Suppose that $E = [a, b]$ and that each f_k is integrable on $[a, b]$. If $f = \sum_{k=1}^{\infty} f_k$ converges uniformly on $[a, b]$, then f is integrable on $[a, b]$ and
$$\int_a^b \sum_{k=1}^{\infty} f_k(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,\mathrm{d}x.$$
3. *Term-by-term differentiation.* Suppose that E is a bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^{\infty} f_k$ converges at some $x_0 \in E$, and $\sum_{k=1}^{f} {}_k'$ converges uniformly on E, then $f := \sum_{k=1}^{\infty} f_k$ converges uniformly on E, f is differentiable on E, and
$$\left(\sum_{k=1}^{\infty} f_k(x)\right)' = \sum_{k=1}^{\infty} f_k'(x)$$
for $x \in E$.

Theorem 7.15 *Weierstrass M-Test*
Let E be a nonempty subset of $\mathbb{R}$, let $f_k : E \to \mathbb{R}, k \in \mathbb{N}$, and suppose that $M_k \geq 0$ satisfies $\sum_{k=1}^{\infty} M_k < \infty$. If $|f_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in E$, then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on E.

7.3 Power Series

Definition *Power Series*
Let (a_n) be a sequence of real numbers, and $c \in \mathbb{R}$. A *power series* is a series of the form
$\sum_{n=1}^{\infty} a_n (x - c)^n$
With a_n being the *coefficients* and c its centre.

Definition *Radius of Convergence*
The *radius of convergence* R of the power series
$\sum_{n=1}^{\infty} a_n (x - c)^n$

is defined by
$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$
unless $(a_n r^n)$ is bounded for all $r \geq 0$, in which case we declare $R = \infty$.

Theorem 1
Suppose the radius of convergence R satisfies $0 < R < \infty$. If $|x - c| < R$, the power series converges absolutely. If $|x - c| > R$, the power series diverges.

Theorem 2
Assume that $R > 0$. Suppose that $0 < r < R$. Then the series converges uniformly and absolutely on $|x - c| \leq r$ to a continuous function f. Hence
$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$
defines a continuous function $f : (c - R, c + R) \to \mathbb{R}$.

Lemma
The two power series $\sum_{n=1}^{\infty} a_n (x - c)^n$ and $\sum_{n=1}^{\infty} n a_n (x - c)^{n-1}$ have the same radius of convergence.

Theorem 3
Suppose the radius of convergence of the power series is R. Then the function
$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$
is infinitely differentiable on $|x - c| < R$, and for such x,
$f'(x) = \sum_{n=0}^{\infty} n a_n (x - c)^{n-1}$
and the series converges absolutely, and also uniformly on $[c - r, c + r]$ $\forall r < R$. Moreover $a_n = \frac{f^{(n)}(c)}{n!}$

8 Metric Spaces

8.1 Introduction

Definition 10.1 *Metric Space*
A *metric space* is a set X together with a function $\rho : X \times X \to \mathbb{R}$ (called the *metric* of ρ) which satisfies the following properties for all $x, y, z \in X$:

$$\begin{aligned} \text{Positive Definite} \quad & \rho(x, y) \geq 0 \text{ with } \rho(x, y) = 0 \iff x = y \\ \text{Symmetric} \quad & \rho(x, y) = \rho(y, x) \\ \text{Triangle Inequality} \quad & \rho(x, y) \leq \rho(x, z) + \rho(z, y) \end{aligned}$$

Definition 10.7 *Ball*
Let $a \in X$ and $r > 0$. Then *open ball* (in X) with *centre* a and *radius* r is the set
$B_r(a) := \{x \in X : \rho(x, a) < r\}$
and the *closed ball* (in X) with *centre* a and *radius* r is the set $\{x \in X : \rho(x, a) \leq r\}$

Definition 10.8 *Open & Closed*

1. A set $V \subseteq X$ is said to be *open* if and only if for every $x \in V$ there is an $\epsilon > 0$ such that the open ball $B_\epsilon(x)$ is contained in V.
2. A set $E \subseteq X$ is set to be *closed* if and only if $E^c := X \setminus E$ is open.

Remark 10.9 Every open ball is open, and every closed ball is closed.

Remark 10.10 If $a \in X$, then $X \setminus \{a\}$ is open, and $\{a\}$ is closed.

Remark (10.11) In an arbitrary metric space, the empty set $\emptyset$ and the whole space X are both open and closed.

Definition 10.13 *Convergence, Cauchy, & Boundedness*
Let $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ *converges* (in X) if there is a point $a \in X$ (called the *limit* of x_n) such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
$n \geq N \implies \rho(x_n, a) < \epsilon$.
2. $\{x_n\}$ is *Cauchy* if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
$n, m \geq N \implies \rho(x_n, x_m) < \epsilon$.
3. $\{x_n\}$ is *bounded* if there is an $M > 0$ and a $b \in X$ such that $\rho(x_n, b) \leq M$ for all $n \in \mathbb{N}$.

Theorem 10.14
Let X be a metric space.

1. A sequence X can have at most one limit.
2. If $x_n \in X$ converges to a and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then x_{n_k} converges to a as $k \to \infty$.
3. Every convergent sequence X is bounded.
4. Every convergent sequence in X is Cauchy.

Remark 10.15
Let $x_n \in X$. Then $x_n \to a$ as $n \to \infty$ if and only if for every open set V which contains a, there is an $N \in \mathbb{N}$ such that $n \geq N \implies x_n \in V$.

Theorem 10.16
Let $E \subseteq X$. Then E is closed if and only if the limit of every convergent sequence $x_k \in E$ satisfies
$\lim_{k\to\infty} x_k \in E$.

Remark 10.17 The discrete space contains bounded sequence which have no convergent subsequences.

Remark 10.18 The metric space $X = \mathbb{Q}$ contains Cauchy sequences which do not converge.

Definition 10.19 *Completeness*
A metric space X is said to be *complete* if and only if every Cauchy sequence $x_n \in X$ converges to some point in X.

Remark 10.20
By *10.19*, a complete metric space X satisfies two properties:

1. Every Cauchy sequence in X converges;
2. the limit of every Cauchy sequence in X stay in X.

Theorem 10.21
Let X be a complete metric space E be a subset of X. Then E (as a subspace) is complete if and only if E as a (subset) is closed.

8.2 Limits of Functions

Definition 10.22 *Cluster Point*
A point $a \in X$ is said to be a *cluster point* (of X) if and only if $B_\delta(a)$ contains infinitely many points for each $\delta > 0$.

Definition 10.25 *Converge*
Let a be a cluster point of X and $f : X \setminus \{a\} \to Y$. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < \rho(x, a) < \delta \implies \tau(f(x), L) < \epsilon.$$

In this case we write $f(x) \to L$ as $x \to a$, or

$$L = \lim_{x\to a} f(x),$$

and call L the *limit* of $f(x)$ as x approaches a.

Theorem 10.26
Let a be a cluster point of X and $f, g : X \setminus \{a\} \to Y$.

1. If $f(x) = g(x)$ $\forall x \in X \setminus \{a\}$ and $f(x)$ has a limit as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and
$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x).$$
2. *Sequential characterisation of limits.* The limit
$$L := \lim_{x\to a} f(x)$$
exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in X \setminus \{a\}$ which converges to a as $n \to \infty$.

3. Suppose that $Y = \mathbb{R}^n$. If $f(x)$ and $g(x)$ have a limit as x approaches a, then so do $(f+g)(x), (f \cdot g)(x), (\alpha f)(x)$, and $(f/g)(x)$ [when $Y = \mathbb{R}$ and the limit of $g(x)$ is nonzero]. In fact,

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x),\\ \lim_{x\to a}(\alpha f)(x) &= \alpha \lim_{x\to a} f(x),\\ \lim_{x\to a}(f\cdot g)(x) &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)\end{aligned}$$

and [when $Y = \mathbb{R}$ and the limit of $g(x)$ is nonzero]

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

4. *Squeeze Theorem for Functions.* Suppose that $Y = \mathbb{R}$. If $h : X \setminus \{a\} \to \mathbb{R}$ satisfies $g(x) \le h(x) \le f(x)\ \forall x \in X \setminus \{a\}$, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$$

then the limit of h exists, as $x \to a$, and

$$\lim_{x\to a} h(x) = L$$

5. *Comparison Theorem for Functions.* Suppose that $Y = \mathbb{R}$. If $f(x) \le g(x)\ \forall X \setminus \{a\}$, and if f and g have a limit as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Definition 10.27 *Continuity*
Let E be a nonempty subset of X and $f : E \to Y$.

1. f is said to be *continuous at a point* $a \in E$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x,a) < \delta \text{ and } \in E \implies \tau(f(x), f(a)) < \epsilon.$$

2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

Theorem 10.28
Let E be a nonempty subset of X and $f, g : E \to Y$.

1. f is continuous at $a \in E$ if and only if $f(x_n) \to f(a)$, as $n \to \infty$, for all sequences $x_n \in E$ which converge to a.

2. Suppose that $Y = \mathbb{R}^n$. If f, g are continuous at a point $a \in E$ (respectively continuous on a set E), then so are $f+g, f\cdot g$, and αf (for any $\alpha \in \mathbb{R}$). Moreover, in the case $Y = \mathbb{R}$, f/g is continuous at $a \in E$ when $g(a) \ne 0$ [respectively, on E when $g(x) \ne 0$, $\forall x \in E$].

Theorem 10.29
Suppose that X, Y, and Z are metric space and that a is a cluster point of X. Suppose further that $f : X \to Y$ and $g : f(X) \to Z$. If $f(x) \to L$ as a $x \to a$ and g is continuous at L, then

$$\lim_{x\to a}(g\circ f)(x) = g\left(\lim_{x\to a} f(x)\right).$$

Definition 10.30 *Bolzano-Weierstrass Property*
X is said to satisfy the *Bolzano-Weierstrass Property* if and only if every bounded sequence $x_n \in X$ has a convergent subsequence.

8.3 Interior, Closure, and Boundary

Theorem 10.31
Let X be a metric space.

1. If $\{V_\alpha\}_{\alpha\in A}$ is any collection of open sets in X, then $\bigcup_{\alpha\in A} V_\alpha$ is open.
2. If $\{V_k : k = 1, 2, \ldots, n\}$ is a finite collection of open sets in X, then $\bigcap_{k=1}^n V_k := \bigcap_{k\in\{1,2,\ldots,n\}} V_k$ is open.
3. If $\{E_\alpha\}_{\alpha\in A}$ is any collection of closed sets in X, then $\bigcap_{\alpha\in A} E_\alpha$ is closed.
4. If $\{E_k : k = 1, 2, \ldots, n\}$ is a finite collection of closed sets in X, then $\bigcup_{k=1}^n E_k := \bigcup_{k\in\{1,2,\ldots,n\}} E_k$ is closed.
5. If V is open in X and E is closed in X, then $V \setminus E$ is open and $E \setminus V$ is closed.

Remark 10.32
Statements 2 and 4 of *Theorem 10.31* are false if arbitrary collections are used in place of finite collections.

Definition 10.33 *Interior & Closure*
Let E be a subset of a metric space X.

1. The *interior* of E is the set

$$E^O := \bigcup\{V : V \subseteq E \text{ and } V \text{ is open in } X\}.$$

2. The *closure* of E is the set

$$\overline{E} := \bigcap\{B : B \supseteq E \text{ and } B \text{ is closed in } X\}.$$

Theorem 10.34
Let $E \subseteq X$. Then

1. $E^O \subseteq E \subseteq \overline{E}$,
2. if V is open and $V \subseteq E$, then $V \subseteq E^0$, and
3. if C is closed and $C \supseteq E$, then $C \supseteq \overline{E}$.

Definition 10.37 *Boundary*
Let $E \subseteq X$. The *boundary* of E is the set

$$\partial E := \{x \in X : \forall r > 0,\ B_r(x) \cap E \ne \emptyset \text{ and } B_r(x) \cup E^c \ne \emptyset\}.$$

[We refer to the last two conditions in the definition of ∂E by saying $B_r(x)$ *intersects* E and E^c.]

Theorem 10.39
Let $E \subseteq X$. Then

$$\partial E = \overline{E} \setminus E^0.$$

Theorem 10.40
Let $A, B \subseteq X$. Then

1. $(A \cup B)^O \supseteq A^O \cup B^O, \quad (A \cap B)^O = A^O \cap B^O$,
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}, \quad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$,
3. $\partial(A \cup B) \subseteq \partial A \cup \partial B$, and $\partial(A \cap B) \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

8.4 Compact Sets

Definition 10.41 *Covering*
Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ be a collection of subsets of a metric space X and suppose that E is a subset of X.

1. $\mathcal{V}$ is said to *cover* E if and only if

$$E \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

2. $\mathcal{V}$ is said to be an *open covering* of E if and only if $\mathcal{V}$ covers E and each V_α is open.
3. Let $\mathcal{V}$ be a covering of E. $\mathcal{V}$ is said to have a *finite* (respectively *countable*) *subcovering* if and only if there is a finite (respectively, countable) subset A_0 of A such that $\{V_\alpha\}_{\alpha\in A_0}$ covers E.

Definition 10.42 *Compact*
A subset H of a metric space X is said to be *compact* if and only if every open covering of H has a finite subcover.

Remark 10.43 The empty set and all finite subsets of a metric space are compact.

Remark 10.44 A compact set is always closed.

Remark 10.45 A closed subset of a compact set is compact.

Theorem 10.46
Let H be a subset of a metric space X. If H is compact, then H is closed and bounded.

Remark 10.47 The converse of *Theorem 10.46* is false for arbitrary metric spaces

Definition 10.48 *Separable*
A metric space X is said to be *separable* if and only if it contains a countable dense subset (i.e. if and only if there is a countable subset Z of X such that for every point $a \in X$ there is a sequence $x_k \in Z$ such that $x_k \to a$ as $k \to \infty$).

Theorem 10.49 *Lindelöf*
Let E be a subset of a separable metric space X. If $\{V_\alpha\}_{\alpha \in A}$ is a collection of open sets and $E \subseteq \bigcup_{\alpha \in A} V_\alpha$, then there is a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ of A such that

$$E \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}$$

Theorem 10.50 *Heine-Borel*
Let X be a separable metric space which satisfies the *Bolzano-Weierstrass Property*, and H be a subset of X. Then H is compact if and only if it is closed and bounded.

Definition 10.51 *Uniform Continuity*
Let X be a metric space, E be a nonempty subset of X, and $f : E \to Y$. Then f is said to be *uniformly continuous* on E if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x, a) < \delta \text{ and } x, a \in E \implies \tau(f(x), f(a)) < \epsilon.$$

Theorem 10.52
Suppose that E is a compact subset of X and that $f : X \to Y$. Then f is uniformly continuous on E if and only if f is continuous on E.

8.5 Connected Sets

Definition 10.53 *Separate & Connected*
Let X be a metric space.

1. A pair of nonempty open sets U, V in X is said to *separate* X if and only if $X = U \cup V$ and $U \cap V = \emptyset$.
2. X is said to be *connected* if and only if X cannot be separated by any pair of open sets U, V.

Definition 10.54 *Relatively open & closed*
Let X be a metric space and $E \subseteq X$.

1. A set $U \subseteq E$ is said to be *relatively open* in E if and only if there is a set V open in X such that $U = E \cap V$.
2. A set $A \subseteq E$ is said to be *relatively closed* in E if and only if there is a set C closed in X such that $A = E \cap C$.

Remark 10.55
Let $E \subseteq X$. If there exists a pair of open sets A, B in X which separate E, then E is not connected.

Theorem 10.56
A subset E of $\mathbb{R}$ is connected if and only if E is an interval.

8.6 Continuous Functions

Theorem 10.58
Suppose that $f : X \to Y$. Then f is continuous if and only if $f^{-1}(V)$ is open in X for every open V in Y.

Corollary 10.59
Let $E \subseteq X$ and $f : E \to Y$. Then f is continuous on E if and only if $f^{-1}(V) \cap E$ is relatively open in E for all open sets V in Y.

Theorem 10.61
If H is compact in X and $f : H \to Y$ is continuous on H, then $f(H)$ is compact in Y.

Theorem 10.62
If E is connected in X and $f : E \to Y$ is continuous on E, then $f(E)$ is connected in Y.

Theorem 10.63 *Extreme Value Theorem*
Let H be a nonempty, compact subset of X and suppose that $f : H \to \mathbb{R}$ is continuous. Then

$$M := \sup\{f(x) : x \in H\} \quad \text{and} \quad m := \inf\{f(x) : x \in H\}$$

are finite real numbers and there exist points $x_M, x_m \in H$ such that $M = f(x_M)$ and $m = f(x_m)$.

Theorem 10.64
If H is a compact subset of X and $f : H \to Y$ is injective and continuous, then f^{-1} is continuous on $f(H)$.

9 Contraction Mapping & ODEs

9.1 Banach's Contraction Mapping Theorem

Definition *Contraction*
Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if there exists a number α with $0 < \alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \; \forall x, y \in X.$$

Note the target space and the domain must be the same.

Remark

1. It is *really* important that α be *strictly less* than 1. It's also really important that we have $d(f(x), f(y)) \leq \alpha d(x, y)$ and *not just* $d(f(x), f(y)) < d(x, y) \; \forall x, y \in X$. So $f(x) = \cos(x)$ is not a contraction on $\mathbb{R}$.
2. The constant $\alpha < 1$ is called the *contraction constant* of f.

Theorem *Banach's Contraction Mapping Theorem*
If (X, d) is a complete metric space and if $f : X \to X$ is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$.

Remarks

1. It's really important that X be complete.
2. It's really important that the image of X under f is contained in X.
3. A point x such that $f(x) = x$ is called a *fixed point* of f.

9.2 Existence and uniqueness for solutions to ODEs

Definition *Lipschitz Condition*
Suppose $A \in \mathbb{R}$, $\rho, r > 0$, and $F : [A - \rho, A + \rho] \times [-r, r] \to \mathbb{R}$ is continuous. Suppose also that for all $x, y \in [A - \rho, A + \rho]$ and all $t \in [-r, r]$ we have, for some $M > 0$

$$|F(x, t) - F(y, t)| \leq M|x - y|$$

Theorem *Picard*
Suppose F satisfies the Lipschitz Condition. Then there exists an $s > 0$ such that the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, t)$$
$$x(0) = A$$

has a unique solution $x(t)$ for $|t| < s$.

Workshop 2 – Uniform convergence of sequences of functions

The purpose of this workshop activity is to provide some practice in the notions of pointwise and uniform convergence of sequences of functions, and in some of the theorems concerning uniform convergence of sequences of functions.

1. Let $f_n(x) = \frac{xn^{1/2}}{1+nx^2}$ for $x \in \mathbb{R}$. Prove that f_n converges pointwise to the zero function. Is the convergence uniform over $\mathbb{R}$? (**Hint:** Fix n and think about $\sup_{x\in\mathbb{R}} |f_n(x)|$. Does this go to zero as $n \to \infty$?)

   Solution. If $x = 0$ we have $f_n(0) = 0$ for all n and so $f_n(0)$ converges to 0. If $x \neq 0$, then $|f_n(x)| \leq \frac{|x|n^{1/2}}{nx^2} = \frac{1}{n^{1/2}|x|}$ which goes to zero as $n \to \infty$. So f_n converges pointwise to 0. But $f_n(n^{-1/2}) = 1/2$ for all n so the convergence is not uniform over $\mathbb{R}$. (If you hadn't spotted that $n^{-1/2}$ is an interesting point, you could have used calculus to find the maximum of the function $|f_n| \ldots$)

2. Let $f_n : [0, 1) \to \mathbb{R}$ be defined by $f_n(x) = nx^n$. Show that $f_n \to 0$ pointwise but $\int_0^1 f_n \to 1$. What does this demonstrate?

   Solution. From FPM we know that for $0 \leq x < 1$ we have $nx^n \to 0$ as $n \to \infty$. However $\int_0^1 f_n = n/(n+1) \to 1$ as $n \to \infty$. So pointwise convergence on an interval does not imply the corresponding convergence of definite integrals.

3. Consider the sequence of functions on $\mathbb{R}$ given by $f_n(x) = (x - 1/n)^2$. Prove that it converges pointwise and find the limit function. Is the convergence uniform on $\mathbb{R}$? Is the convergence uniform on bounded intervals?

   Solution. For each fixed x we have $x_n := x - 1/n \to x$ as $n \to \infty$; hence, by FPM, $x_n^2 \to x^2$. So the sequence of functions f_n converges pointwise to the function $f(x) = x^2$. What about uniform convergence? We need to consider whether the sequence $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$ goes to zero as $n \to \infty$. Let's look at

$$|f_n(x) - f(x)| = |(x - 1/n)^2 - x^2| = \frac{|2x - 1/n|}{n}.$$

The values of this expression as x ranges over $\mathbb{R}$ are not even *bounded* (to see this let $x \to \infty$), and so the sup does not even *exist*, let alone go to 0 as $n \to \infty$. So we do not have uniform convergence of f_n to f. (Another way of seeing this is to note that $|f_n(n) - f(n)| = \frac{|2n - n^{-1}|}{n} \geq 1$ and so $|f_n(n) - f(n)|$ does not go to zero.)

If however we work on $[-M, M]$ instead of on the whole of $\mathbb{R}$, the above calculation shows that for $|x| \leq M$ we have

$$|f_n(x) - f(x)| = \frac{|2x - 1/n|}{n} \leq \frac{2M + 1/n}{n}$$

so that $\sup_{x\in[-M,M]} |f_n(x) - f(x)|$ goes to zero as $n \to \infty$, and hence the convergence is uniform on bounded intervals.

4. Let $f_n(x) = x - x^n$. Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? Is the convergence uniform on $[0, 1)$?

   Solution. For $0 \leq x < 1$ we have $f_n(x) \to x$ and for $x = 1$ we have $f_n(1) = 0$. So the limit function is $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 0$. Since each f_n is continuous on $[0, 1]$ and f isn't, the convergence can't be uniform on $[0, 1]$. As for uniform convergence on $[0, 1)$, for $0 \leq x < 1$ we have $|f_n(x) - f(x)| = |x^n|$ so that $\sup_{0\leq x<1} |f_n(x) - f(x)| = \sup_{0\leq x<1} |x^n| = 1$, so once again the convergence is not uniform.

5. Consider the sequence of functions defined on $[0, \infty)$ defined by $f_n(x) = \frac{x^n}{1+x^n}$. Prove that (f_n) converges pointwise and find the limit function. Is the convergence uniform on $[0, \infty)$? Is the convergence uniform on bounded intervals of the form $[0, a)$?

   Solution. If $0 \leq x < 1$ we have $x^n \to 0$ and so $f_n(x) \to 0$. If $x = 1$ we have $f_n(1) = 1/2$ for all n so $f_n(1) \to 1/2$. If $x > 1$ we have $f_n(x) = \frac{1}{1+x^{-n}}$ which tends to 1 as $n \to \infty$. So the limit function is f where $f(x) = 0$ for $0 \leq x < 1$, $f(1) = 1/2$ and $f(x) = 1$ for $x > 1$. Each f_n is continuous on $[0, \infty)$ but f is not: hence the convergence cannot have been uniform. The same argument applies if the point of discontinuity 1 belongs in $[0, a)$, that is, when $a > 1$. For $a \leq 1$ we have to argue from first principles: for each n, $\sup_{0\leq x<a} |f_n(x) - f(x)| = \sup_{0\leq x<a} |\frac{x^n}{1+x^n}| = \sup_{0\leq x<a} |\frac{1}{1+x^{-n}}| = \frac{1}{1+a^{-n}}$ as the function $x \mapsto \frac{1}{1+x^{-n}}$ is increasing on $[0, a)$. Now if $a < 1$, $\frac{1}{1+a^{-n}} \to 0$ as $n \to \infty$, while if $a = 1$, $\frac{1}{1+a^{-n}} = 1/2$ which does not tend to 0. So we get uniform convergence to 0 on $[0, a)$ if and only if $a < 1$.

6. Let $f_n(x) = nx(1 - x^2)^n$ for $0 \leq x \leq 1$. Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? (**Hint:** Consider the integrals $\int_0^1 f_n$.) Is the convergence uniform on $[a, 1]$ where $0 < a < 1$?

   Solution. If $x = 0$ or 1 then $f_n(0) = 0$. If $0 < x < 1$ then $0 < 1 - x^2 < 1$ and so $n(1-x^2)^n \to 0$. So f_n converges pointwise to the zero function. If the convergence were uniform we'd have to have $\int_0^1 f_n \to \int_0^1 0 = 0$. But $\int_0^1 nx(1 - x^2)^n \mathrm{d}x = n/2 \int_0^1 u^n \mathrm{d}u = \frac{n}{2(n+1)} \to 1/2 \neq 0$. So the convergence cannot be uniform on $[0, 1]$. If however $a \leq x \leq 1$, then $1 - x^2 \leq 1 - a^2$ so that $|f_n(x)| \leq n(1 - a^2)^n \to 0$ as $n \to \infty$ since $a > 0$. So the convergence is uniform on such intervals.

7. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions which converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Let (x_n) be a sequence of real numbers which converges to $x \in \mathbb{R}$. Show that $f_n(x_n) \to f(x)$.

   Solution. Note that $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$. [It's also true that $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)|$ but this turns out to be not so helpful for this particular problem.] Note also that uniform convergence of (f_n) and continuity of each f_n implies that f is continuous. Let $\epsilon > 0$. Then, by uniform convergence of f_n to f, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(y) - f(y)| < \epsilon/2$ for *all* y, in particular for $y = x_n$. So if $n \geq N$ we have $|f_n(x_n) - f(x_n)| < \epsilon/2$. Since $x_n \to x$, and since f is continuous, there is an $M \in \mathbb{N}$ such that $n \geq M$ implies $|f(x_n) - f(x)| < \epsilon/2$. So if we take $n \geq \max\{N, M\}$ we have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Workshop 3 – Uniform continuity

The purpose of this workshop is to study an auxiliary topic that we won't cover in the lectures, but which provides one very important result that we shall need in our study of integration.

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. We know that it is continuous at a for all $a \in \mathbb{R}$. So, for every a, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ imples $|f(x) - f(a)| < \epsilon$. For $a > 1$ and $\epsilon = 1$ find the best possible δ. Is this best possible δ independent of $a > 1$? (**Hint:** Draw the graph of the function and include the horizontal lines $y = a^2 \pm 1$.)

   Solution: Our δ will be the best one such that $x \in (a - \delta, a + \delta)$ imples that $x^2 \in (a^2 - 1, a^2 + 1)$, that is, $x \in ((a^2 - 1)^{1/2}, (a^2 + 1)^{1/2})$. Clearly very small δ's are fine, until *either* $a - \delta$ hits $(a^2 - 1)^{1/2}$ or $a + \delta$ hits $(a^2 + 1)^{1/2}$. So we have to work out $a - (a^2 - 1)^{1/2}$ and $(a^2 + 1)^{1/2} - a$ and our best δ will be whichever is the smaller of these two numbers. The first is $1/(a + (a^2 - 1)^{1/2})$ and the second is $1/((a^2 + 1)^{1/2} + a)$, of which $1/((a^2 + 1)^{1/2} + a)$ is the smaller. So any $\delta \leq 1/((a^2 + 1)^{1/2} + a)$ will work, and any $\delta > 1/((a^2 + 1)^{1/2} + a)$ won't work. Thus we clearly can't find a δ which works uniformly for all $a > 1$. The best δ goes to 0 when a gets larger and larger.

2. Consider the same function, but now on $[0, 1]$. Prove that for all $\epsilon > 0$, if we take $\delta = \epsilon/2$ we have that $|x - a| < \delta$ (and $x, a \in [0, 1]$) implies $|f(x) - f(a)| < \epsilon$. So the "best" δ in the definition of continuity at $a \in [0, 1]$ can be taken to be independent of a in this case.

   Solution: Well, $|f(x) - f(a)| = |x^2 - a^2| = |x - a||x + a| \leq 2|x - a|$ since $|x + a| \leq |x| + |a| \leq 1 + 1 = 2$. So $|x - a| < \epsilon/2$ implies $|f(x) - f(a)| < 2 \times \epsilon/2 = \epsilon$.

Definition. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** on I if for every $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in I$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

So with $f(x) = x^2$ and $I = [0, 1]$ we have that f is uniformly continuous, while with the same f but $I = (1, \infty)$, we have that it isn't uniformly continuous. (Note that uniform continuity only makes sense for functions which are already continuous!)

3. Let $f(x) = 1/x$ on $(0, \infty)$. Is f uniformly continuous?

   Solution: No. Take $\epsilon = 1$. Consider the sequences $x_n = 1/n$ and $y_n = 1/(n+1)$. Then $|f(x_n) - f(y_n)| = 1$ so there is no $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$.

4. Let $f(x) = 1/x$ on $[a, \infty)$ where $a > 0$. Is f uniformly continuous?

   Solution: Yes, either by the next exercise, or, more directly from the definition as follows. Let $\epsilon > 0$. Consider $|f(x) - f(y)|$ for $a \leq x, y$. This equals $|x - y|/|xy| \leq a^{-2}|x - y|$ for such x, y. So if $a > 0$ and we take $\delta < \epsilon a^2$ we have $|x - y| < \delta$ implies $|f(x) - f(y)| < a^{-2}\epsilon a^2 = \epsilon$.

5. Let I be an open interval in $\mathbb{R}$. Suppose $f : I \to \mathbb{R}$ is differentiable and its derivative f' is bounded on I. Prove that f is uniformly continuous on I.

   Solution. Suppose that $|f'(\xi)| \leq M$ for all $\xi \in I$. By the mean value theorem we have, for each x and y in I, that $f(x) - f(y) = (x - y)f'(\xi)$ for some ξ between x and y. So, $|f(x) - f(y)| = |(x - y)||f'(\xi)| \leq M|x - y|$. Let $\epsilon > 0$ and let $\delta = \epsilon/M$. If now $|x - y| < \delta$ we have $M|x - y| < M\delta = \epsilon$, as required.

6. Show that $f(x) = \sin x$ is uniformly continous on $\mathbb{R}$.

   Solution. Note that the derivative satisfies $|f'(x)| = |\cos x| \leq 1$ on $\mathbb{R}$ and use the previous exercise.

7. Let I be any interval in $\mathbb{R}$. Prove that a continuous function $f : I \to \mathbb{R}$ is uniformly continuous on I if and only if whenever $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$, then $|f(s_n) - f(t_n)| \to 0$.

Solution. Let us suppose that $f : I \to \mathbb{R}$ is uniformly continuous on I and that $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$ as $n \to \infty$. Let $\epsilon > 0$. By uniform continuity of f there is a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$. Since $|s_n - t_n| \to 0$ there is an N such that $n \geq N$ implies $|s_n - t_n| < \delta$. So if $n \geq N$ we have $|f(s_n) - f(t_n)| < \epsilon$.

Now suppose that f is continuous but not uniformly continuous. So there exists an $\epsilon > 0$ such that taking $\delta = 1/n$ there are $s_n, t_n \in I$ with $|s_n - t_n| < \delta$ but $|f(s_n) - f(t_n)| \geq \epsilon$. So $|s_n - t_n| \to 0$ but $|f(s_n) - f(t_n)| \not\to 0$.

A fundamental fact is that if we are working on a **closed, bounded** interval $[a, b]$, then any continuous function $f : [a, b] \to \mathbb{R}$ is automatically uniformly continuous.

Theorem. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then it is uniformly continuous.

8. Prove this theorem by arguing by contradiction, using the the previous question and the Bolzano–Weierstrass theorem.

Solution. Assume f is not uniformly continuous. By the previous question there is an $\epsilon > 0$ and there are sequences x_n and y_n with $|x_n - y_n| \to$ but $|f(x_n) - f(y_n)| \geq \epsilon$. The Bolzano–Weierstrass theorem tells us that (x_n) has a subsequence x_{n_k} convergent to some $x \in [a, b]$. Since $|x_n - y_n| \to 0$ we see that y_{n_k} also converges to x as $k \to \infty$. Continuity of f at x gives that $\lim_{k\to\infty} f(x_{n_k}) = f(x)$ and similarly $\lim_{k\to\infty} f(y_{n_k}) = f(x)$. So $\lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| = 0$. But this contradicts the fact that $|f(x_n) - f(y_n)| \geq \epsilon$ for all n.

9. Find an example of an $f : (0, 1) \to \mathbb{R}$ which is continous but not uniformly continuous. Where exactly did we use the fact that $[a, b]$ was a closed and bounded interval in the proof of the theorem?

Solution. We've already noted that $f(x) = 1/x$ is one such. When we used the Bolzano–Weierstrass theorem it was important that the subsequence converged to some point **inside** the interval in question. If we were working with (a, b) rather than $[a, b]$ the subsequence might well converge to a or b, which would be useless.

Workshop 5 – Riemann Integration

This workshop provides some practice in questions concerning Riemann integration. We've basically covered Questions 1-4 in the lectures, so this is an opportunity to consolidate your understanding of them. Then move right on to Question 5. Recall that we're following the lecture notes issued on Learn, not the treatment in Wade, so all arguments should be made with reference to the lecture notes issued.

1. Show that if $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable, then f must be bounded and have bounded support.

Solution: If f is Riemann-integrable, then, taking $\epsilon = 1$ in the definition, there exist step functions ϕ and ψ such that $\phi \leq f \leq \psi$. Then $|f| \leq \max\{|\phi|, |\psi|\}$, which, as a step function itself, takes on only finitely many values and is therefore bounded. [Or, if you like, the step function $\sum_{j=1}^n \alpha_j \chi_{I_j}$ has maximum value at most $\sum_{j=1}^n |\alpha_j|$ and so is bounded.] So f is bounded. Moreover, there are $M, N \in \mathbb{R}_+$ such that $\phi(x) = 0$ for $|x| > M$ and $\psi(x) = 0$ for $|x| > N$, so that $\phi(x) = \psi(x) = 0$ for $|x| > \max\{M, N\}$. Since $\phi \leq f \leq \psi$ this forces $f(x) = 0$ for $|x| > \max\{M, N\}$, and so f has bounded support.

2. Show that $\chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann-integrable.

Solution: Let ϕ and ψ be any step functions such that $\phi \leq \chi_{\mathbb{Q}\cap[0,1]} \leq \psi$. Then on any interval of positive length on which ϕ is constant, the value of ϕ must in fact be non-positive. This is because any interval of positive length must contain irrationals, and we have that $\chi_{\mathbb{Q}\cap[0,1]}(x) = 0$ for irrational x. Thus $\phi(x) \leq 0$ except for possibly finitely many values of x, and therefore $\int \phi \leq 0$. Similarly, since any interval of positive length must contain rationals, ψ must be at least 1 on any interval of positive length meeting $[0, 1]$ on which it is constant. Therefore $\int \psi \geq 1$. Hence $\int \psi - \int \phi \geq 1$. So it is *not* true that for every $\epsilon > 0$ there exist step functions ϕ and ψ such that $\phi \leq \chi_{\mathbb{Q}\cap[0,1]} \leq \psi$ and $\int \psi - \int \phi < \epsilon$.

3. Show that the function defined by $f(x) = 1$ if $x = 1/n$ for some $n \in \mathbb{N}$, and $f(x) = 0$ otherwise, is Riemann-integrable, and calculate $\int f$.

Solution: Let $\phi_n = 0$ for all n and let $\psi_n(x) = 1$ if $0 \leq x \leq 1/n$, *or* if $x = 1/(n-1), 1/(n-2), \ldots, 2$ or 1. Then ϕ_n and ψ_n are step functions, $\phi_n \leq f \leq \psi_n$, $\int \phi_n = 0$ for all n and $\int \psi_n = 1/n$ for all n. So the sequences $\int \phi_n$ and $\int \psi_n$ have the same common limit 0. By Theorem 2 from the lecture notes, f is Riemann-integrable, and $\int f = 0$.

4. Give an example demonstrating the falsehood of the statement "$|f|$ Riemann-integrable implies f Riemann-integrable".

Solution: Let $f(x) = 1$ if $0 \leq x \leq 1$ and x is rational, $f(x) = -1$ if $0 \leq x \leq 1$ and x is irrational, and $f(x) = 0$ otherwise. Then $|f| = \chi_{[0,1]}$ is a step function, so Riemann-integrable. On the other hand, f is not Riemann-integrable for the same reasons as for $\chi_{\mathbb{Q}\cap[0,1]}$ in Question 2.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable, and that $f = 0$ outside $[a, b]$ where $a < b$. Show that $(\exp f)\chi_{[a,b]}$ is also Riemann-integrable.

We first note that since f is integrable, f is bounded, so there exists an M such that $|f(x)| \leq M$ for all x, and has bounded support $[a, b]$. Therefore $(\exp f)\chi_{[a,b]}$ is also bounded by e^M and has bounded support $[a, b]$ too.

Solution 1: Since f is Riemann-integrable we have by Lemma 1 from the lecture notes that for all $\epsilon > 0$ there exist $a = x_0 < x_1 < \ldots < x_n = b$ such that

$$\sum_{j=1}^{n} \sup_{x,y\in(x_{j-1},x_j)} |f(x) - f(y)|(x_j - x_{j-1}) < \epsilon.$$

Now the derivative of $\exp : [-M, M] \to \mathbb{R}$ is bounded by e^M and therefore for all x and y we have $|\exp f(x) - \exp f(y)| \leq e^M |f(x) - f(y)|$ by the Mean Value theorem. Hence

$$\sum_{j=1}^{n} \sup_{x,y\in(x_{j-1},x_j)} |\exp f(x) - \exp f(y)|(x_j - x_{j-1})$$

$$\leq e^M \sum_{j=1}^{n} \sup_{x,y\in(x_{j-1},x_j)} |f(x) - f(y)|(x_j - x_{j-1}) < e^M \epsilon.$$

Now $(\exp f)\chi_{[a,b]}$ is bounded and has bounded support, so by Lemma 1 it is Riemann-integrable.

Solution 2: We have

$$\exp f(x)\chi_{[a,b]}(x) = \sum_{n=0}^{\infty} \frac{f(x)^n}{n!} \chi_{[a,b]}(x) := \sum_{n=0}^{\infty} g_n(x).$$

Now each g_n is Riemann-integrable since it is the product of Riemann integrable functions, (g_0 is integrable as $\chi_{[a,b]}$ is.) Moreover, the series $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on $[a, b]$ by the Weierstrass M-test: if $|f(x)| \leq M$ we have $|g_n(x)| \leq M^n/n!$ for all x and $\sum_n M^n/n! = e^M$ converges. So by the theorem on integrating uniformly convergent series of functions, $\exp f(x)\chi_{[a,b]} = \sum_{n=0}^{\infty} g_n(x)$ is integrable on $[a, b]$.

Solution 3: Let $\epsilon > 0$. Then there exist step functions ϕ, ψ with $\phi \leq f \leq \psi$ and $\int \psi - \int \phi < \epsilon$. Let $\Phi = e^{\phi}\chi_{[a,b]}$ and $\Psi = e^{\psi}\chi_{[a,b]}$. Then Φ, Ψ are step functions and $\Phi \leq \exp(f)\chi_{[a,b]} \leq \Psi$. If now M is a number such that $-M \leq \phi \leq \psi \leq M$ the mean value theorem gives us that $\Psi(x) - \Phi(x) \leq e^M(\psi(x) - \phi(x))$. Integrate this to get $\int \Psi - \int \Phi < e^M \epsilon$.

6. For $x > 0$ define $L(x) = \int_1^x \frac{dt}{t}$. Show that $L(xy) = L(x) + L(y)$. Show that $L'(x) = 1/x$. Show that L is the inverse to the exponential function $E(x)$ defined by the power series $E(x) = \sum_{k=0}^{\infty} x^k/k!$ (**Hint:** Consider the derivative of the composition $L \circ E$. Be careful with domains and ranges. You may use any properties of the function E that we developed in the lectures.)

Solution: First of all the integral defining $L(x)$ exists because $t \mapsto 1/t$ is continuous on $(0, \infty)$. Now

$$L(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_1^{y} \frac{dt}{t} = L(x) + L(y)$$

by changing variables (legally!) in the second integral. By the Fundamental Theorem of Calculus (Theorem 5 of the Lecture notes) $L'(x) = 1/x$. Consider $\phi(x) = L \circ E(x)$ for $x \in \mathbb{R}$ (valid since $E(x) > 0$ for all x); by the chain rule ϕ is differentiable on $\mathbb{R}$ and $\phi'(x) = L'(E(x))E'(x) = \frac{E(x)}{E(x)} = 1$. So by Rolle's theorem or the Mean Value theorem, $\phi(x) - x$ is constant on $\mathbb{R}$ and since $L \circ E(0) = L(1) = 0$ we have that $\phi(x) = x$ on $\mathbb{R}$, that is $L \circ E = \mathbb{I}_{\mathbb{R}}$. Similarly $E \circ L = \mathbb{I}_{\mathbb{R}_+}$.

7. Let $g : [a, b] \to \mathbb{R}$ (with $a < b$) be continuous and nonnegative. If $\int_a^b g = 0$, show that $g = 0$ on $[a, b]$.

Solution: If there is an $x_0 \in [a, b]$ such that $g(x_0) > 0$, then by the sign-preserving property of continuous functions from FPM, there is an interval $I \subseteq [a, b]$ of positive length, such that $g(x) > g(x_0)/2$ on I. So $\phi := \frac{1}{2}g(x_0)\chi_I$ is a step function such that $\phi \leq g$ but $\int \phi > 0$. This contradicts the fact that $\int_a^b g = 0$.

8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be Riemann-integrable. Prove the Cauchy–Schwarz inequality

$$\int fg \leq \left(\int f^2\right)^{1/2} \left(\int g^2\right)^{1/2}.$$

Hint: $\int (f - \alpha g)^2 \geq 0$ for any constant $\alpha \in \mathbb{R}$.

Solution: We know from Theorem 3 of the lecture notes that $f - \alpha g$ and hence $(f - \alpha g)^2$ are integrable. Moreover $\int (f - \alpha g)^2 \geq 0$, by the same theorem. Multiplying out we have

$$\int f^2 - 2\alpha \int fg + \alpha^2 \int g^2 \geq 0$$

(where fg is also integrable by Theorem 3). For this to happen the discriminant $(-2\int fg)^2 - 4\int f^2 \int g^2$ of the quadratic expression must be nonpositive (at most one real root). Hence

$$\left(\int fg\right)^2 \leq \int f^2 \int g^2.$$

Workshop 6 – Examples of Metric Spaces

In this workshop we shall meet a variety of examples of metric spaces. In most – but not all – cases the subtlety involved in checking that a given function is a metric lies in verification of the triangle inequality. You'll notice that there are several different metrics called d_1, several called d_2 and several called d_∞, depending on what the underlying set X is. Why do we use these notations?

1. Show that both

$$d_1(x,y) := \sum_{i=1}^n |x_i - y_i| \quad \text{and} \quad d_\infty(x,y) := \max_{1\le i\le n} |x_i - y_i|$$

define metrics on $\mathbb{R}^n$. **Remark.** So it is quite possible for a given set to have many distinct metrics defined on it.

Solution: In both cases, only the triangle inequality is maybe not obvious. We have $|x_i - y_i| \le |x_i - z_i| + |z_i - y_i|$ for each i. We can sum this to obtain the triangle inequality for d_1; and for d_∞ we have $|x_i - y_i| \le |x_i - z_i| + |z_i - y_i| \le \max_i |x_i - z_i| + \max_i |z_i - y_i| = d_\infty(x,z) + d_\infty(z,y)$ so that $d_\infty(x,y) = \max_i |x_i - y_i| \le d_\infty(x,z) + d_\infty(z,y)$.

2. Show that

$$d_1(f,g) := \int_0^1 |f-g|$$

defines a metric on the class $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

3. Show that

$$d_2(f,g) := \left(\int_0^1 |f-g|^2\right)^{1/2}$$

defines a metric on the class $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$. (**Hint:** Recall that in the last workshop we proved that if $F, G : [0,1] \to \mathbb{R}$ are continuous functions, then we have the Cauchy–Schwarz inequality $|\int_0^1 FG| \le \left(\int_0^1 F^2\right)^{1/2} \left(\int_0^1 G^2\right)^{1/2}$ which is analogous to the Cauchy–Schwarz inequality $|\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}||\mathbf{y}|$ for euclidean space $\mathbb{R}^n$ – which we used in lectures to establish the triangle inequality for $\mathbb{R}^n$ with the usual metric. So use the Cauchy–Schwarz inequality for integrals to deduce $\left(\int |F+G|^2\right)^{1/2} \le \left(\int |F|^2\right)^{1/2} + \left(\int |G|^2\right)^{1/2}$.)

Solutions: In both cases we need to worry about whether $d(f,g) = 0$ implies $f = g$. If $d(f,g) = 0$ we have that $|f-g|$ (or $|f-g|^2$) has integral 0, so by last week's assignment, the nonnegative continuous function $|f-g|$ (or $|f-g|^2$) is identically 0; that is, $f = g$. The triangle inequality for d_1 follows by integrating the inequality $|f(s) - g(s)| \le |f(s) - h(s)| + |h(s) - g(s)|$ on $[0,1]$. The triangle inequality for d_2 follows since we have

$$\int (F+G)^2 = \int F^2 + \int G^2 + 2\int FG \le \left(\left(\int F^2\right)^{1/2} + \left(\int G^2\right)^{1/2}\right)^2$$

and from this we deduce that $d_2(f,g) \le d_2(f,h) + d_2(h,g)$ upon setting $F = f - h$ and $G = h - g$.

4. Let $\mathcal{R}$ denote the vector space of Riemann-integrable functions $f : \mathbb{R} \to \mathbb{R}$. For $f, g \in \mathcal{R}$ let

$$d_1(f,g) := \int |f-g|.$$

Does d_1 define a metric on $\mathcal{R}$? (If you don't know what a vector space is, don't worry.)

Solution: No: the function $f(x) = 0$ for $x \ne 0$ and $f(0) = 1$ is a step function, hence is in $\mathcal{R}$, is not the zero function, yet $d_1(f,0) = \int |f| = 0$.

5. Which of the follwing are metrics on $\mathbb{R}$?

(i) $d(x,y) = \sin|x-y|$

(ii) $d(x,y) = |\sin(x-y)|$

(iii) $d(x,y) = \log(1 + |x-y|)$

(iv) $d(x,y) = |x-y|^2$

(v) $d(x,y) = |x-y|^{1/2}$.

Solution:

(i) No, because $d(3\pi/2, 0) = -1 \not\ge 0$.

(ii) No, because $d(2\pi, 0) = 0$ but $2\pi \ne 0$.

(iii) This is positive, symmetric and $d(x,y) = 0$ implies $x = y$. So we only have to consider the triangle inequality. Do we have

$$\log(1 + |x-y|) \le \log(1 + |x-z|) + \log(1 + |z-y|)?$$

Exponentiating both sides reduces this to

$$1 + |x-y| \le (1 + |x-z|)(1 + |z-y|)?$$

and upon multiplying out the RHS and applying the triangle inequality for $\mathbb{R}$ we see that this is true, so d is a metric on $\mathbb{R}$.

(iv) No, the triangle inequality fails. If we take $x = 0$, $z = 1$ and $y = 2$ then we have $d(x,y) = 4$ while $d(x,z) + d(z,y) = 2$.

(v) This is positive, symmetric and $d(x,y) = 0$ implies $x = y$. So we only have to consider the triangle inequality. Do we have

$$|x-y|^{1/2} \le |x-z|^{1/2} + |z-y|^{1/2}?$$

Yes, since

$$|x-y| \le |x-z| + |z-y| \le (|x-z|^{1/2} + |z-y|^{1/2})^2$$

as is seen by multiplying out the last term.

6. a) On the same picture, sketch the unit balls $B(0,1)$ in $\mathbb{R}^2$ with respect to each of the metrics d_1, d_2 (i.e. the usual metric) and d_∞. Also sketch $B(0,2)$ for d_1. [$B(0,2)$ is the ball centred at 0 with radius2.]
b) Show that for $\mathbb{R}^n$ with the metrics d_1, d_2 and d_∞ we have

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \le n d_\infty(x,y).$$

What does this have to do with part a)?
c) Show that $d_1(x,y) \le \sqrt{n} d_2(x,y)$ and that $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$.

Solution: a) They are, in order, the "diamond" with vertices $(\pm1, 0), (0, \pm1)$, the usual unit disc, and the square with vertices $(\pm1, \pm1)$. Notice the containments, and that the square with vertices $(\pm1, \pm1)$ is contained in the "diamond" with vertices $(\pm2, 0), (0, \pm2)$.
b)

$$\max_j |x_j - y_j|^2 \le \sum_j |x_j - y_j|^2 \le (\sum_j |x_j - y_j|)^2$$

and

$$\sum_{j=1}^n |x_j - y_j| \le n \max_j |x_j - y_j|.$$

Part a) demonstrates visually that if $d_1(0,y) < 1$ then $d_2(0,y) < 1$ and if $d_2(0,y) < 1$ then $d_\infty(0,y) < 1$ and if $d_\infty(0,y) < 1$ then $d_1(0,y) < 2$. These represent special cases of the inequalities from part b).
c) By the Cauchy–Schwarz inequality we have

$$d_1(x,y) = \sum_{j=1}^n |x_j - y_j| \times 1 \le (\sum_{j=1}^n |x_j - y_j|^2)^{1/2} (\sum_{j=1}^n 1^2)^{1/2} = n^{1/2} d_2(x,y)$$

and

$$d_2(x,y)^2 = \sum_{j=1}^n |x_j - y_j|^2 \le n(\max_j |x_j - y_j|)^2.$$

7. a) Let $f : \mathbb{R} \to \mathbb{R}$. What conditions on f ensure that $d(x,y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$?
b) (Harder.) Let $g : [0,\infty) \to \mathbb{R}$. What conditions on g ensure that $\rho(x,y) = g(|x-y|)$ defines a metric on $\mathbb{R}$?

Solution: For f we require it to be injective in order that $d(x,y) = 0$ implies $x = y$. The only other nonobvious thing is the triangle inequality, and this is okay since

$$d(x,y) = |f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| = d(x,z) + d(z,y)$$

by the ordinary triangle inequality for $\mathbb{R}$. So all we need is injectivity of f. For g we require that it map into $[0,\infty)$ in order that the metric be nonnegative, and we require that $g(0) = 0$ and that 0 is the only value of t such that $g(t) = 0$ in order that $\rho(x,x) = 0$ and that $\rho(x,y) = 0$ implies $x = y$. Symmetry is obvious. The triangle inequality is more delicate: if g is increasing we have $\rho(x,y) = g(|x-y|) \le g(|x-z| + |z-y|)$ by the ordinary triangle inequality for $\mathbb{R}$. So if we had

$$g(p+q) \le g(p) + g(q) \text{ for all } p, q \ge 0$$

we'd then have

$$\rho(x,y) = g(|x-y|) \le g(|x-z| + |z-y|) \le g(|x-z|) + g(|z-y|) = \rho(x,z) + \rho(z,y)$$

Now if g is differentiable with continuous derivative which is nonnegative and decreasing, then we have

$$g(p+q) - g(p) = \int_0^q g'(t+p)dt \le \int_0^q g'(t)dt = g(q) - g(0) = g(q).$$

So a sufficient condition is that g map into $[0,\infty)$, $g(0) = 0$, g be strictly increasing and g' being continuous and decreasing. **Any reasonable argument is acceptable. The idea is to get you to think a bit...** Note that $g(x) = \sqrt{x}$ satisfies these conditions.

8. Let X be the set of strings of 0's and 1's of length 2^{1000}. For a pair of strings consider the two quantities

(i) the number of entries in which the two strings differ;

(ii) 2^{-j} where j is the first entry in which two strings differ (taken to be 0 if the two strings are identical);

does either define a metric on X?

Solution: Both are metrics. Positivity, symmetry and $d(x,y)=0$ implies $x=y$ are clear. Suppose x,y and z are strings and that x and z differ in j entries and that z and y differ in k entries. Then x and y can differ in at most $j+k$ entries. So we have a metric for item (i). Suppose now that the first entry where x and z differ is j and that the first entry where z and y differ is k. Then x and y have the same entries at least until the $\min\{j,k\}$'th one. So $d(x,y) \leq 2^{-\min\{j,k\}} \leq 2^{-j}+2^{-k}$.

9. Consider a graph whose vertices are the students of the University of Edinburgh and whose edges (of length 1) link each pair of students who have shaken hands. Define $d(x,y)$ to be the length of the shortest path in the graph which joins student x to student y. (Of course we set $d(x,x)=0$.) Show that d defines a metric on the set of students of the University of Edinburgh. (We assume that the University of Edinburgh is sufficiently sociable so that each pair of students is joined by a path of *some* finite length.)

Solution: The only thing to check is the triangle inequality. If x and z can be joined by a chain of acquaintances of length j and if z and y can be joined by a chain of acquaintances of length k, then mainfestly x and y can be joined by a chain of acquaintances of length at most $j+k$.

Some supplementary questions:

A. Let ℓ^1 be the set of all absolutely convergent series of real numbers, that is,

$$\ell^1 = \{(x_n)_{n=1}^\infty \;:\; x_n \in \mathbb{R},\ \sum_{n=1}^\infty |x_n| < \infty\}.$$

Show that for (x_n) and $(y_n) \in \ell^1$, $\sum_{n=1}^\infty |x_n - y_n|$ converges and that

$$d((x_n),(y_n)) = \sum_{n=1}^\infty |x_n - y_n|$$

defines a metric on ℓ^1.

B. Let (X,d) be a metric space. Show that $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$ defines a metric on X.

C. Does $\sigma(x,y) = \left(|x_1-y_1|^{1/2} + |x_2-y_2|^{1/2}\right)^2$ define a metric on $\mathbb{R}^2$?

D. Let X be a vector space with an inner product $\langle x,y\rangle$. Let

$$d(x,y) = \langle x-y, x-y\rangle^{1/2}.$$

Show that d defines a metric on X.

Workshop 7 – More on Metric Spaces

In this workshop we shall develop some of the aspects of the theory of metric spaces that we haven't had time to cover in the lectures. In particular we want to understand sensible ways of comparing two metrics on a given set, and to further understand the notion of completeness in metric spaces.

A. Comparison of metrics

In last week's workshop we considered the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ and we proved that

$$d_\infty(x,y) \leq d_2(x,y) \leq d_1(x,y) \leq n d_\infty(x,y)$$

for all $x,y \in \mathbb{R}^n$.

1. What was the picture that went along with this set of inequalities?

We say that two metrics d and ρ on a set X are **strongly equivalent** if there exist positive numbers A and B such that

$$d(x,y) \leq A\rho(x,y) \text{ and } \rho(x,y) \leq Bd(x,y) \text{ for all } x,y \in X.$$

2. Show that each pair from $\{d_1, d_2, d_\infty\}$ is a pair of strongly equivalent metrics on $\mathbb{R}^n$.

Consider the metrics d_1, d_2 and d_∞, but now on the space $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

3. Show that

$$d_1(f,g) \leq d_2(f,g) \leq d_\infty(f,g) \text{ for all } f,g \in C([0,1]).$$

Solution: By the Cauchy–Schwarz inequality we have $\int_0^1 (|f-g| \times 1) \leq \left(\int_0^1 |f-g|^2\right)^{1/2} \times \left(\int_0^1 1\right)^{1/2}$. Also, $|f(s)-g(s)|^2 \leq d_\infty(f,g)^2$ for all $0 \leq s \leq 1$ so integration gives $\int_0^1 |f(s)-g(s)|^2 \leq d_\infty(f,g)^2$.

4. Which pairs of metrics from $\{d_1, d_2, d_\infty\}$ are strongly equivalent?

Solution: No pair is strongly equivalent. For example, with $f_n(x) = x^n$ we have $d_1(f_n, 0) = 1/(n+1)$ but $d_\infty(f_n, 0) = 1$ for all n, showing that d_1 and d_∞ are not strongly equivalent.

We say that two metrics d and ρ on a set X are **equivalent** if for every $x \in X$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(x,y) < \delta \text{ implies } \rho(x,y) < \epsilon$$

and such that

$$\rho(x,y) < \delta \text{ implies } d(x,y) < \epsilon.$$

5. Show that if d and ρ are strongly equivalent then they are also equivalent.

Solution: If $\epsilon > 0$ then choosing $\delta = \epsilon/B$ works (for all x) for the first statement and $\delta = \epsilon/A$ works (for all x) for the second statement. So $\delta = \min\{\epsilon/A, \epsilon/B\}$ works for both.

6. Interpret the definition of equivalence between two metrics d and ρ on a set X in terms of balls $B_d(x,r)$ and $B_\rho(y,s)$ in the corresponding metric spaces (X,d) and (X,ρ). **Hint:** Question 1.

Solution. Fix $x \in X$. Suppose that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $\rho(x,y) < \epsilon$. Then, given $\epsilon > 0$ there exists a $\delta > 0$ such that $B_d(x,\delta) \subseteq B_\rho(x,\epsilon)$. So, inside every ρ-open ball we can inscribe a d-open ball with the same centre. And vice-versa.

7. Suppose that d and ρ are metrics on a set X. Prove the following statement: The metrics d and ρ are equivalent if and only if every subset of X which is open with respect to d is also open with respect to ρ, and every subset of X which is open with respect to ρ is also open with respect to d.

Solution. Suppose first that d and ρ are equivalent and that E is open with respect to d. Then, for every $a \in E$ there exists an $r > 0$ such that $B_d(a,r) \subseteq E$. By the previous question there exists an $s > 0$ such that $B_\rho(a,s) \subseteq B_d(a,r)$ and hence such that $B_\rho(a,s) \subseteq E$. Hence E is open with respect to ρ. By symmetry, if E is open with respect to ρ, it is also E is open with respect to d.

Now suppose that the d-open sets of X exactly coincide with the ρ-open ones. Consider a ρ-open ball of X, say $B_\rho(a,r) = \{x \in X : \rho(x,a) < r\}$. According to the hypothesis, $B_\rho(a,r)$ is also d-open, and since a belongs to it, there exists a d-open ball centred at a which is contained in $B_\rho(a,r)$. Likewise every d-open ball contains a ρ-open ball with the same centre. Question 6 now shows that d and ρ are equivalent.

8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and injective and its inverse $f^{-1} : f(\mathbb{R}) \to \mathbb{R}$ is also continuous. Show that the metric $\rho(x,y) = |f(x) - f(y)|$ is equivalent to the usual metric $d(x,y) = |x-y|$ on $\mathbb{R}$. What conditions on f are needed so that d and ρ are *strongly* equivalent?

Solution: Fix $x \in \mathbb{R}$ and let $\epsilon > 0$. We first wish to find a $\delta_1 > 0$ such that $d(x,y) = |x-y| < \delta_1$ implies that $\rho(x,y) = |f(x)-f(y)| < \epsilon$. But this is immediate since f is continuous at x. Similarly continuity of f^{-1} ensures we can find a $\delta_2 > 0$ such that $\rho(x,y) = |f(x)-f(y)| < \delta_2$ implies $d(x,y) = |x-y| < \epsilon$. Then take $\delta = \min\{\delta_1, \delta_2\}$.

In order to have strong equivalence we need that there exist positive numbers c and C such that

$$|f(x)-f(y)| \leq C|x-y| \text{ and } |f(x)-f(y)| \geq c|x-y| \text{ for all } x,y \in X.$$

This happens, for example if both f and f^{-1} have continuous derivatives which are bounded.

B. Completeness

9. Show that the metrics d_1 and d_2 on $C([0,1])$ do not give rise to complete metric spaces.

Solution: For d_1, and $n \geq 3$, take $f_n(x)$ to be 1 if $0 \leq x \leq 1/2 - 1/n$, -1 if $1/2 + 1/n \leq x \leq 1$ and to be linear and continuous on $1/2 - 1/n \leq x \leq 1/2 + 1/n$. Then f_n is continuous and for $m \geq n$, $|f_n(x) - f_m(x)| \leq 2$ and is zero outside the interval $[1/2 - 1/n, 1/2 + 1/n]$ which has length $2/n$. So

$$d_1(f_m, f_n) = \int_0^1 |f_m - f_n| \leq 2 \times 2/n = 4/n$$

which goes to zero as $n \to \infty$. So (f_n) is Cauchy. However, there is no continuous function $f : [0,1] \to \mathbb{R}$ such that $d_1(f_n, f) \to 0$ as $n \to \infty$. To see this, suppose for a contradiction that there is such an f.

Then $d_1(f_n, f) \geq \int_0^{1/2-1/n} |f_n - f| = \int_0^{1/2-1/n} |f-1|$. So on the one hand $\int_0^{1/2-1/n} |f-1|$ is an increasing sequence of nonnegative numbers and on the other hand it converges to zero. This can only mean that $\int_0^{1/2-1/n} |f-1| = 0$ for all n, forcing $|f(t)-1| = 0$ for $0 \leq t < 1/2$ since $|f(t)-1|$ is continuous and nonnegative. Therefore $f(t) = 1$ for $0 \leq t < 1/2$ and similarly $f(t) = -1$ for $1/2 < t \leq 1$. But then f cannot be continuous.

10. Let (X,d) be a discrete metric space. Show it is complete.

Solution: Suppose (x_n) is Cauchy in (X,d). Take $\epsilon = 1$. Then there exists some N such that for $m,n \geq N$ we have $d(x_m, x_n) < 1$. But this means that for $m,n \geq N$ we have that $x_m = x_n$, i.e. that x_n is constant for $n \geq N$. Hence the sequence (x_n) converges, and so (X,d) is complete.

Workshop 10 – Contraction mappings

The purpose of this workshop is to give some practice in problems related to contraction mappings and the contraction mapping theorem.

1. (You may **not** use a calculator for this question.) Prove that the function $f(x) = 2 + x^{-2}$ is a contraction mapping of $[2,\infty)$ into itself. Deduce that the equation

$$x^3 - 2x^2 - 1 = 0$$

has a unique real solution a. Describe how this solution may be obtained by an iterative procedure, giving a bound for the error at the n'th stage in terms of the inital point of the iteration. If the initial guess is $x_0 = 2$, what is the first value of n for which the iteration guarantees that x_n approximates a to 3 decimal places?

Solution: For $x \geq 2$ we have $f(x) \geq 2$. Moreover $f'(x) = -2/x^3$ so that on $[2, \infty)$, $|f'(x)| \leq 1/4$, from which it follows that f is a contraction with constant $\alpha = 1/4$. Since $[2, \infty)$ is complete, there is a unique fixed point in $[2, \infty)$. Now for $x \geq 2$, $f(x) = x$ iff $2 + x^{-2} = x$ iff $x^3 - 2x^2 - 1 = 0$. So we deduce that $x^3 - 2x^2 - 1 = 0$ has a unique real solution in $[2, \infty)$. Since the derivative of $x^3 - 2x^2 - 1$ is $3x^2 - 4x = x(3x - 4)$ we have a local maximum value of -2 at 0 and a local minimum at $4/3$ meaning that $x^3 - 2x^2 - 1$ has only the single real zero.

Let's set $x_0 = 2$, so that $x_1 = f(x_0) = 2 + 1/4 = 9/4$, $x_2 = f(x_1) = 2 + 16/81$ etc. We know that if a is the unique fixed point of f, then $|a - x_n| \leq \alpha^n/(1-\alpha)|x_1 - x_0| = 4^{-n} \times 4/3 \times 1/4 = \frac{1}{3.4^n}$. So if we want a to 3 decimal places this means we want $\frac{1}{3.4^n} < 1/10000$, i.e. $4^n > 3334$. Clearly $n = 6$ is good enough but $n = 5$ isn't. So x_6 does the job.

2. Show that there is a unique continuous function $\phi : [0, 1] \to \mathbb{R}$ such that

$$\phi(t) = t + \int_0^1 e^{-s}\phi(st)\mathrm{d}s.$$

(**Hint:** Find a suitable complete metric space X and a suitable contraction $F : X \to X$ such that the fixed points of F correspond precisely to solutions of the displayed equation. You may assume that if ϕ is continuous on $[0, 1]$, so is $\int_0^1 e^{-s}\phi(st)\mathrm{d}s$.)

Solution: Let $X = C([0, 1])$ with the d_∞ metric (which is complete) and let $F(\phi)(t) = t + \int_0^1 e^{-s}\phi(st)\mathrm{d}s$ which is a continuous function of t for $\phi \in X$.[1] Moreover

$$F(\phi)(t) - F(\psi)(t) = \int_0^1 e^{-s}[\phi(st) - \psi(st)]\mathrm{d}s$$

so that

$$|F(\phi)(t) - F(\psi)(t)| = \int_0^1 e^{-s}|\phi(st) - \psi(st)|\mathrm{d}s \leq \int_0^1 e^{-s}\mathrm{d}s \;\; d_\infty(\phi, \psi)$$

and hence

$$d_\infty(F(\phi), F(\psi)) \leq (1 - e^{-1})d_\infty(\phi, \psi).$$

Since $(1 - e^{-1}) < 1$ we have that F is a contraction and so there is a unique fixed point in X.

3. Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Show that there is a unique continuous function $x : [0, 1] \to \mathbb{R}$ such that

$$x(t) = \int_0^t K(t, s)x(s)\mathrm{d}s \text{ for } 0 \leq t \leq 1$$

by following the following steps:

(i) Let $Tx(t) = \int_0^t K(t, s)x(s)\mathrm{d}s$ and suppose that $\sup_{0\leq t,s\leq 1} |K(t, s)| = M$. Assuming that Tx is a continuous function for x continuous, and letting $T^{(n)}$ be the n-fold composition of T with itself, show by induction that for $n \in \mathbb{N}$ and $0 \leq t \leq 1$ we have

$$|T^{(n)}x(t) - T^{(n)}y(t)| \leq \frac{M^n t^n}{n!} \sup_{0\leq s\leq 1} |x(s) - y(s)|.$$

(ii) Show that for n sufficiently large, $T^{(n)}$ is a contraction on $C([0, 1])$ with the uniform metric.

[1] Indeed, continuity of ϕ implies ϕ is uniformly continuous on $[0, 1]$ so that for all $\epsilon > 0$ there is a $\delta > 0$ such that $\delta < \epsilon$ and $|x - y| < \delta$ implies $|\phi(x) - \phi(y)| < \epsilon$. So for $|t - t'| < \delta$ and $0 \leq s \leq 1$ we have $|\phi(st) - \phi(st')| < \epsilon$. Therefore, for $|t - t'| < \delta$,

$$|F(\phi)(t) - F(\phi)(t')| \leq |t - t'| + \int_0^1 e^{-s}|\phi(st) - \phi(st')|\mathrm{d}s < \epsilon + \epsilon\int_0^1 e^{-s}\mathrm{d}s = \epsilon(2 - e^{-1}).$$

(iii) Apply a previously established result.

4. Consider the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2tx, \qquad x(0) = 1.$$

Let $x_0(t) = 1$ and

$$x_n(t) = 1 + \int_0^t 2sx_{n-1}(s)\mathrm{d}s.$$

Find a formula for x_1, x_2 and x_3, and then, by induction, for x_n. Find $\lim_{n\to\infty} x_n(t)$ and show that this agrees with the solution of the ordinary differential equation obtained by separation of variables. Why is this not surprising?

Solution:

$$x_1(t) = 1 + \int_0^t 2s\mathrm{d}s = 1 + 2t^2/2;$$

$$x_2(t) = 1 + \int_0^t 2sx_1(s)\mathrm{d}s = 1 + \int_0^t 2s(1 + 2s^2/2)\mathrm{d}s = 1 + 2t^2/2 + \frac{2^2t^4}{4.2}.$$

Similarly

$$x_3(t) = 1 + \int_0^t 2sx_2(s)\mathrm{d}s = 1 + \int_0^t 2s(1 + 2s^2/2 + 2^2s^4/4.2)\mathrm{d}s = 1 + t^2 + \frac{2t^4}{4} + \frac{2^3t^6}{6.4.2}.$$

This suggests that

$$x_n(t) = 1 + \sum_{j=1}^{n} \frac{2^jt^{2j}}{2j(2j-2)\ldots 1} = \sum_{j=0}^{n} \frac{t^{2j}}{j!}.$$

which is true for $n = 0, 1$. Assume it holds for n. Then

$$x_{n+1}(t) = 1 + \int_0^t 2s\left(\sum_{j=0}^{n} \frac{s^{2j}}{j!}\right)\mathrm{d}s = 1 + \sum_{j=0}^{n} \frac{1}{j!}\int_0^t 2ss^{2j}\mathrm{d}s$$

$$= 1 + \sum_{j=0}^{n} \frac{1}{j!}\frac{2t^{2j+2}}{2j+2} = 1 + \sum_{j=0}^{n} \frac{1}{(j+1)!}t^{2j+2} = 1 + \sum_{j=1}^{n+1} \frac{1}{j!}t^{2j} = \sum_{j=0}^{n+1} \frac{1}{j!}t^{2j},$$

establishing the inductive step. So

$$x_n(t) = \sum_{j=0}^{n} \frac{t^{2j}}{j!}$$

which has limit e^{t^2} as $n \to \infty$.

Returning to the ODE, separation of variables gives $\frac{\mathrm{d}x}{x} = 2t\mathrm{d}t$, and integration gives $\log|x| = t^2 + c$ and fitting the initial condition we see that $\log x = t^2$ and so $x = e^{t^2}$.

Let $F(x)(t) = 1 + \int_0^t 2sx(s)\mathrm{d}s$. A fixed point x of F is a solution to $x(t) = 1 + \int_0^t 2sx(s)\mathrm{d}s$; and differentiating we see that this corresponds to $\frac{\mathrm{d}x}{\mathrm{d}t} = 2tx$ and $x(0) = 1$. The iterative process $x_0 = 1$ and $x_n = F(x_{n-1})$ is guaranteed to find the fixed point of F in any complete metric space for which F is a contraction. Such a space is $C([-A, A])$ for any $A < 1$; to see that F is a contraction we have to look at $F(x(t)) - F(y(t)) = \int_0^t 2s[x(s) - y(s)]\mathrm{d}s$ for $|t| \leq A$ and observe that

$$|F(x(t)) - F(y(t))| \leq \int_0^t 2s\mathrm{d}sd_\infty(x, y) = t^2d_\infty(x, y) \leq A^2d_\infty(x, y)$$

so the contraction constant is $A^2 < 1$. (This does not explain why x_n seeks out the solution to the ODE for *all* time.)

5. Consider the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (e^x - 1)\cos(x^3 - [x^2 + t^2 + 1]^{-1}) \text{ with } x(0) = 0.$$

Find the unique solution to this equation near $t = 0$. (**Hint:** Don't look too far.)

Solution: The RHS satisfies the conditions of the existence and uniquess theorem. Observing that $x(t) \equiv 0$ *is* a solution, it is therefore the *unique* one near $t = 0$.

6. Consider the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^{1/3} \text{ with } x(0) = 0.$$

Find a solution to this equation. Find another. Why does this not contradict the theorem about existence and uniqueness of solutions to ordinary differential equations? Can you find all the solutions to this equation? (**Beware:** Using MAPLE on this problem will likely lead to a nonsensical answer.)

Solution: $x(t) \equiv 0$ is a solution, as is $x(t) = \left(\frac{2}{3}|t|\right)^{3/2}$ as is $x(t) = 0$ for $t \leq A$ and $x(t) = \left(\frac{2}{3}(t - A)\right)^{3/2}$ for $t \geq A$, for any $A \in \mathbb{R}$. This does not contradict the theorem because the function $x \mapsto x^{1/3}$ is not Lipschitz near 0.

7. Consider the integral equation

$$x(s) = s^2 + s\int_0^s x(t)\mathrm{d}t \text{ for } 0 \leq s \leq 1.$$

For $x \in C([0, 1])$ let

$$Tx(s) = s\int_0^s x(t)\mathrm{d}t.$$

Let $y(s) = s^2$. Calculate Ty, T^2y, T^3y, etc., and show that the series

$$y + Ty + T^2y + \ldots$$

converges uniformly on $[0, 1]$ to a solution of the integral equation. Why is this not surprising?

8. Let (X, d) be a *compact* metric space, and suppose $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for all $x \neq y \in X$. Show that f has a unique fixed point. (**Hint:** Consider the function $\phi(x) = d(x, f(x))$.)

Solution: The map $\phi : X \to \mathbb{R}$ is continuous and so its image is a closed and bounded subset of $\mathbb{R}$ since X is compact. If f has no fixed point, then ϕ does not take the value 0, and so the infimum of its values is strictly positive, say $k > 0$, and there exists an $x \in X$ such that $d(x, f(x)) = k$. But now, with $y = f(x)$ we have $d(y, f(y)) = d(f(x), f(f(x))) < d(x, f(x)) = k$. This contradicts the definition of k as the infimum of the values of ϕ. So f must have a fixed point. Uniqueness was covered in lectures.

Workshop 11 – Connected sets

The purpose of this workshop is to further discuss the notions of a connected and a path-connected set.

Recall that a set $E \subset X$ is *connected* if there do not exist open sets $U, V \subset X$ satisfying simultaneously these properties:

$$E \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap E \neq \emptyset, \quad V \cap E \neq \emptyset.$$

Recall that a set $E \subset \mathbb{R}^n$ is *path connected* if for any two points $x, y \in E$ there exists

$$\gamma : [0,1] \to E \text{ continuous such that } \gamma(0) = x, \quad \gamma(1) = y.$$

1. We have established that every path-connected set is connected. Is the reverse statement also true? As a hint, consider the following set $E \subset \mathbb{R}^2$.

$$E = \{(x,0) : x \in [0,1]\} \cup \{(0,1)\} \cup \bigcup_{n\in\mathbb{N}} \{(1/n, y) : y \in [0,1]\}.$$

(The set looks like an upside down "comb" with teeth that accumulate near y-axis.)

(i) Is E connected? Suppose it is not and consider that the point $(0,1)$ must belong to one of the opens sets U or V.

(ii) Is E path-connected? Focus again on the point $(0,1)$ and its path-connectivness with the rest of the set.

Solution:

(i) Suppose U, V are open sets such that

$$E \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap E \neq \emptyset.$$

Since the point $(0,1)$ belongs to one of these sets, we may assume that $(0,1) \in U$. Since U is open there exists $r > 0$ such that $B((0,1), r) \subset U$.

The ball $B((0,1), r)$ will therefore contain points $(1/n, 1)$ for n large. Let's fix one such point. It follows that $(1/n, 1) \in U$.

We claim that any point $y \in E$ that is path-connected with $(1/n, 1)$ must also lie in U. If this wasn't the case, i.e., $y \in V$ then looking at $\gamma^{-1}(U), \gamma^{-1}(V)$ we see that these are disjoint, open in $[0,1]$ and cover $[0,1]$ which would imply that the interval $[0,1]$ is not connected which is false. Therefore $y \in U$.

As all points of E except $(0,1)$ are path-connected with $(1/n, 1)$ and we already know that $(0,1), (1/n, 1) \in U$ we can conclude that $E \subset U$ and therefore $V \cap E = \emptyset$. Hence E is a connected set.

(ii) We have already observed in (i) that all points of E except $(0,1)$ are path-connected with each other. Let us make an argument that $(0,1)$ is not path-connected with a point $(0,0)$. By contradiction assume that there is

$$\gamma : [0,1] \to E \text{ continuous such that } \gamma(0) = (0,1), \quad \gamma(1) = (0,0).$$

Since $\gamma : [0,1] \to \mathbb{R}^2$ we may think of γ as (γ_1, γ_2) where these are functions $[0,1] \to \mathbb{R}$ (the first and second coordinate, respectively). We claim that there must be a point $t \in [0,1]$ such that $\gamma_1(t) > 0$. Indeed, if $\gamma_1(t) = 0$ everywhere then γ_2 must be constant by the intermediate value theorem since the points $(0,s)$ do not belong to E unless $s = 0$ or $s = 1$.

Using continuity of γ it therefore follows that there exists $t_0 > 0$ such that

$$\gamma(t) \in B((0,1), 1/2) \text{ for all } 0 \le t \le t_0 \text{ and } \gamma_1(t_0) > 0.$$

The condition $\gamma(t) \in B((0,1), 1/2)$ implies that $\gamma_2(t) > 0$ and hence

$$\gamma_1(t) \in \{1/n : n \in \mathbb{N}\} \cup \{0\}, \quad \text{for } 0 \le t \le t_0.$$

This is impossible as γ_1 is continuous, $\gamma_1(0) = 0$ and hence by the intermediate value theorem γ_1 must attain all values in the interval $[0, \gamma_1(t_0)]$.

Hence we have a contradiction and therefore the points $(0,1)$ and $(0,0)$ are not path-connected.

We aim to show below that every *open, connected* set in $\mathbb{R}^n$ is path-connected (this can be generalized to vector spaces like $C[a,b]$). In the exercises below $E \subset \mathbb{R}^n$ will always be open and connected.

2. Choose and arbitrary point $x \in E$. Define the open sets U, V as follows:

$$U = \{y \in E : \text{there exists continuous path inside } E \text{ connecting } x \text{ and } y.\},$$

$$V = \{y \in E : \text{there does not exist continuous path inside } E \text{ connecting } x \text{ an}$$

Show that

$$E \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap E \neq \emptyset.$$

Solution: $U \cap E \neq \emptyset$ since $x \in U$. Also clearly, U, V are disjoint as the condition for V is the negation of the condition for U. This also implies that every point $y \in E$ satisfies one of the conditions that defines U or V and therefore $E = U \cup V$.

3. Show that any two points in U are *path-connected.*

Solution: Let $y, z \in U$. As x is path-connected with both y and z once can write a formula for a path joining y and z through the point x. Hence the claim holds.

4. Show that the sets U, V are open. Hint consider $y \in U$ and show that there exists $r > 0$ such that all points inside $B(y, r)$ are path-connected with y by a path that lies inside E. Repeat a similar argument for V.

Solution: Let $y \in U$. Since E is open there exists $r > 0$ such that $B(y, r) \subset E$. Consider any $z \in B(y, r)$. We claim that x and z are path-connected. Indeed, we know that x and y are path-connected, i.e., there is $\gamma : [0,1] \to E$ continuous such that $\gamma(0) = x$ and $\gamma(1) = y$. Consider now $\widetilde{\gamma} : [0,1] \to E$:

$$\widetilde{\gamma}(t) = \begin{cases} \gamma(2t), & \text{for } 0 \le t \le 1/2, \\ 2y(1-t) + 2z(t - 1/2), & \text{for } 1/2 < t \le 1. \end{cases}$$

This is clearly a continuous path joining x and z. Therefore $z \in U$.

The argument for the set V is similar. Once again for $y \in V$ we find $r > 0$ such that $B(y, r) \subset E$. We claim that $z \in V$. Indeed, if $z \in U$ then x and z are path-connected. Since y and z are path-connected (by a straight line segment from y to z) it would follow that x and y must be path-connected via a path through z. But that contradicts the fact that $y \in V$. Therefore $z \in V$.

5. Conclude that the set V is empty and make the conclusion on path-connectivness of E.

Solution: If $V \neq \emptyset$ then the sets U, V we have defined satisfy:

$$E \subset U \cup V, \quad U \cap V = \emptyset, \quad U \cap E \neq \emptyset, \quad V \cap E \neq \emptyset$$

and both U, V are open. Hence E is not connected which contradicts out assumption we have made on E. Therefore $V = \emptyset$. It follows that $E = U$ and therefore by the result of 3) any two points of E are path-connected.

6. In this question we consider the metric space $\mathbb{R}^2$ equipped with the usual metric.

In each case below state whether the subset $E \subseteq \mathbb{R}^2$ is (i) open (ii) closed (iii) compact and (iv) connected? Give brief reasons.

1. $E = \{(x,y) : x^2 + y^2 = 2\}$
2. $E = \{(x,y) : x^2 - y^2 = 2\}$
3. $E = \{(x,y) : x^2 + y^2 \le 2\}$
4. $E = \{(x,y) : x^2 - y^2 \ge 2\}$
4. $E = \{(x,y) : x^2 - y^2 \ge 2\}$
5. $E = \mathbb{R}^2$
6. $E = \{(x,y) : (x-1)^2 + y^2 \le 1\} \cup \{(x,y) : (x+1)^2 + y^2 < 1\}$
7. $E = \mathbb{Q}^2$
8. $E = \{0\} \times \mathbb{Q}$.

Solution:

1. $E = \{(x,y) : x^2 + y^2 = 2\}$ – not open; closed. Compact since closed and bounded; connected since path-connected.
2. $E = \{(x,y) : x^2 - y^2 = 2\}$ – not open; closed. Not compact since unbounded; not connected since $\{(x,y) : x > 0\}$ and $\{(x,y) : x < 0\}$ provide a disconnection.
3. $E = \{(x,y) : x^2 + y^2 \le 2\}$ – not open; closed. Compact since closed and bounded; connected since path-connected.
4. $E = \{(x,y) : x^2 - y^2 \ge 2\}$ – not open; closed. Not compact since unbounded; not connected since$\{(x,y) : x > 0\}$ and $\{(x,y) : x < 0\}$ provide a disconnection.
5. $E = \mathbb{R}^2$ – open and closed. Not compact since unbounded; connected since path-connected.
6. $E = \{(x,y) : (x-1)^2 + y^2 \le 1\} \cup \{(x,y) : (x+1)^2 + y^2 < 1\}$ – the union of a closed unit disc centred at $(1,0)$ and an open unit disc centred at $(-1,0)$ – not open; not closed. Not compact since not closed; connected since path-connected (the point $(0,0) \in E$ providing the gateway from the left to the right half-planes).
7. $E = \mathbb{Q}^2$ – not open; not closed. Not compact since not closed (nor bounded); not connected since $\{(x,y) : x > \sqrt{2}\}$ and $\{(x,y) : x > \sqrt{2}\}$ provide a disconnection.
8. $E = \{0\} \times \mathbb{Q}$ – not open; not closed. Not compact since not closed (nor bounded); not connected since $\{(x,y) : y > \sqrt{2}\}$ and $\{(x,y) : y > \sqrt{2}\}$ provide a disconnection.

07.02

Proof. By Definition 7.1, $f_n \to f$ pointwise on E if and only if $f_n(x) \to f(x)$ for all $x \in E$. This occurs, by Definition 2.1, if and only if for every $\varepsilon > 0$ and $x \in E$ there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$. ■

07.03a

Proof. Let $f_n(x) = x^n$ and set

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

07.03b

Then $f_n \to f$ pointwise on $[0, 1]$ (see Example 2.20), each f_n is continuous and differentiable on $[0, 1]$, but f is neither differentiable nor continuous at $x = 1$. ■

07.04

Proof. Set

$$f_n(x) = \begin{cases} 1 & x = p/m \in \mathbf{Q},\ \text{written in reduced form, where } m \leq n \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbf{N}$ and

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is integrable on $[0, 1]$ (with integral zero), but f is not integrable on $[0, 1]$ (see Example 5.11). ■

07.05

Proof. Let $f_n(x) = x^n/n$ and set $f(x) = 0$. Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is differentiable with $f_n'(x) = x^{n-1}$. Thus the left side of (1) is 1 at $x = 1$ but the right side of (1) is zero. ■

07.06

Proof. Let $f_1(x) = 1$ and, for $n > 1$, let f_n be a sequence of functions whose graphs are triangles with bases $2/n$ and altitudes n (see Figure 7.1). By the point-slope form, formulas for these f_n's can be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x < 1/n \\ 2n - n^2 x & 1/n \leq x < 2/n \\ 0 & 2/n \leq x \leq 1. \end{cases}$$

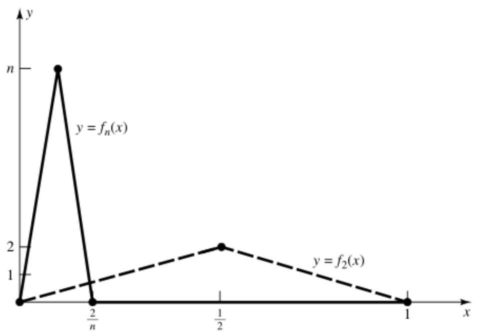

FIGURE 7.1

Then $f_n \to 0$ pointwise on $[0, 1]$ and, since the area of a triangle is one-half base times altitude, $\int_0^1 f_n(x)\,dx = 1$ for all $n \in \mathbf{N}$. Thus, the left side of (2) is 1 but the right side is zero. ■

07.09

Proof. Let $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{and} \quad x \in E \quad \text{imply} \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

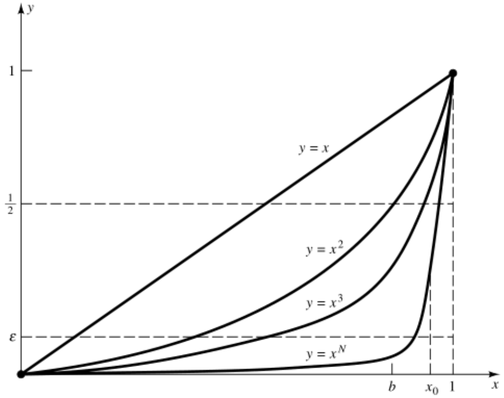

FIGURE 7.3

Since f_N is continuous at $x_0 \in E$, choose $\delta > 0$ such that

$$|x - x_0| < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

Suppose that $|x - x_0| < \delta$ and that $x \in E$. Then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon.$$

Thus f is continuous at $x_0 \in E$. ■

07.10

Proof. By Exercise 7.1.3, f is bounded on $[a, b]$. To prove that f is integrable, let $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad |f(x) - f_n(x)| < \frac{\varepsilon}{3(b-a)} \tag{3}$$

for all $x \in [a, b]$. Using this inequality for $n = N$, we see that by the definition of upper and lower sums,

$$U(f - f_N, P) \leq \frac{\varepsilon}{3} \quad \text{and} \quad L(f - f_N, P) \geq -\frac{\varepsilon}{3}$$

for any partition P of $[a, b]$. Since f_N is integrable, choose a partition P such that

$$U(f_N, P) - L(f_N, P) < \frac{\varepsilon}{3}.$$

It follows that

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f - f_N, P) + U(f_N, P) - L(f_N, P) - L(f - f_N, P) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon; \end{aligned}$$

that is, f is integrable on $[a, b]$. We conclude by Theorem 5.22 and (3) that

$$\left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right| \leq \int_a^x |f_n(t) - f(t)|\,dt \leq \frac{\varepsilon(x-a)}{3(b-a)} < \varepsilon$$

for all $x \in [a, b]$ and $n \geq N$. ■

07.11

Proof. Suppose first that $f_n \to f$ uniformly on E as $n \to \infty$. Let $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

for $x \in E$. Since $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$, it is clear that (4) holds for all $x \in E$.

Conversely, if (4) holds for $x \in E$, then $\{f_n(x)\}_{n \in \mathbf{N}}$ is Cauchy for each $x \in E$. Hence, by Cauchy's Theorem for sequences (Theorem 2.29),

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exists for each $x \in E$. Take the limit of the second inequality in (4) as $m \to \infty$. We obtain $|f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon$ for all $n \geq N$ and $x \in E$. Hence, by definition, $f_n \to f$ uniformly on E. ■

07.12

Proof. Fix $c \in (a, b)$ and define

$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} & x \neq c \\ f_n'(c) & x = c \end{cases}$$

for $n \in \mathbf{N}$. Clearly,

$$f_n(x) = f_n(c) + (x - c) g_n(x) \tag{5}$$

for $n \in \mathbf{N}$ and $x \in (a, b)$.

We claim that for any $c \in (a, b)$, the sequence g_n converges uniformly on (a, b). Let $\varepsilon > 0$, $n, m \in \mathbf{N}$, and $x \in (a, b)$ with $x \neq c$. By the Mean Value Theorem, there is a ξ between x and c such that

$$g_n(x) - g_m(x) = \frac{f_n(x) - f_m(x) - (f_n(c) - f_m(c))}{x - c} = f_n'(\xi) - f_m'(\xi).$$

Since f_n' converges uniformly on (a, b), it follows that there is an $N \in \mathbf{N}$ such that

$$n, m \geq N \quad \text{implies} \quad |g_n(x) - g_m(x)| < \varepsilon$$

for $x \in (a, b)$ with $x \neq c$. This implication also holds for $x = c$ because $g_n(c) = f_n'(c)$ for all $n \in \mathbf{N}$. This proves the claim.

To show that f_n converges uniformly on (a, b), notice that by the claim, g_n converges uniformly as $n \to \infty$ and (5) holds for $c = x_0$. Since $f_n(x_0)$ converges as $n \to \infty$ by hypothesis, it follows from (5) and $b - a < \infty$ that f_n converges uniformly on (a, b) as $n \to \infty$.

Fix $c \in (a, b)$. Define f, g on (a, b) by $f(x) := \lim_{n \to \infty} f_n(x)$ and $g(x) := \lim_{n \to \infty} g_n(x)$. We need to show that

$$f'(c) = \lim_{n \to \infty} f_n'(c). \tag{6}$$

Since each g_n is continuous at c, the claim implies g is continuous at c. Since $g_n(c) = f_n'(c)$, it follows that the right side of (6) can be written as

$$\lim_{n \to \infty} f_n'(c) = \lim_{n \to \infty} g_n(c) = g(c) = \lim_{x \to c} g(x).$$

On the other hand, if $x \neq c$ we have by definition that

$$\frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} = \lim_{n \to \infty} g_n(x) = g(x).$$

Therefore, the left side of (6) also reduces to

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} g(x).$$

This verifies (6), and the proof of the theorem is complete. ■

07.15

Proof. Let $\varepsilon > 0$ and use the Cauchy Criterion to choose $N \in \mathbf{N}$ such that $m \ge n \ge N$ implies $\sum_{k=n}^{m} M_k < \varepsilon$. Thus, by hypothesis,

$$\left|\sum_{k=n}^{m} f_k(x)\right| \le \sum_{k=n}^{m} |f_k(x)| \le \sum_{k=n}^{m} M_k < \varepsilon$$

for $m \ge n \ge N$ and $x \in E$. Hence, the partial sums of $\sum_{k=1}^{\infty} f_k$ are uniformly Cauchy and the partial sums of $\sum_{k=1}^{\infty} |f_k(x)|$ are Cauchy for each $x \in E$. ■

10.09

Proof. Let $B_r(a)$ be an open ball. By definition, we must prove that given $x \in B_r(a)$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq B_r(a)$. Let $x \in B_r(a)$ and set $\varepsilon = r - \rho(x,a)$. (Look at Figure 8.5 to see why this choice of ε should work.) If $y \in B_\varepsilon(x)$, then by the Triangle Inequality, assumption, and the choice of ε,

$$\rho(y,a) \le \rho(y,x) + \rho(x,a) < \varepsilon + \rho(x,a) = r.$$

Thus, by Definition 10.7, $y \in B_r(a)$. In particular, $B_\varepsilon(x) \subseteq B_r(a)$. Similarly, we can show that $\{x \in X : \rho(x,a) > r\}$ is also open. Hence, every closed ball is closed. ■

10.10

Proof. By Definition 10.8, it suffices to prove that the complement of every *singleton* $E := \{a\}$ is open. Let $x \in E^c$ and set $\varepsilon = \rho(x,a)$. Then, by Definition 10.7, $a \notin B_\varepsilon(x)$, so $B_\varepsilon(x) \subseteq E^c$. Therefore, E^c is open by Definition 10.8. ■

10.11

Proof. Since $X = \emptyset^c$ and $\emptyset = X^c$, it suffices by Definition 10.8 to prove that $\emptyset$ and X are both open. Because the empty set contains no points, "every" point $x \in \emptyset$ satisfies $B_\varepsilon(x) \subseteq \emptyset$. (This is called the *vacuous implication*.) Therefore, $\emptyset$ is open. On the other hand, since $B_\varepsilon(x) \subseteq X$ for all $x \in X$ and all $\varepsilon > 0$, it is clear that X is open. ■

10.15

Proof. Suppose that $x_n \to a$, and let V be an open set which contains a. By Definition 10.8, there is an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq V$. Given this ε, use Definition 10.13 to choose an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in B_\varepsilon(a)$. By the choice of ε, $x_n \in V$ for all $n \ge N$.

Conversely, let $\varepsilon > 0$ and set $V = B_\varepsilon(a)$. Then V is an open set which contains a; hence, by hypothesis, there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in V$. In particular, $\rho(x_n, a) < \varepsilon$ for all $n \ge N$. ■

10.16

Proof. The theorem is vacuously satisfied if E is the empty set.

Suppose that $E \ne \emptyset$ is closed but some sequence $x_n \in E$ converges to a point $x \in E^c$. Since E is closed, E^c is open. Thus, by Remark 10.15, there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in E^c$, a contradiction.

Conversely, suppose that E is a nonempty set such that every convergent sequence in E has its limit in E. If E is not closed, then, by Remark 10.11, $E \ne X$, and, by definition, E^c is nonempty and not open. Thus, there is at least one point $x \in E^c$ such that no ball $B_r(x)$ is contained in E^c. Let $x_k \in B_{1/k}(x) \cap E$ for $k = 1, 2, \ldots$. Then $x_k \in E$ and $\rho(x_k, x) < 1/k$ for all $k \in \mathbf{N}$. Now $1/k \to 0$ as $k \to \infty$, so it follows from the Squeeze Theorem (these are real sequences) that $\rho(x_k, x) \to 0$ as $k \to \infty$ (i.e., $x_k \to x$ as $k \to \infty$). Thus, by hypothesis, $x \in E$, a contradiction. ■

10.17

Proof. Let $X = \mathbf{R}$ be the discrete metric space introduced in Example 10.3. Since $\sigma(0,k) = 1$ for all $k \in \mathbf{N}$, $\{k\}$ is a bounded sequence in X. Suppose that there exist integers $k_1 < k_2 < \ldots$ and an $x \in X$ such that $k_j \to x$ as $j \to \infty$. Then there is an $N \in \mathbf{N}$ such that $\sigma(k_j, x) < 1$ for $j \ge N$ (i.e., $k_j = x$ for all $j \ge N$). This contradiction proves that $\{k\}$ has no convergent subsequences. ■

10.18

Proof. Choose (by the Density of Rationals) points $q_k \in \mathbf{Q}$ such that $q_k \to \sqrt{2}$. Then $\{q_k\}$ is Cauchy (by Theorem 10.14iv) but does not converge in X since $\sqrt{2} \notin X$. ■

10.21

Proof. Suppose that E is complete and that $x_n \in E$ converges. By Theorem 10.14iv, $\{x_n\}$ is Cauchy. Since E is complete, it follows from Definition 10.19 that the limit of $\{x_n\}$ belongs to E. Thus, by Theorem 10.16, E is closed.

Conversely, suppose that E is closed and that $x_n \in E$ is Cauchy in E. Since the metrics on X and E are identical, $\{x_n\}$ is Cauchy in X. Since X is complete, it follows that $x_n \to x$, as $n \to \infty$, for some $x \in X$. But E is closed, so x must belong to E. Thus E is complete by definition. ■

10.31

Proof. i) Let $x \in \bigcup_{\alpha \in A} V_\alpha$. Then $x \in V_\alpha$ for some $\alpha \in A$. Since V_α is open, it follows that there is an $r > 0$ such that $B_r(x) \subseteq V_\alpha$. Thus $B_r(x) \subseteq \bigcup_{\alpha \in A} V_\alpha$ (i.e., this union is open).

ii) Let $x \in \bigcap_{k=1}^{n} V_k$. Then $x \in V_k$ for $k = 1, 2, \ldots, n$. Since each V_k is open, it follows that there are numbers $r_k > 0$ such that $B_{r_k}(x) \subseteq V_k$. Let $r = \min\{r_1, \ldots, r_n\}$. Then $r > 0$ and $B_r(x) \subseteq V_k$ for all $k = 1, 2, \ldots, n$; that is, $B_r(x) \subseteq \bigcap_{k=1}^{n} V_k$. Hence, this intersection is open.

iii) By DeMorgan's Law (Theorem 1.36) and part i),

$$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$$

is open, so $\bigcap_{\alpha \in A} E_\alpha$ is closed.

iv) By DeMorgan's Law and part ii),

$$\left(\bigcup_{k=1}^{n} E_k\right)^c = \bigcap_{k=1}^{n} E_k^c$$

is open, so $\bigcup_{k=1}^{n} E_k$ is closed.

v) Since $V \setminus E = V \cap E^c$ and $E \setminus V = E \cap V^c$, the former is open by part ii), and the latter is closed by part iii). ■

10.32

Proof. In the metric space $X = \mathbf{R}$,

$$\bigcap_{k \in \mathbf{N}} \left(-\frac{1}{k}, \frac{1}{k}\right) = \{0\}$$

is closed and

$$\bigcup_{k \in \mathbf{N}} \left[\frac{1}{k+1}, \frac{k}{k+1}\right] = (0,1)$$

is open. ■

10.34

Proof. Since every open set V in the union defining E^o is a subset of E, it is clear that the union of these V's is a subset of E. Thus $E^o \subseteq E$. A similar argument establishes $E \subseteq \overline{E}$. This proves i).

By Definition 10.33, if V is an open subset of E, then $V \subseteq E^o$ and if C is a closed set containing E, then $\overline{E} \subseteq C$. This proves ii) and iii). ■

10.39

Proof. By Definition 10.37, it suffices to show

$$x \in \overline{E} \text{ if and only if } B_r(x) \cap E \ne \emptyset \text{ for all } r > 0, \text{ and} \qquad (2)$$

$$x \notin E^o \text{ if and only if } B_r(x) \cap E^c \ne \emptyset \text{ for all } r > 0. \qquad (3)$$

We will provide the details for (2) and leave the proof of (3) as an exercise. Suppose that $x \in \overline{E}$ but $B_{r_0}(x) \cap E = \emptyset$ for some $r_0 > 0$. Then $(B_{r_0}(x))^c$ is a closed set which contains E; hence, by Theorem 10.34iii, $\overline{E} \subseteq (B_{r_0}(x))^c$. It follows that $\overline{E} \cap B_{r_0}(x) = \emptyset$ (e.g., $x \notin \overline{E}$, a contradiction).

Conversely, suppose that $x \notin \overline{E}$. Since $(\overline{E})^c$ is open, there is an $r_0 > 0$ such that $B_{r_0}(x) \subseteq (\overline{E})^c$. In particular, $\emptyset = B_{r_0}(x) \cap \overline{E} \supseteq B_{r_0}(x) \cap E$ for some $r_0 > 0$. ■

10.40

Proof. i) Since the union of two open sets is open, $A^o \cup B^o$ is an open subset of $A \cup B$. Hence, by Theorem 10.34ii, $A^o \cup B^o \subseteq (A \cup B)^o$.

Similarly, $(A \cap B)^o \supseteq A^o \cap B^o$. On the other hand, if $V \subset A \cap B$, then $V \subset A$ and $V \subset B$. Thus, $(A \cap B)^o \subseteq A^o \cap B^o$.

ii) Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, it is clear that by Theorem 10.34iii), $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Similarly, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. To prove the reverse inequality for union, suppose that $x \notin \overline{A \cup B}$. Then there is a closed set E which contains $A \cup B$ such that $x \notin E$. Since E contains both A and B, it follows that $x \notin \overline{A}$ and $x \notin \overline{B}$. This proves part ii).

iii) Let $x \in \partial(A \cup B)$; that is, suppose that $B_r(x)$ intersects both $A \cup B$ and $(A \cup B)^c$ for all $r > 0$. Since $(A \cup B)^c = A^c \cap B^c$, it follows that $B_r(x)$ intersects both A^c and B^c for all $r > 0$. Thus, $B_r(x)$ intersects A and A^c for all $r > 0$, or $B_r(x)$ intersects B and B^c for all $r > 0$ (i.e., $x \in \partial A \cup \partial B$). This proves the first set inequality in part iii).

To prove the second set inequality, fix $x \in \partial(A \cap B)$ [i.e., suppose that $B_r(x)$ intersects $A \cap B$ and $(A \cap B)^c$ for all $r > 0$]. If $x \in (A \cap \partial B) \cup (B \cap \partial A)$, then there is nothing to prove. If $x \notin (A \cap \partial B) \cup (B \cap \partial A)$, then $x \in (A^c \cup (\partial B)^c) \cap (B^c \cup (\partial A)^c)$. Hence, it remains to prove that $x \in A^c \cup (\partial B)^c$ implies $x \in \partial A$ and $x \in B^c \cup (\partial A)^c$ implies $x \in \partial B$. By symmetry, we need only prove the first implication.

Case 1. $x \in A^c$. Since $B_r(x)$ intersects A, it follows that $x \in \partial A$.

Case 2. $x \in (\partial B)^c$. Since $B_r(x)$ intersects B, it follows that $B_r(x) \subseteq B$ for small $r > 0$. Since $B_r(x)$ also intersects $A^c \cup B^c$, it must be the case that $B_r(x)$ intersects A^c. In particular, $x \in \partial A$. ■

10.43

Proof. These statements follow immediately from Definition 10.42. The empty set needs no set to cover it, and any finite set H can be covered by finitely many sets, one set for each element in H. ■

10.44

Proof. Suppose that H is compact but not closed. Then H is nonempty and (by Theorem 10.16) there is a convergent sequence $x_k \in H$ whose limit x does not belong to H. For each $y \in H$, set $r(y) := \rho(x,y)/2$. Since x does not belong to H, $r(y) > 0$; hence, each $B_{r(y)}(y)$ is open and contains y; that is, $\{B_{r(y)}(y) : y \in H\}$ is an open covering of H. Since H is compact, we can choose points y_j and radii $r_j := r(y_j)$ such that $\{B_{r_j}(y_j) : j = 1, 2, \ldots, N\}$ covers H.

Set $r := \min\{r_1, \ldots, r_N\}$. (This is a finite set of positive numbers, so r is also positive.) Since $x_k \to x$ as $k \to \infty$, $x_k \in B_r(x)$ for large k. But $x_k \in B_r(x) \cap H$ implies $x_k \in B_{r_j}(y_j)$ for some $j \in \mathbf{N}$. Therefore, it follows from the choices of r_j and r, and from the Triangle Inequality, that

$$\begin{aligned} r_j &\ge \rho(x_k, y_j) \ge \rho(x, y_j) - \rho(x_k, x) \\ &= 2r_j - \rho(x_k, x) > 2r_j - r \ge 2r_j - r_j = r_j, \end{aligned}$$

a contradiction. ■

10.45

Proof. Let E be a closed subset of H, where H is compact in X and suppose that $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ is an open covering of E. Now $E^c = X \setminus E$ is open; hence, $\mathcal{V} \cup \{E^c\}$ is an open covering of H. Since H is compact, there is a finite set $A_0 \subseteq A$ such that

$$H \subseteq E^c \cup \left(\bigcup_{\alpha \in A_0} V_\alpha\right).$$

But $E \cap E^c = \emptyset$. Therefore, E is covered by $\{V_\alpha\}_{\alpha \in A_0}$. ■

10.46

Proof. Suppose that H is compact. By Remark 10.44, H is closed. It is also bounded. Indeed, fix $b \in X$ and observe that $\{B_n(b) : n \in \mathbf{N}\}$ covers X. Since H is compact, it follows that

$$H \subset \bigcup_{n=1}^{N} B_n(b)$$

for some $N \in \mathbf{N}$. Since these balls are nested, we conclude that $H \subset B_N(b)$ (i.e., H is bounded). ■

10.47

Proof. Let $X = \mathbf{R}$ be the discrete metric space introduced in Example 10.3. Since $\sigma(0, x) \leq 1$ for all $x \in \mathbf{R}$, every subset of X is bounded. Since $x_k \to x$ in X implies $x_k = x$ for large k, every subset of X is closed. Thus [0, 1] is a closed, bounded subset of X. Since $\{x\}_{x\in[0,1]}$ is an uncountable open covering of [0, 1], which has no finite subcover, we conclude that [0, 1] is closed and bounded, but not compact. ■

10.49

Proof. Let Z be a countable dense subset of X, and consider the collection $\mathcal{T}$ of open balls with centers in Z and rational radii. This collection is countable. Moreover, it "approximates" all other open sets in the following sense:

Claim: Given any open ball $B_r(x) \subset X$, there is a ball $B_q(a) \in \mathcal{T}$ such that $x \in B_q(a)$ and $B_q(a) \subseteq B_r(x)$.

Proof of Claim: Let $B_r(x) \subset X$ be given. By Definition 10.48, choose $a \in Z$ such that $\rho(x, a) < r/4$, and choose by Theorem 1.18 a rational $q \in \mathbf{Q}$ such that $r/4 < q < r/2$. Since $r/4 < q$, we have $x \in B_q(a)$. Moreover, if $y \in B_q(a)$, then

$$\rho(x, y) \leq \rho(x, a) + \rho(a, y) < q + \frac{r}{4} < \frac{r}{2} + \frac{r}{4} < r.$$

Therefore, $B_q(a) \subseteq B_r(x)$. This establishes the claim.

To prove the theorem, let $x \in E$. By hypothesis, $x \in V_\alpha$ for some $\alpha \in A$. Hence, by the claim, there is a ball $B_x \in \mathcal{T}$ such that

$$x \in B_x \subseteq V_\alpha. \tag{4}$$

The collection $\mathcal{T}$ is countable; hence, so is the subcollection

$$\{U_1, U_2, \dots\} := \{B_x : x \in E\}. \tag{5}$$

By (4), for each $k \in \mathbf{N}$ there is at least one $\alpha_k \in A$ such that $U_k \subseteq V_{\alpha_k}$. Hence, by (5),

$$E \subseteq \bigcup_{x\in E} B_x = \bigcup_{k=1}^{\infty} U_k \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}.$$ ■

10.50

Proof. By Theorem 10.46, every compact set is closed and bounded.

Conversely, suppose to the contrary that H is closed and bounded but not compact. Let $\mathcal{V}$ be an open covering of H which has no finite subcover of H. By Lindelöf's Theorem, we may suppose that $\mathcal{V} = \{V_k\}_{k\in\mathbf{N}}$; that is,

$$H \subseteq \bigcup_{k\in\mathbf{N}} V_k. \tag{6}$$

By the choice of $\mathcal{V}$, $\cup_{j=1}^{k} V_j$ cannot contain H for any $k \in \mathbf{N}$. Thus we can choose a point

$$x_k \in H \setminus \bigcup_{j=1}^{k} V_j \tag{7}$$

for each $k \in \mathbf{N}$. Since H is bounded, the sequence x_k is bounded. Hence, by the Bolzano–Weierstrass Property, there is a subsequence x_{k_ν} which converges to some x as $\nu \to \infty$. Since H is closed, $x \in H$. Hence, by (6), $x \in V_N$ for some $N \in \mathbf{N}$. But V_N is open; hence, there is an $M \in \mathbf{N}$ such that $\nu \geq M$ implies $k_\nu > N$ and $x_{k_\nu} \in V_N$. This contradicts (7). We conclude that H is compact. ■

10.52

Proof. If f is uniformly continuous on a set, then it is continuous whether or not the set is compact.

Conversely, suppose that f is continuous on E. Given $\varepsilon > 0$ and $a \in E$, choose $\delta(a) > 0$ such that

$$x \in B_{\delta(a)}(a) \quad \text{and} \quad x \in E \quad \text{imply} \quad \tau(f(x), f(a)) < \frac{\varepsilon}{2}.$$

Since $a \in B_\delta(a)$ for all $\delta > 0$, it is clear that $\{B_{\delta(a)/2}(a) : a \in E\}$ is an open covering of E. Since E is compact, choose finitely many points $a_j \in E$ and numbers $\delta_j := \delta(a_j)$ such that

$$E \subseteq \bigcup_{j=1}^{N} B_{\delta_j/2}(a_j). \tag{8}$$

Set $\delta := \min\{\delta_1/2, \dots, \delta_N/2\}$.

Suppose that $x, a \in E$ with $\rho(x, a) < \delta$. By (8), x belongs to $B_{\delta_j/2}(a_j)$ for some $1 \leq j \leq N$. Hence,

$$\rho(a, a_j) \leq \rho(a, x) + \rho(x, a_j) < \frac{\delta_j}{2} + \frac{\delta_j}{2} = \delta_j;$$

that is, a also belongs to $B_{\delta_j}(a_j)$. It follows, therefore, from the choice of δ_j that

$$\tau(f(x), f(a)) \leq \tau(f(x), f(a_j)) + \tau(f(a_j), f(a)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that f is uniformly continuous on E. ■

10.55

Proof. Set $U = A \cap E$ and $V = B \cap E$. It suffices to prove that U and V are relatively open in E and separate E. It is clear by hypothesis and the remarks above that U and V are nonempty, they are both relatively open in E, and $U \cap V = \emptyset$. It remains to prove that $E = U \cup V$. But E is a subset of $A \cup B$, so $E \subseteq U \cup V$. On the other hand, both U and V are subsets of E, so $E \supseteq U \cup V$. We conclude that $E = U \cup V$. ■

10.56

Proof. Let E be a connected subset of $\mathbf{R}$. If E is empty or contains only one point, then E is a degenerate interval. Hence we may suppose that E contains at least two points.

Set $a = \inf E$ and $b = \sup E$. Notice that $-\infty \leq a < b \leq \infty$. Suppose for simplicity that $a, b \notin E$; that is, $E \subseteq (a, b)$. If $E \neq (a, b)$, then there is an $x \in (a, b)$ such that $x \notin E$. By the Approximation Property, $E \cap (a, x) \neq \emptyset$ and $E \cap (x, b) \neq \emptyset$, and, by assumption, $E \subseteq (a, x) \cup (x, b)$. Hence, E is separated by the open sets (a, x), (x, b), a contradiction.

Conversely, suppose that E is an interval which is not connected. Then there are sets U, V, relatively open in E, which separate E (i.e., $E = U \cup V$, $U \cap V = \emptyset$), and there exist points $x_1 \in U$ and $x_2 \in V$. We may suppose that $x_1 < x_2$. Since $x_1, x_2 \in E$ and E is an interval, $I_0 := [x_1, x_2] \subseteq E$. Define f on I_0 by

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V. \end{cases}$$

Since $U \cap V = \emptyset$, f is well defined. We claim that f is continuous on I_0. Indeed, fix $x_0 \in [x_1, x_2]$. Since $U \cup V = E \supseteq I_0$, it is evident that $x_0 \in U$ or $x_0 \in V$. We may suppose the former. Let $y_k \in I_0$ and suppose that $y_k \to x_0$ as $k \to \infty$. Since U is relatively open, there is an $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap E \subset U$. Since $y_k \in E$ and $y_k \to x_0$, it follows that $y_k \in U$ for large k. Hence $f(y_k) = 0 = f(x_0)$ for large k. Therefore, f is continuous at x_0 by the Sequential Characterization of Continuity.

We have proved that f is continuous on I_0. Hence, by the Intermediate Value Theorem (Theorem 3.29), f must take on the value 1/2 somewhere on I_0. This is a contradiction, since by construction, f takes on only the values 0 or 1. ■

10.58

Proof. Suppose that f is continuous on X and that V is open in Y. We may suppose that $f^{-1}(V)$ is nonempty. Let $a \in f^{-1}(V)$; that is, $f(a) \in V$. Since V is open, choose $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq V$. Since f is continuous at a, choose $\delta > 0$ such that (10) holds. Evidently,

$$B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a))) \subseteq f^{-1}(V). \tag{11}$$

Since $a \in f^{-1}(V)$ was arbitrary, we have shown that every point in $f^{-1}(V)$ is interior to $f^{-1}(V)$. Thus $f^{-1}(V)$ is open.

Conversely, let $\varepsilon > 0$ and $a \in X$. The ball $V = B_\varepsilon(f(a))$ is open in Y. By hypothesis, $f^{-1}(V)$ is open. Since $a \in f^{-1}(V)$, it follows that there is a $\delta > 0$ such that $B_\delta(a) \subseteq f^{-1}(V)$. This means that if $\rho(x, a) < \delta$, then $\tau(f(x), f(a)) < \varepsilon$. Therefore, f is continuous at $a \in X$. ■

10.61

Proof. Suppose that $\{V_\alpha\}_{\alpha\in A}$ is an open covering of $f(H)$. By Theorem 1.37,

$$H \subseteq f^{-1}(f(H)) \subseteq f^{-1}\left(\bigcup_{\alpha\in A} V_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(V_\alpha).$$

Hence, by Corollary 10.59, $\{f^{-1}(V_\alpha)\}_{\alpha\in A}$ is a covering of H whose sets are all relatively open in H. Since H is compact, there are indices $\alpha_1, \alpha_2, \dots, \alpha_N$ such that

$$H \subseteq \bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})$$

(see Exercise 10.5.7). It follows from Theorem 1.37 that

$$f(H) \subseteq f\left(\bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})\right) = \bigcup_{j=1}^{N} (f \circ f^{-1})(V_{\alpha_j}) = \bigcup_{j=1}^{N} V_{\alpha_j}.$$

Therefore, $f(H)$ is compact. ■

10.62

Proof. Suppose that $f(E)$ is not connected. By Definition 10.53, there exists a pair $U, V \subset Y$ of relatively open sets in $f(E)$ which separates $f(E)$. By Exercise 10.6.4, $f^{-1}(U) \cap E$ and $f^{-1}(V) \cap E$ are relatively open in E. Since $f(E) = U \cup V$, we have

$$E = (f^{-1}(U) \cap E) \cup (f^{-1}(V) \cap E).$$

Since $U \cap V = \emptyset$, we also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Thus $f^{-1}(U) \cap E$, $f^{-1}(V) \cap E$ is a pair of relatively open sets which separates E. Hence, by Definition 10.53, E is not connected, a contradiction. ■

10.63

Proof. By symmetry, it suffices to prove the result for M. Since H is compact, $f(H)$ is compact. Hence, by the Theorem 10.46, $f(H)$ is closed and bounded. Since $f(H)$ is bounded, M is finite. By the Approximation Property, choose $x_k \in H$ such that $f(x_k) \to M$ as $k \to \infty$. Since $f(H)$ is closed, $M \in f(H)$. Therefore, there is an $x_M \in H$ such that $M = f(x_M)$. A similar argument shows that m is finite and attained on H. ■

10.64

Proof. By Exercise 10.6.4a, it suffices to show that $(f^{-1})^{-1}$ takes closed sets in X to relatively closed sets in $f(H)$. Let E be closed in X. Then $E \cap H$ is a closed subset of H, so by Remark 10.45, $E \cap H$ is compact. Hence, by Theorem 10.61, $f(E \cap H)$ is compact, in particular, closed. Since f is 1–1, $f(E \cap H) = f(E) \cap f(H)$ (see Exercise 1.5.7). Since $f(E \cap H)$ and $f(H)$ are closed, it follows that $f(E) \cap f(H)$ is relatively closed in $f(H)$. Since $(f^{-1})^{-1} = f$, we conclude that $(f^{-1})^{-1}(E) \cap f(H)$ is relatively closed in $f(H)$. ■

Riemann Prop 1

Proof. List all the potential jump points of either ϕ or ψ together as $\{x_0 < \ldots < x_n\}$. Suppose $\phi(x) = \sum_{j=1}^n c_j \chi_{(x_{j-1},x_j)}(x)$ and $\psi(x) = \sum_{j=1}^n d_j \chi_{(x_{j-1},x_j)}(x)$ for $x \neq x_0, x_1, \ldots x_n$. Then the left hand side is $\sum_{j=1}^n (\alpha c_j + \beta d_j)(x_j - x_{j-1}) = \alpha \sum_{j=1}^n c_j (x_j - x_{j-1}) + \beta \sum_{j=1}^n d_j (x_j - x_{j-1}) = \alpha \int \phi + \beta \int \psi$. ■

Riemann Theorem 1

Proof of Theorem 1. Suppose first that f is Riemann-integrable. Then, given $\epsilon > 0$, there exist step functions ϕ_0 and ψ_0 such that $\phi_0 \leq f \leq \psi_0$ and $\int \psi_0 - \int \phi_0 < \epsilon$. Consider the sets of real numbers

$$A = \{\int \phi \ : \ \phi \text{ is a step function and } \phi \leq f\}$$

and

$$B = \{\int \psi \ : \ \psi \text{ is a step function and } \psi \geq f\}.$$

Then A is a nonempty subset of $\mathbb{R}$ (since $\int \phi_0$ belongs to it) which is bounded above by $\int \psi_0$ or indeed any member of B. So it has a least upper bound U which satisfies $\int \phi_0 \leq U \leq \int \psi_0$. Similarly, the greatest lower bound L of B satisfies $\int \psi_0 \geq L \geq \int \phi_0$, and moreover $U \leq L$. Hence $0 \leq L - U < \epsilon$. Since this is true for arbitrary $\epsilon > 0$ we deduce that $U = L$, i.e. $\sup A = \inf B$.

Now suppose that $\sup A = \inf B := I$. By the approximation property of sup and inf, given any $\epsilon > 0$ there exist step functions ϕ and ψ with $\phi \leq f$ and $\psi \geq f$ and $\int \phi > I - \epsilon/2$, and $\int \psi < I + \epsilon/2$. Hence $\int \psi - \int \phi < \epsilon$. ■

Riemann Theorem 1

Proof. Supose first that f is Riemann-integrable. Then, taking $\epsilon = 1/n$ in the definition, there exist step functions ϕ_n and ψ_n such that $\phi_n \leq f \leq \psi_n$ and $\int \psi_n - \int \phi_n < 1/n$. Hence $\int \psi_n - \int \phi_n \to 0$.

Now suppose that there exist ϕ_n and ψ_n as in the statement of the theorem. Given $\epsilon > 0$ choose N such that $\int \psi_n - \int \phi_n < \epsilon$ for all $n \geq N$; then ϕ_N and ψ_N do the job.

Finally, in such a case, by the definition of $\int f$ (in Definition 4) we have $\int \phi_n \leq \int f \leq \int \psi_n$, and so $|\int \phi_n - \int f| = \int f - \int \phi_n \leq \int \psi_n - \int \phi_n \to 0$. Similary $\int \psi_n \to \int f$. ■

Riemann Lemma 1

Proof.

(i) implies (ii). Suppose first that f is Riemann-integrable, let $\epsilon > 0$ and let ϕ and ψ be step functions as in the definition of Riemann-integrability, with $\int \psi - \int \phi < \epsilon$. We may assume that ϕ and ψ are zero outside $[a, b]$. Enumerate the potential jump points of ϕ and ψ together as $a = x_0 < \ldots < x_n = b$. Let ϕ_* and ϕ^* be the lower and upper step functions of f with respect to $\{x_0, \ldots, x_n\}$. Then

$$\phi \leq \phi_* \leq f \leq \phi^* \leq \psi \text{ and } \int \phi^* - \int \phi_* \leq \int \psi - \int \phi < \epsilon.$$

But $\int \phi^* - \int \phi_* = \sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1})$, so (ii) holds.

(ii) implies (iii). Since for a nonempty bounded subset $A \subseteq \mathbb{R}$ we have $\sup\{|a - b| \ : \ a, b \in A\} = \sup A - \inf A$, it follows that $\sup_{x,y \in I_j} |f(x) - f(y)| = M_j - m_j$. So, assuming (ii),

$$\sum_{j=1}^n \sup_{x,y \in I_j} |f(x) - f(y)||I_j| = \int \sum_j M_j \chi_{I_j} - \int \sum_j m_j \chi_{I_j} = \int \phi^* - \int \phi_* < \epsilon.$$

(iii) implies (i). If there exist $a = x_0 < \ldots < x_n = b$ such that

$$\sum_{j=1}^n \sup_{x,y \in I_j} |f(x) - f(y)||I_j| < \epsilon$$

holds, then the lower and upper step functions ϕ_* and ϕ^* of f with respect to $\{x_0, \ldots, x_n\}$ verify the definition of Riemann-integrability of f. ■

Riemann Theorem 3

Proof. By Theorem 2 there are sequences $\phi_n, \psi_n, \varphi_n$ and θ_n of step functions such that

$$\phi_n \leq f \leq \psi_n \text{ and } \varphi_n \leq g \leq \theta_n$$

and

$$\int \phi_n, \int \psi_n \to \int f \text{ and } \int \varphi_n, \int \theta_n \to \int g.$$

(a) Let us first prove it for α and $\beta \geq 0$. Then

$$\alpha\phi_n + \beta\varphi_n \leq \alpha f + \beta g \leq \alpha\psi_n + \beta\theta_n$$

and both $\int(\alpha\phi_n + \beta\varphi_n)$ and $\int(\alpha\psi_n + \beta\theta_n)$ converge to $\alpha \int f + \beta \int g$. In particular, the difference $\int(\alpha\psi_n + \beta\theta_n) - \int(\alpha\phi_n + \beta\varphi_n) \to 0$. By Theorem 2, $\alpha f + \beta g$ is Riemann-integrable and $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$.

Now let us prove it for $\alpha = -1$ and $\beta = 0$. In this case we have $-\psi_n \leq -f \leq -\phi_n$ and $\int(-\psi_n)$ and $\int(-\phi_n)$ both converge to $-\int f$. So $-f$ is Riemann-integrable and $\int -f = -\int f$.

Combining these two cases gives the general case. (Why?)

(b) If $f \geq 0$ then each $\psi_n \geq 0$, so that each $\int \psi_n \geq 0$ and hence $\lim \int \psi_n = \int f \geq 0$. For the second part apply what's just been proved to $g - f$.

(c) If f is Riemann-integrable, then, by Lemma 1, so is $|f|$ since by the triangle inequality we have $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$. Now use part (b). (Why is this enough?)

(d) $\max\{f, 0\} = (f + |f|)/2$, then use (a) and (c), while $\min\{f, 0\} = (f - |f|)/2$ likewise; then $\max\{f, g\} = \max\{f - g, 0\} + g$ – use the firts part of this item, and part (a) – and $\min\{f, g\} = -\max\{-f, -g\}$ – use the result for max and part (a).

(e) Let us first see that f^2 is Riemann-integrable. Since f is bounded there is an M such that $|f(x)| \leq M$ for all x. We have $|f(x)^2 - f(y)^2| = |f(x) - f(y)||f(x) + f(y)| \leq 2M|f(x) - f(y)|$. Using Lemma 1 we see that f Riemann-integrable implies f^2 Riemann-integrable. (Why?) Finally, $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$. Now use part (a) and what we have just proved about squares. ■

Riemann Theorem 4

Proof. Note that f is bounded (by the extreme value theorem) and has bounded support. Let $\epsilon > 0$. Uniform continuity of f on $[a, b]$ tells us that there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$. Choose $a = x_0 < \ldots < x_n = b$ such that $x_i - x_{i-1} < \delta$. Then for all x and y in (x_{i-1}, x_i) we have $|f(x) - f(y)| < \epsilon/(b - a)$. Turning to Lemma 1, we see that

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| \leq \sum_{j=1}^{n} \frac{\epsilon}{(b-a)}|I_j| \leq \epsilon,$$

and so by Lemma 1, f is Riemann-integrable. ■

Riemann Theorem 5

Proof. We will deal with the case $a < x < b$. Let $h > 0$ be sufficiently small so that $x + h < b$ and consider $|\frac{G(x+h) - G(x)}{h} - g(x)|$. (The argument for $h < 0$ is similar.) This quantity equals

$$|\frac{1}{h}\int_x^{x+h} [g(t) - g(x)]\mathrm{d}t| \leq \frac{1}{h}\int_x^{x+h} |g(t) - g(x)|\mathrm{d}t$$

by Theorem 3 (b). Now, as g is continuous at x, if $\epsilon > 0$, there exists a $\delta > 0$ such that if $x < t < x + h$ and $h < \delta$, then $|g(t) - g(x)| < \epsilon$. So for such h,

$$\frac{1}{h}\int_x^{x+h} |g(t) - g(x)|\mathrm{d}t \leq \epsilon$$

by Theorem 3 (b) once again. Thus $h < \delta$ implies $\left|\frac{G(x+h) - G(x)}{h} - g(x)\right| < \epsilon$, and so $G'(x)$ exists and equals $g(x)$. ■

Riemann Theorem 6

Proof. Let $G(x) = \int_a^x f'$. Then by Theorem 5, $G'(x)$ exists for all x in (a, b) and $G'(x) = f'(x)$. Thus $G - f$, being continuous on $[a, b]$ and differentiable on (a, b), must be constant on $[a, b]$ by Rolle's theorem. So $\int_a^b f' = G(b) = G(a) + f(b) - f(a) = f(b) - f(a)$. ■

Riemann Theorem 7

Proof. Let $\epsilon > 0$. There is an N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. For each such n there exist step functions ϕ_n and ψ_n such that

$$\phi_n \leq f_n \leq \psi_n \text{ and } \int \psi_n - \int \phi_n < 1/n.$$

Since

$$f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon \tag{2}$$

we have

$$\phi_n(x) - \epsilon < f(x) < \psi_n(x) + \epsilon$$

for $x \in [a, b]$ and $n \geq N$. Let $\varphi_n = \phi_n - \epsilon\chi_{[a,b]}$ and $\theta_n = \psi_n + \epsilon\chi_{[a,b]}$. Then φ_n and θ_n are step functions and

$$\varphi_n \leq f \leq \theta_n \text{ and } \int \theta_n - \int \varphi_n = \int \psi_n - \int \phi_n + 2\epsilon(b - a) < \frac{1}{n} + 2\epsilon(b - a).$$

So if $n \geq \max\{N, \epsilon^{-1}\}$,

$$\int \theta_n - \int \varphi_n < (1 + 2(b - a))\epsilon.$$

Hence f is Riemann-integrable. By integrating (2) we have, for $n \geq N$,

$$\int f_n - \epsilon(b - a) \leq \int f \leq \int f_n + \epsilon(b - a)$$

that is,

$$|\int f_n - \int f| \leq \epsilon(b - a)$$

which shows that $\lim_{n\to\infty} \int f_n = \int f$. ■

Power Theorem 1

Proof. Let us first suppose that $|x-c|<R$. Consider a number ρ such that $|x-c|<\rho<R$. So we have that $(a_n\rho^n)$ is bounded, say $|a_n\rho^n|\le K$ for all n. Then

$$|a_n||x-c|^n = |a_n|\left(\frac{|x-c|}{\rho}\right)^n \rho^n = \left(\frac{|x-c|}{\rho}\right)^n \times |a_n|\rho^n \le K\left(\frac{|x-c|}{\rho}\right)^n,$$

and the geometric series

$$\sum\left(\frac{|x-c|}{\rho}\right)^n$$

converges since $\frac{|x-c|}{\rho}<1$. Thus, by comparison, $(*)$ converges absolutely for such x, and hence also converges for such x.

Now suppose that $|x-c|>R$. For such x we have that the sequence $(a_n(x-c)^n)$ of individual terms in $(*)$ is unbounded, so the series cannot converge. ■

Power Theorem 2

Proof. We have already seen the absolute convergence. With the same notation and argument as in the proof of Theorem 1 above, we have (for $r<\rho<R$) that

$$|a_n||x-c|^n \le K\left(\frac{r}{\rho}\right)^n := M_n$$

for all x with $|x-c|\le r$. Since $\sum M_n$ converges, the Weierstrass M-test tells us the convergence is uniform on $[c-r,c+r]$. Since each $a_n(x-c)^n$ is a continuous function, so is the limiting function $f:[c-r,c+r]\to\mathbb{R}$. Since $r<R$ was arbitrary, we see that f is defined and continuous on $(c-R,c+R)$. ■

Power Lemma 1

Proof. Let the radii of convergence be R_1 and R_2 respectively. Since $|a_nr^n|\le|na_nr^n|$ (for $n\ge1$) we see that $R_2\le R_1$. [As the terms of the second series are "bigger", there's in principle a smaller chance that it'll converge.] Suppose now for a contradiction that $R_2<R_1$. Then we can choose ρ and r such that $R_2<\rho<r<R_1$ and such that (a_nr^n) is bounded, say $|a_nr^n|\le K$. Then

$$|na_n\rho^n| = |a_nr^n|\times n(\rho/r)^n \le K\times n(\rho/r)^n.$$

But the sequence $n(\rho/r)^n$ converges to zero since $\rho<r$, and therefore $(na_n\rho^n)$ is also bounded. This contradicts the definition of R_2, and so $R_1=R_2$. ■

Power Theorem 3

Proof. Consider the series $\sum_{n=0}^{\infty} na_n(x-c)^{n-1}$ which has radius of convergence R and so converges uniformly on $[c-r,c+r]$ for any $r<R$. Since $na_n(x-c)^{n-1}$ is the derivative of $a_n(x-c)^n$ and since the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges at at least one point, we can apply Theorem 7.14 (iii) from Wade to conclude that $f'(x)=\sum_{n=0}^{\infty} na_n(x-c)^{n-1}$. Clearly $f(c)=a_0$ and $f'(c)=a_1$ and by repeatedly differentiating the formula $f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$ (using what we have already proved!) and substituting $x=c$ we obtain $f^{(n)}(c)=a_nn!$. ■

Contraction Theorem 1

Proof. Pick $x_0\in X$ and let $x_1=f(x_0)$, $x_2=f(x_1)$ etc. so that $x_{n+1}=f(x_n)$. Consider $d(x_{n+1},x_n)=d(f(x_n),f(x_{n-1})\le\alpha d(x_n,x_{n-1}))$ by hypothesis. Repeating, we have

$$d(x_{n+1},x_n)\le\alpha^n d(x_1,x_0)$$

so that, when $m\ge n$

$$d(x_m,x_n)\le d(x_m,x_{m-1})+\ldots+d(x_{n+1},x_n)\le(\alpha^{m-1}+\ldots+\alpha^n)d(x_1,x_0)$$
$$\le\frac{\alpha^n}{1-\alpha}d(x_1,x_0)$$

since $\alpha<1$. This shows that (x_n) is a Cauchy sequence in X and hence, by completeness of X, there is an $x\in X$ to which it converges.

Now a contraction map is continuous, so continuity of f at x shows that $f(x)=f(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}x_{n+1}=x$, so that indeed $f(x)=x$.

Finally, if there are $x,y\in X$ with $f(x)=x$ and $f(y)=y$, we have $d(x,y)=d(f(x),f(y))\le\alpha d(x,y)$, which, since $\alpha<1$, forces $d(x,y)=0$, i.e. $x=y$.