1 The Real Number System

1.1 Introduction

1.2 Ordered Field Axioms

Remark 1.1
We will assume that the sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties:

1. If $n, m \in \mathbb{Z}$, then $n+m, n-m$ and mn belong to $\mathbb{Z}$
2. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$
3. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$

Definition 1.4 *Absolute Value*
The *absolute value* of a number $a \in \mathbb{R}$ is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

Remark 1.5 The *absolute value* is multiplicative; that is, $|ab| = |a||b| \; \forall a, b, \in \mathbb{R}$

Theorem 1.6 *Fundamental Theorem of Absolute Values*
Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M \iff -M \leq a \leq M$.

Theorem 1.7 The absolute value satisfies the following three properties:

1. *Positive Definite*: For all $a \in \mathbb{R}$, $|a| > 0$ with $|a| = 0$ if and only if $a = 0$.
2. *Symmetric*: For all $a, b, \in \mathbb{R}$, $|a-b| = |b-a|$,
3. *Triangle Inequalities*: For all $a, b \in \mathbb{R}$
   $|a+b| \leq |a| + |b|$ and $||a| - |b|| \leq |a-b|$

Theorem 1.9 Let $x, y, a \in \mathbb{R}$

1. $x < y + \epsilon \; \forall \epsilon > 0 \iff x \leq y$
2. $x > y - \epsilon \; \forall \epsilon > 0 \iff x \geq y$
3. $|a| < \epsilon \; \forall \epsilon > 0 \iff a = 0$

1.3 Completeness Axiom

Definition 1.10 *Upper bounds*
Let $E \subset \mathbb{R}$ be non-empty

1. The set E is said to be *bounded above* if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$, in which case M is called an *upper bound* of E.
2. A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E has a *finite supremum* s and write $s = \sup E$.)

Remark 1.12 If a set has one upper bound, it has infinitely many upper bounds.

Remark 1.13 If a set has a supremum, then it has only one supremum.

Theorem *Approximation Property for Suprema*
If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that
$\sup E - \epsilon < a \leq \sup E$

Theorem 1.15
If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

Postulate 3 *Completeness Axiom*
If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

Theorem 1.16 *The Archimedean Principle*
Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

Theorem 1.18 *Density of Rationals*
If $a, b, \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

Definition 1.19 *Upper bounds*
Let $E \in \mathbb{R}$ be nonempty

1. The set E is said to be *bounded below* if and only if there is an $m \in \mathbb{R}$ such that $a \geq E$, in which case m is called a *lower bound* of the set E.
2. A number t is called an *infimum* of the set E if and only if t is a lower bound of E and $t \geq m$ and write $t = \inf E$.
3. E is said to be *bounded* if and only if it is bounded both above and below.

Theorem 1.20 *Reflection Principle*
Let $E \in \mathbb{R}$ be nonempty

1. E has a supremum if and only if $-E$ has an infimum, in which case $\inf(-E) = -\sup E$.
2. E has an infimum if and only if $-E$ has a supremum, in which case $\sup(-E) = -\inf E$

Theorem 1.21 *Monotone Property*
Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.

1. If B has a supremum, then $\sup A \leq \sup B$.
2. If B has an infimum, then $\inf A \geq \inf B$.

1.4 Mathematical Induction

Theorem 1.22 *Well-Ordering Principle*
If E is a nonempty subset of $\mathbb{N}$, then E has a least element (i.e. E has a finite infimum and $\inf E \in E$).

Theorem 1.23
Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition which satisfies the following two properties:

1. $A(1)$ is true.
2. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.

Then $A(n)$ is true for all $n \in \mathbb{N}$.

Theorem 1.26 *Binomial Formula*
If $a, b, \in \mathbb{R}, n \in \mathbb{N}$ and 0^0 is interpreted to be 1, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

1.5 Inverse Functions and Images

Definition 1.29 *Injection, Surjection, Bijection*
Let X and Y be sets and $f : X \to Y$

1. f is said to be *injective* if and only if
   $$x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \implies x_1 = x_2$$
2. f is said to be *surjective* if and only if
   $$\forall y \in Y \; \exists x \in X \ni y = f(x)$$
3. f is called *bijective* if and only if it is both injective and surjective

Theorem 1.30
Let X and Y be sets and $f : X \to Y$. Then the following three statements are equivalent.

1. f has an inverse;
2. f is injective from X onto Y;

3. There is a function $g : Y \to X$ such that
   $g(f(x)) = x \quad \forall x \in X$ and
   $f(g(y)) = y \quad \forall y \in Y$

Moreover, for each $f : X \to Y$, there is only one function g that satisfies these. It is the inverse function f^{-1}.

Remark 1.31
Let I be an interval and let $f : I \to \mathbb{R}$. If the derivative of f is either always positive on I, or always negative on I, then f is injective on I.

Definition 1.33 *Image*
Let X and Y be sets and $f : X \to Y$. The *image* of a set $E \subseteq X$ under f is the set

$$f(E) := \{y \in Y : y = f(x) \text{ for some } x \in E\}$$

The *inverse image* of a set $E \subseteq Y$ under f is the set

$$f^{-1}(E) := \{x \in X : f(x) = y \text{ for some } y \in E\}$$

Definition 1.35 *Union, Intersection*
Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ be a collection of sets.

1. The *union* of the collection $\mathcal{E}$ is the set

$$\bigcup_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$$

2. The *intersection* of the collection $\mathcal{E}$ is the set

$$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

Theorem 1.36 *DeMorgan's Laws*
Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. If for each $E \subseteq X$ the symbol E^c represents the set $X \setminus E$, then $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$ and $\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$

Theorem 1.37
Let X and Y be sets and $f : X \to Y$.

1. If $\{E_\alpha\}\alpha \in A$ is a collection of subsets of X, then
   $f\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f(E_\alpha)$ and
   $f\left(\bigcap_{\alpha \in A} E_\alpha\right) \subseteq \bigcap_{\alpha \in A} f(E_\alpha)$
2. If B and C are subsets of X, then $f(C) \setminus f(B) \subseteq f(C \setminus B)$
3. If $\{E_\alpha\}_{\alpha \in A}$ is a collection of subsets of Y, then
   $f^{-1}\left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(E_\alpha)$ and
   $f^{-1}\left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(E_\alpha)$
4. If B and C are subsets of Y, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.
5. If $E \subseteq f(x)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$.

1.6 Countable and Uncountable Sets

Definition 1.38 *Countable & Uncountable*
Let E be a set.

1. E is said to be *finite* if and only if either $E = \emptyset$ or there exists an injective function which takes $\{1, 2, \ldots, n\}$ onto E, for some $n \in \mathbb{N}$.
2. E is said to be *countable* if and only if there exists and injective function which takes $\mathbb{N}$ onto E.
3. E is said to be *at most countable* if and only if E is either finite or countable.
4. E is said to be *uncountable* if and only if E is neither finite nor countable.

Remark 1.39 *Cantor's Diagonalisation Argument*
The open interval $(0, 1)$ is uncountable.

Lemma 1.40
A nonempty set E is at most countable if and only if there is a function g from $\mathbb{N}$ onto E.

Theorem 1.41
Suppose A and B are sets.

1. If $A \subseteq B$ and B is at most countable, then A is at most countable.
2. If $A \subseteq B$ and A is uncountable, then B is uncountable.
3. $\mathbb{R}$ is uncountable.

Theorem 1.42
Let $A_1, A_2, \ldots$ be at most countable sets.

1. Then $A_1 \times A_2$ is at most countable.
2. If

$$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j \in \mathbb{N}} A_j := \{x : x \in A_j \text{ for some } j \in \mathbb{N}\},$$

   then E is at most countable.

Remark 1.43
The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but the set of irrationals is uncountable.

2 Sequences in $\mathbb{R}$

2.1 Limits of Sequences

Definition 2.1 *Convergence*
A sequence of real numbers $\{x_n\}$ is set to *converge* to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

$$n \geq N \implies |x_n - a| < \epsilon$$

Remark 2.4 A sequence can have at most one limit.

Definition 2.5 *Subsequence*
By a *subsequence* of a sequence $\{x_n\}_{n \in \mathbb{N}}$, we shall mean a sequence of the form $\{x_{n_k}\}_{k \in \mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$.

Remark 2.6
If $\{x_n\}_{n \in \mathbb{N}}$ converges to a and $\{x_{nk}\}k \in \mathbb{N}$ is any subsequence of $\{x_n\}_{n \in \mathbb{N}}$, then x_{nk} converges to a as $k \to \infty$.

Definition 2.7 *Bounded Sequences*
Let $\{x_n\}$ be a sequence of real numbers.

1. The sequence $\{x_n\}$ is said to be *bounded above* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above.
2. The sequence $\{x_n\}$ is said to be *bounded below* if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded below.
3. $\{x_n\}$ is said to be *bounded* if and only if it is bounded both above and below.

Theorem 2.8 Every convergent sequence is bounded.

2.2 Limit Theorems

Theorem 2.9 *Squeeze Theorem*
Suppose that $\{x_n\}, \{y_n\}$, and $\{w_n\}$ are real sequences.

1. If $x_n \to a$ and $y_n \to a$ as $n \to \infty$, and if there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq w_n \leq y_n \text{ for } n \geq N_0$$

   then $w_n \to a$ as $n \to \infty$.
2. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Theorem 2.11
Let $E \subset \mathbb{R}$. If E has a finite supremum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ (respectively, a sequence $y_n \in E$ such that $y_n \to \inf E$) as $n \to \infty$.

Theorem 2.12
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then

1. $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$
2. $\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$ and
3. $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$
   If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then
4. $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$
   (In particular, all these limits exist.)

Definition 2.14 *Divergence*
Let $\{x_n\}$ be a sequence of real numbers.

1. $\{x_n\}$ is said to *diverge* to $+\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
   $n \geq N \implies x_n > M$
2. $\{x_n\}$ is said to *diverge* to $-\infty$ if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
   $n \geq N \implies x_n < M$

Theorem 2.15
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \to +\infty$ (respectively, $x_n \to -\infty$) as $n \to \infty$.

1. If y_n is bounded below (respectively, y_n is bounded above), then $\lim_{n\to\infty}(x_n + y_n) = +\infty$
2. If $\alpha > 0$, then $\lim_{n\to\infty}(\alpha x_n) = +\infty$
3. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$, then $\lim_{n\to\infty}(x_n y_n) = +\infty$
4. If $\{y_n\}$ is bounded and $x_n \neq 0$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$

Corollary 2.16
Let $\{x_n\}$, $\{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$, as $n \to \infty$, then
$\lim_{n\to\infty}(x_n + y_n) = x + y$ provided that the right side is not of the form $\infty - \infty$, and
$\lim_{n\to\infty}(\alpha x_n) = \alpha x, \quad \lim_{n\to\infty}(x_n y_n) = xy$
provided that none of these products is of the form $0 \cdot \pm\infty$.

Theorem 2.17 *Comparison Theorem*
Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that $x_n \leq y_n$ for $n \geq N_0$ then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.
In particular, if $x_n \in [a, b]$ converges to some point c, then c must belong to $[a, b]$.

2.3 Bolzano-Weierstrass Theorem

Definition 2.18 *Increasing, Decreasing*
Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

1. $\{x_n\}$ is said to be *increasing* (respectively, *strictly increasing*) if and only if $x_1 \leq x_2 \leq \cdots$ (respectively, $x_1 < x_2 < \cdots$).
2. $\{x_n\}$ is said to be *decreasing* (respectively, *strictly decreasing*) if and only if $x_1 \geq x_2 \geq \cdots$ (respectively, $x_1 > x_2 > \cdots$).
3. $\{x_n\}$ is said to be *monotone* if and only if it is either increasing or decreasing.

Theorem 2.19 *Monotone Convergence Theorem*
If $\{x_n\}$ is increasing and bounded above, or if $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

Definition 2.22 *Nested*
A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be *nested* if and only if $I_1 \supseteq I_2 \supseteq \cdots$.

Theorem 2.23 *Nested Interval Property*
If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then $E := \bigcap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$ then E is a single point.

Remark 2.24 The Nested Interval Property might not hold if "closed" is omitted.

Remark 2.25 The Nested Interval Property might not hold if "bounded" is omitted.

Theorem 2.26 *Bolzano—Weierstrass Theorem*
Every bounded sequence of real numbers has a convergent subsequence.

2.4 Cauchy Sequences

Definition 2.27 *Cauchy*
A sequence of points $x_n \in \mathbb{R}$ is said to be *Cauchy* (in R) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
$n, m \geq N \implies |x_n - x_m| < \epsilon$

Remark 2.28 If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

Theorem 2.29 *Cauchy*
Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges (to some point $a \in \mathbb{R}$).

Remark 2.31 A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

2.5 Limits Supremum and Infimum

Definition 2.32 *Limit Supremum & Infimum*
Let $\{x_n\}$ be a real sequence. Then the *limit supremum* of $\{x_n\}$ is the extended real number
$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}(\sup_{k\geq n} x_k)$
and the *limit infimum* of $\{x_n\}$ is the extended real number
$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}(\inf_{k\geq n} x_k)$

Theorem 2.35
Let $\{x_n\}$ be a sequence of real numbers,
$s = \limsup_{n\to\infty} x_n$, and $t = \liminf_{n\to\infty} x_n$.
Then there are subsequences $\{x_{nk}\}_{k\in\mathbb{N}}$ and $\{x_{\ell j}\}_{j\in\mathbb{N}}$ such that $x_{nk} \to s$ as $k \to \infty$ and $x_{\ell j} \to t$ as $j \to \infty$.

Theorem 2.36
Let $\{x_n\}$ be a real sequence and x be an extended real number. Then $x_n \to x$ as $n \to \infty$ if and only if
$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.

Theorem 2.37
Let $\{x_n\}$ be a sequence of real numbers. Then $\limsup_{n\to\infty} x_n$ (respectively, $\liminf_{n\to\infty}$) is the largest value (respectively, the smallest value) to which some subsequences of $\{x_n\}$ converges. Namely, if $x_{nk} \to x$ as $k \to \infty$, then
$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n$.

Remark 2.38 If $\{x_n\}$ is any sequence of real numbers, then $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Remark 2.39 A real sequence $\{x_n\}$ is bounded above if and only if $\limsup_{n\to\infty} x_n < \infty$, and is bounded below if and only if $\liminf_{n\to\infty} x_n > -\infty$.

Theorem 2.40
If $x_n \leq y_n$ for n large, then
$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$ and
$\liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty} y_n$

3 Functions on R

3.1 Two-Sided Limits

Definition 3.1 *Limits*
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let

f be a real function defined everywhere on I except possibly at a. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$$

In this case we write

$$L = \lim_{x \to a} f(x) \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

and call L the *limit* of $f(x)$ as x approaches a.

Remark 3.4
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined everywhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \setminus \{a\}$ and $f(x) \to L$ as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

Theorem 3.6 *Sequential Characterisation of Limits*
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then

$$L = \lim_{x \to a} f(x)$$

exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $\{x_n\} \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

Theorem 3.8
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, are real functions defined everywhere on I except possibly at a. If $f(x)$ and $g(x)$ converge as x approaches a, then so do $(f+g)(x), (fg)(x), (\alpha f)(x)$, and $(f/g)(x)$ (when the limit of $g(x)$ is nonzer). In fact,

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} + \lim_{x \to a} g(x)$$
$$\lim_{x \to a} (\alpha f)(x) = \lim_{x \to a} f(x)$$
$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} \lim_{x \to a} g(x)$$

and (when the limit of $g(x)$ is nonzero)

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Theorem 3.9 *Squeeze Theorem for Functions*
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined everywhere on I except possibly at a.

1. If $g(x) \leq h(x) \leq f(x)\ \forall x \in I \setminus \{a\}$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L,$$

then the limit of $h(x)$ exists, as $x \to a$, and

$$\lim_{x \to a} h(x) = L.$$

2. If $|g(x)| \leq M\ \forall x \in I \setminus \{a\}$ and $f(x) \to 0$ as $x \to a$, then

$$\lim_{x \to a} f(x) g(x) = 0$$

Theorem 3.10 *Comparison Theorem for Functions*
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined everywhere on I except possibly at a. If f and g have a limit as x approaches a and $f(x) \leq g(x)\ \forall x \in I \setminus \{a\}$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

3.2 One-Sided Limits and Limits at Infinity

Definition 3.12 *Converge from left & right*
Let $a \in \mathbb{R}$ and f be a real function.

1. $f(x)$ is said to *converge to L as x approaches a from the right* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *right-hand limit* of f at a, and denote it by

$$f(a+) := L =: \lim_{x \to a+} f(x)$$

2. $f(x)$ is said to *converge to L as x approaches a from the left* if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$a + \delta \in I \quad \text{and} \quad a < x < a + \delta \implies |f(x) - L| < \epsilon$$

in this case we call L the *left-hand limit* of f at a, and denote it by

$$f(a-) := L =: \lim_{x \to a-} f(x)$$

Theorem 3.14
Let f be a real function. Then the limit

$$\lim_{x \to a} f(x)$$

exists and equals L if and only if

$$L = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$$

Definition 3.15 *Convergence*
Let $a, L, \in \mathbb{R}$ and let f be a real function.

1. $f(x)$ is said to *converge* to L as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subset \text{Dom}(f)$ and given $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

Similarly, $f(x)$ is said to *converge* to L as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(infty, -c) \subset \text{Dom}(f)$ and given $\epsilon > 0$ there is $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write

$$\lim_{x \to \infty} = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty$$

2. The function $f(x)$ is said to converge to ∞ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \text{Dom}(f)$ and given $M \in$ R there is a $\delta > 0$ such that $0 \leq |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to a$$

Similarly, $f(x)$ is said to *converge* to $-\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \text{Dom}(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$, in which case we shall write

$$\lim_{x \to a} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to a$$

Theorem 3.17
Let a be an extended real number, and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then

$$\lim_{x \to a; x \in I} f(x)$$

exists and equals L if and only if $f(x_n) \to L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \to a$ as $n \to \infty$.

3.3 Continuity

Definition 3.19 *Continuous*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.

1. f is said to be *continuous at a point* $a \in \mathbb{E}$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, and a) such that
$$|x - a| < \delta \quad \text{and} \quad x \in E \implies |f(x) - f(a)| < \epsilon$$
2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

Remark 3.20
Let I be an open interval which contains a point a and $f : I \to \mathbb{R}$. Then f is continuous at $a \in I$ if and only if
$$f(a) = \lim_{x \to a} f(x)$$

Theorem 3.21
Suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f : E \to \mathbb{R}$. Then the following statements are equivalent:

1. f is continuous at $a \in E$.
2. If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

Theorem 3.22
Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (respectively continuous on the set E), then so are $f + g$, fg, and αf (for any $\alpha \in \mathbb{R}$). Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$ (respectively, on E when $g(x) \neq 0 \ \forall x \in E$).

Definition 3.23 *Composition*
Suppose that A and B are subsets of R, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $F(A) \subseteq B$ for every $x \in A$, then the *composition* of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by
$$(g \circ f)(x) := g(f(x)), \quad x \in A$$

Theorem 3.24
Suppose that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that $f(x) \in B \ \forall x \in A$.

1. If $A := I \backslash \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if
$$L := \lim_{x \to a; x \in I} f(x)$$
exists and belongs to B, and if g is continuous and $L \in B$, then
$$\lim_{x \to a; x \in I} (g \circ f)(x) = g\left(\lim_{x \to a; x \in I} f(x)\right)$$
2. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.

Definition 3.25 *Bounded*
Let E be a nonempty subset of $\mathbb{R}$. A function $f : E \to \mathbb{R}$ is said to be *bounded* on E if and only if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$, in which case we shall say that f is *dominated* by M on E.

Theorem 3.26 *Extreme Value Theorem*
If I a is closed, bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover if
$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x)$$
then there exist points $x_m, x_M \in I$ such that
$$f(x_M) = M \quad \text{and} \quad f(x_m) = m$$

Remark 3.27 The Existence Value Theorem is false if either "closed" or "bounded" is dropped from the hypotheses.

Lemma 3.28
Suppose that $a < B$ and that $f : [a, b) \to \mathbb{R}$. If f is continuous at a point $x_0 \in [a, b)$ and $f(x_0) > 0$, then there exist a positive number ϵ and a point $x_1 \in [a, b)$ such that $x_1 > x_0$ and $f(x) > \epsilon \ \forall x \in [x_0, x_1]$.

Theorem 3.29 *Intermediate Value Theorem*
Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

Remark 3.34 The composition of two functions $g \circ f$ can be nowhere continuous, even though f is discontinuous only on $\mathbb{Q}$ and g is discontinuous at only one point.

3.4 Uniform Continuity

Definition 3.35 *Uniform continuity*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$. Then f is said to be *uniformly continuous* on E if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that
$$|x - a| < \delta \quad \text{and} \quad x, a, \in E \implies |f(x) - f(a)| < \epsilon \quad \forall a \in E$$

Lemma 3.38
Suppose that $E \subseteq \mathbb{R}$ and that $f : E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, the $f(x_n)$ is Cauchy.

Theorem 3.39
Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

Theorem 3.40
Suppose that $a < b$ and that $f : (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if f can be continuously extended to $[a, b]$; that is, if and only if there is a continuous function $g : [a, b] \to \mathbb{R}$ which satisfies
$$f(x) = g(x), \quad x \in (a, b)$$

4 Differentiability on R

4.1 The Derivative

Definition 4.1 *Differentiable*
A real function f is said to be *differentiable* at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and
$$f'(a) := \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
exists. In this case $f'(a)$ is called the *derivative* of f at a.

Theorem 4.2
A real function f is differentiable at some point $a \in \mathbb{R}$ if and only if there exist an open interval I and a function $F : I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at a, and
$$f(x) = F(x)(x - a) + f(a)$$
holds for all $x \in I$ in which case $F(a) = f'(a)$.

Theorem 4.3
A real function f is differentiable at a if and only if there is a function T of the form $T(x) := m(x)$ such that
$$\lim_{h \to 0} \frac{f(a + h) - f(a) - t(h)}{h} = 0$$

Theorem 4.4
If f is differentiable at a, then f is continuous at a.

Definition 4.6 *Continuously differentiable*
Let I be a nondegenerate interval.

1. A function $f : I \to \mathbb{R}$ is said to be *differentiable* on I if and only if

$$f_i'(a) := \lim_{x \to a; x \in I} \frac{f(x) - f(a)}{x - a}$$

exists and is finite for every $a \in I$.

2. f is said to be *continuously differentiable* on I if and only if f_I' exists and is continuous on I.

Remark 4.9
$f(x) = |x|$ is differentiable on $[0, 1]$ and on $[-1, 0]$ but not on $[-1, 1]$.

4.2 Differentiability Theorems

Theorem 4.10
Let f and g be real functions and $\alpha \in \mathbb{R}$. If f and g are differentiable at a, then $f + g$, αf, $f \cdot g$, and [when $g(a) \neq 0$] f/g are all differentiable at a. In fact,

$$\begin{aligned}(f + g)'(a) &= f'(a) + g'(a) \\ (\alpha f)'(a) &= \alpha f'(a) \\ (f \cdot g)'(a) &= g(a)g'(a) + f(a)g'(a) \\ \left(\frac{f}{g}\right)'(a) &= \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}\end{aligned}$$

Theorem 4.11 *Chain Rule*
Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

4.3 Mean Value Theorem

Lemma 4.12 *Rolle's Theorem*
Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Remark 4.13
The continuity hypothesis on Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.

Remark 4.14
The differentiability hypothesis on Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.

Theorem 4.15
Suppose that $a, b, \in \mathbb{R}$ with $a < b$.

1. *Generalised Mean Value Theorem*: If f, g are continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a))$$

2. *Mean Value Theorem*: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - A)$$

Definition 4.16 *Increasing, Monotone, Decreasing*
Let E be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$.

1. f is said to be *increasing* (respectively, *strictly increasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \leq f(x_2)$ [respectively, $f(x_1) < f(x_2)$].
2. f is said to be *decreasing* (respectively, *strictly decreasing*) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2 \implies f(x_1) \geq f(x_2)$ [respectively, $f(x_1) > f(x_2)$].
3. f is said to be *monotone* (respectively, *strictly monotone*) on E if and only if f is either decreasing or increasing (respectively, either strictly decreasing or strictly increasing) on E.

Theorem 4.17
Suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a, b]$, and that f is differentiable on (a, b).

1. If $f'(x) > 0$ [respectively $f'(x) < 0$] for all $x \in (a, b)$, then f is strictly increasing (respectively, strictly decreasing) on $[a, b]$.
2. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
3. If g is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on $[a, b]$.

Theorem 4.18
Suppose that f is increasing on $[a, b]$

1. If $c \in [a, b)$, then $f(c+)$ exists and $f(c) \leq f(c+)$.
2. If $c \in (a, b]$, then $f(c-)$ exists and $f(c-) \leq f(c)$.

Theorem 4.19
If f is monotone on an interval I, then f has at most countable many points of discontinuity on I.

Theorem 4.21 *Bernoulli's Inequality*
Let α be a positive real number. If $0 < \alpha < 1$, then $(1 + x)^\alpha \leq 1 + \alpha x \ \forall x \in [-1, \infty)$, and if $\alpha \geq 1$, then $(1 + x)^\alpha \geq 1 + \alpha x \ \forall x \in [-1, \infty)$.

Theorem 4.23 *Intermediate Value Theorem for Derivatives*
Suppose that f is differentiable on $[a, b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a, b)$ such that $f'(x_0) = y_0$.

4.4 Taylor's Theorem and L'Hopital's Rule

Theorem 4.24 *Taylor's Formula*
Let $n \in \mathbb{N}$ and let a, b be extended real numbers with $a < b$. If $f : (a, b) \to \mathbb{R}$, and if $f^{(n+1)}$ exists on (a, b), then for each pair of points $x, x_0 \in (a, b)$ there is a number c between x and x_0 such that
$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$

Theorem 4.27 *L'Hopital's Rule*
Let a be an extended real number and I be an open interval which either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I \setminus \{a\}$ and that $g(x) \neq 0 \neq g'(x) \ \forall x \in I \setminus \{a\}$. Suppose further that

$$A := \lim_{x \to a; x \in I} f(x) = \lim_{x \to a; x \in I} g(x)$$

is either 0 or ∞. If

$$B := \lim_{x \to a; x \in I} \frac{f'(x)}{g'(x)}$$

exists as an extended real number, then

$$\lim_{x \to a; x \in I} \frac{f(x)}{g(x)} = \lim_{x \to a; x \in I} \frac{f'(x)}{g'(x)}$$

4.5 Inverse Function Theorems

Theorem 4.32
Let I be a nondegenerate interval and suppose that $f : I \to \mathbb{R}$ is injective. If f is continuous on I, then $J := f(I)$ is an interval, f is strictly monotone on I, and f^{-1} is continuous and strictly monotone on J.

Theorem 4.33 *Inverse Function Theorem*
Let I be an open interval and $f : I \to \mathbb{R}$ be injective and continuous. If $b = f(a)$ for some $a \in I$ and if $f'(a)$ exists and is nonzero, then f^{-1} is differentiable at b and $(f^{-1})'(b) = \frac{1}{f'(a)}$.

5 Riemann Integration

5.1 Introduction

5.2 Step functions and their integrals

Definition 1 *Step function*
We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a *step function* if there exist real numbers $x_0 < x_1 < \cdots < x_n$ (for some $n \in \mathbb{N}$) such that

1. $\phi(x) = 0$ for $x < x_0$ and $x > x_n$
2. ϕ is constant on $(x_{j-1}, x_j) 1 \leq j \leq n$.

Definition 2
If ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ which takes the value c_j on (x_{j-1}, x_j), then

$$\int \phi := \sum_{j=1}^{n} c_j (x_j - x_{j-1})$$

Proposition 1
If ϕ and ψ are step functions and α and $\beta \in \mathbb{R}$, then

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi.$$

5.3 Riemann-integrable functions and their integrals

Definition 3 *Riemann-integrable*
Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is *Riemann-integrable* if for every $\epsilon > 0$ there exist step functions ϕ and ψ such that $\phi \leq f \leq \psi$ and $\int \psi - \int \phi < \epsilon$

Theorem 1
A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if
$\sup\{\int \phi : \phi$ is a step function and $\phi \leq f\} =$
$\inf\{\int \psi : \psi$ is a step function and $\psi \geq f\}$.

Definition 4
If f is Riemann-integrable we define its integral $\int f$ as the common value
$\int f := \sup\{\int \phi : \phi$ is a step function and $\phi \leq f\}$
$= \inf\{\int \psi : \psi$ is a step function and$\psi \geq f\}$.

Theorem 2
A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if there exist sequences of step functions ϕ_n and ψ_n such that $\phi_n \leq f \leq \psi_n \ \forall n$, and $\int \psi_n - \int \phi_n \to 0$
If ϕ_n and ψ_n are any sequences of step functions satisfying above, then $\int \phi_n \to \int f$ and $\int \psi_n \to \int f$ as $n \to \infty$.

Lemma 1
Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with bounded support $[a, b]$. The following are equivalent:

1. f is Riemann-integrable.
2. for every $\epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ such that, if M_j and m_j denote the supremum and infimum values of$f$ on $[x_{j-1}, x_j]$ respectively, then $\sum_{j-1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon$
3. for every $\epsilon > 0$ there exist $a = x_0 < \cdots < x_n = b$ such that, with $I_j = (x_{j-1}, x_j)$ for $j \geq 1$, $\sum_{j=1}^{n} \sup_{x,y,\in I_j} |f(x) - f(y)||I_j| < \epsilon$.
   For $f : \mathbb{R} \to \mathbb{R}$ a bounded function with bounded support $[a, b]$ and for $a = x_0 < \cdots < x_n = b$, let $I_j = (x_{j-1}, x_j), m_j := \inf_{x \in I_j} f(x)$ and $M_j := \sup_{x \in I_j} f(x)$.
   Define the *lower step function of f with respect to* $\{x_0, \ldots, x_n\}$ as
   $\phi_*(x) = \sum_{j=1}^{n} m_j \chi_{I_j} + \sum_{j=0}^{n} \chi_{x_j}$
   and the *upper step function of f with respect to* $\{x_0, \ldots, x_n\}$ as
   $\phi^*(x) = \sum_{j=1}^{n} M_j \chi_{I_j} + \sum_{j=0}^{n} \chi_{x_j}$
   Note that ϕ_* and ϕ^* are step functions, and that $\phi_* \leq f \leq \phi^*$.

Theorem 3
Suppose f and g are Riemann-integrable and α and β are real numbers. Then

1. $\alpha f + \beta g$ is Riemann-integrable and
   $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$
2. If $f \geq 0$ then $\int f \geq 0$; if $f \leq g$ then $\int f \leq \int g$.
3. $|f|$ is Riemann-integrable and $|\int f| \leq \int |f|$
4. $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann-integrable.
5. fg is Riemann-integrable.

Theorem 4
If $g : [a, b] \to \mathbb{R}$ is continuous, and f defined by $f(x) = g(x)$ for $a \leq x \leq b$, $f(x) = 0$ for $x \notin [a, b]$ then f is Riemann-integrable.

5.4 Fundamental Theorem of Calculus, and Practical Integration

Theorem 5
Let $g : [a, b] \to \mathbb{R}$ be Riemann-integrable. For $a \leq x \leq b$ let $G(x) = \int_a^x g$. Suppose g is continuous at x for some $x \in [a, b]$. [If x is an endpoint, we mean one-sided continuous.] Then G is differentiable at x and $G'(x) = g(x)$. [If x is an endpoint, we mean one-sided differentiable.]

Theorem 6
Suppose $f : [a, b] \to \mathbb{R}$ has continuous derivative f' on $[a, b]$. Then $\int_a^b f' = f(b) - f(a)$.

5.5 Integrals and uniform limits of sequences and series of functions

Theorem 7
Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence if Riemann-integrable functions which converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then f is Riemann-integrable and $\int f = \lim_{n\to\infty} \int f_n$.

6 Infinite Series of Real Numbers

6.1 Introduction

Definition 6.1 *Partial sum*
Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series with terms a_k.

1. For each $n \in \mathbb{N}$, the *partial sum of S of order n* is defined by
   $s_n := \sum_{k=1}^{n} a_k$
2. S is said to *converge* if and only if its sequence of partial sums $\{s_n\}$ converges to some $s \in \mathbb{R}$ as $n \to \infty$; that is, if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N \implies |s_n - s| < \epsilon$. In this case we shall write $\sum_{k=1}^{\infty} a_k = s$
   and call s the *sum*, or *value*, of the series $\sum_{k=1}^{\infty} a_k$
3. S is said to *diverge* if and only if its sequence of partial sums $\{s_n\}$ does not converge as $n \to \infty$. When s_n diverges to $+\infty$ as $n \to \infty$, we shall also write
   $\sum_{k=1}^{\infty} a_k = s$

Theorem 6.5 *Divergence Test*
Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence of real numbers. If a_k does not converge to zero, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 6.6 *Telescoping Series*
If $\{a_k\}$ is a convergent real sequence, then

$$\sum_{k=1}^{\infty} (a_k - a_{k+1}) = a_1 - \lim_{k\to\infty} a_k$$

Theorem 6.7 *Geometric Series*
Suppose that $x \in \mathbb{R}$, that $N \in \{0, 1, \ldots\}$, and that 0^0 is in-

terpreted to be 1. Then the series $\sum_{k=N}^{\infty} x^k$ converges if and only if $|x| < 1$, in which case

$$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}$$

In particular,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1.$$

Theorem 6.8 *The Cauchy Criterion*
Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m \geq n \geq N \implies \left|\sum_{k=n}^{m} a_k\right| < \epsilon$$

Corollary 6.9
Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \left|\sum_{k=n}^{\infty} a_k\right| < \epsilon$$

Theorem 6.10
Let $\{a_k\}$ and $\{b_k\}$ be real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent series, then
$\sum_{i=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$
and
$\sum_{k=1}^{\infty}(\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$
for any $\alpha \in \mathbb{R}$.

6.2 Series with Nonnegative Terms

Theorem 6.11
Suppose that $a_k \geq 0$ for large k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if its sequence of partial sums $\{s_n\}$ is bounded; that is, if and only if there exists a finite number $M > 0$ such that $|\sum_{i=1}^{n} a_k| \leq M \ \forall n \in \mathbb{N}$.

Theorem 6.12 *Integral Test*
Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly integrable on $[1, \infty)$; that is if and only if

$$\int_1^{\infty} f(x)\,\mathrm{d}x < \infty$$

Corollary 6.13 *p-Series Test* The series

$$\sum_{i=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Theorem 6.14 *Comparison Test*
Suppose that $0 \leq a_k \leq b_k$ for large k.

> If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$. If $\sum_{k=1}^{\infty} b_k = \infty$, then $\sum_{k=1}^{\infty} a_k = \infty$.

Theorem 6.16 *Limit Comparison Test*
Suppose that $a_k \geq 0$, that $b_k > 0$ for large k, and that $L := \lim_{n\to\infty} \frac{a_n}{b_n}$ exists as an extended real number.

1. If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.
2. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.
3. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

6.3 Absolute Convergence

Definition 6.18 *Absolute & Conditional Convergence*
Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series.

1. S is said to *converge absolutely* if and only if $\sum_{k=1}^{\infty} |a_k| < \infty$
2. S is said to *converge conditionally* if and only if S converges but not absolutely.

Remark 6.19
A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m > n \geq N \implies \sum_{k=n}^{m} |a_k| < \epsilon$$

Remark 6.20
If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k$ converges, but not conversely. In particular, there exist conditionally convergent series.

Definition 6.21 *Limit supremum*
The *limit supremum* of a sequence of real numbers $\{x_k\}$ is defined to be

$$\limsup_{k\to\infty} x_k := \lim_{n\to\infty} \left(\sup_{k>n} x_k\right).$$

Remark 6.22
Let $x \in \mathbb{R}$ and $\{x_k\}$ be a real sequence.

1. If $\limsup_{k\to\infty} x_k < x$, then $x_k < x$ for large k.
2. If $\limsup_{k\to\infty} x_k > x$, then $x_k > x$ for infinitely many ks.
3. If $x_k \to x$ as $x \to \infty$, then $\limsup_{k\to\infty} x_k = x$.

Theorem 6.23 *Root Test*
Let $a_k \in \mathbb{R}$ and $r := \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

1. If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
2. If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 6.24 *Ratio test*
Let $a_k \in \mathbb{R}$ with $a_k \neq 0$ for large k and suppose that

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

exists as an extended real number.

1. If $r < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.
2. If $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.25 The Root and Ratio tests are inconclusive when $r = 1$.

Definition 6.26 *Rearrangement*
A series $\sum_{j=1}^{\infty} b_j$ is called a *rearrangement* of a series $\sum_{k=1}^{\infty} a_k$ if and only if there is an injection $f : \mathbb{N} \to \mathbb{N}$ such that

$$b_{f(k)} = a_k, \quad k \in \mathbb{N}$$

Theorem 6.27
If $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\sum_{j=1}^{\infty} b_j$ is any rearrangement of $\sum_{k=1}^{\infty} a_k$, then $\sum_{j=1}^{\infty} b_j$ converges and

$$\sum_{k=1}^{\infty} a_k = \sum_{j=1}^{\infty} b_j$$

6.4 Alternating Series

Theorem 6.30 *Abel's Formula*
Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be real sequences, and for each pair of integers $n \geq m \geq 1$ set
$A_{n,m} := \sum_{k=m}^{n} a_k$ Then
$\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$ for all integers $n > m \geq 1$.

Theorem 6.31 *Dirichlet's Test*
Let $a_k, b_k \in \mathbb{R}$ for $k \in \mathbb{N}$. If the sequence of partial sums $s_n = \sum_{k=1}^n a_k$ is bounded and $b_k \to 0$ as $k \to \infty$, then $\sum_{k=1}^\infty a_k b_k$ converges.

Corollary 6.32 *Alternating Series Test*
If $a_k \to 0$ as $k \to \infty$, then
$\sum_{k=1}^\infty (-1)^k a_k$
converges.

7 Infinite Series of Functions

7.1 Uniform Convergence of Sequences

Definition 7.1 *Pointwise Convergence*
Let E be a nonempty subset of $\mathbb{R}$. A sequence of functions $f_n : E \to \mathbb{R}$ is said to *converge pointwise* on E if and only if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in E$.

Remark 7.2
Let E be a nonempty subset of $\mathbb{R}$. Then a sequence of functions f_n converges pointwise on E, as $n \to \infty$ if and only if for every $\epsilon > 0$ and $x \in E$ there is an $N \in \mathbb{N}$ (which may depend on x as well as ϵ) such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

Remark 7.3
The pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable).

Remark 7.4
The pointwise limit of integrable functions is not necessarily integrable.

Remark 7.5
There exist differentiable functions f_n and f such that $f_n \to f$ pointwise on $[0,1]$ but

$$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)'$$

for $x = 1$.

Remark 7.6
There exist continuous functions f_n and f such that $f_n \to f$ pointwise on $[0,1]$ but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x$$

Definition 7.7 *Uniform Convergence*
Let E be a nonempty subset of $\mathbb{R}$. A sequence of function $f_n : E \to \mathbb{R}$ is said to *converge uniformly* on E to a function f if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon$$

for all $x \in E$.

Theorem 7.9
Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_n \to f$ uniformly on E, as $n \to \infty$. If f_n is continuous at some $x_0 \in E$, then f is continuous at $x_0 \in E$.

Theorem 7.10
Suppose that $f_n \to f$ uniformly on a closed interval $[a,b]$. If each f_n is integrable on $[a,b]$, then so is f and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \left(\lim_{n\to\infty} f_n(x)\right)\,\mathrm{d}x$$

In fact, $\lim_{n\to\infty} \int_a^x f_n(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t$ uniformly for $x \in [a,b]$.

Lemma 7.11 *Uniform Cauchy Criterion*
Let E be a nonempty subset of $\mathbb{R}$ and let $f_n : E \to \mathbb{R}$ be a sequence of functions. Then f_n converges uniformly on E if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |f_n(x) - f_m(x)| < \epsilon$$

for all $x \in E$.

Theorem 7.12
Let (a,b) be a bounded interval and suppose that f_n is a sequence of funtions which converges at some $x_0 \in (a,b)$. If each f_n is differentiable on (a,b), and f_n' converges uniformly on (a,b) as $n \to \infty$, the f_n converges uniformly on (a,b) and $\lim_{n\to\infty} f_n'(x) = (\lim_{n\to\infty} f_n(x))'$
for each $x \in (a,b)$.

7.2 Uniform Convergence of Series

Definition 7.13 *Convergence*
Let f_k be a sequence of real functions defined on some set E and set

$$s_n(k) := \sum_{k=1}^n f_k(x), \quad x \in E, n \in \mathbb{N}$$

1. The series $\sum_{k=1}^n f_k(x)$ is said to *converge pointwise* on E if and only if the sequence $s_n(x)$ converges pointwise on E as $n \to \infty$.
2. The series $\sum_{k=1}^n f_k(x)$ is said to *converge uniformly* on E if and only if the sequence $s_n(x)$ converges uniformly on E as $n \to \infty$.
3. The series $\sum_{k=1}^n f_k(x)$ is said to *converge absolutely (pointwise)* on E if and only if $\sum_{k=1}^n |f_k(x)|$ converges for each $x \in E$.

Theorem 7.14
Let E be a nonempty subset of $\mathbb{R}$ and let $\{f_k\}$ be a sequence of real functions defined on E.

1. Suppose that $x_0 \in E$ and that each f_k is continuous at $x_0 \in E$. If $f = \sum_{k=1}^\infty f_k$ converges uniformly on E, then f is continuous at $x_0 \in E$.
2. *Term-by-term integration.* Suppose that $E = [a,b]$ and that each f_k is integrable on $[a,b]$. If $f = \sum_{k=1}^\infty f_k$ converges uniformly on $[a,b]$, then f is integrable on $[a,b]$ and
$$\int_a^b \sum_{k=1}^\infty f_k(x)\,\mathrm{d}x = \sum_{k=1}^\infty \int_a^b f_k(x)\,\mathrm{d}x.$$
3. *Term-by-term differentiation.* Suppose that E is a bounded, open interval and that each f_k is differentiable on E. If $\sum_{k=1}^\infty f_k$ converges at some $x_0 \in E$, and $\sum_{k=1}^f {}'_k$ converges uniformly on E, then $f := \sum_{k=1}^\infty f_k$ converges uniformly on E, f is differentiable on E, and
$$\left(\sum_{k=1}^\infty f_k(x)\right)' = \sum_{k=1}^\infty f_k'(x)$$
for $x \in E$.

Theorem 7.15 *Weierstrass M-Test*
Let E be a nonempty subset of $\mathbb{R}$, let $f_k : E \to \mathbb{R}, k \in \mathbb{N}$, and suppose that $M_k \geq 0$ satisfies $\sum_{k=1}^\infty M_k < \infty$. If $|f_k(x)| \leq M_k$ for $k \in \mathbb{N}$ and $x \in E$, then $\sum_{k=1}^\infty f_k$ converges absolutely and uniformly on E.

7.3 Power Series

Definition *Power Series*
Let (a_n) be a sequence of real numbers, and $c \in \mathbb{R}$. A *power series* is a series of the form
$\sum_{n=1}^\infty a_n (x-c)^n$
With a_n being the *coefficients* and c its centre.

Definition *Radius of Convergence*
The *radius of convergence* R of the power series
$\sum_{n=1}^\infty a_n (x-c)^n$

is defined by
$R = \sup\{r \geq 0 : (a_n r^n) \text{ is bounded}\}$
unless $(a_n r^n)$ is bounded for all $r \geq 0$, in which case we declare $R = \infty$.

Theorem 1
Suppose the radius of convergence R satisfies $0 < R < \infty$. If $|x - c| < R$, the power series converges absolutely. If $|x - c| > R$, the power series diverges.

Theorem 2
Assume that $R > 0$. Suppose that $0 < r < R$. Then the series converges uniformly and absolutely on $|x - c| \leq r$ to a continuous function f. Hence
$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$
defines a continuous function $f : (c - R, c + R) \to \mathbb{R}$.

Lemma
The two power series $\sum_{n=1}^{\infty} a_n (x - c)^n$ and $\sum_{n=1}^{\infty} n a_n (x - c)^{n-1}$ have the same radius of convergence.

Theorem 3
Suppose the radius of convergence of the power series is R. Then the function
$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$
is infinitely differentiable on $|x - c| < R$, and for such x,
$f'(x) = \sum_{n=0}^{\infty} n a_n (x - c)^{n-1}$
and the series converges absolutely, and also uniformly on $[c - r, c + r]$ $\forall r < R$. Moreover $a_n = \frac{f^{(n)}(c)}{n!}$

8 Metric Spaces

8.1 Introduction

Definition 10.1 *Metric Space*
A *metric space* is a set X together with a function $\rho : X \times X \to \mathbb{R}$ (called the *metric* of ρ) which satisfies the following properties for all $x, y, z \in X$:

Positive Definite	$\rho(x, y) \geq 0$ with $\rho(x, y) = 0 \iff x = y$
Symmetric	$\rho(x, y) = \rho(y, x)$
Triangle Inequality	$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$

Definition 10.7 *Ball*
Let $a \in X$ and $r > 0$. Then *open ball* (in X) with *centre* a and *radius* r is the set
$B_r(a) := \{x \in X : \rho(x, a) < r\}$
and the *closed ball* (in X) with *centre* a and *radius* r is the set $\{x \in X : \rho(x, a) \leq r\}$

Definition 10.8 *Open & Closed*

1. A set $V \subseteq X$ is said to be *open* if and only if for every $x \in V$ there is an $\epsilon > 0$ such that the open ball $B_\epsilon(x)$ is contained in V.
2. A set $E \subseteq X$ is set to be *closed* if and only if $E^c := X \setminus E$ is open.

Remark 10.9 Every open ball is open, and every closed ball is closed.

Remark 10.10 If $a \in X$, then $X \setminus \{a\}$ is open, and $\{a\}$ is closed.

Remark (10.11) In an arbitrary metric space, the empty set $\emptyset$ and the whole space X are both open and closed.

Definition 10.13 *Convergence, Cauchy, & Boundedness*
Let $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ *converges* (in X) if there is a point $a \in X$ (called the *limit* of x_n) such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
   $n \geq N \implies \rho(x_n, a) < \epsilon$.
2. $\{x_n\}$ is *Cauchy* if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that
   $n, m \geq N \implies \rho(x_n, x_m) < \epsilon$.
3. $\{x_n\}$ is *bounded* if there is an $M > 0$ and a $b \in X$ such that $\rho(x_n, b) \leq M$ for all $n \in \mathbb{N}$.

Theorem 10.14
Let X be a metric space.

1. A sequence X can have at most one limit.
2. If $x_n \in X$ converges to a and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then x_{n_k} converges to a as $k \to \infty$.
3. Every convergent sequence X is bounded.
4. Every convergent sequence in X is Cauchy.

Remark 10.15
Let $x_n \in X$. Then $x_n \to a$ as $n \to \infty$ if and only if for every open set V which contains a, there is an $N \in \mathbb{N}$ such that $n \geq N \implies x_n \in V$.

Theorem 10.16
Let $E \subseteq X$. Then E is closed if and only if the limit of every convergent sequence $x_k \in E$ satisfies
$\lim_{k\to\infty} x_k \in E$.

Remark 10.17 The discrete space contains bounded sequence which have no convergent subsequences.

Remark 10.18 The metric space $X = \mathbb{Q}$ contains Cauchy sequences which do not converge.

Definition 10.19 *Completeness*
A metric space X is said to be *complete* if and only if every Cauchy sequence $x_n \in X$ converges to some point in X.

Remark 10.20
By *10.19*, a complete metric space X satisfies two properties:

1. Every Cauchy sequence in X converges;
2. the limit of every Cauchy sequence in X stay in X.

Theorem 10.21
Let X be a complete metric space E be a subset of X. Then E (as a subspace) is complete if and only if E as a (subset) is closed.

8.2 Limits of Functions

Definition 10.22 *Cluster Point*
A point $a \in X$ is said to be a *cluster point* (of X) if and only if $B_\delta(a)$ contains infinitely many points for each $\delta > 0$.

Definition 10.25 *Converge*
Let a be a cluster point of X and $f : X \setminus \{a\} \to Y$. Then $f(x)$ is said to *converge to L, as x approaches a*, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$0 < \rho(x, a) < \delta \implies \tau(f(x), L) < \epsilon.$$

In this case we write $f(x) \to L$ as $x \to a$, or

$$L = \lim_{x\to a} f(x),$$

and call L the *limit* of $f(x)$ as x approaches a.

Theorem 10.26
Let a be a cluster point of X and $f, g : X \setminus \{a\} \to Y$.

1. If $f(x) = g(x)$ $\forall x \in X \setminus \{a\}$ and $f(x)$ has a limit as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and
   $$\lim_{x\to a} g(x) = \lim_{x\to a} f(x).$$
2. *Sequential characterisation of limits.* The limit
   $$L := \lim_{x\to a} f(x)$$
   exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in X \setminus \{a\}$ which converges to a as $n \to \infty$.

3. Suppose that $Y = \mathbb{R}^n$. If $f(x)$ and $g(x)$ have a limit as x approaches a, then so do $(f+g)(x), (f\cdot g)(x), (\alpha f)(x)$, and $(f/g)(x)$ [when $Y = \mathbb{R}$ and the limit of $g(x)$ is nonzero]. In fact,

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$
$$\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x),$$
$$\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

and [when $Y = \mathbb{R}$ and the limit of $g(x)$ is nonzero]

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

4. *Squeeze Theorem for Functions.* Suppose that $Y = \mathbb{R}$. If $h : X\setminus\{a\} \to \mathbb{R}$ satisfies $g(x) \le h(x) \le f(x)$ $\forall x \in X\setminus\{a\}$, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$$

then the limit of h exists, as $x \to a$, and

$$\lim_{x\to a} h(x) = L$$

5. *Comparison Theorem for Functions.* Suppose that $Y = \mathbb{R}$. If $f(x) \le g(x)$ $\forall X\setminus\{a\}$, and if f and g have a limit as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Definition 10.27 *Continuity*
Let E be a nonempty subset of X and $f : E \to Y$.

1. f is said to be *continuous at a point* $a \in E$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x,a) < \delta \text{ and } \in E \implies \tau(f(x), f(a)) < \epsilon.$$

2. f is said to be *continuous on* E if and only if f is continuous at every $x \in E$.

Theorem 10.28
Let E be a nonempty subset of X and $f, g : E \to Y$.

1. f is continuous at $a \in E$ if and only if $f(x_n) \to f(a)$, as $n \to \infty$, for all sequences $x_n \in E$ which converge to a.
2. Suppose that $Y = \mathbb{R}^n$. If f, g are continuous at a point $a \in E$ (respectively continuous on a set E), then so are $f+g, f\cdot g$, and αf (for any $\alpha \in \mathbb{R}$). Moreover, in the case $Y = \mathbb{R}$, f/g is continuous at $a \in E$ when $g(a) \neq 0$ [respectively, on E when $g(x) \neq 0$, $\forall x \in E$].

Theorem 10.29
Suppose that X, Y, and Z are metric space and that a is a cluster point of X. Suppose further that $f : X \to Y$ and $g : f(X) \to Z$. If $f(x) \to L$ as a $x \to a$ and g is continuous at L, then

$$\lim_{x\to a}(g\circ f)(x) = g\left(\lim_{x\to a} f(x)\right).$$

Definition 10.30 *Bolzano-Weierstrass Property*
X is said to satisfy the *Bolzano-Weierstrass Property* if and only if every bounded sequence $x_n \in X$ has a convergent subsequence.

8.3 Interior, Closure, and Boundary

Theorem 10.31
Let X be a metric space.

1. If $\{V_\alpha\}_{\alpha\in A}$ is any collection of open sets in X, then $\bigcup_{\alpha\in A} V_\alpha$ is open.
2. If $\{V_k : k = 1, 2, \ldots, n\}$ is a finite collection of open sets in X, then $\bigcap_{k=1}^n V_k := \bigcap_{k\in\{1,2,\ldots,n\}} V_k$ is open.
3. If $\{E_\alpha\}_{\alpha\in A}$ is any collection of closed sets in X, then $\bigcap_{\alpha\in A} E_\alpha$ is closed.
4. If $\{E_k : k = 1, 2, \ldots, n\}$ is a finite collection of closed sets in X, then $\bigcup_{k=1}^n E_k := \bigcup_{k\in\{1,2,\ldots,n\}} E_k$ is closed.
5. If V is open in X and E is closed in X, then $V\setminus E$ is open and $E\setminus V$ is closed.

Remark 10.32
Statements 2 and 4 of *Theorem 10.31* are false if arbitrary collections are used in place of finite collections.

Definition 10.33 *Interior & Closure*
Let E be a subset of a metric space X.

1. The *interior* of E is the set

$$E^O := \bigcup\{V : V \subseteq E \text{ and } V \text{ is open in } X\}.$$

2. The *closure* of E is the set

$$\overline{E} := \bigcap\{B : B \supseteq E \text{ and } B \text{ is closed in } X\}.$$

Theorem 10.34
Let $E \subseteq X$. Then

1. $E^O \subseteq E \subseteq \overline{E}$,
2. if V is open and $V \subseteq E$, then $V \subseteq E^0$, and
3. if C is closed and $C \supseteq E$, then $C \supseteq \overline{E}$.

Definition 10.37 *Boundary*
Let $E \subseteq X$. The *boundary* of E is the set

$$\partial E := \{x \in X : \forall r > 0,\ B_r(x)\cap E \neq \emptyset \text{ and } B_r(x)\cup E^c \neq \emptyset\}.$$

[We refer to the last two conditions in the definition of ∂E by saying $B_r(x)$ *intersects* E and E^c.]

Theorem 10.39
Let $E \subseteq X$. Then

$$\partial E = \overline{E}\setminus E^0.$$

Theorem 10.40
Let $A, B \subseteq X$. Then

1. $(A\cup B)^O \supseteq A^O \cup B^O$, $\quad (A\cap B)^O = A^O \cap B^O$,
2. $\overline{A\cup B} = \overline{A}\cup\overline{B}$, $\quad \overline{A\cap B} \subseteq \overline{A}\cap\overline{B}$,
3. $(A\cup B) \subseteq A\cup B$, and $(A\cap B) \subseteq (A\cap B)\cup(B\cap A)\cup(A\cap B)$.

8.4 Compact Sets

Definition 10.41 *Covering*
Let $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ be a collection of subsets of a metric space X and suppose that E is a subset of X.

1. $\mathcal{V}$ is said to *cover* E if and only if

$$E \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

2. $\mathcal{V}$ is said to be an *open covering* of E if and only if $\mathcal{V}$ covers E and each V_α is open.
3. Let $\mathcal{V}$ be a covering of E. $\mathcal{V}$ is said to have a *finite* (respectively *countable*) *subcovering* if and only if there is a finite (respectively, countable) subset A_0 of A such that $\{V_\alpha\}_{\alpha\in A_0}$ covers E.

Definition 10.42 *Compact*
A subset H of a metric space X is said to be *compact* if and only if every open covering of H has a finite subcover.

Remark 10.43 The empty set and all finite subsets of a metric space are compact.

Remark 10.44 A compact set is always closed.

Remark 10.45 A closed subset of a compact set is compact.

Theorem 10.46
Let H be a subset of a metric space X. If H is compact, then H is closed and bounded.

Remark 10.47 The converse of *Theorem 10.46* is false for arbitrary metric spaces

Definition 10.48 *Separable*
A metric space X is said to be *separable* if and only if it contains a countable dense subset (i.e. if and only if there is a countable subset Z of X such that for every point $a \in X$ there is a sequence $x_k \in Z$ such that $x_k \to a$ as $k \to \infty$).

Theorem 10.49 *Lindelöf*
Let E be a subset of a separable metric space X. If $\{V_\alpha\}_{\alpha \in A}$ is a collection of open sets and $E \subseteq \bigcup_{\alpha \in A} V_\alpha$, then there is a countable subset $\{\alpha_1, \alpha_2, \ldots\}$ of A such that

$$E \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}$$

Theorem 10.50 *Heine-Borel*
Let X be a separable metric space which satisfies the *Bolzano-Weierstrass Property*, and H be a subset of X. Then H is compact if and only if it is closed and bounded.

Definition 10.51 *Uniform Continuity*
Let X be a metric space, E be a nonempty subset of X, and $f : E \to Y$. Then f is said to be *uniformly continuous* on E if and only if given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x, a) < \delta \text{ and } x, a \in E \implies \tau(f(x), f(a)) < \epsilon.$$

Theorem 10.52
Suppose that E is a compact subset of X and that $f : X \to Y$. Then f is uniformly continuous on E if and only if f is continuous on E.

8.5 Connected Sets

Definition 10.53 *Separate & Connected*
Let X be a metric space.

1. A pair of nonempty open sets U, V in X is said to *separate* X if and only if $X = U \cup V$ and $U \cap V = \emptyset$.
2. X is said to be *connected* if and only if X cannot be separated by any pair of open sets U, V.

Definition 10.54 *Relatively open & closed*
Let X be a metric space and $E \subseteq X$.

1. A set $U \subseteq E$ is said to be *relatively open* in E if and only if there is a set V open in X such that $U = E \cap V$.
2. A set $A \subseteq E$ is said to be *relatively closed* in E if and only if there is a set C closed in X such that $A = E \cap C$.

Remark 10.55
Let $E \subseteq X$. If there exists a pair of open sets A, B in X which separate E, then E is not connected.

Theorem 10.56
A subset E of $\mathbb{R}$ is connected if and only if E is an interval.

8.6 Continuous Functions

Theorem 10.58
Suppose that $f : X \to Y$. Then f is continuous if and only if $f^{-1}(V)$ is open in X for every open V in Y.

Corollary 10.59
Let $E \subseteq X$ and $f : E \to Y$. Then f is continuous on E if and only if $f^{-1}(V) \cap E$ is relatively open in E for all open sets V in Y.

Theorem 10.61
If H is compact in X and $f : H \to Y$ is continuous on H, then $f(H)$ is compact in Y.

Theorem 10.62
If E is connected in X and $f : E \to Y$ is continuous on E, then $f(E)$ is connected in Y.

Theorem 10.63 *Extreme Value Theorem*
Let H be a nonempty, compact subset of X and suppose that $f : H \to \mathbb{R}$ is continuous. Then

$$M := \sup\{f(x) : x \in H\} \quad \text{and} \quad m := \inf\{f(x) : x \in H\}$$

are finite real numbers and there exist points $x_M, x_m \in H$ such that $M = f(x_M)$ and $m = f(x_m)$.

Theorem 10.64
If H is a compact subset of X and $f : H \to Y$ is injective and continuous, then f^{-1} is continuous on $f(H)$.

9 Contraction Mapping & ODEs

9.1 Banach's Contraction Mapping Theorem

Definition *Contraction*
Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if there exists a number α with $0 < \alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \; \forall x, y \in X.$$

Note the target space and the domain must be the same.

Remark

1. It is *really* important that α be *strictly less* than 1. It's also really important that we have $d(f(x), f(y)) \leq \alpha d(x, y)$ and *not just* $d(f(x), f(y)) < d(x, y) \; \forall x, y \in X$. So $f(x) = \cos(x)$ is not a contraction on $\mathbb{R}$.
2. The constant $\alpha < 1$ is called the *contraction constant* of f.

Theorem *Banach's Contraction Mapping Theorem*
If (X, d) is a complete metric space and if $f : X \to X$ is a contraction, then there is a unique point $x \in X$ such that $f(x) = x$.

Remarks

1. It's really important that X be complete.
2. It's really important that the image of X under f is contained in X.
3. A point x such that $f(x) = x$ is called a *fixed point* of f.

9.2 Existence and uniqueness for solutions to ODEs

Definition *Lipschitz Condition*
Suppose $A \in \mathbb{R}$, $\rho, r > 0$, and $F : [A - \rho, A + \rho] \times [-r, r] \to \mathbb{R}$ is continuous. Suppose also that for all $x, y \in [A - \rho, A + \rho]$ and all $t \in [-r, r]$ we have, for some $M > 0$

$$|F(x, t) - F(y, t)| \leq M|x - y|$$

Theorem *Picard*
Suppose F satisfies the Lipschitz Condition. Then there exists an $s > 0$ such that the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x, t)$$
$$x(0) = A$$

has a unique solution $x(t)$ for $|t| < s$.