Proof. By Definition 7.1, $f_n \to f$ pointwise on E if and only if $f_n(x) \to f(x)$ for all $x \in E$. This occurs, by Definition 2.1, if and only if for every $\varepsilon > 0$ and $x \in E$ there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$. ■

Proof. Let $f_n(x) = x^n$ and set

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

Then $f_n \to f$ pointwise on $[0, 1]$ (see Example 2.20), each f_n is continuous and differentiable on $[0, 1]$, but f is neither differentiable nor continuous at $x = 1$. ■

Proof. Set

$$f_n(x) = \begin{cases} 1 & x = p/m \in \mathbf{Q}, \text{ written in reduced form, where } m \leq n \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbf{N}$ and

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is integrable on $[0, 1]$ (with integral zero), but f is not integrable on $[0, 1]$ (see Example 5.11). ■

Proof. Let $f_n(x) = x^n/n$ and set $f(x) = 0$. Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is differentiable with $f_n'(x) = x^{n-1}$. Thus the left side of (1) is 1 at $x = 1$ but the right side of (1) is zero. ■

Proof. Let $f_1(x) = 1$ and, for $n > 1$, let f_n be a sequence of functions whose graphs are triangles with bases $2/n$ and altitudes n (see Figure 7.1). By the point-slope form, formulas for these f_n's can be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x < 1/n \\ 2n - n^2 x & 1/n \leq x < 2/n \\ 0 & 2/n \leq x \leq 1. \end{cases}$$

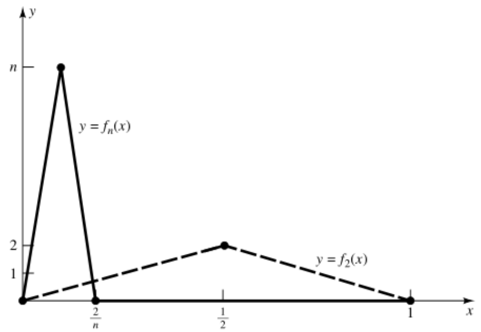

FIGURE 7.1

Then $f_n \to 0$ pointwise on $[0, 1]$ and, since the area of a triangle is one-half base times altitude, $\int_0^1 f_n(x)\,dx = 1$ for all $n \in \mathbf{N}$. Thus, the left side of (2) is 1 but the right side is zero. ■

Proof. Let $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{and} \quad x \in E \quad \text{imply} \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

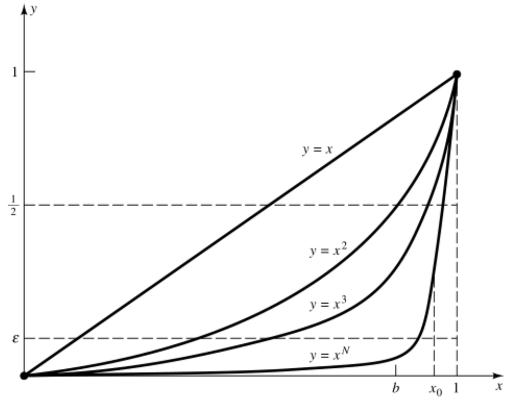

FIGURE 7.3

Since f_N is continuous at $x_0 \in E$, choose $\delta > 0$ such that

$$|x - x_0| < \delta \quad \text{and} \quad x \in E \quad \text{imply} \quad |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

Suppose that $|x - x_0| < \delta$ and that $x \in E$. Then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \varepsilon.$$

Thus f is continuous at $x_0 \in E$. ■

Proof. By Exercise 7.1.3, f is bounded on $[a, b]$. To prove that f is integrable, let $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad |f(x) - f_n(x)| < \frac{\varepsilon}{3(b-a)} \tag{3}$$

for all $x \in [a, b]$. Using this inequality for $n = N$, we see that by the definition of upper and lower sums,

$$U(f - f_N, P) \leq \frac{\varepsilon}{3} \quad \text{and} \quad L(f - f_N, P) \geq -\frac{\varepsilon}{3}$$

for any partition P of $[a, b]$. Since f_N is integrable, choose a partition P such that

$$U(f_N, P) - L(f_N, P) < \frac{\varepsilon}{3}.$$

It follows that

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f - f_N, P) + U(f_N, P) - L(f_N, P) - L(f - f_N, P) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon; \end{aligned}$$

that is, f is integrable on $[a, b]$. We conclude by Theorem 5.22 and (3) that

$$\left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right| \leq \int_a^x |f_n(t) - f(t)|\,dt \leq \frac{\varepsilon(x-a)}{3(b-a)} < \varepsilon$$

for all $x \in [a, b]$ and $n \geq N$. ■

Proof. Suppose first that $f_n \to f$ uniformly on E as $n \to \infty$. Let $\varepsilon > 0$ and choose $N \in \mathbf{N}$ such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

for $x \in E$. Since $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$, it is clear that (4) holds for all $x \in E$.

Conversely, if (4) holds for $x \in E$, then $\{f_n(x)\}_{n \in \mathbf{N}}$ is Cauchy for each $x \in E$. Hence, by Cauchy's Theorem for sequences (Theorem 2.29),

$$f(x) := \lim_{n \to \infty} f_n(x)$$

exists for each $x \in E$. Take the limit of the second inequality in (4) as $m \to \infty$. We obtain $|f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon$ for all $n \geq N$ and $x \in E$. Hence, by definition, $f_n \to f$ uniformly on E. ■

Proof. Fix $c \in (a, b)$ and define

$$g_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} & x \neq c \\ f_n'(c) & x = c \end{cases}$$

for $n \in \mathbf{N}$. Clearly,

$$f_n(x) = f_n(c) + (x - c)g_n(x) \tag{5}$$

for $n \in \mathbf{N}$ and $x \in (a, b)$.

We claim that for any $c \in (a, b)$, the sequence g_n converges uniformly on (a, b). Let $\varepsilon > 0$, $n, m \in \mathbf{N}$, and $x \in (a, b)$ with $x \neq c$. By the Mean Value Theorem, there is a ξ between x and c such that

$$g_n(x) - g_m(x) = \frac{f_n(x) - f_m(x) - (f_n(c) - f_m(c))}{x - c} = f_n'(\xi) - f_m'(\xi).$$

Since f_n' converges uniformly on (a, b), it follows that there is an $N \in \mathbf{N}$ such that

$$n, m \geq N \quad \text{implies} \quad |g_n(x) - g_m(x)| < \varepsilon$$

for $x \in (a, b)$ with $x \neq c$. This implication also holds for $x = c$ because $g_n(c) = f_n'(c)$ for all $n \in \mathbf{N}$. This proves the claim.

To show that f_n converges uniformly on (a, b), notice that by the claim, g_n converges uniformly as $n \to \infty$ and (5) holds for $c = x_0$. Since $f_n(x_0)$ converges as $n \to \infty$ by hypothesis, it follows from (5) and $b - a < \infty$ that f_n converges uniformly on (a, b) as $n \to \infty$.

Fix $c \in (a, b)$. Define f, g on (a, b) by $f(x) := \lim_{n\to\infty} f_n(x)$ and $g(x) := \lim_{n\to\infty} g_n(x)$. We need to show that

$$f'(c) = \lim_{n \to \infty} f_n'(c). \tag{6}$$

Since each g_n is continuous at c, the claim implies g is continuous at c. Since $g_n(c) = f_n'(c)$, it follows that the right side of (6) can be written as

$$\lim_{n \to \infty} f_n'(c) = \lim_{n \to \infty} g_n(c) = g(c) = \lim_{x \to c} g(x).$$

On the other hand, if $x \neq c$ we have by definition that

$$\frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} = \lim_{n \to \infty} g_n(x) = g(x).$$

Therefore, the left side of (6) also reduces to

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} g(x).$$

This verifies (6), and the proof of the theorem is complete. ■

Proof. Let $\varepsilon > 0$ and use the Cauchy Criterion to choose $N \in \mathbf{N}$ such that $m \geq n \geq N$ implies $\sum_{k=n}^m M_k < \varepsilon$. Thus, by hypothesis,

$$\left| \sum_{k=n}^m f_k(x) \right| \leq \sum_{k=n}^m |f_k(x)| \leq \sum_{k=n}^m M_k < \varepsilon$$

for $m \geq n \geq N$ and $x \in E$. Hence, the partial sums of $\sum_{k=1}^\infty f_k$ are uniformly Cauchy and the partial sums of $\sum_{k=1}^\infty |f_k(x)|$ are Cauchy for each $x \in E$. ■

Proof. Let $B_r(a)$ be an open ball. By definition, we must prove that given $x \in B_r(a)$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq B_r(a)$. Let $x \in B_r(a)$ and set $\varepsilon = r - \rho(x,a)$. (Look at Figure 8.5 to see why this choice of ε should work.) If $y \in B_\varepsilon(x)$, then by the Triangle Inequality, assumption, and the choice of ε,

$$\rho(y,a) \le \rho(y,x) + \rho(x,a) < \varepsilon + \rho(x,a) = r.$$

Thus, by Definition 10.7, $y \in B_r(a)$. In particular, $B_\varepsilon(x) \subseteq B_r(a)$. Similarly, we can show that $\{x \in X : \rho(x,a) > r\}$ is also open. Hence, every closed ball is closed. ■

Proof. By Definition 10.8, it suffices to prove that the complement of every *singleton* $E := \{a\}$ is open. Let $x \in E^c$ and set $\varepsilon = \rho(x,a)$. Then, by Definition 10.7, $a \notin B_\varepsilon(x)$, so $B_\varepsilon(x) \subseteq E^c$. Therefore, E^c is open by Definition 10.8. ■

Proof. Since $X = \emptyset^c$ and $\emptyset = X^c$, it suffices by Definition 10.8 to prove that $\emptyset$ and X are both open. Because the empty set contains no points, "every" point $x \in \emptyset$ satisfies $B_\varepsilon(x) \subseteq \emptyset$. (This is called the *vacuous implication*.) Therefore, $\emptyset$ is open. On the other hand, since $B_\varepsilon(x) \subseteq X$ for all $x \in X$ and all $\varepsilon > 0$, it is clear that X is open. ■

Proof. Suppose that $x_n \to a$, and let V be an open set which contains a. By Definition 10.8, there is an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq V$. Given this ε, use Definition 10.13 to choose an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in B_\varepsilon(a)$. By the choice of ε, $x_n \in V$ for all $n \ge N$.

Conversely, let $\varepsilon > 0$ and set $V = B_\varepsilon(a)$. Then V is an open set which contains a; hence, by hypothesis, there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in V$. In particular, $\rho(x_n,a) < \varepsilon$ for all $n \ge N$. ■

Proof. The theorem is vacuously satisfied if E is the empty set.

Suppose that $E \ne \emptyset$ is closed but some sequence $x_n \in E$ converges to a point $x \in E^c$. Since E is closed, E^c is open. Thus, by Remark 10.15, there is an $N \in \mathbf{N}$ such that $n \ge N$ implies $x_n \in E^c$, a contradiction.

Conversely, suppose that E is a nonempty set such that every convergent sequence in E has its limit in E. If E is not closed, then, by Remark 10.11, $E \ne X$, and, by definition, E^c is nonempty and not open. Thus, there is at least one point $x \in E^c$ such that no ball $B_r(x)$ is contained in E^c. Let $x_k \in B_{1/k}(x) \cap E$ for $k = 1, 2, \ldots$. Then $x_k \in E$ and $\rho(x_k, x) < 1/k$ for all $k \in \mathbf{N}$. Now $1/k \to 0$ as $k \to \infty$, so it follows from the Squeeze Theorem (these are real sequences) that $\rho(x_k, x) \to 0$ as $k \to \infty$ (i.e., $x_k \to x$ as $k \to \infty$). Thus, by hypothesis, $x \in E$, a contradiction. ■

Proof. Let $X = \mathbf{R}$ be the discrete metric space introduced in Example 10.3. Since $\sigma(0,k) = 1$ for all $k \in \mathbf{N}$, $\{k\}$ is a bounded sequence in X. Suppose that there exist integers $k_1 < k_2 < \ldots$ and an $x \in X$ such that $k_j \to x$ as $j \to \infty$. Then there is an $N \in \mathbf{N}$ such that $\sigma(k_j, x) < 1$ for $j \ge N$ (i.e., $k_j = x$ for all $j \ge N$). This contradiction proves that $\{k\}$ has no convergent subsequences. ■

Proof. Choose (by the Density of Rationals) points $q_k \in \mathbf{Q}$ such that $q_k \to \sqrt{2}$. Then $\{q_k\}$ is Cauchy (by Theorem 10.14iv) but does not converge in X since $\sqrt{2} \notin X$. ■

Proof. Suppose that E is complete and that $x_n \in E$ converges. By Theorem 10.14iv, $\{x_n\}$ is Cauchy. Since E is complete, it follows from Definition 10.19 that the limit of $\{x_n\}$ belongs to E. Thus, by Theorem 10.16, E is closed.

Conversely, suppose that E is closed and that $x_n \in E$ is Cauchy in E. Since the metrics on X and E are identical, $\{x_n\}$ is Cauchy in X. Since X is complete, it follows that $x_n \to x$, as $n \to \infty$, for some $x \in X$. But E is closed, so x must belong to E. Thus E is complete by definition. ■

Proof. i) Let $x \in \bigcup_{\alpha\in A} V_\alpha$. Then $x \in V_\alpha$ for some $\alpha \in A$. Since V_α is open, it follows that there is an $r > 0$ such that $B_r(x) \subseteq V_\alpha$. Thus $B_r(x) \subseteq \bigcup_{\alpha\in A} V_\alpha$ (i.e., this union is open).

ii) Let $x \in \bigcap_{k=1}^n V_k$. Then $x \in V_k$ for $k = 1, 2, \ldots, n$. Since each V_k is open, it follows that there are numbers $r_k > 0$ such that $B_{r_k}(x) \subseteq V_k$. Let $r = \min\{r_1, \ldots, r_n\}$. Then $r > 0$ and $B_r(x) \subseteq V_k$ for all $k = 1, 2, \ldots, n$; that is, $B_r(x) \subseteq \bigcap_{k=1}^n V_k$. Hence, this intersection is open.

iii) By DeMorgan's Law (Theorem 1.36) and part i),

$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$$

is open, so $\bigcap_{\alpha\in A} E_\alpha$ is closed.

iv) By DeMorgan's Law and part ii),

$$\left(\bigcup_{k=1}^n E_k\right)^c = \bigcap_{k=1}^n E_k^c$$

is open, so $\bigcup_{k=1}^n E_k$ is closed.

v) Since $V \setminus E = V \cap E^c$ and $E \setminus V = E \cap V^c$, the former is open by part ii), and the latter is closed by part iii). ■

Proof. In the metric space $X = \mathbf{R}$,

$$\bigcap_{k\in\mathbf{N}} \left(-\frac{1}{k}, \frac{1}{k}\right) = \{0\}$$

is closed and

$$\bigcup_{k\in\mathbf{N}} \left[\frac{1}{k+1}, \frac{k}{k+1}\right] = (0,1)$$

is open. ■

Proof. Since every open set V in the union defining E^o is a subset of E, it is clear that the union of these V's is a subset of E. Thus $E^o \subseteq E$. A similar argument establishes $E \subseteq \overline{E}$. This proves i).

By Definition 10.33, if V is an open subset of E, then $V \subseteq E^o$ and if C is a closed set containing E, then $\overline{E} \subseteq C$. This proves ii) and iii). ■

Proof. By Definition 10.37, it suffices to show

$$x \in \overline{E} \text{ if and only if } B_r(x) \cap E \ne \emptyset \text{ for all } r > 0, \text{ and} \tag{2}$$

$$x \notin E^o \text{ if and only if } B_r(x) \cap E^c \ne \emptyset \text{ for all } r > 0. \tag{3}$$

We will provide the details for (2) and leave the proof of (3) as an exercise. Suppose that $x \in \overline{E}$ but $B_{r_0}(x) \cap E = \emptyset$ for some $r_0 > 0$. Then $(B_{r_0}(x))^c$ is a closed set which contains E; hence, by Theorem 10.34iii, $\overline{E} \subseteq (B_{r_0}(x))^c$. It follows that $\overline{E} \cap B_{r_0}(x) = \emptyset$ (e.g., $x \notin \overline{E}$, a contradiction).

Conversely, suppose that $x \notin \overline{E}$. Since $(\overline{E})^c$ is open, there is an $r_0 > 0$ such that $B_{r_0}(x) \subseteq (\overline{E})^c$. In particular, $\emptyset = B_{r_0}(x) \cap \overline{E} \supseteq B_{r_0}(x) \cap E$ for some $r_0 > 0$. ■

Proof. i) Since the union of two open sets is open, $A^o \cup B^o$ is an open subset of $A \cup B$. Hence, by Theorem 10.34ii, $A^o \cup B^o \subseteq (A \cup B)^o$.

Similarly, $(A \cap B)^o \supseteq A^o \cap B^o$. On the other hand, if $V \subset A \cap B$, then $V \subset A$ and $V \subset B$. Thus, $(A \cap B)^o \subseteq A^o \cap B^o$.

ii) Since $\overline{A} \cup \overline{B}$ is closed and contains $A \cup B$, it is clear that by Theorem 10.34iii), $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Similarly, $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. To prove the reverse inequality for union, suppose that $x \notin \overline{A \cup B}$. Then there is a closed set E which contains $A \cup B$ such that $x \notin E$. Since E contains both A and B, it follows that $x \notin \overline{A}$ and $x \notin \overline{B}$. This proves part ii).

iii) Let $x \in \partial(A \cup B)$; that is, suppose that $B_r(x)$ intersects both $A \cup B$ and $(A \cup B)^c$ for all $r > 0$. Since $(A \cup B)^c = A^c \cap B^c$, it follows that $B_r(x)$ intersects both A^c and B^c for all $r > 0$. Thus, $B_r(x)$ intersects A and A^c for all $r > 0$, or $B_r(x)$ intersects B and B^c for all $r > 0$ (i.e., $x \in \partial A \cup \partial B$). This proves the first set inequality in part iii).

To prove the second set inequality, fix $x \in \partial(A \cap B)$ [i.e., suppose that $B_r(x)$ intersects $A \cap B$ and $(A \cap B)^c$ for all $r > 0$]. If $x \in (A \cap \partial B) \cup (B \cap \partial A)$, then there is nothing to prove. If $x \notin (A \cap \partial B) \cup (B \cap \partial A)$, then $x \in (A^c \cup (\partial B)^c) \cap (B^c \cup (\partial A)^c)$. Hence, it remains to prove that $x \in A^c \cup (\partial B)^c$ implies $x \in \partial A$ and $x \in B^c \cup (\partial A)^c$ implies $x \in \partial B$. By symmetry, we need only prove the first implication.

Case 1. $x \in A^c$. Since $B_r(x)$ intersects A, it follows that $x \in \partial A$.

Case 2. $x \in (\partial B)^c$. Since $B_r(x)$ intersects B, it follows that $B_r(x) \subseteq B$ for small $r > 0$. Since $B_r(x)$ also intersects $A^c \cup B^c$, it must be the case that $B_r(x)$ intersects A^c. In particular, $x \in \partial A$. ■

Proof. These statements follow immediately from Definition 10.42. The empty set needs no set to cover it, and any finite set H can be covered by finitely many sets, one set for each element in H. ■

Proof. Suppose that H is compact but not closed. Then H is nonempty and (by Theorem 10.16) there is a convergent sequence $x_k \in H$ whose limit x does not belong to H. For each $y \in H$, set $r(y) := \rho(x,y)/2$. Since x does not belong to H, $r(y) > 0$; hence, each $B_{r(y)}(y)$ is open and contains y; that is, $\{B_{r(y)}(y) : y \in H\}$ is an open covering of H. Since H is compact, we can choose points y_j and radii $r_j := r(y_j)$ such that $\{B_{r_j}(y_j) : j = 1, 2, \ldots, N\}$ covers H.

Set $r := \min\{r_1, \ldots, r_N\}$. (This is a finite set of positive numbers, so r is also positive.) Since $x_k \to x$ as $k \to \infty$, $x_k \in B_r(x)$ for large k. But $x_k \in B_r(x) \cap H$ implies $x_k \in B_{r_j}(y_j)$ for some $j \in \mathbf{N}$. Therefore, it follows from the choices of r_j and r, and from the Triangle Inequality, that

$$\begin{aligned} r_j &\ge \rho(x_k, y_j) \ge \rho(x, y_j) - \rho(x_k, x) \\ &= 2r_j - \rho(x_k, x) > 2r_j - r \ge 2r_j - r_j = r_j, \end{aligned}$$

a contradiction. ■

Proof. Let E be a closed subset of H, where H is compact in X and suppose that $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ is an open covering of E. Now $E^c = X \setminus E$ is open; hence, $\mathcal{V} \cup \{E^c\}$ is an open covering of H. Since H is compact, there is a finite set $A_0 \subseteq A$ such that

$$H \subseteq E^c \cup \left(\bigcup_{\alpha\in A_0} V_\alpha\right).$$

But $E \cap E^c = \emptyset$. Therefore, E is covered by $\{V_\alpha\}_{\alpha\in A_0}$. ■

Proof. Suppose that H is compact. By Remark 10.44, H is closed. It is also bounded. Indeed, fix $b \in X$ and observe that $\{B_n(b) : n \in \mathbf{N}\}$ covers X. Since H is compact, it follows that

$$H \subset \bigcup_{n=1}^N B_n(b)$$

for some $N \in \mathbf{N}$. Since these balls are nested, we conclude that $H \subset B_N(b)$ (i.e., H is bounded). ■

Proof. Let $X = \mathbf{R}$ be the discrete metric space introduced in Example 10.3. Since $\sigma(0, x) \le 1$ for all $x \in \mathbf{R}$, every subset of X is bounded. Since $x_k \to x$ in X implies $x_k = x$ for large k, every subset of X is closed. Thus $[0, 1]$ is a closed, bounded subset of X. Since $\{x\}_{x\in[0,1]}$ is an uncountable open covering of $[0, 1]$, which has no finite subcover, we conclude that $[0, 1]$ is closed and bounded, but not compact. ■

Proof. Let Z be a countable dense subset of X, and consider the collection $\mathcal{T}$ of open balls with centers in Z and rational radii. This collection is countable. Moreover, it "approximates" all other open sets in the following sense:

Claim: Given any open ball $B_r(x) \subset X$, there is a ball $B_q(a) \in \mathcal{T}$ such that $x \in B_q(a)$ and $B_q(a) \subseteq B_r(x)$.

Proof of Claim: Let $B_r(x) \subset X$ be given. By Definition 10.48, choose $a \in Z$ such that $\rho(x, a) < r/4$, and choose by Theorem 1.18 a rational $q \in \mathbf{Q}$ such that $r/4 < q < r/2$. Since $r/4 < q$, we have $x \in B_q(a)$. Moreover, if $y \in B_q(a)$, then

$$\rho(x, y) \le \rho(x, a) + \rho(a, y) < q + \frac{r}{4} < \frac{r}{2} + \frac{r}{4} < r.$$

Therefore, $B_q(a) \subseteq B_r(x)$. This establishes the claim.

To prove the theorem, let $x \in E$. By hypothesis, $x \in V_\alpha$ for some $\alpha \in A$. Hence, by the claim, there is a ball $B_x \in \mathcal{T}$ such that

$$x \in B_x \subseteq V_\alpha. \tag{4}$$

The collection $\mathcal{T}$ is countable; hence, so is the subcollection

$$\{U_1, U_2, \dots\} := \{B_x : x \in E\}. \tag{5}$$

By (4), for each $k \in \mathbf{N}$ there is at least one $\alpha_k \in A$ such that $U_k \subseteq V_{\alpha_k}$. Hence, by (5),

$$E \subseteq \bigcup_{x\in E} B_x = \bigcup_{k=1}^{\infty} U_k \subseteq \bigcup_{k=1}^{\infty} V_{\alpha_k}. \quad ■$$

Proof. By Theorem 10.46, every compact set is closed and bounded.

Conversely, suppose to the contrary that H is closed and bounded but not compact. Let $\mathcal{V}$ be an open covering of H which has no finite subcover of H. By Lindelöf's Theorem, we may suppose that $\mathcal{V} = \{V_k\}_{k\in\mathbf{N}}$; that is,

$$H \subseteq \bigcup_{k\in\mathbf{N}} V_k. \tag{6}$$

By the choice of $\mathcal{V}$, $\cup_{j=1}^{k} V_j$ cannot contain H for any $k \in \mathbf{N}$. Thus we can choose a point

$$x_k \in H \setminus \bigcup_{j=1}^{k} V_j \tag{7}$$

for each $k \in \mathbf{N}$. Since H is bounded, the sequence x_k is bounded. Hence, by the Bolzano–Weierstrass Property, there is a subsequence x_{k_ν} which converges to some x as $\nu \to \infty$. Since H is closed, $x \in H$. Hence, by (6), $x \in V_N$ for some $N \in \mathbf{N}$. But V_N is open; hence, there is an $M \in \mathbf{N}$ such that $\nu \ge M$ implies $k_\nu > N$ and $x_{k_\nu} \in V_N$. This contradicts (7). We conclude that H is compact. ■

Proof. If f is uniformly continuous on a set, then it is continuous whether or not the set is compact.

Conversely, suppose that f is continuous on E. Given $\varepsilon > 0$ and $a \in E$, choose $\delta(a) > 0$ such that

$$x \in B_{\delta(a)}(a) \quad \text{and} \quad x \in E \quad \text{imply} \quad \tau(f(x), f(a)) < \frac{\varepsilon}{2}.$$

Since $a \in B_\delta(a)$ for all $\delta > 0$, it is clear that $\{B_{\delta(a)/2}(a) : a \in E\}$ is an open covering of E. Since E is compact, choose finitely many points $a_j \in E$ and numbers $\delta_j := \delta(a_j)$ such that

$$E \subseteq \bigcup_{j=1}^{N} B_{\delta_j/2}(a_j). \tag{8}$$

Set $\delta := \min\{\delta_1/2, \dots, \delta_N/2\}$.

Suppose that $x, a \in E$ with $\rho(x, a) < \delta$. By (8), x belongs to $B_{\delta_j/2}(a_j)$ for some $1 \le j \le N$. Hence,

$$\rho(a, a_j) \le \rho(a, x) + \rho(x, a_j) < \frac{\delta_j}{2} + \frac{\delta_j}{2} = \delta_j;$$

that is, a also belongs to $B_{\delta_j}(a_j)$. It follows, therefore, from the choice of δ_j that

$$\tau(f(x), f(a)) \le \tau(f(x), f(a_j)) + \tau(f(a_j), f(a)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that f is uniformly continuous on E. ■

Proof. Set $U = A \cap E$ and $V = B \cap E$. It suffices to prove that U and V are relatively open in E and separate E. It is clear by hypothesis and the remarks above that U and V are nonempty, they are both relatively open in E, and $U \cap V = \emptyset$. It remains to prove that $E = U \cup V$. But E is a subset of $A \cup B$, so $E \subseteq U \cup V$. On the other hand, both U and V are subsets of E, so $E \supseteq U \cup V$. We conclude that $E = U \cup V$. ■

Proof. Let E be a connected subset of $\mathbf{R}$. If E is empty or contains only one point, then E is a degenerate interval. Hence we may suppose that E contains at least two points.

Set $a = \inf E$ and $b = \sup E$. Notice that $-\infty \le a < b \le \infty$. Suppose for simplicity that $a, b \notin E$; that is, $E \subseteq (a, b)$. If $E \ne (a, b)$, then there is an $x \in (a, b)$ such that $x \notin E$. By the Approximation Property, $E \cap (a, x) \ne \emptyset$ and $E \cap (x, b) \ne \emptyset$, and, by assumption, $E \subseteq (a, x) \cup (x, b)$. Hence, E is separated by the open sets (a, x), (x, b), a contradiction.

Conversely, suppose that E is an interval which is not connected. Then there are sets U, V, relatively open in E, which separate E (i.e., $E = U \cup V$, $U \cap V = \emptyset$), and there exist points $x_1 \in U$ and $x_2 \in V$. We may suppose that $x_1 < x_2$. Since $x_1, x_2 \in E$ and E is an interval, $I_0 := [x_1, x_2] \subseteq E$. Define f on I_0 by

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V. \end{cases}$$

Since $U \cap V = \emptyset$, f is well defined. We claim that f is continuous on I_0. Indeed, fix $x_0 \in [x_1, x_2]$. Since $U \cup V = E \supseteq I_0$, it is evident that $x_0 \in U$ or $x_0 \in V$. We may suppose the former. Let $y_k \in I_0$ and suppose that $y_k \to x_0$ as $k \to \infty$. Since U is relatively open, there is an $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap E \subset U$. Since $y_k \in E$ and $y_k \to x_0$, it follows that $y_k \in U$ for large k. Hence $f(y_k) = 0 = f(x_0)$ for large k. Therefore, f is continuous at x_0 by the Sequential Characterization of Continuity.

We have proved that f is continuous on I_0. Hence, by the Intermediate Value Theorem (Theorem 3.29), f must take on the value 1/2 somewhere on I_0. This is a contradiction, since by construction, f takes on only the values 0 or 1. ■

Proof. Suppose that f is continuous on X and that V is open in Y. We may suppose that $f^{-1}(V)$ is nonempty. Let $a \in f^{-1}(V)$; that is, $f(a) \in V$. Since V is open, choose $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq V$. Since f is continuous at a, choose $\delta > 0$ such that (10) holds. Evidently,

$$B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a))) \subseteq f^{-1}(V). \tag{11}$$

Since $a \in f^{-1}(V)$ was arbitrary, we have shown that every point in $f^{-1}(V)$ is interior to $f^{-1}(V)$. Thus $f^{-1}(V)$ is open.

Conversely, let $\varepsilon > 0$ and $a \in X$. The ball $V = B_\varepsilon(f(a))$ is open in Y. By hypothesis, $f^{-1}(V)$ is open. Since $a \in f^{-1}(V)$, it follows that there is a $\delta > 0$ such that $B_\delta(a) \subseteq f^{-1}(V)$. This means that if $\rho(x, a) < \delta$, then $\tau(f(x), f(a)) < \varepsilon$. Therefore, f is continuous at $a \in X$. ■

Proof. Suppose that $\{V_\alpha\}_{\alpha\in A}$ is an open covering of $f(H)$. By Theorem 1.37,

$$H \subseteq f^{-1}(f(H)) \subseteq f^{-1}\left(\bigcup_{\alpha\in A} V_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(V_\alpha).$$

Hence, by Corollary 10.59, $\{f^{-1}(V_\alpha)\}_{\alpha\in A}$ is a covering of H whose sets are all relatively open in H. Since H is compact, there are indices $\alpha_1, \alpha_2, \dots, \alpha_N$ such that

$$H \subseteq \bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})$$

(see Exercise 10.5.7). It follows from Theorem 1.37 that

$$f(H) \subseteq f\left(\bigcup_{j=1}^{N} f^{-1}(V_{\alpha_j})\right) = \bigcup_{j=1}^{N} (f \circ f^{-1})(V_{\alpha_j}) = \bigcup_{j=1}^{N} V_{\alpha_j}.$$

Therefore, $f(H)$ is compact. ■

Proof. Suppose that $f(E)$ is not connected. By Definition 10.53, there exists a pair $U, V \subset Y$ of relatively open sets in $f(E)$ which separates $f(E)$. By Exercise 10.6.4, $f^{-1}(U) \cap E$ and $f^{-1}(V) \cap E$ are relatively open in E. Since $f(E) = U \cup V$, we have

$$E = (f^{-1}(U) \cap E) \cup (f^{-1}(V) \cap E).$$

Since $U \cap V = \emptyset$, we also have $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Thus $f^{-1}(U) \cap E$, $f^{-1}(V) \cap E$ is a pair of relatively open sets which separates E. Hence, by Definition 10.53, E is not connected, a contradiction. ■

Proof. By symmetry, it suffices to prove the result for M. Since H is compact, $f(H)$ is compact. Hence, by the Theorem 10.46, $f(H)$ is closed and bounded. Since $f(H)$ is bounded, M is finite. By the Approximation Property, choose $x_k \in H$ such that $f(x_k) \to M$ as $k \to \infty$. Since $f(H)$ is closed, $M \in f(H)$. Therefore, there is an $x_M \in H$ such that $M = f(x_M)$. A similar argument shows that m is finite and attained on H. ■

Proof. By Exercise 10.6.4a, it suffices to show that $(f^{-1})^{-1}$ takes closed sets in X to relatively closed sets in $f(H)$. Let E be closed in X. Then $E \cap H$ is a closed subset of H, so by Remark 10.45, $E \cap H$ is compact. Hence, by Theorem 10.61, $f(E \cap H)$ is compact, in particular, closed. Since f is 1–1, $f(E \cap H) = f(E) \cap f(H)$ (see Exercise 1.5.7). Since $f(E \cap H)$ and $f(H)$ are closed, it follows that $f(E) \cap f(H)$ is relatively closed in $f(H)$. Since $(f^{-1})^{-1} = f$, we conclude that $(f^{-1})^{-1}(E) \cap f(H)$ is relatively closed in $f(H)$. ■