Fundamentals of Optimization Notes

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September 29, 2019

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2.1 Convexity

- **Definition** *Convexity*
  A set $C \subset \mathbb{R}^n$ is convex $\iff \forall x, y \in C, \forall \lambda \in [0, 1]\ f(\lambda x + (1 - \lambda)) \leq \lambda f(x) + (1 - \lambda) f(y)$
- **Definition** *Epigraph*
  The epigraph of a function is the set lying **above** the function. i.e. If C is a subset of $\mathbb{R}^n$, and function $f : C \mapsto \mathbb{R}$, then the epigraph of f is
  $$\text{epi}(f) = \{(x, z) | x \in C, z \geq f(x)\} \in \mathbb{R}^{n+1}$$
- **Definition** *Convex set*
  A set $C \subset \mathbb{R}^n$ is convex
  $$\iff \forall x, y \in C, \forall \lambda \in [0, 1], \quad \exists z = \lambda x + (1 - \lambda) y \in C$$
- **Definition** Function $f : C \mapsto \mathbb{R}$ is *convex* if and only if
  $$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y),\ \forall x, y \in C, \forall \lambda \in [0, 1]$$
- **Definition** Function $f : C \mapsto \mathbb{R}$ is *concave* if and only if
  $$f(\lambda x + (1 - \lambda) y) \geq \lambda f(x) + (1 - \lambda) f(y),\ \forall x, y \in C, \forall \lambda \in [0, 1]$$

- **Definition** Function $f : C \mapsto \mathbb{R}$ is *strictly convex* if and only if

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y),\ \forall x, y \in C, \forall \lambda \in [0,1]$$

- **Definition** Function $f : C \mapsto \mathbb{R}$ is *strictly concave* if and only if

$$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y),\ \forall x, y \in C, \forall \lambda \in [0,1]$$

- **Definition** A vector $\vec{x}$ is a **local** minimum of f if

$$\exists \epsilon > 0 \text{ such that } f(\vec{x}) \leq f(x), \forall x |x - \vec{x}| < \epsilon$$

- **Definition** A vector $\vec{x}$ is a **global** minimum of f if

$$f(\vec{x}) \leq f(x),\ \forall x \in X$$

- If X is a convex set and $f : X \mapsto \mathbb{R}$ is a convex function, then the minimization problem is said to be *convex*.
- **Lemma** In a convex minimization problem a **local** minimum is a global minimum.
- **Definition** A linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ satisfies
  1. $L(x+y) = L(x) + L(y)$
  2. $\alpha L(x) = L(\alpha x)$ for scalar α.
- **Definition** An affine map is a linear map *plus* a shift of the origin. It takes the form $F(x) = d + Mx$, where $d \in \mathbb{R}^m$.
- **Lemma** An affine function (and hence also a linear function) is both convex and concave.

2.2 Properties of Convex Sets and Functions

1. For any collection $\{C_i | i \in I\}$ of convex sets, the intersection $\cap_{i \in I} C_i$ is convex.
2. The vector sum $\{x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$ of two convex sets C_1 and C_2 is convex.
3. The image of a convex set under a linear map is convex. i.e. If C is a convex set and L a linear map $\{L(x) | x \in C\}$ is convex. (True also for affine map.)
4. For C a convex set, $f : C \mapsto \mathbb{R}$ a convex function and $L : C \mapsto \mathbb{R}$ and affine function, the **level sets** $\{x \in C | f(x) \leq \alpha\}$ and $\{x \in C | L(x) = \alpha\}$ are convex for all scalars α.
5. For C a convex set and any collection $f_i : C \mapsto \mathbb{R} | i \in I$ of convex functions, a non-negative weighted sum, i.e. $f = \sum_{i \in I} w_f f_i : C \mapsto \mathbb{R}$, where $w_i \geq 0$, $i \in I$, is convex.
6. For I a (finite) index set, C a convex set, and $f_i : C \mapsto \mathbb{R}$ a convex function $\forall i \in I$, the function $h : C \mapsto \mathbb{R}$ defined by $h(x) = \max_{i \in I} f_i(x)$ is convex.
7. For C a convex set, $f : C \mapsto \mathbb{R}$ a convex function, and $g : \mathbb{R} \mapsto \mathbb{R}$ a convex and non-decreasing function, the function $h : C \mapsto \mathbb{R}$ defined by $h(x) = g(f(x))$ is convex.
8. Let $C \subseteq \mathbb{R}^n$ be a convex set and $f : C \mapsto \mathbb{R}$ be differentiable over C.
   (a) The function f is convex if and only if

   $$f(y) \geq f(x) + \nabla^T f(x)(y - x), \quad \forall x, y \in C$$

   (b) if the inequality is strict $\forall x \neq y$, then f is strictly convex.

9. Let $C \subseteq \mathbb{R}^n$ be a convex set and $f : C \mapsto \mathbb{R}$ be twice continuously differentiable over C.
   (a) If f is convex in C and C has an interior (i.e. a point not on its boundary), then $\nabla^2 f(x)$ is positive semi-definite at all $x \in C$.
   (b) If $\nabla^2 f(x)$ is positive semi-definite for all $x \in C$, then f is convex in C.
   (c) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex.
10. Let H be a square symmetric matrix and let $f(x) = x^T Hx$ (so f is a quadratic function).
    (a) f is convex if and only if H is positive semi-definite.
    (b) f is strictly convex if and only if H is positive definite.