

A Comparison of Secret Sharing Schemes Based on Latin Squares and RSA

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Abstract

In recent years there has been a great deal of work done on secret sharing schemes. Secret Sharing Schemes allow for the division of keys so that an authorised set of users may access information. In this paper we wish to present a critical comparison of two of these Schemes based on Latin Squares [3] and RSA [9]. These two protocols will be examined in terms of their positive and negative aspects of their security.

Keywords: Cryptography and Secure Communication; Secret Sharing Schemes; Distributed Systems; RSA Digital Signature Algorithm; Latin Squares.

1 Introduction

In communications networks which require security, it is important that secrets be protected by more than one key. Furthermore a system of several keys with more than one way for their combination may allow for the unique recovery of a secret. Schemes that have a group of participants that could recover a secret are known as *Secret Sharing Schemes*.

The idea of secret sharing is to start with a secret, divide it into pieces called *shares*, which are then distributed amongst users such that the pooled shares are specific subsets of users allowed to reconstruct the original secret, [6].

Threshold Schemes

Shamir [8], describes threshold schemes as being very helpful in the management of cryptographic keys. The most secure key management scheme keeps the key in a single place. This sort of scheme may not always be appropriate, and an obvious solution to this may be to make multiple copies of the key. This may increase the risk associated in keeping multiple keys secret. By using Shamir's [8] threshold scheme concept we can get a very robust key management scheme.

Threshold schemes are well suited to applications in which a group of individuals with conflicting interests must cooperate [8]. By following Shamir's [8] protocol and choosing the correct t and w parameters we can give any sufficiently large majority the authority to take some action while giving any sufficiently large minority veto powers. We shall now use the definition outlined in [10] to describe what a threshold secret sharing scheme is.

Definition 1.1. *Let t and w be positive integers, $t \leq w$. A (t, w) -threshold scheme is a method of sharing a key K among a set of w players (denoted by \mathcal{P}), in such a way that any t participants can compute the value of K , but no group of $t - 1$ participants can do so.*

The value of K is chosen by a special participant which is referred to by [10] as the *dealer*. The dealer is denoted by D and we must assume that $D \notin \mathcal{P}$. When D wants to share the key K

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among the participants in \mathcal{P} , D gives each participant some partial information referred to earlier as a share. The shares should be distributed secretly, so no participant knows the share given to any other participant. At some later time, a subset of participants $B \subseteq \mathcal{P}$ will pool their shares or return them to the dealer in an attempt to compute the key K . If $|B| \geq t$, then they should be able to compute the value of K as a function of the shares they collectively hold. Furthermore if $|B| < t$, then they should not be able to compute K . If we follow the notation of Stinson [10],

$$\mathcal{P} = \{P_i : 1 \leq i \leq w\} \quad (1.1)$$

as the set of participants, \mathcal{K} is the set of keys and \mathcal{S} as the set of secrets. A useful point proposed by Shamir [8] is that a hierarchical scheme may be created, so that some players may have shares which are of more importance (weight).

1.1 Access Structures

In our outline of threshold schemes, we wanted t out of w players to be able to determine the key. A more general situation is to specifically exactly which subsets of players should be able to determine the key and those that should not [10]. If we describe Γ as being a set of subsets of \mathcal{P} , and the subsets in Γ as being the subset of players that should be able to compute the key. Γ is denoted as being the access structure and the subsets in Γ are called authorised subsets.

Furthermore if we let \mathcal{K} be the set of keys and \mathcal{S} be the share set. We shall continue to use the dealer D who wants to share a key $k \in \mathcal{K}$, and then gives each player a share $S \in \mathcal{S}$. Some time later a subset of players will attempt to determine K from the shares they collectively hold. If we notice that a (t, w) -threshold scheme creates the access structure $\{B \subseteq \mathcal{P} \mid |B| \geq t\}$, which is referred to by Stinson [10] as the *threshold access structure*.

If Γ is an access structure, then $B \in \Gamma$ is a minimal authorized subset and $A \notin \Gamma$ whenever $A \subseteq B, A \neq B$. The set of minimal authorized subsets of Γ is denoted by Γ_0 and is called the basis of Γ . Since Γ consists of all subsets of \mathcal{P} that are supersets of a subset in the basis Γ_0 . Thus Γ is determined uniquely as a function of Γ_0 such that:

$$\Gamma = \{C \subseteq \mathcal{P}, B \subseteq C, B \in \Gamma_0\} \quad (1.2)$$

2 Latin Squares

In their 1994 paper Cooper, Donovan and Seberry [3] laid the foundation for the use of critical sets as a combinatorial structure which could be used to construct a secret sharing scheme. We should begin this section by defining a Latin Square and the concept of a critical set.

Definition 2.1. A $n \times n$ Latin Square is an $n \times n$ matrix whose entries are taken from a set of n objects so that no object occurs twice in any row or column.

Definition 2.2. A critical set of a Latin Square L defined over the set $X = \{1, \dots, n\}$ where,

$$C = \{(i, j, k) \in X \times X \times X\} \quad (2.1)$$

such that L is the only square of order n with i in the (j, k) th for every $(i, j, k) \in C$. Furthermore no proper subset of C may satisfies this condition

An important construction which we need to define is the concept of a strong critical set for a Latin Square.

Definition 2.3. A critical set L is a strong critical set if there exists a set $\{P_1, \dots, P_m\}$ of $m = n^2 - \|A\|$ partitions of order n , which satisfy the following properties:

- $L \supset P_m \supset P_{m-1} \supset \dots \supset P_2 \supset P_1 = A$
- $\forall i, 1 \leq i \leq m-1, P_i \cup \{(r_i, c_i, e_i)\} = P_{i+1}$
- $P_i \cup \{(r_i, c_i, e_i)\}$ is not a partial Latin Square such that $\nexists e \in N \setminus \{e_i\}$

Definition 2.4. A critical set is referred to as being semi-strong, if there exists a set $\{P_1, \dots, P_m\}$ of $m = n^2 - |A|$ partial Latin Squares, of order n , which satisfy the following properties:

1. $L \supset P_m \supset P_{m-1} \supset \dots \supset P_2 \supset P_1 = A$
2. $\forall i, 1 \leq i \leq m-1, P_i \cup \{(r_i, c_i, e_i)\} = P_{i+1}$ such that one of $P_i \cup \{(r_i, c_i, e_i)\}$ or $P_i \cup \{(r_i, c, e_i)\}$ or $P_i \cup \{(r, c_i, e_i)\}$ is not a partial Latin Square for any $e \in N \setminus \{e_i\}$ or $c \in N \setminus \{c_i\}$ or $r \in N \setminus \{r_i\}$ respectively.

2.1 The Proposed Scheme

In Cooper [3] a secret sharing scheme is constructed with a secret key made from a Latin Square L , of order n . Furthermore [3] notes the following characteristics:

- The Latin Square L is kept private, but its order however is made public.
- The Shares are based on a partial Latin Square $S = \{\cup A_i | A_i \in L\}$ where A_i is a critical set. With the union is taken over all possible critical sets in L over some subset of critical sets.
- The number of critical sets used depends on the size of the Latin Square and the number of shares.
- The access structure is defined as $\Gamma = \{B | B \subseteq S \text{ \& } A \subseteq B\}$ where A is some critical set in L . Where Γ is monotone

We shall now outline the basic protocol presented by Cooper [3]:

- A Latin Square L of order n is chosen. The number n is made public, but the Latin Square L is kept secret and taken to be the key.
- The set S which is the union of a number of critical sets in L
- For each $(i, j; k) \in S$, the share $(i, j; k)$ is distributed privately to a participant.
- When a critical set of shares are brought together, they can reconstruct the Latin Square L and thus the secret key.

2.2 The Ranking Problem

The constructions proposed by [3, 11, 1], are such that each user is given one element from a Latin Square and a subset of these elements may be combined to form a critical set. In Donovan [4], a more general construction is given such that, a set S is the union of a number of critical sets in a Latin Square. Elements from the set S are dealt out to each player, so that a group of players wish to reconstruct the critical set and the secret can be recovered. This gives rise to the question to that complex issue in Latin Squares of there being some positions which are more important than others.

An intruder who knew C's share and the location of the other shares, what the player did next would depend upon their knowledge of the concurrence scheme. If our player knew the scheme then one would start by guessing at two of the other shares (A and B, or D and E, or A and D) in which case it is an disadvantage compared to an intruder who knows a share other than C's.

If our player does not know the scheme, it would seem most logical to try to guess D's share before trying to guess two other shares at once. Again, in this case, our player is at a disadvantage compared to an intruder who knows a share other than C's.

2.3 Security of a Latin Square Based Scheme

The main security issues with this type of scheme were investigated heavily by Cooper [3]. We shall now examine these vulnerabilities:

- An unauthorized players knows one n th of the critical set.
- A group of unauthorized players have a greater chance of reconstructing the critical set with their group of shares.
- The security of this scheme is based on the number of possible latin squares which contain the partial Latin Square defined by a disloyal group of players. It has been estimated that the number of Latin Squares containing the set C for $\{(i, j; k)\}$ such that for a square of order $n = 11$, ≥ 19000000

The complexity of completing partial Latin Squares has been investigated by Colbourn [2]. The computational complexity of this problem is NP-Complete. However even for a Latin Square of order $n = 11$ there are still a measurable number of solutions which can be generated by brute force.

3 RSA Threshold System

Threshold schemes however are by no means perfect despite their proponents [9]. Many of these schemes have a great many short falls which include at least one of the following:

1. The scheme has no rigorous security proof
2. Share generation and verification is interactive and requires synchronous communications network
3. The size of each share increases linearly with respect to the number of players.

In an effort to rectify this situation [9] presents a new RSA threshold scheme that exhibits the following:

1. Unforgeable and robust if we assume that the RSA problem is hard [7]
2. Share generation and verification is completely non-interactive [7]
3. The size of the share is bounded by a constant and the size of the discrete logarithm problem [8] and [6]

Shoup [9] further stresses the fact that the share is a standard RSA signature. This is underpinned by the fact that the public key and verification algorithm are the same as for an RSA signature [8, 7]. The refined model examined in this paper and in [9] where there is one threshold t for the maximum number of traitors and k is the minimum quorum size.

3.1 The RSA threshold Scheme

We must first establish a set of players w , denoted $1, \dots, w$, a trusted designer/dealer, and traitor. This systems also has a signature verification, a share verification and share combining algorithms. Shoup [9] only uses 2 variables, however in our investigation we must remain consistent with the majority of the literature and consider 3 parameters. So we denote the number of corrupted players as c , the number of shares needed to produce a signature as t and the set of all users w . We also mention the requirement for these parameters is, $t \geq c + 1$ and $w - c \geq t$.

The dealing phase is initiated by the dealer generating a public key, along with a set of secret key shares and a set of verification keys. The corrupt player obtains the secret key shares of the corrupted players, the public and verification keys. The post dealing phase is when the corrupt player acts by submitting a signing request to the loyal players for a message. After the request has been submitted, a player outputs a signature share for the submitted message.

The signature verification algorithm takes an input message, a signature and a public key to determine if the signature is valid. The signature verification algorithm takes an input message, and signature share on that message from player i , to determine if that signature share is valid. The share combining algorithm takes a message and t valid signature shares on the message with the public key and the verification keys. The algorithm then outputs a valid signature on that message. The non-forgeability of signatures protocol dictates that if an adversary forges a signature at the end of the protocol our player outputs a valid signature on a message that was not submitted as a signing request to at least $t - c$ loyal players. Furthermore we must stress that the threshold signature scheme is non-forgeable if it is computationally infeasible for the corrupt adversary to forge a signature.

3.2 Security of RSA Threshold

Theorem 3.1. *For $t = w + 1$, in the random oracle model for H' , the above protocol is a secure threshold signature scheme which is robust and non-forgeable. Thus we assume that the standard RSA signature scheme is secure.*

We shall only outline a very short comment on the proof for this theorem. One should consult Shoup [9] for a more detailed approach. The robustness of the threshold signature scheme is cemented in its non-forgeable. We assume that the standard RSA signature scheme is secure because of the

difficulty in solving the adaptive message attack. This statement can be justified by the random oracle model of [6] such that given some random $x \in \mathbf{Z}_n^*$, it is hard to compute y such that $y^e = x$

4 Analysis

We shall put forward the merits of a Latin Square SSS and the RSA based system to examine

- A Latin Square Scheme, can provide good security when the critical set on which the scheme is founded is not based on strong critical or semi-strong critical partial Latin Squares.
- Latin Squares of large order i.e. ≥ 11 provide for a relatively secure system.
- The current literature believes that the RSA problem is hard to compute
- The Decision Diffie-Hellman (DDH) assumption-given some random $g, h \in Q_n$, along with g^n and h^b , it is hard to decide if $a \equiv b \bmod m$.
- Finding a correct authorized group of shares from one given share is computationally difficult.

If we were to look at a computational attack against the Latin Square Scheme, one would need only to find one disloyal player and simply generate a completion for that share [1]. Although the prospect of finding a solution to this problem becomes more difficult as the size of the scheme increases beyond 11 players [5], it is still possible. Without a scheme that allows for a disenrollment procedure [5], a brute force attack for computing the completion of the Latin Square is a viable attack.

If one already holds one of the other share then, there is a 1 in 4 chance of completing the critical set and discovering the secret by simply picking one share at random [3]. A 25 percent chance of completing a critical set given one player is disloyal, is a risk not worth taking in our view. If one player were somehow compelled or convinced that becoming disloyal was appropriate then a scheme that placed so much trust in one player is too risky.

Although the Latin Square model is entirely theoretical, it must be asked why one would use such a scheme that has two major faults. Unless one can ensure that no players will defect and become disloyal, then this scheme is far from desirable. In contrast RSA based protocols are one of the best methods available to ensure the security of a multiparty scheme for digital signatures [6, 9].

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