Probabilistic Default Reasoning with Conditional Constraints

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Abstract

We propose a combination of probabilistic reasoning from conditional constraints with approaches to default reasoning from conditional knowledge bases. In detail, we generalize the notions of Pearl's entailment in system Z, Lehmann's lexicographic entailment, and Geffner's conditional entailment to conditional constraints. We give some examples that show that the new notions of z-, lexicographic, and conditional entailment have similar properties like their classical counterparts. Moreover, we show that the new notions of z-, lexicographic, and conditional entailment are proper generalizations of both their classical counterparts and the classical notion of logical entailment for conditional constraints.

Introduction

In this paper, we elaborate a combination of probabilistic reasoning from conditional constraints with approaches to default reasoning from conditional knowledge bases. As a main result, this combination provides new notions of entailment for conditional constraints, which respect the ideas of classical default reasoning from conditional knowledge bases, and which are generally much stronger than the classical notion of logical entailment based on conditioning. Moreover, the results of this paper can also be applied for handling inconsistencies in probabilistic knowledge bases.

Informally, the ideas behind this paper can be described as follows. Assume that we have the following knowledge at hand: "all penguins are birds" (G1), "between 90 and 95% of all birds fly" (G2), and "at most 5% of all penguins fly" (G3). Moreover, assume a first scenario in which "Tweety is a bird" (E1) and second one in which "Tweety is a penguin" (E2). What do we conclude about Tweety's ability to fly?

A closer look at this example shows that the statements G1–G3 describe statistical knowledge (or objective knowledge), while E1 and E2 express degrees of belief (or subjective knowledge). One way of handling such combinations of statistical knowledge and degrees of belief is *reference class reasoning*, which goes back to Reichenbach (1949) and was further refined by Kyburg (1974; 1983) and Pollock (1990).

Another related field is *default reasoning from conditional knowledge bases*, where we have generic statements of the form "all penguins are birds", "generally, all birds fly", and "generally, no penguin flies" in addition to some concrete evidence as E1 and E2. The literature contains several different approaches to default reasoning and extensive work on the desired properties. The core of these properties are the rationality postulates proposed by Kraus *et al.* (1990). These rationality postulates constitute a sound and complete axiom system for several classical model-theoretic entailment relations under uncertainty measures on worlds. In detail, they characterize classical model-theoretic entailment under preferential structures (Shoham 1987; Kraus *et al.* 1990), infinitesimal probabilities (Adams 1975; Pearl 1989), possibility measures (Dubois & Prade 1991), and world rankings (Spohn 1988; Goldszmidt & Pearl 1992). . They also characterize an entailment relation based on conditional objects (Dubois & Prade 1994). A survey of all these relationships is given in (Benferhat *et al.* 1997). Recently, Friedman and Halpern (2000) showed that many approaches describe to the same notion of inference, since they are all expressible as plausibility measures.

Mainly to solve problems with irrelevant information, the notion of rational closure as a more adventurous notion of entailment has been introduced by Lehmann (Lehmann 1989; Lehmann & Magidor 1992). This notion of entailment is equivalent to entailment in system Z by Pearl (1990), to the least specific possibility entailment by Benferhat *et al.* (1992), and to a conditional (modal) logic-based entailment by Lamarre (1992). Finally, mainly in order to solve problems with property inheritance from classes to exceptional subclasses, the maximum entropy approach to default entailment was proposed by Goldszmidt *et al.* (1993); the notion of lexicographic entailment was introduced by Lehmann (1995) and Benferhat *et al.* (1993); the notion of conditional entailment was proposed by Geffner (Geffner 1992; Geffner & Pearl 1992); and an infinitesimal belief function approach was suggested by Benferhat *et al.* (1995).

Coming back to our introductory example, we realize that G1–G3 and E1–E2 represent *interval restrictions for conditional probabilities*, also called *conditional constraints* (Lukasiewicz 1999b). The literature contains extensive work on reasoning about conditional constraints (Dubois & Prade 1988; Dubois *et al.* 1990; 1993; Amarger *et al.* 1991; Jaumard *et al.* 1991; Thöne *et al.* 1992; Frisch & Haddawy 1994; Heinsohn 1994; Luo *et al.* 1996; Lukasiewicz 1999a; 1999b) and their generalizations, for example, to probabilistic logic programs (Lukasiewicz 1998).

Now, the main idea of this paper is to use techniques for default reasoning from conditional knowledge bases in order to perform probabilistic reasoning from statistical knowledge and degrees of beliefs. More precisely, we extend the notions of entailment in system Z, Lehmann's lexicographic entailment, and Geffner's conditional entailment to the framework of conditional constraints.

Informally, in our introductory example, the statements G2 and G3 are interpreted as "generally, a bird flies with a probability between 0.9 and 0.95" (G2*) and "generally, a penguin flies with a probability of at most 0.05" (G3*), respectively. In the first scenario, we then simply use the whole probabilistic knowledge $\{$G1, G2*, G3*, E1$\}$ to conclude under classical logical entailment that "Tweety flies with a probability between 0.9 and 0.95". In the second scenario, it turns out that the whole probabilistic knowledge $\{$G1, G2*, G3*, E2$\}$ is unsatisfiable. More precisely, $\{$G1, G2*, G3$^\star\}$ is inconsistent in the context of a penguin. In fact, the main problem is that G2* should not be applied anymore to penguins. That is, we can easily resolve the inconsistency by removing G2*, and then conclude from $\{$G1, G3*, E2$\}$ under classical logical entailment that "Tweety flies with a probability of at most 0.05".

Hence, the results of this paper can also be used for handling inconsistencies in probabilistic knowledge bases. More precisely, the new notions of nonmonotonic entailment coincide with the classical notion of logical entailment as far as *satisfiable* sets of conditional constraints are concerned. Furthermore, they allow desirable conclusions from certain kinds of *unsatisfiable* sets of conditional constraints.

We remark that this inconsistency handling is guided by the principles of default reasoning from conditional knowledge bases. It is thus based on a natural preference relation on conditional constraints, and not on the assumption that all conditional constraints are equally weighted (as, for example, in the work by Jaumard *et al.* (1991)).

The work closest in spirit to this paper is perhaps the one by Bacchus *et al.* (1996), which suggests to use the *random worlds method* (Grove *et al.* 1994) to induce degrees of beliefs from quite rich statistical knowledge bases. However, differently from (Bacchus *et al.* 1996), we do not make use of a strong principle such as the random worlds method (which is closely related to probabilistic reasoning under maximum entropy). Moreover, we restrict our considerations to the propositional setting.

The main contributions of this paper are as follows:

- We illustrate that the classical notion of logical entailment for conditional constraints is not very well-suited for default reasoning with conditional constraints.
- We introduce the notions of z-entailment, lexicographic entailment, and conditional entailment for conditional constraints, which are a combination of the classical notions of entailment in system Z (Pearl 1990), Lehmann's lexicographic entailment (Lehmann 1995), and Geffner's conditional entailment (Geffner 1992; Geffner & Pearl 1992), respectively, with the classical notion of logical entailment for conditional constraints.
- We give some examples that analyze the nonmonotonic properties of the new notions of entailment for default reasoning with conditional constraints. It turns out that the new notions of z-entailment, lexicographic entailment, and conditional entailment have similar properties like their classical counterparts.
- We show that the new notions of z-entailment, lexicographic entailment, and conditional entailment for conditional constraints properly extend the classical notions of entailment in system Z, lexicographic entailment, and conditional entailment, respectively.
- We show that the new notions of z-entailment, lexicographic entailment, and conditional entailment for conditional constraints properly extend the classical notion of logical entailment for conditional constraints.

Note that all proofs are given in (Lukasiewicz 2000).

Preliminaries

We now introduce some necessary technical background.

We assume a finite nonempty set of *basic propositions* (or *atoms*) Φ. We use $\bot$ and $\top$ to denote the propositional constants *false* and *true*, respectively. The set of *classical formulas* is the closure of $\Phi \cup \{\bot, \top\}$ under the Boolean operations $\neg$ and $\wedge$. A *strict conditional constraint* is an expression $(\psi|\phi)[l,u]$ with real numbers $l,u \in [0,1]$ and classical formulas ψ and ϕ. A *defeasible conditional constraint* (or *default*) is an expression $(\psi\|\phi)[l,u]$ with real numbers $l,u \in [0,1]$ and classical formulas ψ and ϕ. A *conditional constraint* is a strict or defeasible conditional constraint. The set of *strict probabilistic formulas* (resp., *probabilistic formulas*) is the closure of the set of all strict conditional constraints (resp., conditional constraints) under the Boolean operations $\neg$ and $\wedge$. We use $(F \vee G)$, $(F \Rightarrow G)$, and $(F \Leftrightarrow G)$ to abbreviate $\neg(\neg F \wedge \neg G)$, $\neg(F \wedge \neg G)$, and $(\neg(\neg F \wedge G)) \wedge (\neg(F \wedge \neg G))$, respectively, and adopt the usual conventions to eliminate parentheses.

A *probabilistic default theory* is a pair $T = (P, D)$, where P is a finite set of strict conditional constraints and D is a finite set of defeasible conditional constraints. A *probabilistic knowledge base* KB is a strict probabilistic formula. Informally, default theories represent strict and defeasible generic knowledge, while probabilistic knowledge bases express some concrete evidence.

A *possible world* is a truth assignment $I \colon \Phi \rightarrow \{\mathbf{true}, \mathbf{false}\}$, which is extended to classical formulas as usual. We use $\mathcal{I}_\Phi$ to denote the set of all possible worlds for Φ. A possible world I *satisfies* a classical formula ϕ, or I is a *model* of ϕ, denoted $I \models \phi$, iff $I(\phi) = \mathbf{true}$.

A *probabilistic interpretation* Pr is a probability function on $\mathcal{I}_\Phi$ (that is, a mapping $Pr \colon \mathcal{I}_\Phi \rightarrow [0,1]$ such that all $Pr(I)$ with $I \in \mathcal{I}_\Phi$ sum up to 1). The *probability* of a classical formula ϕ in the probabilistic interpretation Pr, denoted $Pr(\phi)$, is defined as follows:

$$Pr(\phi) \;=\; \sum_{I \in \mathcal{I}_\Phi,\, I \models \phi} Pr(I)\,.$$

For classical formulas ϕ and ψ with $Pr(\phi) > 0$, we use $Pr(\psi|\phi)$ to abbreviate $Pr(\psi \wedge \phi) \,/\, Pr(\phi)$. The *truth* of probabilistic formulas F in a probabilistic interpretation Pr, denoted $Pr \models F$, is inductively defined as follows:

- $Pr \models (\psi|\phi)[l,u]$ iff $Pr(\phi)=0$ or $Pr(\psi|\phi) \in [l,u]$.
- $Pr \models (\psi\|\phi)[l,u]$ iff $Pr(\phi)=0$ or $Pr(\psi|\phi) \in [l,u]$.
- $Pr \models \neg F$ iff not $Pr \models F$.
- $Pr \models (F \wedge G)$ iff $Pr \models F$ and $Pr \models G$.

We remark that there is no difference between strict and defeasible conditional constraints as far as the notion of truth in probabilistic interpretations is concerned.

A probabilistic interpretation Pr *satisfies* a probabilistic formula F, or Pr is a *model* of F, iff $Pr \models F$. Pr *satisfies* a set of probabilistic formulas $\mathcal{F}$, or Pr is a *model* of $\mathcal{F}$, denoted $Pr \models \mathcal{F}$, iff Pr is a model of all $F \in \mathcal{F}$. We say $\mathcal{F}$ is *satisfiable* iff a model of $\mathcal{F}$ exists.

We next define the notion of logical entailment as follows. A strict probabilistic formula F is a *logical consequence* of a set of probabilistic formulas $\mathcal{F}$, denoted $\mathcal{F} \models F$, iff each model of $\mathcal{F}$ is also a model of F. A strict conditional constraint $(\psi|\phi)[l,u]$ is a *tight logical consequence* of $\mathcal{F}$, denoted $\mathcal{F} \models_{\mathit{tight}} (\psi|\phi)[l,u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all models Pr of $\mathcal{F}$ with $Pr(\phi) > 0$ (note that we canonically define $l=1$ and $u=0$, when $\mathcal{F} \models (\phi|\top)[0,0]$).

We remark that every notion of entailment for conditional constraints is associated with a notion of consequence and a notion of tight consequence. Informally, the notion of consequence describes entailed intervals, while the notion of tight consequence characterizes the tightest entailed interval. That is, if $(\psi|\phi)[l,u]$ is a tight consequence of $\mathcal{F}$, then $[l',u'] \supseteq [l,u]$ for all consequences $(\psi|\phi)[l',u']$ of $\mathcal{F}$.

Motivating Examples

What should a probabilistic knowledge base entail under a probabilistic default theory? To get a rough idea on the reply to this question, we now introduce two natural notions of entailment and analyze their properties. It will turn out that neither of these two notions is fully adequate for probabilistic default reasoning with conditional constraints.

In the sequel, let $T=(P,D)$ be a probabilistic default theory. We first define the notion of 0-entailment, which applies to probabilistic knowledge bases of the form $KB=(\varepsilon|\top)[1,1]$. In detail, a strict conditional constraint $(\psi|\phi)[l,u]$ is a 0-*consequence* of KB, denoted $KB \Vdash\!\sim^{0} (\psi|\phi)[l,u]$, iff $P \cup D \models (\psi|\phi \wedge \varepsilon)[l,u]$. It is a *tight* 0-*consequence* of KB, denoted $KB \Vdash\!\sim^{0}_{\mathit{tight}} (\psi|\phi)[l,u]$, iff $P \cup D \models_{\mathit{tight}} (\psi|\phi \wedge \varepsilon)[l,u]$. Informally, we use the concrete evidence in KB to fix our "point of interest" and the generic knowledge in T to draw the requested conclusion. That is, we perform classical conditioning.

We next define the notion of 1-entailment, which applies to all probabilistic knowledge bases KB. A strict probabilistic formula F is a 1-*consequence* of KB, denoted $KB \Vdash\!\sim^{1} F$, iff $P \cup D \cup \{KB\} \models F$. A strict conditional constraint $(\psi|\phi)[l,u]$ is a *tight* 1-*consequence* of KB, denoted $KB \Vdash\!\sim^{1}_{\mathit{tight}} (\psi|\phi)[l,u]$, iff $P \cup D \cup \{KB\} \models_{\mathit{tight}} (\psi|\phi)[l,u]$. Informally, we draw our conclusion from the union of the concrete evidence in KB and the generic knowledge in T.

We now analyze the properties of these two notions of entailment. Our first example concentrates on the aspects of *ignoring irrelevant information* and *property inheritance*.

Example 1 The knowledge "all penguins are birds" and "at least 95% of all birds have legs" can be expressed by the following probabilistic default theory $T_1=(P_1,D_1)$:

$$P_1 = \{(\mathit{bird} \,|\, \mathit{penguin})[1,1]\},$$
$$D_1 = \{(\mathit{legs} \,\|\, \mathit{bird})[.95,1]\}\,.$$

Now, T_1 should entail that "generally, birds have legs with a probability of at least 0.95" (that is, e.g., if we know that Tweety is a bird, and we do not have any other knowledge, then we should conclude that the probability of Tweety having legs is at least 0.95). Indeed, this conclusion is drawn under both 0- and 1-entailment (see item (1) in Table 1).

Moreover, T_1 should entail that "generally, yellow birds have legs with a probability of at least 0.95" (as the property "yellow" is not mentioned at all in T_1 and thus *irrelevant*), and that "generally, penguins have legs with a probability of at least 0.95" (as the set of all penguins is a *nonexceptional subclass* of the set of all birds, and thus penguins should *inherit* all properties of birds). However, while 1-entailment still allows the desired conclusions, 0-entailment just yields the interval $[0,1]$ (see items (2)–(3) in Table 1). $\Box$

We next concentrate on the principle of *specificity* and the problem of *inheritance blocking*.

Example 2 Let us consider the following probabilistic default theory $T_2=(P_2,D_2)$:

$$P_2 = \{(\mathit{bird} \,|\, \mathit{penguin})[1,1]\},$$
$$D_2 = \{(\mathit{legs} \,\|\, \mathit{bird})[.95,1],\ (\mathit{fly} \,\|\, \mathit{bird})[.9,.95],\ (\mathit{fly} \,\|\, \mathit{penguin})[0,.05]\}\,.$$

This default theory should entail that "generally, penguins fly with a probability of at most 0.05" (as properties of more specific classes should override inherited properties of less specific classes). Indeed, 0-entailment yields the desired conclusion, while 1-entailment reports an unsatisfiability (see item (7) in Table 1).

Moreover, T_2 should entail that "generally, penguins have legs with a probability of at least 0.95", since penguins are exceptional birds w.r.t. to the ability of being able to fly, but not w.r.t. the property of having legs. However, 0-entailment provides only the interval $[0,1]$, and 1-entailment reports even an unsatisfiability (see item (5) in Table 1). $\Box$

The following example deals with the *drowning problem* (Benferhat *et al.* 1993).

Example 3 Let us consider the following probabilistic default theory $T_3=(P_3,D_3)$:

$$P_3 = \{(\mathit{bird} \,|\, \mathit{penguin})[1,1]\},$$
$$D_3 = \{(\mathit{fly} \,\|\, \mathit{bird})[.9,.95],\ (\mathit{fly} \,\|\, \mathit{penguin})[0,.05],\ (\mathit{easy\_to\_see} \,\|\, \mathit{yellow})[.95,1]\}\,.$$

This default theory should entail that "generally, yellow penguins are easy to see", as the set of all yellow penguins

Table 1: Examples of 0- and 1-entailed tight intervals.

	T	KB	$(\psi\|\phi)$	$\Vert\!\sim^{0}_{\mathit{tight}}$	$\Vert\!\sim^{1}_{\mathit{tight}}$
(1)	T_1	$(\mathit{bird}\,\|\,\top)[1,1]$	$(\mathit{legs}\|\top)$	$[.95,1]$	$[.95,1]$
(2)	T_1	$(\mathit{bird}\wedge \mathit{yellow}\,\|\,\top)[1,1]$	$(\mathit{legs}\|\top)$	$\mathbf{[0,1]}$	$[.95,1]$
(3)	T_1	$(\mathit{penguin}\,\|\,\top)[1,1]$	$(\mathit{legs}\|\top)$	$\mathbf{[0,1]}$	$[.95,1]$
(4)	T_2	$(\mathit{bird}\,\|\,\top)[1,1]$	$(\mathit{legs}\|\top)$	$[.95,1]$	$[.95,1]$
(5)	T_2	$(\mathit{penguin}\,\|\,\top)[1,1]$	$(\mathit{legs}\|\top)$	$\mathbf{[0,1]}$	$\mathbf{[1,0]}$
(6)	T_2	$(\mathit{bird}\,\|\,\top)[1,1]$	$(\mathit{fly}\|\top)$	$[.9,.95]$	$[.9,.95]$
(7)	T_2	$(\mathit{penguin}\,\|\,\top)[1,1]$	$(\mathit{fly}\|\top)$	$[0,.05]$	$\mathbf{[1,0]}$
(8)	T_3	$(\mathit{penguin}\wedge \mathit{yellow}\,\|\,\top)[1,1]$	$(\mathit{easy\_to\_see}\|\top)$	$\mathbf{[0,1]}$	$\mathbf{[1,0]}$
(9)	T_4	$(\mathit{magpie}\,\|\,\top)[1,1]$	$(\mathit{chirp}\|\top)$	$[.7,.8]$	$[.7,.8]$
(10)	T_5	$(\mathit{penguin}\wedge \mathit{metal\_wings}\,\|\,\top)[1,1]$	$(\mathit{fly}\|\top)$	$[0,1]$	$\mathbf{[1,0]}$
(11)	T_2	$(\mathit{bird}\,\|\,\top)[.9,1]\wedge(\mathit{penguin}\,\|\,\top)[.1,1]$	$(\mathit{fly}\|\top)$	**undefined**	$[.86,.91]$
(12)	T_2	$(\mathit{bird}\,\|\,\top)[.9,1]\wedge(\mathit{penguin}\,\|\,\top)[.9,1]$	$(\mathit{fly}\|\top)$	**undefined**	$\mathbf{[1,0]}$

is a nonexceptional subclass of the set of all yellow objects. But, 0-entailment gives only the interval $[0,1]$, and 1-entailment reports an unsatisfiability (see item (8) in Table 1). □

The next example is taken from (Bacchus *et al.* 1996).

Example 4 Let us consider the following probabilistic default theory $T_4=(P_4,D_4)$:

$$P_4 = \{(\mathit{bird}\,|\,\mathit{magpie})[1,1]\},$$
$$D_4 = \{(\mathit{chirp}\,\|\,\mathit{bird})[.7,.8],\ (\mathit{chirp}\,\|\,\mathit{magpie})[0,.99]\}\,.$$

This default theory should entail "generally, the probability that magpies chirp is between 0.7 and 0.8", since we know more about birds w.r.t. the property of being able to chirp than about magpies. Indeed, both 0- and 1-entailment yield the desired conclusion (see item (9) in Table 1). □

The following example concerns *ambiguity preservation* (Benferhat *et al.* 1995).

Example 5 Let us consider the following probabilistic default theory $T_5=(P_5,D_5)$:

$$P_5 = \{(\mathit{bird}\,|\,\mathit{penguin})[1,1]\},$$
$$D_5 = \{(\mathit{fly}\,\|\,\mathit{metal\_wings})[.95,1],\ (\mathit{fly}\,\|\,\mathit{bird})[.95,1],\ (\mathit{fly}\,\|\,\mathit{penguin})[0,.05]\}\,.$$

Assume now that Oscar is a penguin with metal wings. As Oscar is a penguin, we should conclude that the probability that Oscar flies is at most 0.05. However, as Oscar has also metal wings, we should conclude that the probability that Oscar flies is at least 0.95. As argued in the literature on default reasoning (Benferhat *et al.* 1995), such ambiguities should be preserved. Indeed, 0-entailment yields the desired interval $[0,1]$, while 1-entailment reports an unsatisfiability (see item (10) in Table 1). □

What about handling *purely probabilistic evidence*?

Example 6 Let us consider again the probabilistic default theory T_2 of Example 2. Assume a first scenario in which our belief is "the probability that Tweety is a bird is at least 0.9" and "the probability that Tweety is a penguin is at least 0.1" and a second scenario in which our belief is "the probability that Tweety is a bird is at least 0.9" and "the probability that Tweety is a penguin is at least 0.9". What do we conclude about Tweety's ability to fly in these scenarios?

The notion of 0-entailment is undefined for such purely probabilistic evidence, whereas the notion of 1-entailment yields the probability interval $[.86,.91]$ in the first scenario, and reports an unsatisfiability in the second scenario (see items (11)–(12) in Table 1). □

Summarizing the results, 0-entailment is too weak, while 1-entailment is too strong. In detail, 0-entailment often yields the trivial interval $[0,1]$ and is even undefined for purely probabilistic evidence, while 1-entailment often reports unsatisfiabilities (in fact, in the most interesting scenarios, as 1-entailment is actually *monotonic*).

Roughly speaking, our ideal notion of entailment for probabilistic knowledge bases under probabilistic default theories should lie somewhere between 0- and 1-entailment. One idea to obtain such a notion could be to strengthen 0-entailment by adding some inheritance mechanism. Another idea is to weaken 1-entailment by handling unsatisfiabilities. In the rest of this paper, we will focus on the second idea.

Probabilistic Default Reasoning

In this section, we extend the classical notions of entailment in system Z (Pearl 1990), Lehmann's lexicographic entailment (1995), and Geffner's conditional entailment (Geffner 1992; Geffner & Pearl 1992) to conditional constraints.

The main idea behind these extensions is to use the following two interpretations of defaults. As far as default rankings and priority orderings are concerned, we interpret a default $(\psi\|\phi)[l,u]$ as "generally, if ϕ is true, then the probability of ψ is between l and u". Whereas, as far as notions of entailment are concerned, we interpret $(\psi\|\phi)[l,u]$ as "the conditional probability of ψ given ϕ is between l and u".

Preliminaries

A probabilistic interpretation Pr *verifies* a default $(\psi\|\phi)[l,u]$ iff $Pr(\phi)=1$ and $Pr\models(\psi|\phi)[l,u]$. It *falsifies* a default $(\psi\|\phi)[l,u]$ iff $Pr(\phi)=1$ and $Pr\not\models(\psi|\phi)[l,u]$. A set of defaults D *tolerates* a default d *under* a set of strict conditional constraints P iff $P\cup D$ has a model that verifies d. A set of defaults D is *under* P *in conflict* with d iff no model of $P\cup D$ verifies d.

A *default ranking* σ on D maps each $d\in D$ to a nonnegative integer. It is *admissible* with $T=(P,D)$ iff each set of defaults $D'\subseteq D$ that is under P in conflict with some default $d\in D$ contains a default d' such that $\sigma(d')<\sigma(d)$. A probabilistic default theory $T=(P,D)$ is σ*-consistent* iff there exists a default ranking on D that is admissible with T. It is σ*-inconsistent* iff no such default ranking exists.

A *probability ranking* κ maps each probabilistic interpretation on $\mathcal{I}_\Phi$ to a member of $\{0,1,\dots\}\cup\{\infty\}$ such that $\kappa(Pr)=0$ for at least one interpretation Pr. It is extended to all strict probabilistic formulas F as follows. If F is satisfiable, then $\kappa(F)=\min\{\kappa(Pr)\,|\,Pr\models F\}$; otherwise, $\kappa(F)=\infty$. We say κ is *admissible* with F iff $\kappa(\neg F)=\infty$. It is *admissible* with a default $(\psi\|\phi)[l,u]$ iff

$$\kappa((\phi|\top)[1,1])<\infty \ \text{ and}$$
$$\kappa((\phi|\top)[1,1]\wedge(\psi|\phi)[l,u])<\kappa((\phi|\top)[1,1]\wedge\neg(\psi|\phi)[l,u])\,.$$

Roughly speaking, the intuition behind this definition is to interpret $(\psi\|\phi)[l,u]$ as "generally, if ϕ is true, then the probability of ψ is between l and u". A probability ranking κ is *admissible* with a probabilistic default theory $T=(P,D)$ iff κ is admissible with all $F\in P$ and all $d\in D$.

System Z

We now extend the notion of entailment in system Z (Pearl 1990; Goldszmidt & Pearl 1996) to conditional constraints.

In the sequel, let $T=(P,D)$ be a σ-consistent probabilistic default theory. The notion of z-entailment is linked to an ordered partition of D, a default ranking z, and a probability ranking κ^z.

We first define the z-partition of D. Let $(D_0,\dots,D_k)$ be the unique ordered partition of D such that, for $i=0,\dots,k$, each D_i is the set of all defaults in $D-\bigcup\{D_j\,|\,0\le j<i\}$ that are tolerated under P by $D-\bigcup\{D_j\,|\,0\le j<i\}$ (note that we define $D-\bigcup\{D_j\,|\,0\le j<i\}=D$ for $i=0$). We call this $(D_0,\dots,D_k)$ the z*-partition* of D.

Example 7 The z-partition for the probabilistic default theory $T_2=(P_2,D_2)$ of Example 2 is given as follows:

$$(\{(\mathit{legs}\,\|\,\mathit{bird})[.95,1],(\mathit{fly}\,\|\,\mathit{bird})[.9,.95]\},$$
$$\{(\mathit{fly}\,\|\,\mathit{penguin})[0,.05]\})\,.\ \Box$$

We now define the default ranking z. For $j=0,\dots,k$, each $d\in D_j$ is assigned the value j under z. The probability ranking κ^z on all probabilistic interpretations Pr is then defined as follows:

$$\kappa^z(Pr)\ =\ \begin{cases}\infty & \text{if } Pr\not\models P\\ 0 & \text{if } Pr\models P\cup D\\ 1+\max\limits_{d\in D:\ Pr\not\models d} z(d) & \text{otherwise.}\end{cases}$$

The following result shows that, in fact, z is a default ranking that is admissible with T, and κ^z is a probability ranking that is admissible with T.

Lemma 8 *a) z is a default ranking admissible with T.*
b) κ^z is a probability ranking admissible with T.

We next define a preference relation on probabilistic interpretations. For probabilistic interpretations Pr and Pr', we say Pr is z*-preferable* to Pr' iff $\kappa^z(Pr)<\kappa^z(Pr')$. A model Pr of a set of probabilistic formulas $\mathcal{F}$ is a z*-minimal model* of $\mathcal{F}$ iff no model of $\mathcal{F}$ is z-preferable to Pr.

We are now ready to define the notion of z-entailment as follows. A strict probabilistic formula F is a z*-consequence* of KB, denoted $KB\,|\!\!\sim^z F$, iff each z-minimal model of $P\cup\{KB\}$ satisfies F. A strict conditional constraint $(\psi|\phi)[l,u]$ is a *tight* z*-consequence* of KB, denoted $KB\,|\!\!\sim^z_{\mathit{tight}}(\psi|\phi)[l,u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all z-minimal models Pr of $P\cup\{KB\}$ with $Pr(\phi)>0$.

Coming back to Examples 1–6, it turns out that the nonmonotonic properties of z-entailment differ from the ones of 0- and 1-entailment (see Table 2).

In detail, in the given examples, z-entailment ignores irrelevant information, shows property inheritance to globally nonexceptional subclasses, and respects the principle of specificity. Moreover, it may also handle purely probabilistic evidence. However, properties are still not inherited to more specific classes that are exceptional with respect to some other properties. Moreover, z-entailment still has the drowning problem and does not preserve ambiguities.

The following examples illustrate how z-entailed tight intervals are determined.

Example 9 Given T_2 of Example 2, we get:

$$(\mathit{penguin}\,|\,\top)[1,1]\,|\!\!\sim^z_{\mathit{tight}}(\mathit{legs}\,|\,\top)[0,1]$$

Here, the interval "$[0,1]$" comes from the tight logical consequence $P_2\cup\{(\mathit{fly}\,\|\,\mathit{penguin})[0,.05],(\mathit{penguin}\,|\,\top)[1,1]\}\models_{\mathit{tight}}(\mathit{legs}\,|\,\top)[0,1]$. $\Box$

Example 10 Given T_5 of Example 5, we get:

$$(\mathit{penguin}\wedge\mathit{metal\_wings}\,|\,\top)[1,1]\,|\!\!\sim^z_{\mathit{tight}}(\mathit{fly}\,|\,\top)[0,.05]\,.$$

Here, the interval "$[0,.05]$" comes from the tight logical consequence $P_5\cup\{(\mathit{fly}\,\|\,\mathit{penguin})[0,.05],(\mathit{penguin}\wedge\mathit{metal\_wings}|\top)[1,1]\}\models_{\mathit{tight}}(\mathit{fly}\,|\,\top)[0,.05]$. $\Box$

Lexicographic Entailment

We now extend Lehmann's lexicographic entailment (Lehmann 1995) to conditional constraints.

In the sequel, let $T=(P,D)$ be a σ-consistent probabilistic default theory. We now use the z-partition $(D_0,\dots,D_k)$ of D to define a lexicographic preference relation on probabilistic interpretations.

Table 2: Examples of z-, lexicographically, and conditionally entailed tight intervals.

	T	KB	$(\psi\|\phi)$	$\Vert\!\sim^{z}_{tight}$	$\Vert\!\sim^{lex}_{tight}$	$\Vert\!\sim^{ce}_{tight}$
(1)	T_1	$(\mathit{bird}\,\|\,\top)[1,1]$	$(\mathit{legs}\,\|\top)$	$[.95,1]$	$[.95,1]$	$[.95,1]$
(2)	T_1	$(\mathit{bird}\wedge\mathit{yellow}\,\|\,\top)[1,1]$	$(\mathit{legs}\,\|\top)$	$[.95,1]$	$[.95,1]$	$[.95,1]$
(3)	T_1	$(\mathit{penguin}\,\|\,\top)[1,1]$	$(\mathit{legs}\,\|\top)$	$[.95,1]$	$[.95,1]$	$[.95,1]$
(4)	T_2	$(\mathit{bird}\,\|\,\top)[1,1]$	$(\mathit{legs}\,\|\top)$	$[.95,1]$	$[.95,1]$	$[.95,1]$
(5)	T_2	$(\mathit{penguin}\,\|\,\top)[1,1]$	$(\mathit{legs}\,\|\top)$	$\mathbf{[0,1]}$	$[.95,1]$	$[.95,1]$
(6)	T_2	$(\mathit{bird}\,\|\,\top)[1,1]$	$(\mathit{fly}\,\|\top)$	$[.9,.95]$	$[.9,.95]$	$[.9,.95]$
(7)	T_2	$(\mathit{penguin}\,\|\,\top)[1,1]$	$(\mathit{fly}\,\|\top)$	$[0,.05]$	$[0,.05]$	$[0,.05]$
(8)	T_3	$(\mathit{penguin}\wedge\mathit{yellow}\,\|\,\top)[1,1]$	$(\mathit{easy\_to\_see}\,\|\top)$	$\mathbf{[0,1]}$	$[.95,1]$	$[.95,1]$
(9)	T_4	$(\mathit{magpie}\,\|\,\top)[1,1]$	$(\mathit{chirp}\,\|\top)$	$[.7,.8]$	$[.7,.8]$	$[.7,.8]$
(10)	T_5	$(\mathit{penguin}\wedge\mathit{metal\_wings}\,\|\,\top)[1,1]$	$(\mathit{fly}\,\|\top)$	$\mathbf{[0,.05]}$	$\mathbf{[0,.05]}$	$[0,1]$
(11)	T_2	$(\mathit{bird}\,\|\,\top)[.9,1]\wedge(\mathit{penguin}\,\|\,\top)[.1,1]$	$(\mathit{fly}\,\|\top)$	$[.86,.91]$	$[.86,.91]$	$[.86,.91]$
(12)	T_2	$(\mathit{bird}\,\|\,\top)[.9,1]\wedge(\mathit{penguin}\,\|\,\top)[.9,1]$	$(\mathit{fly}\,\|\top)$	$[0,.15]$	$[0,.15]$	$[0,.15]$

For probabilistic interpretations Pr and Pr', we say Pr is *lexicographically preferable* to Pr' iff there exists some $i \in \{0,\ldots,k\}$ such that $|\{d \in D_i \mid Pr \models d\}| > |\{d \in D_i \mid Pr' \models d\}|$ and $|\{d \in D_j \mid Pr \models d\}| = |\{d \in D_j \mid Pr' \models d\}|$ for all $i < j \leq k$. A model Pr of a set of probabilistic formulas $\mathcal{F}$ is a *lexicographically minimal model* of $\mathcal{F}$ iff no model of $\mathcal{F}$ is lexicographically preferable to Pr.

We now define the notion of lexicographic entailment as follows. A strict probabilistic formula F is a *lexicographic consequence* of KB, denoted $KB \Vert\!\sim^{lex} F$, iff each lexicographically minimal model of $P \cup \{KB\}$ satisfies F. A strict conditional constraint $(\psi|\phi)[l,u]$ is a *tight lexicographic consequence* of KB, denoted $KB \Vert\!\sim^{lex}_{tight} (\psi|\phi)[l,u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all lexicographically minimal models Pr of $P \cup \{KB\}$ with $Pr(\phi) > 0$.

Coming back to Examples 1–6, it turns out that lexicographic entailment has nicer nonmonotonic features than z-entailment (see Table 2).

In detail, in the given examples, lexicographic entailment ignores irrelevant information, shows property inheritance to nonexceptional subclasses, and respects the principle of specificity. Moreover, it does not block property inheritance, it does not have the drowning problem, and it may also handle purely probabilistic evidence. However, lexicographic entailment still does not preserve ambiguities.

The following examples illustrate how lexicographically entailed tight intervals are determined.

Example 11 Given T_2 of Example 2, we get:

$$(\mathit{penguin}\,|\,\top)[1,1] \Vert\!\sim^{lex}_{tight} (\mathit{legs}\,|\,\top)[.95,1]\,.$$

Here, the interval "$[.95,1]$" comes from the tight logical consequence $P_2 \cup \{(\mathit{legs} \,\|\, \mathit{bird})[.95,1], (\mathit{fly} \,\|\, \mathit{penguin})[0,.05], (\mathit{penguin}\,|\,\top)[1,1]\} \models_{tight} (\mathit{legs}\,|\,\top)[.95,1]$. □

Example 12 Given T_5 of Example 5, we get:

$$(\mathit{penguin} \wedge \mathit{metal\_wings}\,|\,\top)[1,1] \Vert\!\sim^{lex}_{tight} (\mathit{fly}\,|\,\top)[0,.05]\,.$$

Here, the interval "$[0,.05]$" comes from the tight logical consequence $P_5 \cup \{(\mathit{fly} \,\|\, \mathit{penguin})[0,.05], (\mathit{penguin} \wedge \mathit{metal\_wings}|\top)[1,1]\} \models_{tight} (\mathit{fly}\,|\,\top)[0,.05]$. □

Conditional Entailment

We next extend Geffner's conditional entailment (Geffner 1992; Geffner & Pearl 1992) to conditional constraints.

In the sequel, let $T = (P, D)$ be a probabilistic default theory.

We first define priority orderings on D as follows. A *priority ordering* $\prec$ on D is an irreflexive and transitive binary relation on D. We say $\prec$ is *admissible* with T iff each set of defaults $D' \subseteq D$ that is under P in conflict with some default $d \in D$ contains a default d' such that $d' \prec d$. We say T is *$\prec$-consistent* iff there exists a priority ordering on D that is admissible with T.

Example 13 Consider the probabilistic default theory $T_2 = (P_2, D_2)$ of Example 2. A priority ordering $\prec$ on D_2 that is admissible with T_2 is given by $(\mathit{fly} \,\|\, \mathit{bird})[.9,.95] \prec (\mathit{fly} \,\|\, \mathit{penguin})[0,.05]$. □

The existence of an admissible default ranking implies the existence of an admissible priority ordering.

Lemma 14 *If T is σ-consistent, then T is $\prec$-consistent.*

We next define a preference ordering on probabilistic interpretations as follows. Let Pr and Pr' be two probabilistic interpretations and let $\prec$ be a priority ordering on D. We say that Pr is *$\prec$-preferable* to Pr' iff $\{d \in D \mid Pr \not\models d\} \neq \{d \in D \mid Pr' \not\models d\}$ and for each $d \in D$ such that $Pr \not\models d$ and $Pr' \models d$, there exists some default $d' \in D$ such that $d \prec d'$, $Pr \models d'$, and $Pr' \not\models d'$. A model Pr of a set of probabilistic formulas $\mathcal{F}$ is a *$\prec$-minimal model* of $\mathcal{F}$ iff no model of $\mathcal{F}$ is $\prec$-preferable to Pr. A model Pr of a set of probabilistic formulas $\mathcal{F}$ is a *conditionally minimal model* of $\mathcal{F}$ iff Pr is a $\prec$-*minimal model* of $\mathcal{F}$ for some priority ordering $\prec$ admissible with T.

We finally define the notion of conditional entailment. A strict probabilistic formula F is a *conditional consequence* of KB, denoted $KB \,\|\!\!\sim^{ce} F$, iff each conditionally minimal model of $P \cup \{KB\}$ satisfies F. A strict conditional constraint $(\psi|\phi)[l,u]$ is a *tight conditional consequence* of KB, denoted $KB \,\|\!\!\sim^{ce}_{tight} (\psi|\phi)[l,u]$, iff l (resp., u) is the infimum (resp., supremum) of $Pr(\psi|\phi)$ subject to all conditionally minimal models Pr of $P \cup \{KB\}$ with $Pr(\phi) > 0$.

Coming back to Examples 1–6, we see that among all introduced notions of entailment, conditional entailment is the one with the nicest nonmonotonic properties (see Table 2).

In detail, in the given examples, conditional entailment ignores irrelevant information, shows property inheritance to nonexceptional subclasses, and respects the principle of specificity. Moreover, it does not block property inheritance, and it does not have the drowning problem. Finally, conditional entailment preserves ambiguities and may also handle purely probabilistic evidence.

The following examples illustrate how conditionally entailed tight intervals are determined.

Example 15 Given T_2 of Example 2, we get:

$$(\textit{penguin} \mid \top)[1,1] \,\|\!\!\sim^{ce}_{tight} (\textit{legs} \mid \top)[.95,1]\,.$$

Here, the interval "$[.95,1]$" comes from the tight logical consequence $P_2 \cup \{(\textit{legs} \,\|\, \textit{bird})[.95,1], (\textit{fly} \,\|\, \textit{penguin})[0,.05], (\textit{penguin} \mid \top)[1,1]\} \models_{tight} (\textit{legs} \mid \top)[.95,1]$. □

Example 16 Given T_5 of Example 5, we get:

$$(\textit{penguin} \wedge \textit{metal\_wings} \mid \top)[1,1] \,\|\!\!\sim^{ce}_{tight} (\textit{fly} \mid \top)[0,1]\,.$$

Here, the interval "$[0,1]$" is the convex hull of the intervals "$[0,.05]$" and "$[.95,1]$", which come from the tight logical consequences $P_5 \cup \{(\textit{fly} \,\|\, \textit{penguin})[0,.05], (\textit{penguin} \wedge \textit{metal\_wings} \mid \top)[1,1]\} \models_{tight} (\textit{fly} \mid \top)[0,.05]$ and $P_5 \cup \{(\textit{fly} \,\|\, \textit{bird})[.95,1], (\textit{fly} \,\|\, \textit{metal\_wings})[.95,1], (\textit{penguin} \wedge \textit{metal\_wings} \mid \top)[1,1]\} \models_{tight} (\textit{fly} \mid \top)[.95,1]$, respectively. □

Relationship to Classical Formalisms

We now analyze the relationship to classical default reasoning from conditional knowledge bases and to classical probabilistic reasoning with conditional constraints.

A *logical formula* is a probabilistic formula that contains only conditional constraints of the kind $(\psi|\phi)[1,1]$ or $(\psi\|\phi)[1,1]$. A *strict logical formula* is a strict probabilistic formula that contains only strict conditional constraints of the form $(\psi|\phi)[1,1]$. A *logical default theory* T is a probabilistic default theory that contains only logical formulas. A *logical knowledge base* KB is a strict logical formula.

We use the operator γ on logical formulas, sets of logical formulas, and logical default theories, which replaces each strict conditional constraint $(\psi|\phi)[1,1]$ (resp., defeasible conditional constraint $(\psi\|\phi)[1,1]$) by the classical implication $\phi \Rightarrow \psi$ (resp., classical default $\phi \rightarrow \psi$). Given a logical default theory T, we use $\mid\!\sim^{z}$ (resp., $\mid\!\sim^{lex}$, $\mid\!\sim^{ce}$) to denote the classical notion of z-, (resp., lexicographic, conditional) entailment with respect to $\gamma(T)$.

The following result shows that the introduced notions of z-, lexicographic, and conditional entailment are generalizations of their classical counterparts.

Theorem 17 *Let $T = (P, D)$ be a logical default theory and let KB be a logical knowledge base. Then, for every semantics $s \in \{z, lex, ce\}$:*

$$KB \,\|\!\!\sim^{s} (\psi|\top)[1,1] \;\text{ iff }\; \gamma(KB) \mid\!\sim^{s} \psi.$$

The next result shows that, when the union of generic and concrete probabilistic knowledge is satisfiable, the notions of z-, lexicographic, and conditional entailment coincide with the notion of 1-entailment.

Theorem 18 *Let $T = (P, D)$ be a probabilistic default theory and let KB be a probabilistic knowledge base such that $P \cup D \cup \{KB\}$ is satisfiable. Then, for every semantics $s \in \{z, lex, ce\}$:*

1. *$KB \,\|\!\!\sim^{s} F$ iff $P \cup D \cup \{KB\} \models F$.*
2. *$KB \,\|\!\!\sim^{s}_{tight} (\psi|\phi)[l,u]$ iff $P \cup D \cup \{KB\} \models_{tight} (\psi|\phi)[l,u]$.*

Summary and Outlook

We presented the notions of z-entailment, lexicographic entailment, and conditional entailment for conditional constraints, which combine the classical notions of entailment in system Z, Lehmann's lexicographic entailment, and Geffner's conditional entailment with the classical notion of logical entailment for conditional constraints. We showed that the introduced notions for probabilistic default reasoning with conditional constraints have similar properties like their classical counterparts. Moreover, they properly extend both their classical counterparts and the classical notion of logical entailment for conditional constraints.

An interesting topic of future research is to extend other formalisms for classical default reasoning to the probabilistic framework of conditional constraints.

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References

Adams, E. W. 1975. *The Logic of Conditionals*, volume 86 of *Synthese Library*. Dordrecht, Netherlands: D. Reidel.

Amarger, S.; Dubois, D.; and Prade, H. 1991. Constraint propagation with imprecise conditional probabilities. In *Proceedings UAI-91*, 26–34. Morgan Kaufmann.

Bacchus, F.; Grove, A.; Halpern, J.; and Koller, D. 1996. From statistical knowledge bases to degrees of beliefs. *Artif. Intell.* 87:75–143.

Benferhat, S.; Cayrol, C.; Dubois, D.; Lang, J.; and Prade, H. 1993. Inconsistency management and prioritized syntax-based entailment. In *Proceedings IJCAI-93*, 640–645. Morgan Kaufmann.

Benferhat, S.; Dubois, D.; and Prade, H. 1992. Representing default rules in possibilistic logic. In *Proceedings KR-92*, 673–684. Morgan Kaufmann.

Benferhat, S.; Dubois, D.; and Prade, H. 1997. Nonmonotonic reasoning, conditional objects and possibility theory. *Artif. Intell.* 92(1–2):259–276.

Benferhat, S.; Saffiotti, A.; and Smets, P. 1995. Belief functions and default reasoning. In *Proceedings UAI-95*, 19–26. Morgan Kaufmann.

Carnap, R. 1950. *Logical Foundations of Probability*. Chicago: University of Chicago Press.

de Finetti, B. 1974. *Theory of Probability*. New York: Wiley.

Dubois, D., and Prade, H. 1988. On fuzzy syllogisms. *Computational Intelligence* 4(2):171–179.

Dubois, D., and Prade, H. 1991. Possibilistic logic, preferential models, non-monotonicity and related issues. In *Proceedings IJCAI-91*, 419–424. Morgan Kaufmann.

Dubois, D., and Prade, H. 1994. Conditional objects as non-monotonic consequence relationships. *IEEE Trans. Syst. Man Cybern.* 24:1724–1740.

Dubois, D.; Prade, H.; Godo, L.; and de Màntaras, R. L. 1993. Qualitative reasoning with imprecise probabilities. *Journal of Intelligent Information Systems* 2:319–363.

Dubois, D.; Prade, H.; and Touscas, J.-M. 1990. Inference with imprecise numerical quantifiers. In Ras, Z. W., and Zemankova, M., eds., *Intelligent Systems*. Ellis Horwood. chapter 3, 53–72.

Eiter, T., and Lukasiewicz, T. 2000. Complexity results for default reasoning from conditional knowledge bases. In *Proceedings KR-2000*. Morgan Kaufmann. To appear.

Fagin, R.; Halpern, J. Y.; and Megiddo, N. 1990. A logic for reasoning about probabilities. *Inf. Comput.* 87:78–128.

Friedman, N., and Halpern, J. Y. 2000. Plausibility measures and default reasoning. *J. ACM*. To appear.

Frisch, A. M., and Haddawy, P. 1994. Anytime deduction for probabilistic logic. *Artif. Intell.* 69:93–122.

Geffner, H., and Pearl, J. 1992. Conditional entailment: Bridging two approaches to default reasoning. *Artif. Intell.* 53(2–3):209–244.

Geffner, H. 1992. *Default Reasoning: Causal and Conditional Theories*. Cambridge, MA: MIT Press.

Goldszmidt, M., and Pearl, J. 1992. Rank-based systems: A simple approach to belief revision, belief update and reasoning about evidence and actions. In *Proceedings KR-92*, 661–672. Morgan Kaufmann.

Goldszmidt, M., and Pearl, J. 1996. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artif. Intell.* 84(1–2):57–112.

Goldszmidt, M.; Morris, P.; and Pearl, J. 1993. A maximum entropy approach to nonmonotonic reasoning. *IEEE Trans. Pattern Anal. Mach. Intell.* 15(3):220–232.

Grove, A.; Halpern, J.; and Koller, D. 1994. Random worlds and maximum entropy. *J. Artif. Intell. Res.* 2:33–88.

Heinsohn, J. 1994. Probabilistic description logics. In *Proceedings UAI-94*. Morgan Kaufmann.

Jaumard, B.; Hansen, P.; and de Aragão, M. P. 1991. Column generation methods for probabilistic logic. *ORSA J. Comput.* 3:135–147.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intell.* 14(1):167–207.

Kyburg, Jr., H. E. 1974. *The Logical Foundations of Statistical Inference*. Dordrecht, Netherlands: D. Reidel.

Kyburg, Jr., H. E. 1983. The reference class. *Philos. Sci.* 50:374–397.

Lamarre, P. 1992. A promenade from monotonicity to nonmonotonicity following a theorem prover. In *Proceedings KR-92*, 572–580. Morgan Kaufmann.

Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artif. Intell.* 55(1):1–60.

Lehmann, D. 1989. What does a conditional knowledge base entail? In *Proceedings KR-89*, 212–222. Morgan Kaufmann.

Lehmann, D. 1995. Another perspective on default reasoning. *Ann. Math. Artif. Intell.* 15(1):61–82.

Lukasiewicz, T. 1998. Probabilistic logic programming. In *Proceedings ECAI-98*, 388–392. J. Wiley & Sons.

Lukasiewicz, T. 1999a. Local probabilistic deduction from taxonomic and probabilistic knowledge-bases over conjunctive events. *Int. J. Approx. Reas.* 21(1):23–61.

Lukasiewicz, T. 1999b. Probabilistic deduction with conditional constraints over basic events. *J. Artif. Intell. Res.* 10:199–241.

Lukasiewicz, T. 2000. Probabilistic default reasoning with strict and defeasible conditional constraints. Technical Report INFSYS RR-1843-00-02, Institut für Informationssysteme, Technische Universität Wien.

Luo, C.; Yu, C.; Lobo, J.; Wang, G.; and Pham, T. 1996. Computation of best bounds of probabilities from uncertain data. *Computational Intelligence* 12(4):541–566.

Pearl, J. 1989. Probabilistic semantics for nonmontonic reasoning: A survey. In *Proceedings KR-89*, 505–516. Morgan Kaufmann.

Pearl, J. 1990. System Z: A natural ordering of defaults with tractable applications to default reasoning. In *Proceedings TARK-90*, 121–135. Morgan Kaufmann.

Pollock, J. L. 1990. *Nomic Probabilities and the Foundations of Induction*. Oxford: Oxford University Press.

Reichenbach, H. 1949. *Theory of Probability*. Berkeley, CA: University of California Press.

Shoham, Y. 1987. A semantical approach to nonmonotonic logics. In *Proceedings of the 2nd IEEE Symposium on Logic in Computer Science*, 275–279.

Spohn, W. 1988. Ordinal conditional functions: A dynamic theory of epistemic states. In Harper, W., and Skyrms, B., eds., *Causation in Decision, Belief Change, and Statistics*, volume 2. Dordrecht, Netherlands: Reidel. 105–134.

Thöne, H.; Güntzer, U.; and Kießling, W. 1992. Towards precision of probabilistic bounds propagation. In *Proceedings UAI-92*, 315–322. Morgan Kaufmann.