

Privacy-Compatibility For General Utility Metrics

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Abstract

In this note, we present a complete characterization of the utility metrics that allow for non-trivial differential privacy guarantees.

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1 Introduction

The field of data privacy is, at its heart, the study of tradeoffs between utility and privacy. The theoretical computer science community has embraced a strong and compelling definition of privacy — differential privacy [2, 3] — but utility definitions, quite naturally, depend on the application at hand. For a given function f , can we achieve arbitrarily close to perfect utility by relaxing the privacy parameter sufficiently? We show that this question has a satisfyingly simple answer: yes, if and only if the image of f has compact completion. Furthermore, in this case there exists a single base measure μ such that conventional exponential mechanisms based on μ are capable of achieving arbitrarily good utility.

2 Definitions

We are given two metric spaces (\mathbf{X}, ρ) and (\mathbf{Y}, σ) and a continuous function $f : \mathbf{X} \rightarrow \mathbf{Y}$. We think of the input database as being an element $x \in \mathbf{X}$, and our goal is to disclose an approximation to the value of $f(x)$ while preserving privacy. To allow for a cleaner exposition, we will assume throughout this paper that f has Lipschitz constant 1, i.e. $\sigma(f(x), f(z)) \leq \rho(x, z)$ for all $x, z \in \mathbf{X}$. All of our results generalize to arbitrary Lipschitz continuous functions, an issue that we return to in Remark 2.4.

Definition 2.1. A *mechanism* is a function $\mathcal{M} : \mathbf{X} \rightarrow \Delta(\mathbf{Y})$, where $\Delta(\mathbf{Y})$ denotes the set of all Borel probability measures on \mathbf{Y} . For a point $x \in \mathbf{X}$, we will often denote the probability measure $\mathcal{M}(x)$ using the alternate notation \mathcal{M}_x .

Definition 2.2. For $\varepsilon > 0$, we say that a mechanism \mathcal{M} *achieves ε -differential privacy* if the following relation holds for every $x, z \in \mathbf{X}$ and every Borel set $T \subseteq \mathbf{Y}$:

$$\mathcal{M}_x(T) \leq e^{\varepsilon \rho(x, z)} \mathcal{M}_z(T).^1 \quad (1)$$

For $\gamma, \delta > 0$, we say that \mathcal{M} *achieves γ -utility with probability at least $1 - \delta$* if the following relation holds for every $x \in \mathbf{X}$:

$$\mathcal{M}_x(B_\sigma(f(x), \gamma)) \geq 1 - \delta. \quad (2)$$

We abbreviate this relation by saying that \mathcal{M} achieves (γ, δ) -utility.

Definition 2.3. Given a function $f : \mathbf{X} \rightarrow \mathbf{Y}$, the *privacy-utility tradeoff* of f is the function

$$\varepsilon^*(\gamma, \delta) = \inf\{\varepsilon > 0 \mid \exists \text{ a mechanism } \mathcal{M} \text{ satisfying } \varepsilon\text{-differential privacy and } (\gamma, \delta)\text{-utility}\},$$

where the right side is interpreted as ∞ if the set in question is empty.

Remark 2.4. In prior work on differential privacy, it is more customary to express differential privacy guarantees in terms of an *adjacency relation* on inputs, rather than a metric space on the inputs. In this framework, the sensitivity of f (the maximum of $|f(a) - f(b)|$ over all adjacent pairs a, b) plays a pivotal role in determining the privacy achieved by a mechanism. The Lipschitz constant of f plays the equivalent role in our setting.

¹A number of results in the literature, including recent work of Roth and Roughgarden [6] on mechanisms for predicate queries, achieve only a weakened definition of privacy known as (ε, δ) -differential privacy; such results do not fit in the framework presented here.

One could of course equate the two frameworks by defining the privacy metric ρ to be the shortest-path metric in the graph defined by the adjacency relation. This would equate the Lipschitz constant of f with its sensitivity. However, it is much more convenient to describe our mechanisms and their analysis under the assumption that f has Lipschitz constant 1; for any Lipschitz continuous f this can trivially be achieved by rescaling both ρ and the corresponding privacy bound by C , the Lipschitz constant of f .

Thus, for example, if one is given a function f and wishes to know whether there exists a mechanism achieving ε -differential privacy and (γ, δ) -utility, the answer is yes if and only if $\varepsilon/\varepsilon^*(\gamma, \delta)$ is greater than the Lipschitz constant (i.e., sensitivity) of f . In cases where the sensitivity Δ_f depends on the number of points in an input database, N , the relation $\varepsilon/\varepsilon^*(\gamma/\delta) \geq \Delta_f$ can be used to solve for N in terms of the parameters $\varepsilon, \gamma, \delta$. For example, in many papers (e.g. [1]) $\Delta_f = 1/N$ and then we find that $N = \varepsilon^*(\gamma, \delta)/\varepsilon$ is the minimum number of points in the input database necessary to achieve ε -differential privacy and (γ, δ) -utility.

Remark 2.5. Our definition of utility captures many prior formulations. For settings where the output space is simply \mathbb{R} , the traditional utility metric reflecting the difference between the given answer and the true answer is easily captured in our framework. A variety of prior work on problems involving more complex outputs can also be cast as measuring utility in a metric space. For example, Blum et al. [1] propose utility with respect to a concept class \mathcal{H} , and define the utility of a candidate output database y on an input x as $\max_{h \in \mathcal{H}} |h(x) - h(y)|$. This setup can be viewed as mapping input databases x to vectors $(h_1(x), h_2(x), \dots)$ and taking the utility metric σ to be the L^∞ metric on output vectors. Hardt and Talwar [4] use L^2 as their utility metric, but whereas they compute the mean square (or p -th moment) of its distribution, we define disutility to be the probability that the σ value exceeds γ .

Definition 2.6. Given a measure μ on \mathbf{X} , and a scalar $\beta > 0$, the (*conventional*) *exponential mechanism* $\mathcal{C}^{\mu; \beta}$ is given by the formula:

$$\mathcal{C}_x^{\mu; \beta}(T) = \frac{\int_T e^{-\beta \sigma(f(x), y)} d\mu(y)}{\int_{\mathbf{Y}} e^{-\beta \sigma(f(x), y)} d\mu(y)}, \quad (3)$$

provided that the denominator is finite. Otherwise $\mathcal{C}_x^{\mu; \beta}$ is undefined.²

The differential privacy guarantee for exponential mechanisms is given by the following theorem, whose proof parallels the original proof of McSherry and Talwar [5] and is given in the Appendix.

Theorem 2.7. *If f has Lipschitz constant C then the conventional exponential mechanism $\mathcal{C}^{\mu; \beta}$ is $(2C\beta)$ -differentially private for every μ .*

3 A topological criterion for privacy-compatibility

A surprising result of Blum et al. [1] shows that, in the natural setting of one-dimensional range queries over continuous domains, *no* mechanism can simultaneously achieve non-trivial privacy and utility guarantees. What is it about this application that makes privacy fundamentally impossible? In this section, we introduce a definition of *privacy-compatibility* and give a complete characterization of the applications that satisfy this definition.

Definition 3.1. We say that f is *privacy-compatible* if $\varepsilon^*(\gamma, \delta) < \infty$ for all $\gamma, \delta > 0$.

²We use the word “conventional” here to refer to the rich subclass of exponential mechanisms whose score function is σ ; however, not all exponential mechanisms fall in this class.

Suppose that f is Lipschitz continuous and that the metric space (\mathbf{X}, ρ) is bounded. We now prove that f is privacy-compatible if and only if the completion of the metric space $f(\mathbf{X})$ is compact. Observe that rescaling the metrics ρ, σ does not affect the question of whether f is privacy-compatible nor whether $f(\mathbf{X})$ has compact completion, but it does rescale the Lipschitz constant of f and the diameter of \mathbf{X} . Accordingly, we may assume without loss of generality that the Lipschitz constant of f and the diameter of \mathbf{X} are both bounded above by 1, i.e.

$$\sigma(f(x_1), f(x_2)) \leq \rho(x_1, x_2) \leq 1 \quad (4)$$

for all $x_1, x_2 \in \mathbf{X}$.

Definition 3.2. A probability measure μ on a metric space (\mathbf{X}, σ) is *uniformly positive* if it is the case that for all $r > 0$,

$$\inf_{x \in X} \mu(B_\sigma(x, r)) > 0.$$

Example 3.3. The uniform measure on $[0, 1]$ is uniformly positive. The Gaussian measure on \mathbb{R} is not uniformly positive because one can find intervals of width $2r$ with arbitrarily small measure by taking the center of the interval to be sufficiently far from 0.

Theorem 3.4. *If the Lipschitz constant of f and the diameter of X are both bounded above by 1, then the following are equivalent:*

1. f is privacy-compatible;
2. For every $\gamma, \delta > 0$, there is a conventional exponential mechanism that achieves (γ, δ) -utility;
3. There exists a uniformly positive measure on $(f(\mathbf{X}), \sigma)$;
4. The completion of $(f(\mathbf{X}), \sigma)$ is compact.

Proof. For simplicity, throughout the proof we assume without loss of generality that $\mathbf{Y} = f(\mathbf{X})$. The notation $B(y, r)$ denotes the ball of radius r around y in the metric space (\mathbf{Y}, σ) .

(2) \Rightarrow (1) The exponential mechanism $\mathcal{M}^{\mu; \beta}$ achieves (2β) -differential privacy.

(3) \Rightarrow (2) For μ a uniformly positive measure on (Y, σ) , and $\gamma, \delta > 0$, let $m = \inf_{y \in \mathbf{Y}} \mu(B(y, \gamma/2))$ and let $\beta = \frac{2}{\gamma} \ln\left(\frac{1}{\delta m}\right)$. We claim that the exponential mechanism $\mathcal{M} = \mathcal{M}^{\mu; \beta}$ achieves (γ, δ) -utility. To see this, let $x \in \mathbf{X}$ be an arbitrary point, let $z = f(x)$, and let

$$a = \int_{B(x, \gamma)} e^{-\beta \sigma(z, y)} d\mu(y) \quad b = \int_{\mathbf{X} \setminus B(x, \gamma)} e^{-\beta \sigma(z, y)} d\mu(y).$$

We have

$$\begin{aligned} a &\geq \int_{B(z, \gamma/2)} e^{-\beta \sigma(z, y)} d\mu(y) \geq \int_{B(z, \gamma/2)} e^{-\beta \gamma/2} d\mu(y) = e^{-\beta \gamma/2} \mu(B(z, \gamma/2)) \geq e^{-\beta \gamma/2} m \\ b &< \int_{\mathbf{Y}} e^{-\beta \gamma} d\mu(y) = e^{-\beta \gamma}. \end{aligned}$$

Hence, for every $x \in \mathbf{X}$,

$$\mathcal{M}_x(B(f(x), \gamma)) = \frac{a}{a+b} = 1 - \frac{b}{a+b} > 1 - \frac{e^{-\beta \gamma}}{e^{-\beta \gamma/2} m} = 1 - \frac{1}{e^{\beta \gamma/2} m} = 1 - \delta.$$

(4) \Rightarrow (3) We use the following fact from the topology of metric spaces: a complete metric space is compact if and only, for every r , if it has a finite covering by balls of radius r . (See Theorem A.2 in the Appendix.) For $i = 1, 2, \dots$, let $C_i = \{y_{i,1}, \dots, y_{i,n(i)}\}$ be a finite set of points such that the balls of radius 2^{-i} centered at the points of C_i cover \mathbf{Y} . Now define a probability measure μ supported on the countable set $C = \cup_{i=1}^{\infty} C_i$, by specifying that for $y \in C$, $\mu(y) = \sum_{i: y \in C_i} \left(\frac{1}{2^{i n(i)}} \right)$. Equivalently, one can describe μ by saying that a procedure for randomly sampling from μ is to flip a fair coin until heads comes up, let i be the number of coin flips, and sample a point of C_i uniformly at random. We claim that μ is uniformly positive. To see this, given any $r > 0$ let $i = \lceil \log_2(1/r) \rceil$, so that $2^{-i} \leq r$. For any point $y \in \mathbf{Y}$, there exists some j ($1 \leq j \leq n(i)$) such that $y \in B(y_{i,j}, 2^{-i})$. This implies that $B(y, r)$ contains $y_{i,j}$, hence $\mu(B(y, r)) \geq \mu(y_{i,j}) \geq \frac{1}{2^{i n(i)}}$. The right side depends only on r (and not on y), hence $\inf_{y \in \mathbf{Y}} \mu(B(y, r))$ is strictly positive, as desired.

(1) \Rightarrow (4) We prove the contrapositive. Suppose that the completion of \mathbf{Y} is not compact. Once again using point-set topology (Theorem A.2) this implies that there exists an infinite collection of pairwise disjoint balls of radius r , for some $r > 0$. Let y_1, y_2, \dots , be the centers of these balls. By our assumption that $\mathbf{Y} = f(\mathbf{X})$, we may choose points x_i such that $y_i = f(x_i)$ for all $i \geq 1$. Suppose we are given a mechanism \mathcal{M} that achieves r -utility with probability at least $1/2$. For every $\alpha > 0$ we must show that \mathcal{M} does not achieve α -differential privacy. The relation $\sum_{i=1}^{\infty} \mathcal{M}_{x_1}(B(y_i, r)) \leq 1$ implies that there exists some i such that

$$\mathcal{M}_{x_1}(B(y_i, r)) < e^{-\alpha}/2. \quad (5)$$

The fact that \mathcal{M} achieves r -utility with probability at least $1/2$ implies that

$$\mathcal{M}_{x_i}(B(y_i, r)) > 1/2. \quad (6)$$

Combining (5) with (6) leads to

$$\mathcal{M}_{x_i}(B(y_i, r)) > e^{\alpha} \mathcal{M}_{x_1}(B(y_i, r)) \geq e^{\alpha \rho(x_i, x_1)} \mathcal{M}_{x_1}(B(y_i, r)), \quad (7)$$

hence \mathcal{M} violates α -differential privacy. \square

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A Appendix

Lemma A.1. *If $f : \mathbf{X} \rightarrow \mathbf{Y}$ has Lipschitz constant 1, then the conventional exponential mechanism $\mathcal{M}^{\mu;\beta}$ achieves (2β) -differential privacy.*

Proof. The proof follows the original proof of McSherry and Talwar [5]. The triangle inequality implies that for any x, z

$$\begin{aligned} \int_T e^{-\beta\sigma(f(x),y)} d\mu(y) &\leq \int_T e^{-\beta[\sigma(f(z),y)-\sigma(f(x),f(z))]} d\mu(y) \\ &= e^{\beta\sigma(f(x),f(z))} \int_T e^{-\beta\sigma(f(x),y)} d\mu(y) \\ &\leq e^{\beta\rho(x,z)} \int_T e^{-\beta\sigma(f(z),y)} d\mu(y) \\ \int_Y e^{-\beta\sigma(f(x),y)} d\mu(y) &\geq \int_Y e^{-\beta[\sigma(f(z),y)+\sigma(f(x),f(z))]} d\mu(y) \\ &= e^{-\beta\sigma(f(x),f(z))} \int_Y e^{-\beta\sigma(f(z),y)} d\mu(y) \\ &\geq e^{-\beta\rho(x,z)} \int_Y e^{-\beta\sigma(f(z),y)} d\mu(y). \end{aligned}$$

The inequality $\mathcal{M}_x(T) \leq e^{2\beta\rho(x,z)} \mathcal{M}_z(T)$ follows upon taking the quotient of these two inequalities. \square

Theorem A.2. *For a metric space (\mathbf{X}, σ) , the following are equivalent:*

1. *The completion of \mathbf{X} is a compact topological space.*
2. *For every $r > 0$, \mathbf{X} can be covered by a finite collection of balls of radius r .*
3. *For every $r > 0$, \mathbf{X} does not contain an infinite collection of pairwise disjoint balls of radius r .*

Proof. **(2) \Rightarrow (1)** Assume that property (2) holds. Recall that a metric space is compact if and only if every infinite sequence of points has a convergent subsequence, and it is complete if and only if every Cauchy sequence is convergent. Thus, we must prove that every infinite sequence x_1, x_2, \dots in \mathbf{X} has a Cauchy subsequence. We can use a pigeonhole-principle argument to construct the Cauchy subsequence. In fact, the construction will yield a sequence of points z_1, z_2, \dots and sets S_1, S_2, \dots such that the diameter of S_k is at most $1/k$ and $z_i \in S_k$ for all $i \geq k$; these two properties immediately imply that z_1, z_2, \dots is a Cauchy sequence as desired.

The construction begins by defining $S_0 = \mathbf{X}$. Now, for any $k > 0$, assume inductively that we have a set S_{k-1} such that the relation $x_i \in S_{k-1}$ is satisfied by infinitely many i . Let $B_1, B_2, \dots, B_{n(k)}$ be a finite collection of balls of radius $\frac{1}{2k}$ that covers \mathbf{X} . There must be at least one value of j such that the relation $x_i \in S_{k-1} \cap B_j$ is satisfied by infinitely many i . Let $S_k = S_{k-1} \cap B_j$ and let z_k be any point in the sequence x_1, x_2, \dots that belongs to S_k and occurs strictly later in the sequence than z_{k-1} . This completes the construction of the Cauchy subsequence and establishes that the completion of \mathbf{X} is compact.

(1) \Rightarrow (3) If \mathbf{X} contains an infinite collection of pairwise disjoint balls of radius r , then the centers of these balls form an infinite set with no limit point in \mathbf{X} , violating compactness.

(3) \Rightarrow (2) Given $r > 0$, let $B(x_1, r/2), \dots, B(x_n, r/2)$ be a maximal collection of disjoint balls of radius $r/2$. (Such a collection must be finite, by property (3).) The balls $B(x_1, r), \dots, B(x_n, r)$

cover \mathbf{X} , because if there were a point $y \in \mathbf{X}$ not covered by these balls, then $B(y, r/2)$ would be disjoint from $B(x_i, r/2)$ for $i = 1, \dots, n$, contradicting the maximality of the collection. \square