On Theorem 2.3 in "Prediction, Learning, and Games" by Cesa-Bianchi and Lugosi.

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This note proves a loss bound for the exponentially weighted average forecaster with time-varying potential, see [1, § 2.3] for context and definitions. The present proof gives a better constant in the regret term than Theorem 2.3 in [1]. This proof first appeared in [2] (Theorem 2), where a more general algorithm is considered. Here the proof is rewritten using the notation of [1].

Theorem 1. *Assume that the loss function ℓ is convex in the first argument and $\ell(p,y) \in [0,1]$ for all $p \in \mathcal{D}$ and $y \in \mathcal{Y}$. For any positive reals $\eta_1 \geq \eta_2 \geq \ldots$, for any $n \geq 1$ and for any $y_1, \ldots, y_n \in \mathcal{Y}$, the regret of the exponentially weighted average forecaster with time-varying learning rate η_t satisfies*

$$\widehat{L}_n - \min_{i=1,\ldots,N} L_{i,n} \leq \frac{\ln N}{\eta_n} + \frac{1}{8}\sum_{t=1}^{n} \eta_t \,. \tag{1}$$

In particular, for $\eta_t = \sqrt{\frac{4\ln N}{t}}$, $t = 1, \ldots, n$, we have

$$\widehat{L}_n - \min_{i=1,\ldots,N} L_{i,n} \leq \sqrt{n \ln N}\,.$$

Proof. The forecaster at step t predicts $\widehat{p}_t = \sum_{i=1}^{N} \frac{w_{i,t-1}}{W_{t-1}} f_{i,t}$, where $w_{i,t-1} = e^{-\eta_t L_{i,t-1}}$ and $W_{t-1} = \sum_{j=1}^{N} w_{j,t-1}$. Due to convexity of ℓ we have

$$\ell(\widehat{p}_t, y_t) \leq \sum_{i=1}^{N} \frac{w_{i,t-1}}{W_{t-1}}\, \ell(f_{i,t}, y_t)\,.$$

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Using the Hoeffding inequality ([1, Lemma A.1]), we get

$$e^{-\eta_t \sum_{i=1}^N \frac{w_{i,t-1}}{W_{t-1}} \ell(f_{i,t},y_t)} \geq \sum_{i=1}^N \frac{w_{i,t-1}}{W_{t-1}} e^{-\eta_t \ell(f_{i,t},y_t) - \eta_t^2/8}$$

and thus

$$e^{-\eta_t \ell(\widehat{p}_t,y_t)} \geq \sum_{i=1}^N \frac{w_{i,t-1}}{W_{t-1}} e^{-\eta_t \ell(f_{i,t},y_t) - \eta_t^2/8} . \tag{2}$$

Consider the values

$$s_{i,t-1} = e^{-\eta_{t-1} L_{i,t-1} + \eta_{t-1} \widehat{L}_{t-1} - \frac{1}{8}\eta_{t-1} \sum_{k=1}^{t-1} \eta_k}$$

and note that

$$\frac{w_{i,t-1}}{W_{t-1}} = \frac{\frac{1}{N}\left(s_{i,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}}}{\sum_{j=1}^N \frac{1}{N}\left(s_{j,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}}} . \tag{3}$$

Let us show that $\sum_{j=1}^N \frac{1}{N} s_{j,t} \leq 1$ by induction over t. For $t = 0$ this is trivial, since $s_{j,0} = 1$ for all j. Assume that $\sum_{j=1}^N \frac{1}{N} s_{j,t-1} \leq 1$. Then

$$\sum_{j=1}^N \frac{1}{N}\left(s_{j,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}} \leq \left(\sum_{j=1}^N \frac{1}{N} s_{j,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}} \leq 1 , \tag{4}$$

since the function $x \mapsto x^\alpha$ is concave and monotone for $x \geq 0$ and $\alpha \in [0,1]$ and since $\eta_{t-1} \geq \eta_t > 0$. Using (4) to bound the right-hand side of (3), we get $\frac{w_{i,t-1}}{W_{t-1}} \geq \frac{1}{N}\left(s_{i,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}}$; and combining with (2), we get

$$e^{-\eta_t \ell(\widehat{p}_t,y_t)} \geq \sum_{i=1}^N \frac{1}{N}\left(s_{i,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}} e^{-\eta_t \ell(f_{i,t},y_t) - \eta_t^2/8} .$$

It remains to note that

$$s_{i,t} = \left(s_{i,t-1}\right)^{\frac{\eta_t}{\eta_{t-1}}} e^{-\eta_t \ell(f_{i,t},y_t) + \eta_t \ell(\widehat{p}_t,y_t) - \eta_t^2/8}$$

and we get $\sum_{i=1}^N \frac{1}{N} s_{i,t} \leq 1$.

For any i, we have $\frac{1}{N} s_{i,n} \leq \sum_{j=1}^N \frac{1}{N} s_{j,n} \leq 1$, thus

$$-\eta_n L_{i,n} + \eta_n \widehat{L}_n - \frac{1}{8}\eta_n \sum_{k=1}^n \eta_k \leq \ln N ,$$

and (1) follows. $\square$

Theorem 1 recommends the learning rate $\eta_t = \sqrt{(4\ln N)/t}$ instead of $\sqrt{(8\ln N)/t}$ used in Theorem 2.3 in [1] and achieves the regret term $\sqrt{n \ln N}$ instead of $\sqrt{2n\ln N} + \sqrt{0.125\ln N}$.

To compare the bounds for arbitrary learning rates, let us observe that the proof of Theorem 2.3 in [1] actually implies (under the assumptions of Theorem 1):

$$\widehat{L}_n - \min_{i=1,\ldots,N} L_{i,n} \le \left(\frac{2}{\eta_n} - \frac{1}{\eta_1}\right)\ln N + \frac{1}{8}\sum_{t=1}^{n}\eta_t\,.$$

The right-hand side of this inequality is larger than the right-hand side of (1) if $\eta_n \neq \eta_1$. If η_t are equal for all t, the bounds coincide and give the bound of Theorem 2.2 in [1].

References

[1] N. Cesa-Bianchi, G. Lugosi. *Prediction, Learning, and Games*. Cambridge University Press, Cambridge, England, 2006.

[2] A. Chernov, F. Zhdanov. Prediction with expert advice under discounted loss. Proc. of ALT 2010, LNCS 6331, pp. 255-269. See also: arXiv:1005.1918v1 [cs.LG].