Discrete Schemes for Gaussian Curvature and Their Convergence

Zhiqiang Xu[*]   Guoliang Xu [†]

Institute of Computational Math.,

Academy of Mathematics and System Sciences,

Chinese Academy of Sciences, Beijing, 100080, China

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Abstract

In this paper, a new discrete scheme for Gaussian curvature is presented. We show that this new scheme converges at the regular vertex with valence not less than 5. By constructing a counterexample, we also show that it is impossible for building a discrete scheme for Gaussian curvature which converges over the regular vertex with valence 4. Moreover, the convergence property of a modified discrete scheme for the Gaussian curvature on certain meshes is presented. Finally, asymptotic errors of several discrete schemes for Gaussian curvature are compared.

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1 Introduction

Some applications from computer vision, computer graphics, geometric modeling and computer aided design require estimating intrinsic geometric invariants. It is well known that Gaussian curvature is one of the most essential

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[*]Email:xuzq@lsec.cc.ac.cn

[†]Email:xuguo@lsec.cc.ac.cn

geometric invariants for surfaces. However, in the classical differential geometry, this invariant is well defined only for C^2 smooth surfaces. In modern computer-related geometry fields, one often uses C^0 continuous discrete triangular meshes to represent smooth surfaces approximately. Hence, the problem of estimating accurately Gaussian curvature for triangular meshes is raised naturally.

In the past years, a wealth of different estimations have been proposed in the vast literature of applied geometry. These methods for estimating Gaussian curvature can be divided into two classes. The first class is based on the local fitting or interpolation technique [2, 5, 6, 7, 20], while the second class is based on discretization formulations which represent the information about the Gaussian curvature [1, 3, 7, 12, 16, 18]. In this paper, our focus is on the methods in the second class. The main aim of the paper is to present a new discrete scheme for the discrete Gaussian curvature which converges at the regular vertex with valence not less than 5.

Let M be a triangulation of smooth surface S in $\mathbb{R}^3$. For a vertex $\mathbf{p}$ of M, suppose $\{\mathbf{p}_i\}_{i=1}^n$ is the set of the one-ring neighbor vertices of $\mathbf{p}$. The set $\{\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}\}$ $(i = 1, \cdots, n)$ of n Euclidean triangles forms a piecewise linear approximation of S around $\mathbf{p}$. Throughout the paper, we use the following conventions $\mathbf{p}_{n+1} = \mathbf{p}_1$ and $\mathbf{p}_0 = \mathbf{p}_n$. Let γ_i denote the angle $\angle\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ and the angular defect at $\mathbf{p}$ be $2\pi - \sum_i \gamma_i$.

A popular discrete scheme for computing Gaussian curvature is in the form of $\frac{2\pi - \sum_i \gamma_i}{E}$, where E is a geometry quantity. In general, one selects E as $A(\mathbf{p})/3$ and obtain the following approximation

$$G^{(1)} := \frac{3(2\pi - \sum_i \gamma_i)}{A(\mathbf{p})}, \tag{1}$$

where $A(\mathbf{p})$ is the sum of the areas of triangles $\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$. In [1], another scheme

$$G^{(2)} := \frac{2\pi - \sum_i \gamma_i}{S_p} \tag{2}$$

is given, where

$$S_p := \sum_i \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1} - \frac{\cos\gamma_i}{2}(\eta_i^2 + \eta_{i+1}^2)\right]$$

is called the module of the mesh at $\mathbf{p}$. In [16], the discrete approximation

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$G^{(1)}$ is modified as

$$G^{(3)} := \frac{2\pi - \sum_i \gamma_i}{\frac{1}{2} \sum_i \text{area}(\mathbf{p}_i \mathbf{p} \mathbf{p}_{i+1}) - \frac{1}{8} \sum_i \cot(\gamma_i) d_i^2}, \tag{3}$$

where d_i is the length of edges $\mathbf{p}_i \mathbf{p}_{i+1}$. There are several different points of view for explaining the reason why the angular defect is closely related to the Gaussian curvature with including the viewpoints of Gaussian-Bonnet theorem, Gaussian map and Legendre's formula (see the next section for details).

Asymptotic analyses for the discrete schemes have been given in [1, 7, 18]. In [7], the authors show that for the non-uniform data, the discrete scheme $G^{(1)}$ does not convergent to true Gaussian curvature always. In [1], Borrelli et al. prove that the angular defect is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures and show that if $\mathbf{p}$ is a regular vertex with valence six, then the scheme $G^{(2)}$ converges to the exact Gaussian curvature in a linear rate. Moreover, Borrelli et al. show that 4 is the only value of the valence such that the angular defect depends upon the principal directions. In [18], Xu proves that the discrete scheme $G^{(1)}$ has quadratic convergence rate if the mesh satisfies the so-called parallelogram criterion, which requires valence 6. Therefore, one hopes to construct a discrete scheme which converges over any discrete mesh. But in [21], Xu et al. show that it is impossible to construct a discrete scheme which is convergent for any discrete mesh. Hence, we have to be content with the discrete schemes which converge under some conditions. According to the past experience [1, 8, 21], we regard a discrete scheme desirable if it has the following properties

1. It converges at a regular vertex, at least for sufficiently large valence (the definition of the regular vertex will be given in Section 2);

2. It converges at the umbilical point, i.e., the points satisfying $k_m = k_M$ where k_m and k_M are two principal curvatures.

As stated before, the previous discrete schemes, including $G^{(1)}, G^{(2)}$ and $G^{(3)}$, only converge at the regular vertex with valence 6. In [1], a method for computing the Gaussian curvature at the regular vertex with valence unequal to 4 is described. But the method requires two meshes with valences n_1 and n_2 ($n_1 \neq 4, n_2 \neq 4, n_1 \neq n_2$). In this paper, we will construct a discrete scheme which converges at the regular vertex with valence not less than 5,

and also at the umbilical points with any valence. Moreover, the discrete scheme requires only a single mesh. Hence, the new discrete scheme is more desirable. Furthermore, we show that it is impossible to construct a discrete scheme which is convergent at the regular vertex with valence 4. Therefore, for the regular vertex, the convergence problem remains open for the case of vertex with valence 3. We also want to point out that we discuss the pointwise convergence in this paper, which is different with the convergence in norm as discussed in [4, 10].

The rest of the paper is organized as follows. Section 2 describes some notations and definitions and Section 3 shows three different viewpoints for expressing the relation between the angular defect and Gaussian curvature. In Section 4, we study the convergence property of the modified discrete Gaussian curvature scheme. We present in Section 5 a new discrete scheme and prove that the scheme has good convergence property. In Section 6, for the regular vertex with valence 4, we show that it is impossible to build a discrete scheme which is convergent to the real Gaussian curvature. Some numerical results are given in Section 7.

2　Preliminaries
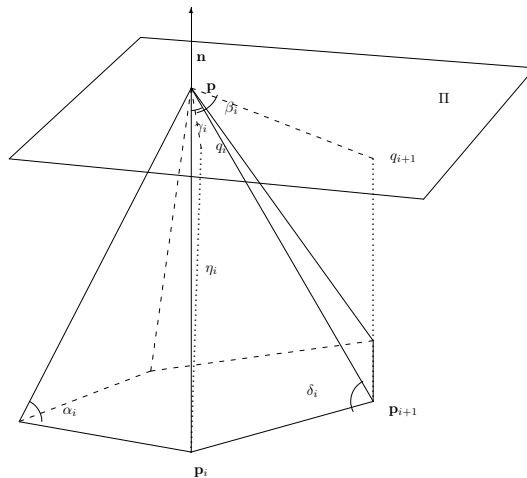
Fig.1. Notations.

In this section, we introduce some notations and definitions used throughout the paper (see Fig. 1). Let S be a given smooth surface and $\mathbf{p}$ be a point

over S. Suppose the set $\{\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}\}$, $i = 1, \cdots, n$, of n Euclidean triangles form a piecewise linear approximation of S around $\mathbf{p}$. The vector from $\mathbf{p}$ to $\mathbf{p}_i$ is denoted as $\overrightarrow{\mathbf{p}\mathbf{p}_i}$. The normal vector and tangent plane of S at the point $\mathbf{p}$ is denoted by $\mathbf{n}$ and Π, respectively. We denote the projection of $\mathbf{p}_i$ onto Π as $\mathbf{q}_i$, and define the plane containing $\mathbf{n}, \mathbf{p}$ and $\mathbf{p}_i$ as Π_i. Then we let κ_i denote the curvature of the plane curve $S \cap \Pi_i$ at $\mathbf{p}$. The distances from $\mathbf{p}$ to $\mathbf{p}_i$ and $\mathbf{q}_i$ are denoted as η_i and l_i, respectively. The angles $\angle\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}$ and $\angle\mathbf{q}_i\mathbf{p}\mathbf{q}_{i+1}$ are denoted as γ_i and β_i, respectively. The two principal curvatures at $\mathbf{p}$ are denoted as k_m and k_M. Let $\eta = \max_i \eta_i$. The following results are presented in [1, 8, 18]:

$$\frac{l_i}{\eta_i} = 1 + O(\eta), \quad \beta_i = \gamma_i + O(\eta^2), \tag{4}$$

$$\left\| \sum_i w_i \overrightarrow{\mathbf{p}\mathbf{p}_i} \right\| = \sum_i \frac{w_i \kappa_i \eta_i^2}{2} + O(\eta^3), \tag{5}$$

where $w_i \in \mathbb{R}$.

Now we give the definition of the regular vertex using the notations introduced above.

Definition 2.1. Let $\mathbf{p}$ be a point of a smooth surface S and let $\mathbf{p}_i, i = 1, \cdots, n$ be its one ring neighbors. The point $\mathbf{p}$ is called a regular vertex if it satisfies the following conditions
   (1) the $\beta_i = \frac{2\pi}{n}$,
   (2) the η_is all take the same value η.

Remark 2.2. We can replace (1) in Definition 2.1 by requiring the γ_i all take the same value. Since $\beta_i = \gamma_i + O(\eta^2)$, all the results in the paper hold also for the alternative definition.

3   Angular Defect and Gaussian Curvature

In this section, we summarize three different viewpoints for expressing the relation between angular defect and Gaussian curvature. These viewpoints have been described in different literature [7, 16, 18]. We collect them there. Throughout the section, we use $G^{(1)}(\mathbf{p})$ to denote the discrete Gaussian curvature at $\mathbf{p}$, which is obtained using $G^{(1)}$.

3.1 Gaussian-Bonnet theorem viewpoint

Let D be a region of surface S, whose boundary consists of piecewise smooth curves Γ_js. Then the local Gaussian-Bonnet theorem is as follows

$$\iint_D G(p)dA + \sum_j \int_{\Gamma_j} k_g(\Gamma_j)ds + \sum_j \alpha_j = 2\pi,$$

where $G(p)$ is the Gaussian curvature at p, $k_g(\Gamma_j)$ is the geodesic curvature of the boundary curve Γ_j and α_j is the exterior angle at the jth corner point $\mathbf{p}_j$ of the boundary. If all the Γ_js are the geodesic curves, the above formula reduces to

$$\iint_D G(p)dA = 2\pi - \sum_j \alpha_j. \tag{6}$$

Let M be a triangulation of surface S. For vertex $\mathbf{p}$ of valence n, each triangle $\mathbf{p}_i\mathbf{pp}_{i+1}$ can be partitioned into three equal parts, one corresponding to each of its vertices. We let D be the union of the part corresponding to $\mathbf{p}$ of triangles $\mathbf{p}_i\mathbf{pp}_{i+1}$. Note that $\sum_i \gamma_i = \sum_j \alpha_j$. Assuming $G(\mathbf{p})$ is a constant on D, and using (6), we have $G(\mathbf{p})$ can be approximated by $G^{(1)}(\mathbf{p})$.

3.2 Spherical image viewpoint

We now introduce another definition of Gaussian curvature. Let D be a small patch of area A including point $\mathbf{p}$ on the surface S. There will be a corresponding patch of area I on the Gaussian map. Gaussian curvature at $\mathbf{p}$ is the limit of ratio $\lim_{A \to 0} \frac{I}{A}$.

Let us consider a discrete version of the definition. The Gaussian map image, i.e. the spherical image, of the triangle $\mathbf{p}_i\mathbf{pp}_{i+1}$ is the point $\frac{(\mathbf{p}-\mathbf{p}_i)\times(\mathbf{p}-\mathbf{p}_{i+1})}{\|(\mathbf{p}-\mathbf{p}_i)\times(\mathbf{p}-\mathbf{p}_{i+1})\|}$. Join these points by great circle forming a spherical polygon on the unit sphere. The area of this spherical polygon is $2\pi - \sum_i \gamma_i$. Same as the above, each triangle is partitioned into three parts, one corresponding to each vertex. Then the Gaussian curvature can be approximated by $G^{(1)}(\mathbf{p})$.

3.3 Geodesic triangles viewpoint

Let $T = ABC$ be a geodesic triangle on the surface S with angles α, β, γ and geodesic edge lengths a, b, c. Let $A'B'C'$ be a corresponding Euclidean

triangle with edge lengths a, b, c and angles α', β', γ'. Legendre presents the following formulation

$$\alpha - \alpha' = \text{area}(T)\frac{G(A)}{3} + o(a^2 + b^2 + c^2),$$

where $\text{area}(T)$ is the area of the geodesic triangle ABC, $G(A)$ is the Gaussian curvature at A.

Using Legendre's formulation for each triangles with $\mathbf{p}$ as a vertex, we arrive at the estimating formula $G^{(1)}(\mathbf{p})$ again.

4 Convergence of Angular Defect Schemes

In [18], Xu gives an analysis about the scheme $G^{(1)}$ and proves that the scheme converges at the vertex satisfying so-called parallelogram criterion. A numerical test shows that the scheme does not converge at the regular vertex with valence unequal to 6 and at umbilical points. In [1], Borrelli et. al. give an elegant analysis about the angular defect. They show that if the vertex $\mathbf{p}$ is regular, then the angular deficit is asymptotically equivalent to a homogeneous polynomial of degree two in the principal curvatures with closed form coefficients. Moreover, they present another angular scheme $G^{(2)} := \frac{2\pi - \sum_i \gamma_i}{S_p}$. In fact, using the law of cosine, we have

$$\frac{1}{2}\sum_i \text{area}(\mathbf{p}_i\mathbf{p}\mathbf{p}_{i+1}) - \frac{1}{8}\sum_i \cot(\gamma_i)d_i^2$$
$$= \sum_i \left[\frac{1}{4}\eta_i\eta_{i+1}\sin\gamma_i - \frac{1}{8}\frac{\cos\gamma_i}{\sin\gamma_i}(\eta_i^2 + \eta_{i+1}^2 - 2\eta_i\eta_{i+1}\cos\gamma_i)\right]$$
$$= \sum_i \frac{1}{4\sin\gamma_i}\left[\eta_i\eta_{i+1} - \frac{\cos\gamma_i}{2}(\eta_i^2 + \eta_{i+1}^2)\right] = S_p.$$

This shows that $G^{(2)}$ and $G^{(3)}$ are equivalent, which means these two schemes obtain the same value for the same triangular mesh.

In [18], the author proves that the discrete scheme $G^{(1)}$ has quadratic convergence rate under the parallelogram criterion. In the following theorem, we shall show that the discrete scheme $G^{(3)}$ has also quadratic convergence rate under the same criterion.

Theorem 4.1. *Let* $\mathbf{p}$ *be a vertex of* M *with valence six, and let* $\mathbf{p}_j, j = 1, \cdots, 6$ *be its neighbor vertices. Suppose* $\mathbf{p}$ *and* $\mathbf{p}_j, j = 1, \cdots, 6$ *are on a sufficiently smooth parametric surface* $\mathbf{F}(\xi_1, \xi_2) \in \mathbb{R}^3$, *and there exist* $\mathbf{u}, \mathbf{u}_j \in \mathbb{R}^2$ *such that*

$$\mathbf{p} = \mathbf{F}(\mathbf{u}), \ \ \mathbf{p}_j = \mathbf{F}(\mathbf{u}_j) \ and \ \mathbf{u}_j - \mathbf{u} = (\mathbf{u}_{j-1} - \mathbf{u}) + (\mathbf{u}_{j+1} - \mathbf{u}), \ j = 1, \cdots, 6.$$

Then

$$\frac{2\pi - \gamma_i}{\frac{1}{2}A(\mathbf{p}, r) - \frac{1}{8}\sum_i \cot(\gamma_i(r))d_i^2(r)} = G(\mathbf{p}) + O(r^2),$$

where, $G(\mathbf{p})$ *is the real Gaussian curvature of* $\mathbf{F}(\mathbf{u})$ *at* $\mathbf{p}$,

$$A(\mathbf{p}, r) := \sum_i \operatorname{area}[\mathbf{p}_i(r)\mathbf{p}\mathbf{p}_{i+1}(r)], \ \ \mathbf{p}_i(r) := \mathbf{F}(\mathbf{u}_i(r)),$$

and $\mathbf{u}_i(r) = \mathbf{u} + r(\mathbf{u}_i - \mathbf{u}), i = 1, \cdots, 6.$

Proof. Let

$$A(\mathbf{p}, r) = a_0 r^2 + a_1 r^3 + O(r^4) \tag{7}$$

and

$$\frac{A(\mathbf{p}, r)}{2} - \frac{1}{8}\sum_i \cot(\gamma_i(r))d_i^2(r) = b_0 r^2 + b_1 r^3 + O(r^4)$$

be the Taylor expansions with respect to r. According to Theorem 4.1 in [18],

$$\frac{3(2\pi - \gamma_i)}{A(\mathbf{p}, r)} = G(\mathbf{p}) + O(r^2).$$

Hence, to prove the theorem, we need to show $b_0 = a_0/3, b_1 = a_1/3$. According to [18], we have $a_1 = 0$, which implies that we only need to prove $b_0 = a_0/3, b_1 = 0$.

Note that the $\mathbf{u}, \mathbf{u}_j, j = 1, \ldots 6$, satisfy the parallelogram criterion. Without loss of generality, we may assume $\mathbf{u} = [0, 0]^T, \mathbf{u}_1 = [1, 0]^T$. Then there exists a constant $a > 0$ and an angle θ such that

$$\mathbf{u}_2 = [a\cos\theta, a\sin\theta]^T.$$

Hence, $\mathbf{u}_3 = [a\cos\theta - 1, a\sin\theta]^T, \mathbf{u}_{j+3} = -\mathbf{u}_j, j = 1, 2, 3.$ Let

$$\mathbf{u}_j = s_j\mathbf{d}_j = s_j[g_j, l_j]^T, \ j = 1, \cdots, 6,$$

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where $s_j = \|\mathbf{u}_j\|$ and $\|\mathbf{d}_j\| = 1$. Then, we have

$$s_1 = 1, \ s_2 = a, \ s_3 = \sqrt{a^2 - 2ac + 1}, \ s_4 = s_1, \ s_5 = s_2, \ s_6 = s_3,$$
$$g_1 = 1, \ g_2 = c, \ g_3 = (ac - 1)/s_3, \ g_4 = -g_1, \ g_5 = -g_2, \ g_6 = -g_3,$$
$$l_1 = 0, \ l_2 = t, \ l_3 = at/s_3, \ l_4 = -l_1, \ l_5 = -l_2, \ l_6 = -l_3,$$

where $(c, t) := (\cos\theta, \sin\theta)$. Note that

$$A(\mathbf{p}, r) = \frac{1}{2}\sum_{j=1}^{6} \sqrt{\|\mathbf{p}_j(r) - \mathbf{p}\|^2 \|\mathbf{p}_{j+1}(r) - \mathbf{p}\|^2 - \langle \mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle^2}, \ (8)$$

$$\cot(\gamma_j(r)) = \frac{\langle \mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle}{\sqrt{\|\mathbf{p}_j(r) - \mathbf{p}\|^2 \|\mathbf{p}_{j+1}(r) - \mathbf{p}\|^2 - \langle \mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle^2}},$$
$$\tag{9}$$
$$d_j^2(r) = \|\mathbf{p}_j(r) - \mathbf{p}\|^2 + \|\mathbf{p}_{j+1}(r) - \mathbf{p}\|^2 - 2\langle \mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle. \quad (10)$$

Let $\mathbf{F}_{\mathbf{d}_j}^k$ denote the kth order directional derivative of $\mathbf{F}$ in the direction $\mathbf{d}_j$. Then using Taylor expansion with respect to r, we have

$$\|\mathbf{p}_j(r) - \mathbf{p}_j\|^2 = s_j^2 r^2 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_j}\rangle + s_j^3 r^3 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_j}^2\rangle + \frac{1}{4}s_j^4 r^4 \langle \mathbf{F}_{d_j}^2, \mathbf{F}_{d_j}^2\rangle$$
$$+\frac{1}{3}s_j^4 r^4 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_j}^3\rangle + \frac{1}{6}s_j^5 r^5 \langle \mathbf{F}_{d_j}^2, \mathbf{F}_{d_j}^3\rangle + \frac{1}{12}s_j^5 r^5 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_j}^4\rangle + O(r^6), \quad (11)$$

and

$$\langle \mathbf{p}_j(r) - \mathbf{p}, \mathbf{p}_{j+1}(r) - \mathbf{p}\rangle$$
$$= \ s_j s_{j+1} r^2 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_{j+1}}\rangle + \frac{1}{2}s_j s_{j+1}^2 r^3 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_{j+1}}^2\rangle + \frac{1}{2}s_j^2 s_{j+1} r^3 \langle \mathbf{F}_{d_{j+1}}, \mathbf{F}_{d_j}^2\rangle$$
$$+ \ \frac{1}{4}s_j^2 s_{j+}^2 r^4 \langle \mathbf{F}_{d_{j+1}}^2, \mathbf{F}_{d_j}^2\rangle + \frac{1}{6}s_j s_{j+}^3 r^4 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_{j+1}}^3\rangle + \frac{1}{6}s_j^3 s_{j+1} r^4 \langle \mathbf{F}_{d_{j+1}}, \mathbf{F}_{d_{j+1}}\rangle$$
$$+ \ \frac{1}{12}s_j^2 s_{j+1}^3 r^5 \langle \mathbf{F}_{d_j}^2, \mathbf{F}_{d_{j+1}}^3\rangle + \frac{1}{12}s_{j+1}^2 s_j^3 r^5 \langle \mathbf{F}_{d_j}^2, \mathbf{F}_{d_{j+1}}^3\rangle$$
$$+ \ \frac{1}{24}s_{j+1}^4 s_j r^5 \langle \mathbf{F}_{d_j}, \mathbf{F}_{d_{j+1}}^4\rangle + \frac{1}{24}s_{j+1} s_j^4 r^5 \langle \mathbf{F}_{d_j}^4, \mathbf{F}_{d_{j+1}}\rangle + O(r^6). \quad (12)$$

To compute all the inner products in the two equations above, we let

$$\mathbf{t}_i = \frac{\partial \mathbf{F}(\xi_1, \xi_2)}{\partial \xi_i}, \ \mathbf{t}_{ij} = \frac{\partial^2 \mathbf{F}(\xi_1, \xi_2)}{\partial \xi_i \xi_j}, \ \mathbf{t}_{ijk} = \frac{\partial^3 \mathbf{F}}{\partial \xi_i \partial \xi_j \partial \xi_k}, \ \mathbf{t}_{ijkl} = \frac{\partial^4 \mathbf{F}}{\partial \xi_i \partial \xi_j \partial \xi_k \xi_l}$$

for $i, j, k, l = 1, 2$ and

$$g_{ij} = \mathbf{t}_i^T \mathbf{t}_j, \ g_{ijk} = \mathbf{t}_i^T \mathbf{t}_{jk}, \ e_{ijkl} = \mathbf{t}_i^T \mathbf{t}_{jkl}, \ e_{ijklm} = \mathbf{t}_i^T \mathbf{t}_{jklm}, \ f_{ijklm} = \mathbf{t}_{ij}^T \mathbf{t}_{klm}.$$

Since $\mathbf{F}_{\mathbf{d}_j}^k$ can be written as the linear combinatorics of $\mathbf{t}_i, \mathbf{t}_{ij}, \mathbf{t}_{ijk}$ and $\mathbf{t}_{ijkl}$, all the inner products in (11) and (12) can be expressed as linear combinations of $g_{ij}, g_{ijk}, g_{ijkl}, e_{ijkl}, e_{ijklm}$ and f_{ijklm}.

Substituting (11) and (12) into (8), (9) and (10), and then substituting (8), (9) and (10) into the expression $\frac{1}{2} A(\mathbf{p}, r) - \frac{1}{8} \sum_i \cot(\gamma_i(r)) d_i^2(r)$, and using Maple to conduct all the symbolic calculation, we have

$$b_0 = a_0/3 = \sqrt{a^2 t^2 (g_{11} g_{22} - g_{12}^2)}, \ b_1 = 0.$$

The theorem is proved. □

Remark 4.2. The calculation of b_0, b_1 involves a huge number of terms. It is almost impossible to finish the derivation by hand. Maple completes all the computation in 26 seconds on a PC equipped with a 3.0GHZ Intel(R) CPU. The Maple code that conducts all derivation of the theorem is available in `http://lsec.cc.ac.cn/~xuzq/maple.html`. The interested readers are encouraged to perform the computation.

Remark 4.3. It should be pointed out that there is another discrete scheme

$$G^{(4)} := \frac{2\pi - \sum_i \gamma_i}{A_M(\mathbf{p})},$$

where $A_M(\mathbf{p})$ is the area of Voronoi region. Since $\sum_i \text{area}(\mathbf{p}_i \mathbf{p} \mathbf{p}_{i+1})$ could be approximated by $3A_M(p)$ under some conditions, for example the conditions of Theorem 4.1, $G^{(4)}$ is easily derived from $G^{(1)}$.

5   A New Discrete Scheme of the Gaussian Curvature and Its Convergence

In this section, we introduce a new discrete scheme for Gaussian curvature which converges over the umbilical points and regular vertices with valence greater than 4. This is the main result of the paper. We firstly discuss some

properties about the discrete mean curvature. Setting $\alpha_i = \angle \mathbf{p}_i \mathbf{p}_{i-1} \mathbf{p}$ and $\delta_i = \angle \mathbf{p}_i \mathbf{p}_{i+1} \mathbf{p}$, we let

$$H^{(1)} := 2 \left\| \frac{\sum_i (\cot \alpha_i + \cot \delta_i) \overrightarrow{\mathbf{p}\mathbf{p}_i}}{\sum_i (\cot \alpha_i + \cot \delta_i) \eta_i^2} \right\|, \tag{13}$$

which is a popular discrete scheme for the mean curvature at vertex $\mathbf{p}$ (c.f. [13]). Moreover, the real mean curvature and the real Gaussian curvature at $\mathbf{p}$ are denoted as H and G respectively. Then, we have

Lemma 5.1. *At the regular vertex $\mathbf{p}$, or the umbilical points, the discrete scheme $H^{(1)}$ converges linearly to the mean curvature H as $\eta = \eta_i \to 0$.*

Proof. Firstly, let us consider the convergence property at the regular vertex. Since $\mathbf{p}$ is a regular vertex, $\frac{\cot \alpha_i + \cot \delta_i}{\cot \alpha_j + \cot \delta_j} = 1 + O(\eta^2)$, for any different i and j. It follows from equation (5), we have

$$\left\| \sum_i (\cot \alpha_i + \cot \delta_i) \overrightarrow{\mathbf{p}\mathbf{p}_i} \right\| = \sum_i \frac{(\cot \alpha_i + \cot \delta_i) \eta_i^2 k_i}{2} + O(\eta^3).$$

Hence,

$$H^{(1)} = \sum_i \frac{(\cot \alpha_i + \cot \delta_i) \eta_i^2}{\sum_j (\cot \alpha_j + \cot \delta_j) \eta_j^2} \kappa_i + O(\eta) = \frac{1}{n} \sum_i \kappa_i + O(\eta) = H + O(\eta).$$

Secondly, we study the convergence properties at the umbilical points. Over the umbilical points, $k_i = k_j = H$ for any i and j. Hence,

$$\begin{aligned} H^{(1)} &:= 2 \left\| \frac{\sum_i (\cot \alpha_i + \cot \delta_i) \overrightarrow{\mathbf{p}\mathbf{p}_i}}{\sum_i (\cot \alpha_i + \cot \delta_i) \eta_i^2} \right\| \\ &= \frac{\sum_i (\cot \alpha_i + \cot \delta_i) \eta_i^2 k_i + O(\eta^3)}{\sum_i (\cot \alpha_i + \cot \delta_i) \eta_i^2} = H + O(\eta). \end{aligned}$$

Combining the two results above, the theorem holds. $\qquad\square$

Now, we turn to a new discrete scheme for Gaussian curvature. Let $\varphi_i := \sum_{j=1}^i \gamma_j$ and

$$G^{(5)} := \frac{2\pi - \sum_i \gamma_i - 2(S_p - A)(H^{(1)})^2}{2A - S_p},$$

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where

$$A \quad := \quad \sum_i \frac{1}{4\sin\gamma_i}\Big(\frac{\eta_i\eta_{i+1}}{2}(1 - \cos 2\varphi_i \cos 2\varphi_{i+1})$$

$$-\frac{\cos\gamma_i}{4}(\eta_i^2 \sin^2 \varphi_i + \eta_{i+1}^2 \sin^2 \varphi_{i+1})),$$

$$S_p := \sum_i \frac{1}{4\sin(\gamma_i)}\left[\eta_i\eta_{i+1} - \frac{\cos(\gamma_i)}{2}(\eta_i^2 + \eta_{i+1}^2)\right].$$

Then, we have

Theorem 5.2. *For the regular vertices with valence not less than 5, or the umbilical points, $G^{(5)}$ converges towards the Gaussian curvature G as $\eta_i \to 0$.*

Proof. We firstly consider the regular vertex case. We set $\theta(n) := \frac{2\pi}{n}$. Since $\mathbf{p}$ is a regular vertex, $\gamma_i = \theta(n) + O(\eta^2)$ for any i according to (4). After a brief calculation, we have $A = A' + O(\eta^4)$, $S_p = S_p' + O(\eta^4)$, where

$$A' \quad = \quad \frac{1}{16\sin(\theta(n))}[2n - n\cos(2\theta(n)) - n\cos(\theta(n))]\eta^2,$$

$$S_p' \quad = \quad \frac{n}{4\sin(\theta(n))}[1 - \cos(\theta(n))]\eta^2.$$

Hence, we have

$$(2\pi - \sum_i \gamma_i - 2(S_p - A)(H^{(1)})^2)/(2A - S_p)$$

$$= \quad (2\pi - \sum_i \gamma_i - 2(S_p' - A')(H^{(1)})^2)/(2A' - S_p') + O(\eta^2).$$

Note that $\frac{\eta_{max}}{\eta_{min}} = 1 + O(\eta)$. According to Theorem 3 in [1], we have

$$2\pi - \sum_i \gamma_i = A'G + B'(k_M^2 + k_m^2) + o(\eta^2),$$

where, $B' = \frac{1}{16\sin(\theta(n))}[n + \frac{n}{2}\cos(2\theta(n)) - \frac{3n}{2}\cos(\theta(n))]\eta^2$.
    Note that $S_p' = A' + 2B'$ and

$$A'G + B'(k_M^2 + k_m^2) \quad = \quad A'G + B'[(k_M + k_m)^2 - 2k_M k_m]$$

$$= \quad A'G + 4B'H^2 - 2B'G$$

$$= \quad (A' - 2B')G + 4B'H^2.$$

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Hence, $2\pi - \sum_i \gamma_i = (A'-2B')G+4B'H^2+o(\eta^2)$. Note that $A' = O(\eta^2), B' = O(\eta^2)$ and $A' - 2B' \neq 0$ provided $n \neq 3$. According to Lemma 5.1, $H^{(1)}$ converges to the real mean curvature. Hence, we have, when $n \geq 5$,

$$
\begin{aligned}
G &= \frac{2\pi - \sum_i \gamma_i - 4B'H^2}{A' - 2B'} + o(1) \\
&= \frac{2\pi - \sum_i \gamma_i - 2(S'_p - A')(H^{(1)})^2}{2A' - S'_P} + o(1) = G^{(5)} + o(1).
\end{aligned}
$$

Therefore, $G^{(5)}$ converges to the Gaussian curvature.

Now, let us consider the umbilical point case. For umbilical points, each directional is the principal direction. According to Lemma 4 in [1], we have

$$
2\pi - \sum_i \gamma_i = (AG + (S_p - A)k_m^2) + o(\eta^2)
$$

over the umbilical points. Since $k_m^2 = H^2 = G$, we have

$$
\begin{aligned}
2\pi - \sum_i \gamma_i &= (AG + (S_p - A)k_m^2) + o(\eta^2) \\
&= (AG + 2(S_p - A)H^2 - (S_p - A)G) + o(\eta^2).
\end{aligned}
$$

Hence,

$$
G = \frac{2\pi - \sum_i \gamma_i - 2(S_p - A)(H^{(1)})^2}{2A - S_p} + o(1) = G^{(5)} + o(1).
$$

The theorem holds. $\qquad\square$

Remark 5.3. Theorem 5.2 shows that the new scheme $G^{(5)}$ converges over the regular vertex with valence greater than 4. As shown before, the previous schemes only converge over the regular vertex with valence 6, and hence the new scheme has better convergence properties over the available scheme.

Remark 5.4. In [8], the authors also prove that the discrete scheme $H^{(1)}$ converges to the real mean curvature at the regular vertex. However, the definition of the regular vertex in [8] is different with our definition.

Remark 5.5. According to the conclusions above, the Gaussian curvature and mean curvature can be approximated over the regular vertex with valence greater than 4. Hence, using the formulation $k_m = H - \sqrt{H^2 - G}, k_M = H + \sqrt{H^2 - G}$, one can approximate the principal curvatures over the regular vertex with valence greater than 4.

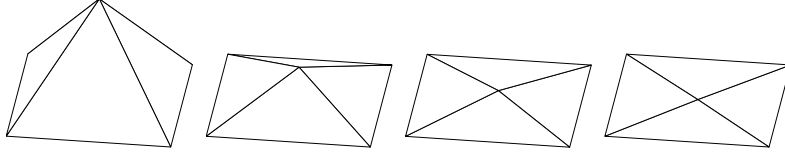# 6 A Counterexample for the Regular Vertex with Valence 4

In [21], we have constructed a triangular mesh and shown that it is impossible to construct a discrete Gaussian curvature scheme which converges for that mesh. But the vertex in the mesh is not regular. In this section, we shall show that it is also impossible to build a discrete Gaussian curvature scheme which converges over the regular vertex with valence 4.

Suppose the xy plane is triangulated around $(0,0)$ by choosing 4 points $\mathbf{q}_1 = (r_1, 0), \mathbf{q}_2 = (0, r_1), \mathbf{q}_3 = (-r_1, 0)$ and $\mathbf{q}_4 = (0, -r_1)$. For a bivariate function $f(x, y)$, the graph of $f(x, y)$, i.e. $\mathbf{F}(x, y) = [x, y, f(x, y)]^T$, can be regarded as a parametric surface. Let $\mathbf{p}_0 = \mathbf{F}(0, 0)$ and $\mathbf{p}_i = \mathbf{F}(\mathbf{q}_i), i = 1, 2, 3, 4$. The set of triangles $\mathbf{p}_i \mathbf{p}_0 \mathbf{p}_{i+1}$ forms a triangular mesh approximation of $\mathbf{F}$ at $\mathbf{p}_0$. The triangular mesh is denoted as M_f. When $f(x, y)$ is in the form of $x^2 + cxy + y^2$ where $c \in \mathbb{R}$, it is easy to prove that $\mathbf{p}_0 := (0, 0, 0)^T$ is a regular vertex with valence 4. Moreover, $\mathbf{p}_1 = (r_1, 0, r_1^2)^T, \mathbf{p}_2 = (0, r_1, r_1^2)^T, \mathbf{p}_3 = (-r_1, 0, r_1^2)^T, \mathbf{p}_4 = (0, -r_1, r_1^2)^T$. Now we show that it is impossible to construct a discrete scheme for Gaussian curvature which converges over the vertex $\mathbf{p}_0$ (See Fig. 2).

We assume that the discrete scheme for Gaussian curvature involving one-ring neighbor vertices of $\mathbf{p}_0$, which is denoted as $G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$, is convergent for the regular vertex with valence 4 over triangular mesh surface M_f, where $f(x, y)$ is in the form of $x^2 + cxy + y^2$.

It is easy to calculate that the Gaussian curvature of $\mathbf{F}(x, y, z)$ at p_0 is $4 - c^2$. By the convergence property of $G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ we have $\lim_{r_1 \to 0} G(M_f, \mathbf{p}_0; \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 4 - c^2$. Note that the triangular mesh M_f is independent of c, i.e. for any function $f(x, y)$ which is in the form of $x^2 + cxy + y^2$, the triangular mesh M_f is the same. Hence, $\lim_{r_1 \to 0} G(M_f, p_0; p_1, p_2, p_3, p_4)$ is independent of c. A contradiction occurs.

Hence, the assumption of $G(M_f, p_0; p_1, p_2, p_3, p_4)$ being convergent for the triangular mesh M_f does not hold.

Fig.2. A sequence of regular vertex with valence $n = 4$ for the function $f(x, y) = x^2 + xy + y^2$. At the regular vertex, it is impossible to construct a discrete Gaussian curvature scheme which converges to the correct value.

Remark 6.1. The counterexample in this section justifies the conclusion in [1], which says that 4 is the only value of valence such that $2\pi - \sum_i \gamma_i$ depends upon the principal directions.

Remark 6.2. An open problem is to find a discrete scheme for Gaussian curvature which converges at the regular vertex with valence 3.

7 Numerical Experiments

The aim of this section is to exhibit the numerical behaviors of the discrete schemes mentioned above. For a real vector $\mathbf{a} = (a_{20}, a_{11}, a_{02})$, we define a bivariate function $f_{\mathbf{a}}(x, y) := a_{20}x^2 + a_{11}xy + a_{02}y^2$, and regard the graph of the function $f_{\mathbf{a}}(x, y)$ as a parametric surface

$$\mathbf{F_a}(x, y) = [x, y, f_{\mathbf{a}}(x, y)]^T \in \mathbb{R}^3.$$

The Gaussian curvature of $\mathbf{F_a}(x, y)$ at the origin is $4a_{20}a_{02} - a_{11}^2$. The domain around $(0, 0)$ is triangulated locally by choosing n points:

$$\mathbf{q}_k = l_k(\cos\theta_k, \sin\theta_k), \quad \theta_k = 2(k-1)\pi/n, \quad k = 1, \cdots, n.$$

Let $\mathbf{p}_k = \mathbf{F_a}(\mathbf{q}_k)$ and $\mathbf{p}_0 = (0, 0, 0)^T$. Hence, the set of triangles $\{\mathbf{p}_k\mathbf{p}_0\mathbf{p}_{k+1}\}$ forms a piecewise linear approximation of $\mathbf{F_a}$ around $\mathbf{p}_0$. We set $e_k := f_a(\cos\theta_k, \sin\theta_k)$ and select

$$l_k = \sqrt{\frac{\sqrt{1 + 4e_k^2(l_{k-1}^2 + l_{k-1}^4 e_{k-1}^2)} - 1}{2e_k^2}}, \quad k \geq 2 \tag{14}$$

Table 1: The asymptotic maximal error $\varepsilon^{(i)}(n)$.

n	$\varepsilon^{(1)}(n)$	$\varepsilon^{(2)}(n)$	$\varepsilon^{(4)}(n)$	$\varepsilon^{(5)}(n)$
4	$4.6016e+01$	$3.3571e+01$	$3.3570e+01$	$3.3593e+01$
5	$8.2000e+00$	$9.3792e+00$	$9.3792e+00$	$4.1631e+01\eta$
6	$1.2226e+01\eta$	$1.2903e+01\eta$	$1.2903e+01\eta$	$1.1488e+01\eta$
7	$3.8464e+00$	$4.5783e+00$	$4.5783e+00$	$9.0676e-01\eta$
8	$5.8387e+00$	$7.7628e+00$	$7.7628e+00$	$6.5630e+01\eta$

so that $\mathbf{p}_0$ is a regular vertex.

We let $G^{(i)}(\mathbf{p}_0 : \mathbf{F_a})$ denote the approximated Gaussian curvatures of $\mathbf{F_a}$ at $\mathbf{p}_0$, which is obtained by using the discrete scheme $G^{(i)}$. Suppose $\mathcal{A}$ is a set consisting of M randomly chosen vectors $\mathbf{a}$. Then, we let

$$\varepsilon^{(i)}(n) = \sum_{\mathbf{a} \in \mathcal{A}} |G^{(i)}(\mathbf{p}_0 : \mathbf{F_a}) - (4a_{20}a_{02} - a_{11}^2)|/M.$$

In fact, $\varepsilon^{(i)}(n)$ measures the error of the discrete scheme $G^{(i)}$ at the regular vertex with valence n. The convergence property and the convergence rate are checked by taking $l_1 = 1/8, 1/16, 1/32, \cdots$. (when $k \geq 2$, l_k can be obtained by (14).) Since $\mathbf{p}$ is regular, each edge has the same length η. Table 1 shows the asymptotic maximal error $\varepsilon^{(i)}(n)$ for $M = 10^4$. Here, the vertex valences n are taken to be $4, 5, \cdots, 8$.

From table 1, we can see that all methods work well on valence 6 but only new method works well for valence ≥ 5.

We compute the Gaussian curvature over a randomly triangulated unit sphere by the discrete schemes to test their convergent property at the umbilical points. The vertexes of the random triangulation are uniform distribution on the sphere. Fig. 3 shows the random triangulation for the unit sphere. Denote the vertices in the random triangulation as $\mathbf{p}_i, i = 1, \cdots, N$ where N is the number of the vertices in the random triangulation. We let $G^{(j)}(\mathbf{p}_i)$ denote the approximate Gaussian curvature at the vertex $\mathbf{p}_i$ which is calculated by $G^{(j)}$. Similarly to the above, we use $\varepsilon^{(j)} = \sum_{i=1}^{N} |G^{(j)}(\mathbf{p}_i) - 1)|/N$ to measure the error of discrete scheme $G^{(j)}$ and use η to denote the average length of the edges. Table 2 lists $\varepsilon^{(j)}$ for different N. Moreover, we also use $\varepsilon^{(H)}$ to denote the error of the discrete scheme $H^{(1)}$ for the mean curvature.

Table 2: The asymptotic error $\varepsilon^{(i)}$ over a sphere with very irregular connectivity.

N	η	$\varepsilon^{(1)}$	$\varepsilon^{(2)}$	$\varepsilon^{(4)}$	$\varepsilon^{(5)}$	$\varepsilon^{(H)}$
30	0.710	$3.798e-01$	$1.905e-01$	$1.905e-01$	$2.126e-01$	2.840e-02
100	0.383	$3.517e-01$	$5.480e-02$	$5.480e-02$	$1.192e-01$	1.301e-02
400	0.196	$2.673e-01$	$1.280e-02$	$1.280e-02$	$1.730e-02$	2.600e-03
1300	0.109	$2.812e-01$	$3.801e-03$	$3.801e-03$	$6.500e-03$	7.540e-04
5000	0.056	$2.669e-01$	$9.648e-04$	$9.648e-04$	$2.703e-03$	1.893e-04
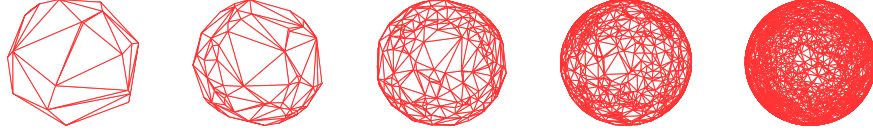
Fig. 3. Our test random triangulations. From left to right, the number of vertices is 30, 100, 400, 1300, 5000 respectively.

From these numerical results, we can draw the following conclusions:

1. For the regular vertices with the valence greater than 4, or the umbilical points, the discrete scheme $G^{(5)}$ converges to the real Gaussian curvature. This agrees with the theoretical result.

2. At the regular vertices and the umbilical points, the difference between $G^{(2)}$ and $G^{(4)}$ is very small.

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