KALMAN FILTER CONTROL IN THE REINFORCEMENT LEARNING FRAMEWORK

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ABSTRACT. There is a growing interest in using Kalman-filter models in brain modelling. In turn, it is of considerable importance to make Kalman-filters amenable for reinforcement learning. In the usual formulation of optimal control it is computed off-line by solving a backward recursion. In this technical note we show that slight modification of the linear-quadratic-Gaussian Kalman-filter model allows the on-line estimation of optimal control and makes the bridge to reinforcement learning. Moreover, the learning rule for value estimation assumes a Hebbian form weighted by the error of the value estimation.

1. MOTIVATION

Kalman filters and their various extensions are well studied and widely applied tools in both state estimation and control. Recently, there is an increasing interest in Kalman-filters or Kalman-filter like structures as models for neurobiological substrates. It has been suggested that Kalman-filtering (i) may occur at sensory processing [6, 7], (ii) may be the underlying computation of the hippocampus, and may be the underlying principle in control architectures [8, 9]. Detailed architectural similarities between Kalman-filter and the entorhinal-hippocampal loop as well as between Kalman-filters and the neocortical hierarchy have been described recently [2, 3]. Interplay between the dynamics of Kalman-filter-like architectures and learning of parameters of neuronal networks has promising aspects for explaining known and puzzling phenomena, such as priming, repetition suppression and categorization [4, 1].

As it is well known, Kalman-filter provides an on-line estimation of the state of the system. On the other hand, optimal control cannot be computed on-line, because it is typically given by a backward recursion (the Ricatti-equations). For on-line parameter estimations without control aspects, see [5].

The aim of this paper is to derive an on-line control method for the Kalman-filter and achieve optimal performance asymptotically. Slight modification of the linear-quadratic-Gaussian (LQG) Kalman-filter model is introduced for treating the LQG model as a reinforcement learning (RL) problem.

2. THE KALMAN FILTER AND THE LQG MODEL

Consider a linear dynamical system with state $\mathbf{x}_t \in \mathbb{R}^n$, control $\mathbf{u}_t \in \mathbb{R}^m$, observation $\mathbf{y}_t \in \mathbb{R}^k$, noises $\mathbf{w}_t \in \mathbb{R}^n$ and $\mathbf{e}_t \in \mathbb{R}^k$ (which are assumed to be Gaussian

and white, with covariance matrix Ω^w and Ω^e , respectively), in discrete time t:

$$\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t + \mathbf{w}_t$$

$$\mathbf{y}_t = H\mathbf{x}_t + \mathbf{e}_t,$$

the initial state has mean $\hat{\mathbf{x}}_1$ and covariance Σ_1 . Executing the control step \mathbf{u}_t in \mathbf{x}_t costs

(3)
$$c(\mathbf{x}_t, \mathbf{u}_t) := \mathbf{x}_t^T Q \mathbf{x}_t + \mathbf{u}_t^T R \mathbf{u}_t,$$

and after the Nth step the controller halts and receives a final cost of $\mathbf{x}_N^T Q_N \mathbf{x}_N$. This problem has the well known solution

$$(4) \qquad \hat{\mathbf{x}}_{t+1} = F\hat{\mathbf{x}}_t + G\mathbf{u}_t + K_t(\mathbf{y}_t - H\hat{\mathbf{x}}_t)$$

$$(5) K_t = F\Sigma_t H^T (H\Sigma_t H^T + \Omega^e)^{-1}$$

(6)
$$\Sigma_{t+1} = \Omega^w + F \Sigma_t F^T - K_t H \Sigma_t A^T$$
 (state estimation)

and

$$(7) \quad \mathbf{u}_t = -L_t \hat{\mathbf{x}}_t$$

(8)
$$L_t = (G^T S_{t+1} G + R)^{-1} G^T S_{t+1} F$$

$$(9) S_t = Q_t + F^T S_{t+1} F - F^T S_{t+1} G L_t. (optimal control)$$

Unfortunately, the optimal control equations are not on-line, because they can be solved only by stepping backward from the final, Nth step.

3. Kalman Filtering in the Reinforcement Learning Framework

First of all, we slightly modify the problem: the run time of the controller will not be a fixed number N. Instead, after each time step, the process will be stopped with some fixed probability p (and then the controller incurs the final cost $c_f(\mathbf{x}_f) := \mathbf{x}_f^t Q_f \mathbf{x}_f$).

3.1. The cost-to-go function. Let $V_t^*(\mathbf{x})$ be the optimal cost-to-go function at time step t, i.e.

(10)
$$V_t^*(\mathbf{x}) := \inf_{\mathbf{u}_t, \mathbf{u}_{t+1}, \dots} E\left[c(\mathbf{x}_t, \mathbf{u}_t) + c(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) + \dots + c_f(\mathbf{x}_f) \middle| \mathbf{x}_t = \mathbf{x}\right].$$

Clearly, for any \mathbf{x} ,

$$(11) V_t^*(\mathbf{x}) = p \cdot c_f(\mathbf{x}) + (1-p) \cdot \inf_{\mathbf{u}} \left(c(\mathbf{x}, \mathbf{u}) + E_w \left[V_{t+1}^* (F\mathbf{x} + G\mathbf{u} + w) \right] \right)$$

It can be easily shown that the optimal cost-to-go function is time-independent, furthermore, it is a quadratic function of \mathbf{x} , that is, it is of the form

$$(12) V^*(\mathbf{x}) = \mathbf{x}^T \Pi^* \mathbf{x}.$$

Our task is to estimate V^* (in fact, the parameter matrix Π^*) on-line. This will be done by value iteration.

3.2. Value iteration, greedy action selection and the temporal differencing error. Value iteration starts with an arbitrary initial cost-to-go function $V_0(\mathbf{x}) = \mathbf{x}^T \Pi_0 \mathbf{x}$. After this, control actions are selected according to the current value function estimate, the value function is updated according to the experience, and these two steps are iterated.

The tth estimate of V^* is $V_t(\mathbf{x}) = \mathbf{x}^T \Pi_t \mathbf{x}$. The greedy control action according to this is given by

(13)
$$\mathbf{u}_t = \arg\min_{\mathbf{u}} \left(c(\mathbf{x}_t, \mathbf{u}) + E \left[V_t(F\mathbf{x}_t + G\mathbf{u} + w) \right] \right)$$

(14)
$$= \arg\min_{\mathbf{u}} \left(\mathbf{u}^T R \mathbf{u} + (F \mathbf{x}_t + G \mathbf{u})^T \Pi_t (F \mathbf{x}_t + G \mathbf{u}) \right)$$

$$= -(R + G^T \Pi_t G)^{-1} (G^T \Pi_t F) \mathbf{x}_t.$$

For the sake of simplicity, the cost-to-go function will be updated by using the 1-step temporal differencing (TD) method. Naturally, it can be substituted with more sophisticated methods like multi-step TD or eligibility traces. The TD error is

$$\delta_t = \begin{cases} V_t(\mathbf{x}_t) - c_f(\mathbf{x}_t) & \text{if the controller was stopped at the } t\text{th time step,} \\ \left(c(\mathbf{x}_t, \mathbf{u}_t) + V_t(\mathbf{x}_{t+1})\right) - V_t(\mathbf{x}_t), & \text{otherwise.} \end{cases}$$

and the update rule for the parameter matrix Π_t is

(17)
$$\Pi_{t+1} = \Pi_t + \alpha_t \cdot \delta_t \cdot \nabla_{\Pi_t} V_t(\mathbf{x}_t)$$

$$= \Pi_t + \alpha_t \cdot \delta_t \cdot \mathbf{x}_t \mathbf{x}_t^T,$$

where α_t is the learning rate. Note that value-estimation error weighted Hebbian learning rule has emerged.

4. Concluding remarks

The Kalman-filter control problem was slightly modified to fit the RL framework and an on-line control rule was achieved. The well-founded theory of reinforcement learning ensures asymptotic optimality for the algorithm. The described method is highly extensible. There are straightforward generalizations to other cases, e.g., to extended Kalman filters, dynamics with unknown parameters, non-quadratic cost functions, or more advanced RL algorithms, e.g. eligibility traces. For quadratic loss functions, we have found that learning is Hebbian and it is weighted by the error of value-estimation.

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