Properties of the Discrete Pulse Transform for Multi-Dimensional Arrays

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ISBN: 978-1-86854-785-2

Technical Report 2010/01

1 Introduction

This report presents properties of the Discrete Pulse Transform on multi-dimensional arrays introduced earlier in [1]. The main result given here in Lemma 2.1 is also formulated in [4, Lemma 21]. However, the proof, being too technical, was omitted there and hence it appears in full in this publication.

2 The Lemma

The lemma which follows deals with two technical aspects of the Discrete Pulse Transform of a function $f \in \mathcal{A}(\mathbb{Z}^d)$ (where $\mathcal{A}(\mathbb{Z}^d)$ denotes a vector lattice). The first is that the Discrete Pulse representation of a function f, given by

$$f = \sum_{n=1}^{N} D_n(f),$$

can be written as the sum of individual pulses of each resolution layer $D_n(f)$. The second result in the lemma below indicates a form of linearity for the nonlinear LULU operators.

Lemma 2.1
Let $f \in \mathcal{A}(\mathbb{Z}^d)$, $\mathrm{supp}(f) < \infty$, be such that f does not have local minimum sets or local maximum sets of size smaller than n, for some $n \in \mathbb{N}$. Then we have the following two results.

a)

$$(id - P_n)f = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} + \sum_{j=1}^{\gamma^+(n)} \varphi_{nj}, \tag{1}$$

where $V_{ni} = \mathrm{supp}(\phi_{ni}), i = 1, 2, ..., \gamma^-(n)$, are local minimum sets of f of size n, $W_{nj} = \mathrm{supp}(\varphi_{nj}), j = 1, 2, ..., \gamma^+(n)$, are local maximum sets of f of size n, ϕ_{ni} and φ_{nj} are negative and positive discrete pulses respectively, and we also have that

- $V_{ni} \cap V_{nj} = \emptyset$ and $\mathrm{adj}(V_{ni}) \cap V_{nj} = \emptyset,\ i, j = 1, ..., \gamma^-(n),\ i \neq j,$ (2)
- $W_{ni} \cap W_{nj} = \emptyset$ and $\mathrm{adj}(W_{ni}) \cap W_{nj} = \emptyset,\ i, j = 1, ..., \gamma^+(n), i \neq j,$ (3)
- $V_{ni} \cap W_{nj} = \emptyset\ i = 1, ..., \gamma^-(n)\ , j = 1, ..., \gamma^+(n).$ (4)

b) For every fully trend preserving operator A

$$U_n(id - AU_n) = U_n - AU_n,$$
$$L_n(id - AL_n) = L_n - AL_n.$$

Proof.

a) Let $V_{n1}, V_{n2}, ..., V_{n\gamma^-(n)}$ be all local minimum sets of size n of the function f. Since f does not have local minimum sets of size smaller than n, then f is a constant on each of these sets, by [4, Theorem 14]. Hence, the sets are disjoint, that is $V_{ni} \cap V_{nj} = \emptyset$, $i \neq j$. Moreover, we also have

$$\text{adj}(V_{ni}) \cap V_{nj} = \emptyset,\ i,j = 1, ..., \gamma^-(n). \tag{5}$$

Indeed, let $x \in \text{adj}(V_{ni}) \cap V_{nj}$. Then there exists $y \in V_{ni}$ such that $(x, y) \in r$. Hence $y \in V_{ni} \cap \text{adj}(V_{nj})$. From the local minimality of the sets V_{ni} and V_{nj} we obtain respectively $f(y) < f(x)$ and $f(x) < f(y)$, which is clearly a contradiction. For every $i = 1, ..., \gamma^-(n)$ denote by y_{ni} the point in $\text{adj}(V_{ni})$ such that

$$f(y_{ni}) = \min_{y \in \text{adj}(V_{ni})} f(y). \tag{6}$$

Then we have

$$U_n f(x) = \begin{cases} f(y_{ni}) & \text{if } x \in V_{ni},\ i = 1, ..., \gamma^-(n) \\ \\ f(x) & \text{otherwise (by [4, Theorem 9])} \end{cases}$$

Therefore

$$(id - U_n)f = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} \tag{7}$$

where ϕ_{ni} is a discrete pulse with support V_{ni} and negative value (down pulse).

Let $W_{n1}, W_{n2}, ..., W_{n\gamma^+(n)}$ be all local maximum sets of size n of the function $U_n f$. By [4, Theorem 12(b)] every local maximum set of $U_n f$ contains a local maximum set of f. Since f does not have local maximum sets of size smaller than n, this means that the sets W_{nj}, $j = 1, ..., \gamma^+(n)$, are all local maximum sets of f and f is constant on each of them. Similarly to the local minimum sets of f considered above we have $W_{ni} \cap W_{nj} = \emptyset$, $i \neq j$, and $\text{adj}(W_{ni}) \cap W_{nj} = \emptyset$, $i, j = 1, ..., \gamma^+(n)$. Moreover, since $U_n(f)$ is constant on any of the sets $V_{ni} \cup \{y_{ni}\}$, $i = 1, ..., \gamma^-(n)$, see [4, Theorem 14], we also have

$$(V_{ni} \cup \{y_{ni}\}) \cap W_{nj} = \emptyset,\ i = 1, ..., \gamma^-(n),\ j = 1, ..., \gamma^+(n), \tag{8}$$

which implies (4).

Further we have

$$L_n U_n f(x) = \begin{cases} U_n f(z_{nj}) & \text{if } x \in W_{nj},\ j = 1, ..., \gamma^+(n) \\ \\ U_n f(x) & \text{otherwise} \end{cases}$$

where $z_{nj} \in \text{adj}(W_{nj})$, $j = 1, ..., \gamma^+(n)$, are such that $U_n f(z_{nj}) = \max_{z \in \text{adj}(W_{nj})} U_n f(z)$. Hence

$$(id - L_n)U_n f = \sum_{j=1}^{\gamma^+(n)} \varphi_{nj} \tag{9}$$

where φ_{nj} is a discrete pulse with support W_{nj} and positive value (up pulse). Thus we have shown that

$$(id - P_n)f = (id - U_n)f + (id - L_n)U_n f = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} + \sum_{j=1}^{\gamma^+(n)} \varphi_{nj}.$$

b) Let the function $f \in \mathcal{A}(\mathbb{Z}^d)$ be such that it does not have any local minimum or local maximum sets of size less than n. Denote $g = (id - AU_n)(f)$. We have

$$g = (id - AU_n)(f) = (id - U_n)(f) + ((id - A)U_n)(f). \tag{10}$$

As in a) we have that (7) holds, that is we have

$$(id - U_n)(f) = \sum_{i=1}^{\gamma^-(n)} \phi_{ni}, \tag{11}$$

where the sets $V_{ni} = \mathrm{supp}(\phi_{ni})$, $i = 1, ..., \gamma^-(n)$, are all the local minimum sets of f of size n and satisfy (2). Therefore

$$g = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} + ((id - A)U_n)(f). \tag{12}$$

Furthermore,

$$U_n(f)(x) = \begin{cases} f(x) & \text{if} \quad x \in \mathbb{Z}^d \setminus \bigcup_{i=1}^{\gamma^-(n)} V_{ni} \\ & \\ v_i & \text{if} \quad x \in V_{ni} \cup \{y_{ni}\},\ i = 1, ..., \gamma^-(n), \end{cases}$$

where $v_i = f(y_{ni}) = \min_{y \in \mathrm{adj}(V_{ni})} f(y)$. Using that A is fully trend preserving, for every $i = 1, ..., \gamma^-(n)$ there exists w_i such that $((id - A)U_n)(f)(x) = w_i$, $x \in V_{ni} \cup \{y_{ni}\}$. Moreover, using that every adjacent point has a neighbor in V_{ni} we have that $\min_{y \in \mathrm{adj}(V_{ni})} ((id - A)U_n)(f)(y) = w_i$. Considering that the value of the pulse ϕ_{ni} is negative, we obtain through the representation (12) that V_{ni}, $i = 1, ..., \gamma^-(n)$, are local minimum sets of g.

Next we show that g does not have any other local minimum sets of size n or less. Indeed, assume that V_0 is a local minimum set of g such that $\mathrm{card}(V_0) \leq n$. Since $V_0 \cup \mathrm{adj}(V_0) \subset \mathbb{Z}^d \setminus \bigcup_{i=1}^{\gamma^-(n)} V_{ni}$ it follows from (12) that V_0 is a local minimum set of $((id - A)U_n)(f)$. Then using that $(id - A)$ is neighbor trend preserving and using [4, Theorem 17] we obtain that there exists a local minimum set W_0 of $U_n(f)$ such that $W_0 \subseteq V_0$. Then applying again [4, Theorem 17] or [4, Theorem 12] we obtain that there exists a local minimum set $\tilde{W}_0$ of f such that $\tilde{W}_0 \subseteq W_0 \subseteq V_0$. This inclusion implies that $\mathrm{card}(\tilde{W}_0) \leq n$. Given that f does not have local minimum sets of size less than n we have $\text{card}(\tilde{W}_0) = n$, that is $\tilde{W}_0$ is one of the sets V_{ni} - a contradiction. Therefore, V_{ni}, $i = 1, ..., \gamma^-(n)$, are all the local minimum sets of g of size n or less. Then using again (7) we have

$$(id - U_n)(g) = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} \tag{13}$$

Using (11) and (13) we obtain

$$(id - U_n)(g) = (id - U_n)(f)$$

Therefore

$$\begin{aligned}(U_n(id - AU_n))(f) &= U_n(g) \ = \ g - (id - U_n)(f) \\ &= (id - AU_n)(f) - (id - U_n)(f) \\ &= (U_n - AU_n)(f).\end{aligned}$$

This proves the first identity. The second one is proved in a similar manner. ■

References

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