

Where Fail-Safe Default Logics Fail

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Abstract

Reiter's original definition of default logic allows for the application of a default that contradicts a previously applied one. We call *failure* this condition. The possibility of generating failures has been in the past considered as a *semantical* problem, and variants have been proposed to solve it. We show that it is instead a *computational* feature that is needed to encode some domains into default logic.

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1 Introduction

Since the introduction of default logic [Rei80], semantical problems of the original definition have been identified, and variants have been proposed to solve them [Luk88, Bre91, Ryc91, DSJ94, GM94, MT95]. One of the problems with Reiter’s definition is that the application of a sequence of defaults may lead to failure. The following example shows this problem.

$$T = \left\langle \left\{ \frac{a : b}{c}, \frac{c : a}{\neg b} \right\}, \{a\} \right\rangle$$

The default $\frac{a:b}{c}$ entails c whenever a is true and b is consistent with our current knowledge. The application of this default makes c true, therefore making the second default $\frac{c:a}{\neg b}$ applicable, which makes b false. This result is however in contradiction with the assumption of consistency of b we made for applying the first default.

We call *failure* the condition in which the application of another default makes a default that has already been applied inapplicable. In other words, if the application of a default contradicts the assumption of a default that has already been applied, this is a failure. The possibility of failures has been in the past considered a drawback, especially because failures may make the evaluation of theories like T impossible (i.e., these theories have no extensions.)

A typical solution to this problem is to refrain from applying defaults that would lead to failure. This is done by justified default logic [Luk88], constrained default logic [DSJ94], and cumulative default logic [Bre91, GM94]. For the theory T above, the application of the first default makes c true, but the second default is not applied because it would lead to failure. We therefore conclude that T is equivalent to the propositional theory $\{a, c\}$.

This solution is *semantically* good, as it allows for the evaluation of theories like T (i.e., it assigns extensions to theories that would otherwise have none.) On the other hand, having forbidden failures can be seen as a *computational* problem, as the encoding of some domains requires exactly this “ability to fail.” An example is the translation from reasoning about actions to default logic proposed by Turner [Tur97]. This translation generates all possible evaluations of a variable x by means of a pair of defaults $\frac{x}{x}$ and $\frac{\neg x}{\neg x}$, and then removes the unwanted evaluations by generating failures. For example, the simple theory $\{Holds(Alive, S_0)\}$, telling that Fred is alive in the initial state, is translated into the following default theory:

$$\left\langle \left\{ \frac{: Holds(Alive, S_0)}{Holds(Alive, S_0)}, \frac{: \neg Holds(Alive, S_0)}{\neg Holds(Alive, S_0)}, \frac{\neg Holds(Alive, S_0) :}{false} \right\}, \emptyset \right\rangle$$

Either the first or the second default can be applied, but not both. Depending on which one we decide to apply, we obtain $Holds(Alive, S_0)$ or $\neg Holds(Alive, S_0)$. The third default generates a failure whenever $\neg Holds(Alive, S_0)$ is true. The only possible remaining case is therefore that in which $Holds(Alive, S_0)$ is true. In other words, the first two defaults generate all possible evaluations of $Holds(Alive, S_0)$ and the third default deletes the one we do not want. In general, some defaults generate all possible evaluations of the fluents and some other defaults delete the ones that are not possible. The latter are called *killing defaults* by Cholewinski, Marek, Mikitiuk, and Truszczyński [CMMT95], who presented other reductions where killing defaults are used to delete unwanted solutions. Since this deletion is realized by generating a failure, these translations do not work for semantics where failure is impossible, such as justified default logic [Luk88]. The inability to fail can be therefore seen as a limitation, as some translations from other formalisms into default logics require failures.

In this paper, we consider the problem of translating default semantics that can fail (fail-prone) into default semantics where failure is impossible (fail-safe). In order for the results to abstract over the specific semantics, we consider a sufficiently general definition of “semantics of default logic”, based on the concept of process [FM92, AS94, FM94, Ant99]. We formalize the concept of fail-safeness in this framework, and investigate the translability from fail-prone into fail-safe semantics.

The translations we consider are polynomial either in time or size of the result. We consider translations that preserve the skeptical consequences, or the extensions, or the processes of a default theory.

The least constrained form of translation is that translating the inference problem: given a theory $\langle D, W \rangle$ and a formula p we require the translation to produce another theory $\langle D', W' \rangle$ and another formula p' in such a way $\langle D, W \rangle \models p$ holds in one semantics if and only if $\langle D', W' \rangle \models p'$ holds in the other one. Such translations are possible in polynomial time between all semantics that have the same complexity. For example, we can translate Reiter’s default logic into justified default logic in this way as both semantics are Π_2^P -complete [Got92, Sti92, CS93].

We extend this result by simplifying the translation of p into p' : we indeed show a translation such that $\langle D, W \rangle \models p$ holds if and only if $\langle D', W' \rangle \models a \vee p$ holds, where a is a new variable. Intuitively, the new variable expresses the condition of failure in a semantics that cannot fail: by setting a to true whenever a failure should be necessary, the generated extension implies $a \vee p$, and is thus irrelevant to skeptical entailment.

Using this translation, the extensions of the theories $\langle D, W \rangle$ and $\langle D', W' \rangle$ are not the same. We therefore consider translations that preserve the extensions: such translations are called faithful [Kon88, Got95]. Clearly, no faithful translation is possible from a semantics that may not have extensions to one that always have. This is how Delgrande and Schaub [DS03], for example, have shown that Reiter's default logic cannot be translated into justified default logic. However, the question remains if this impossibility is only due to a possible lack of extensions. We therefore restrict to the case in which the original theory has at least one extension and prove that an extension-preserving translation exists or not depending on what we assume to be polynomial: the running time of the translation or the size of the result.

Finally, we consider translations preserving the processes of a default theory. Processes are the basic semantic notion of the operational semantics for default logic [FM92, AS94, FM94, Ant99]; they are sequences of defaults that can be applied in a default theory. Since two or more processes may correspond to the same extension, two default theories may have the same extensions but different processes. Therefore, translations that preserve the processes can be considered as the “most preserving” ones.

2 Definitions

2.1 Default Logics

We use the operational semantics for default logics. Two slightly different operational semantics for default logics have been given independently by Antoniou and Sperschneider [AS94, Ant99] and by Froidevaux and Mengin [FM92, FM94]. A default is a rule of the form:

$$d = \frac{\alpha : \beta}{\gamma}$$

The formulae α , β , and γ are called the precondition, the justification, and the consequence of d , and are denoted as $prec(d)$, $just(d)$, and $cons(d)$,

respectively. This notation is extended to sets and sequences of defaults in the obvious way. A default is applicable if its precondition is true and its justification is consistent; if this is the case, its consequence should be considered true.

A default theory is a pair $\langle D, W \rangle$ where D is a set of defaults and W is a consistent theory, called the background theory. We assume that all formulae are propositional, the alphabet and the set D are finite, and all defaults have a single justification. The assumption that W is consistent is not standard; however, all known semantics give the same evaluation when the background theory is inconsistent. We define a process to be a sequence of defaults that can be applied starting from the background theory.

Definition 1 *A process of a default theory $\langle D, W \rangle$ is a sequence of defaults Π such that $W \cup \text{cons}(\Pi)$ is consistent and $W \cup \text{cons}(\Pi[d]) \models \text{prec}(d)$ for every default $d \in \Pi$, where $\Pi[d]$ denotes the sequence of defaults preceeding d in Π .*

The definition of processes only takes into account the preconditions and the consequences of defaults. This is because the interpretation of the justifications depends on the semantics. All semantics select a set of processes that satisfy two conditions: successfulness and closure. Intuitively, successfulness means that the justifications of the applied defaults are not contradicted; closure means that no other default should be applied.

The particular definition of successfulness and closure depend on the specific semantics. The following are the definitions used by Reiter's and constrained default logic.

Successfulness:

Local: for each $d \in \Pi$, the set $W \cup \text{cons}(\Pi) \cup \text{just}(d)$ is consistent;

Global: $W \cup \text{cons}(\Pi) \cup \text{just}(\Pi)$ is consistent.

Closure:

Inapplicability: no default $d \notin \Pi$ is applicable to Π , i.e., either $W \cup \text{cons}(\Pi) \not\models \text{prec}(d)$ or $W \cup \text{cons}(\Pi) \cup \text{just}(d)$ is inconsistent;

Maximality: for any $d \notin \Pi$, the sequence $\Pi \cdot [d]$ is not a globally successful process.

We abstract the notions of successfulness and closure from the particular semantics, and define them to be two conditions on Π and $\langle D, W \rangle$ such that:

1. successfulness is antimonotonic: if $\Pi \cdot \Pi'$ is successful, then Π is successful;
2. $[\]$ is a successful process, if W is consistent;
3. successfulness and closure can be expressed as the combination of a number of consistency tests over the formulae in Π and $\langle D, W \rangle$;
4. these consistency tests are independent on the order of defaults in Π ;
5. the combination of the results of these consistency tests results can be done in polynomial time.

A default logic semantics defined in terms of two conditions of successfulness and closure that satisfy these assumptions is called *regular*. Most default logic semantics are regular: the only exception known to the author is the semantics of concise extensions, which is not regular because the condition of subsumption requires checking all possible default orderings [Ryc91].

We remark that the last condition does not imply that successfulness and closure can be checked in polynomial time: they can only be checked in polynomial time once the consistency/entailment tests have been done. This is typically the case: for example, Reiter's default logic closure condition amounts to check whether, for every $d \notin \Pi$, either $W \cup \text{cons}(\Pi) \not\models \text{prec}(d)$ or $W \cup \text{cons}(\Pi) \cup \text{just}(d)$ is inconsistent. Once we have checked the consistency of $W \cup \text{cons}(\Pi) \cup \neg \text{prec}(d)$ and $W \cup \text{cons}(\Pi) \cup \text{just}(d)$ for every $d \notin \Pi$, determining whether closure is satisfied can be done in linear time.

An *extension* of a default theory is the deductive closure of $W \cup \text{cons}(\Pi)$ where Π is a successful and closed process. Note that more than one process may generate the same extension. The *skeptical* consequences of a default theory are the formulae that are entailed by all its extensions. The *credulous* consequences are those implied by some of its extensions.

Fail-safeness of a semantics is formalized as follows.

Definition 2 (Fail-Safe Semantics) *A regular semantics for default logic is fail-safe if, for every default theory, any successful process is the prefix of a successful and closed process.*

This definition formalizes the idea that a sequence of defaults cannot generate a failure: if we can apply a sequence of defaults, then an extension will be eventually generated, possibly after applying some other defaults. In other words, the situation in which we apply some defaults but then find out that we do not generate an extension never occurs. Fail-safeness is a form of commitment to defaults: if we apply a default, we never end up with contradicting its assumption.

Fail-safeness can also be seen as a form of monotonicity of processes w.r.t. to sets of defaults: if a semantics is fail-safe, then adding some defaults to a theory may only extend the successful and closed process of the theory and create new ones. However, this form of monotonicity is not the same as that typically used in the literature, which is defined in terms of consequences, not processes. Froidevaux and Mengin [FM94, Theorem 29] have proved a result that essentially states that every semantics in which closure is defined as maximal successfulness is fail-safe. As a result, justified and constrained default logics are fail-safe.

We recall that we assume that the background theory W is consistent. In this case, if a semantics is fail-safe, then every default theory has a successful and closed process: since the process $[]$ is successful, a process Π that is successful and closed exists. The condition of antimonotonicity provides an algorithm for finding this successful and closed process: if Π is successful and closed, all its initial fragments are successful as well. We can therefore obtain a successful and closed process by iteratively adding to $[]$ a default that lead to a successful process.

While all fail-safe semantics give extensions to theories, the converse is only true in some cases. According to Reiter's and rational default logics the simple default theory $\langle \{\frac{a}{\neg a}\}, \emptyset \rangle$ has no extension, and these two semantics are therefore not fail-safe. On the other hand, every default theory has at least one concise extension [Ryc91], while the semantics of concise extensions is not fail-safe, as shown by the following example.

$$\begin{aligned}
T &= \langle \{d_1, d_2\}, \emptyset \rangle \\
&\text{where} \\
d_1 &= \frac{:x}{x} \\
d_2 &= \frac{:x \wedge y}{x \wedge y}
\end{aligned}$$

The default d_1 is applicable to $W = \emptyset$; the default d_2 is applicable to $[d_1]$ because d_2 is not subsumed by d_1 . On the other hand, the process $[d_1, d_2]$ is not concise, as d_1 is subsumed by d_2 . As a result, $[d_1]$ is a successful process, but is not the initial part of any successful and closed process under the semantics of concise extensions.

2.2 Translations

In this paper, we investigate the extent to which fail-safe semantics are less expressive than fail-prone ones. In particular, we study whether one of the two facts below can be proved:

1. every theory under an arbitrary regular default semantics can be translated into a normal default theory under Reiter’s semantics;
2. there are theories under Reiter’s semantics that cannot be translated into whichever fail-safe semantics.

Since Reiter’s default logic is fail-safe when restricting to normal defaults, the existence of a translation of the first kind implies that every regular default semantics can be translated into at least one fail-safe semantics. However, the restriction to normal defaults makes such a result more general. Since most of the semantics for default logic behave like Reiter’s on normal defaults (an exception is Rychlik’s concise semantics [Ryc91], which is however not fail-safe), this result extends to all these semantics. In order to simplify the terminology, we formally define “normal default logic” as a semantics for default logic.

Definition 3 *Normal default logic is the restriction of Reiter’s default logic to the case of normal defaults.*

Ideally, we would like results of the first kind to be exactly the converse of the second one, i.e., every regular semantics can be translated into every fail-safe semantics. This general question has however an easy (and of little significance) negative answer: the semantics that has $[\]$ as the only successful and closed process is fail-safe, but none of the considered semantics can be translated into it. Indeed, consequence-preserving translations are impossible in polynomial time because entailment is coNP-complete in this semantics but Π_2^P -complete in most semantics; faithful translations are

impossible because this semantics always gives a single extension to a theory while the other one may give more.

The results about the existence of translations depend on what we require from the translations. A minimal requirement is that the consequences are preserved. At the other extreme, we may require a translation to preserve the set of processes. In between, and this is perhaps the most interesting case, we have the preservation of the extensions.

Consequence-Preserving: we want to obtain the same consequences of the original theory;

Extension-Preserving (Faithful): we want a bijective correspondence between the extensions;

Process-Preserving: bijective correspondence between the processes.

In all three cases, we assume that new variables can be introduced, as it is common in translations between logics. Technically, this is possible thanks to the concept of var-equivalence [LLM03].

Definition 4 *Two formulae α and β are var-equivalent w.r.t. variables X if and only if $\alpha \models \gamma$ iff $\beta \models \gamma$ for every formula γ that only contains variables in X .*

In plain terms, two formulae are var-equivalent if and only if their consequences, if restricted to be formulae on a given alphabet, are the same.

The translations we consider may introduce new variables: a theory $\langle D, W \rangle$ that only contains variables X is translated into a default theory $\langle D', W' \rangle$ that contains variables $X \cup Y$. Preservation is assumed to hold modulo var-equivalence: preserving the extensions means that each extension of $\langle D, W \rangle$ is var-equivalent to an extension of $\langle D', W' \rangle$ w.r.t. X , and vice versa; preserving the consequences means that $\langle D, W \rangle \models \gamma$ iff $\langle D', W' \rangle \models \gamma$ for each formula γ that only contains variables in X . Faithful translations based on var-equivalence of extensions have been considered by Delgrande and Schaub [DS03] and by Janhunen [Jan98, Jan03].

The condition of process preservation requires a suitable correspondence of sequences of defaults. We assume that the defaults are numbered, both in the original and in the generated theory. We could then enforce the consequences being the same or being var-equivalent; this is however not necessary

as polynomial process-preserving translations will be proved not to exist anyway.

A condition on translations that has been considered by several authors is that of modularity [Imi87, Got95, Jan03], which requires only the defaults to be translated. Formally, a default theory $\langle D, W \rangle$ should be translated into a theory $\langle D', W' \cup W \rangle$, where $\langle D', W' \rangle$ is the result of translating $\langle D, \emptyset \rangle$. Another condition on translations that has been considered in the past is that of locality [Kon88]: each sentence and each default is translated separately. We do not consider modularity and locality in this paper.

A requirement we impose on the translations is that of being polynomial. There are two possible definitions of polynomiality, depending on what is required to be polynomial: the running time or the produced output. This difference is important, as some translations require exponential time but still output a polynomially large theory. Formally, two kinds of translations are considered.

polynomial: run in polynomial time;

polysize: produce a polynomially large result.

The existence of translations depends both on what we want to preserve (consequences, extensions, or processes), and also on which computational condition we set (polynomial time or size.) The results proved in this paper are summarized in Table 1.

	Polynomial-time	Polysize
Almost consequence-preserving	yes	yes
Consequence-preserving	no	yes
Faithful (extension-preserving)	no	yes
Process-preserving	no	no

Table 1: Translatability from regular to fail-safe semantics.

2.3 Theories Having No Extensions

Some semantics for default logics, like Reiter's, may not have extensions even if W is consistent. On the other hand, all fail-safe default semantics have

extensions. As a simple result, there is no way for translating all theories from Reiter’s default logics into any fail-safe semantics in general. This argument has been used to prove that Reiter’s default logics cannot be translated into justified or constrained default logic by Delgrande and Schaub [DS03], and that semi-normal default theories cannot always be translated into normal default theories by Janhunen [Jan03].

These results are however only consequences of the property of having or not having extensions, not of the property of fail-safeness. For example, we could define a default theory to always have $[]$ as a successful and closed process if no other one exists, and this change would not affect the fail-safeness of the semantics. As a result, we do not use the possible lack of extensions to prove the impossibility of translations.

In facts, the problem of the possible lack of extensions can be ignored by simply assuming that the default theory to translate has extensions. This simple assumption makes it possible some translations that are otherwise impossible: for example, there exists a poly-size faithful translation from Reiter’s default logics into normal or justified default logic if we restrict to default theories having extensions. In the rest of this paper, **we only consider theories having extensions.**

3 Translations

In this section, we show two translations from an arbitrary regular semantics into a normal default theory. The first one is a poly-size faithful translation; the second one is a polynomial translation that is “almost” consequence-preserving. Both translations are based on the idea of “simulating” the construction of processes of the original theory. We first show how this simulation can be done, and then apply it for obtaining the two translations.

3.1 Simulation of Defaults

The reductions we show are based on simulating a regular semantics using only normal defaults. Before going into the technical details, we explain the basic idea of the translation. For each default of the original theory, we introduce two variables that represent the application of the default in the original theory. Once the value of these variables are set, we can use other defaults for checking successfulness and closure of the original process.

The fact that extensions are generated only when the process is known to be successful and closed is taken into account in two ways:

1. we use new variables, and draw conclusions on the original variables only when we know that the simulated process is successful and closed;
2. if the simulated process is not successful or not closed, the extension we generate is a formula F ; the specific choice of F depends on whether we want a faithful or an almost-consequence-preserving translation, as will be explained in the next sections.

Given a default theory $\langle D, W \rangle$ and a formula F , both built over the alphabet X , we generate a normal default theory $\langle D_u^F, W_u \rangle$ such that each extension of $\langle D, W \rangle$ is var-equivalent to an extension of $\langle D_u^F, W_u \rangle$ w.r.t. X , and each extension of $\langle D_u^F, W_u \rangle$ is var-equivalent either to F or to an extension of $\langle D, W \rangle$ w.r.t. X . Since all var-equivalences are w.r.t. X , we often omit the part “w.r.t. X ” in what follows.

Given $\langle D, W \rangle$ and a specific regular semantics, the conditions of successfulness and closure of a process Π can be expressed as a number of consistency checks over the formulae of $\langle D, W \rangle$ and Π . Let $\omega_1, \dots, \omega_u$ be the formulae to be checked for consistency. These formulae may depend on which defaults are in Π , i.e., they are not exactly boolean formulae, as they may include the propositions like $(d_i \in \Pi)$, where $d_i \in D$. The successfulness and closure of a process can be checked in polynomial time given the results of these consistency checks. By a well-known result in circuit complexity [BS90], every polynomial boolean function can be expressed by a circuit of polynomial size.

The first steps of the translations are:

1. for each $d_i \in D$, we introduce two new variables c_i and e_i ;
2. we introduce $m + 1$ sets of new variables X_0, X_1, \dots, X_m , where $m = |D|$; each of these sets X_i is in bijective correspondence with X ;
3. for each formula ω_i , we consider the formula δ_i that is obtained by replacing each term $(d_j \in \Pi)$ with c_j in ω_i , i.e., $\delta_i = \omega_i[(d_j \in \Pi)/c_j]$;
4. we build the circuit $C(o_1, \dots, o_u, c_1, \dots, c_m)$ that encodes the successfulness and closure tests, where the input variables o_1, \dots, o_m represent the results of the consistency checks and each c_i encodes the presence of d_i in the process to be checked.

We give an example of the formulae δ_i and the circuit C used for translating $\langle D, W \rangle$ from Reiter's semantics into normal default logic. The consistency checks to be done are $u = 2m$, where m is the number of defaults. Namely, for each default $d_i = \frac{\alpha_i:\beta_i}{\gamma_i}$ we have to check the consistency of the following formulae:

$$\begin{aligned}\omega_i &= W \wedge \left(\bigwedge_{d_j \in D} (d_j \in \Pi) \rightarrow \gamma_j \right) \wedge \beta_i \\ \omega_{m+i} &= W \wedge \left(\bigwedge_{d_j \in D} (d_j \in \Pi) \rightarrow \gamma_j \right) \wedge \alpha_i\end{aligned}$$

The formulae δ_i are obtained by replacing $d_i \in \Pi$ with c_i , and are therefore the following ones:

$$\begin{aligned}\delta_i &= W \wedge \left(\bigwedge_{d_j \in D} c_j \rightarrow \gamma_j \right) \wedge \beta_i \\ \delta_{m+i} &= W \wedge \left(\bigwedge_{d_j \in D} c_j \rightarrow \gamma_j \right) \wedge \alpha_i\end{aligned}$$

The circuit $C(o_1, \dots, o_u, c_1, \dots, c_m)$ encodes the successfulness and closure of a process, provided that the consistency of δ_i is represented by the value of o_i and the presence of d_i in the process is represented by the value of c_i . For Reiter's default logic, we have:

$$C(o_1, \dots, o_u, c_1, \dots, c_m) = \bigwedge_{d_i \in D} \left[(c_i \rightarrow o_i) \wedge (\neg c_i \rightarrow (o_{m+1} \vee \neg o_i)) \right]$$

In words, for each default $d_i \in D$, if d_i is in Π (i.e., c_i is true) then the justification of d_i must be consistent with the consequences of all defaults in Π , (i.e., o_i is true). If $d_i \notin D$ (i.e., c_i is false), then either the precondition of d_i is not entailed (i.e., o_{m+1}) or its justification is not consistent (i.e., $\neg o_i$).

What has been shown are the formulae δ_i and the circuit C for the particular case of Reiter's semantics. Similar definitions can be given for every regular semantics. The default theory that results from the translation is the following one.

$$\langle D_u^F, W_u \rangle = \langle A \cup N \cup V \cup G \cup Z, W[X/X_0] \rangle$$

Let the defaults of the original theory be $D = \{d_1, \dots, d_m\}$, and let $d_i = \frac{\alpha_i : \beta_i}{\gamma_i}$. The defaults of the translated theory D_u^F are defined as follows.

$A = \{a_i \mid d_i \in D\}$; each default a_i represents the application of the default d_i in the simulated process:

$$a_i = \frac{\alpha_i[X/X_0] : \gamma_i[X/X_0] \wedge c_i \wedge e_i}{\gamma_i[X/X_0] \wedge c_i \wedge e_i}$$

$N = \{n_i \mid d_i \in D\}$; each default n_i represents the choice of not applying the default d_i :

$$n_i = \frac{: \neg c_i \wedge e_i}{\neg c_i \wedge e_i}$$

$V = \{v_1, \dots, v_u\}$; each v_i relates the consistency of δ_i with the value of the variable o_i :

$$v_i = \frac{e_1 \wedge \dots \wedge e_m : \delta_i[X/X_i] \wedge o_i \wedge t_i}{\delta_i[X/X_i] \wedge o_i \wedge t_i}$$

$G = \{g_1, \dots, g_u\}$; each g_i relates the inconsistency of δ_i with the value of o_i :

$$g_i = \frac{e_1 \wedge \dots \wedge e_m \wedge \neg \delta_i[X/X_i] : \neg o_i \wedge t_i}{\neg o_i \wedge t_i}$$

$Z = \{z_1, z_2\}$; these two defaults are used to compute the result of the circuit C and to “output” the generated extension or F accordingly:

$$\begin{aligned} z_1 &= \frac{e_1 \wedge \dots \wedge e_m \wedge t_1 \wedge \dots \wedge t_u \wedge C(o_1, \dots, o_u, c_1, \dots, c_m) : W \wedge \bigwedge_{d_i \in D} (c_i \rightarrow \gamma_i)}{W \wedge \bigwedge_{d_i \in D} (c_i \rightarrow \gamma_i)} \\ z_2 &= \frac{e_1 \wedge \dots \wedge e_m \wedge t_1 \wedge \dots \wedge t_u \wedge \neg C(o_1, \dots, o_u, c_1, \dots, c_m) : F}{F} \end{aligned}$$

We use the following abbreviations: E for $e_1 \wedge \dots \wedge e_m$ and T for $t_1 \wedge \dots \wedge t_u$. The preconditions of the defaults in D_u^F have been defined so that the defaults of $A \cup N$ have to be applied first; then, the defaults of $V \cup G$ can be applied; the defaults of Z can be applied only at the end. We prove a number of lemmas relating the processes of the theory $\langle D_u^F, W_u \rangle$ with the processes of $\langle D, W \rangle$.

Lemma 1 *If Π is a successful and closed process of $\langle D_u^F, W_u \rangle$, then Π contains exactly one among a_i or n_i for each i , and contains them only in the first m positions.*

Proof. Since all defaults in $D_u^F \setminus (A \cup N)$ contain E as a precondition, and E is only made true once m defaults of $A \cup N$ are applied, the first m defaults of Π are in $A \cup N$. If both a_i and n_i are in Π , then Π is not successful, as the consequences of these defaults contradicts each other. If neither a_i nor n_i are in Π , then the default n_i is applicable; therefore, Π is not a closed process. Finally, since exactly one between a_i and n_i is in the first m positions of Π , no other defaults of $A \cup N$ can be in a position of Π after the m -th. \square

In words, the processes of $\langle D_u^F, W_u \rangle$ begin with the application of exactly one between a_i or n_i for each i . After that, no other default of $A \cup N$ can be applied. The idea is that the truth value of c_i reflects the application of d_i in the default theory $\langle D, W \rangle$. Given a process Π of $\langle D_u^F, W_u \rangle$, we define the “simulated process” $O(\Pi)$ to be the following process of $\langle D, W \rangle$:

$$O(\Pi) = [d_{i_1}, \dots, d_{i_k}] \text{ where } a_{i_1}, \dots, a_{i_k} \text{ is the sequence of defaults } a_i \text{ of } \Pi$$

We prove that $O(\Pi)$ is a process of the original theory $\langle D, W \rangle$.

Lemma 2 *For every process Π of $\langle D_u^F, W_u \rangle$, $O(\Pi)$ is a process of $\langle D, W \rangle$.*

Proof. By assumption, Π is a process of $\langle D_u^F, W_u \rangle$. As a result, $W_u \cup \text{cons}(\Pi)$ is consistent and $W_u \cup \text{cons}(\Pi[d]) \models \text{prec}(d)$ for every $d \in \Pi$. These two conditions imply the same ones on W and $O(\Pi)$ once the inverse substitution $[X_0/X]$ has been applied. \square

The converse of this lemma also holds.

Lemma 3 *For each process Π' of $\langle D, W \rangle$, there exists a successful and closed process Π of $\langle D_u^F, W_u \rangle$ such that $O(\Pi) = \Pi'$.*

Proof. Let Π_1 be the process obtained by replacing each d_i with a_i in Π' and adding all n_i 's such that $d_i \notin \Pi'$. This process Π_1 is successful, and $O(\Pi_1) = \Pi'$. Since $\langle D_u^F, W_u \rangle$ is normal, Π_1 is the prefix of a successful and closed process $\Pi_1 \cdot \Pi_2$. Since Π_1 already contains m defaults, no default of Π_2 is in $A \cup N$. As a result, $O(\Pi_1 \cdot \Pi_2)$ is equal to $O(\Pi_1)$, which is in turn equal to Π' . \square

Together, these lemmas prove that the processes of $\langle D, W \rangle$ are in correspondence with the successful and closed processes of $\langle D_u^F, W_u \rangle$. Namely, each process Π' of $\langle D, W \rangle$ corresponds to some successful and closed processes Π of $\langle D_u^F, W_u \rangle$ such that $O(\Pi) = \Pi'$, and for each successful and closed processes Π of $\langle D_u^F, W_u \rangle$, it holds that $O(\Pi)$ is a process of $\langle D, W \rangle$. What is still missing is the effect of the successfulness and closure of the process of $\langle D, W \rangle$ on the corresponding processes of $\langle D_u^F, W_u \rangle$. We prove two preliminary lemmas

Lemma 4 *Every successful and closed process Π of $\langle D_u^F, W_u \rangle$ contains either v_i or g_i , but not both, for every i , in the positions of Π from the $m+1$ -th to the $m+u$ -th.*

Proof. As for Lemma 1, after applying the first m defaults the formula E is true but T is not. As a result, we can only apply defaults in $V \cup G$. The rest of the proof is like that of Lemma 1. \square

Lemma 5 *If Π' is a process of $\langle D, W \rangle$ and Π is a successful and closed process of $\langle D_u^F, W_u \rangle$ such that $O(\Pi) = \Pi'$, then $\text{cons}(\Pi)$ entails o_i or $\neg o_i$ depending on whether Π' satisfies ω_i .*

Proof. By Lemma 1 and Lemma 4, the first m defaults of Π are in $A \cup N$ and the next u defaults are in $V \cup G$. Since $O(\Pi) = \Pi'$, $a_j \in \Pi$ if and only if $d_j \in \Pi'$, and $n_j \in \Pi$ if and only if $d_j \notin \Pi'$. As a result, $\text{cons}(\Pi)$ contains either c_j or $\neg c_j$, depending on whether $d_j \in \Pi'$. Since the value of each c_j indicates the presence of $d_j \in \Pi'$, the satisfiability of the formulae δ_i corresponds to the satisfaction of the conditions ω_i .

By construction, the default v_i is only applicable if δ_i is consistent, which means that Π' satisfies the condition ω_i . For the same reason, g_i is applicable only if ω_i is not satisfied. As a result, $\text{cons}(\Pi)$ contains either o_i or $\neg o_i$ depending on whether Π' satisfies the condition ω_i . \square

We now establish a correspondence on the conditions of successfulness and closure.

Lemma 6 *If Π' is a successful and closed process of $\langle D, W \rangle$ and Π is a successful and closed process of $\langle D_u^F, W_u \rangle$ such that $O(\Pi) = \Pi'$, then $W_u \wedge \text{cons}(\Pi)$ and $W \wedge \text{cons}(\Pi')$ are var-equivalent w.r.t. X .*

Proof. The precondition of z_1 and z_2 include $E \wedge T$ and either $C(\dots)$ or $\neg C(\dots)$, respectively. By Lemma 1 and Lemma 4, $E \wedge T$ is entailed by $\text{cons}(\Pi)$. By Lemma 5, the truth value of each o_i is related to the process Π' satisfying the condition ω_i ; since Π' is successful and closed, C evaluates to true. Therefore, z_1 is applicable while z_2 is not. Since Π is successful and closed, z_1 is in Π . The var-equivalence of $W_u \wedge \text{cons}(\Pi)$ with $W \wedge \text{cons}(\Pi')$ is due to the fact that the consequence of z_1 is equivalent the latter formula after having replaced each c_i with either true or false, depending on whether $a_i \in \Pi$. \square

The converse of this lemma also holds.

Lemma 7 *If Π' is a process of $\langle D, W \rangle$ that is either not successful or not closed and Π is a successful and closed process of $\langle D_u^F, W_u \rangle$ such that $O(\Pi) = \Pi'$, then $W_u \wedge \text{cons}(\Pi)$ is var-equivalent to F w.r.t. X .*

Proof. Same as the proof of the previous theorem, but C this time evaluates to false. Therefore Π includes z_2 , which has F as a consequence. \square

These lemmas establish a correspondence between the extensions of the original and generated theories.

Theorem 1 *Every extension of $\langle D, W \rangle$ is var-equivalent to an extension of $\langle D_u^F, W_u \rangle$ and every extension of $\langle D_u^F, W_u \rangle$ is var-equivalent either to F or to an extension of $\langle D, W \rangle$.*

Proof. If Π' is a successful and closed process of $\langle D, W \rangle$, then there exists a closed and successful process Π of $\langle D_u^F, W_u \rangle$ such that $O(\Pi) = \Pi'$ by Lemma 3. By Lemma 6, it holds that the extension generated by Π is var-equivalent to that generated by Π' .

If Π is a closed and successful process of $\langle D_u^F, W_u \rangle$, by Lemma 2 $O(\Pi)$ is a process of $\langle D, W \rangle$. If $O(\Pi)$ is successful and closed, by Lemma 6, the extension generated by Π is var-equivalent to that generated by Π' . If $O(\Pi)$ is either not successful or not closed, by Lemma 7, the extension generated by Π is var-equivalent to F . \square

3.2 Faithful (Extension-Preserving) Translations

Delgrande and Schaub [DS03] have proved that Reiter’s default logic cannot be translated into justified default logic. Janhunen [Jan03] has proved that semi-normal defaults cannot be translated into normal defaults under Reiter’s semantics. Both results imply the impossibility of translating Reiter’s semantics into a fail-safe semantics. However, both proofs are based on the possible lack of extensions in Reiter’s semantics. We therefore investigate whether theories having extensions under Reiter’s semantics can be faithfully translated into a fail-safe default theory, namely, normal default logic.

A faithful, but exponential, translation from every regular default logic semantics into normal default logics always exists: if the extensions of the original theories are obtained by the deductive closure of the formulae in $\{E_1, \dots, E_m\}$, the following theory is a faithful translation of it into normal default logic, where the e_i ’s are new variables.

$$T = \left\langle \left\{ \frac{e_i \wedge E_i \wedge \bigwedge_{j=1, \dots, m} j \neq i \neg e_j}{e_i \wedge E_i \wedge \bigwedge_{j=1, \dots, m} j \neq i \neg e_j} \mid i = 1, \dots, m \right\}, \emptyset \right\rangle$$

This theory contains exactly one default for each extension of the original theory. Since these defaults are normal, and their justifications are inconsistent with each other, the successful and closed processes of this theory are exactly the sequences composed of a single default. The generated extensions are exactly the same (modulo var-equivalence) of the original theory.

The problem with this translation is not only that it is exponential: even worst, once the set of all extensions $\{E_i\}$ has been determined, there is no reason for using default logic, as we can simply use propositional logic instead.

Polynomial translations can be of two kinds: either the new theory can be built in polynomial time, or it has polynomial size. The first condition implies the second, but not vice versa. A translation from a regular semantics into normal default logic exists if we only require the result of the translation to be polynomial in size.

Theorem 2 *For every regular semantics there exists a poly-size faithful (extension-preserving) translation that maps all default theories that have extensions into normal default theories.*

Proof. The simulation shown in Section 3.1 is “almost” an extension-preserving translation. Indeed, all extensions of the original theory are translated into

extensions of the generated theory. On the other hand, processes of the original theory that do not generate extensions correspond to processes that generate F as an extension.

We can build a faithful translation as follows: first, we determine a single successful and closed process Π of the original theory. We then translate the default theory $\langle D, W \rangle$ into the simulating theory $\langle D_u^F, W_u \rangle$ in which $F = W \wedge \text{cons}(\Pi)$. This is a faithful translation because each successful and closed process of the original theory $\langle D, W \rangle$ corresponds to a successful and closed process of the theory $\langle D_u^F, W_u \rangle$ generating the same extension (modulo var-equivalence). The processes of $\langle D, W \rangle$ that are either not successful or not closed correspond to processes of $\langle D_u^F, W_u \rangle$ that generate an extension that is var-equivalent to F . Since F is an extension of the original theory, this translation is faithful. \square

The translation of this theorem is not polynomial-time, as it requires the generation of at least one successful and closed process of the original theory. On the other hand, such a process has always polynomial size. The result of the translation is therefore always of polynomial size. As will be shown later in the paper, no consequence-preserving polynomial-time translation exists. This implies that no faithful and polynomial-time translation exists as well.

3.3 Almost-Consequence-Preserving Translations

A simple computational argument shows that any default theory can be translated into a normal default theory in polynomial time if we admit the queries to be translated as well, i.e., $\langle D, W \rangle \models q$ in a regular semantics if and only if $\langle D', W' \rangle \models q'$, where D' is normal. Indeed, query answering is in Π_2^P for all regular default logics and Π_2^P -hard for Reiter's default logic even in the restriction to normal theories [Got92, Sti92]. However, for a translation to be “exactly” consequence-preserving q' should be the same as q .

A consequence-preserving translation is easy to give if we allow an exponential blow up of the theory: if the extensions of $\langle D, W \rangle$ are $\{E_1, \dots, E_m\}$, the skeptical consequences of $\langle D, W \rangle$ are exactly the classical consequences of $E_1 \vee \dots \vee E_m$, which are also the skeptical consequences of the default theory $\langle \emptyset, \{E_1 \vee \dots \vee E_m\} \rangle$. Ben-Eliyahu and Dechter [BED96] defined a better translation from default logic into propositional logic, which can be polynomial even if the number of extensions is exponential. Translations from default logic into propositional logic are however known to be exponen-

tial in the worst case due to the different complexity of the semantics. We now concentrate on polynomial translations.

The case of default theories having no extensions has already been considered, so we restrict to theories that have extensions. We have already shown a faithful translation that is poly-size: this translation clearly preserves the consequences as well. In this section, we show a translation from every regular semantics into normal default logic that is:

1. polynomial-time;
2. “almost” consequence preserving: if $\langle D, W \rangle$ is the original theory and $\langle D', W' \rangle$ is the result of the translation, then $\langle D, W \rangle \models q$ if and only if $\langle D', W' \rangle \models a \vee q$, where a is a new variable created by the translation.

The translation is based on the theory that simulates the process construction of the original theory. As we have already noticed, the only problem with this simulation is that the processes of the original theory that either are not successful or not closed correspond to successful and closed processes in the simulating theory. Since these processes generate $Cn(F)$ as an extension, all we have to do is to specify a value of F that do not affect entailment.

The trick we use is to translate q into $a \vee q$ and to set $F = a$. If we use the skeptical semantics, the extensions of the simulating theory that do not correspond to extensions of the original theory imply a , which in turn implies $a \vee q$; as a result, they do not affect the consequences of the theory.

Theorem 3 *For every regular default logic there exists a polynomial-time translation that maps a default theory $\langle D, W \rangle$ that has extensions into a normal default theory $\langle D', W' \rangle$ such that $\langle D, W \rangle \models q$ if and only if $\langle D', W' \rangle \models a \vee q$, where a is a new variable.*

The formula F and the way in which queries are translated are chosen in such a way the extensions of $\langle D', W' \rangle$ that do not correspond to extensions of the original theory $\langle D, W \rangle$ are irrelevant to the specific query evaluation mechanism. As a result, if we are interested into credulous entailment, we can use $F = \neg a$ and translate a query q into $a \wedge q$.

The question of whether the addition of a to the queries is necessary depends on the kind of translation used: we have already shown a faithful (and, therefore, consequence-preserving) translation that is poly-size. We will show that no polynomial-time consequence-preserving (i.e., that do not modify queries at all) translation exists unless part of the polynomial hierarchy collapses.

4 Impossibility of Translations

In this section, we show that some translations are impossible: namely, there is no polynomial-time exact consequence-preserving translation and no polynomial-time or polysize process-preserving translation from Reiter's default logics into any fail-safe default logic.

4.1 Consequence-Preserving Translations

We have already shown a poly-size faithful translation and a polynomial-time almost-consequence-preserving translation. We prove that no polynomial-time reduction that preserves the consequences exactly exists. To this end, we show a problem that is hard for Reiter's default logic but easy for all fail-safe default semantics. We cannot use a problem that has already been analyzed in the past (such as entailment or model checking) because these problems have the same complexity for Reiter's and for some fail-safe semantics.

For all fail-safe semantics, generating an extension is relatively easy, as it can be done by applying defaults until the process is closed. This property can be used to define a problem that is hard for Reiter's semantics but easy for all fail-safe ones: if it is known that either all extensions imply a or all extensions imply $\neg a$, then a single arbitrary extension suffices to check whether a is entailed. In turns, the assumption that all extensions imply a or all extensions imply $\neg a$ is equivalent to the assumption that the default theory implies either a or $\neg a$. We prove that entailment is hard for Reiter's default logic even under this assumption.

Theorem 4 *The problem of checking whether $\langle D, W \rangle \models a$ in Reiter's default logic is $\Sigma_2^p \cap \Pi_2^p$ -hard even if $\langle D, W \rangle$ implies either a or $\neg a$, and it has extensions.*

Proof. Let P be a problem in $\Sigma_2^p \cap \Pi_2^p$. We reduce the problem of telling whether $x \in P$ to the problem $T \models a$, where T is a default theory that either implies a or it implies $\neg a$.

Since P is in Σ_2^p , the question $x \in P$ can be reduced to the problem of checking the existence of extensions of a default theory with an empty background theory $\langle D_p, \emptyset \rangle$ [Got92]. Since P is in Π_2^p , its complementary problem is in Σ_2^p as well. As a result, the question $x \notin P$ can therefore be reduced to the existence of extensions of another theory $\langle D_n, \emptyset \rangle$. Therefore,

$x \in P$ if and only if $\langle D_p, \emptyset \rangle$ has extensions while $\langle D_n, \emptyset \rangle$ has not, and vice versa if $x \notin P$.

Let a be a variable that is mentioned neither in D_p nor in D_n . The default theory we use is the following one:

$$\begin{aligned} T &= \langle D, \emptyset \rangle \\ \text{where} \\ D &= \left\{ \frac{:a}{a}, \frac{:\neg a}{\neg a} \right\} \cup \left\{ \frac{a \wedge \text{prec}(d) : \text{just}(d)}{\text{cons}(d)} \middle| d \in D_p \right\} \cup \left\{ \frac{\neg a \wedge \text{prec}(d) : \text{just}(d)}{\text{cons}(d)} \middle| d \in D_n \right\} \end{aligned}$$

The only two defaults that can be applied from the background theory are the first two. They cannot be applied together, however. Once the first one is applied, the theory becomes equivalent to $\langle D_p, \{a\} \rangle$, while the application of the second one makes it equivalent to $\langle D_n, \{\neg a\} \rangle$. Since the existence of extensions for these two theories are related to the question $x \in P$, we have:

1. if $x \in P$, all extensions of T imply a ;
2. if $x \notin P$, all extensions of T imply $\neg a$.

As a result, either $T \models a$ or $T \models \neg a$; in particular, $T \models a$ if and only if $x \in P$. \square

The same problem is relatively easy for every fail-safe default semantics. Indeed, to solve it we only need to generate an extension and to check whether it implies a or $\neg a$: since all extensions are the same as for the entailment of a and $\neg a$, checking one extension suffices. Generating an arbitrary extension can be done easily in every fail-safe semantics.

Theorem 5 *Checking whether $T \models a$ is in Δ_2^p for every fail-safe semantics, if either $T \models a$ or $T \models \neg a$.*

Proof. Since either all extensions of T imply a or all extensions of T imply $\neg a$, we can check whether $T \models a$ by finding a single extension E of T and then checking whether $E \models a$. Finding one extension E is easy because the semantics is fail-safe. If Π is a successful process that is not closed, then there exists Π' such that $\Pi \cdot \Pi'$ is successful and closed. By the antimonotonicity of successfulness, if d is the first default of Π' , i.e., $\Pi' = [d] \cdot \Pi''$, then $\Pi \cdot [d]$ is successful. As a result, if Π is successful then it is either closed or there

exists a default d such that $\Pi \cdot [d]$ is successful. We can therefore start with $\Pi = []$, which is successful. At each step, if Π is closed, we can check whether $\text{cons}(\Pi) \models a$. Otherwise, there exists d such that $\Pi \cdot [d]$ is successful. We set $\Pi = \Pi \cdot [d]$, and continue. This algorithm must necessarily end up with a successful and closed process.

This algorithm only takes a polynomial number of steps if we have access to an NP-oracle. Indeed, all we have to do is to check closure of Π and successfulness of $\Pi \cdot [d]$ at each step; these conditions can be verified in polynomial time by letting the NP-oracle perform the consistency tests. \square

As a result of these two theorems, no polynomial-time consequence-preserving translation exists from Reiter's semantics to an arbitrary fail-safe semantics, unless $\Sigma_2^p \cap \Pi_2^p = \Delta_2^p$. The following theorem shows that even a polynomial number of calls to an NP-oracle do not suffice to translate from Reiter's semantics to any fail-safe one.

Theorem 6 *If there exists an (exact) consequence-preserving translation from Reiter's semantics into any fail-safe semantics that only requires a polynomial number of calls to an NP-oracle, then $\Sigma_2^p \cap \Pi_2^p = \Delta_2^p$.*

Proof. If such a translation exists, then for any $P \in \Sigma_2^p \cap \Pi_2^p$ we could translate the question $x \in P$ into the question $T \models a$ under Reiter's semantics where either $T \models a$ or $T \models \neg a$. In turns, with a polynomial number of calls to the oracle we can translate the question into the same question for a fail-safe semantics, where it can be solved with a polynomial number of other calls to the oracle. \square

Since no translation employing a polynomial number of calls to an NP-oracle exists, no polynomial-time translation exists either. Since the theorem has been proved using only theories in which all extensions have the same behavior w.r.t. the query (either they all entail it, or they all entail their negation,) this result holds for both skeptical and credulous reasoning.

The impossibility of polynomial-time faithful translations is a consequence of the above theorem: a faithful translation is also a consequence-preserving translation, and cannot therefore be polynomial-time.

Theorem 7 *If there exists a faithful translation from Reiter's default logic into any fail-safe default logic that only requires a polynomial number of calls to an NP-oracle, then $\Sigma_2^p \cap \Pi_2^p = \Delta_2^p$.*

4.2 Process-Preserving Translations

A process-preserving translation is a translation that not only preserves the extensions, but also the processes of a default theory. Clearly, we cannot enforce the processes to be exactly the same, otherwise the two theories would have the same defaults. Therefore, we only impose that there is a one-to-one correspondence between the defaults of the original and generated theories, such that the processes of the original theory matches the processes of the generated theory thanks to this correspondence.

An easy way for creating this correspondence is to assume that the first part D of a default theory $\langle D, W \rangle$ is a sequence of defaults rather than a set. In other words, we add an enumeration on the defaults so that we can write $D = \{d_1, \dots, d_m\}$. A process-preserving translation is a function that maps a default theory $\langle \{d_1, \dots, d_m\}, W \rangle$ into another default theory $\langle \{d'_1, \dots, d'_m\}, W' \rangle$ with the same number of defaults, and such that $[d_{i_1}, \dots, d_{i_r}]$ is a successful and closed process of the first theory if and only if $[d'_{i_1}, \dots, d'_{i_r}]$ is a successful and closed process of the second one.

We prove that there is no polynomial-time or polysize process-preserving translation from Reiter's default logics into any fail-safe default logic. To this aim, we show a problem that is hard in Reiter's default logic but easy in all fail-safe semantics.

Definition 5 (Completeness of Process) *Given a default theory $\langle D, W \rangle$ and a sequence of defaults Π , check whether there exists a successful a complete process $\Pi \cdot \Pi'$.*

The following theorem characterizes the complexity of completeness of processes for Reiter's semantics.

Theorem 8 *The problem of completeness of processes is Σ_2^P -complete in Reiter's default logic even for theories that have extensions.*

Proof. Membership: guess a sequence of defaults Π' and check whether $\Pi \cdot \Pi'$ is a successful and closed process.

Hardness: we reduce the problem of existence of extensions to this one. Given a theory $\langle D, W \rangle$ we build the theory $\langle D', W \rangle$, where $D' = \{d_n, d_p\} \cup D''$ and d_n , d_p , and D'' are defined as follows.

$$\begin{aligned}
d_n &= \frac{: \neg a}{\neg a} \\
d_p &= \frac{: a}{a} \\
D'' &= \left\{ \frac{a \wedge prec(d) : just(d)}{cons(d)} \middle| d \in D \right\}
\end{aligned}$$

This theory, as required, has one extension: the one generated by the process $[d_n]$. The other processes, if any, are made of d_p followed by the defaults that corresponds to the successful and closed processes of $\langle D, W \rangle$. As a result, the consistent process $[d_p]$ can be extended to form a consistent and closed process if and only if $\langle D, W \rangle$ has extensions. \square

The problem of completability of extensions is relatively easy for all fail-safe semantics, as it amounts to checking whether Π is a successful process. By definition, indeed, any successful process is either closed or can be extended to form a successful process. Moreover, if Π is not a process, or it is not successful, then it cannot be extended to generate a successful process thanks to the anti-monotonicity of successfulness.

As a result, completability of processes is equivalent to verifying whether a sequence of defaults is a successful process, which is a problem in Δ_2^P . For the fail-safe semantics defined in the literature, the problem is even simpler, as it is in D^P . We prove that is hard for the same class for justified default logic for the sake of completeness.

Theorem 9 *Checking whether Π is a successful process is in Δ_2^P for every regular default semantics, and is D^P -complete for justified default logic.*

Proof. The conditions of Π being a process and being successful can be computed in polynomial time once a number of consistency tests have been performed. The problem is therefore in Δ_2^P . For the case of justified default logic, these consistency tests are independent to each other, that is, the formulae to check do not depend to the results of the other tests. As a result, the problem is in D^P .

The hardness result is an obvious consequence of the fact that the applicability of a single default is hard: the problem **sat-unsat**, i.e., checking whether a pair of formulae $\langle \alpha, \beta \rangle$ is composed of a satisfiable formula α and an unsatisfiable formula β , is D^P -hard. This problem can indeed be reduced

to the problem of checking whether $[d]$ is a successful and closed process of the theory below:

$$\left\langle \left\{ \frac{\neg\beta : \alpha}{\alpha} \right\}, \emptyset \right\rangle$$

The sequence of defaults $[d]$ is indeed a successful process if and only if $\neg\beta$ is valid (that is, β is inconsistent) and α is consistent. As a result, the problem is D^p -hard. \square

Suppose that there exists a process-preserving translation from Reiter's default logic to any fail-safe default logics. We can then solve the problem of completability of a process Π in Reiter's default logic by simply translating both the theory and the process, and then solving the problem in the fail-safe default semantics. This would imply that $\Sigma_2^p = \Delta_2^p$.

This result can be strengthened to polysize translations. We can indeed prove that the problem of completability of processes does not simplify thanks to a preprocessing phase. This is proved by showing that the problem of completability of processes is $\|\rightarrow\Sigma_2^p$ -hard for Reiter's default logics, and cannot therefore be "compiled to" Δ_2^p . The class $\|\rightarrow\Sigma_2^p$ has been introduced by Cadoli et al. [CDLS02, Lib01] to characterize the complexity of problems when preprocessing of problem is allowed. We omit the details here, and refer the reader to the papers by Cadoli et al. [CDLS02, Lib01].

Theorem 10 *The problem of completability of processes is $\|\rightarrow\Sigma_2^p$ -complete in Reiter's default logic, where the default theory is the fixed part of the instance.*

Proof. We adapt the reduction by Gottlob, as follows: given a formula $\exists X \forall Y . \neg\phi$, where $|X| = |Y| = n$ and ϕ contains only clauses of three literals, let $A = \{\gamma_1, \dots, \gamma_m\}$ be the set of all clauses of three literals over the alphabet $X \cup Y$; we build a default theory and a successful process of it as follows:

$$\begin{aligned} T &= \langle \{p_i, n_i \mid 1 \leq i \leq m\} \cup \{a_i, b_i \mid 1 \leq i \leq n\} \cup \{f\}, W \rangle \\ &\text{where :} \\ p_i &= \frac{c_i \wedge e_i}{c_i \wedge e_i} \end{aligned}$$

$$\begin{aligned}
n_i &= \frac{: \neg c_i \wedge e_i}{\neg c_i \wedge e_i} \\
a_i &= \frac{: x_i \wedge f_i}{x_i \wedge f_i} \\
b_i &= \frac{: \neg x_i \wedge f_i}{\neg x_i \wedge f_i} \\
f &= \frac{\wedge e_i \wedge \wedge f_i : \{c_i \rightarrow \gamma_i \mid \gamma_i \in A\}}{\text{false}} \\
W &= \emptyset \\
\Pi &= [d_1, \dots, d_m] \\
&\text{where } d_i \text{ is } p_i \text{ if } \gamma \in \phi \text{ and } n_i \text{ otherwise}
\end{aligned}$$

The sequence Π is always a successful process. Moreover, $W \wedge \text{cons}(\Pi)$ implies all variables e_i and either c_i or $\neg c_i$. Namely, c_i is entailed if p_i is in Π , and $\neg c_i$ is entailed otherwise. We can therefore replace each c_i such that $\gamma_i \in \phi$ with **true** and each c_i such that $\gamma_i \notin \phi$ with **false**. The default f therefore simplifies to:

$$f' = \frac{\wedge f_i : \{\gamma_i \mid \gamma_i \in \phi\}}{\text{false}}$$

As shown by Gottlob, the resulting theory has extensions if and only if $\exists X \forall Y . \neg \phi$. This is therefore a polynomial translation from $\exists \forall \text{QBF}$ into the problem of completability of a process. Moreover, the default theory only depends on the number of variables of the QBF, while the process is the only part that depends on the specific ϕ . As a result, this is a $\|\rightsquigarrow$ -reduction, and proves that the problem of completability of processes is $\|\rightsquigarrow \Sigma_2^p$ -hard. Since the problem is in Σ_2^p , it is in $\|\rightsquigarrow \Sigma_2^p$ as well. As a result, the problem is $\|\rightsquigarrow \Sigma_2^p$ -complete. \square

This theorem proves that no polysize process-preserving translation from Reiter's semantics into any fail-safe semantics exists. Indeed, if this were the case, we could translate any theory from Reiter's default logics into a theory that has the same successful and closed processes under a fail-safe semantics. This would prove that the problem is in $\|\rightsquigarrow \Delta_2^p$, which implies that $\|\rightsquigarrow \Pi_2^p = \|\rightsquigarrow \Sigma_2^p$, and therefore the polynomial hierarchy collapses [CDLS02].

5 Conclusions

The possibility for a default logic semantics to make a sequence of applicable defaults to fail can be seen as a semantical drawback or as a computational feature. In this paper, we have studied how much a semantics gains from this ability to generate failures. In particular, we have restricted to the case of theories having extensions, and have shown that translations from fail-prone to fail-safe semantics are possible or not depending on the constraints that are imposed on the translation. In particular, a translation that preserves the extensions or the skeptical consequences and produces a polynomial sized result exists, while a polynomial-time translation does not. We have also considered more liberal (almost-consequence-preserving) and more restrictive (process-preserving) constraints on the translations.

The main results of this paper imply that the ability of failing can only give an advantage in terms of translation time (e.g., not all Reiter's theories can be faithfully translated into normal or justified theories in polynomial time), but not in terms of expressibility (e.g., for every Reiter's theory there exists an equivalent normal or justified theory of polynomial size.) This distinction is important, because it shows that fail-prone semantics are better than fail-safe ones in solving problems by translating them into default logic, but are not in terms of which domains can be encoded in polynomial space. In short, the possible failure of processes is a computational advantage, but not an expressiveness advantage.

These results hold only under some assumptions: the reductions are constrained to be polynomial (either in time or space) but can introduce new variables. Moreover, we only consider theories that have extensions, and prove the existence of translations only for normal default logic, which we considered a prototypical fail-safe semantics. These assumptions make the results of this paper incomparable to what proved in two similar works:

1. Delgrande and Schaub [DS03] have shown reductions from other variants of default logics into Reiter's; they take into account theories having no extensions, and limit to the specific case of justified, constrained, and rational default logic;
2. Janhunen [Jan03] has shown that Reiter's default logic can be translated into semi-normal default logic; translations are assumed not only faithful but also modular, and theories having no extensions are taken into account.

Both these two works consider polynomial faithful translations with new variables. The results presented in our paper are more general than the ones above in the sense that we proved the existence of translations from an arbitrary regular default theory into a fail-safe one and the non-existence of a translation from Reiter’s semantics into an arbitrary fail-safe one.

Some apparent contradictions between the results proved in this paper and those by Delgrande and Schaub and by Janhunen are due to the fact that we only consider default theories having extensions. This is why, for example, some results about the impossibility of translations by Delgrande and Schaub [DS03, Theorem 6] and by Janhunen [Jan03, Theorem 5], which relies on the possible non-existence of extensions in Reiter’s semantics, are not in contradiction with our results on the existence of such reductions. These apparent contradictions show that some translations are only impossible because of the possible lack of extensions, and become possible as soon as theories having no extensions are excluded from consideration.

An interesting question left open by the present work is whether the comparison between fail-safe and fail-prone semantics can be extended to logics that are not based on defaults. Clearly, a suitable definition of failure is needed; however, it seems somehow natural to consider propositional circumscription [Lif94] as a fail-safe non-monotonic logic (we add negative literals to a theory as far as possible, but never retract an added literal) and autoepistemic logic [Moo85] as a fail-prone one (we can “generate” a conclusion x by means of a formula like $\Box x \rightarrow x$ but then retract the conclusion, if a condition F is met, by means of a formula like $F \rightarrow \neg x$.)

Finally, we note that frameworks for comparing propositional knowledge representation formalisms have been given by Cadoli et al. [CDLS00] and by Penna [Pen00]. The translations considered in these framework are allowed to translate queries (or models), while the only translation of queries admitted in this paper is the addition of a literal to queries, i.e., q is translated into $a \vee q$.

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