

Good Pairs of Adjacency Relations in Arbitrary Dimensions

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Abstract

In this text we show that the notion of a “good pair” that was introduced in the paper [6] has actually known models. We will show, how to choose cubical adjacencies, the generalizations of the well known 4- and 8-neighborhood to arbitrary dimensions, in order to find good pairs. Furthermore, we give another proof for the well known fact that the Khalimsky-topology [7] implies good pairs. The outcome is consistent with the known theory as presented by T.Y. Kong, A. Rosenfeld [11], G.T. Herman [4] and M. Khachan et.al [8] and gives new insights in higher dimensions.

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1 Introduction

In the text [6] the author has given a new framework to define $(n - 1)$ -manifolds in \mathbb{Z}^n together with a notion of “good pairs” of adjacency relations. Such a good pair makes it possible for a $(n - 1)$ -manifold to satisfy a discrete analog of the Theorem of Jordan-Brouwer. This Theorem is a generalization of the Jordan-curve Theorem, which states that every simple closed curve in \mathbb{R}^2 separates its complement in exactly two connected components and is itself the boundary of both of them. Brouwer showed that the statement is true for simple $(n - 1)$ -manifolds in \mathbb{R}^n for all $n \geq 2$. It has been an open question since the beginnings of digital image analysis, if this is true in a discrete setting, so to speak in \mathbb{Z}^n .

As the figure 1 shows, it is not even clear what a simple closed curve should look like in a discrete setting. And really, this depends on the adjacency we impose on the points of \mathbb{Z}^n . We also see from the figure, that it is not enough to use only one adjacency for the base-set (background / white points) and the objects (foreground / black points), we have to use pairs of them. Unfortunately, not every pair of adjacencies is suitable because some even fail to make a $(n - 1)$ -manifold out of the neighbors of a given point, and so they do not even satisfy the Theorem of Jordan-Brouwer. On these grounds the notion of a good pair arose and good pairs are the central topic of this article.

A solution for the points in the figure would be, to equip the black points with the 8-adjacency and use the 4-adjacency for the white ones. Then is clear that a discrete notion of the Jordan-theorem is true for this example.

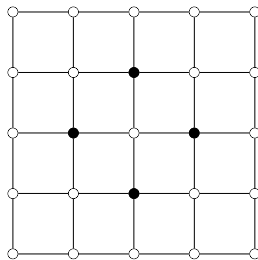


Figure 1: Depending on the adjacency relations we use for the black and the white points, respectively, the set of black points is connected (8-adjacency) or disconnected (4-adjacency). Also the set of white points may be connected (8-adjacency) or disconnected (4-adjacency). Only 4-adjacency is depicted.

For a long time adjacencies like the 4- and 8-neighborhood have been used, and of course, it is possible to generalize them to higher dimensions. This is done in this paper and we will see, which pairs of such relations give us good pairs. To do so, we will use the gridcube model of \mathbb{Z}^n which is widely accepted and may be found in the book of A. Rosenfeld and R. Klette [10]. It gives us a basic understanding of how these adjacencies may be build in high dimensions and once we have a good mathematical description for them, we may use it for the study of pairs of the adjacencies that we will call “cubical” because of the relation to this model.

In the 1980s E. Khalimsky [7] proposed a topological motivated approach with the so called Khalimsky-neighborhood. This topological notion gives also rise to graph-theoretic adjacencies and so it seems interesting to study it. Since it is already known,

that these relations form good pairs, as seen in [7] and [4], we can use it as a test for the theory that also shows, how we are able to combine topological and graph-theoretic concepts.

The paper is organized as follows: We start with some basic definitions in section 2 where we do a tour through basic discrete topology and the graph-theoretic knowledge we use in this text, in section 3 the important concepts of the paper [6] are given and in section 4 we apply the theory to the aforementioned adjacency relations. We end the text with some conclusions in section 5.

2 Basic Definitions

2.1 Topological Basics

We use this section to introduce some basic topological notions. These stem from the usual set-theoretic topology as it might be found in any textbook on topology like the one of Stöcker and Zieschang [13], but we also introduce some facts given by P.S. Alexandrov in his text [1]¹.

Definition 1 A pair $(\mathcal{P}, \mathcal{T})$ is called **topological space** for a set \mathcal{P} and set $\mathcal{T} \subset \mathfrak{P}(\mathcal{P})$, the so called **open sets** or **topology** on \mathcal{P} , with the following properties:

1. $O_i \in \mathcal{T}, i \in I \Rightarrow \bigcup_{i \in I} O_i \in \mathcal{T}$
2. $O_1, \dots, O_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^n O_i \in \mathcal{T}$
3. $\mathcal{P}, \emptyset \in \mathcal{T}$

A trivial topology on \mathcal{P} is the **discrete** topology $\mathfrak{P}(\mathcal{P})$. Please do not mistake the special “discrete” topology with the “discrete” setting we are working in. Even the \mathbb{R}^n may be equipped with a discrete topology and almost none of the discrete topologies we are referring to in this text are powersets of the base-set.

The subsets of \mathcal{P} , which have an open complement are called **closed**. An open set $U \in \mathcal{T}$ is called **neighborhood** of a point $x \in \mathcal{P}$ if x is contained in U .

A topological space that satisfies the following stronger claim instead of property (2), is called **Alexandrov-space**

$$2'. \quad O_i \in \mathcal{T}, i \in I \Rightarrow \bigcap_{i \in I} O_i \in \mathcal{T}.$$

All results for topological spaces are also true in Alexandrov-spaces. Topological spaces may be classified concerning the following separation properties:

Definition 2 A topological space may satisfy some of the separation axioms:

$$T_0: \bigwedge_{x,y \in \mathcal{P}, x \neq y} \bigvee_{U \in \mathcal{T}} : (x \in U \not\supset y) \vee (x \notin U \supset y)$$

$$T_1: \bigwedge_{x,y \in \mathcal{P}, x \neq y} \bigvee_{U \in \mathcal{T}} : x \in U \not\supset y$$

¹Actually, Paul Alexandroff is the same person as Pavel Sergeyevitch Alexandrov. The different names origin in a different transcription of the cyrillic letters in German and English.

$$T_2: \bigwedge_{x,y \in \mathcal{P}, x \neq y} \bigvee_{U,V \in \mathcal{T}} : (x \in U \not\supset y) \wedge (x \notin V \ni y) \wedge (U \cap V = \emptyset)$$

One can see, that every T_i -space is also a T_{i-1} -space. It is also true, that considering property (2') interesting only for T_0 -spaces:

Lemma 1 *An Alexandrov-space that satisfies the separation axioms T_1 or T_2 necessarily has the discrete topology.*

Proof. Let \mathcal{P} be a T_1 -space, $p \in \mathcal{P}$ and U a neighborhood of p . If $U = \{p\}$, then we are done. Otherwise, there exists a $q \neq p$ in U and by property (1), we may find a neighborhood U' , the contains p but not q . The intersection of all these sets is open and so, \mathcal{P} has to be discrete.

The proof for T_2 -spaces is analog. □

To give a topology on a set \mathcal{P} , it is enough to give a certain family \mathcal{B} of open sets that can be used to generate all the open sets of \mathcal{P} by using set-theoretic union. This family is then called **base** of the topology \mathcal{T} . A topological space is called **locally finite**, if for any point p in \mathcal{P} exists a finite open set and a finite closed set that both contain p . In the following, we define how we can build new topological spaces from given ones.

Definition 3 *Let $(\mathcal{P}_i, \mathcal{T}_i)$, $i \in I$, be a family of topological spaces and let $\mathcal{P} = \prod_{i \in I} \mathcal{P}_i$ be their product and $p_i : \mathcal{P} \rightarrow \mathcal{P}_i$ projections. The **product topology** \mathcal{T} is defined by the base*

$$\mathcal{B} = \left\{ \bigcap_{k \in K} p_k^{-1}(O_k) : O_k \in \mathcal{T}_k, K \subset I, K \text{ finite} \right\}.$$

The space $(\mathcal{P}, \mathcal{T})$ is called topological product of the $(\mathcal{P}_i, \mathcal{T}_i)$.

Definition 4 *Let $(\mathcal{P}, \mathcal{T})$ be a topological space and $A \subset \mathcal{P}$. With the topology*

$$\mathcal{T}|_A = \{O \cap A : O \in \mathcal{T}\}$$

The set A can be turned into a topological space $(A, \mathcal{T}|_A)$. The topology $\mathcal{T}|_A$ is called **subspace topology** of A with respect \mathcal{P} .

Definition 5 *A mapping $f : \mathcal{P} \rightarrow \mathcal{Q}$ between two topological spaces $(\mathcal{P}, \mathcal{T})$, $(\mathcal{Q}, \mathcal{U})$ is called **continuous**, if for every $O \in \mathcal{U}$ the set $f^{-1}(O)$ is in \mathcal{T} .*

Definition 6 *A topological space (X, \mathcal{T}) is called **connected**, if it cannot be decomposed into two nonempty open sets:*

$$\mathcal{P} = O_1 \cup O_2, O_1, O_2 \in \mathcal{T}, O_1 \neq \emptyset \neq O_2 \Rightarrow O_1 \cap O_2 \neq \emptyset.$$

A set $A \subset \mathcal{P}$ is called **connected**, if it is connected in the subspace topology.

Lemma 2 *Let $(\mathcal{P}, \mathcal{T})$ be a connected topological space and let $(\mathcal{Q}, \mathcal{T}')$ be a topological space. If $f : \mathcal{P} \rightarrow \mathcal{Q}$ is a continuous mapping, then \mathcal{Q} is connected.* □

From the continuity of the projections p_i in the definition of the product topology we can deduce the following

Lemma 3 *A topological space is connected if and only if all of its factors are connected.* \square

We define a **path** of length $m \in \mathbb{N}$ to be a continuous mapping $w : \{0, \dots, m\} \rightarrow \mathcal{P}$. A path is **closed** if $w(0) = w(m)$.

Definition 7 *A topological space $(\mathcal{P}, \mathcal{T})$ is called **path-connected**, if for any two points $p, q \in \mathcal{P}$ exists a path w of length m depending only on p and q , such that $w(0) = p$ and $w(m) = q$.*

*The topological space $(\mathcal{P}, \mathcal{T})$ is called **locally path-connected** if for every point $p \in \mathcal{P}$ and every neighborhood U of p a path-connected neighborhood $V \subset U$ exists.*

Corollary 1 *The following holds:*

1. *Path-connected spaces are connected.*
2. *Connected and locally path-connected spaces are path-connected.* \square

Definition 8 *Let X and Y be topological spaces. A homotopy from X to Y is a family of mappings $h_t : X \rightarrow Y$, $t \in I = [0, 1]$ with the following property: The mapping $H : X \times I \rightarrow Y$, $H(x, t) = h_t(x)$, is continuous. The set $X \times I$ has the product topology.*

*Two functions are called **homotopic**, $f \cong g : X \rightarrow Y$ if a homotopy $h_t : X \rightarrow Y$ exists with $h_0 = f$ and $h_1 = g$. If g is constant then f is called nullhomotopic.*

*A homotopy is called **linear**, if it is linear in t .*

Just like in the definition of paths, the set I does not need to be the set $[0, 1]$ in the discrete setting we are going to use arbitrary connected subsets of \mathbb{Z} for instance $\{0, \dots, m\} \subset \mathbb{N}$ with a fitting topology.

Definition 9 *A topological space is called simply connected if any closed path is nullhomotopic.*

This means that we continuously contract every closed path into one point.

Lemma 4 *If $(\mathcal{P}, \mathcal{T})$ is a union of two open simply connected subspaces with contractible intersection, then it is simply connected.* \square

2.2 Alexandrov-Spaces

Every Alexandrov-space has an unique base that is given by the set of minimal neighborhoods of all points in the base-set. The minimal neighborhoods are easily identified as the intersections of all neighborhoods of a given point. Let p be a point in an Alexandrov-space $(\mathcal{P}, \mathcal{T})$. We write $U_{\mathcal{T}}(p)$ to denote its minimal neighborhood. Analog we may find a minimal closed set containing a given point p . We denote this

set by $C_{\mathcal{T}}(p)$. To create an analogy to the graph-theoretic background of most of this theory, we define

$$A_{\mathcal{T}}(p) := (U_{\mathcal{T}}(p) \cup C_{\mathcal{T}}(p)) \setminus \{p\} \quad (1)$$

to be the **adjacency** of the point p in $(\mathcal{P}, \mathcal{T})$. The set $A_{\mathcal{T}}(p)$ can be made to an Alexandrov-space in the subspace-topology.

Given a set $M \subset \mathcal{P}$ we may analog define the sets:

$$U_{\mathcal{T}}(M) := \left\{ p \in \mathcal{P} : \bigvee_{q \in M} p \in U_{\mathcal{T}}(q) \right\} \quad (2)$$

$$C_{\mathcal{T}}(M) := \left\{ p \in \mathcal{P} : \bigvee_{q \in M} p \in C_{\mathcal{T}}(q) \right\} \quad (3)$$

Lemma 5 *The set functions $U_{\mathcal{T}}$ and $C_{\mathcal{T}}$ are closure operators, they satisfy:*

1. $U_{\mathcal{T}}(\emptyset) = \emptyset$.
2. $M \subset N \Rightarrow U_{\mathcal{T}}(M) \subset U_{\mathcal{T}}(N)$.
3. $U_{\mathcal{T}}(U_{\mathcal{T}}(M)) = U_{\mathcal{T}}(M)$.

Proof. The first property is trivial. To show the second one let $p \in U_{\mathcal{T}}(M)$. Therefore, it exists a $q \in M$ such that $p \in U_{\mathcal{T}}(q)$. By the precondition we have $q \in N$ and therefore $p \in U_{\mathcal{T}}(N)$.

To prove property 3, let $p \in U_{\mathcal{T}}(U_{\mathcal{T}}(M))$, therefore, a q exists in $U_{\mathcal{T}}(M)$ such that $p \in U_{\mathcal{T}}(q)$. If $q \in M$ holds, then holds $p \in U_{\mathcal{T}}(M)$. Otherwise, a $q' \in M$ exists such that q is in $U_{\mathcal{T}}(q')$. By the property T_0 of an Alexandrov-space, the point p has to be in $U_{\mathcal{T}}(q')$ and therefore in $U_{\mathcal{T}}(M)$. The other inclusion follows from 2. \square

Lemma 6 *Let $(\mathcal{P}, \mathcal{T})$ be an Alexandrov-space that contains one point p such that the only open neighborhood of p is the set \mathcal{P} itself. Then $(\mathcal{P}, \mathcal{T})$ is contractible.*

Proof. We define a homotopy $F : \mathcal{P} \times I \rightarrow \mathcal{P}$ by $F(q, t) = q$ for $0 \leq t < 1$ and $F(q, 1) = p$ for each $q \in \mathcal{P}$.

We show, that F is continuous. Let $M \subset \mathcal{P}$ be open.

Case 1: The point p is in M . W.l.o.g. $M = \mathcal{P}$. Therefore, the set $F^{-1}(M) = \mathcal{P} \times I$ is open.

Case 2: The point p is not in M . The set $F^{-1}(M) = M \times [0, 1)$ is open. \square

Lemma 7 *Let $(\mathcal{P}, \mathcal{T})$ be an Alexandrov-space and $p \in \mathcal{P}$, then the set $U_{\mathcal{T}}(p)$ is contractible. Therefore, the Alexandrov-space $(\mathcal{P}, \mathcal{T})$ has a base of contractible open sets. In particular, the set $(\mathcal{P}, \mathcal{T})$ is local contractible.*

Proof. We utilize Lemma 6 together with $Y = U(x)$ and $\omega = x$. \square

It is possible to establish a notion of dimension in Alexandrov-spaces. It can also be found in Evako et.al. [3]:

Definition 10 Let $(\mathcal{P}, \mathcal{T})$ be a Alexandrov-space and $p \in \mathcal{P}$.

- $\dim_{\mathcal{P}}(p) := 0$, if $U_{\mathcal{T}}(p) \setminus \{p\} = \emptyset$.
- $\dim(\mathcal{P}) := n$, if there is a point p in \mathcal{P} such that $\dim_{\mathcal{P}}(p) = n$ and for all $q \in \mathcal{P}$ exists a $k \leq n$ with $\dim_{\mathcal{P}}(q) = k$.
- $\dim_{\mathcal{P}}(p) := n + 1$, if $\dim(U_{\mathcal{T}}(p) \setminus \{p\}) = n$. The set $U_{\mathcal{T}}(p) \setminus \{p\}$ has the subspace topology.
- If no $k \in \mathbb{N}$ exists such that $\dim_{\mathcal{P}}(p) = k$ then define $\dim_{\mathcal{P}}(p) = \infty$.

Definition 11 We call $(\mathcal{P}, \mathcal{T})$ a 0-surface, if \mathcal{P} has two points and is disconnected under \mathcal{T} .

The set $(\mathcal{P}, \mathcal{T})$ is called n -surface for $n > 0$, if \mathcal{P} is connected under \mathcal{T} and for all $p \in \mathcal{P}$ the set $A_{\mathcal{T}}(p)$ is a $(n - 1)$ -surface.

A n -surface $(\mathcal{P}, \mathcal{T})$ is called n -sphere, if \mathcal{P} is finite and it is simply connected for $n > 1$.

By Evako et.al.[3] gilt:

Theorem 1 Let $(\mathcal{P}, \mathcal{T})$ be a Alexandrov-space that is a n -surface for $n > 2$. Then, for any point $p \in \mathcal{P}$ holds, that $A_{\mathcal{T}}(p)$ is simply connected. \square

Theorem 2 Every Alexandrov-space is a partial order and every partial order defines an Alexandrov-space. \square

2.3 The Khalimsky-Topology

In this section we study an important Alexandrov-topologies. To define it we start with a topologization of the set \mathbb{Z} which we can interpret as a discrete line. What possibilities do we have to define a non-trivial topology on this set such that it is connected?

One can see, that the sets

$$\mathcal{B} = \{\{x\} : x \in \mathbb{Z}, x \equiv 0(2)\} \cup \{\{x-1, x, x+1\} : x \in \mathbb{Z}, x \equiv 1(2)\} \quad (4)$$

and

$$\mathcal{B}' = \{\{x\} : x \in \mathbb{Z}, x \equiv 1(2)\} \cup \{\{x-1, x, x+1\} : x \in \mathbb{Z}, x \equiv 0(2)\} \quad (5)$$

are bases of topologies. They differ only by a translation. Therefore, it seems reasonable to just choose one of them both. We will use the base \mathcal{B} and denote its generated topology by κ .

Lemma 8 The Alexandrov-space (\mathbb{Z}, κ) is connected. \square

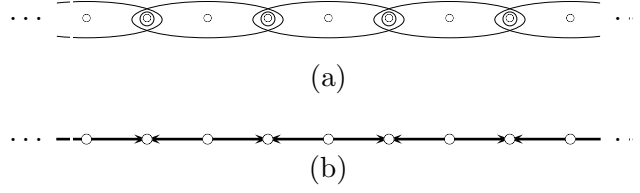


Figure 2: Figure (a) shows a section of (\mathbb{Z}, κ) . The base of the topology is represented by ellipses. The base of the topology may also be depicted as a digraph. Figure (b) shows this.

To go from here to the higher-dimensional case, we may view \mathbb{Z}^n as a n -fold topological product of \mathbb{Z} . We denote the product topology with κ_n . By all we know so far, it is clear, that (\mathbb{Z}^n, κ_n) is connected. We call this class of spaces **Khalimsky-spaces** after E. Khalimsky [7].

Lemma 9 *The Alexandrov-space (\mathbb{Z}^n, κ_n) is connected for all $n \geq 1$.*

Proof. This follows from Lemma 3. □

Theorem 3 *All Khalimsky-spaces (\mathbb{Z}^n, κ_n) , $n \geq 2$ satisfy the separation theorem of Jordan-Brouwer.*

Proof. The proof is easy if one uses the methods of algebraic topology, because (\mathbb{Z}^n, κ_n) is isomorphic to a cell-decomposition of \mathbb{R}^n :

$$\mathbb{R}^n = (\{\{i\} : i \in \mathbb{Z}, i \equiv 0(2)\} \cup \{(i-1, i+1) : i \in \mathbb{Z}, i \equiv 1(2)\})^n \quad (6)$$

The set $(i-1, i+1)$ denotes the open real interval between the integers $i-1$ and $i+1$. Since the Theorem of Jordan-Brouwer is true for any \mathbb{R}^n , $n \geq 2$, it has to hold for n -dimensional Khalimsky-space.

We give another proof in section 4.3. □

2.4 Adjacency Relations

To establish structure on the points of the set \mathbb{Z}^n we have to define some kind of connectivity relation. This might be done in terms of a (set-theoretic) topology as in the last section, or we may develop a graph-theoretic framework as in the following part of the text.

Definition 12 *Given a set \mathcal{P} , a relation $\alpha \subset \mathcal{P} \times \mathcal{P}$ is called **adjacency** if it has the following properties:*

1. α is finitary: $\forall p \in \mathcal{P} : |\alpha(p)| < \infty$.

2. \mathcal{P} is connected under α .
3. Every finite subset of \mathcal{P} has at most one infinite connected component as complement.

A set $M \subset \mathcal{P}$ is called **connected** if for any two points p, q in M exist points p_0, \dots, p_m and a positive integer m such that $p_0 = p$, $p_m = q$ and $p_{i+1} \in A(p_i)$ for all $i \in \{0, \dots, m-1\}$. Compare this definition to the topological one we gave above.

The property 3 of an adjacency-relation is in \mathbb{Z}^n for $n \geq 2$ always satisfied.

In the text we will consider pairs (α, β) of adjacencies on the set \mathbb{Z}^n . In this pair α represents the adjacency on a set $M \subset \mathbb{Z}^n$, while β represents the adjacency on $M^C = \mathbb{Z}^n \setminus M$.

Let \mathcal{T} be the set of all translations on the set \mathbb{Z}^n . The generators $\tau_1, \dots, \tau_n \in \mathcal{T}$ of \mathbb{Z}^n induce a adjacency π in a natural way:

Definition 13 Two points p, q of \mathbb{Z}^n are called **proto-adjacent**, in terms $p \in \pi(q)$, if there exists a $i \in \{1, \dots, n\}$ such that $p = \tau_i(q)$ or $p = \tau_i^{-1}(q)$.

We can view the generators of \mathbb{Z}^n as the standard base of \mathbb{R}^n .

Another important adjacency on \mathbb{Z}^n is ω .

$$\omega(p) := \{q \in \mathbb{Z}^n : |p_i - q_i| \leq 1, 0 \leq i \leq n\} \quad (7)$$

In the rest of the text let α and β be two adjacencies on \mathbb{Z}^n such that for any $p \in \mathbb{Z}^n$ holds

$$\pi(p) \subset \alpha(p), \beta(p) \subset \omega(p) \quad (8)$$

Lemma 10 The set \mathbb{Z}^n is connected under π . □

3 Digital Manifolds

If we want to talk about $(n-1)$ -Manifolds in \mathbb{Z}^n we have to give a proper definition. Unfortunately, all the definitions known to the author from the literature are not usable in terms of generalization to higher dimension or for the unification of the topological and graph-theoretic approach. So it is necessary, to give a new definition that satisfies this two criteria. This is done in [6]. The new definition is mainly based on the so called *separation property*. It gives a description on how a discrete $(n-1)$ -manifold should look like locally.

3.1 The Separation Property

We call the set

$$C^k = \{0, 1\}^k \times \{0\}^{n-k} \subset \mathbb{Z}^n \quad (9)$$

the k -dimensional standard **cube** in \mathbb{Z}^n . The set C^k can be embedded in $\binom{n}{k}$ different ways in \mathbb{Z}^n . A general k -cube in \mathbb{Z}^n is defined by a translation of a standard cube.

Indeed, we can construct any k -cube C from one point p with k generators in the following way:

$$C = \{\tau_1^{e_1} \cdot \tau_k^{e_k}(p) : e_i \in \{0, 1\}, i = 1, \dots, k\} \quad (10)$$

The dimension of C' is then $k + l$. We use this construction in the next definition.

Definition 14 Let $M \subset \mathbb{Z}^n$, $n \geq 2$ and C be a k -cube, $2 \leq k \leq n$. The complement of M is in C **not separated** by M under the pair (α, β) , if for every α -component M' of $C \cap M$ and every $(k-2)$ -subcube C^* of C the following is true:

If C^* is such that $C^* \cap M' \neq \emptyset$ has maximal cardinality among all sets of this form, and the sets $\tau_1(C^*) \setminus M$ and $\tau_2(C^*) \setminus M$ are both nonempty and lie in one common β -component of M^C , then holds

$$(\tau_1 \tau_2)^{-1}(\tau_1 \tau_2(C^*) \cap M') \subset \tau_1^{-1}(\tau_1(C^*) \cap M') \cap \tau_2^{-1}(\tau_2(C^*) \cap M') . \quad (11)$$

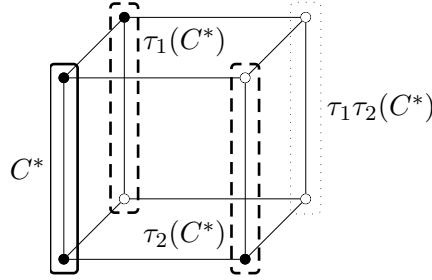


Figure 3: C^* has an intersection of maximal cardinality with the α -component M' . The sets $\tau_1(C^*) \setminus M'$ and $\tau_2(C^*) \setminus M'$ are nonempty and belong to a β -component of M^C . Since $\tau_1 \tau_2(C^*) \cap M' = \emptyset$, the property of definition 14 is satisfied for this C^* . But the set M' separates M^C in the cube C^* . Why?

In the following, we only consider the case when $C \cap M$ has at most one α -component. This can be justified by viewing any other α -component besides the one considered as part of the background, since there is no α -connection anyway. This property also gets important if we study the construction of the simplicial complex.

A set M has the **separation property** under a pair (α, β) , if for every k -cube C , $2 \leq k \leq n$ as in the definition 14 the set M^C is in C not separated by M

The meaning of the separation property is depicted in the figure 4.

Definition 15 An α -connected set $M \subset \mathbb{Z}^n$, for $n \geq 2$, is a **(digital) $(n-1)$ -manifold** under the pair (α, β) , if the following properties hold:

1. In any n -cube C the set $C \cap M$ is α -connected.
2. For every $p \in M$ the set $\omega(p) \setminus M$ has exactly two β -components C_p and D_p .
3. For every $p \in M$ and every $q \in \alpha(p) \cap M$ the point q is β -adjacent to C_p and D_p .
4. M has the separation property.

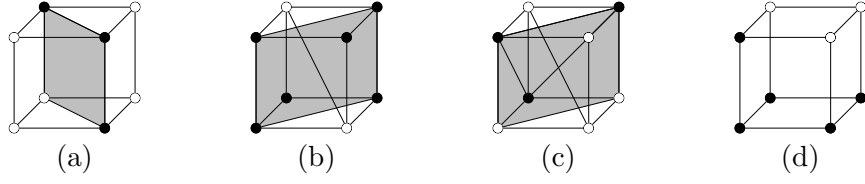


Figure 4: The black points represent the set M in the given 3-cubes. The white points represent the complement of M . In the cases (a) to (c) the complement, which is connected, is separated by M . This separation is depicted by the gray plane spanned by C^* and $\tau_1\tau_2(C^*)$. In Figure (d) occurs no separation, since the only choice for C^* would be a 1-cube, that contains only black points.

How should a $(n-1)$ -manifold look like globally in general? We do not know. But we might say, that a single point in \mathbb{Z}^n might be considered as the inside of some object, i.e. that it might be separated by the other points. The way to do this is to require the set of neighbors of a point to be a $(n-1)$ -manifold. This justifies the following:

Definition 16 A pair (α, β) of adjacency relations on \mathbb{Z}^n is a **separating pair** if for all $p \in \mathbb{Z}^n$ the set $\beta(p)$ is a $(n-1)$ -manifold under α .

3.2 Double Points

Definition 17 A point $p \in \beta(z)$ $z \in \mathbb{Z}^n$ is a **double point** under the pair (α, β) , if there exist points $q \in \pi(z) \cap \alpha(p)$ and $r \in \beta(z) \cap \pi(p)$ and a simple² translation $\tau \in \mathcal{T}$ with $\tau(p) = q$, $\tau(r) = z$ and $q \in \alpha(r)$.

This concept is the key to a local characterization of the good pairs (α, β) . Without it, one could not consistently define topological invariants like the Euler-characteristic. It means that an edge between points in a set M can be crossed by an edge between points of its complement and these four points lie in a square defined by the corresponding adjacencies. This crossing can be seen as a double point, belonging both to the foreground *and* to the background. Also, mention the close relationship to the separation property, which is a more general concept of similar interpretation. For further insight, refer to the text [6].

Definition 18 A separating pair of adjacencies (α, β) in \mathbb{Z}^n is a **good pair**, if for every $p \in \mathbb{Z}^n$ the set $\beta(p)$ contains no double points.

4 Good Pairs of Adjacency Relations

4.1 Cubical Adjacencies

We will study adjacencies in the sense of the gridcube-model. This is a common model in computer graphics literature and has nothing to do with the n -cubes we talked about

²A translation τ is called **simple** if no other translation σ exists with $\sigma^n = \tau$, $n \in \mathbb{Z}$, $|n| \neq 1$.

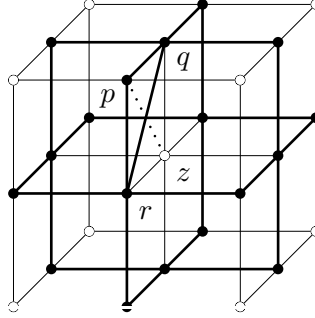


Figure 5: A double point p . The fat edges represent the α -adjacency. Only the relevant edges have been drawn for clarity. The dotted edge represents the β -adjacency of p and z . The black points are the β -neighbors of z .

earlier. We use this model here to make it easy to study the adjacency relations in this section. For more on this topic refer to the Book of Rosenfeld and Klette [10]

We identify the points of \mathbb{Z}^n with n -dimensional unit-cubes with barycenters in the points of the lattice \mathbb{Z}^n . The cube W that represents the point $0 \in \mathbb{Z}^n$ can be expressed in euclidean space as $[-\frac{1}{2}, \frac{1}{2}]^n$. Those gridcubes may be interpreted as union of (polytopal) faces of different dimension. Any of its faces is again a gridcube, only with a lower dimension. Take, for instance, a 3-dimensional gridcube $[-\frac{1}{2}, \frac{1}{2}]^3$. It has, among others, the 0-dimensional face $(\frac{1}{2}, \frac{1}{2})$, the 1-dimensional face $[(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})]$ and the 2-dimensional face with the vertices $(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$.

Two given gridcubes may share a k -dimensional face for $0 \leq k < n$. This k -face is just the intersection of both of them. So we might say that the elements $(0, \dots, 0)$ and $(1, \dots, 1)$ of \mathbb{Z}^n intersect in a common vertex (0-face) with the coordinates $(\frac{1}{2}, \dots, \frac{1}{2})$. However, the elements $(0, \dots, 0)$ and $(1, 0, \dots, 0)$ share a common $(n-1)$ -face.

In the rest of the text we will no longer make the gridcube model explicit. It just serves as an introduction to visualize the concepts that we use to analyze the discrete geometry even in higher dimensions³.

Definition 19 Two points $p, q \in \mathbb{Z}^n$ are called *k -adjacent* for $0 \leq k < n$, denoted by $p \in \alpha_k(q)$, if their corresponding gridcubes share a common k -face. We call this adjacencies **cubical**.

Clearly, this kind of relation we just defined is an adjacency-relation in the sense of definition 12:

Lemma 11 The relation α_k is an adjacency-relation on \mathbb{Z}^n for every $n \geq 2$ and all integers k between 0 and $n-1$.

Proof. First, we have to check that for any $p \in \mathbb{Z}^n$ the set $\alpha_k(p)$ has only finite cardinality. It is easy to check, that $\alpha_0(p)$ is just $\omega(p)$ as defined earlier and every $\alpha_k(p)$ for

³I find it a lot easier to imagine a four-dimensional cube, than a four-dimensional grid...

$0 \leq k < n$ is a subset of $\omega(p)$. Since $\omega(p)$ has $3^n - 1$ Elements in \mathbb{Z}^n , the relations α_k must be finitary.

To see that \mathbb{Z}^n is connected under any α_k , $0 \leq k < n$, we observe that α_{n-1} is just another interpretation for the relation π defined earlier. Since \mathbb{Z}^n is π -connected as proven in [6] and every α_k is a superset of α_{n-1} , we conclude that \mathbb{Z}^n is α_k -connected.

The last property is in \mathbb{Z}^n with $n \geq 2$ trivially satisfied. \square

Lemma 12 *The cubical adjacency $\alpha_k(x_1, \dots, x_n)$ may be represented in \mathbb{Z}^n as the set:*

$$\left\{ (y_1, \dots, y_n) \in \mathbb{Z}^n : \max_{i=1, \dots, n} \{|x_i - y_i|\} = 1, 1 \leq \sum_{i=1}^n |x_i - y_i| \leq n - k \right\} \quad (12)$$

Proof. Let p and q be two points of \mathbb{Z}^n such that $p \in \alpha_k(q)$. This means, the gridcubes corresponding to p and q share a common k -face. Their distance in the maximum-metric may not be greater than 1. Furthermore, p and q may not share a single common l -face for $0 \leq l < k$. That means, all of that l -faces must be faces of common k -faces. Therefore, the two points may not have more than k coordinates in common. \square

Lemma 13 *Let α be a cubical adjacency on \mathbb{Z}^n . It holds:*

1. α is invariant under translations
2. α is invariant under permutations of coordinates.

Proof. Let τ be any translation on \mathbb{Z}^n . We need to show $\tau(\alpha(p)) = \alpha(\tau(p))$ for any $p \in \mathbb{Z}^n$. From the representation of $\alpha(p)$ we may deduce:

$$\tau(\alpha(p)) = \tau(\{q \in \mathbb{Z}^n : q \in \alpha(p)\}) \quad (13)$$

$$= \{\tau(q) : q \in \mathbb{Z}^n, q \in \alpha(p)\} \quad (14)$$

$$= \{q' \in \mathbb{Z}^n : q' \in \alpha(\tau(p))\} \quad (15)$$

$$= \alpha(\tau(p)) \quad (16)$$

The proof of the second part is analog. \square

What is the structure of the cubical adjacencies in \mathbb{Z}^n ? We take a closer look at n -dimensional cubes.

Lemma 14 *The number of k -faces of a n -dimensional cube is*

$$\binom{n}{k} \cdot 2^{n-k} . \quad (17)$$

Proof. We use induction on the dimension n of the cube.

For $n = 0$ we observe, that a 0-dimensional cube is just a point and has only one 0-face. Therefore, the induction base is correct.

In the case $n > 0$, we notice that a n -dimensional cube may be created from a $(n - 1)$ -dimensional one by doubling the cube and inserting a k -face for every $(k - 1)$ -face in the original cube. Therefore, we get by induction hypothesis and Pascals Theorem:

$$2 \binom{n-1}{k} \cdot 2^{n-1-k} + \binom{n-1}{k-1} \cdot 2^{n-1-(k-1)} = \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] \cdot 2^{n-k} \quad (18)$$

$$= \binom{n}{k} \cdot 2^{n-k} \quad (19)$$

This proves the Lemma. \square

Lemma 15 For every $p \in \mathbb{Z}^n$, the number of k -neighbors is

$$|\alpha_k(p)| = \sum_{i=k}^n \binom{n}{i} \cdot 2^{n-i} . \quad (20)$$

Proof. Obviously, any l -face σ of a cube contains at least one k -face τ for $0 \leq k \leq l \leq n$. Therefore, k -adjacent cubes exist, that are also l -adjacent. Since that are those, that share more than one common k -face, the set $\alpha_k(p)$ for $p \in \mathbb{Z}^n$ may be decomposed into the following disjoint sets:

$$\begin{aligned} \alpha_k(p) = & \{q \in \mathbb{Z}^n : p, q \text{ have at most one } k\text{-face in common} \} \\ & \cup \{q \in \mathbb{Z}^n : p, q \text{ have at most one } (k+1)\text{-face in common} \} \\ & \vdots \\ & \cup \{q \in \mathbb{Z}^n : p, q \text{ have at most one } (n-1)\text{-face in common} \} \end{aligned} \quad (21)$$

By adding the cardinalities of these sets, which we can easily compute with the last Lemma we get the result $|\alpha_k(p)| = \sum_{i=k}^n \binom{n}{i} 2^{n-i}$. This proves the Lemma. \square

By this technique we get as examples of cubical adjacencies in \mathbb{Z}^2 the known 4- and 8-adjacencies, in \mathbb{Z}^3 the 6-, 18- and 26-adjacencies and in \mathbb{Z}^4 the 8-, 32-, 64- and 80-adjacencies.

4.2 Good Pairs of Cubical Adjacencies

In this section we will study, how we have to choose two cubical adjacencies to get to a good pair. We first will see, that it does not matter at which point of \mathbb{Z}^n we study the adjacency, since the neighborhoods of all points look the same.

Lemma 16 Let α be a cubical adjacency in \mathbb{Z}^n . For any $p \in \mathbb{Z}^n$ the set $\alpha(p)$ is graph-theoretical isomorphic to $\alpha(0)$.

Proof. This follows from the invariance under translations and the symmetry of the cubical adjacencies. \square

Lemma 17 *Let $M \subset \mathbb{Z}^n$ be α_k -connected. Then M is also α_l -connected for $0 \leq l < k \leq n-1$.*

Proof. Let M be α_k -connected. Thus, we have for any two $p, q \in M$ a path $p = p^{(0)}, \dots, p^{(a)} = q$ such that $p^{(i)} \in M$ for $i \in \{0, \dots, a\}$ and $p^{(i-1)} \in \alpha_k(p^{(i)})$ for $i \in \{1, \dots, a\}$. By definition of α_k , Lemma 12 and $l < k$ holds for $p^{(i-1)}$ and $p^{(i)}$: $|p_j^{(i)} - p_j^{(i-1)}| \leq 1$ and

$$1 \leq \sum_{j=1}^n |p_j^{(i)} - p_j^{(i-1)}| \leq n - k < n - l . \quad (22)$$

Therefore, we have $p^{(i-1)} \in \alpha_l(p^{(i)})$ for $i \in \{1, \dots, a\}$ and the path $p = p^{(0)}, \dots, p^{(a)} = q$ is also a α_l -path. \square

The next Lemmata help us understand, which adjacencies may be used as good pair.

Lemma 18 *Let (α_l, α_k) be a pair of cubical adjacencies on \mathbb{Z}^n , $n \geq 2$. For any n -cube C as in section 3.1, the set $C \cap \alpha_k(0)$ is connected under α_l if the following holds:*

1. $0 \leq k \leq n-2$ and $0 \leq l \leq n-1$, or
2. $k = n-1$ and $0 \leq l \leq n-2$.

Proof. 1. We use Lemma 17 and prove the proposition for $l = n-1$

Let C' be any subcube of C , that does not contain the point 0. We first show that $C' \cap M$ is α_l -connected. Suppose w.l.o.g. that the point $p = (1, 0, \dots, 0)$ is in C' and choose any other point $r \in C' \setminus M$. The point r then has the form $r = (1, r_2, \dots, r_n)$ with

$$\max_{i=1, \dots, n} |r_i| = 1 \text{ and } 1 \leq 1 + \sum_{i=2}^n |r_i| \leq n - k . \quad (23)$$

We select the smallest index $i \in \{2, \dots, n\}$ such that $r_i \neq 0$ and define

$$r' = (1, r_2, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_n) . \quad (24)$$

The point r' is in $\alpha_{n-1}(r)$:

$$\max_{i=1, \dots, n} |r'_i| = 1 \text{ and } 1 \leq \sum_{i=1}^n |r'_i - r_i| = |r'_i - r_i| = 1 \leq n - (n-1) . \quad (25)$$

By iterating this process we get an α_{n-1} -path from r to p .

Let now be C' and C'' be two different $(n-1)$ -cubes. We may suppose w.l.o.g. that $p = (1, 0, \dots, 0) \in C'$ and $q = (0, 1, 0, \dots, 0) \in C''$. The two cubes contain a common

point $t = (1, 1, 0, \dots, 0)$ in M since this point is α_{n-1} -adjacent to p and q and it is in $\alpha_k(0)$ for $0 \leq k \leq n-2$:

$$\max_{i=1, \dots, n} |t_i| = 1 \text{ and } \sum_{i=1}^n |t_i| = 2 \leq n - k . \quad (26)$$

Therefore, the set $C \cap \alpha_k(0)$ is α_{n-1} -connected.

2. We show, that $C \cap \alpha_{n-1}$ is connected under α_{n-2} . By Lemma 17 this is enough.

The set $C \cap \alpha_{n-1}$ contains all points $p^{(i)} = (p_1, \dots, p_n)$, such that exactly one $i \in \{1, \dots, n\}$ exists with $p_i^{(i)} \neq 0$ and $|p_i^{(i)}| = 1$. Let $p^{(i)}$ and $p^{(j)}$ be two such points with $i \neq j$. We have

$$\max_{m=1, \dots, n} |p_m^{(i)} - p_m^{(j)}| = 1 \text{ and } \sum_{m=1}^n |p_m^{(i)} - p_m^{(j)}| = 2 \leq n - (n-2) . \quad (27)$$

Therefore, $p^{(i)}$ and $p^{(j)}$ are α_{n-2} -connected. \square

Corollary 2 *Given a pair (α_l, α_k) , then $\alpha_k(0)$ is α_l -connected, if the following holds:*

1. $0 \leq k \leq n-2$ and $0 \leq l \leq n-1$ or
2. $k = n-1$ and $0 \leq l \leq n-2$.

Proof. This follows from the configuration of the n -cubes in $\omega(0)$ and the distribution of the π -neighbors of 0 in those n -cubes \square

Lemma 19 *Let (α_l, α_k) be a pair of cubical adjacencies on \mathbb{Z}^n with $n \geq 2$. Then the set $\omega(p) \setminus \alpha_k(0)$ has exactly two α_k -components for all $p \in \alpha_k 0$.*

Proof. Obviously, 0 is in $\omega(p)$ for any $p \in \alpha_k(0)$ and it has no other α_k -neighbors in $\alpha_k(0) \setminus \omega(p)$.

We choose any point p in $\alpha_k(0)$. Then, $\omega(p)$ contains points s with $\max_{i=1, \dots, n} |s_i| = 2$. Those are not contained in $\alpha_k(0)$ and form a π -connected set. Therefore they are also α_k -connected.

Define the set:

$$\omega(p)_i := \{s \in \omega(p) : |s_i| = 2\} . \quad (28)$$

W.l.o.g. we consider $\omega(p)_1$ that contains the point $p' = (2, p_2, \dots, p_n)$. It is easy to see, that either the point $p'' = (-2, p_2, \dots, p_n)$ or the point p' is in $\omega(p)_1$.

Let $s = (2, s_2, \dots, s_n)$ be any point in $\omega(p)_i$. We construct a π -path from s to p' by defining the point

$$s' = (2, \dots, p_2, \dots, p_{i-1}, p_i, s_{i+1}, \dots, s_n) \quad (29)$$

with the smallest index $i \in \{2, \dots, n\}$ such that $s_i \neq p_i$. The point s' is a π -neighbor of s and after a finite number of iterations we have the π -path from s to p' . The sets $\omega(p)_i$ and $\omega(p)_j$ contain the points

$$(p_1, \dots, p_{i-1}, 2, p_{i+1}, \dots, p_n) \quad (30)$$

and

$$(p_1, \dots, p_{j-1}, 2, p_{j+1}, \dots, p_n) , \quad (31)$$

respectively. In both sets the point

$$t = (p_1, \dots, p_{i-1}, 2, p_{i+1}, \dots, p_{j-1}, 2, p_{j+1}, \dots, p_n) \quad (32)$$

is contained and therefore, the sets are π -connected.

It remains to show, that points in $\omega(0) \setminus \alpha_k(0) \cap \omega(p)$ are π -adjacent to one of the $\omega(p)_i$. Let $i \in \{1, \dots, n\}$, such that $s_i = p_i \neq 0$. In the case $p_i > 0$, then we have

$$s' = (s_0, \dots, s_{i-1}, s_i + 1, s_{i+1}, \dots, s_n) \in \omega(p)_i \quad (33)$$

and in the case $p_i < 0$, it holds

$$s' = (s_0, \dots, s_{i-1}, s_i - 1, s_{i+1}, \dots, s_n) \in \omega(p)_i . \quad (34)$$

Finally, we have to observe the case of the point s with $0 = s_i \neq p_i$. Then, the point

$$s' = (s_1, \dots, s_{i-1}, p_i, s_{i+1}, \dots, s_n) \quad (35)$$

is a π -neighbor of s , that is not in $\alpha_k(0)$. This follows from

$$n - k < \sum_{j=1}^n |s_j| < \sum_{j=1}^{i-1} |s_j| + |p_i| + \sum_{j=1}^n |s_j| = \sum_{j=1}^n |s_j| + 1 . \quad (36)$$

Thus, in the set $\alpha_k(0) \setminus \omega(p)$, there is only one α_k -component different from 0. \square

Lemma 20 *Let (α_l, α_k) be a pair of cubical adjacencies on \mathbb{Z}^n with $n \geq 2$. For any $p \in \alpha_k(0)$ any point $q \in \alpha_l(p) \cup \alpha_k(0)$ is α_k -adjacent to both of the α_k -components of $\omega(p) \setminus \alpha_k(0)$, if the following holds:*

1. $1 \leq k \leq n-1$ and $0 \leq l \leq n-1$
2. $0 \leq k \leq n-2$ and $l = n-1$

Proof. 1. Let $p \in \alpha_k(0)$ and $q \in \alpha_l \cap \alpha_k(0)$ be arbitrary chosen. Since $q \in \alpha_k(0)$, the point q is α_k -adjacent to the α_k -component $\{0\}$.

We define the set

$$I(p, q) := \{1 \leq i \leq n : p_i = q_i = 0\} . \quad (37)$$

This set is non-empty since $1 \leq k \leq n-1$. Let $q' = (q'_1, \dots, q'_n)$ be the point with the following coordinates

$$q'_i = \begin{cases} q_i & \text{if } q_i \neq 0 \\ p_i & \text{if } q_i = 0 \text{ and } p_i \neq 0 \end{cases} \quad (38)$$

The remaining q'_i for $i \in I(p, q)$ will be assigned with the values ± 1 and 0 such that exactly $n - k + 1$ of n coordinates are different from 0.

The point q' is not contained in $\alpha_k(0)$, since

$$\sum_{i=1}^n |q'_i| = n - k + 1 > n - k . \quad (39)$$

Because of $|q'_i - p_i| \leq 1$, the point q' must be in $\omega(p)$. Therefore, the point q' is in $\omega(p) \setminus (\alpha_k(0) \cup \{0\})$. Since q is not 0, we have

$$\sum_{i=1}^n |q'_i - q_i| = \sum_{i=1}^n |q'_i| - \sum_{i=1}^n |q_i| \leq n - k . \quad (40)$$

Therefore, the point q' is in $\alpha_k(q)$ and the set $\omega(p) \setminus (\alpha_k(0) \cup \{0\})$ is α_k -adjacent to q .

For $k = n - 1$ and $l = n - 1$ the set $\alpha_l(p) \cap \alpha_k(0)$ is empty. Thus the proposition is true.

We may not choose k as 0 for $0 \leq l \leq n - 2$, as the following example shows: Let $p \in \pi(0)$ and $q \in \alpha_l(p) \cap \pi(0)$, then

$$\sum_{i=1}^n |p_i - q_i| = 2 \leq n - l , \quad (41)$$

and thus $q \in \alpha_l(p)$. The point q has no α_0 -neighbors in $\omega(p) \setminus \alpha_0(0)$, because of $\alpha_0 = \omega$ and because $r_i = \pm 2$ for $p_i = \pm 1$, hold for all $r \in \omega(p) \setminus (\omega(0) \cup \{0\})$.

2. Let p be a point in $\alpha_k(0)$ and let q be any point in $\alpha_{n-1}(p) \cap \alpha_k(0)$. Obviously, the point q is α_k -adjacent to $\{0\}$.

We choose

$$i \in \{1 \leq i \leq n : p_i \neq 0 \text{ and } q_i \neq 0\} . \quad (42)$$

This is a non-empty set, because the points q and p coincide in at least one non-zero coordinate, since $q \in \alpha_{n-1}(p) \cap \gamma_k(0)$. We define

$$q' = \begin{cases} (q_1, \dots, q_{i-1}, q_i + 1, q_{i+1}, \dots, q_n) & \text{if } p_i = 1 \\ (q_1, \dots, q_{i-1}, q_i - 1, q_{i+1}, \dots, q_n) & \text{if } p_i = -1 \end{cases} \quad (43)$$

We have $|q'_j - p_j| \leq 1$ for all $j \in \{1, \dots, n\}$. Therefore, the point q' is in $\omega(p)$. Thus, $|q_i| = 2$, otherwise $|q_i - p_i|$ would not be smaller or equal to 1. We conclude that q' is no point in $\alpha_k(0)$ and $q' \neq 0$. The point q' is therefore a member of $\omega(p) \setminus (\alpha_k(0) \cup \{0\})$ and it is a π -neighbor of q . Which means it is an α_k -neighbor, too. Thus, the point q is α_k -adjacent to $\omega(p) \setminus (\alpha_k(0) \cup \{0\})$. \square

Lemma 21 *Let (α_l, α_k) be a pair of cubical adjacencies. Then the set $\alpha_k(0)$ satisfies the separation property under this pair.*

Proof. Instead of $\alpha_k(0)$, we consider the set $\bar{\alpha}_k(0) = \alpha_k(0) \cup \{0\}$. We may do so, because the point 0 is a different α_k -component of $\alpha_k(0)^C$. It is not separable and therefore has no influence of the separability of the other points. If we know whether $\bar{\alpha}(0)$ has the separation property, then we also know that $\alpha_k(0)$ has it too.

Let C be a m -cube, $2 \leq m \leq n$, containing a point of $\bar{\alpha}_k(0)$ and let C^* be a $(m-2)$ -subcube of C that has a maximal intersection with $\bar{\alpha}_k(0)$. There exist two translations τ_1 and τ_2 such that we may decompose C in the following way:

$$C = C^* \cup \tau_1(C^*) \cup \tau_2(C^*) \cup \tau_1\tau_2(C^*) . \quad (44)$$

Case 1: The translated cubes $\tau_i(C^*)$, $i = 1, 2$ and $\tau_1\tau_2(C^*)$ each contain a point p such that $\max_{i=1,\dots,n} |p_i| = 2$. Then, one can easily see that $\tau_1\tau_2(C^*)$ is fully in M^C . And so we have

$$(\tau_1\tau_2)^{-1}(\tau_1\tau_2(C^*) \cap M) \subset \tau_1^{-1}(\tau_1(C^*) \cap M) \cap \tau_2^{-1}(\tau_2(C^*) \cap M) \subset C^* \cap M . \quad (45)$$

This is true, especially if $\tau_i(C^*) \setminus \bar{\alpha}_k(0) \neq \emptyset$, $i = 1, 2$ and both of the set are α_k -connected.

Case 2: The cube C contains no point p such that $\max_{i=1,\dots,n} |p_i| = 2$. Let q be the point in C^* that satisfies $\sum_{i=1}^n |q_i| = x$ and $0 \leq x \leq n-k$ be minimal in C . It is sufficient to claim this minimality as the following consideration shows: We have:

$$C^* \cap M = \left\{ p : \sum_{i=1}^n |p-i| \leq \min(m-2, n-k) - x \right\} \quad (46)$$

$$\tau_{1,2}C^* \cap M = \left\{ p : \sum_{i=1}^n |p-i| \leq \min(m-3, n-k-1) - x \right\} \quad (47)$$

$$\tau_1\tau_2(C^*) \cap M = \left\{ p : \sum_{i=1}^n |p-i| \leq \min(m-4, n-k-2) - x \right\} \quad (48)$$

The cube C^* has always a maximal number of points in $\bar{\alpha}_k(0)$. If $\tau_{1,2}(C^*)$ and $\tau_1\tau_2(C^*)$, respectively contain a maximal number of points in $\bar{\alpha}_k(0)$, so they are both contained in $\bar{\alpha}_k(0)$.

Therefore, we have the following inclusions:

$$(\tau_1\tau_2)^{-1}(\tau_1\tau_2(C^*) \cap M) \subset \tau_1^{-1}(\tau_1(C^*) \cap M) \cap \tau_2^{-1}(\tau_2(C^*) \cap M) \subset C^* \cap M . \quad (49)$$

This chain is correct especially if $\tau_i(C^*) \setminus \bar{\alpha}_k(0) \neq \emptyset$, $i = 1, 2$ and both set are α_k -connected.

In both cases the separation property follows. \square

Lemma 22 *It holds:*

1. The set $\alpha_{n-1}(0) \subset \mathbb{Z}^n$ contains no α_k -double points for $0 \leq k \leq n-1$.
2. The set $\alpha_k(0) \subset \mathbb{Z}^n$ contains no α_{n-1} -double points for $0 \leq k \leq n-2$.

Proof. 1. Let p be in $\alpha_{n-1}(0)$. Then, the point p has the form $(0, \dots, 0, \pm 1, 0, \dots, 0)$. It cannot contain any π -neighbors $r = (r_1, \dots, r_n)$ in $\alpha_{n-1}(0)$, because these satisfy

$$\sum_{i=1}^n |p_i - r_i| = 1 . \quad (50)$$

The point r cannot be 0 and satisfies:

$$\sum_{i=1}^n |r_i| = 2 . \quad (51)$$

Therefore, no neighbor of p can be contained in $\alpha_{n-1}(0)$ and no p exists, which satisfies the definition 17.

2. We need to show, that for no $p \in \alpha_k(0)$ with $0 \leq k \leq n-2$, exist two points $r \in \pi(0) \cap \alpha_{n-2}(p)$ and $q \in \pi(0) \cap \alpha_k(0)$ and a translation σ with $\sigma(r) = 0$ and $\sigma(q) = p$ such that $r \in \alpha_{n-1}(p)$.

Assume for contradiction that such a configuration exists. Then, the two points q and r are α_{n-1} -adjacent. Therefore, it holds:

$$\sum_{i=1}^n |r_i - q_i| = 1, |r_i - q_i| \leq 1 \text{ for } 1 \leq i \leq n . \quad (52)$$

It follows the existence of a j in $\{1, \dots, n\}$ such that $r_j \neq q_j$ and $r_i = q_i$ for all other indices i . Furthermore, the point q is in $\pi(0)$ and it can be written as $(0, \dots, \pm 1, 0, \dots, 0)$ with $q_l = \pm 1$ and we know that $q_l = r_l$. From $\sigma(r) = 0$ it follows that $(-\sigma)(q) = p = (q_1 - r_1, \dots, q_n - r_n)$ and therefore, the point p has the form

$$p = (0, \dots, 0, q_j - r_j, 0, \dots, 0) . \quad (53)$$

In addition, r is an element of $\pi(p)$ and $|p_i - r_i| \leq 1$ for all $1 \leq i \leq n$. But this cannot be the case, since

$$|p_j - r_j| = |q_l - 2r_l| = |-2r_l| = 2 \text{ since } r_l \neq 0 . \quad (54)$$

This contradicts the assumption and the Lemma is proven. \square

Lemma 23 Given a pair (α_l, α_k) on \mathbb{Z}^n with $n \geq 2$, the set $\alpha_k(0) \subset \mathbb{Z}^n$ contains α_l -double points for all $0 \leq k \leq n-2$ and $0 \leq l \leq n-2$.

Proof. Consider the point $p = (1, 1, 0, \dots, 0) \in \alpha_k(0)$ with k conforming the precondition. The point $q = (1, 0, \dots, 0)$ is in $\pi(0)$ and the point $r = (0, 1, 0, \dots, 0)$ is in $\pi(p)$. In addition a translation σ exists such that $\sigma(z) = r$ and $\sigma(p) = q$.

Because of $q \in \alpha_l(r)$ for $0 \leq l \leq n-2$, the Lemma is true. \square

We now have all the tools in our hands to state the final Theorem on the good pairs of cubical ajacencies. This Theorem gives us a complete characterization of this kind of good pairs in \mathbb{Z}^n for all dimensions n at least 2.

Theorem 4 A pair of cubical adjacencies (α_l, α_k) in \mathbb{Z}^n is a good pair, if

1. $k = n - 1$ and $0 \leq l \leq n - 2$,
2. $0 \leq k \leq n - 2$ and $l = n - 1$.

There are no other good pairs of cubical adjacencies.

Proof. We need to show that $\alpha_k(0)$ is a $(n - 1)$ -manifold in \mathbb{Z}^n that contains no α_l -double points. Lemma 22 gives the pairs (α_l, α_k) without double points. Corollary 2 shows that $\alpha_k(0)$ is a $(n - 1)$ -manifold under α_l , this is enough because of the invariance under translation of α_k . And from Lemma 23 we know which pairs of cubical adjacencies have double points. \square

4.3 The Khalimsky-Topology as Good Pair of Adjacencies

In this section we will show, that the notion of an Alexandrov-space and the graph-theoretic framework common to digital geometry may be put under a common umbrella. We will see, that the Khalimsky-topology κ_n on the set \mathbb{Z}^n might be considered as a pair of adjacencies (κ_n, κ_n) , and that these pairs are good ones.

Basing on Theorem 2 we may consider a graph structure on \mathbb{Z}^n given by the topology κ_n . We denote this graphical adjacency also with κ_n . Also, remember the equations 2 and 3.

Lemma 24 For any $p, q \in \mathbb{Z}^n$ holds:

$$p \in C_{\kappa_n}(q) \Leftrightarrow \bigwedge_{i=1}^n (p_i \geq q_i \pmod{2}) :\Leftrightarrow p \succeq q \quad (55)$$

$$p \in U_{\kappa_n}(q) \Leftrightarrow \bigwedge_{i=1}^n (p_i \leq q_i \pmod{2}) :\Leftrightarrow p \preceq q \quad (56)$$

Proof. This is Theorem 8 in Evako et al. [3]. \square

The Khalimsky-adjacency κ_n may now be represented in the following way:

$$\kappa_n(p) := \left\{ q \in \mathbb{Z}^n : \max_{i=1, \dots, n} |q_i - p_i| = 1, p \preceq q \vee q \preceq p \right\}. \quad (57)$$

Lemma 25 For all $n \geq 1$ holds: $\pi \subset \kappa_n$.

Proof. Let p, q be two points in \mathbb{Z}^n such that $p \in \pi(q)$. By definition of π we have

$$\max_{i=1, \dots, n} |p_i - q_i| = 1 \text{ and } \sum_{i=1}^n |p_i - q_i| \leq 1 \quad (58)$$

Therefore, exactly one $i \in \{0, \dots, n\}$ exists with $q_i = p_i + 1$ or $q_i = p_i - 1$. For all $j \in \{1, \dots, n\}$, $j \neq i$ is $p_i = q_i$. We have:

$$\bigwedge_{i=1}^n (p_i \leq q_i \pmod{2}) \text{ or } \bigwedge_{i=1}^n (p_i \geq q_i \pmod{2}) . \quad (59)$$

And so, $p \in \kappa_n(q)$. \square

We are not in the convenient position to find a reference point like 0 for the cubical adjacencies. The next Lemma clarifies this fact.

Lemma 26 For each $p \in \mathbb{Z}^n$ exists a translation τ such that $\kappa_n(\tau(p)) \neq \tau(\kappa_n(p))$.

Proof. By construction of the Khalimsky-topology this Lemma is obviously true: Let τ be any translation of the form $(0, \dots, 0, 1, 0, \dots, 0)$ in \mathbb{Z}^n . Then, $\tau(p)$ is either odd in a component where p is even or vice versa. In both cases, the point $\tau(p)$ has a neighborhood different from the one of p . \square

We are able to make some statements about the interaction of certain translations and κ_n .

Lemma 27 Let p and q be two points in \mathbb{Z}^n , $I = \{i : p_i = q_i\}$ and let τ be a translation with $|\tau(0)_i| \leq 1$ for $i \in I$ and $\tau(0)_i = 0$ otherwise. Then holds

$$p \preceq q \Leftrightarrow \tau(p) \preceq \tau(q) . \quad (60)$$

Proof. (\Rightarrow) Let $p \preceq q$. Then holds $p_j \leq q_j \pmod{2}$ for all $j \notin I$. Because of $p_i = q_i$ we deduce $p_i \pm 1 = q_i \pm 1 \pmod{2}$. Therefore, it holds $\tau(p) \preceq \tau(q)$. (\Leftarrow) Analog. \square

Lemma 28 For all $p \in \mathbb{Z}^n$, $n \geq 2$, the set $\kappa_n(p)$ is κ_n -connected.

Proof. The Lemma follows by Definition 4 and Theorem 11 in Evako et al. [3]. \square

From the proof of Theorem 11 in Evako et al. [3] we get

Lemma 29 For all $p \in \mathbb{Z}^n$, $n \geq 2$, every n -cube, that contains points from $\kappa_n(p)$, is κ_n -connected. \square

Lemma 30 For all $p \in \mathbb{Z}^n$, $n \geq 2$, and all $q \in \kappa_n(p)$ the set $\omega(q) \setminus \kappa_n(p)$ has exactly two κ_n -components C_q and D_q .

Proof. Let p and q be the same as in the last Lemma. A κ_n -component of $\omega(q) \setminus \kappa_n(p)$ is $\{p\}$, because p has in $\omega(q)$ only neighbors $\kappa_n(p)$. We denote this component by C_q .

Now define

$$\omega(q)_i := \{r \in \omega(q) : |r_i - p_i| = 2\} \quad (61)$$

We will show that this set is π -connected for all $1 \leq i \leq n$. We prove the result w.l.o.g. for $i = 1$.

The point

$$r = (p_1 + 2, p_2, \dots, p_n) \quad (62)$$

is in $\omega(q)_1$. Let $r' \neq r$ be any point in $\omega(q)_i$ and let $i \in \{2, \dots, n\}$ be the smallest index such that $r_i \neq r'_i$. We construct a π -path from r' to r . The point

$$r'' = (r_1, \dots, r_i, r'_{i+1}, \dots, r'_n) \quad (63)$$

is a π -neighbor of r' , because, both points differ according to the choice of i only in the i -th coordinate by 1. If $r'' = r$ the path is constructed, otherwise we iterate the algorithm with r'' in place of r' . After at most $n - 1$ steps the π -path is constructed.

If the intersection of two sets $\omega(q)_i$ and $\omega(q)_j$ is non-empty, then it is π -connected, too.

Let r be a point in the set

$$D_q := \omega(q) \cap (\omega((p) \setminus (\kappa_n(p) \cup \{p\}))) . \quad (64)$$

If $\omega(q)_i \neq \emptyset$ and $r_i = q_i$ for this $i \in \{1, \dots, n\}$, then r_i is the π -neighbor of some point in $\omega(q)_i$.

Otherwise, an i exists such that $\omega(q)_i \neq \emptyset$. From $r_i \neq q_i$ follows $|r_i - q_i| = 1$ and therefore $r_i = p_i$. We may define the point

$$s = (r_1, \dots, r_{i-1}, q_i, r_{i+1}, \dots, r_n) . \quad (65)$$

The points r, s are in $\omega(p)$, so we have

$$\max_{i=1, \dots, n} |r_i - p_i| = 1 \text{ and } \max_{i=1, \dots, n} |s_i - p_i| = 1 . \quad (66)$$

Since the point r is no member of $\kappa_n(p)$, it follows:

$$\bigvee_{j_1} (r_{j_1} > p_{j_1}) \text{ and } \bigvee_{j_2} (r_{j_2} < p_{j_2}) . \quad (67)$$

The indices j_1 and j_2 are distinct. From $r_i = p_i$ follows, that j_1, j_2 are both dissimilar to i . Therefore we have for s :

$$s_{j_1} = r_{j_1} > p_{j_1} \text{ and } s_{j_2} = r_{j_2} < p_{j_2} , \quad (68)$$

which gives $s \notin \kappa_n(p)$. Thereby, s is in D_q and D_q is the second κ_n -component of the set $\omega(q) \setminus \kappa_n(p)$. \square

Lemma 31 For all $p \in \mathbb{Z}^n$ with $n \geq 2$ and any $q \in \kappa_n(p)$, all the points $r \in \kappa_n(p) \cap \kappa_n(q)$ are κ_n -adjacent to the sets C_q and D_q .

Proof. It obvious, that all points $r \in \kappa_n(p) \cap \kappa_n(q)$ are κ_n -adjacent to the set $C_q = \{p\}$. So it remains to show, that r is also κ_n -adjacent to D_q .

Case 1: For some index i in $\{0, \dots, n\}$ holds that $r_i = q_i$ and the set $\omega(q)_i$ is not empty. Then, the point r is π -adjacent to D_q .

Case 2: It is $r_i \neq q_i$ for all i such that $\omega(q)_i \neq \emptyset$. Consider the set $I = \{i : \omega(q)_i \neq \emptyset\}$. We show that the point

$$s \text{ with } s_i = \begin{cases} q_i & \text{if } i \in I \\ r_i & \text{if } i \in \{1, \dots, n\} \setminus I \end{cases} \quad (69)$$

is no member of $\kappa_n(p)$ under this preconditions. Since s can be identified as κ_n -adjacent to r , the point r is κ_n -adjacent to D_p .

The point s is distinct from q by definition of I and $r \neq p$. We know that $r_i = p_i$ for $i \in I$ because of $r_i \neq q_i$ and $|r_i - p_i| < 2$ and r is a member of $\omega(q)$. The set $\omega(q)_i$ is non-empty if and only if $q_i \neq p_i$. Therefore, a translation τ exists such that

$$\tau(0)_i = \begin{cases} \pm 1 & \text{if } i \in \{1, \dots, n\} \setminus I \\ 0 & \text{if } i \in I \end{cases} \quad (70)$$

We may choose τ such that $\tau(q) = p$ and $\tau(s) = r$.

The point q is in $\kappa_n(p)$. Suppose w.l.o.g. that $q \preceq p$. Therefore, we get $q_i < p_i = r_i \pmod{2}$ for $i \in I$. Then follows $q \preceq r$ with $r \in \kappa_n(q)$. By definition of s and the fact $r \neq p$, it holds that $q \preceq s$ and by Lemma 27 we get

$$\tau(q) = p \preceq r = \tau(s) . \quad (71)$$

So we can find a j with $s_j = r_j > p_j \pmod{2}$ and $j \notin I$. But at the same time $q_i = s_i < p_i \pmod{2}$ for all $i \in I$. Therefore the point s cannot be contained in $\kappa_n(p)$. We have to show that r and s are κ_n -neighbors: For $i \notin I$ we have $r_i = s_i$ and for $i \in I$ it holds

$$s_i = q_i < p_i = r_i \pmod{2} \quad (72)$$

and so follows $s \preceq r$ which means $s \in \kappa_n(r)$. \square

For the proof of the separation property we consider the set $\bar{\kappa}_n(p) = \kappa_n(p) \cup \{p\}$ for all $p \in \mathbb{Z}^n$, $n \geq 2$ instead of $\kappa_n(p)$. This is reasonable, since the point p lies in no separable component of the complement of $\kappa_n(p)$ in \mathbb{Z}^n . If we have the result for the modified set we may easily translate it for the original one.

Lemma 32 *Let C be any k -cube $\omega(p) \cup \{p\}$, $0 \leq k \leq n$ and let q be the point in C with minimal π -distance⁴ to p . For $q \preceq p$ and all $q' \in C \setminus \{q\}$ holds*

$$q' \preceq q \Leftrightarrow q' \preceq p . \quad (73)$$

An analog claim holds for $q \succeq p$.

⁴The π -distance of two points p and q is the infimum over the length of all π -paths from p to q .

Proof. (\Rightarrow) This direction of the proof follows by transitivity of the partial order \preceq .
(\Leftarrow) The points q' and q are contained in the same k -cube C and it holds that

$$\sum_{i=1}^n |q_i - p_i| = k < l = \sum_{i=1}^n |q'_i - p_i| . \quad (74)$$

After rearranging the coordinates of q, q' and p , we get

$$q' = (q_1, \dots, q_k, q'_{k+1}, \dots, q'_l, p_{l+1}, \dots, p_n) \quad (75)$$

$$q = (q_1, \dots, q_k, p_{k+1}, \dots, p_l, p_{l+1}, \dots, p_n) . \quad (76)$$

From $q' \preceq p$ now follows $q \preceq p$ by the definition of \preceq . \square

Lemma 33 *Let C be a k -cube $2 \leq k \leq n$ and q be a point in C with minimal π -distance to p and $q \preceq p$. For all $q' \in \pi(q) \cap C$ holds $q' \preceq q$ if and only if C is contained in the set $\overline{\kappa}_n(p)$.*

An analog claim holds for $q \succeq p$.

Proof. (\Leftarrow) Since $C \subset \overline{\kappa}_n(p)$, all the points $q' \in C \cap \pi(q)$ are in $\kappa_n(p)$. Therefore, they satisfy $q' \preceq q$ or $q' \succeq q$. If there exists a $q' \preceq q$ and a $q'' \succeq q$, so we have

$$\bigvee_{i_1} (q'_{i_1} < q_{i_1} \pmod{2}) \text{ and } \bigvee_{i_2} (q''_{i_2} > q_{i_2} \pmod{2}) . \quad (77)$$

The point $q''' = \tau_{i_1} \tau_{i_2}(q)$ then satisfies

$$\bigvee_{i_1} (q'''_{i_1} < q_{i_1} \pmod{2}) \text{ and } \bigvee_{i_2} (q'''_{i_2} > q_{i_2} \pmod{2}) . \quad (78)$$

Therefore holds $q''' \not\preceq q$ and $q''' \not\succeq q$ and the point q''' no member of $\overline{\kappa}_n(p)$. So for all points $q' \in C \cap \pi(q)$ the relation $q' \preceq q$ holds.

(\Rightarrow) We prove by induction on k . In the case $k = 2$ holds $q \preceq p$ and for all $q' \in C \cap \pi(q)$ holds $q' \preceq q \preceq p$. The two π -neighbors q_1 and q_2 of q in C are in $\overline{\kappa}_n(p)$. This means that $q_1 = \tau_1(q) \preceq q$ and $q_2 = \tau_2(q) \preceq q$. Therefore, we have

$$\tau_1(\tau_2(q)) \preceq \tau_1(q) \preceq p . \quad (79)$$

We conclude, that C is contained in $\overline{\kappa}_n(p)$.

For the induction step $k > 2$ we let $C = C' \cup \tau(C')$ for certain $(k-1)$ -cubes $C', \tau(C')$ and a translation τ . Let q be in C w.l.o.g. Since all the points q' in $\pi(q) \cap C$ satisfy the relation $q' \preceq q$, the $(k-1)$ -cube C' has to be contained by induction hypothesis in $\overline{\kappa}_n(p)$. For all $q'' \in C'$ holds $q'' \preceq q$. Therefore, by Lemma 27, we find for all $\tau(q'') \in \tau(C')$:

$$\tau(q'') \preceq \tau(q) \preceq q \preceq p . \quad (80)$$

It follows that $C \subset \overline{\kappa}_n(p)$. \square

Corollary 3 Let C be a k -cube, $2 \leq k \leq n$ and q be the point with minimal π -distance to p . Then, all the subcubes C' of C such that $q' \preceq q \preceq p$ or $q' \succeq q \succeq p$ for all $q' \in C' \cap \pi(q)$, are contained in $\bar{\kappa}_n(p)$.

For $q \neq p$ only one of these cases applies. \square

Lemma 34 The set $\bar{\kappa}_n(p)$ has the separation property under the pair (κ_n, κ_n) for any cube $C \subset (\omega(p) \cup \{p\})$ with $p \notin C$.

Proof. We consider three cases. The first case is, that C is contained in $\bar{\kappa}_n(p)$. The separation property is obviously satisfied in this case.

Case 2: Let the k -cube C be of the form $C' \cup \tau(C')$ with C' a $(k-1)$ -cube contained in $\bar{\kappa}_n(p)$. In this case the set $\tau(C')$ contains no points q' in $\bar{\kappa}_n(p)$, since otherwise these points would satisfy $q' \leq q \leq p$. Particularly, the point $\tau(q)$ is not in $\bar{\kappa}_n(p)$.

Every $(k-2)$ -cube C'' with $C'' \cap \bar{\kappa}_n(p)$ is in C' . Then, the set $\tau_1(C'') \subset C'$ is also contained in $\bar{\kappa}_n(p)$. Therefore the separation property holds in C .

Case 3: There is only one $(k-2)$ -subcube C' of C that contains all points of $C \cap \bar{\kappa}_n(p)$. Then we get $\tau_1 \tau_2(C') \cap \bar{\kappa}_n(p) = \emptyset$. Therefore it holds

$$(\tau_1 \tau_2)^{-1}(\tau_1 \tau_2(C') \cap \bar{\kappa}_n(p)) \subset (\tau_1^{-1}(\tau_1(C') \cap \bar{\kappa}_n(p))) \cap (\tau_2^{-1}(\tau_2(C') \cap \bar{\kappa}_n(p))) . \quad (81)$$

And so, the separation property holds. \square

Lemma 35 The set $\bar{\kappa}_n(p)$ has the separation property for the pairs (κ_n, κ_n) for cubes $C \subset \omega(p) \cup \{p\}$ with $p \in C$.

Proof. Case 1: The separation property is satisfied for $C \subset \bar{\kappa}_n(p)$.

Case 2: For a k -cube C of the form $C' \cup \tau(C')$ such that $C' \subset \bar{\kappa}_n(p)$ only the point $\tau(p)$ is in $\bar{\kappa}_n(p)$, because, if for all $q \in C'$ the relation $q \succeq p$ is true, then it holds for $\tau(q) \in \tau(C')$ that

$$\tau(q) \succeq \tau(p) \preceq p . \quad (82)$$

Since $\tau(p)$ has minimal π -distance to p in $\tau(C')$, none of the aforementioned $\tau(q)$ can be contained in $\bar{\kappa}_n(p)$.

Let $C'' \subset C'$ be any $(k-2)$ -cube. Then, the set $C'' \cap \bar{\kappa}_n(p)$ is maximal with respect to inclusion in C . In turn, the set $\tau_1(C'') \setminus \bar{\kappa}_n(p)$ is empty and the separation property holds for C .

Case 3: Consider the k -cube $C = C' \cup \tau_1(C') \cup \tau_2(C') \cup \tau_1 \tau_2(C')$ and let $C' \cap \bar{\kappa}_n(p)$ be maximal with respect to inclusion. Since we are not in case 2, we have $\tau_1(C') \setminus \bar{\kappa}_n(p) \neq \emptyset$. The $(k-1)$ -cube C' has a l -subcube, $0 \leq l < k-1$, that is contained in $\bar{\kappa}_n(p)$, the point p has to be in C' liegen. Now, either all points $q \in C'$ are in relation $q \preceq p$ or they satisfy $q \succeq p$. W.l.o.g. we use the first relation.

All the points in $q \in \tau_i(C')$, $i = 1, 2$, are in the relation $q \succeq p$, since otherwise, we had $C' \cup \tau_i(C') \subset \bar{\kappa}_n(p)$. Now, we have $\tau_1 \tau_2(p) \succeq \tau_1(p) \tau_2(p) \succeq p$. Likewise, all the translations τ , that generate the $(k-1)$ -cube C' , satisfy by Lemma 27:

$$\tau_1 \tau_2(\tau(p)) \succeq \tau_1(\tau(p)), \tau_2(\tau(p)) \succeq \tau(p) \quad (83)$$

Therefore, only the points $\tau_1\tau_2(\tau(p))$ and $\tau_i(\tau(p))$, $i = 1, 2$ are in $\bar{\kappa}_n(p)$, if $\tau(p) \succeq p$ holds. So we have

$$(\tau_1\tau_2)^{-1}(\tau_1\tau_2(C') \cap \bar{\kappa}_n(p)) \subset (\tau_1^{-1}(\tau_1(C') \cap \bar{\kappa}_n(p))) \cap (\tau_2^{-1}(\tau_2(C') \cap \bar{\kappa}_n(p))) \quad (84)$$

and the separation property holds in C . \square

Corollary 4 *The set $\bar{\kappa}_n(p)$ has the separation property under the pair (κ_n, κ_n) .*

Proof. The claim follows from the Lemmata 34 and 35 for cubes $C \subset \omega(p) \cup \{p\}$.

For any cube C that is not contained in $\omega(p) \cup \{p\}$, the separation property holds, because C has the form $C' \cup \tau_1(C') \cup \tau_2(C') \cup \tau_1\tau_2(C')$ and the set $\tau_1\tau_2(C') \cap \bar{\kappa}_n(p)$ is always empty, since $C' \subset \omega(p) \cup \{p\}$ is true if we maximize the set $C' \cap \bar{\kappa}_n(p)$ with respect to inclusion. In the case $\tau_1(C) \setminus \bar{\kappa}_n(p) = \emptyset$ the separation property holds trivially. For $\tau_1(C) \setminus \bar{\kappa}_n(p) \neq \emptyset$ this is also true because of

$$(\tau_1\tau_2)^{-1}(\tau_1\tau_2(C') \cap \bar{\kappa}_n(p)) = \emptyset \subset (\tau_1^{-1}(\tau_1(C') \cap \bar{\kappa}_n(p))) \cap (\tau_2^{-1}(\tau_2(C') \cap \bar{\kappa}_n(p))) \quad (85)$$

And so the separation property holds again. \square

Theorem 5 *For all $p \in \mathbb{Z}^n$, $n \geq 2$ the set $\kappa_n(p)$ is a $(n-1)$ -manifold.*

Proof. The first three properties of a digital $(n-1)$ -manifold are shown in the Lemmata 29 to 31 and the separation property is proven in Corollary 4. \square

Lemma 36 *Given the pair (κ_n, κ_n) on \mathbb{Z}^n , $n \geq 2$, and any point $p \in \mathbb{Z}^n$, the set $\kappa_n(p)$ contains no κ_n -double points.*

Proof. Assume for contradiction, we have the points $z \in \kappa_n(p)$, $q \in \kappa_n(p) \cap \pi(z)$ and $r \in \kappa_n(z) \cap \pi(p)$, and $q = \sigma(p)$ and $z = \sigma(r)$ for a simple translation σ .

The point z is in $\kappa_n(p)$ and so we have $z \preceq p$ or $p \preceq z$. We consider w.l.o.g. the case $z \preceq p$. We have exactly one $i \in \{1, \dots, n\}$ such that

$$z_i = r_i < p_i \pmod{2} \quad (86)$$

Therefore, it holds that

$$q_i = p_i > r_i \pmod{2} \quad (87)$$

Furthermore, we can find a $j \in \{1, \dots, n\}$ such that

$$z_j = q_j < p_j = r_j \pmod{2} \quad (88)$$

It follows that $q_i > r_i \pmod{2}$ and $q_j < r_j \pmod{2}$. Therefore neither $q \preceq r$ nor $r \preceq q$ may be true. This contradicts the assumption that $q \in \kappa(p)$ and so no double points may occur. \square

Theorem 6 *The pair (κ_n, κ_n) is a good pair on \mathbb{Z}^n for all $n \geq 2$.*

Proof. The proof follows with Theorem 5 and Lemma 36. □

5 Conclusions

We have shown that the cubical adjacencies and the khalimsky-topology give good pairs. This was already known, for instance G.T. Herman proved this in his book [4]. The difference here is, that our theory resembles more closely the euclidean case and surfaces are really subsets of the given space. We also could give a slight unification of the topological with the graph-theoretic setting, although this was already present in the disguise of Alexandrov-spaces, for these have an graph-theoretic interpretation via partial orders. It is possible to give proofs for other adjacency relations to be good pairs, for instance the hexagonal adjacencies also give good pairs, as G.T. Herman shows in the same book. It may be also possible to give good pairs of more complicated adjacency relations, but then, the proofs might tend to get even more technical than the ones we saw we saw in this paper.

References

- [1] Paul Alexandroff. *Diskrete Räume*. Math. Sb. 1937.
- [2] P.S. Alexandrov. *Combinatorial Topology. Volumes 1,2 and 3*. Reprint, Dover Publications, Inc. Mineola, New York, 1998.
- [3] A.V.Evako, R.Kopperman, Y.V.Mukhin. *Dimensional Properties of Graphs and Digital Spaces*. Journal of Mathematical Imaging and Vision 6, 109-119 (1996).
- [4] Gabor T. Herman, *Geometry of Digital Spaces*. Birkhäuser Boston, 1998.
- [5] Gabor T. Herman, Enping Zhao. *Jordan Surfaces in Simply Connected Digital Spaces*. Journal of Mathematical Imaging and Vision 6, 121-138, 1996.
- [6] Martin Hünninger. *Digital Manifolds and the Theorem of Jordan-Brouwer* <http://arxiv.org/abs/1111.3000>, 2011.
- [7] Efim Khalimsky. *Topological structures in computer science*. J. Appl. Math. and Sim. Vol 1, 25-40, 1987.
- [8] Mohammed Khachan, Patrik Chenin, Hafsa Deddi. *Digital pseudomanifolds, digital weakmanifolds and Jordan-Brouwer separation theorem*. Disc. Appl. Mathematics 125, 45-57, 2003.
- [9] Christer O. Kiselman. *Digital Jordan Curve Theorems. Discrete Geometry for Computer Imagery*. Gunilla Borgefors, Ingela Nyström, Gabriella Sanniti di Baja (Eds.), 9th International Conference, DGCI 2000, Uppsala, Sweden, December 1315, 2000, pp. 4655. Lecture Notes in Computer Science 1953. Springer, 2000.
- [10] Reinhardt Klette, Azriel Rosenfeld. *Digital Geometry*. Morgan Kaufmann Publishers, 2004.

- [11] Azriel Rosenfeld, T.Y. Kong. *Digital topology: introduction and survey*. Computer Vision Graphics and Image Processing 48 (3), 357-393, 1989.
- [12] Azriel Rosenfeld. *Arcs and Curves in digital pictures*. J. ACM 20, 81-87, 1973.
- [13] Ralph Stöcker, Heiner Zieschang. *Algebraische Topologie*. 2. Überarbeitete Auflage. Stuttgart: Teubner, 1994.