Completeness and Decidability Properties for Functional Dependencies in XML

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Abstract

XML is of great importance in information storage and retrieval because of its recent emergence as a standard for data representation and interchange on the Internet. However XML provides little semantic content and as a result several papers have addressed the topic of how to improve the semantic expressiveness of XML. Among the most important of these approaches has been that of defining integrity constraints in XML. In a companion paper we defined strong functional dependencies in XML(XFDs). We also presented a set of axioms for reasoning about the implication of XFDs and showed that the axiom system is sound for arbitrary XFDs. In this paper we prove that the axioms are also complete for unary XFDs (XFDs with a single path on the l.h.s.). The second contribution of the paper is to prove that the implication problem for unary XFDs is decidable and to provide a linear time algorithm for it.

1 Introduction

The eXtensible Markup Language (XML) [6] has recently emerged as a standard for data representation and interchange on the Internet [23, 1]. While providing syntactic flexibility, XML provides little semantic content and as a result several papers have addressed the topic of how to improve the semantic expressiveness of XML. Among the most important of these approaches has been that of defining integrity constraints in XML [8, 16]. Several different classes of integrity constraints for XML have been defined including key constraints [7, 8, 9], path constraints [2, 12, 8, 11], and inclusion constraints [15, 14] and properties such as axiomatization and satisfiability have been investigated for these constraints. One observation to make on this research is that the flexible structure of XML makes the investigation of integrity constraints in XML more complex and subtle than in relational databases. However, one topic that has been identified as an open problem in XML research [23] and which has been little investigated is how to extended the oldest and most well studied integrity constraint in relational databases, namely *functional dependencies* (FDs), to XML and then how to develop a normalization theory for XML. This problem is not of just theoretical interest. The theory of FDs and normalization forms the cornerstone of practical relational database design and the development of a similar theory for XML will similarly lay the foundation for understanding how to design XML documents. In addition, the study of FDs in XML is important because of the close connection between XML and relational databases. With current technology, the source of XML data is typically a relational database [1] and relational databases are also normally used to store XML data [20]. Hence, given that FDs are the most important constraint in relational databases, the study of FDs in XML assumes heightened importance over other types of constraints which are unique to XML [10]. The only papers that have specifically addressed this problem are the recent papers [3, 22]. Before presenting the contributions of [3, 22], we briefly outline the approaches to defining FD satisfaction in incomplete relational databases.

There are two approaches, the first called the *weak satisfaction* approach and the other called the *strong satisfaction* approach [5]. In the weak satisfaction approach, a relation is defined to weakly satisfy a FD if there exists *at least one* completion of the relation, obtained by replacing all occurrences of nulls by data values, which satisfies the FD. A relation is said to strongly satisfy a FD if *every* completion of the relation satisfies the FD. Both approaches have their advantages and disadvantages (a more complete discussion of this issue can be found in [22]). The weak satisfaction approach has the advantage of allowing a high degree of uncertainty to be represented in a database but at the expense of making maintenance of integrity constraints much more difficult. In contrast, the strong satisfaction approach restricts the amount of uncertainty that can be represented in a database but makes the maintenance of integrity constraints much easier. However, as argued in [18], both approaches have their place in real world applications and should be viewed as complementary rather than competing approaches. Also, it is possible to combine the two approaches by having some FDs in a relation strongly satisfied and others weakly satisfied [17].

The contribution of [3] was, for the first time, to define FDs in XML (what we call XFDs) and then to define a normal form for a XML document based on the definition of a XFD. However, there are some difficulties with the definition of a XFD given in [3]. The most fundamental problem is that although it is explicitly recognized in the definitions that XML documents have missing information, the definitions in [3], while having some elements of the weak instance approach, are not a strict extension of this approach since there are XFDs that are violated according to the definition in [3] yet there are completions of the tree that satisfy the XFDs (see [22] for an example). As a result of this it is not clear that there is any correspondence between FDs in relations and XFDs in XML documents. The other difficulty is that the approach to defining XFDs is not straightforward and is based on the complex and non-intuitive notion of a "tree tuple".

In [22] a different approach was taken to defining XFDs which overcomes the difficulties just discussed with the approach adopted in [3]. The definition in [22] is based on extending the strong satisfaction approach to XML. The definition of a XFD given in [22] was justified formally by two main results. The first result showed that for a very general class of mappings from an incomplete relation into a XML document, a relation strongly satisfies a unary FD (only one attribute on the l.h.s. of the FD) if and only if the corresponding XML document strongly satisfies the corresponding XFD. The second result showed that a XML document strongly satisfies a XFD if and only if every completion of the XML document also satisfies the XFD. The other contributions in [22] were firstly to define a set of axioms for reasoning about the implication of XFDs and show that the axioms are sound for arbitrary XFDs. The final contribution was to define a normal form, based on a modification of the one proposed in [3], and prove that it is a necessary and sufficient condition for the elimination of redundancy in a XML document.

The contribution of this paper is to extend the work in [22] in two important ways. As just mentioned, in [22] a set of axioms for XFDs were provided and shown to be sound. In this paper we prove that the axioms are also complete for unary XFDs. The second contribution of the paper is to prove that the implication problem for unary XFDs is decidable and to provide a linear time algorithm for it. These results have considerable significance in the development of a theory of normalization for XML documents. In relational databases, the classic results on soundness and completeness of Armstrong's axioms [4] and the resulting closure algorithm for FD implication play an essential role in determining whether a relation is in one of the classic normal forms. Similarly, the results in this paper are an important first step in the development of algorithms for testing the normal form proposed in [22]. In addition, the result on completeness is of theoretical interest in itself since it ensures that there are no other 'hidden' axioms for reasoning about the implication of XFDs.

The rest of this paper is organized as follows. Section 2 contains some preliminary definitions. In Section 3 a XFD is defined. In Section 4 axioms for XFDs are presented and are shown to be sound for arbitrary XFDs and complete for unary XFDs. In Section 5 the implication problem for unary XFDs is investigated and a linear time algorithm for the implication problem is presented and shown to be correct. Finally, Section 6 contains concluding comments.

2 Preliminary definitions

In this section we present some preliminary definitions that we need before defining XFDs. We firstly present the definition of a XML tree adapted from the definition given in [8].

Definition 1 Assume a countably infinite set $\mathbf{E}$ of element labels (tags), a countable infinite set $\mathbf{A}$ of attribute names and a symbol $\mathtt{S}$ indicating text. An *XML tree* is defined to be $T = (V, lab, ele, att, val, v_r)$ where V is a *finite* set of nodes in T; lab is a function from V to $\mathbf{E} \cup \mathbf{A} \cup \{\mathtt{S}\}$; ele is a partial function from V to a sequence of V nodes such that for any $v \in V$, if $ele(v)$ is defined then $lab(v) \in \mathbf{E}$; att is a partial function from $V \times \mathbf{A}$ to V such that for any $v \in V$ and $l \in \mathbf{A}$, if $att(v, l) = v_1$ then $lab(v) \in \mathbf{E}$

and $lab(v_1) = l$; val is a function such that for any node in $v \in V, val(v) = v$ if $lab(v) \in \mathbf{E}$ and $val(v)$ is a string if either $lab(v) = \mathtt{S}$ or $lab(v) \in \mathbf{A}$; v_r is a distinguished node in V called the *root* of T and we define $lab(v_r) = root$. Since node identifiers are unique, a consequence of the definition of val is that if $v_1 \in \mathbf{E}$ and $v_2 \in \mathbf{E}$ and $v_1 \neq v_2$ then $val(v_1) \neq val(v_2)$. We also extend the definition of val to sets of nodes and if $V_1 \subseteq V$, then $val(V_1)$ is the set defined by $val(V_1) = \{val(v)|v \in V_1\}$.

For any $v \in V$, if $ele(v)$ is defined then the nodes in $ele(v)$ are called *subelements* of v. For any $l \in \mathbf{A}$, if $att(v, l) = v_1$ then v_1 is called an *attribute* of v. Note that a XML tree T must be a tree. Since T is a tree the ancestors of a node v, denote by $Ancestor(v)$ are defined as in Definition 1. The children of a node v are also defined as in Definition 1 and we denote the parent of a node v by $Parent(v)$.

We note that our definition of val definition differs slightly from that in [8] since we have extended the definition of the val function so that it is also defined on element nodes. The reason for this is that we want to include in our definition paths that do not end at leaf nodes, and when we do this we want to compare element nodes by node identity, i.e. node equality, but when we compare attribute or text nodes we want to compare them by their contents, i.e. value equality. This point will become clearer in the examples and definitions that follow.

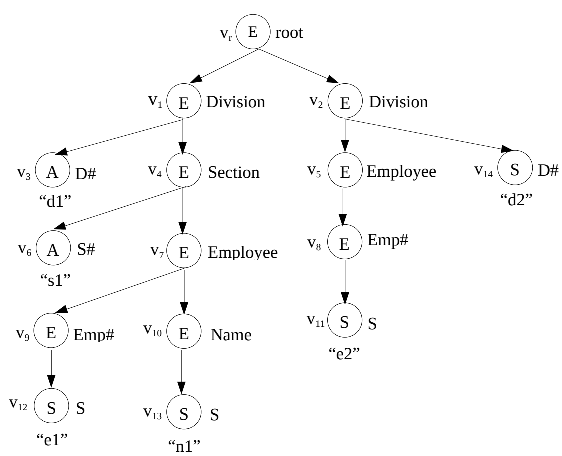

Figure 1: A XML tree

We now give some preliminary definitions related to paths.

Definition 2 A *path* is an expression of the form $l_1. \cdots .l_n$, $n \geq 1$, where $l_i \in \mathbf{E} \cup \mathbf{A} \cup \{\mathtt{S}\}$ for all $i, 1 \leq i \leq n$ and $l_1 = root$. If p is the path $l_1. \cdots .l_n$ then $Last(p)$ is l_n.

For instance, in Figure 1, `root` and `root.Division` are paths.

Definition 3 Let p denote the path $l_1. \cdots .l_n$. The function $Parnt(p)$ is the path $l_1. \cdots .l_{n-1}$. Let p denote the path $l_1. \cdots .l_n$ and let q denote the path $q_1. \cdots .q_m$. The path p is said to be a *prefix* of the path q if $n \leq m$ and $l_1 = q_1, \ldots, l_n = q_n$. Two paths p and q are equal, denoted by $p = q$, if p is a prefix of q and q is a prefix of p. The path p is said to be a *strict prefix* of q if p is a prefix of q and $p \neq q$. We also define the intersection of two paths p_1 and p_2, denoted but $p_1 \cap p_2$, to be the maximal common prefix of both paths. It is clear that the intersection of two paths is also a path.

For example, in Figure 1, `root.Division` is a strict prefix of `root.Division.Section` and `root.Division.d#` $\cap$ `root.Division.Employee.Emp#.S` $=$ `root.Division`.

Definition 4 A *path instance* in a XML tree T is a sequence $\bar{v}_1.\cdots.\bar{v}_n$ such that $\bar{v}_1 = v_r$ and for all $\bar{v}_i, 1 < i \leq n, v_i \in V$ and $\bar{v}_i$ is a child of $\bar{v}_{i-1}$. A path instance $\bar{v}_1.\cdots.\bar{v}_n$ is said to be *defined over the path* $l_1.\cdots.l_n$ if for all $\bar{v}_i, 1 \leq i \leq n$, $lab(\bar{v}_i) = l_i$. Two path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ are said to be *distinct* if $v_i \neq v'_i$ for some i, $1 \leq i \leq n$. The set of path instances over a path p in a tree T is denoted by $Paths(p)$

Definition 5 An *extended XML tree* is a tree $(V \cup \mathbf{N}, lab, ele, att, val, v_r)$ where $\mathbf{N}$ is a set of marked nulls that is disjoint from V and if $v \in \mathbf{N}$ and $v \notin \mathbf{E}$ then $val(v)$ is undefined.

Definition 6 Let T be a XML tree and let P be a set of paths. Then (T, P) is *consistent* if:

(i) For any two paths $l_1.\cdots.l_n$ and $l'_1.\cdots.l'_m$ in P such that $l'_m = l_i$ for some i, $1 \leq i \leq n$ then $l_1.\cdots.l_i = l'_1.\cdots.l'_m$;

(ii) If v_1 and v_2 are two nodes in T such that v_1 is the parent of v_2, then there exists a path $l_1.\cdots.l_n$ in P such that there exists i and j, where $1 \leq i \leq n$ and $1 \leq j \leq n$ and $i < j$ and $label(v_1) = l_i$ and $label(v_2) = l_j$.

Definition 7 Let T be a XML tree and let P be a set of paths and such that (T, P) is consistent. Then a minimal extension of T, denoted by T_P, is an extended XML tree constructed as follows. Initially let T_P be T. Process each path p in P in an arbitrary order as follows. For every node in v in T such that $lab(v)$ appears in p and there does not exist a path instance containing v which is defined over p, construct a path instance over p by adding nodes from $\mathbf{N}$ as ancestors and descendants of v.

The next lemma follows easily from the construction procedure.

Lemma 1 *T_P is unique up to the labelling of the null nodes.*

For instance, the minimal extension of the tree in Figure 1 is shown in Figure 2.

Definition 8 A path instance $\bar{v}_1.\cdots.\bar{v}_n$ in T is defined to be *complete* if $\bar{v}_1.\cdots.\bar{v}_n \in T_P$. A tree T is defined to be complete w.r.t. a set of paths P if (T, P) is consistent and $T = T_P$. Also we often do not need to distinguish between nulls and so the statement $v = \perp$ is shorthand for $\exists j(v = \perp_j)$ and $v \neq \perp$ is shorthand for $\nexists j(v = \perp_j)$.

The next function returns all the final nodes of the path instances of a path p.

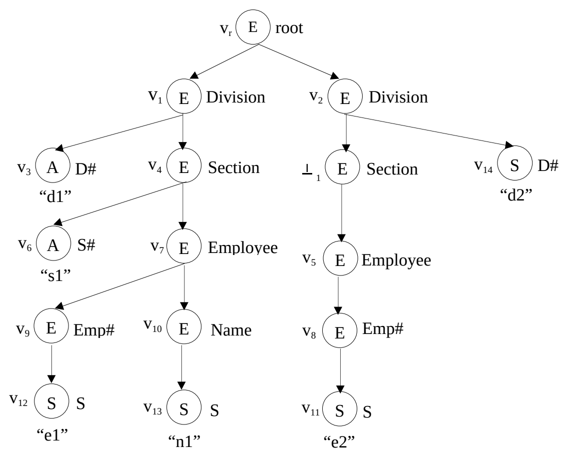

Figure 2: The minimal extension of a XML tree.

Definition 9 Let T_P be the minimal extension of T. The function $N(p)$, where p is the path $l_1.\cdots.l_n$, is defined to be the set $\{\bar{v}|\bar{v}_1.\cdots.\bar{v}_n \in Paths(p) \wedge \bar{v} = \bar{v}_n\}$.

For example, in Figure 2, $N(\texttt{root.Division.Section.Employee}) = \{v_7, v_5\}$ and $N(\texttt{root.Division.Section}\ ) = \{v_4, \perp_1\}$.

We now need to define a function that is related to ancestor.

Definition 10 Let T_P be the minimal extension of T. The function $AAncestor(v, p)$ where $v \in V \cup \mathbf{N}$, p is a path and $v \in N(p)$, is defined by $AAncestor(v, p) = \{v' | v' \in \{\bar{v}'_1, \cdots, \bar{v}'_n\} \wedge v = \bar{v}'_n \wedge \bar{v}'_1.\cdots.\bar{v}'_n \in Paths(p)\}$.

For example, in Figure 2, $AAncestor(v_5, \texttt{root.Division.Section.Employee}) = \{v_r, v_2, \perp_1, v_5\}$. The next function returns all nodes that are the final nodes of path instances of p and are descendants of v.

Definition 11 Let T_P be the minimal extension of T.The function $Nodes(v, p)$, where $v \in V \cup \mathbf{N}$ and p is a path, is the set defined by $Nodes(v, p) = \{x | x \in N(p) \wedge v \in AAncestor(x, p)\}$. Note that $Nodes(v, p)$ may be empty.

For example, in Figure 2, $Nodes(v_r, \texttt{root.Division.Section.Employee}) = \{v_5, v_7\}$, $Nodes(v_1, \texttt{root.Division.Section.Employee}) = \{v_7\}$, $Nodes(v_7, \texttt{root.Division}) = \phi$.

Definition 12 The partial ordering $>$ on the set of nodes V in a XML tree T is defined by $v_1 > v_2$ iff $v_2 \in Ancestor(v_1)$, where v_1 and v_2 are in V.

In a similar fashion, we define a partial ordering on paths as follows.

Definition 13 The partial ordering $>$ on a set of paths P is defined by $p_2 > p_1$ if p_1 is a prefix of p_2, where p_1 and p_2 are paths in P.

For example, in Figure 2, `root.Division.D#` $>$ `root.Division`. Also, `root.Division.D#` and `root.Division.Section` are incomparable.

Lastly we extend the definition of the *val* function so that $val(\perp_j) = \perp_j$. Note that different unmarked nulls are not considered to be equal and so $val(\perp_i) \neq val(\perp_j)$ if $i \neq j$.

3 Strong Functional Dependencies in XML

This leads us to the main definition of our paper.

Definition 14 Let T be a XML tree and let P be a set of paths such that (T, P) is consistent. A XML functional dependency (XFD) is a statement of the form: $p_1, \cdots, p_k \rightarrow q$ where $p_1, \cdots, p_k$ and q are paths in P. T *strongly satisfies* the XFD if $p_i = q$ for some $i, 1 \leq i \leq k$ or for any two distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ in T_P, $((\bar{v}_n = \perp \wedge \bar{v}'_n = \perp) \vee (\bar{v}_n \neq \perp \wedge \bar{v}'_n = \perp) \vee (\bar{v}_n = \perp \wedge \bar{v}'_n \neq \perp) \vee (\bar{v}_n \neq \perp \wedge \bar{v}'_n \neq \perp \wedge val(\bar{v}_n) \neq val(\bar{v}'_n) \Rightarrow \exists i, 1 \leq i \leq k$, such that $x_i \neq y_i$ if $Last(p_i) \in \mathbf{E}$ else $\perp \notin Nodes(x_i, p_i)$ and $\perp \notin Nodes(y_i, p_i)$ and $val(Nodes(x_i, p_i)) \cap val(Nodes(y_i, p_i)) = \phi$,where $x_i = \{v | v \in \{\bar{v}_1, \cdots, \bar{v}_n\} \wedge v \in N(p_i \cap q)\}$ and $y_i = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(p_i \cap q)\}$.

We note that since the path $p_i \cap q$ is a prefix of q, there always exists one and only one node in $\{v'_1, \cdots, \bar{v}'_n\}$ that is also in $N(p_i \cap q)$ and so x_i is always defined and unique. Similarly for y_i.

We now outline the thinking behind the above definition firstly for the simplest case where the l.h.s. of the XFD contains a single path. In the relational model, if we are given a relation r and a FD $A \rightarrow B$, then to see if $A \rightarrow B$ is satisfied we have to check the B values and their corresponding A values. In the relational model the correspondence between B values and A values is obvious - the A value corresponding to a B value is the A value in the same tuple as the B value. However, in XML there is no concept of a tuple so it is not immediately clear how to generalize the definition of an FD to XML. Our solution is based on the following observation. In a relation r with tuple t, the value $t[A]$ can be seen as the 'closest' A value to the B value $t[B]$. In Definition 14 we generalize this observation and given a path instance $\bar{v}_1.\cdots.\bar{v}_n$ in $Paths(q)$, we first compute the 'closest' ancestor of $\bar{v}_n$ that is also an ancestor of a node in $N(p)$ (x_1 in the above definition) and then compute the 'closest p-nodes' to be the set of nodes which terminate a path instance of p and are descendants of x_1. We then proceed in a similar fashion for the other path $\bar{v}'_1.\cdots.\bar{v}'_n$ and compute the 'p-nodes' which are closest to $\bar{v}'_n$. We note that in this definition, as opposed to the relational case, there will be in general more than one 'closest p - node' and so $Nodes(x_1, p)$ and $Nodes(y_1, p)$ will in general contain more than one node. Having computed the 'closest p-nodes' to $\bar{v}_n$ and $\bar{v}'_n$, if $val(\bar{v}_n) \neq val(\bar{v}'_n)$ we then require, generalizing on the relational case, that the $val's$ of the sets of corresponding 'closest p-nodes' be disjoint.

The rationale for the case where there is more than one path on the l.h.s. is similar. Given a XFD $p_1, \cdots, p_k \rightarrow q$ and two paths $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ which end in nodes with different val, we firstly compute, for each p_i, the set of 'closest p_i nodes' to $\bar{v}_n$ in the same fashion as just outlined.

Then extending the relational approach to FD satisfaction, we require that in order for $p_1, \cdots, p_k \to q$ to be satisfied there is at least one p_i for which the $val's$ of the set of 'closest p_i nodes' to $\bar{v}_n$ is disjoint from the $val's$ of the set of 'closest p_i nodes' to $\bar{v}'_n$. We now illustrate the definition by some examples.

Example 1 Consider the XML tree shown in Figure 3 and the XFD

`root.Department.Lecturer.Subject.Subject#` $\to$ `root.Department.Lecturer.Subject.SubjName.S`.

Then $v_r.v_1.v_5.v_{13}.v_{17}.v_{22}$ and $v_r.v_2.v_9.v_{15}.v_{21}.v_{24}$ are two distinct path instances in

$Paths($`root.Department.Lecturer.Subject.SubjName.S`$)$ and $val(v_{22})$ = `"n1"` and $val(v_{24})$ = `"n2"`. So $N($`root.Department.Lecturer.Subject.Subject#`$\cap$

`root.Department.Lecturer.Subject.SubjName.S`$)) = \{v_{13}, v_{14}, v_{15}\}$ and so $x_1 = v_{13}$ and $y_1 = v_{15}$. Thus $val(Nodes(x_1,$ `root.Department.Lecturer.Subject.Subject#`$)) = \{$`"s1"`$\}$ and

$val(Nodes(y_1,$ `root.Department.Lecturer.Subject.Subject#`$)) = \{$`"s2"`$\}$. Similarly for the paths $v_r.v_1.v_6.v_{13}.v_{14}.v_{23}$ and $v_r.v_2.v_9.v_{15}.v_{21}.v_{24}$ and so the XFD is satisfied. We note that if we change val of node v_{18} in Figure 3 to `"s1"` then the XFD is violated.

Consider next the XFD `root.Department.Head` $\to$ `root.Department`. Then $v_r.v_1$ and $v_r.v_2$ are two distinct paths instances in $Paths($`root.Department`$)$ and $val(v_1) = v_1$ and $val(v_2) = v_2$. Also

$N($`root.Department.Head` $\cap$ `root.Department`$) = \{v_1, v_2\}$ and so $x_1 = v_1$ and $y_1 = v_2$. Thus $val(Nodes(x_1,$ `root.Department.Head`$)) = \{$`"h1"`$\}$ and $Val(Nodes(y_1,$ `root.Department.Head`$)) = \{$`"h2"`$\}$ and so the XFD is satisfied. We note that if we change val of node v_8 in Figure 3 to `"h1"` then the XFD is violated.

Consider next the XFD `root.Department.Lecturer.Lname`, `root.Department.Dname` $\to$

`root.Department.Lecturer.Subject.Subject#`. Then $v_r.v_1.v_5.v_{13}.v_{16}$ and $v_r.v_2.v_9.v_{15}.\perp_1$ are two distinct path instances in $Paths($`root.Department.Lecturer.Subject.Subject#`$)$ and $val(v_{16})$ = `"s1"` and the final node in $v_r.v_2.v_9.v_{15}.\perp_1$ is null.

Then $N($`root.Department.Lecturer.Lname`$\cap$ `root.Department.Lecturer.Subject.Subject#`$) = \{v_5, v_6, v_9\}$ and so $x_1 = v_5$ and $y_1 = v_9$ and so $val(Nodes(x_1,$ `root.Department.Lecturer.Lname`$))$ = `"l1"` and $val(Nodes(y_1,$.`root.Department.Lecturer.Lname`$))$ = `"l1"`. We then compute

$N($`root.Department.Dname` $\cap$ `root.Department.Lecturer.Subject.Subject#`$) = \{v_1, v_2\}$ and so $x_2 = v_1$ and $y_2 = v_2$ and so $val(Nodes(x_2,$.`root.Department.Dname`$))$ = `"d1"` and

$val(Nodes(y_2,$.`root.Department.D name`$))$ = `"d2"`. Similarly, for the paths $v_r.v_1.v_6.v_{14}.v_{18}$, we derive that $x_2 = v_6$ and $y_2 = v_9$ and so

$val(Nodes(x_2,$.`root.Department.Dname`$)) \neq val(Nodes(y_2,$.`root.Department.D name`$))$ = `"d2"` and so the XFD is satisfied. Thus $x_1 \neq y_1$. Similarly for $v_r.v_1.v_6$ and $v_r.v_2.v_9.v_{15}.\perp_1$ and so the XFD is satisfied.

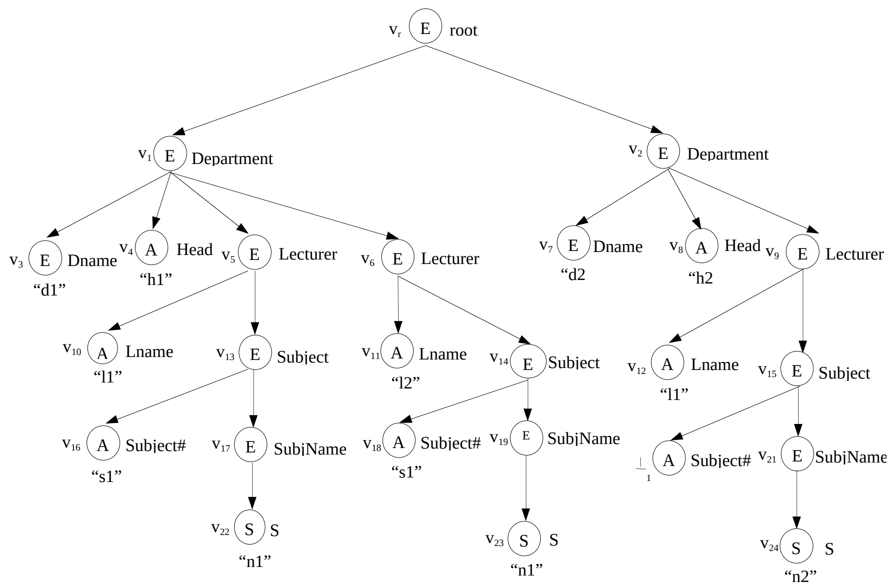

Figure 3: A XML tree illustrating the definition of a XFD

4 Axiomatization for XFDs

In this section we address the issues of completeness of the axiom system for reasoning about implication of XFDs that was presented in [22]. The axiom system is the following.

Axiom A1. $p_1, \cdots, p_k \rightarrow p_i$ for any p_i, $1 \leq i \leq k$.

Axiom A2. If $p_1, \cdots, p_k \rightarrow q$, then $p, p_1, \cdots, p_k \rightarrow q$ for any path p.

Axiom A3. If $p_1, \cdots, p_k \rightarrow q$, and $q \rightarrow s$ then $p_1, \cdots, p_k \rightarrow s$.

Axiom A4. If $p_1, \cdots, p_k \rightarrow q$ and $\forall i, 1 \leq i \leq k, p_i \cap q = root$, then $p \rightarrow q$ for any path p.

Axiom A5. If $p \rightarrow q$ then $p' \rightarrow q$ for all paths p' such that $p \cap q$ is prefix of p' and either p' is a prefix of p or p' is a prefix of q.

Axiom A6. If $Last(p) \in \mathbf{E}$ and q is a prefix of p then $p \rightarrow q$.

Axiom A7. If $Last(q) \in \mathbf{A}$ then $Parnt(q) \rightarrow q$.

Axiom A8. $p \rightarrow root$ for any path p.

Theorem 1 *Axioms A1 - A8 are sound for implication of arbitrary XFDs.*

Proof. For the sake of the completeness of this paper, the proof from [22] is reproduced in the Appendix.

We now illustrate these axioms by an example.

Example 2 Consider the XML tree show in Figure 4 and the set Σ of XFDs {`root.A.B.C.C#` $\rightarrow$ `root.A.D.E`, `root.A.D.E` $\rightarrow$ `root.A.D.E.F.F#`, `root.A` $\rightarrow$ `root.G`}. It can be easily verified that the XML tree in Figure 4 satisfies Σ. Then from Σ and the axioms we can deduce that the following XFDs

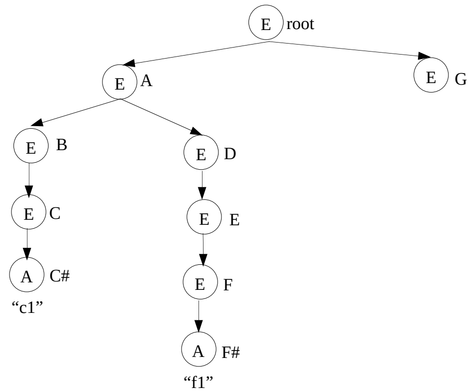

Figure 4: XML tree illustrating axioms for XFDs.

are implied by Σ[1] : from A1 we can derive `root.A` $\rightarrow$ `root.A`, from A2 and `root.A` $\rightarrow$ `root.G` we can derive that `root.A, root.A.B.C` $\rightarrow$ `root.G`, from A3 and `root.A.B.C.C#` $\rightarrow$ `root.A.D.E` and `root.A.D.E` $\rightarrow$ `root.A.D.E.F.F#` we can derive that `root.A.B.C.C#` $\rightarrow$`root.A.D.E.F.F#`, from A4 and `root.A` $\rightarrow$ `root.G` we can derive that `root.A.D.E` $\rightarrow$ `root.G`, from A5 and `root.A.B.C.C#` $\rightarrow$ `root.A.D.E` we can derive that `root.A.B` $\rightarrow$ `root.A.D.E` and that `root.A.D` $\rightarrow$ `root.A.D.E`, from A6 we can derive that `root.A.D.E` $\rightarrow$ `root.A`, from A7 we can derive that `root.A.D.E.F` $\rightarrow$ `root.A.D.E.F.F#` and from A8 we derive that `root.A.D` $\rightarrow$ `root`.

This now leads to the first major result of the paper.

Theorem 2 *Axioms A1 - A8 are complete for unary XFDs*

Proof. See Appendix.

5 Decidability Of Implication for Unary XFDs

In this section we derive the second main result of the paper by showing that the implication problem for unary XFDs is decidable. We do this by constructing an algorithm for generating P^+, the set of all paths q such that $q \in P^+$ if and only if $p \rightarrow q \in \Sigma^+$ and then prove that the algorithm is correct. We note also that the running time of the algorithm is linear in the number of XFDs in Σ. Firstly we present an algorithm which is analogous to the classic chase procedure for relations [19].

Before presenting the next algorithm, we define two functions.

Definition 15 The function $Anc(p)$, where p is a path, is the set defined by $Anc(p) = \{q | q$ is a *strict prefix* of $p\}$. The function $Att(p)$ is the set defined by $Att(p) = \{q | p = Parnt(q) \wedge Last(q) \in \mathbf{A}\}$.

[1]We do not show all the XFDs that can be derived from the axioms

Algorithm 1

INPUT: A set Σ of unary XFDs and a tree T which is complete w.r.t. the set of paths in Σ.

OUTPUT: A XML tree $\bar{T}$ satisfying the set of XFDs and which is complete w.r.t. the set of paths in Σ.

```
T̄ = T;
Repeat until no more changes can be made to T̄
  For each p → q ∈ Σ do
    If Last(q) ∉ E then
      If there exist v1, v2, v3, v4 ∈ T̄ such that
        v1, v2 ∈ N(p), v3, v4 ∈ N(q) and val(v1) = val(v2) and
        val(v3) < val(v4) then
          val(v4) := val(v3);
    If Last(q) ∈ E then
      If there exist v1, v2, v3, v4 in T̄ such that
        v1, v2 ∈ N(p), v3, v4 ∈ N(q) and val(v1) = val(v2) then
          attach all descendants of v4 to v3;
          DeleteSameAtts(v3);
          vl := Parent(v3); vr := Parent(v4);
          repeat until vl = vr
            attach all descendants of vr to vl except for v4;
            deleteSameAtts(vl);
            delete(v4);
            v4 := vr;
            vl:= Parent(vl); vr := Parent(vr);
          endrepeat;
  endfor
endrepeat

procedure DeleteSameAtts (node v);
   For any pair of nodes v5 and v6 such that v5 and v6 are children
   of v and lab(v5) = lab(v6) and lab(v5) ∈ A and
   val(v5) ≤ val(v6) then delete v6;
return;
```

We now illustrate Algorithm 1 by an example.

Example 3 Let Σ be the set of XFDs {`root.A.A#` $\rightarrow$ `root.A.B.B#`, `root.A.B.B#` $\rightarrow$ `root.A.B.C.C#`, `root.A.B` $\rightarrow$ `root.A.B.D`} and let the initial tree T be as shown in Figure 5. Then if we apply the XFD `root.A.A#` $\rightarrow$ `root.A.B.B#` the resulting tree is shown in Figure 6. If we then apply `root.A.B.B#` $\rightarrow$ `root.A.B.C.C#` the resulting tree is shown in Figure 7. Finally, if we apply `root.A.B` $\rightarrow$ `root.A.B.D` then the tree is shown in Figure 8.

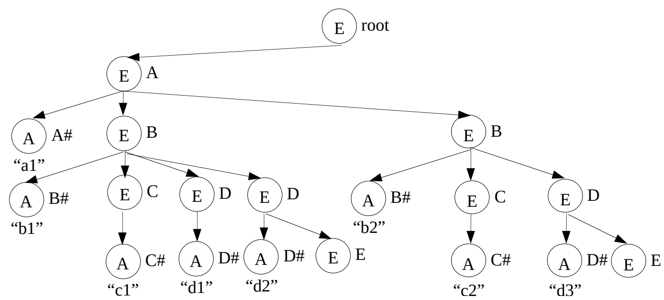

Figure 5: Initial XML tree

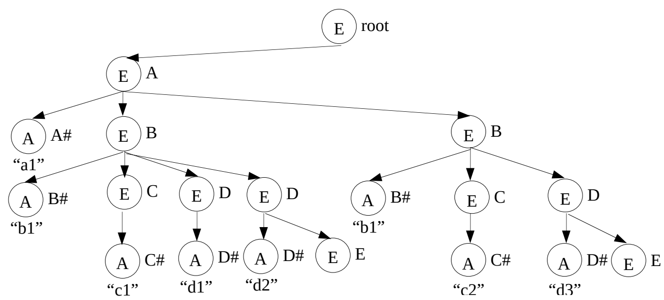

Figure 6: XML tree after applying `root.A.A#` $\rightarrow$ `root.A.B.B#`

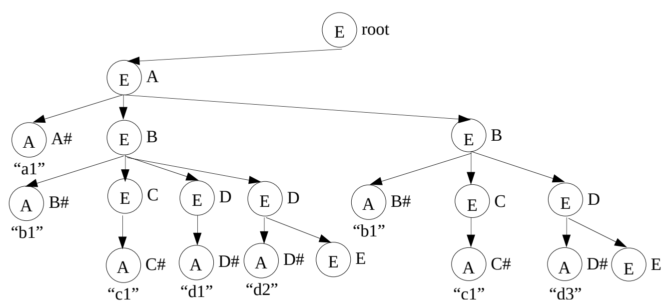

Figure 7: XML tree after applying `root.A.B.B#` $\rightarrow$ `root.A.B.C.C#`

Lemma 2 *Algorithm1 always terminates.*

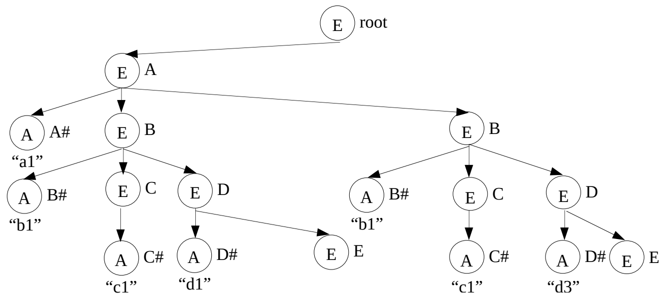

Figure 8: XML tree after applying `root.A.B` $\rightarrow$ `root.A.B.D`

Proof. The function $Count(p)$, where p is a path and $Last(p) \in \mathbf{E}$, is defined to be $|N(p)|$, were $||$ denotes cardinality. The function $Sum(p)$, where $p \notin \mathbf{E}$ is defined as follows. For any text string value $\mathtt{t}$ define the function $int(\mathtt{t})$ to be the integer value obtained by considering $\mathtt{t}$ to be an integer to base 256 and then define $Sum(p)$ to be $\sum_{v \in N(p)} int(val(v))$. At each iteration of the repeat loop either $Count(p)$ or $Sum(p)$ strictly decreases for at least one path p. Hence since both $Count(p)$ and $Sum(p)$ are both bounded below by 0 Algorithm 1 must terminate. $\square$

Firstly, let us denote by P_Σ the set of paths that appear on the l.h.s. or r.h.s. of any XFD in a set of unary XFDs Σ.

Lemma 3 *The tree $\bar{T}$ produced by Algorithm 1 is complete w.r.t. P_Σ.*

Proof. The proof is by induction. Initially the result is true because of the restriction placed on the input tree T by Algorithm 1. Assume then the result is true after iteration $k-1$. Then during iteration k, the only path instances which can possibly be changed are those in $Paths(q)$ or $Paths(Anc(q))$ for some $q \in \mathbf{E}$. However, if we merge two nodes in $N(q)$ then we also merge their ancestor nodes and so after iteration k, $Paths(q)$ and $Paths(Anc(q))$ will again contain only complete paths and so the result is established. $\square$

Lemma 4 *The tree generated by Algorithm 1 satisfies Σ.*

Proof. From the definition of the algorithm, the algorithm terminates only when there is no XFD that is violated. $\square$

Next, we introduce an algorithm for calculating the closure of a set of XFDs.

Algorithm 2

```
INPUT: A set Σ of unary XFDs and a path p.
OUTPUT:  P+ the set of paths such that q ∈ P+ iff
                 p → q  is implied by Σ.
  1:P+ = {p, root};
  2:If Last(p) ∈ E then P+ = P+ ∪ Anc(p);
  3: for each q ∈ P+ do P+ := P+ ∪ Att(q);
  Unused = Σ;
  repeat until no more changes to P+
    Choose arbitrarily r → s from Σ;
    4:(If ∃ p1 ∈ P+ such that r ∩ s is prefix of p1 and
    either p1 is a prefix of r or p1 is a prefix of s)
    5:∨ (r ∈ P+)
    6:∨ (r ∩ s = root) then
      P+ = P+ ∪ {s};
      Unused = Unused − {r → s};
      7:if Last(s) ∈ E then P+ = P+ ∪ Anc(s) ∪ Att(s);
  endrepeat
```

We note that it is easily seen that since each XFD in Σ is used only once, the running time of Algorithm 2 is linear in the number of XFDs in Σ. We now proceed to prove that Algorithm 2 is correct. Two cases are considered separately.

Case 1: $Last(p) \notin \mathbf{E}$

First construct a tree T_0 with the following properties. T_0 is complete w.r.t. P_Σ and for every path appearing in P_Σ, except for the root, there are exactly two path instances of the path in T_0. Also, the path instances for p have the property that the *val* of the final nodes in the path instances are the same whereas the *val* of the end nodes for the path instances of any other path in P_Σ are distinct. Such a tree can always be constructed. We now illustrate the construction by an example.

Example 4 Let $\Sigma = \{\texttt{root.A.B.B\#} \rightarrow \texttt{root.A.A\#}, \texttt{root.A.C.C\#} \rightarrow \texttt{root.A.B}, \texttt{root.A.A\#} \rightarrow \texttt{root.D.D\#}\}$ and let p be the path `root.A.B.B#`. Then the tree T_0 is shown in Figure 9.

The next step is using as input the set of XFDs returned in P^+ by Algorithm 2 and the tree T_0, generate the tree $\bar{T}_0$ using Algorithm 1. We note that it follows from Lemma 4 that $\bar{T}_0$ satisfies P^+. We now prove some preliminary lemma before establishing the main result.

Lemma 5 *Let $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ be two distinct path instances in $Paths(q)$ in T_0 for any path q in P_Σ. Then the only common node to both path instances is root.*

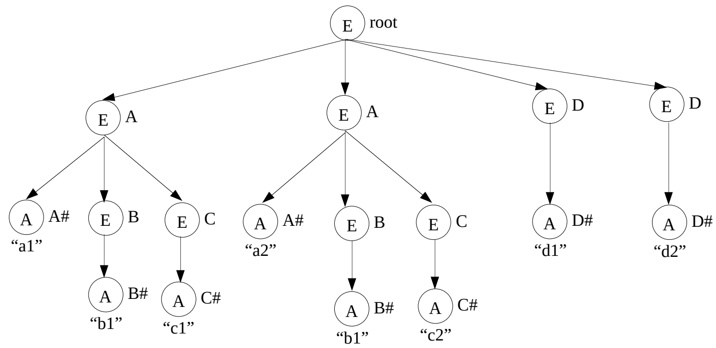

Figure 9: A XML tree

Proof. Suppose to the contrary that node $\bar{v}_j$ is common to both path instances. Then $\bar{v}_j \in N(s)$ for some path s that is a prefix of q. So because of the definition of T_0 there must exist two path instances in $Paths(s)$ and so there exists another node $\bar{v}_j^1$ in $N(s)$ that is distinct from $\bar{v}_j$. There are then two possibilities. The first is that there exists another path instance $\bar{v}_1''.\cdots.\bar{v}_n''$ in $Paths(q)$ that contains $\bar{v}_j^1$. If this is the case then since $\bar{v}_j^1$ and $\bar{v}_j$ are distinct the paths $\bar{v}_1.\cdots.\bar{v}_n$, $\bar{v}_1'.\cdots.\bar{v}_n'$ and $\bar{v}_1''.\cdots.\bar{v}_n''$ are distinct which contradicts the fact that T_0 has only two path instances for any path. The other possibility is that there is no path instance in $Paths(q)$ in T_0 that contains $\bar{v}_j^1$ but this contradicts the fact that T_0 is complete w.r.t. P_Σ. So either possibility leads to a contradiction so we conclude that the only common node to $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}_1'.\cdots.\bar{v}_n'$ is *root*. □

Lemma 6 *Let q be any path in P_Σ. Then if there exist two distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}_1'.\cdots.\bar{v}_n'$ in $Paths(q)$ in $\bar{T}_0$ such that $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}_1'.\cdots.\bar{v}_n'$ have a common node that is not the root then there exists s' such that $s' \in Anc(q)$ and $s' \in P^+$.*

Proof. We prove the result by induction on the number of steps in constructing $\bar{T}_0$. Initially the result is true for T_0 by Lemma 5. Assume inductively then that it is true after iteration $k-1$. The only way that we can have that $\bar{v}_1.\cdots.$ and $\bar{v}_1'.\cdots.\bar{v}_n'$ in $Paths(q)$ have a common non root node after iteration k is if we merge two ancestor nodes of $\bar{v}_n$ and $\bar{v}_n'$. For this to happen we have to have s such that $s \in Anc(q)$ and $s \in P^+$. □

Lemma 7 *If a XFD $r \to s$ is violated in $\bar{T}_0$ then $p \to s$ is violated in $\bar{T}_0$.*

Proof. If $r \to s$ is violated then there exist distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}_1'.\cdots.\bar{v}_n'$ in $Paths(s)$ such that $val(\bar{v}_n) \neq val(\bar{v}_n')$. However by the construction of $\bar{T}_0$, $N(p)$ contains only two nodes, say v_1 and v_2, such that $val(v_1) = val(v_2)$. Let us then compute $x_1 = \{v | v \in \{\bar{v}_1, \cdots, \bar{v}_n\} \wedge v \in N(p \cap s)\}$ and $y_1 = \{v | v \in \{v_1', \cdots, \bar{v}_n'\} \wedge v \in N(p \cap s)\}$. If $x_1 = y_1$ then $Nodes(x_1, p) = Nodes(y_1, p)$

and so $Nodes(x_1, p) \cap Nodes(y_1, p) \neq \phi$ and so $p \rightarrow s$ is violated. If $x_1 \neq y_1$ then we must have that $val(Nodes(x_1, p)) \cap val(Nodes(y_1, p)) \neq \phi$ because$v_1 \in Nodes(x_1, p)$ and $v_2 \in Nodes(y_1, p)$ and $val(v_1) = val(v_2)$ and so $p \rightarrow s$ is again violated. □

Lemma 8 *If there is a path q in P_Σ such that $Last(q) \notin \mathbf{E}$ then $q \in P^+$ iff there exist two distinct nodes v_1 and v_2 in $N(q)$ in $\bar{T}_0$ such that $val(v_1) = val(v_2)$.*

Proof.

If: We prove the result again by induction on the number of steps in constructing $\bar{T}_0$. Initially the result is true since $p \in P^+$ and the val of the two nodes in $N(p)$ is the same. Suppose then that there exist two nodes in in $N(q)$ such that $val(v_1) \neq val(v_2)$ before step k and $val(v_1) = val(v_2)$ after step k. By the definition of Algorithm 1 the only way for this to happen is if $q \in P^+$.

Only If: We shall show the contrapositive that if there exist two nodes v_1 and v_2 in $N(q)$ such that $val(v_1) \neq val(v_2)$ then $q \notin P^+$. If there exist two nodes v_1 and v_2 in $N(q)$ such that $val(v_1) \neq val(v_2)$ then using the same reasoning as in Lemma 7 it follows that $\bar{T}_0$ violates $p \rightarrow q$ and so by Lemma 4 we must have that $q \notin P^+$. □

Lemma 9 *If there is a path q in P_Σ such that $Last(q) \in \mathbf{E}$ then $q \in P^+$ iff $|N(q)| = 1$ in $\bar{T}_0$.*

Proof.

If: We prove the result by induction on the number of steps in generating $\bar{T}_0$. Initially the result is true since p is the only node in P^+ and $Last(p) \notin \mathbf{E}$. Suppose then it is true after iteration k. Then by definition of the algorithm, the only case where we can have a path q such that $|N(q)| \neq 1$ after step k but $|N(q)| = 1$ after step $k+1$ is if $q \in P^+$.

Only If: Suppose $|N(q)| \neq 1$. By the construction of $\bar{T}_0$ it can easily be seen that $N(p)$ contains only two nodes and val of the nodes are equal. Thus using the same arguments as used in Lemma 7 that $p \rightarrow q$ is violated in $\bar{T}_0$ which is a contradiction since $q \in P^+$ and by Lemma 4 $\bar{T}_0$ satisfies $p \rightarrow q$. Hence we conclude that $|N(q)| = 1$. □

Theorem 3 *Algorithm 2 correctly computes P^+ when $Last(p) \notin \mathbf{E}$.*

Proof. We firstly show that if $q \in P^+$ then $p \rightarrow q$ is in Σ^+. We show this by induction on the number of iterations in computing P^+. At line 1 P^+ contains p and *root* and $p \in P^+$ by axiom A1 and *root* $\in P^+$ by axiom A8. At line 2 the result follows by axiom A6 and at line 3 by axiom A7. Hence the inductive hypothesis is true at the commencement of the loop. Let P_j^+ denote the computation of P^+ after iteration j. Assume then that the hypothesis is true after iteration $j-1$. If the q is added to P_j^+

because of line 4 then $p \to q$ is in P^+ because of axiom A5 and the induction hypothesis. If q is added at line 5 then $r \in P^+$ by the induction hypothesis and axiom A3. If q is added because line 6 then $q \in P^+$ by axiom A4. If q is added as a result of line 7 then $q \in P_j^+$ because of axioms A6 and A7.

Next we show that if $p \to q \in \Sigma^+$ then $q \in P^+$. We firstly claim that $\bar{T}_0$ satisfies Σ (note that this does not follows from Lemma 4 since we are using P^+ as input to Algorithm 1 rather than Σ). Let $r \to s$ be any XFD in Σ. Suppose firstly that $r = root$. If $r \to s$ is violated in $\bar{T}_0$ then by Lemma 7 $p \to s$ must be violated. However since $root \to s \in \Sigma$ then we must have that $s \in P^+$ or else s could be added at line 6 contradicting the definition of P^+. So by Lemma 4 $p \to s$ is satisfied in $\bar{T}_0$ hence we conclude that $r \to s$ is satisfied in $\bar{T}_0$ or else by Lemma 7 $p \to s$ is violated which is a contradiction. Suppose then that $r \to s$ is violated in $\bar{T}_0$ and $r \neq root$. The first way for this to happen is if there exist two path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(s)$ such that $x_1 \neq y_1$, where $x_1 = \{v | v \in \{\bar{v}_1, \cdots, \bar{v}_n\} \wedge v \in N(r \cap s)\}$ and $y_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(r \cap s)\}$, and there exists $v_1 \in Nodes(x_1, r)$ and $v_2 \in Nodes(y_1, r)$ such that $val(v_1) = val(v_2)$. For this to happen it follows from Lemma 8 that $r \in P^+$. We must also have that $s \in P^+$ or else s could be added to P^+ by line 5 thus contradicting the definition of P^+. However, if $s \in P^+$ then by Lemma 4 $p \to s$ is satisfied in $\bar{T}_0$ which contradicts the assumption that $r \to s$ is violated in $\bar{T}_0$ by Lemma 7. We conclude that in this case $r \to s$ is satisfied. The second way that $r \to s$ could be violated in $\bar{T}_0$ is if there exist two path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(s)$ such that $x_1 = y_1$. If $x_1 = root$ then $r \cap s = root$ and so $s \in P^+$ or else it could be added at line 6 contradicting the definition of P^+. If $r \to s$ is violated, then by Lemma 7 $p \to s$ is violated which contradicts Lemma 4 and so we conclude that $r \to s$ is satisfied. Assume then that $x_1 \neq root$ and so by Lemma 6 and since $x_1 = y_1$ there exists s' such that $s' \in Anc(s)$ and $s' \in P^+$. It follows that $r \cap s$ is a prefix of s' and so $r \cap s \in Anc(s)$ and thus $r \cap s \in P^+$ or else it could be added at line 7 of Algorithm 2 which contradicts the definition of P^+. Next, since $r \cap s \in P^+$ it follows that s must be in P^+ or else it could be added at line 4 since $r \cap s$ is a prefix of r which contradicts the definition of P^+. However, by Lemma 4 $\bar{T}_0$ satisfies $p \to s$ and if $r \to s$ is violated in $\bar{T}_0$ then $p \to s$ is violated in $\bar{T}_0$ by Lemma 7 which is a contradiction and so $r \to s$ must be satisfied in $\bar{T}_0$.

To complete the proof suppose that $p \to q \in \Sigma^+$. Since $\bar{T}_0$ satisfies Σ then $\bar{T}_0$ also satisfies $p \to q$. Suppose firstly that $Last(q) \in \mathbf{E}$. If $|N(q)| \neq 1$ then using a similar argument to Lemma 7 this would imply that $p \to q$ is violated in $\bar{T}_0$ which is a contradiction and so $|N(q)| = 1$. Then by Lemma 9 $q \in P^+$. Suppose instead that that $Last(q) \notin \mathbf{E}$. By definition of Algorithm 1, there exists two nodes in $N(q)$, say v_1 and v_2, since $Last(q) \notin \mathbf{E}$ and non element nodes are not removed in the Algorithm. If $val(v_1) \neq val(v_2)$ then using similar arguments to those used in Lemma 7 it follows that $\bar{T}_0$ violates $p \to q$ which is a contradiction. Thus we must have that $val(v_1) = val(v_2)$ and so by Lemma 8 $q \in P^+$. This completes the proof. $\square$

Case 2: $Last(p) \in \mathbf{E}$

First construct a tree T_1 with the following properties. T_1 is complete w.r.t. P_Σ. For p and every path q such that q is a prefix of p or q is an attribute node and its parent is a prefix of p, T_1 contains exactly one path instance. For every other path in P_Σ, T_1 contains exactly two path instances. Also, the *val* of any node in T_1 is distinct. Such a tree always exists. The construction procedure is illustrated by the following example.

Example 5 Let $\Sigma = \{\texttt{root.A.B.B\#} \rightarrow \texttt{root.A.B.C.C\#}, \texttt{root.A.B.C.C\#} \rightarrow \texttt{root.A.A\#}, \texttt{root.A.D.D\#} \rightarrow \texttt{root.E.E\#}\}$ and let p be the path `root.A.B`. Then the tree T_1 is shown in Figure 10.

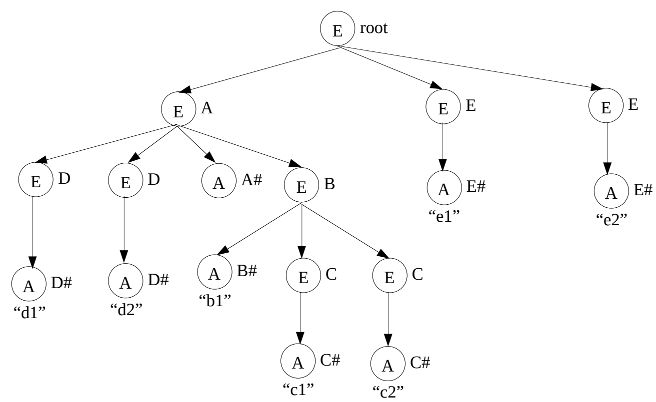

Figure 10: A XML tree

The next step is using as input the set of XFDs returned in P^+ by Algorithm 2 and the tree T_1 generate the tree $\bar{T}_1$ using Algorithm 1. We note that it follows from Lemma 4 that $\bar{T}_1$ satisfies P^+. We now prove some preliminary lemmas before establishing the main result.

Lemma 10 *Let $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ be two distinct path instances in $Paths(q)$ in T_1 for any path q appearing in P_Σ. Then $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ have a common node that is not the root only if $q \cap p \neq root$.*

Proof. Suppose to the contrary that $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ have a common node $\bar{v}_j$ that is not the root and $q \cap p = root$. Then $\bar{v}_j \in N(s)$ for some path s such that s is a prefix of q. So because of the definition of T_1 there must exist two path instances in $Paths(s)$ and so there exists another node $\bar{v}^1_j$ in $N(s)$ that is distinct from $\bar{v}_j$. Then since $q \cap p = root$, it follows that s is not a prefix of p. Hence there must be two path instances of s in T_1 and using the same reasoning as in Lemma 5 shows that this leads to a contradiction. □

Lemma 11 *Let q be any path in P_Σ. Then if there exist two distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ in $\bar{T}_1$ such that $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ have a common node that is not the root then either (there exists s' such that $s' \in Anc(q)$ and $s' \in P^+$) or $q \cap p \neq root$.*

Proof. We prove the result by induction on the number of steps in constructing $\bar{T}_1$. Initially the result is true for T_0 by Lemma 10. Assume inductively then that it is true after iteration $k-1$. The only way that we can have that $\bar{v}_1.\cdots.$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ have a common non root node after iteration k is if we merge two ancestor nodes of $\bar{v}_n$ and $\bar{v}'_n$. For this to happen by definition of Algorithm 1 we have that s such that $s \in Anc(q)$ and $s \in P^+$. □

Lemma 12 *If a XFD $r \to s$ is violated in $\bar{T}_1$ then $p \to s$ is violated in $\bar{T}_1$.*

Proof. If $r \to s$ is violated then there exist distinct paths $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(s)$ such that $val(\bar{v}_n) \neq val(\bar{v}'_n)$. However by the construction of $\bar{T}_1$, $N(p)$ contains only one node and so if we compute $x_1 = \{v | v \in \{\bar{v}_1, \cdots, \bar{v}_n\} \wedge v \in N(p \cap s)\}$ and $y_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(p \cap s)\}$. Then $x_1 = y_1$ since T is a tree and so by definition of a XFD $p \to s$ is violated. □

Lemma 13 *If there is a path q in P_Σ such that $Last(q) \notin \mathbf{E}$ then $q \in P^+$ iff there exist two distinct nodes v_1 and v_2 in $N(q)$ in $\bar{T}_1$ such that $val(v_1) = val(v_2)$.*

Proof.

If: We prove the result again by induction on the number of steps in constructing $\bar{T}_1$. Initially the result is true since there is no path q and two nodes v_1 and v_2 in $N(q)$ in $\bar{T}_1$ such that $val(v_1) = val(v_2)$. Assume then it is true after iteration $k-1$. Suppose then that there exist two nodes in in $N(q)$ such that $val(v_1) \neq val(v_2)$ before step k and $val(v_1) = val(v_2)$ after step k. By the definition of Algorithm 1 the only way for this to happen is if $q \in P^+$.

Only If: We shall show the contrapositive that if there exist two nodes v_1 and v_2 in $N(q)$ such that $val(v_1) \neq val(v_2)$ then $q \notin P^+$. If there exist two nodes v_1 and v_2 in $N(q)$ such that $val(v_1) \neq val(v_2)$ then using the same reasoning as in Lemma 12 it follows that $\bar{T}_1$ violates $p \to q$ and so by Lemma 4 we must have that $q \notin P^+$. □

Lemma 14 *If there is a path q in P_Σ such that $Last(q) \in \mathbf{E}$ then $q \in P^+$ iff $|N(q)| = 1$ in $\bar{T}_1$.*

Proof.

If: We prove the result by induction on the number of steps in generating $\bar{T}_1$. Initially, by definition of T_1, if $|N(q)| = 1$ then either $q = p$ or q is a prefix of p or $Last(q) \in \mathbf{A}$ and the $Parnt(q)$ is a prefix of p. If $q = p$ then $q \in P^+$ by line 1. If q is a prefix of p then $q \in Anc(p)$ and so $q \in P^+$ by line 2. If $Last(q) \in \mathbf{A}$ and the $Parnt(q)$ is a prefix of p then $q \in P^+$ by line 3. Hence at the start of the repeat loop the result is true. Suppose then it is true after iteration k. Then by definition of the Algorithm 1, the only case where we can have a path q such that $|N(q)| \neq 1$ after step k but $|N(q)| = 1$ after step $k+1$ is if $q \in P^+$.

Only If: Suppose to the contrary that $|N(q)| \neq 1$. By the construction of $\bar{T}_1$ it can easily be seen that $N(p)$ contains only one node. Hence if we define $x_1 = \{v | v \in \{\bar{v}_1, \cdots, \bar{v}_n\} \wedge v \in N(p \cap q))\}$ and $y_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(p \cap q)\}$ then $x_1 = y_1$ since $|N(p)| = 1$ and so by the definition of a XFD $p \to q$ is violated in $\bar{T}_1$. This is a contradiction since $q \in P^+$ and by Lemma 4 $\bar{T}_1$ satisfies $p \to q$ and so we conclude that $|N(q)| = 1$. □

Lemma 15 *Let $r \to s$ be a XFD in Σ, let $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ be path instances in $Paths(s)$ in $\bar{T}_1$, and let $x_1 = \{v | v \in \{v_1, \cdots, v_n\} \wedge v \in N(r \cap s)\}$ and $y_1 = \{v | v \in \{v'_1, \cdots, v'_n\} \wedge v \in N(r \cap s)\}$. Then if $p \cap s$ is a strict prefix of $r \cap s$ and $p \to r \notin \Sigma$ and $p \to s \notin \Sigma$ then $x_1 \neq y_1$.*

Proof. The claim of the lemma can best be illustrated by a diagram. Let s_1 denote the path instance $\bar{v}_1.\cdots.\bar{v}_n$ and let s_2 denote the path instance $\bar{v}'_1.\cdots.\bar{v}'_n$. Then the claim of the lemma is that only the situation illustrated in (b) of Figure 11 can arise, and not the situation illustrated in (a) of Figure 11. We prove the result by induction on the number of steps to generate $\bar{T}_1$. Firstly we claim that T_1 cannot have the structure illustrated in (a) of Figure 11. Suppose that it has. Then since by definition of T_1 there has to be two path instances for every path and $x_1 = y_1$, there must be another distinct node in $N(r \cap s)$. However, using the same argument as in Lemma 5 shows that we then contradict the fact that either there are exactly two path instances for any path in T_1 or we contradict the fact that T_1 is complete. Hence T_1 must have the structure shown in (b) of Figure 11. Assume inductively then that the property holds after iteration $k-1$ of Algorithm 1. The only way that (b) of Figure 11 could possibly arise is if we merged path instances of r or path instances of s but this cannot occur because of the definition of Algorithm 1 and the assumptions that $p \to r \notin \Sigma$ and $p \to s \notin \Sigma$. □

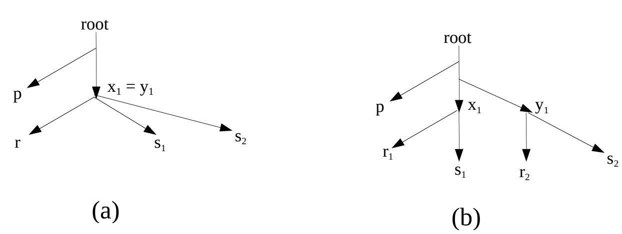

Figure 11: A XML tree illustrating Lemma 15

Theorem 4 *Algorithm 2 correctly computes P^+ when $Last(p) \in \mathbf{E}$.*

Proof. The proof that if $q \in P^+$ then $p \to q$ is in Σ^+ is the same as for Theorem 3.

Next we show that if $p \to q \in \Sigma^+$ then $q \in P^+$. We firstly claim that $\bar{T}_1$ satisfies Σ. Let $r \to s$ be any XFD in Σ. If $r = root$ then it follows from Lemma 12 and Lemma 4 that $r \to s$ is satisfied in $\bar{T}_1$. Suppose then that $r \to s$ is violated in $\bar{T}_1$ and $r \neq root$. The first way for this to happen is if there exist two path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(s)$ such that $x_1 \neq y_1$, where $x_1 = \{v | v \in \{\bar{v}_1, \cdots, \bar{v}_n\} \wedge v \in N(r \cap s)\}$ and $y_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(r \cap s)\}$, and there exist $v_1 \in Nodes(x_1, p)$ and $v_2 \in Nodes(y_1, p)$ such that $val(v_1) = val(v_2)$. Also, since $x_1 \neq y_1$, v_1 and v_2 are distinct. For this to happen it follows from Lemma 13 that $r \in P^+$. We must also have that $s \in P^+$ or else s could be added to P^+ at line 5 thus contradicting the definition of P^+. However, if $s \in P^+$ then by Lemma 4 $p \to s$ is satisfied in $\bar{T}_1$ which contradicts the assumption that $r \to s$ is violated in $\bar{T}_1$ by Lemma 12. We conclude that $r \to s$ is satisfied. The second way that $r \to s$ could be violated in $\bar{T}_1$ is if there exist two path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(s)$ such that $x_1 = y_1$. If $x_1 = root$ then the same arguments as used in Theorem 3 shows that $r \to s$ is satisfied. If $x_1 \neq root$ then by Lemma 11 there either exists s' such that $s' \in Anc(s)$ and $s' \in P^+$ or $s \cap p \neq root$. Consider the first possibility. Using the same arguments as in Theorem 3 shows that $r \to s$ is satisfied. Consider then the second situation where $s \cap p \neq root$. There are two cases to consider: (a) $r \cap s = root$ and (b) $r \cap s \neq root$. Consider (a). Since $r \to s \in \Sigma$ and $r \cap s = root$ then $s \in P^+$ or else it could be added at line 6 thus contradicting the definition of P^+. So if $r \to s$ is violated in $\bar{T}_1$ then by Lemma 12 $p \to s$ is violated which contradicts Lemma 4 and so we conclude that $r \to s$ is satisfied in $\bar{T}_1$. Consider (b). We now consider the subcases: (b.1) $p \cap s$ is a strict prefix of $r \cap s$ and (b.2) $p \cap s$ is not a strict prefix of $r \cap s$. Consider (b.1). Suppose $p \to s \in \Sigma$. Then $s \in P^+$ or else it could be added at line 5 which contradicts the definition of P^+. So by Lemma 4 $p \to s$ is satisfied in $\bar{T}_1$. Suppose then that $p \to r \in \Sigma$. Then $r \in P^+$ or else it could be added at line 5 which contradicts the definition of P^+ and so $s \in P^+$ or else it could be added at line 5 which contradicts the definition of P^+. Assume then that $p \to s \notin \Sigma$ and that $p \to r \notin \Sigma$. Then this case cannot arise because by Lemma 15 this would imply that $x_1 \neq y_1$ which contradicts the assumption that $x_1 = y_1$. Consider (b.2). Since $p \cap s$ is not a strict prefix of $r \cap s$ then $r \cap s$ is a prefix of p and so we must have that $r \cap s \in P^+$ or else it could be added at line 2 which contradicts the definition of P^+. Then since $r \cap s \in P^+$ it follows that $s \in P^+$ or else it could be added at line 4 since $r \cap s$ is a prefix of $r \cap s$ and $r \cap s$ is a prefix of r and s. Hence using the same arguments as previously it follows that $r \to s$ is satisfied.

To complete the proof suppose that $p \to q \in \Sigma^+$. Since $\bar{T}_1$ satisfies Σ then $\bar{T}_1$ also satisfies $p \to q$. Suppose firstly that $Last(q) \in \mathbf{E}$. If $|N(q)| \neq 1$ then using a similar argument to Lemma 12 this would imply that $p \to q$ is violated in $\bar{T}_1$ which is a contradiction and so $|N(q)| = 1$. Then by Lemma 14 $q \in P^+$. Suppose instead that that $Last(q) \notin \mathbf{E}$. By definition of Algorithm 1, there exists two nodes in $N(q)$, say v_1 and v_2, since $Last(q) \notin \mathbf{E}$ and non element nodes are not removed in Algorithm 1. If $val(v_1) \neq val(v_2)$ then using similar arguments to those used in Lemma 12 it follows that $\bar{T}_1$ violates $p \to q$ which is a contradiction. Thus we must have that $val(v_1) = val(v_2)$ and so by Lemma 13 $q \in P^+$. This completes the proof. $\square$

6 Conclusions

In this paper we have investigated issues related to the functional dependencies in XML. Such constraints are important because of the close relationship between XML and relational databases and also because of the importance of functional dependencies in developing a theory of normalization. In an associated paper [22] we defined functional dependencies in XML (XFDs) and provided a set of axioms for reasoning about XFD implication. In this paper we have proven prove that the axioms are also complete for unary XFDs. The second contribution of the paper has been to prove that the implication problem for unary XFDs is decidable and to provide a linear time algorithm for it. These results have considerable significance in the development of a theory of normalization for XML documents. In relational databases, the classic results on soundness and completeness of Armstrong's axioms [4] and the resulting closure algorithm for FD implication play an essential role in determining whether a relation is in one of the classic normal forms. Similarly, the results in this paper are an important first step in the development of algorithms for testing the normal form proposed in [22].

There are several other issues related to the one investigated in this paper that we intend to investigate in the future. The main results of this paper have only been established for unary XFDs and there is a need to extend the results to arbitrary XFDs. Secondly, the approach adopted in this paper is based on the strong satisfaction approach to XFD satisfaction but the techniques we have used can also be extended to defining weak satisfaction. In this case there is a need to a develop complete and sound axiom system for implication of weak XFDs as well as determining if the implication of weak satisfaction is decidable and if so to develop an efficient algorithm for it. Thirdly, there is a need to investigate the extension of the other important class of constraints in relational databases, namely multivalued dependencies (MVDs) [13], to XML We have already completed some preliminary work on this problem [21] and in particular we have defined MVDs in XML and proposed a 4NF for XML and shown that it eliminates redundancy. However, important issues such as axiom systems for MVDs and the interaction between XFDs and MVDs in XML have yet to be investigated.

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7 Appendix

—— **Proof of Theorem 1**

Axiom A1 is immediate from the definition of a XFD.

Consider A2. Suppose that there exists two distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ satisfying the conditions in Definition 14. Then by Definition 14 we must have that $\exists i, 1 \leq i \leq k$, such that $x_i \neq y_i$ if $Last(p_i) \in \mathbf{E}$ else $\exists i, 1 \leq i \leq k$, such that $\perp \notin Nodes(x_i, p_i)$ and $\perp \notin Nodes(y_i, p_i)$ and $val(Nodes(x_i, p_i)) \cap val(Nodes(y_i, p_i)) = \phi$. This condition will still hold for the XFD $p, p_1, \cdots, p_k \rightarrow q$ and so A2 is sound.

Consider A3. Suppose that there exists two distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(s)$ such that. Then since $q \rightarrow s$ is satisfied, we must have that $x'_1 \neq y'_1$ where $x'_1 = \{v | v \in \{v_1, \cdots, \bar{v}_n\} \wedge v \in N(q \cap s)\}$ and $y'_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(q \cap s)\}$.. Then there must exist path instances $\bar{t}_1.\cdots.\bar{t}_n$ and $\bar{t}'_1.\cdots.\bar{t}'_n$ in $Paths(q)$ such that x'_1 is in $\bar{t}_1.\cdots.\bar{t}_n$ and y'_1 is in $\bar{t}'_1.\cdots.\bar{t}'_n$. Since $x'_1 \neq y'_1$ the two path instances must be distinct. Also, since $q \rightarrow s$ is satisfied we must have that $val(\bar{t}_n) \neq val(\bar{t}'_n)$. So by the definition of a XFD and since $p_1, \cdots, p_k \rightarrow q$ is satisfied we must have that $\exists i, 1 \leq i \leq k$, such that $x_i \neq y_i$ (where x_i and y_i are defined as in Definition 14) if $Last(p_i) \in \mathbf{E}$ else $\exists i, 1 \leq i \leq k$, such that $\perp \notin Nodes(x_i, p_i)$ and $\perp \notin Nodes(y_i, p_i)$ and $val(Nodes(x_i, p_i)) \cap val(Nodes(y_i, p_i)) = \phi$ and so $p_1, \cdots, p_k \rightarrow s$ is satisfied.

Consider A4. Suppose that there exists two distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ satisfying the conditions $((\bar{v}_n = \perp \wedge \bar{v}'_n = \perp) \vee (\bar{v}_n \neq \perp \wedge \bar{v}'_n = \perp) \vee (\bar{v}_n = \perp \wedge \bar{v}'_n \neq \perp) \vee (\bar{v}_n \neq \perp \wedge \bar{v}'_n \neq \perp \wedge val(\bar{v}_n) \neq val(\bar{v}'_n))$. Then since $p_i \cap q = root$, it follows that $x_i = y_i = root$ for all i, $1 \leq i \leq k$ and so $p_1, \cdots, p_k \rightarrow q$ is violated which is a contradiction. Hence there cannot exist distinct path instances $\bar{v}_1.\cdots.\bar{v}_n$ and $\bar{v}'_1.\cdots.\bar{v}'_n$ in $Paths(q)$ satisfying the conditions $((\bar{v}_n = \perp \wedge \bar{v}'_n = \perp) \vee (\bar{v}_n \neq \perp \wedge \bar{v}'_n = \perp) \vee (\bar{v}_n = \perp \wedge \bar{v}'_n \neq \perp) \vee (\bar{v}_n \neq \perp \wedge \bar{v}'_n \neq \perp \wedge val(\bar{v}_n) \neq val(\bar{v}'_n))$ and so any XFD $p'_1, \cdots, p'_j \rightarrow q$ is automatically satisfied.

Consider A5. Suppose firstly that $p \cap q$ is prefix of p' and p' is a prefix of p and that $p' \rightarrow q$ is violated. Then since p' is a prefix of p we can ignore the case where $p' = p$ and assume that $Last(p') \in \mathbf{E}$. So for $p' \rightarrow q$ to be violated we must have that $x'_1 = y'_1$ where $x'_1 = \{v | v \in \{v_1, \cdots, \bar{v}_n\} \wedge v \in N(p' \cap q)\}$ and $y'_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(p' \cap q)\}$. However since p' is a prefix of p and $p \cap q$ is prefix of p', it follows that $x_1 = x'_1$ and $y_1 = y'_1$, where $x_1 = \{| v \in \{v_1, \cdots, \bar{v}_n\} \wedge v \in N(p \cap q)\}$ and $y_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(p \cap q)\}$. Thus it follows that $x_1 = y_1$ and so $Nodes(x_1, p) = Nodes(y_1, p)$ and so $p \rightarrow q$ is violated which is a contradiction and so $p' \rightarrow q$ is satisfied. Next suppose that $p \cap q$ is prefix of p' and p' is a prefix of q and that $p' \rightarrow q$ is violated. Then since p' is a prefix of p we can ignore the case where $p' = q$ and assume that $Last(p') \in \mathbf{E}$. So for $p' \rightarrow q$ to be violated we must have that $x'_1 = y'_1$ where $x'_1 = \{v | v \in \{v_1, \cdots, \bar{v}_n\} \wedge v \in N(p' \cap q)\}$ and $y'_1 = \{v | v \in \{v'_1, \cdots, \bar{v}'_n\} \wedge v \in N(p' \cap q)\}$. Then since $p \cap q$ is a prefix of p' this implies that $x_1 = y_1$ which implies a contradiction as before.

Consider A6. Suppose that there exists two distinct path instances $\bar{v}_1. \cdots .\bar{v}_n$ and $\bar{v}'_1. \cdots .\bar{v}'_n$ in $Paths(q)$ such that $val(\bar{v}_n) \neq val(\bar{v}'_n)$. Then because q is a prefix of p, $p \cap q = q$ and so $x_1 = \bar{v}_n$ and $y_1 = \bar{v}'_n$. Thus $p \rightarrow q$ is satisfied since $val(\bar{v}_n) \neq val(\bar{v}'_n)$

Consider A7. Suppose that there exists two distinct path instances $\bar{v}_1. \cdots .\bar{v}_n$ and $\bar{v}'_1. \cdots .\bar{v}'_n$ in $Paths(q)$ such that $val(\bar{v}_n) \neq val(\bar{v}'_n)$. Then because $Last(q) \in \mathbf{A}$ $Parent(\bar{v}_n) \neq Parent(\bar{v}'_n)$. Also, by definition of x_1 and y_1, $x_1 = Parent(\bar{v}_n)$ and $y_1 = Parent(\bar{v}'_n)$ and thus $x_1 \neq y_1$ and so $Parnt(q) \rightarrow q$ is Satisfied since $Last(Parnt(q)) \in \mathbf{E}$.

Axiom A8 is automatic since there is only one path instance that ends with v_r. □

—— **Proof of Theorem 2**

Let Σ be a set of XFDs and let Σ^+ be the set of XFDs obtained by using Axioms A1 - A8. Let $p \rightarrow q$ be a XFD that is not in Σ^+. Then to show completeness it suffices to show that there exists a tree T that satisfies Σ but not $p \rightarrow q$. We consider several cases.

<u>Case A: $Last(p) \in \mathbf{E}$</u>

We now consider several subcases. The only cases that can arise are: (a) $p > q$; (b) $q > p$; (c) $p \not> q$ and $q \not> p$. We firstly note that because of Axiom A6 case (a) cannot arise so the only cases to consider are (b) and (c). We consider (c) first.

<u>Case AA: $p \not> q$ and $q \not> p$</u>

Let $\{p_1 \rightarrow q, \ldots, p_n \rightarrow q\}$ be the set of all XFDs in Σ^+ which have q on the r.h.s (we note that Σ^+ can be computed using Algorithm 2 in Section 5). Consider the paths $\{p_1 \cap q, \ldots, p_n \cap q\}$. Since each of these paths is a prefix of q we can order the set $\{p_1 \cap q, \ldots, p_n \cap q\}$ according to $>$. Let p_{min} be the minimum of $\{p_1 \cap q, \ldots, p_n \cap q\}$. We firstly claim that $p_{min} \neq root$. If not, then there exists $p_i \rightarrow q$ such that $p_i \cap q = root$ and so by A4 $p \rightarrow q \in \Sigma^+$ which is a contradiction. Next we claim that $p_{min} \rightarrow q$. This follows from the definition of p_{min} and axiom A5. Define the node p_{branch} by $p_{branch} = Parnt(p_{min})$.

Construct then a tree T with the following properties. T is complete w.r.t. P_Σ. For all paths p' such that $p' \cap p_{branch}$ is a strict prefix of p_{branch}, T contains one path instance for p'. If $p' \cap p_{branch}$ is not a strict prefix of p_{branch} then T contains exactly two path instances for p'. Moreover, if $Last(p') \notin \mathbf{E}$ then the val of the two nodes in $N(p')$ are distinct if $p' \rightarrow q \in \Sigma^+$ otherwise they are the same. Such a tree always exists. It is also clear from this construction that T violates $p \rightarrow q$. We also note that T has the property that p_{min} (and hence p_{branch}) cannot be a prefix of p. If it was then $p \rightarrow p_{min}$ by A6 and since $p_{min} \rightarrow q$ then by A3 $p \rightarrow q \in \Sigma^+$ which is a contradiction.

We illustrate the construction by an example. Let $p \rightarrow q$ be the XFD `root.X` $\rightarrow$ `root.A.B.C.D.E.E#` and let $\Sigma = \{$`root.A.B.C.C#` $\rightarrow$ `root.A.B.C.D.E.E#`, `root.A.B.C.D.D#` $\rightarrow$ `root.A.B.C.D.E.E#`, `root.X.X#` $\rightarrow$ `root.A`$\}$. Then the above construction procedure yields the tree T shown in Figure 12.

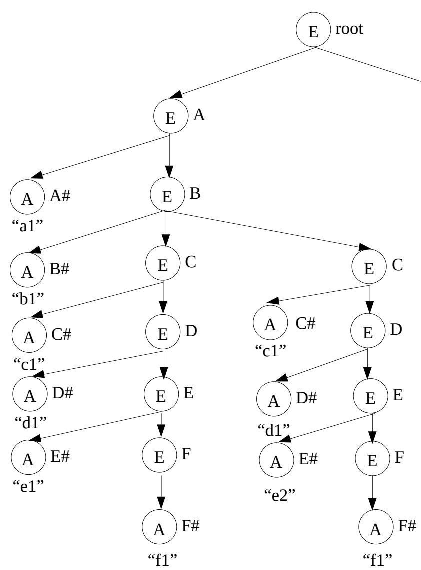


Figure 12: A XML tree

We note that in the construction of T, the correct place to branch the tree is crucial. If the tree branches above or below p_{branch} then T will not satisfy Σ.

It is clear from this construction that T violates $p \rightarrow q$ so it remains to prove that T satisfies Σ. Let $p' \rightarrow q'$ be any XFD (not necessarily in Σ). There are several cases to consider depending on where p' and q' are in the tree T. The different cases can be best illustrated by Figure 13. In this figure we use subscripts to denote different instances of a path. For example, q_1 and q_2 denote different path instances of the path q and q_{v1} and q_{v2} denote different path instances of the path q_v. We shall consider all possible cases where $p' \rightarrow q'$ could be violated in T and show that either $p' \rightarrow q'$ cannot be in Σ or T satisfies $p' \rightarrow q'$.

Case AAA: $p' = p_x$, i.e. $p' \cap q = root$

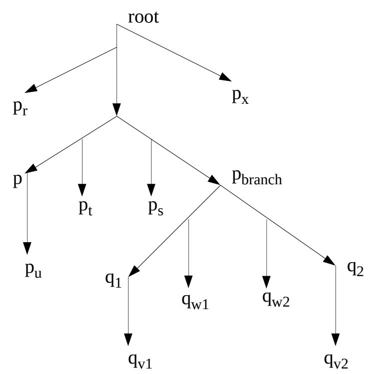


Figure 13: A XML tree

Case AAA.1: $q_w > q' \geq p_{min}$

Suppose that $p' \rightarrow q' \in \Sigma$. Since $q' \geq p_{min}$ and $q' \in \mathbf{E}$ (since $q_w > q'$) it follows from A6 or A1 that $q' \rightarrow p_{min}$. Then since $p_{min} \rightarrow q$ and applying A3 twice we derive that $p' \rightarrow q$. Then since $p' \cap q = root$ by A4 we derive that $p \rightarrow q \in \Sigma^+$ which is a contradiction and so we conclude that $p' \rightarrow q' \notin \Sigma$.

Case AAA.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$ and $p' \rightarrow q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \rightarrow q \in \Sigma^+$ and so by A3 $p' \rightarrow q$. By A4 this implies that $p \rightarrow q \in \Sigma^+$ which is a contradiction and so we conclude that either $p' \rightarrow q'$ is satisfied in T or $p' \rightarrow q' \notin \Sigma$.

Case AAA.3: $q' = q$

If $p' \rightarrow q' \in \Sigma$ then by A4 $p \rightarrow q \in \Sigma^+$ which is a contradiction and so we conclude that $p' \rightarrow q' \notin \Sigma$.

Case AAA.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose $p' \rightarrow q' \in \Sigma$. Since $q' > q$ and $q > p_{min}$, then $q' \rightarrow p_{min} \in \Sigma^+$ by A6 and since $p_{min} \rightarrow q$ by A3 this implies that $q' \rightarrow q$. So if $p' \rightarrow q'$ then by A3 $p' \rightarrow q$ so by A4 $p \rightarrow q \in \Sigma^+$ which is a contradiction and so $p' \rightarrow q' \notin \Sigma$.

Case AAA.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for Case AAA.2

Case AAB: $p_x > p'$

As for Case AAA.

Case AAC: $p' = p_r$, i.e. $p \cap q \geq p' \cap q > root$

Case AAC.1: $q' = q$

If $p' \rightarrow q' \in \Sigma$ then it contradicts the definition of p_{min} and so $p' \rightarrow q' \notin \Sigma$.

Case AAC.2: $q_w > q' \geq p_{min}$

Suppose that $p' \to q' \in \Sigma$. Since $q' \geq p_{min}$ and $q' \in \mathbf{E}$ it follows from A6 that $q' \to p_{min}$. Then since $p_{min} \to q$ and applying A3 twice we derive that $p' \to q \in \Sigma^+$ which contradicts the definition of p_{min} and so $p' \to q' \notin \Sigma$.

Case AAC.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \to q \in \Sigma^+$ and so by A3 $p' \to q$. By A4 this implies that $p \to q \in \Sigma^+$ which contradicts the definition of p_{min} and so we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case AAC.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$. Since $q' > q$ and $q > p_{min}$, then $q' \to p_{min} \in \Sigma^+$ and since $p_{min} \to q$ it follows by A3 that $p' \to q$ which contradicts the definition of p_{min} and so $p' \to q' \notin \Sigma$.

Case AAC.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

If $p' \to q'$ is violated in T, then it follows by construction of T that $q' \to q$ and so by A3 $p' \to q$ which contradicts the definition of p_{min} and so $p' \to q'$ is satisfied.

Case AAD: $p_r > p'$

Case AAD.1: $q' = q$

Assume that $p' \to q' \in \Sigma$. If p' is a prefix of p then by A6 $p \to p'$ and so since $p' \to q$ it follows by A3 that $p \to q \in \Sigma^+$ which is a contradiction and so $p' \to q' \notin \Sigma$. If p' is not a prefix of p then it follows that from the fact that $p' \to q$ and A5 that $p' \cap p \to q$ and since $p \to p' \cap p$ by A6 then applying A3 derives the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case AAD.2: $q_w > q' \geq p_{min}$

Assume that $p' \to q' \in \Sigma$. Then $p' \cap q \to q$ by A4. However by definition of p', $p' \cap q$ is a prefix of p and so by A6 $p \to p' \cap q$ and so by A3 $p \to q$ which is a contradiction and so $p' \to q' \notin \Sigma$.

Case AAD.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \to q \in \Sigma^+$ and so by A3 $p' \to q$. However using the same argument as in AAD.2, if $p' \to q$ then $p \to q \in \Sigma^+$ which is a contradiction and so we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case AAD.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Assume that $p' \to q' \in \Sigma$. As in AAC.4 we derive that $p' \to q$ and so using the same argument as in AAD.2 we derive the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case AAD.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As in AAC.5 we derive that $p' \to q$ and so using the same argument as in AAD.2 we derive the contradiction that $p \to q$.

Case AAE: $p' = p_s$, i.e. $p_{min} > p' \cap q > p \cap q$

As for case for Case AAC.

Case AAF: $p_s > p'$

We can assume that $p \cap q$ is a prefix of p' or else the case reduces to case AAD.

Case AAF.1: $q' = q$

Assume that $p' \rightarrow q' \in \Sigma$. Since $p' \rightarrow q$, by A5 we have that $p' \cap q \rightarrow q$ and since $p_s > p'$ it follows that $p' \cap q$ is a strict prefix of p_{min} which contradicts the definition of p_{min} and so $p' \rightarrow q' \notin \Sigma$.

Case AAF.2: $q_w > q' \geq p_{min}$

Suppose that $p' \rightarrow q' \in \Sigma$. Since $p_{min} \rightarrow q$ it follows from A5 and the definition of q' that $q' \cap q \rightarrow q$. However since $q' \geq p_{min}$ then $Last(q') \in \mathbf{E}$ and so since $q' \cap q$ is prefix of q' it follows that $q' \rightarrow q' \cap q$. So applying A3 twice we derive that $p' \rightarrow q$ which contradicts the definition of p_{min} and so $p' \rightarrow q' \notin \Sigma$.

Case AAF.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$ and $p' \rightarrow q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \rightarrow q \in \Sigma^+$ and so by A3 $p' \rightarrow q \in \Sigma^+$ which again contradicts the definition of p_{min}. So we conclude that either $p' \rightarrow q'$ is satisfied in T or $p' \rightarrow q' \notin \Sigma$.

Case AAF.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$. Since $q' > q$ and $q > p_{min}$, then $q' \rightarrow p_{min} \in \Sigma^+$ by A6. Then since $p_{min} \rightarrow q$, it follows by applying A3 three times that $p' \rightarrow q \in \Sigma^+$ which contradicts the definition of p_{min} and so $p' \rightarrow q' \notin \Sigma$.

Case AAF.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$ and $p' \rightarrow q'$ is violated in T. If $p' \rightarrow q'$ is violated in T then by definition of T we must have that $q' \rightarrow q$ and so by A3 $p' \rightarrow q$ which contradicts the definition of p_{min}. So we conclude that either $p' \rightarrow q'$ is satisfied in T or $p' \rightarrow q' \notin \Sigma$.

Case AAG: $p' = p_t$, i.e. $p > p' \cap q > p \cap q$

Case AAG.1: $q' = q$

If $p' \rightarrow q \in \Sigma$ then by A5 $p' \cap q \rightarrow q \in \Sigma^+$ and by A6 $p \rightarrow p' \cap q$ and so by A3 $p \rightarrow q \in \Sigma^+$ which is a contradiction.and so $p' \rightarrow q \notin \Sigma$.

Case AAG.2: $q_w > q' \geq p_{min}$

If $p' \rightarrow q' \in \Sigma$, then by A6 $q' \rightarrow p_{min}$ and since $p_{min} \rightarrow q$ it follows by applying A3 twice that $p' \rightarrow q$. Then by A5 $p' \cap q \rightarrow q$ and since $p \rightarrow p' \cap q$ by A6 we derive the contradiction $p \rightarrow q \in \Sigma^+$ and so $p' \rightarrow q' \notin \Sigma$.

Case AAG.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$ and $p' \rightarrow q'$ is violated in T. Then for this to happen the two nodes in

$N(q')$ must have different $val's$ and so by the construction of T, $q' \to q \in \Sigma^+$ and so by A3 $p' \to q \in \Sigma^+$ which leads to a contradictionas in Case AAD. So we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case AAG.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Using the same reasoning as in Case AAG.2, if $p' \to q' \in \Sigma$, then we derive the contradiction $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case AAG.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then by the construction of T, for this to happen we must have that $q' \to q$ and so by A3 $p' \to q$. Then using the same reasoning as in AAG.2 we derive the contradiction $p \to q \in \Sigma^+$. So we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case AAH: $p_t > p' \geq p_{min}$

As for case AAG.

Case AAI: $p' = p_u$, i.e. $p' > p$ and $Last(p') \notin \mathbf{E}$

Case AAI.1: $q' = q$

If $p' \to q \in \Sigma$ then since p is a prefix of p' it follows by A5 that $p \to q \in \Sigma^+$ which is a contradiction and so $p' \to q \notin \Sigma$.

Case AAI.2: $q_w > q' \geq p_{min}$

Same as Case AAG.2.

Case AAI.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for Case AAG.3.

Case AAI.4: $q' > q$ and $Last(q') \in \mathbf{E}$

As for AAG.2.

Case AAI.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for case AAG.5.

Case AAJ: $p_u > p' > p$.

As for Case AAI.

Case AAK: $p' = q_w$, i.e. $q > p' \cap q \geq p_{min}$ and $Last(p') \notin \mathbf{E}$

Case AAK.1: $q' = q$

By the construction of T, if $p' \to q' \in \Sigma$ then the two nodes in $N(p')$ have distinct val and so $p' \to q'$ is satisfied in T.

Case AAK.2: $q_w > q' \geq p_{min}$

By A6 it follows that $q' \to p_{min}$ and since $p_{min} \to q$ it follows by A3 that $q' \to q$. So if $p' \to q'$ then by A3 we have that $p' \rightarrow q$. Hence by the construction of T the two nodes in $N(p')$ must have different $val's$ and so by the definition of a XFD $p' \rightarrow q'$ must be satisfied in T.

Case AAK.3: $q' > q$ and $Last(q') \in \mathbf{E}$

As for case AAG.2 it follows that $q' \rightarrow q$, and so from A3 $p' \rightarrow q$ which, using the same reasoning as in case AAK.2, implies that $p' \rightarrow q'$ is satisfied in T.

Case AAK.4: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$ and $p' \rightarrow q'$ is violated in T. For this to happen we must have the two nodes in $N(p')$ have the same val and the two nodes in $N(q')$ have different $val's$. However, by the construction of T if the two nodes in $N(q')$ have different $val's$ then $q' \rightarrow q \in \Sigma^+$. So applying A3 we derive that $p' \rightarrow q \in \Sigma^+$ and by the definition of T this implies that the two nodes in $N(p')$ must have different $val's$ which is a contradiction. So $p' \rightarrow q' \notin \Sigma$ or $p' \rightarrow q'$ is satisfied in T.

Case AAL: $p' = q_v$, i.e. i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

Case AAL.1: $q' = q$

As for case AAK.1.

Case AAL.2: $q_w > q' \geq p_{min}$

As for case AAK.2

Case AAL.3: $q' > q$ and $Last(q') \in \mathbf{E}$

As for case AAK.3

Case AB: $q > p$

We firstly note that because of axiom A7 we can rule out the case where $q \in Att(p)$. Let $\{p_1 \rightarrow q, \ldots, p_n \rightarrow q\}$ be the set of all XFDs in Σ^+ which have q on the r.h.s (we note that Σ^+ can be computed using Algorithm 2 in Section 6). Consider the paths $\{p_1 \cap q, \ldots, p_n \cap q\}$. Since each of these paths is a prefix of q we can order the set $\{p_1 \cap q, \ldots, p_n \cap q\}$ according to $>$. Let p_{min1} be the minimum of $\{p_1 \cap q, \ldots, p_n \cap q\}$ such that $p_{min1} > p$. We note that p_{min1} and p are comparable since both are prefixes of q. Define the node $p_{branch1}$ by $p_{branch1} = Parnt(p_{min1})$. We also note that since $p_i \rightarrow q$, it follows from axiom A5 that $p_{min1} \rightarrow q$.

Construct then a tree T with the following properties T is complete w.r.t. P_Σ. For all paths p' such that $p' \cap p_{branch1}$ is a strict prefix of $p_{branch1}$, T contains one path instance for p'. If $p' \cap p_{branch1}$ is not a strict prefix of $p_{branch1}$ then T contains two path instances for p'. Moreover, if $Last(p') \notin \mathbf{E}$ then the val of the two nodes in $N(p')$ are distinct if $p' \rightarrow q \in \Sigma^+$ otherwise they are the same. Such a tree always exists. It is also clear from this construction that T violates $p \rightarrow q$. As before, we note that the decision of where to branch the tree is critical in the construction of a tree which satisfies Σ. We claim that T satisfies Σ. As before we let $p' \rightarrow q'$ be any XFD (not necessarily in Σ). The various cases that can arise are illustrated in Figure 14. In this figure, as previously, we use subscripts to denote different instances of a path. We shall consider all possible cases where $p' \rightarrow q'$ could be violated in T and show that either $p' \rightarrow q'$ cannot be in Σ or T satisfies $p' \rightarrow q'$.

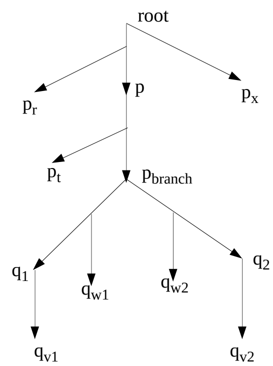


Figure 14: A XML tree

Case ABA: $p' = p_x$, i.e. $p' \cap q = root$

Case ABA.1: $q_w > q' \geq p_{min1}$

Suppose that $p' \rightarrow q' \in \Sigma$. Then by A6 $q' \rightarrow p_{min1}$ and $p_{min1} \rightarrow q$ and so by A3 $p' \rightarrow q \in \Sigma^+$. Then by A4 $p \rightarrow q \in \Sigma^+$ which is a contradiction and so $p' \rightarrow q' \notin \Sigma$.

Case ABA.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for case AAA.2.

Case ABA.3: $q' = q$

As for case AAA.3.

Case ABA.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$. Since $q' > q$ and $q > p_{min1}$, then $q' \rightarrow p_{min1} \in \Sigma^+$ by A6 and since by definition $p_{min1} \rightarrow q$ by A3 this implies that $q' \rightarrow q$. As for case ABA.1 this implies $p \rightarrow q \in \Sigma^+$ which is a contradiction and so $p' \rightarrow q' \notin \Sigma$.

Case ABA.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for Case AAA.2

Case ABB: $p_x > p'$

As for Case AAA.

Case ABC: $p' = p_r$, i.e. $p \cap q \geq p' \cap q > root$

Case ABC.1: $q' = q$

If $p' \rightarrow q \in \Sigma$ then by A5 $p \rightarrow q \in \Sigma^+$ which is a contradiction and so $p' \rightarrow q \notin \Sigma$.

Case ABC.2: $q_w > q' \geq p_{min1}$

Suppose that $p' \rightarrow q' \in \Sigma$. Since $q' \geq p_{min1}$ and since $q' \in \mathbf{E}$ it follows from A6 that $q' \rightarrow p_{min1}$.

Also since $p_{min1} \to q$ by applying A3 twice we derive that $p' \to q$ and so by A5 $p \to q \in \Sigma^+$ which is a contradiction and so $p' \to q' \notin \Sigma$.

Case ABC.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \to q \in \Sigma^+$ and so by A3 $p' \to q \in \Sigma^+$ which, as for case ABC.2, is a contradiction. So we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case ABC.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Since $q' > q$ and $q > p_{min1}$ then from A6 $q' \to p_{min1}$ and since $p_{min1} \to q$ it follows by A3 that $p' \to q$ which, as in case ABC.2, is a contradiction. So $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case ABC.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for case ABC.3

Case ABD: $p' = p_t$, i.e. $p > p' \cap q > p \cap q$

Case ABD.1: $q' = q$

$p' \to q'$ cannot be in Σ or else it contradicts the definition of p_{min1}.

Case ABD.2: $q_w > q' \geq p_{min1}$

If $p' \to q' \in \Sigma$, then by the same reasoning as in AAG.2 $p' \to q$ which contradicts the definition of p_{min1}.

Case ABD.3: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \to q \in \Sigma^+$ and so by A3 $p' \to q$. By A4 this implies that $p \to q \in \Sigma^+$ which contradicts the definition of p_{min1} and so we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case ABD.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$. Since $q' > q$ and $q > p_{min1}$ then by A6 $q' \to p_{min1}$ and since $p_{min1} \to q$ by A3 it follows that $p' \to q \in \Sigma^+$. However this contradicts the definition of p_{min1} and so $p' \to q' \notin \Sigma$.

Case ABD.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for ABC.3

Case ABE: $p_t > p'$

As for case ABD.

Case ABF: $p' = q_w$, i.e. $q > p' \cap q \geq p_{min}$ and $Last(p') \notin \mathbf{E}$

Case ABF.1: $q' = q$

As for case AAK.1.

Case ABF.2: $q_w > q' \geq p_{min1}$

By A6 it follows that $q' \rightarrow p_{min1}$ and since $p_{min1} \rightarrow q$ it follows by A3 that $q' \rightarrow q$. Then following AAK.2, $p' \rightarrow q'$ must be satisfied in T.

Case ABF.3: $q' > q$ and $Last(q') \in \mathbf{E}$

As for case AAG.2 it follows that $q' \rightarrow q$, and so from A3 $p' \rightarrow q$. So by construction of T the two nodes in $Nodes(p')$ have different $val's$ and so $p' \rightarrow q'$ is satisfied.

Case ABF.4: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for AAK.4.

Case ABG: $p' = q_v$, i.e. $p' > q \cap p'$ and $Last(p') \notin \mathbf{E}$

Case ABG.1: $q' = q$

As for case AAK.1.

Case ABG.2: $q_w > q' \geq p_{min1}$

As for case AAK.2

Case ABG.3: $q' > q$ and $Last(q') \in \mathbf{E}$

As for case AAK.3

Case ABH: $p' = p$

Case ABH.1: $q_w > q' \geq p_{min1}$

Suppose that $p' \rightarrow q' \in \Sigma$. By A6 it follows that $q' \rightarrow p_{min1}$ and since $p_{min1} \rightarrow q$ it follows by A3 that $q' \rightarrow q$. Then since $p' \rightarrow q'$ applying A3 means that $p \rightarrow q \in \Sigma^+$ which is a contradiction and so $p' \rightarrow q' \notin \Sigma$.

Case ABH.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \rightarrow q' \in \Sigma$ and $p' \rightarrow q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \rightarrow q \in \Sigma^+$ and so by A3 $p \rightarrow q$ which is a contradiction. So we conclude that either $p' \rightarrow q'$ is satisfied in T or $p' \rightarrow q' \notin \Sigma$.

Case ABH.3: $q' > q$ and $Last(q') \in \mathbf{E}$

As for Case ABH.1.

Case ABH.4: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for Case ABH.2.

Case B: $Last(p) \notin \mathbf{E}$

There are only two cases two consider: (a) $p > q$; (b) $p \not> q$ and $q \not> p$. We consider (b) first.

Case BA: $p \not> q$ and $q \not> p$

Let p_{min} and p_{branch} be defined as in Case AA. We now consider the two subcases where $p > p_{branch}$ and $p \not> p_{branch}$. We now consider the first case.

Case BAA: $p > p_{branch}$

Construct then a tree T with the following properties. Firstly T is complete w.r.t. P_Σ. For all paths p' such that $p' \cap p_{branch}$ is a strict prefix of p_{branch}, T contains one path instance for p'. If $p' \cap p_{branch}$ is not a strict prefix of p_{branch} then T contains two path instances for p' with the following properties. If $p' = p$ then the *val* of the two nodes in $N(p)$ is the same, otherwise if $Last(p') \notin \mathbf{E}$ then the *val* of the two nodes in $N(p')$ are distinct if $p' \to q \in \Sigma^+$ otherwise they are the same. Such a tree always exists

It is clear from this construction that T violates $p \to q$ so it remains to prove that T satisfies Σ. We let $p' \to q'$ be any XFD (not necessarily in Σ). We shall consider all possible cases where $p' \to q'$ may be violated, and show that either $p' \to q'$ is satisfied in T or $p' \to q'$ cannot be Σ. There are several cases to consider depending on where p' and q' are in the tree. The different cases can be best illustrated by Figure 15.

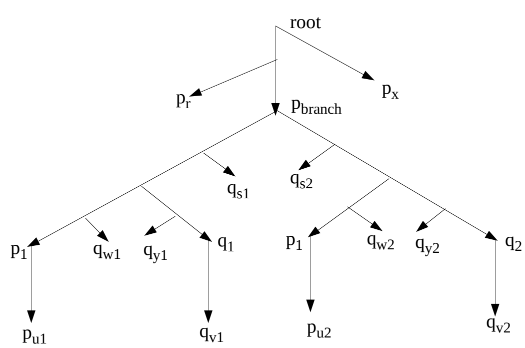


Figure 15: A XML tree

Case BAAA: $p' = p_x$, i.e. $p' \cap q = root$

Case BAAA.1: $q_w > q' \geq p_{min}$

As for Case AAA.1.

Case BAAA.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for Case AAA.2.

Case BAAA.3: $q' = q$

As for Case AAA.3.

Case BAAA.4: $q' > q$ and $Last(q') \in \mathbf{E}$

As for Case AAA.4

Case BAAA.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for Case AAA.2

Case BAAA.6: $q' = q_s$, i.e. $p \cap q \geq p \cap q' \geq p_{min}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then, by construction of T, we have that $q_s \to q \in \Sigma^+$ and so by A3 $p' \to q$ and so by A4 $p \to q \in \Sigma^+$ which is a contradiction. So $p' \to q'$ is satisfied or $p' \to q' \notin \Sigma$.

Case BAAA.7: $q_s > q' \geq p_{min}$

Suppose that $p' \to q' \in \Sigma$. If $q_s > q'$ and $q' \geq p_{min}$ then $q' \to p_{min}$ by A6 and since $p_{min} \to q$ then, by A3, $p' \to q$ which implies a contradiction as in the previous case and so $p' \to q' \notin \Sigma$.

Case BAAA.8: $q' = q_y$, i.e. $q > q' \geq p \cap q$

As for BAAA.6.

Case BAAA.9: $q_y > q' \geq p \cap q$

As for AAA.1.

Case BAAA.10: $q' = p_u$, i.e. $q' > p$ and $Last(q) \notin \mathbf{E}$

If T violates $p' \to q'$, then by construction of T we have that $q' \to q$ and so by A3 $p' \to q$ which implies a contradiction as in case BAAA.6 and so $p' \to q'$ is satisfied.

Case BAAA.11: $p_u > q' > p$

Suppose that $p' \to q' \in \Sigma$. By A6 $q' \to p_{min}$ and, since $p_{min} \to q$ by A3 $p' \to q$ which implies a contradiction as in BAAA.6 and so $p' \to q' \notin \Sigma$.

Case BAAB: $p_x > p'$

As for Case BAAA.

Case BAAC: $p' = p_r$, i.e. $p \cap q \geq p' \cap q > root$

Case BAAC.1: $q_w > q' \geq p_{min}$

As for case AAC.1

Case BAAC.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for case AAC.3

Case BAAC.3: $q' = q$

As for case AAC.1.

Case BAAC.4: $q' > q$ and $Last(q') \in \mathbf{E}$

As for Case AAC.4

Case BAAC.5: $q' = v$

As for Case AAC.5

Case BAAC.6: $q' = q_s$, i.e. $p \cap q \geq p \cap q' \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose $p' \to q' \in \Sigma$. As in case BAAA.6 we can derive that $p' \to q \in \Sigma^+$ which contradicts the definition of p_{min} and so $p' \to q' \notin \Sigma$.

Case BAAC.7: $q_s > q' \geq p_{min}$

Suppose $p' \to q' \in \Sigma$. As in Case BAAA.7 we can derive that $p' \to q$ which contradicts the definition of p_{min} and so $p' \to q' \notin \Sigma$.

Case BAAC.8: $q' = q_y$, i.e. $q > q' \geq p \cap q$

As for BAAC.6.

Case BAAC.9: $q_y > q' \geq p \cap q$

As for BAAC.7.

Case BAAC.10: $q' = p_u$, i.e. $q' > p$ and $Last(q') \notin \mathbf{E}$

Suppose $p' \to q' \in \Sigma$. As in BAAA.10 we can derive that $p' \to q$ which which contradicts the definition of p_{min} and so $p' \to q' \notin \Sigma$.

Case BAAC.11: $p_u > q' > p$

Suppose $p' \to q' \in \Sigma$. As in BAAA.11 we can derive that $p' \to q$ which which contradicts the definition of p_{min} and so $p' \to q' \notin \Sigma$.

Case BAAD: $p_r > p'$

Case BAAD.1: $q' = q$

As for Case AAD.1. The other cases are similar to the corresponding BAAC cases.

Case BAAE: $p' = q_s$, i.e. $p \cap q \geq p \cap q' \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Case BAAE.1: $q_w > q' \geq p_{min}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated . Then since $q' \geq p_{min}$ and $Last(q') \in \mathbf{E}$ then by A6 $q' \to p_{min}$ and $p_{min} \to q$ it follows by applying A3 twice that $p' \to q \in \Sigma^+$. However, by the construction of T if $p' \to q \in \Sigma^+$ then the two nodes in $N(p')$ must have different $val's$. However, for $p' \to q'$ to be violated the two nodes in $N(p')$ must have the same $val's$ which is a contradiction. So we conclude that either $p' \to q'$ is satisfied or $p' \to q' \notin \Sigma$.

Case BAAE.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and that $p' \to q'$ is violated in T. Then for this to happen the two nodes in $N(p')$ must have the same val and the two nodes in $N(q')$ must have different $val's$. So by construction of T, $q' \to q \in \Sigma^+$ and hence by A3 this implies that $p' \to q \in \Sigma^+$. However, by the construction of T, this implies that the two nodes in $N(p')$ must have different $val's$ which is a contradiction and so $p' \to q' \notin \Sigma$ or $p' \to q'$ is satisfied in T.

Case BAAE.3: $q' = q$

If $p' \to q' \in \Sigma$ then by the construction of T the two nodes in $N(p')$ have diffferent $val's$ and so $p' \to q'$ is satisfied.

Case BAAE.4: $q' > q$ and $Last(q') \in \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$. Then by A6 $q' \to p_{min}$ and since $p_{min} \to q$ by A3 we have that $p' \to q \in \Sigma^+$. However, by the construction of T, this implies that the two nodes in $N(p')$ have diffferent

$val's$ and so $p' \to q'$ is satisfied.

Case BAAE.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$. As for case AAC.5 we derive that $p' \to q \in \Sigma^+$ and so, as for case BAAE.4, this implies $p' \to q'$ is satisfied.

Case BAAE.6: $q' = q_y$, i.e. $q > q' \geq p \cap q$

Suppose that $p' \to q'$ is violated in T. For this to happen the two nodes in $N(q')$ must have different $val's$ and so by the definition of T $q' \to q \in \Sigma^+$. Then by A3 $p' \to q \in \Sigma^+$ and so by definition of T the two nodes in $N(p')$ must have different $val's$ which contradicts tha fact that $p' \to q'$ is violated and so $p' \to q'$ is satisfied.

Case BAAE.7: $q_y > q' \geq p \cap q$

As for case BAAE.1.

Case BAAE.8: $q_s > q' \geq p_{min}$

As for BAAA.7 we derive that $p' \to q$. So by the definition of T the two nodes in $N(p')$ must have different $val's$ and so $p' \to q'$ must be satisfied.

Case BAAE.9: $q' = p_u$, i.e. $q' > p$ and $Last(q') \notin \mathbf{E}$

As for case BAAE.2.

Case BAAE.10: $p_u > q' > p$

As for case BAAE.1.

Case BAAF: $p' = q_w$, i.e. $q > p' \cap q \geq p_{min}$ and $Last(p') \notin \mathbf{E}$

Case BAAF.1: $q_w > q' \geq p_{min}$

As for case BAAE.1.

Case BAAF.2: $q' = q_s$, i.e. $p \cap q \geq p \cap q' \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for case BAAE.2.

Case BAAF.3: $q' = q$

As for case BAAE.3.

Case BAAF.4: $q' > q$ and $Last(q') \in \mathbf{E}$

As for case BAAE.4.

Case BAAF.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for case BAAE.5.

Case BAAF.6: $q' = q_y$, i.e. $q > q' \geq p \cap q$

As for BAAE.6.

Case BAAF.7: $q_y > q' \geq p \cap q$

As for case BAAE.7.

Case BAAF.8: $q_s > q' \geq p_{min}$

As for BAAE.8.

Case BAAF.9: $q' = p_u$, i.e. $q' > p$ and $Last(q') \notin \mathbf{E}$

As for case BAAE.2.

Case BAAF.10: $p_u > q' > p$

As for case BAAA.1

Case BAAG: $p' = q_y$, i.e. $q > q' \geq p \cap q$

As for case BAAF.

Case BAAH: $p' = p_u$, i.e. $p' > p$ and $Last(p') \notin \mathbf{E}$

Case BAAH.1: $q_w > q' \geq p_{min}$

As for case BAAE.1.

Case BAAH.2: $q' = q_s$, i.e. $p \cap q \geq p \cap q' \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for case BAAF.2.

Case BAAH.3: $q' = q$

As for case BAAF.3.

Case BAAH.4: $q' > q$ and $Last(q') \in \mathbf{E}$

As for case BAAF.4.

Case BAAH.5: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for case BAAF.5.

Case BAAH.6: $q' = q_y$, i.e. $q > q' \geq p \cap q$

As for BAAF.6.

Case BAAH.7: $q_y > q' \geq p \cap q$

As for case BAAF.7.

Case BAAH.8: $q_s > q' \geq p_{min}$

As for BAAF.8.

Case BAAH.9: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for case BAAE.6.

Case BAAH.10: $p_u > q' > p$

As for case BAAE.1

Case BAAI: $p' = q_v$, i.e. $p' > q$ and $Last(p') \notin \mathbf{E}$

As for case BAAH.

Case BAAJ: $p' = p$

Case BAAJ.1: $q_w > q' \geq p_{min}$

Suppose $p' \to q' \in \Sigma$. Following case BAAE.1 we derive that $p \to q \in \Sigma^+$ which is a contradiction. and so $p' \to q' \notin \Sigma$.

Case BAAJ.2: $q' = q_w$, i.e. $q > q' \cap q \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

Suppose that $p' \to q' \in \Sigma$ and $p' \to q'$ is violated in T. Then for this to happen the two nodes in $N(q')$ must have different $val's$ and so by the construction of T, $q' \to q \in \Sigma^+$ and so by A3 $p' \to q$. By A4 this implies that $p \to q \in \Sigma^+$ which is a contradiction and so we conclude that either $p' \to q'$ is satisfied in T or $p' \to q' \notin \Sigma$.

Case BAAJ.3: $q' > q$ and $Last(q') \in \mathbf{E}$

Assume that $p' \to q' \in \Sigma$. Then as in case BAAE.4 we derive the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case BAAJ.4: $q' = q_v$, i.e. $q' > q$ and $Last(q') \notin \mathbf{E}$

As for case BAAJ.2.

case BAAJ.5: $q' > q$ and $Last(q') \in \mathbf{E}$

Assume that $p' \to q' \in \Sigma$. Then as in case BAAE.4 we derive the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case BAAJ.6: $q' = q_y$, i.e. $q > q' \geq p \cap q$

As for BAAJ.2.

Case BAAJ.7: $q_y > q' \geq p \cap q$

Assume that $p \to q' \in \Sigma$. Then as in case BAAE.1 we derive the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case BAAJ.8: $q' = q_s$, i.e. $p \cap q \geq p \cap q' \geq p_{min}$ and $Last(q') \notin \mathbf{E}$

As for case BAAJ.2.

Case BAAJ.9: $q_s > q'$

Assume that $p \to q' \in \Sigma$. Then as in case BAAC.7 we derive the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case BAAJ.10: $q' = p_u$, i.e. $q' > p$ and $Last(q') \notin \mathbf{E}$

As for case BAAJ.2.

Case BAAJ.11: $p_u > q' > p$

Assume that $p \to q' \in \Sigma$. Then as in case BAAE.1 we derive the contradiction that $p \to q \in \Sigma^+$ and so $p' \to q' \notin \Sigma$.

Case BAB: $p \not> p_{branch}$

Construct a tree T as in Case AB. To show that T satisfies Σ we let $p' \to q'$ be any XFD in Σ. We then consider all possible cases where $p' \to q'$ may be violated, and show that either $p' \to q'$ is satisfied in T or $p' \to q'$ cannot be Σ. The different cases are illustrated in Figure 16. Then the same arguments as in Case AB shows that T satisfies Σ.

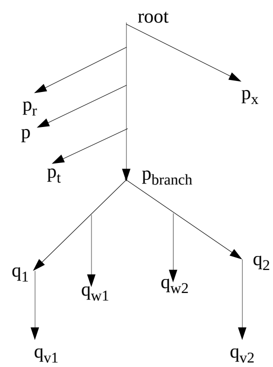


Figure 16: A XML tree

<u>Case BB: $p > q$</u>

The first thing we nore is that $q \neq root$ because of A8. Construct then a tree T as in case BAA. To show that T satisfies Σ we let $p' \rightarrow q'$ be any XFD in Σ. We then consider all possible cases where $p' \rightarrow q'$ may be violated, and show that either $p' \rightarrow q'$ is satisfied in T or $p' \rightarrow q'$ cannot be Σ. The different cases are illustrated in Figure 17. Then the same arguments as in Case BAA shows that T satisfies Σ.

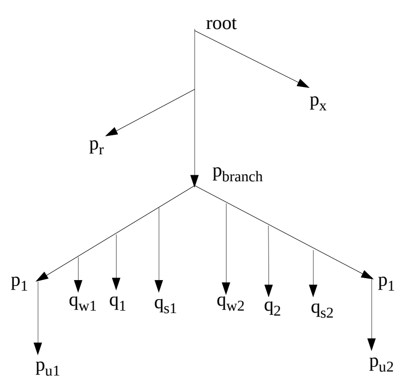
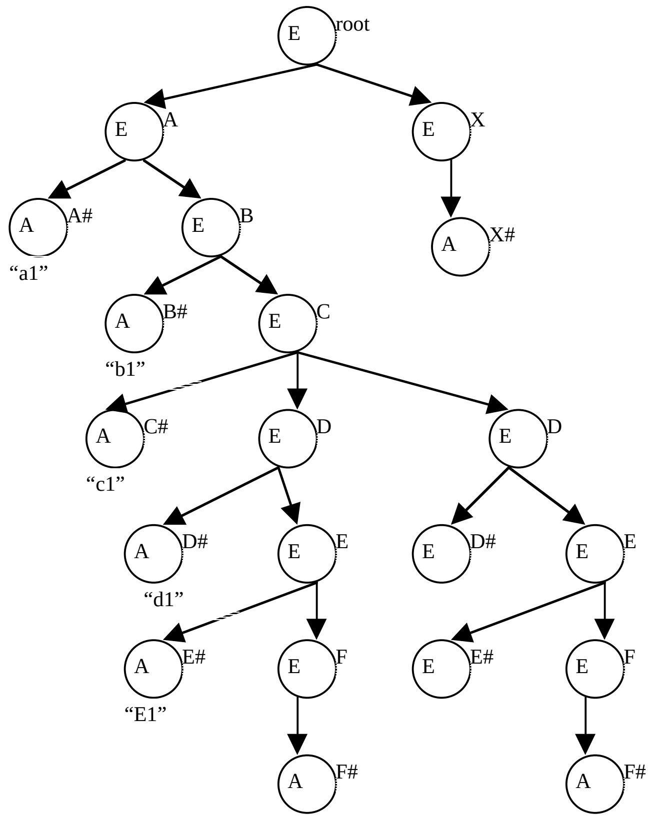


Figure 17: A XML tree

$\square$



E root
E A
E X
A A#
"a1"
E B
A X#
A B#
"b1"
E C
A C#
"c1"
E D
E D
A D#
"d1"
E E
E D#
E E
A E#
"E1"
E F
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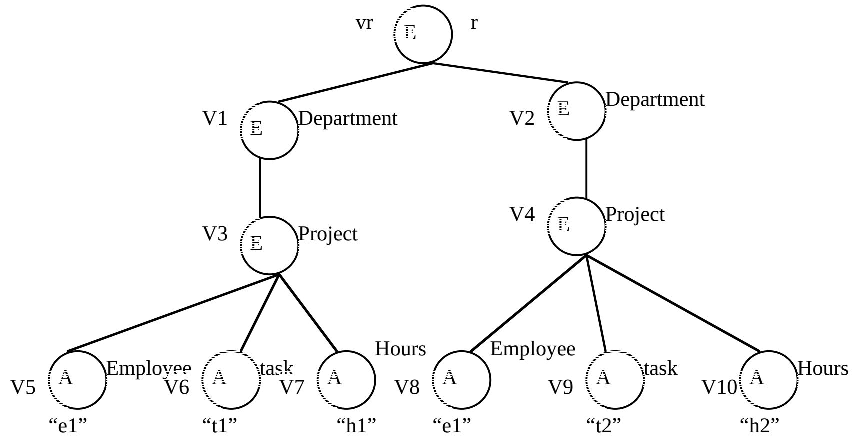
A F#


vr
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V1
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Department
V2
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Department
V3
E
Project
V4
E
Project
V5
A
Employee
"e1"
V6
A
task
"t1"
V7
A
Hours
"h1"
V8
A
Employee
"e1"
V9
A
task
"t2"
V10
A
Hours
"h2"

This is a test

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