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Stochastic Optimization of Smooth Loss

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Let $\phi(z)$ be a smooth loss function, with $|\ell'(z)| \leq L$ and $|\ell'(z) - \ell'(z')| \leq \gamma|z - z'|$. Let $\Omega = \left\{\mathbf{w} \in \mathbb{R}^d : |\mathbf{w}| \leq R\right\}$ be the solution domain. Let $(\mathbf{x}_i, y_i), i = 1, \ldots, n$ be the sequence of i.i.d samples used for training, where $\mathbf{x}_i \in \mathbb{R}^d$ and $y_i \in \{-1, +1\}$. Our goal is to find a solution $\widehat{\mathbf{w}}$ with a good generalization performance. More specifically, let $\ell(\mathbf{w})$ be the expected loss for any solution $\mathbf{w}$, i.e. $\ell(\mathbf{w}) = \mathrm{E}[\ell(y\mathbf{w}^\top\mathbf{x})]$. Our goal is to minimize $\ell(\mathbf{w})$.

A straightforward approach is to optimize $\ell(\mathbf{w})$ by stochastic optimization. Let $w_1 = 0$ be the initial. At each iteration t, we receive a training example $(\mathbf{x}_i, y_i)$, and update the current solution $\mathbf{w}_t$ by

$$\mathbf{w}_{t+1} = \arg\min \pi_\Omega \left(\mathbf{w}_t - \eta\nabla\ell_t(\mathbf{w}_t)\right)$$

where $\eta > 0$ is the stepsize and $\ell_t(\mathbf{w}) = \phi(y_t\mathbf{w}^\top\mathbf{x}_t)$. The final solution $\widehat{\mathbf{w}}$ will be the average of all the solutions, i.e. $\widehat{\mathbf{w}} = \sum_{t=1}^T \mathbf{w}_t/T$. In (Srebro et al., 2010), the authors were able to show that a simple stochastic optimization method, with an appropriate choice of step size η, can achieves the following generalization error bound in expectation, i.e.

$$\mathrm{E}[\ell(\widehat{\mathbf{w}})] \leq \ell(\mathbf{w}_*) + K\left(\frac{t}{n} + \sqrt{\ell(\mathbf{w}_*)\frac{t}{n}}\right)$$

where $t = \gamma|\mathbf{w}_*|^2$.

There are two limitations with the analysis in (Srebro et al., 2010). First, it shows a bound in expectation, not a high probability bound. Second, it requires the knowledge of $\ell(\mathbf{w}_*)$ for tuning the step size in order to achieve the desired bound. In the draft presented in this work, we improve the analysis in (Srebro et al., 2010) by addressing these two limitations.

First, let's address the first limitation by showing a high probability bound. At each iteration, we have

$$\begin{aligned}
\ell_t(\mathbf{w}_t) - \ell_t(\mathbf{w}_*) &\leq \frac{|\mathbf{w}_t - \mathbf{w}_*|^2}{2\eta} - \frac{|\mathbf{w}_{t+1} - \mathbf{w}_*|^2}{2\eta} + \frac{\eta}{2}|\nabla\ell_t(\mathbf{w}_t)|^2 \\
&\leq \frac{|\mathbf{w}_t - \mathbf{w}_*|^2}{2\eta} - \frac{|\mathbf{w}_{t+1} - \mathbf{w}_*|^2}{2\eta} + 2\eta\gamma\ell_t(\mathbf{w}_t)
\end{aligned}$$

where in the last step, we use the property $|\phi'(y_t \mathbf{w}_t^\top \mathbf{x}_t)|^2 \leq 4\gamma\phi(y_t \mathbf{w}_t^\top \mathbf{x}_t)$. By adding the inequalities of all iterations and using the assumption $\eta \leq 1/[2\gamma]$, we have

$$\sum_{t=1}^{T} \ell(\mathbf{w}_t) - \ell(\mathbf{w}_*)) \leq$$
$$\frac{R^2}{2\eta} + 2\eta\gamma \sum_{t=1}^{T} \ell(\mathbf{w}_t) + (-2\eta\gamma + 1) \underbrace{\sum_{t=1}^{T} \ell(\mathbf{w}_t) - \ell_t(\mathbf{w}_t)}_{:=A_T} + \underbrace{\sum_{t=1}^{T} \ell_t(\mathbf{w}_*) - \ell_t(\mathbf{w}_*)}_{:=B_T}$$

To bound A_T and B_T, we need the following bound for martingales.

Theorem 1 *(Bernsteins inequality for martingales). Let $X_1, \ldots, X_n$ be a bounded martingale difference sequence with respect to the filtration $\mathcal{F} = (\mathcal{F}_i)_{1 \leq i \leq n}$ and with $\|X_i\| \leq K$. Let*

$$S_i = \sum_{j=1}^{i} X_j$$

be the associated martingale. Denote the sum of the conditional variances by

$$\Sigma_n^2 = \sum_{t=1}^{n} \mathrm{E}\left[X_t^2 | \mathcal{F}_{t-1}\right]$$

Then for all constants t, $\nu > 0$,

$$\Pr\left[\max_{i=1,\ldots,n} S_i > t \text{ and } \Sigma_n^2 \leq \nu\right] \leq \exp\left(-\frac{t^2}{2(\nu + Kt/3)}\right)$$

and therefore,

$$\Pr\left[\max_{i=1,\ldots,n} S_i > \sqrt{2\nu t} + \frac{\sqrt{2}}{3}Kt \text{ and } \Sigma_n^2 \leq \nu\right] \leq e^{-t}$$

Using the above theorem, with a probability $1 - e^{-t}$, we can bound B_T by

$$B_T \leq \frac{\sqrt{2}t}{3}C + \sqrt{2tC\ell(\mathbf{w}_*)T}$$

where $C = LR + \phi(0)$ and $t = \log(1/\delta)$. To bound A_T, we define martingale difference $X_t = \ell(\mathbf{w}_t) - \ell_t(\mathbf{w}_t)$. Define the conditional variance Σ_T^2 as

$$\Sigma_T^2 = \sum_{t=1}^{T} \mathrm{E}_t\left[X_t^2\right] \leq \sum_{t=1}^{T} \mathrm{E}_t[\ell_t^2(\mathbf{w}_t)] \leq C \sum_{t=1}^{T} \ell(\mathbf{w}_t) = CD_T$$

where $D_T := \sum_{t=1}^{T} \ell(\mathbf{w}_t)$. Using the Berstein inequality for martingale sum, we have

$$
\begin{aligned}
& \Pr\left(A_T \geq 2\sqrt{CD_T\tau} + \sqrt{2}C\tau/3\right) \\
= \quad & \Pr\left(A_T \geq 2\sqrt{CD_T\tau} + \sqrt{2}C\tau/3, \Sigma_T^2 \leq CD_T\right) \\
= \quad & \Pr\left(A_T \geq 2\sqrt{CD_T\tau} + \sqrt{2}C\tau/3, \Sigma_T^2 \leq CD_T, D_T \leq C\right) \\
& + \sum_{i=1}^{m} \Pr\left(A_T \geq 2\sqrt{CD_T\tau} + \sqrt{2}C\tau/3, \Sigma_T^2 \leq CD_T, 2^{i-1}C < D_T \leq 2^i C\right) \\
\leq \quad & \Pr\left(D_T \leq C\right) + \sum_{i=1}^{m} \Pr\left(A_T \geq C\sqrt{2^{i+1}\tau} + \sqrt{2}C\tau/3, \Sigma_T^2 \leq C2^i\right) \\
\leq \quad & \Pr\left(D_T \leq C\right) + me^{-\tau}
\end{aligned}
$$

where $m = \lceil \log_2 T \rceil$. As a result, we have

$$
\Pr\left(A_T \leq 2\sqrt{CD_T t} + \frac{\sqrt{2}}{3}Ct\right) + \Pr(D_T \leq C) \geq 1 - e^{-t}
$$

where $t = \log(1/\delta) + \log m$.

Using the bounds for A_T and B_T, we have, with a probability $1 - 2e^{-t}$,

$$
\begin{aligned}
& (1 - 2\eta\gamma)D_T - \ell(\mathbf{w}_*)T \leq \\
& \quad C + \frac{R^2}{2\eta} + (1 - 2\eta\gamma)\left(2\sqrt{CD_T t} + \frac{\sqrt{2}}{3}Ct\right) + \frac{\sqrt{2}}{3}Ct + \sqrt{2tC\ell(\mathbf{w}_*)T}
\end{aligned}
$$

where $t = \log(1/\delta) + \log m$. Reorganizing the terms in the above inequality, we have

$$
(1 - 2\eta\gamma)\left(D_T - 2\sqrt{CD_T t}\right) - \ell(\mathbf{w}_*)T \leq \frac{R^2}{2\eta} + Ct + \sqrt{2tC\ell(\mathbf{w}_*)T}
$$

where $t = \log(1/\delta) + \log m + 1$. It is easy to verify that $D_T - 2\sqrt{CD_T t}$ is monotonically increasing when $D_T \geq Ct$. Hence, we have, with a probability $1 - 2\delta$,

$$
(1 - 2\eta\gamma)\left(\ell(\widehat{\mathbf{w}}) - 2\sqrt{\frac{Ct}{T}\ell(\widehat{\mathbf{w}})}\right) \leq \ell(\mathbf{w}_*) + \frac{R^2}{2\eta T} + \frac{Ct}{T} + \sqrt{\frac{2Ct}{T}\ell(\mathbf{w}_*)}
$$

or

$$
\ell(\widehat{\mathbf{w}}) - \ell(\mathbf{w}_*) \leq \frac{R^2}{2\eta T} + 2\eta\gamma\ell(\widehat{\mathbf{w}}) + \sqrt{\frac{2Ct}{T}\ell(\mathbf{w}_*)} + 2\sqrt{\frac{Ct}{T}\ell(\widehat{\mathbf{w}})} + \frac{Ct}{T} \tag{1}
$$

By setting $\eta = R/2\sqrt{\gamma T\ell(\widehat{\mathbf{w}})}$, we have, with a probability $1 - \delta$,

$$
\ell(\widehat{\mathbf{w}}) - \ell(\mathbf{w}_*) \leq \sqrt{\frac{2Ct}{T}\ell(\mathbf{w}_*)} + 2\sqrt{\frac{Ct}{T}\ell(\widehat{\mathbf{w}})} + \frac{Ct}{T}
$$

where $t = \log(1/\delta) + \log m + 1 + R^2\gamma/C$. Since $\ell(\mathbf{w}_*) \leq C$ and $\ell(\widehat{\mathbf{w}}) \leq C$, under the assumption $T \geq t$, we have

$$\ell(\widehat{\mathbf{w}}) - \ell(\mathbf{w}_*) \leq \frac{Ct}{T} + 4C\sqrt{\frac{t}{T}} \leq 5C\sqrt{\frac{t}{T}}$$

and therefore, with a probability $1 - 2\delta$

$$\ell(\widehat{\mathbf{w}}) - \ell(\mathbf{w}_*) \leq 4\sqrt{\frac{Ct}{T}\ell(\mathbf{w}_*)} + (2\sqrt{5}+1)\frac{Ct}{T}$$

The above analysis allows us to derive a high probability bound for the proposed algorithm. It however does not resolve the problem of determining the appropriate step size η. We address this limitation by exploring the doubling trick. We divide the learning process into m epoches where the kth epoch is comprised of T_k training examples, with $T_k = T_1 2^{k-1}$. Let $\mathbf{w}_k^1, \ldots, \mathbf{w}_k^{T_k}$ be the sequence of solutions generated by the kth epoch. Define

$$D_k = \frac{1}{T_k}\sum_{i=1}^{T_k} \ell(w_k^i)$$

We assume that, with a probability $1 - \delta$, we have

$$D_k - \ell(\mathbf{w}_*) \leq K\left(\frac{Ct}{T_k} + \sqrt{\frac{Ct}{T_k}\ell(\mathbf{w}_*)}\right)$$

where $t = \log(1/\delta) + \log m + 1 + R^2\gamma/C$. Define $\widehat{\mathcal{D}}_k$ as

$$\widehat{\mathcal{D}}_k = \frac{1}{T_k}\sum_{i=1}^{T_k} \ell_k^i(\mathbf{w}_k^i)$$

where $\ell_k^i(\mathbf{w}) = \phi(y_k^i \mathbf{w}^\top \mathbf{x}_k^i)$. We note that $\widehat{\mathcal{D}}_k$ can be computed from the kth epoch. We would like to bound $\widehat{\mathcal{D}}_k - D_k$ as

$$|\widehat{\mathcal{D}}_k - D_k| = \frac{1}{T_k}\sum_{i=1}^{T} \ell_k^i(\mathbf{w}_k^i) - \ell(\mathbf{w}_k^i)$$

Using the bound for A_T, we have, with a probability $1 - T^2$

$$|\widehat{\mathcal{D}}_k - D_k| \leq 2\sqrt{\frac{Ct}{T_k}D_k} + \frac{\sqrt{2}}{3}\frac{Ct}{T_k}$$

or

$$D_k \leq \frac{C}{T^3}$$

In the second case, since $\mathrm{E}[\widehat{\mathcal{D}}_k] = D_k \leq C/T^3$, using the Markov inequality, we have, with a probability $1 - T^{-2}$,

$$|\widehat{\mathcal{D}}_k - D_k| \leq \frac{C}{T}$$

Combining the above two statements, we have, with a probability $1 - 2T^{-2}$

$$|\widehat{\mathcal{D}}_k - D_k| \leq 2\sqrt{\frac{Ct}{T_k}D_k} + 2\frac{Ct}{T_k}$$

and consequentially,

$$|\widehat{\mathcal{D}}_k - D_k| \leq 6\left(\sqrt{\frac{Ct}{T_k}\widehat{\mathcal{D}}_k} + \frac{Ct}{T_k}\right)$$

We thus will use the following expression as the surrogate for $\ell(\mathbf{w}_*)$

$$\widehat{\ell}_k = \widehat{\mathcal{D}}_k + 6\left(\sqrt{\frac{Ct}{T_k}\widehat{\mathcal{D}}_k} + \frac{Ct}{T_k}\right)$$

Using $\widehat{\ell}_k$, we define the step size η_{k+1} as

$$\eta_{k+1} = \frac{R}{2\sqrt{\gamma T_{k+1}\widehat{\ell}_k}}$$

It is easy to verify that with a probability $1-2T^{-2}$ (i) $\widehat{\ell}_k \geq D_k \geq \ell(\mathbf{w}_*)$ and (ii) $\widehat{\ell}_k - \ell(\mathbf{w}_*) \leq (K+6)\left(\sqrt{\frac{Ct}{T_k}\widehat{\mathcal{D}}_k} + \frac{Ct}{T_k}\right)$. Using the bound in (1), we have

$$(1-2\eta\gamma)(D_{k+1}-\ell(\mathbf{w}_*)) \leq \frac{R^2}{2\eta_{k+1}T_{k+1}} + 2\eta_{k+1}\gamma\ell(\mathbf{w}_*) + 2\sqrt{\frac{Ct}{T_{k+1}}\ell(\mathbf{w}_*)} + 2\sqrt{\frac{Ct}{T_{k+1}}(D_{k+1} - \ell(\mathbf{w}_*))} + \frac{Ct}{T_{k+1}}$$

Using the property $\widehat{\ell}_k \geq \ell(\mathbf{w}_*)$, we have

$$2\eta_{k+1}\gamma\ell(\mathbf{w}_*) \leq R\sqrt{\frac{\gamma}{T_{k+1}}\ell(\mathbf{w}_*)}$$

We also have

$$\frac{R^2}{2\eta_{k+1}T_{k+1}} = R\sqrt{\frac{\gamma}{T_{k+1}}\widehat{\ell}_k} \leq R\sqrt{\frac{\gamma}{T_{k+1}}(K+6)\left[\frac{Ct}{T_k} + \sqrt{\frac{Ct}{T_k}\widehat{\mathcal{D}}_k}\right]}$$

Since

$$\widehat{\mathcal{D}}_k - D_k \leq 2\sqrt{\frac{Ct}{T_k}D_k} + 2\frac{Ct}{T_k}$$

and

$$D_k - \ell(\mathbf{w}_*) \leq K\left(\frac{Ct}{T_k} + \sqrt{\frac{Ct}{T_k}\ell(\mathbf{w}_*)}\right)$$

we have

$$\widehat{\mathcal{D}}_k \leq \ell(\mathbf{w}_*) + K\left(\frac{Ct}{T_k} + \sqrt{\frac{Ct}{T_k}\ell(\mathbf{w}_*)}\right) + 2\frac{Ct}{T_k} + 2\sqrt{K}\frac{Ct}{T_k} + 4\sqrt{K}\frac{Ct}{T_k} + 4\sqrt{K}\sqrt{\frac{Ct}{T_k}\ell(\mathbf{w}_*)}$$

By choosing sufficiently large K, we have

$$\widehat{\mathcal{D}}_k \leq \ell(\mathbf{w}_*) + 2K\left(\frac{Ct}{T_k} + \sqrt{\frac{Ct}{T_k}\ell(\mathbf{w}_*)}\right)$$

Hence,

$$\frac{R^2}{2\eta_{k+1}T_{k+1}} \leq \frac{R\sqrt{2\gamma(K+6)Ct}}{T_{k+1}} + \frac{2R^2\gamma}{T_{k+1}} + 2\sqrt{\frac{2Ct}{T_{k+1}}\ell(\mathbf{w}_*)} + 6\sqrt{2K}\frac{2Ct}{T_{k+1}} + 4\sqrt{2K}\sqrt{\frac{2Ct}{T_{k+1}\ell(\mathbf{w}_*)}}$$

By choosing sufficiently large K, we have

$$\frac{R^2}{2\eta_{k+1}T_{k+1}} \leq \frac{K}{3}\left(\frac{Ct}{T_{k+1}} + \sqrt{\frac{Ct}{T_{k+1}}\ell(\mathbf{w}_*)}\right)$$

We thus have

$$(1-2\eta\gamma)(D_{k+1} - \ell(\mathbf{w}_*)) \leq \frac{2K}{3}\left(\frac{Ct}{T_{k+1}} + \sqrt{\frac{Ct}{T_{k+1}}\ell(\mathbf{w}_*)}\right)$$

By choosing $\eta \leq 1/[6\gamma]$, we have

$$D_{k+1} - \ell(\mathbf{w}_*) \leq K\left(\frac{Ct}{T_{k+1}} + \sqrt{\frac{Ct}{T_{k+1}}\ell(\mathbf{w}_*)}\right)$$

References

Nathan Srebro, Karthik Sridharan, and Ambuj Tewari. Smoothness, low noise and fast rates. In *NIPS*, pages 2199–2207, 2010.