All Roads Lead to Rome

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Legendary mathematician Paul Erdos liked to talk about the perfect proofs of mathematical theorems maintained by God. In [1], six different proofs are given to the infinity of prime numbers. Unlike mathematical proofs which are mentally reproducible objects, signal processing researchers can have an intimate discussion with God by creating computational algorithms which are experimentally reproducible objects. This article presents a class of projection-based solution algorithms to the problem considered in the pioneering work on compressed sensing [2] - perfect reconstruction of a phantom image from 22 radial lines in the frequency domain. Under the framework of projection-based image reconstruction, we will show experimentally that several old and new tools of nonlinear filtering all lead to perfect reconstruction of the phantom image, which suggests that the result achieved by l_1-optimization is less like magic.

I. BACKGROUND

In [2], the authors reported a "puzzling numerical experiment" which obtained the perfect reconstruction of a phantom image at the sampling rate of 50 times smaller than the Nyquist rate[1]. This experiment has motivated the authors of [2] to obtain a nonlinear generalization of Shannon's sampling theorem [3] and others to develop computationally efficient algorithms for l_1-optimization. The rapidly increasing interest in "compressed sensing" (CS) has a significant impact on the community of signal processing - e.g., various special sessions and special issues have been devoted to this topic. At SPIE Conf. on Visual Comm. and Image Proc.'2010, Prof. Changwen Chen at SUNY Buffalo asked an insightful question to the panelists (including the author of this column paper): what new insight do you think CS - the hot computational tool - to signal processing research? This article is based on the follow-up thoughts about Prof. Chen's question; while the author has chosen to indirectly answer his question from an algorithmic perspective - i.e., by demonstrating how the problem of phantom image reconstruction can be solved by a number of standard tools including Perona-Malik (PM) diffusion [4], nonlinear diffusion [5], translation-invariant (TI) thresholding [6] and Shape-adaptive DCT (SA-DCT) [7]. In particular, it will be shown that the classical framework of alternating projection manifests as much magic as l_1-optimization (if not more).

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[1]Strictly speaking, the term of Nyquist rate is only defined for analog signals (not for discrete signals) in the literature of signal processing.

II. Nonlinear Image Filters as Nonexpansive Maps

The limitation of linear filtering on image signals has been recognized as early as 1970s [8]. In late 1980s and early 1990s, two lines of attacks on nonlinear filtering became influential - nonlinear diffusion and nonlinear thresholding. The idea of nonlinear diffusion originated from the scale-space analysis of image signals developed by vision community [9] - the key insight behind Perona and Malik's scheme [4] was to introduce a set of nonlinear diffusion coefficients with edge-stopping capability. Even though the stability of Perona-Malik diffusion has mostly been shown experimentally, it has sparkled a whole new school of thoughts on developing nonlinear tools for image analysis and processing. Two years later, the model of total-variation (TV) diffusion was proposed by Rudin and Osher [10] - the key insight in their work is to replace l_2 by l_1. Another two years later, wavelet shrinkage or thresholding was proposed by Donoho [11] - the key insight there is to recognize the importance of singularities or tails (of heavy tail distributions). The connection between wavelet-shrinkage and TV-diffusion was established much later in 2004 [12]. Similar ideas or extensions related to l_1 have been rediscovered several times including least absolute shrinkage and selection operator (LASSO) [13], basis pursuit (BP) [14] and the latest compressed sensing (CS) [2].

Both diffusion-based and thresholding-based nonlinear filters can be viewed as nonexpansive maps [15] - i.e., $||Pf|| \leq ||f||$. Rigorous proof for the nonexpansiveness of thresholding operators is relatively easy to obtain (e.g., [16]); while similar success has not been achieved for nonlinear diffusion operators (e.g., the convergence of Perona-Malik diffusion has shown to be notoriously difficult to establish analytically, which gives rise to so-called Perona-Malik paradox [17]). Similar sentimental comments can also be made about projection-based texture-synthesis [18] where authors have found experimentally their algorithms converge for all test images despite the lack of convexity of the defined constraint sets. In the mathematical literature, Ekeland seems to be the first one recognizing the importance of nonconvex minimization; at the end of [19], he advocated "to seek some kind of saddle point instead of a minimum". The implication of this proposal into signal processing research seems to be: instead of articulating the objective function for minimization, one could design a constraint set (not necessarily convex) or its associated projection operator (even based on heuristics). Such *projection-first* point of view (in contrast to variational or *energy-first*) arguably better fit the taste of electrical engineers and computational scientists.

III. Image Reconstruction via Alternating Projections

The power of image reconstruction via alternating projections was discovered as early as 1978 by Youla [20]. The idea is extremely simple - as long as we can come up with more than one constraint set for a target, alternating projections onto constraint sets offers a plausible solution to approximate the unknown target. Under the context

of image reconstruction, we often face two kinds of constraint sets: one defined by the observation data (e.g., the Fourier coefficients along 22 radial lines) and the other associated with image prior (a.k.a. regularization functional). Both diffusion-based and thresholding-based nonlinear filters can be interpreted as projection operators onto the prior constraint set; the subtle difference between them is often more on the transient behavior (e.g., the speed of convergence and the route toward it) than their asymptotic one (since both approaches reflect the a priori knowledge about the signals of our interest). A common trick for improving the numerical stability of alternative projection-based algorithms [2] is called deterministic annealing - i.e., one can gradually decrease the diffusion coefficient or threshold parameter as the iteration goes on. Let us denote an image by f, its Fourier transform by F and partial Fourier samples by G. Here is the flow-chart of a generic image reconstruction algorithm via alternating projections.

Algorithm 1. Image Reconstruction via Alternating Projections
Input: observation data G and sampling pattern $\mathbf{S}$;
Output: reconstructed image $\hat{f}$
• Initialization: obtain $\hat{f}^{(0)}$ by ad-hoc back-projection method;
• Main loop: for $k = 0, 1, ..., k_{max}$,
    - Projection onto prior constraint set: apply nonlinear filter to $\hat{f}^{(k)}$ (at the customer's choice);
    - Projection onto observation constraint set: $\hat{F}^{(k)}(m, n) = G(m, n)$ for $\{(m, n) | S(m, n) = 1\}$;

Assuming the function of nonlinear filtering is available as a module, the above algorithm can be easily implemented (fewer than ten lines of MATLAB codes). In our implementation[3], we have reused the tools of PM diffusion, nonlinear diffusion, TI thresholding and SA-DCT thresholding. Fig. 1 includes the evolution of PSNR results for four different nonlinear filters. It can be observed that all of them can achieve perfect reconstruction ($PSNR > 48dB$) - the performance achieved by $l1magic$ (http://www.acm.caltech.edu/l1magic/) with identical experimental setting. The behavior between TV-diffusion and TI thresholding is similar, which supports their theoretic connection established in [12]. What is most interesting observation seems to be the behavior of Perona-Malik diffusion - after a slow start, it rapidly catches up and the PSNR soars into the range of over $80dB$ (lossless up to some rounding errors). A similar "phase-transition" phenomenon has been observed for BM3D-based image reconstruction [21] (BM3D is a more recently-developed and computationally demanding tool of nonlinear filtering).

IV. DISCUSSIONS

So what have we learned? I would argue we - signal processing researchers - can learn three important lessons from the above experiment:

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[2]The deeper reason is only partially known to be connected with overcoming the difficulty with getting trapped by local optimum of a nonconvex function.

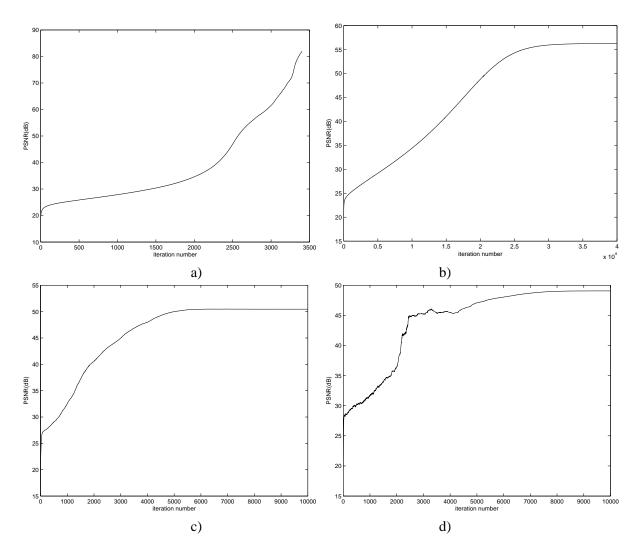[3]The source codes can be accessed at http://www.csee.wvu.edu/~xinl/code/TV_recon.rar

Fig. 1. PSNR Profiles of Alg. 1 with different projection operators: a) PM diffusion [4]; b) nonlinear diffusion [5]; c) TI thresholding [6]; d) SA-DCT thresholding [7].

- All roads lead to Rome. Equipped with powerful analytical skills, mathematicians can prove the conditions for perfect reconstruction of smooth signals; equipped with powerful computational resources, engineers can achieve the same objective by recycling some of old ideas or tools. Mentally reproducible objects such as mathematical theorems and experimentally reproducible objects such as computational algorithms are simply a matter of different taste. Some analytically difficult hard-bones such as Perona-Malik diffusion continues to prevail and offer nice surprises in numerical experiments.

- Understand the limitation of mathematical models. Statistician George Box once said, "All models are wrong; some are useful." The usefulness of any mathematical model relies on how well it matches real-world data. For computer-generated images such as $phantom$, we might find several seemingly-distant models (e.g., diffusion vs. thresholding) are intrinsically connected. It can be further argued that such kind of data represent pathological cases which one do not find in the real world and therefore could mislead our effort on mathematical modeling.

Nevertheless, the model underlying the CS theory is identical to the TV model despite the algorithmic differences between $l1magic$ and TV-diffusion.

- Use computational tools wisely. Like any other tool invented by humans, l_1-optimization is appropriate for certain types of problems. The practice of treating every engineering problem like a nail and attempting to solve it with a hammer (l_1-optimization) has shown to be less fruitful. Of course, we should not blame inventors of those tools by their creation; but realize that it is our own responsibility to choose the right tool to use and use it wisely.

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