A Generalized Disjunctive Paraconsistent Data Model for Negative and Disjunctive Information

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Abstract. This paper presents a generalization of the disjunctive paraconsistent relational data model in which disjunctive positive and negative information can be represented explicitly and manipulated. There are situations where the closed world assumption to infer negative facts is not valid or undesirable and there is a need to represent and reason with negation explicitly. We consider explicit disjunctive negation in the context of disjunctive databases as there is an interesting interplay between these two types of information. *Generalized disjunctive paraconsistent relation* is introduced as the main structure in this model. The relational algebra is appropriately generalized to work on generalized disjunctive paraconsistent relations and their correctness is established.

1 Introduction

Two important features of the relational data model [1] for databases are its value-oriented nature and its rich set of simple, but powerful algebraic operators. Moreover, a strong theoretical foundation for the model is provided by the classical first-order logic [2]. This combination of a respectable theoretical platform, ease of implementation and the practicality of the model resulted in its immediate success, and the model has enjoyed being used by many database management systems.

One limitation of the relational data model, however, is its lack of applicability to nonclassical situations. These are situations involving incomplete or even inconsistent information.

Several types of incomplete information have been extensively studied in the past such as *null* values [3,4], *partial* values [5], *fuzzy* and *uncertain* values [6,7], and *disjunctive* information [8,9].

In this paper, we present a generalization of the disjunctive paraconsistent data model[10]. Our model is capable of representing and manipulating disjunctive positive facts as well as explicit disjunctive negative facts. We introduce *generalized disjunctive paraconsistent relations*, which are the fundamental structures underlying our model. These structures are generalizations of *disjunctive paraconsistent relations* which are capable of representing disjunctive positive and explicit negative definite facts. A generalized disjunctive paraconsistent relation essentially consists of two kinds of information: positive tuple sets representing exclusive disjunctive positive facts (one of which belongs to the relation) and negative tuple sets representing exclusive disjunctive negated facts (one of which does not belong to the relation). Generalized disjunctive paraconsistent relations are strictly more general than disjunctive paraconsistent relations in that for any disjunctive paraconsistent relation, there is a generalized disjunctive paraconsistent relation with the same information content, but not *vice versa*. We define algebraic operators over generalized disjunctive paraconsistent relations that extend the standard operations over disjunctive paraconsistent relations.

2 Motivation

Explicit negation occurs in everyday world where certain values cannot be given to some parameters. In current database systems, negation is implicitly assumed (using closed world assumption [11]) when a particular query has a null answer from the database. But this poses a problem. Consider the following relational database

suppliers

SNUM	SNAME
s1	Haibin
s2	Yuanchun
s3	Raj

parts

PNUM	PNAME
p1	nut
p2	cam
p3	bolt
p4	wheel

supply

SNUM	PNUM
s1	p1
s1	p3
s2	p2
s3	p4

Consider the query "find all suppliers who do not supply part p1 or part p3". Suppose there is a known list of suppliers, then the answer for the query would be { s2, s3 }. This may be a definite answer from the database (augmented with the CWA), but the answer has some indefiniteness because the database may be *incomplete*. Explicit presence of incomplete information in the form of *null values* complicates the problem further. Suppose the tuple (s3,null) is part of the supply relation. Then, we are uncertain whether to include s3 as part of the answer or not. Finally, a similar problem occurs when one allows disjunctive information (such as (s3,p1) or (s3,p2)) as part of the database.

Definite negation can occur without explicit negation in current database systems. The use of functional dependencies provide this facility. Consider the functional dependency that each person can have only one social security number. Hence if we know the SSN for a particular individual then we can explicitly assume the negation of all other possible numbers as the person's social security number.

Sometimes it is important to explicitly include in the database certain negative information. Consider a medical database containing patient information. When a doctor needs to check whether a patient has diabetes, (s)he would be more comfortable with a negative answer generated by the system using definite information (of explicit negative data) than with a negative answer found using the closed world assumption.

We are not considering the use of negative information as an integrity constraint. Rather we are utilizing negative information in query processing to provide definite or disjunctive negation when needed. When a positive data is included in the database, which negates a previous explicit negative information, the new data may be allowed to be entered if the user wishes to enforce it.

In this paper, we consider explicit negation in the context of disjunctive databases. We extend the representation provided in [12] by introducing explicit disjunctive negative facts. There is an interesting interplay between these two kinds of information. Negative facts tend to reduce the amount of incompleteness present in the disjunctive facts as seen by the equivalence $((P \vee Q \vee R) \wedge \neg P) \equiv (Q \vee R) \wedge \neg P$. After introducing generalized disjunctive paraconsistent relations, we present operators to remove redundancies and inconsistencies. We also extend the standard relational algebra to operate on generalized disjunctive paraconsistent relations. The information content of generalized disjunctive paraconsistent relations is characterized in terms of disjucntive paraconsistent relations which we briefly present in the next section.

3 Disjunctive Paraconsistent Relations

In this section, we present a brief overview of disjunctive paraconsistent relations and the algebraic operations on them. For a more detailed description, refer to [10].

Let a *relation scheme* (or just *scheme*) Σ be a finite set of *attribute names*, where for any attribute name $A \in \Sigma$, $dom(A)$ is a non-empty *domain* of values for A. A *tuple* on Σ is any map $t : \Sigma \rightarrow \cup_{A \in \Sigma} dom(A)$, such that $t(A) \in dom(A)$, for each $A \in \Sigma$. Let $\tau(\Sigma)$ denote the set of all tuples on Σ.

Definition 1. A *paraconsistent relation* on scheme Σ is a pair $R = \langle R^+, R^- \rangle$, where R^+ and R^- are any subsets of $\tau(\Sigma)$. We let $\mathcal{P}(\Sigma)$ be the set of all paraconsistent relations on Σ. □

Definition 2. A *disjunctive paraconsistent relation, R,* over the scheme Σ consists of two components $< R^+, R^- >$ where $R^+ \subseteq 2^{\tau(\Sigma)}$ and $R^- \subseteq \tau(\Sigma)$. R^+, the *positive* component, is a set of tuple sets. Each tuple set in this component represents a disjunctive positive fact. In the case where the tuple set is a singleton, we have a definite positive fact. R^-, the *negative* component consists of tuples that we refer to as definite negative tuples. Let $\mathcal{D}(\Sigma)$ represent all disjunctive paraconsistent relations over the scheme Σ. □

Definition 3. Let R be a disjunctive paraconsistent relation over Σ. Then,
$\mathbf{norm}(R)^+ = \{w | w \in R^+ \wedge w \not\subseteq R^-\}$
$\mathbf{norm}(R)^- = R^- - \{t | t \in R^- \wedge (\exists w)(w \in R^+ \wedge t \in w \wedge w \subseteq R^-)\}$ □

A disjunctive paraconsistent relation is called *normalized* if it does not contain any inconsistencies. We let $\mathcal{N}(\Sigma)$ denote the set of all normalized disjunctive paraconsistent relations over scheme Σ.

Definition 4. Let R be a normalized disjunctive paraconsistent relation. Then, $\mathbf{reduce}(R)$ is defined as follows*:*

$$\mathbf{reduce}(R)^+ = \{w' \mid (\exists w)(w \in R^+ \wedge w' = w - R^- \wedge \neg(\exists w_1)(w_1 \in R^+ \wedge (w_1 - R^-) \subset w'))\}$$

$$\mathbf{reduce}(R)^- = R^-$$ □

Definition 5. Let $U \subseteq \mathcal{P}(\Sigma)$. Then,

$$\mathbf{normrep}_\Sigma(U) = U - \{R | R \in U \wedge R^+ \cap R^- \neq \emptyset\}$$ □

The **normrep** operator removes all inconsistent paraconsistent relations from its input.

Definition 6. Let $U \subseteq \mathcal{P}(\Sigma)$. Then,

$$\mathbf{reducerep}_\Sigma(U) = \{R | R \in U \wedge \neg(\exists S)(S \in U \wedge R \neq S \wedge S^+ \subseteq R^+ \wedge S^- \subseteq R^-)\}$$

□

The **reducerep** operator keeps only the "minimal" paraconsistent relations and eliminates any paraconsistent relation that is "subsumed" by others.

Definition 7. The information content of disjunctive paraconsistent relations is defined by the mapping $\mathbf{rep}_\Sigma : \mathcal{N}(\Sigma) \rightarrow \mathcal{P}(\Sigma)$. Let R be a normalized disjunctive paraconsistent relation on scheme Σ with $R^+ = \{w_1, \ldots, w_k\}$. Let $U = \{< \{t_1, \ldots, t_k\}, R^- > | (\forall i)(1 \leq i \leq k \rightarrow t_i \in w_i)\}$. Then,

$$\mathbf{rep}_\Sigma(R) = \mathbf{reducerep}_\Sigma(\mathbf{normrep}_\Sigma(U))$$ □

Definition 8. Let R and S be two normalized disjunctive paraconsistent relations on scheme Σ with $\mathbf{reduce}(R)^+ = \{v_1, \ldots, v_n\}$ and $\mathbf{reduce}(S)^+ = \{w_1, \ldots, w_m\}$. Then, $R \widehat{\cup} S$ is a disjunctive paraconsistent relation over scheme Σ given by $R \widehat{\cup} S = \mathbf{reduce}(T)$, where $T^+ = \mathbf{reduce}(R)^+ \cup \mathbf{reduce}(S)^+$ *and* $T^- = \mathbf{reduce}(R)^- \cap \mathbf{reduce}(S)^-$.
and $R \widehat{\cap} S$ is a disjunctive paraconsistent relation over scheme Σ given by $R \widehat{\cap} S = \mathbf{reduce}(T)$, where T is defined as follows. Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in v_i)\}$ and $F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \leq i \leq m \rightarrow t_i \in w_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \cap F_j$ for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij}*s*. Then,

$$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$$

$$T^- = R^- \cup S^-.$$ □

Definition 9. Let R be a normalized disjunctive paraconsistent relation on scheme Σ, and let F be any logic formula involving attribute names in Σ, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the *selection* of R by F, denoted $\widehat{\sigma}_F(R)$, is a disjunctive paraconsistent relation on scheme Σ, given by $\widehat{\sigma}_F(R) = \mathbf{reduce}(T)$, where $T^+ = \{w | w \in \mathbf{reduce}(R)^+ \wedge (\forall t \in w) F(t)\}$ and $T^- = \mathbf{reduce}(R)^- \cup \sigma_{\neg F}(\tau(\Sigma))$, where σ_F is the usual selection of tuples. □

If Σ and Δ are relation schemes such that $\Sigma \subseteq \Delta$, then for any tuple $t \in \tau(\Sigma)$, we let t^{Δ} denote the set $\{t' \in \tau(\Delta) \mid t'(A) = t(A), \text{ for all } A \in \Sigma\}$ of all extensions of t. We extend this notion for any $T \subseteq \tau(\Sigma)$ by defining $T^{\Delta} = \cup_{t \in T}\ t^{\Delta}$.

Definition 10. Let R be a normalized disjunctive paraconsistent relation on scheme Σ, and $\Delta \subseteq \Sigma$. Then, the *projection* of R onto Δ, denoted $\widehat{\pi}_{\Delta}(R)$, is a disjunctive paraconsistent relation on scheme Δ, given by $\widehat{\pi}_{\Delta}(R) = \mathbf{reduce}(T)$, where $T^{+} = \{\pi_{\Delta}(w) | w \in \mathbf{reduce}(R)^{+}\}$ and $T^{-} = \{t \in \tau(\Delta) | t^{\Sigma \cup \Delta} \subseteq (\mathbf{reduce}(R)^{-})^{\Sigma \cup \Delta}\}$, where π_{Δ} is the usual projection over Δ of tuples. □

Definition 11. Let R and S be normalized disjunctive paraconsistent relations on schemes Σ and Δ, respectively with $\mathbf{reduce}(R)^{+} = \{v_1, \ldots, v_n\}$ and $\mathbf{reduce}(S)^{+} = \{w_1, \ldots, w_m\}$. Then, the *natural join* of R and S, denoted $R \,\widehat{\bowtie}\, S$, is a disjunctive paraconsistent relation on scheme $\Sigma \cup \Delta$, given by $R \,\widehat{\bowtie}\, S = \mathbf{reduce}(T)$, where T is defined as follows. Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in v_i)\}$ and $F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \leq i \leq m \rightarrow t_i \in w_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \bowtie F_j$ for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij}s. Then,

$$T^{+} = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_i \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$$
$$T^{-} = (\mathbf{reduce}(R)^{-})^{\Sigma \cup \Delta} \cup (\mathbf{reduce}(S)^{-})^{\Sigma \cup \Delta}.$$

□

4 Generalized Disjunctive Paraconsistent Relations

In this section, we present the main structure underling our model, the *generalized disjunctive paraconsistent relations*. We identify several types of redundancies and inconsistencies that may appear and provide operators to remove them. Finally, we present the information content of generalized paraconsistent relations.

Definition 12. A *generalized disjunctive paraconsistent relation*, R, over the scheme Σ consists of two components $\langle R^{+}, R^{-} \rangle$ where $R^{+} \subseteq 2^{\tau(\Sigma)}$ and $R^{-} \subset 2^{\tau(\Sigma)}$. R^{+}, the *positive* component, is a set of tuple sets. Each tuple set in this component represens a disjunctive positive fact. In the case where the tuple set is a singleton, we have a definite positive fact. R^{-}, the *negative* component consists of a set of tuple sets. Each tuple set in this component represents a disjunctive negative fact. In the case where the tuple set is a singleton, we have a definite negated fact. Let $\mathcal{GD}(\Sigma)$ represent all generalized disjunctive paraconsistent relatios over the scheme Σ. □

Example 1. Consider the following generalized disjunctive paraconsistent relation:
$supply^{+} = \{\{< s1, p1 >\}, \{< s2, p1 >, < s2, p2 >\}, \{< s3, p3 >, < s3, p4 >\}\}$
$supply^{-} = \{\{< s1, p2 >\}, \{< s1, p3 >\}, \{< s2, p3 >, < s2, p4 >\}\}$. The *positive component* corresponds to the statement $s1$ supplies $p1$, $s2$ supplies $p1$ or $p2$, and $s3$ supplies $p3$ or $p4$ and the *negative component* corresponds to $s1$ does not supply $p2$ and $s1$ does not supply $p3$ and $s2$ does not supply $p3$ or $s2$ does

not supply $p4$. It should be noted that the status of tuples that do not appear anywhere in the generalized disjunctive paraconsistent relation, such as $(s3, p2)$, is unknown. □

Inconsistences can be present in a genearlaized disjunctive paraconsistent relation in two situations. On the one hand, if all the tuples of a tuple set of the posistive component are also present in the union of the singleton tuple set of the negative component. In such a case, the tuple set states that at least one of the tuples in the tuple set must be in the relation whereas the negative component states that all the tuples in the tuple set must not be in the relation. We deal with this inconsistency by removing both the positive tuple set and all its corresponding singleton tuple sets from the negative component. On the other hand, if all the tuples of a tuple set of the negative component are also present in the union of the singleton tuple set of the positive component. In such a case, the tuple set states that at least one of the tuples in the tuple set must not be in the relation whereas the positive component states that all the tuples in the tuple set must be in the relation. We deal with this inconsistency by removing both the negative tuple set and all its corresponding singleton tuple sets from the positive component. This is done by the **g_norm** operator defined as follows:

Definition 13. Let R be a generalized disjunctive paraconsistent relation over Σ. $R^+ = \{w_1, w_2, \cdots, w_n\}$ and $R^- = \{u_1, u_2, \cdots, u_m\}$. Then,

$$\begin{aligned}\mathbf{g\_norm}(R)^+ = R^+ - &\\ &\{w \mid w \in R^+ \wedge w \subseteq \cup u_i \wedge 1 \le i \le m \rightarrow u_i \in R^- \wedge |u_i| = 1\} - \\ &\{w_i \mid 1 \le i \le n \rightarrow w_i \in R^+ \wedge |w_i| = 1 \wedge (\exists u)(u \in R^- \wedge u \subseteq \cup w_i \wedge w_i \subseteq u)\}\\ \mathbf{g\_norm}(R)^- = R^- - &\\ &\{u \mid u \in R^- \wedge u \subseteq \cup w_i \wedge 1 \le i \le n \rightarrow w_i \in R^+ \wedge |w_i| = 1\} - \\ &\{u_i \mid 1 \le i \le m \rightarrow u_i \in R^- \wedge |u_i| = 1 \wedge (\exists w)(w \in R^+ \wedge w \subseteq \cup u_i \wedge u_i \subseteq w)\}\end{aligned}$$

□

A generalized disjunctive paraconsistent relation is called *normalized* if it does not contain any inconsistencies. We let $\mathcal{GN}(\pm)$ denote the set of all normalized generalized disjunctive paraconsistent relations over scheme Σ. We now identify the following four types of redundancies in a normalized generalized disjunctive paraconsistent relation R:

1. $\underline{w_1 \in R^+,\ w_2 \in R^+, \text{ and } w_1 \subset w_2}$. In this case, w_1 subsumes w_2. To eliminate this redundancy, we delete w_2 from R^+.
2. $\underline{u_1 \in R^-,\ u_2 \in R^-, \text{ and } u_1 \subset u_2}$. In this case, u_1 subsumes u_2. To eliminate this redundancy, we delete u_2 from R^-.
3. $\underline{1 \le i \le n,\ w_i \in R^+,\ |w_i| = 1,\ u \in R^-, \text{ and } \cup w_i \subset u}$. This redundancy is eliminated by deleting the tuple set u from R^- and adding the tuple set $u - \cup w_i$ to R^-. Since we are dealing with normalized generalized disjunctive paraconsistent relations, $u - \cup w_i$ cannot be empty.
4. $\underline{1 \le i \le m,\ u_i \in R^-,\ |u_i| = 1,\ w \in R^+, \text{ and } \cup u_i \subset w}$. This redundancy is eliminated by deleting the tuple set w from R^+ and adding the tuple set $w - \cup u_i$

to R^+. Since we are dealing with normalized generalized disjunctive paraconsistent relations, $w - \cup u_i$ cannot be empty.

We now introduce an operator called **g_reduce** to take care of redundancies.

Definition 14. Let R be a normalized generalized disjunctive paraconsistent relation. Then,

$$\begin{aligned}\mathbf{g\_reduce}(R)^+ = \{w' | (\exists w)(w \in R^+ \wedge w' = w - U \wedge \\ \neg(\exists w_1)(w_1 \in R^+ \wedge (w_1 - U) \subset w'))\}\end{aligned}$$

$$\begin{aligned}\mathbf{g\_reduce}(R)^- = \{u' | (\exists u)(u \in R^- \wedge u' = u - W \wedge \\ \neg(\exists u_1)(u_1 \in R^- \wedge (u_1 - W) \subset u'))\}\end{aligned}$$

where, $U = \{u_i | u_i \in R^- \wedge |u_i| = 1\}$ and $W = \{w_i | w_i \in R^+ \wedge |w_i| = 1\}$. □

Example 2. Consider the following generalized disjunctive paraconsistent relation: $R^+ = \{\{< a >\}, \{< b >, < c >\}, \{< c >, < d >\}, \{< a >, < e >\}, \{< f >, < g >\}\}$ and $R^- = \{\{< b >\}, \{< c >, < e >\}, \{< i >\}, \{< d >, < e >, < f >\}\}$. The disjunctive tuple $\{< a >, < e >\}$ is subsumed by $\{< a >\}$ and hence removed. In the disjunctive tuple set $\{< b >, < c >\}$, $< b >$ is redundant due to the presence of the negative singleton tuple set $\{< b >\}$ resulting in the positive tuple $\{< c >\}$ which in turn subsumes $\{< c >, < d >\}$ and makes $\{< c >, < e >\}$ redundant and resulting in $\{< e >\}$ which subsumes the $\{< d >, < e >, < f >\}$. The reduced generalized disjunctive paraconsistent relation is: $\mathbf{g\_reduce}(R)^+ = \{\{< a >\}, \{< c >\}, \{< f >, < g >\}\}$ and $\mathbf{g\_reduce}(R)^- = \{\{< b >\}, \{< e >\}, \{< i >\}\}$ □

The information content of a generalized disjunctive paraconsistent relation can be defined to be a collection of disjunctive paraconsistent relations. The different possible disjunctive paraconsistent relations are constructed by selecting one of the several tuples within a tuple set for each tuple set in the the negative component. In doing so, we may end up with non-minimal disjunctive paraconsistent relations or even with inconsistent disjunctive paraconsistent relations. These would have to be removed in order to obtain the exact information content of generalized disjunctive paraconsistent relations. The formal definitions follow:

Definition 15. Let $U \subseteq \mathcal{D}(\Sigma)$. Then, $\mathbf{g\_normrep}_\Sigma(U) = \{R | R \in U \wedge \neg(\exists w)(w \in R^+ \wedge w \subseteq R^-)\}$ □

The **g_normrep** operator removes all inconsistent disjunctive paraconsistent relations from its input.

Definition 16. Let $U \subseteq \mathcal{D}(\Sigma)$. Then, $\mathbf{g\_reducerep}_\Sigma(U) = \{R | R \in U \wedge \neg(\exists S)(S \in U \wedge R \neq S \wedge S^+ \subseteq R^+ \wedge S^- \subseteq R^-)\}$ □

The **g_reducerep** operator keeps only the "minimal" disjunctive paraconsistent relations and eliminates any disjunctive paraconsistent relation that is "subsumed" by others.

Definition 17. The information content of generalized disjunctive paraconsistent relations is defined by the mapping $\mathbf{g\_rep}_\Sigma : \mathcal{GN}(\Sigma) \to \mathcal{D}(\Sigma)$. Let R be a normalized generalized disjunctive paraconsistent relation on scheme Σ *with* $R^- = \{u_1, \ldots, u_m\}$. Let $U = \{R^+, < \{t_1, \ldots, t_m\} > |(\forall i)(1 \leq i \leq m \to t_i \in u_i)\}$. Then, $\mathbf{g\_rep}_\Sigma(R) = \mathbf{g\_reducerep}_\Sigma(\mathbf{g\_normrep}_\Sigma(U))$ $\square$

Note that the information content is defined only for normalized generalized disjunctive paraconsistent relations.

Example 3. Consider the following generalized disjunctive paraconsistent relation on a single attribute scheme Σ: $R^+ = \{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\} and R^- = \{\{< b >\}, \{< c >, < e >\}, \{< c >, < d >, < g >\}\}$ The process of selecting tuples from tuple sets produces the following disjunctive paraconsistent relations:

$U = \{< \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < c >\}\} >, < \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < c >, < d >\}\} >, < \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < c >, < g >\}\} >, < \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < e >, < c >\}\} >, < \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < e >, < d >\}\} >, < \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < e >, < d >\}\} >\}$.

Normalizing the above set of disjunctive paraconsistent relations using **g_normrep** gives us: $U' = \{< \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < c >\}\} >, < \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < c >, < g >\}\} >\}$.

Finally, removing the non-minimal disjunctive paraconsistent relations using the **g_reducerep** operator, we get the information content $\mathbf{g\_rep}_\Sigma(R)$ as follows: $\mathbf{g\_rep}_\Sigma(R) = \{< \{\{\{< b >, < e >\}, \{< c >, < d >\}, \{< e >, < g >\}\}, \{< b >, < c >\}\} >\}$. $\square$

The following important theorem states that information is neither lost nor gained by removing the redundancies in a generalized disjunctive paraconsistent relations.

Theorem 1. *Let R be a generalized disjunctive paraconsistent relation on scheme Σ. Then,*
$\mathbf{g\_rep}_\Sigma(\mathbf{g\_reduce}(R)) = \mathbf{g\_rep}_\Sigma(R)$ $\square$

5 Generalized Relational Algebra

In this section, we first develop the notion of *precise generalizations* of algebraic operators. This is an important property that must be satisfied by any new operator defined for generalized disjunctive paraconsistent relations. Then, we present several algebraic operators on generalized disjunctive paraconsistent relations that are precise generalizations of their counterparts on disjunctive paraconsistent relations.

Precise Generalization of Operations

It is easily seen that generalized disjunctive paraconsistent relations are a generalization of disjunctive paraconsistent relations, in that for each disjunctive paraconsistent relation there is a generalized disjunctive paraconsistent relation with the same information content, but not *vice versa*. It is thus natural to think of generalising the operations on disjunctive paraconsistent relations, such as union, join, projection etc., to generalized disjunctive paraconsistent relations. However, any such generalization should be intuitive with respect to the *belief system* model of generalized disjunctive paraconsistent relations. We now construct a framework for operators on both kinds of relations and introduce the notion of the precise generalization relationship among their operators.

An n-ary *operator on disjunctive paraconsistent relations with signature* $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a function $\Theta : \mathcal{D}(\Sigma_1) \times \cdots \times \mathcal{D}(\Sigma_n) \to \mathcal{D}(\Sigma_{n+1})$, where $\Sigma_1, \ldots, \Sigma_{n+1}$ are any schemes. Similarly, an n-ary *operator on generalized disjunctive paraconsistent relations with signature* $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a function: $\Psi : \mathcal{GD}(\Sigma_1) \times \cdots \times \mathcal{GD}(\Sigma_n) \to \mathcal{GD}(\Sigma_{n+1})$.

We now need to extend operators on disjunctive paraconsistent relations to sets of disjunctive paraconsistent relations. For any operator $\Theta : \mathcal{D}(\Sigma_1) \times \cdots \times \mathcal{D}(\Sigma_n) \to \mathcal{D}(\Sigma_{n+1})$ on disjunctive paraconsistent relations, we let $\mathcal{S}(\Theta) : 2^{\mathcal{D}(\Sigma_1)} \times \cdots \times 2^{\mathcal{D}(\Sigma_n)} \to 2^{\mathcal{D}(\Sigma_{n+1})}$ be a map on sets of disjunctive paraconsistent relations defined as follows. For any sets $M_1, \ldots, M_n$ of disjunctive paraconsistent relations on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively,

$$\mathcal{S}(\Theta)(M_1, \ldots, M_n) = \{\Theta(R_1, \ldots, R_n) | R_i \in M_i, \text{for all } i, 1 \leq i \leq n\}.$$

In other words, $\mathcal{S}(\Theta)(M_1, \ldots, M_n)$ is the set of Θ-images of all tuples in the cartesian product $M_1 \times \cdots \times M_n$. We are now ready to lead up to the notion of precise operator generalization.

Definition 18. An operator Ψ on generalized disjunctive paraconsistent relations with signature $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is *consistency preserving* if for any normalized generalized disjunctive relations $R_1, \ldots, R_n$ on schemes $\Sigma_1, \ldots, \Sigma_n$, respectively, $\Psi(R_1, \ldots, R_n)$ is also normalized. $\square$

Definition 19. A consistency preserving operator Ψ on generalized disjunctive paraconsistent relations with signature $\langle \Sigma_1, \ldots, \Sigma_{n+1} \rangle$ is a *precise generalization* of an operator Θ on disjunctive paraconsistent relations with the same signature, if for any normalized generalized disjunctive paraconsistent relations $R_1, \ldots, R_n$ on schemes $\Sigma_1, \ldots, \Sigma_n$, we have
$\mathbf{g\_rep}_{\Sigma_{n+1}}(\Psi(R_1, \ldots, R_n)) = \mathcal{S}(\Theta)(\mathbf{g\_rep}_{\Sigma_1}(R_1), \ldots, \mathbf{g\_rep}_{\Sigma_n}(R_n))$.
$\square$

We now present precise generalizations for the usual relation operators, such as union, join, projection. To reflect generalization, a line is placed over an ordinary operator. For example, $\bowtie$ denotes the natural join among ordinary relations, $\dot{\bowtie}$ denotes natural join on paraconsistent relations, $\widehat{\bowtie}$ denotes natural join on disjunctive paraconsistent relations and $\overline{\bowtie}$ denotes natural join on generalized disjunctive paraconsistent relations.

Definition 20. Let R and S be two normalized generalized disjunctive paraconsistent relations on scheme Σ with $\mathbf{g\_reduce}(R)^+ = \{v_1, \ldots, v_n\}$, $\mathbf{g\_reduce}(R)^- = \{u_1, \ldots, u_k\}$ and $\mathbf{g\_reduce}(S)^+ = \{w_1, \ldots, w_m\}$, $\mathbf{g\_reduce}(S)^- = \{x_1, \ldots, x_j\}$. Then, *$R \overline{\cup} S$ is a generalized disjunctive paraconsistent relation over scheme Σ given by $R \overline{\cup} S = \mathbf{g\_reduce}(T)$*, where T is defined as follows. Let $E = \{\{t_1, \ldots, t_k\} | (\forall i)(1 \leq i \leq k \rightarrow t_i \in u_i)\}$ and $F = \{\{t_1, \ldots, t_j\} | (\forall i)(1 \leq i \leq j \rightarrow t_i \in x_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \cap F_j$, for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij}s. Then,
$T^+ = \mathbf{g\_reduce}(R)^+ \cup \mathbf{g\_reduce}(S)^+$
$T^- = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$.
and *$R \overline{\cap} S$ is a generalized disjunctive paraconsistent relation over scheme Σ given by $R \overline{\cap} S = \mathbf{g\_reduce}(T)$*, where T is defined as follows.
Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in v_i)\}$ and $F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \leq i \leq m \rightarrow t_i \in w_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \cap F_j$, for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct A_{ij} s. Then,
$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$.
$T^- = \mathbf{g\_reduce}(R)^- \cup \mathbf{g\_reduce}(S)^-$. □

The following theorem establishes the *precise generalization* property for union and intersection:

Theorem 2. *Let R and S be two normalized generalized disjunctive paraconsistent relations on scheme Σ. Then,*
1. $\mathbf{g\_rep}_\Sigma(R \overline{\cup} S) = \mathbf{g\_rep}_\Sigma(R) \mathcal{S}(\widehat{\cup}) \mathbf{g\_rep}_\Sigma(S)$.
2. $\mathbf{g\_rep}_\Sigma(R \overline{\cap} S) = \mathbf{g\_rep}_\Sigma(R) \mathcal{S}(\widehat{\cup}) \mathbf{g\_rep}_\Sigma(S)$. □

Definition 21. Let R be normalized generalized disjunctive paraconsistent relation on scheme Σ. Then, $\overline{-}R$ is a generalized disjunctive paraconsistent relation over scheme Σ given by
$(\overline{-}R)^+ = \mathbf{g\_reduce}(R)^-$ and $(\overline{-}R)^- = \mathbf{g\_reduce}(R)^+$.

□

Definition 22. Let R be a normalized generalized disjunctive paraconsistent relation on scheme Σ, and let F be any logic formula involving attribute names in Σ, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the *selection of R by F, denoted $\overline{\sigma}_F(R)$*, is a generalized disjunctive paraconsistent relation on scheme Σ, given by $\overline{\sigma}_F(R) = \mathbf{g\_reduce}(T)$, where $T^+ = \{w | w \in \mathbf{g\_reduce}(R)^+ \wedge (\forall t \in w)F(t)\}$ and $T^- = R^- \cup \sigma_{\neg F}(\tau(\Sigma))$, where σ_F is the usual selection of tuples.
□

A disjunctive tuple set is either selected as a whole or not at all. All the tuples within the tuple set must satisfy the selection criteria for the tuple set to be selected.

Definition 23. Let R be a normalized generalized disjunctive paraconsistent relation on scheme Σ with $\textbf{g\_reduce}(R)^- = \{v_1, \ldots, v_n\}$., and $\Delta \subseteq \Sigma$. Then, the *projection of* R *onto* Δ, *denoted* $\overline{\pi}_\Delta(R)$, is a generalized disjunctive paraconsistent relation on scheme Δ, given by $\overline{\pi}_\Delta(R) = \textbf{g\_reduce}(T)$, where T is defined as follows. Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in v_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and let $A_i = \{t \in \pi(\Delta) | t^{\Sigma \cup \Delta} \subseteq (E_i)^{\Sigma \cup \Delta}\}$. *Then,*
$T^+ = \{\pi_\Delta(w) | w \in \textbf{g\_reduce}(R)^+\}$
$T^- = \{w | (\exists t_1) \ldots (\exists t_e)(t_1 \in A_i \wedge \ldots \wedge t_e \in A_e \wedge w = \{t_1, \ldots, t_e\})\}$, where π_Δ is the usual projection over Δ of tuples. □

The positive component of the projections consists of the projection of each of the tuple sets onto Δ and $\overline{\pi}_\Delta(R)^-$ consists of those tuple sets in $\tau(\Delta)$, all of whose extensions are in R^-.

Definition 24. Let R and S be normalized generalized disjunctive paraconsistent relations on schemes Σ and Δ, respectively with $\textbf{g\_reduce}(R)^+ = \{v_1, \ldots, v_n\}$, $\textbf{g\_reduce}(R)^- = \{u_1, \ldots, u_k\}$ and $\textbf{g\_reduce}(S)^+ = \{w_1, \ldots, w_m\}$, $\textbf{g\_reduce}(S)^- = \{x_1, \ldots, x_j\}$. Then, the *natural join of* R and S, denoted $R \overline{\bowtie} S$, is a generalized disjunctive paraconsistent relation on scheme $\Sigma \cup \Delta$, given by $R \overline{\bowtie} S = \textbf{g\_reduce}(T)$, where T is defined as follows. Let $E = \{\{t_1, \ldots, t_n\} | (\forall i)(1 \leq i \leq n \rightarrow t_i \in v_i)\}$ and $F = \{\{t_1, \ldots, t_m\} | (\forall i)(1 \leq i \leq m \rightarrow t_i \in w_i)\}$. Let the elements of E be $E_1, \ldots, E_e$ and those of F be $F_1, \ldots, F_f$ and let $A_{ij} = E_i \bowtie F_j$ for $1 \leq i \leq e$ and $1 \leq j \leq f$. Let $A_1, \ldots, A_g$ be the distinct $A_{ij}s$. Then,
$T^+ = \{w | (\exists t_1) \cdots (\exists t_g)(t_1 \in A_1 \wedge \cdots \wedge t_g \in A_g \wedge w = \{t_1, \ldots, t_g\})\}$
Let $G = \{\{t_1, \ldots, t_k\} | (\forall i)(1 \leq i \leq k \rightarrow t_i \in u_i)\}$ and $H = \{\{t_1, \ldots, t_j\} | (\forall i)(1 \leq i \leq j \rightarrow t_i \in x_i)\}$. Let the elements of G be $G_1, \ldots, G_g$ and those of H be $H_1, \ldots, H_h$ and let $B_{ij} = (G_i)^{\Sigma \cup \Delta} \cup (H_j)^{\Sigma \cup \Delta}$ for $1 \leq i \leq g$ and $1 \leq j \leq h$. Let $B_1, \ldots, B_f$ be the distinct $B_{ij}s$. Then,
$T^- = \{w | (\exists t_1) \cdots (\exists t_f)(t_1 \in B_1 \wedge \cdots \wedge t_f \in B_f \wedge w = \{t_1, \ldots, t_f\})\}$. □

Theorem 3. *Let R and S be two normalized generalized disjunctive paraconsistent relations on scheme Σ_1 and Σ_2. Also let F be a selection formula on scheme Σ_1 and $\Delta \subseteq \Sigma_1$. Then,*

1. $\textbf{g\_rep}_{\Sigma_1}(\overline{\sigma}_F(R)) = \mathcal{S}(\widehat{\sigma}_F)(\textbf{g\_rep}_{\Sigma_1}(R))$.
2. $\textbf{g\_rep}_{\Sigma_1}(\overline{\pi}_\Delta(R)) = \mathcal{S}(\widehat{\pi}_\Delta)(\textbf{g\_rep}_{\Sigma_1}(R))$.
3. $\textbf{g\_rep}_{\Sigma_1 \cup \Sigma_2}(R \overline{\bowtie} S) = \textbf{g\_rep}_{\Sigma_1}(R) \mathcal{S}(\widehat{\bowtie}) \textbf{g\_rep}_{\Sigma_2}(S)$.

□

6 Conclusions and Future Work

We have presented a framework for relational databases under which positive disjunctive as well as explicit negative disjunctive facts can be represented and manipulated. It is the generalization of disjunctive paraconsistent relation in [10]. There are at least two directions for future work. One would be to make the model more expressive by considering disjunctive positive and negative facts.

Work is in progress in this direction. The extended model will be more expressive The algebraic operators will have to be extended appropriately. The other direction for future work would be to find applications of the model presented in this paper. There has been some interest in studying extended logic programs in which the head of clauses can have one or more literals [13]. This leads to two notions of negation: *implicit* negation (corresponding to negative literals in the body) and *explicit* negation (corresponding to negative literals in the head). The model presented in this paper could provide a framework under which the semantics of extended logic programs could be constructed in a bottom-up manner.

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