Expressing the cone radius in the relational calculus with real polynomial constraints

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Abstract

We show that there is a query expressible in first-order logic over the reals that returns, on any given semi-algebraic set A, for every point a radius around which A is conical. We obtain this result by combining famous results from calculus and real algebraic geometry, notably Sard's theorem and Thom's first isotopy lemma, with recent algorithmic results by Rannou.

1 Introduction

The framework of constraint databases, introduced by Kanellakis, Kuper and Revesz [8], provides a nice theoretical model for spatial databases [11]. A spatial dataset is modeled using real polynomial inequality constraints; such sets are also known as semi-algebraic sets [1, 3]. The relational calculus (first-order logic) with real polynomial constraints then serves as a basic query language, denoted here as FO.

The study of the expressive power of query languages for constraint databases is an active domain of research [10]. One of the problems in particular that received attention in recent years is that of determining the exact power of FO in expressing topological properties of spatial databases [2, 5, 7, 9, 13]. One such property, which is central in this research, is that locally around each point, a semi-algebraic set has the topology of a cone. A radius at which this behavior shows is called a *cone radius* around the point for the set.

Accordingly, a cone radius query is a query that returns, for a semi-algebraic set A, a set of pairs $(\vec{p}, r)$ giving for every point $\vec{p}$ a cone radius r in $\vec{p}$ for A. In this paper, we show that there exists an FO formula expressing a cone radius query. So far, this was only known in two dimensions [5]. Expressibility of the cone radius, apart from being a natural question in itself, has also applications: for example, it has been linked to the expressibility of piecewise linear approximations [4].

2 Preliminaries

2.1 Spatial databases and Queries

A *semi-algebraic set in* $\mathbb{R}^n$ is a finite union of sets definable by conditions of the form

$$f_1(\vec{x}) = f_2(\vec{x}) = \cdots = f_k(\vec{x}) = 0,\ g_1(\vec{x}) > 0,\ g_2(\vec{x}) > 0,\ \ldots,\ g_\ell(\vec{x}) > 0,$$

with $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and where $f_1(\vec{x}), \ldots, f_k(\vec{x}), g_1(\vec{x}), \ldots, g_\ell(\vec{x})$ are multi-variate polynomials in the variables $x_1, \ldots, x_n$ with real coefficients. A *database schema* $\mathcal{S}$ is a finite set of relation names, each with a given arity. A *database* over $\mathcal{S}$ assigns to each $S \in \mathcal{S}$ a semi-algebraic set S^D in $\mathbb{R}^n$, if n is the arity of S. A *k-ary query over* $\mathcal{S}$ is a function mapping each database over $\mathcal{S}$ to a semi-algebraic set in $\mathbb{R}^k$.
As query language we use *first-order logic* (FO) over the vocabulary $(+, \cdot, 0, 1, <)$ expanded with the relation names in $\mathcal{S}$. A formula $\varphi(x_1, \ldots, x_k)$ expresses the k-ary query defined by

$$\varphi(D) := \{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid \langle \mathbb{R}, D \rangle \models \varphi(a_1, \ldots, a_k)\},$$

for any database D. Note that $\varphi(D)$ is always semi-algebraic because all relations in D are; indeed, by Tarski's theorem [15], the relations that are first-order definable on the real ordered field are precisely the semi-algebraic sets.

An example of a query expressed in FO is the following: Let $\mathcal{S}$ be a schema containing the relation name S. Consider the FO-formula

$$\varphi_{\text{int}}(\vec{x}) := (\exists \varepsilon > 0)(\forall x'_1) \cdots (\forall x'_n)(\|\vec{x} - \vec{x}'\| < \varepsilon \rightarrow S(x'_1, \ldots, x'_n)).$$

For any database D, $\varphi_{\text{int}}(D)$ equals the interior of S^D. However, not every query is first-order expressible: the query which asks whether a set is connected is not expressible in FO. This result and other results related to constraint databases have recently been collected in a single volume [10].

2.2 Cones

Let $A \subseteq \mathbb{R}^n$ be a semi-algebraic set and $\vec{p} \in \mathbb{R}^n$ a point not in A. We define the *cone with base A and top $\vec{p}$* as the union of all closed line segments between $\vec{p}$ and points in A. We denote this set with $\text{Cone}(A, \vec{p}) := \{t\vec{b} + (1-t)\vec{p} \mid \vec{b} \in A,\ 0 \leqslant t \leqslant 1\}$. For a point $\vec{p} \in \mathbb{R}^n$ and $\varepsilon > 0$, denote the closed ball centered at

$\vec{p}$ with radius ε by $B^n(\vec{p}, \varepsilon)$, and denote the sphere centered at $\vec{p}$ with radius ε, by $S^{n-1}(\vec{p}, \varepsilon)$. The following well-known theorem says that, locally around each point, a semi-algebraic set has the topology of a cone.

Theorem ([1, 3]). *Let $A \subseteq \mathbb{R}^n$ be a semi-algebraic set and $\vec{p}$ a point of A. Then there is a real number $\varepsilon > 0$ such that the intersection $A \cap B^n(\vec{p}, \varepsilon)$ is homeomorphic to the set* $\mathrm{Cone}(A \cap S^{n-1}(\vec{p}, \varepsilon), \vec{p})$.

Any real number $\varepsilon > 0$ as in the lemma is called a *cone radius* of A in $\vec{p}$.

Let $\mathcal{S}$ be a schema containing a relation name S of arity n. A *cone radius query* Q_{radius} is a query which maps any database D over $\mathcal{S}$ to a set of pairs $(\vec{p}, r) \in \mathbb{R}^n \times \mathbb{R}$ such that for every point $\vec{p} \in S^D$ there exists at least one pair $(\vec{p}, r) \in Q_{\text{radius}}(D)$, and for every $(\vec{p}, r) \in Q_{\text{radius}}(D)$, r is a cone radius in $\vec{p}$ for S^D.

We will use the following notation: Let $A \subseteq B \subseteq \mathbb{R}^n$, the closure of A in B is denoted by $\mathrm{cl}_B(A)$, and $\mathrm{int}_B(A)$ indicates the interior of A in B. When the ambient space B is $\mathbb{R}^n$, we omit the subscript B. We denote $\mathrm{cl}(A) - A$ (the frontier of A) with ∂A.

2.3 Whitney stratification

Every semi-algebraic set $A \subseteq \mathbb{R}^n$ can be "nicely" decomposed in a finite sequence $\mathcal{Z}$ of $n+1$ semi-algebraic sets $Z_0, \dots, Z_n$, called *strata*, with the following properties. For each $i = 0, \dots, n$:

1. Z_i is either a C^1 semi-algebraic set in $\mathbb{R}^n$ of dimension i, or an empty set; and

2. each triple $(Z_i, \vec{p}, Z_j)$ for $i < j$ and $\vec{p} \in Z_i$ has the *Whitney property*.

Such a decomposition is called a *C^1- Whitney stratification of A*. We will not need the precise definitions of when a set is C^1 and of the Whitney property. We refer to the paper of Rannou [12] for more details.

We remark that in this paper we do not require the *frontier condition*, which says that the frontier of a stratum is the union of lower dimensional strata, and also do not suppose strata to be connected. Both properties are not necessary for Thom's first isotopy lemma [6, 14], which we will use in our proof in Section 4.

3 Constructing a Whitney stratification

Let A be a semi-algebraic set in $\mathbb{R}^n$. We shall construct a C^1-Whitney stratification $\mathcal{Z}$ of the closure $\mathrm{cl}(A)$ such that A is the union of connected components of strata of $\mathcal{Z}$. We then say that $\mathcal{Z}$ is *compatible with A*. This construction will be expressible in FO.

Our construction is an adaptation of the construction given by Shiota [14, Lemma I.2.2].

We define Z_n as the subset of A where A is locally C^1 and of dimension n. Now suppose that the strata $Z_n, \ldots, Z_{k+1}$ have already been constructed. Then the stratum Z_k is constructed as follows. Define $A_0 = A$ and $A_1 = \partial A$. For $i = 0, 1$ construct

$$\begin{aligned} R_k^i := \{\vec{p} \in A_i - \bigcup_{j=k+1}^{n} Z_j \mid A_i \text{ is } C^1 \text{ and of dimension } k \\ \text{in a neighborhood of } \vec{p}\} \end{aligned} \tag{1}$$

$$W_k^i := \bigcap_{j=k+1}^{n} \mathrm{int}_{R_k^i}(\{\vec{p} \in R_k^i \mid (R_k^i, \vec{p}, Z_j) \text{ has the Whitney property}\}) \tag{2}$$

$$Z_k^i := W_k^i - \mathrm{cl}(R_k^{1-i}). \tag{3}$$

Then we define $Z_k := Z_k^0 \cup Z_k^1$.

The stratum Z_k has indeed the desired properties: By (1) it is C^1 and of dimension k, (2) guarantees that for all points in Z_k, and for any $j > k$, the triples $(Z_k, \vec{p}, Z_j)$ have the Whitney property, and (3) ensures that the connected components of Z_k lie either completely in A or completely in ∂A.

It is well known [17, 14] that the dimension of $A_i - \bigcup_{j=k}^{n} Z_j$ is strictly less than the dimension of $A_i - \bigcup_{j=k+1}^{n} Z_j$ for $i = 0, 1$. Hence, the stratification $\mathcal{Z}$ will consists of exactly $n + 1$ strata Z_k, some of which may be empty.

We now show that the above construction is in FO.

Theorem 1. *Let $\mathcal{S}$ be a database schema containing a relation name S of arity n. The n-ary query $Q_{k\text{-stratum}}$, which takes as input a database D over $\mathcal{S}$, and returns the kth stratum Z_k of the stratification $\mathcal{Z}$ constructed above for $A = S^D$, is expressible in FO.*

In order to prove FO-expressibility of these queries, it is sufficient to show that the sets R_k^i, W_k^i, and Z_k^i occurring in the construction are FO-expressible. But this follows immediately from the work of Rannou [12]. Indeed, from that work we can deduce the following lemma, which immediately implies Theorem 1:

Lemma 1. *(i) Let $\mathcal{S}$ be a database schema containing a relation name S of arity n. The $n + 1$ queries of arity n, defined as*

$$\begin{aligned} Q_{k\text{-reg}}(D) := \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \in S^D \wedge (S^D \textit{ is } C^1 \textit{ and of dimension } k \\ \textit{in an open neighborhood of } \vec{x})\}, \end{aligned}$$

for any database D over $\mathcal{S}$, are all expressible in FO.
(ii) Let $\mathcal{S}$ be a database schema containing two relation names S_1 and S_2 of arity

n. The n-ary query, defined as

$$Q_{\mathit{Whitney}}(D) := \{\vec{x} \in \mathbb{R}^n \mid S_1^D, S_2^D \text{ are } C^1 \text{ and } (S_1^D, \vec{x}, S_2^D) \\ \text{has the Whitney property}\},$$

for any database D over $\mathcal{S}$, is expressible in FO.

4 Expressing the cone radius in FO

We are now ready to prove the main result of this paper.

Theorem 2. *There exists an FO-expressible cone radius query.*

Proof. Consider a semi-algebraic set A in $\mathbb{R}^n$, and let $\mathcal{Z}$ be the C^1-Whitney stratification of $\mathrm{cl}(A)$ compatible with A. Let $\vec{p} \in \mathrm{cl}(A)$ and define the C^1-map

$$f_{\vec{p}} : \mathrm{cl}(A) \to \mathbb{R} : \vec{x} \mapsto \|\vec{x} - \vec{p}\|^2.$$

We will need Thom's First Isotopy Lemma [14]. Applied to the C^1-map $f_{\vec{p}}$ and the C^1-Whitney stratification $\mathcal{Z}$, this lemma can be stated as follows: For any $a < b$,

(a) If $f_{\vec{p}}$ is proper, i.e., $f_{\vec{p}}^{-1}([a, b])$ is compact, and

(b) if for each stratum $Z \in \mathcal{Z}$, the restriction

$$f_{\vec{p}}|(Z \cap \mathrm{int}(B^n(\vec{p}, b) - B^n(\vec{p}, a))) \to (a, b) \subseteq \mathbb{R}$$

has no critical points (this will be explained later),

then for any $c \in (a, b)$, there exists a homeomorphism

$$h : \mathrm{cl}(A) \cap \mathrm{int}(B^n(\vec{p}, b) - B^n(\vec{p}, a)) \to (\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (a, b).$$

Moreover, this homeomorphism satisfies the following two properties:

(i) For each $d \in (a, b)$, $h(\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, d)) = (\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times \{d\}$, and

(ii) $h(Z \cap \mathrm{int}(B^n(\vec{p}, b) - B^n(\vec{p}, a))) = (Z \cap S^{n-1}(\vec{p}, c)) \times (a, b)$ is a homeomorphism for every connected component Z of a stratum of $\mathcal{Z}$.

This statement of Thom's First Isotopy Lemma is a specialized form of Theorem II.6.2 in Shiota [14].

Remark that condition (a) is automatically satisfied. Indeed, the inverse image by $f_{\vec{p}}$ of any interval $[a, b] \subset \mathbb{R}$ is equal to $\mathrm{cl}(A) \cap (B^n(\vec{p}, b) - \mathrm{int}(B^n(\vec{p}, a)))$ which is closed and bounded in $\mathbb{R}^n$.

Claim 1. *If condition (b) is satisfied for* $0 < b$ *(so* $a = 0$*), then every* $c \in (0, b)$ *is a cone radius of* A *in* $\vec{p}$.

Proof of Claim. Take an arbitrary real number $c \in (0, b)$. The lemma gives a homeomorphism

$$h_0 : \mathrm{cl}(A) \cap \mathrm{int}(B^n(\vec{p}, b) - \{\vec{p}\}) \to (\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (0, b).$$

By property (i), we obtain a homeomorphism

$$h_1 : \mathrm{cl}(A) \cap (B^n(\vec{p}, c) - \{\vec{p}\}) \to (\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (0, c],$$

which equals the restriction $h_0|\mathrm{cl}(A) \cap (B^n(\vec{p}, c) - \{\vec{p}\})$. Since the cylinder $(\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c)) \times (0, c]$ is clearly homeomorphic to the cone $\mathrm{Cone}(\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$, e.g., by the homeomorphism

$$h_2(\vec{x}, t) := (1 - \frac{t}{c})\vec{p} + (\frac{t}{c})\vec{x} \quad \text{for } t \in (0, c],$$

we obtain a homeomorphism

$$h_3 := h_2 \circ h_1 : \mathrm{cl}(A) \cap (B^n(\vec{p}, c) - \{\vec{p}\}) \to \mathrm{Cone}(\mathrm{cl}(A) \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}.$$

It is easily verified that $h_2 : (Z \cap S^{n-1}(\vec{p}, c)) \times (0, c] \to \mathrm{Cone}(Z \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$ is a homeomorphism for each connected component Z of a stratum of $\mathcal{Z}$. Since h_1 also satisfies property (ii), we have that $h_3 : (Z \cap (B^n(\vec{p}, c) - \{\vec{p}\}) \to \mathrm{Cone}(Z \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$ is a homeomorphism for each connected component Z of a stratum of $\mathcal{Z}$.

This implies that the restriction $h = h_3|A \cap (B^n(\vec{p}, c) - \{\vec{p}\})$ is a homeomorphism from $h(A \cap (B^n(\vec{p}, c) - \{\vec{p}\})$ to $\mathrm{Cone}(A \cap S^{n-1}(\vec{p}, c), \vec{p}) - \{\vec{p}\}$, because A is the union of connected components of strata of $\mathcal{Z}$.

The homeomorphism h can easily be extended to the point $\vec{p}$, hence c is indeed a cone radius as desired. ☐

Let $\mathcal{S}$ be a schema containing a relation name S of arity n, and let D be a database over $\mathcal{S}$. By Claim 1, we can define the following cone radius query

$$Q_{\mathrm{radius}}(D) := \{(\vec{p}, r) \in \mathbb{R}^n \times \mathbb{R} \mid \vec{p} \in S^D \text{ and } r \in (0, b)\},$$

where the interval $(0, b)$ satisfies condition (b) for the map $f_{\vec{p}}$ and semi-algebraic set $A = S^D$. Let us express this query in FO.

We define the critical point query as

$$\begin{aligned} Q_{\mathrm{crit}}(D) := \{(\vec{p}, \vec{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \vec{p} \in S^D &\text{ and } \vec{x} \in Q_{k\text{-stratum}}(D) \text{ for a certain } k \\ &\text{and } \vec{x} \text{ is a critical point of } f_{\vec{p}} \text{ restricted to } Q_{k\text{-stratum}}(D)\}. \end{aligned}$$

The *critical points* of $f_{\vec{p}}$ restricted to a stratum Z, are the points $\vec{x} \in Z$ where the differential map $d_{\vec{x}}(f_{\vec{p}}|Z)$ is not surjective.

Claim 2. *A point $\vec{x} \in \mathbb{R}^n$ is a critical point of $f_{\vec{p}}|Z$ if and only if the tangent space of Z in $\vec{x}$ is orthogonal to the vector $\vec{x} - \vec{p}$.*

Proof of Claim. We compute the differential $d_{\vec{x}}(f_{\vec{p}}|Z)$ as follows: Locally around $\vec{x}$, we may assume that the projection on the first k coordinates $\Pi : Z \to U \subset \mathbb{R}^k$, is a homeomorphism, where k is the dimension of Z in $\vec{x}$. By definition of the differential, $d_{\vec{x}}(f_{\vec{p}}|Z) = (d_{(x_1,\dots,x_k)}g)(d_{(x_1,\dots,x_k)}\Pi^{-1})^{-1}$, where $g = (f|Z) \circ \Pi^{-1}$. By the C^1 Inverse Function Theorem, we may assume that $\Pi^{-1} : U \to Z : (x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, \varphi_{k+1}, \dots, \varphi_n)$, where $\varphi_i(x_1, \dots, x_k)$ are C^1-maps, and hence $g : U \mapsto \mathbb{R} : (x_1, \dots, x_k) = \sum_{i=1}^k (x_i - p_i)^2 + \sum_{j=k+1}^n (\varphi_j(x_1, \dots, x_k) - p_j)^2$. An elementary calculation shows that the differential of $f_{\vec{p}}|Z$ in $\vec{x}$ is the vector

$$d_{\vec{x}}(f_{\vec{p}}|Z) = 2\left(\left((x_i - p_i) + \sum_{j=k+1}^{n} (x_j - p_j)\frac{\partial \varphi_j}{\partial x_i}(x_1, \dots, x_k)\right)_{i=1,\dots,k}, \underbrace{0, \dots, 0}_{n-k \text{ times}}\right).$$

Since $d_{(x_1,\dots,x_k)}\Pi^{-1}$ is an isomorphism between the tangent space $T_{(x_1,\dots,x_k)}U$ of U in the projection $\Pi(\vec{x})$, and the tangent space $T_{\vec{x}}Z$ of Z in $\vec{x}$, any tangent vector $(v_1, \dots, v_n) \in T_{\vec{x}}Z$ is of the form $(d_{(x_1,\dots,x_k)}\Pi^{-1})(v_1, \dots, v_k)$. More specifically, any tangent vector $\vec{v} \in T_{\vec{x}}Z$ can be written as

$$(v_1, \dots, v_n) = (v_1, \dots, v_k, \sum_{i=1}^{k} \frac{\partial \varphi_{k+1}}{\partial x_i}(x_1, \dots, x_k)v_i, \dots, \sum_{i=1}^{k} \frac{\partial \varphi_n}{\partial x_i}(x_1, \dots, x_k)v_i).$$

Hence, the product

$$d_{\vec{x}}(f_{\vec{p}}|Z) \cdot \vec{v} = 2\sum_{i=1}^{k}(x_i - p_i)v_i + 2\sum_{j=k+1}^{n}(x_j - p_j)\left(\sum_{i=1}^{k}\frac{\partial \varphi_j}{\partial x_i}(x_1, \dots, x_k)v_i\right),$$

is equal to $2\sum_{i=1}^n (x_i - p_i)v_i$. This implies that the differential map $d_{\vec{x}}(f_{\vec{p}}|Z)$ is not surjective if and only if $2\sum_{i=1}^n (x_i - p_i)v_i = 0$ for all tangent vectors $\vec{v} \in T_{\vec{x}}Z$. This proves the Claim. □

The proof of the theorem now continues as follows. The tangent space query

$$Q_{\text{tangent}}(D) := \{(\vec{x}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n \mid S^D \text{ is } C^1,\ \vec{x} \in S^D \text{ and } \vec{v} \in T_{\vec{x}}S^D\},$$

is expressible in FO [12, Lemma 2]. Because the orthogonality of two vectors can be easily expressed in FO, the formula

$$\varphi_{\text{crit}}(\vec{p}, \vec{x}) := S(\vec{p}) \wedge \bigvee_{j=0}^{n} \forall \vec{v}\,(\varphi_{\text{tangent}}(\varphi_{j\text{-stratum}}(S))(\vec{x}, \vec{v}) \to (\vec{x} - \vec{p}) \cdot \vec{v} = 0)$$

expresses Q_{crit} correctly by Claim 2. Here, $\varphi_{j\text{-stratum}}$ denotes an FO-formula expressing $Q_{j\text{-stratum}}$ for $j = 0, \ldots, n$, and φ_{tangent} is an FO formula expressing Q_{tangent}.

A *critical value* of $f_{\vec{p}}$ is the image by $f_{\vec{p}}$ of a critical point. The query which returns the set of critical values is expressible in FO by the formula

$$\varphi_{\text{val}}(\vec{p}, r) := \exists \vec{x} \, (\varphi_{\text{crit}}(\vec{p}, \vec{x}) \wedge r = f_{\vec{p}}(\vec{x})) \, .$$

We now observe that $\{r \in \mathbb{R} \mid \varphi_{\text{val}}(\vec{p}, r)\}$ is finite for each $\vec{p}$. Indeed, the set of critical points $\{\vec{x} \in \mathbb{R}^n \mid \varphi_{\text{crit}}(\vec{p}, \vec{x})\}$ is semi-algebraic and hence admits a C^1-cell decomposition $\mathcal{C} = \{C_1, \ldots, C_m\}$ such that $f|C_i$ is C^1 [16]. Sard's Theorem for C^1-maps [18] implies that each $f_{\vec{p}}|C_i$ attains only a finite number of values. Hence the image by $f_{\vec{p}}$ of the set of critical points is finite.

This implies that either there are no critical values, or there exists a minimal critical value. By Claim 1, any value smaller than this minimal value is a cone radius. We therefore conclude that the query expressed in FO as

$$\varphi_{\text{radius}}(\vec{p}, r) := (\forall r')(\varphi_{\text{val}}(\vec{p}, r') \rightarrow r < r'),$$

is a cone radius query, as desired. $\square$

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