Mining for trees in a graph is NP-complete

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The problem of mining patterns in graph-structured data has received considerable attention in recent years, as it has many interesting applications in such diverse areas as biology, the life sciences, the World Wide Web, and social sciences. Kuramochi en Karypis [2] identified the following problem as fundamental to graph mining:

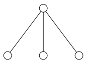
Problem: Disjoint Occurrences of a Graph P

parameter: a small graph P, called the *pattern*.

input: a large graph G and a natural number k.

decide: are there k edge-disjoint subgraphs of G that are all isomorphic to P?

Kuramochi and Karypis relate this problem to Independent Set, a well-known NP-complete problem [1]. They do not go as far, however, as actually proving that Disjoint Occurrences itself can also be NP-complete. The purpose of this short note is to confirm this, even in the very simple case where P is the tree on four nodes, consisting of a node with three children. We denote this tree by T_3:

$T_3 =$ 

As an immediate consequence, the following more general problem, where the pattern is a tree not fixed in advance but part of the input, is NP-complete as well:

Problem: Disjoint Occurrences of a Tree

input: a tree T, a graph G, and a natural number k.

decide: are there k edge-disjoint subgraphs of G that are all isomorphic to T?

Of course, the even more general problem where the pattern is a graph is already well-known to be NP-complete, as it contains the well-known NP-complete Subgraph Isomorphism problem as the special case $k = 1$. We thus see here that restricting to tree patterns does not lower the worst-case complexity.

The NP-completeness of Disjoint Occurrences of T_3 follows immediately from the NP-completeness of the following problem: (a graph is called cubic if every node has degree 3, i.e., has edges to precisely 3 other nodes)

Problem: Independent Set for Cubic Graphs

input: a cubic graph G and a natural number k

decide: are there k nodes in G such that no edge runs between these nodes?

As a matter of fact, for cubic graphs, Indepent Set is precisely the same problem as Disjoint Occurrences of T_3!

To see this, let G be an arbitrary cubic graph. We call a subgraph of G isomorphic to T_3 an *occurrence of* T_3 *in* G. We call the unique node of T_3 that has three children the *center* of T_3. Since G is cubic, we can identify the occurrences of T_3 in G with their centers. Indeed, every occurrence has a unique center, and every node is the center of a unique occurrence. We now easily observe:

> *Two distinct occurrences of* T_3*, with centers* x *and* y*, have an edge in common, if and only if there is an edge between* x *and* y*.*

Consequently:

> G *contains* k *nodes without edges in between, if and only if there are* k *edge-disjoint occurrences of* T_3 *in* G*.*

In other words, Disjoint Occurrences of T_3 is precisely the same problem as Independent Set, restricted to cubic graphs.

To conclude, we point out that the whole reason for NP-completeness is that we want to count disjoint occurrences. Indeed, just counting the occurrences of a fixed pattern P in a graph can be done in polynomial time. Unfortunately, allowing non-disjoint occurrences has problems of its own, as discussed by Kuramochi and Karypis.

References

[1] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, 1979.

[2] M. Kuramochi and G. Karypis. Finding frequent patterns in a large sparse graph. In *Proceedings 4th SIAM International Conference on Data Mining*, 2004.