A fast algorithm for determining the linear complexity of periodic sequences

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Abstract: A fast algorithm is presented for determining the linear complexity and the minimal polynomial of periodic sequences over GF(q) with period $q^n p^m$, where p is a prime, q is a prime and a primitive root modulo p^2 . The algorithm presented here generalizes both the algorithm in [4] where the period of a sequence over GF(q) is p^m and the algorithm in [5] where the period of a binary sequence is $2^n p^m$. When m=0, the algorithm simplifies the generalized Games-Chan algorithm.

Keywords: Cryptography, periodic sequence; linear complexity; minimal polynomial; algorithm

The concept of linear complexity is very useful in the study of the security of stream ciphers for cryptographic applications. A necessary condition for the security of a key stream generator is that it produces a sequence with large linear complexity. Games-Chan algorithm in [1] was proposed to compute the linear complexity of sequences over GF(2) with period 2^n , and was generalized to the sequences over GF(p^m) with period p^n , where p is a prime, by Ding, Xiao and Shan in [2]. Wei, Xiao and Chen in [4] presented an algorithm to compute the linear complexity of sequences over GF(q) with period p^n , where p is a prime, q is a prime and a primitive root modulo p^n , where p is a prime and p^n , where p is a primitive root modulo p^n , where p is a primitive root modulo p^n .

sequences over GF(2) with period 2 ⁿ p ^m, where 2 is a primitive root modulo p².

In this paper, a fast algorithm is presented for determining the linear complexity and the minimal polynomial of periodic sequences over GF(q) with period q ⁿ p ^m, where p is a prime, q is a prime and a primitive root modulo p². The algorithm presented here generalizes both the algorithm in [4] where the period of a sequence over GF(q) is p ^m and the algorithm in [5] where the period of a binary sequence is 2 ⁿ p ^m. When m=0, the algorithm simplifies the generalized Games-Chan algorithm.

1. Preliminaries

We will consider sequences over GF(q). Let $x=(x_1, x_2, \dots, x_n)$ and $y=(y_1, y_2, \dots, y_n)$ be vectors over GF(q). Then define $x+y=(x_1+y_1, x_2+y_2, \dots, x_n+y_n)$.

The generated function of a sequence $s = \{s_0, s_1, s_2, s_3, \dots\}$ is defined by $s(x) = s_0 + s_1 x + s_2 x^2 + s_3 x^3 + \dots = \sum_{i=0}^{\infty} s_i x^i$.

The generated function of a finite sequence $s^N = \{s_0, s_1, s_2, \dots, s_{N-1}\}$ is defined by $s^N(x) = s_0 + s_1x + s_2x^2 + \dots + s_{N-1}x^{N-1}$. If s is a periodic sequence with the first period s^N , then,

$$s(x) = s^{N}(x)(1+x^{N}+x^{2N}+\cdots) = \frac{s^{N}(x)}{1-x^{N}} = \frac{s^{N}(x)/\gcd(s^{N}(x),1-x^{N})}{(1-x^{N})/\gcd(s^{N}(x),1-x^{N})} = \frac{g(x)}{f_{s}(x)}$$

where
$$f_s(x) = (1-x^N)/\gcd(s^N(x),1-x^N)$$
, $g(x) = s^N(x)/\gcd(s^N(x),1-x^N)$

Obviously, $gcd(g(x), f_s(x))=1$, $deg(g(x))< deg(f_s(x))$, $f_s(x)$ is the minimal polynomial of s, and the degree of $f_s(x)$ is the linear complexity of s, that is $deg(f_s(x))=c(s)^{[2]}$.

Let us recall some results in finite field theory^[7] and number theory^[8].

Definition 1.1 Let n be a positive integer. The polynomial $\Phi_n(\mathbf{x}) = \mathbf{p}_{0 < j < n, (j,n)=1}(\mathbf{x} - \mathbf{X}_n^j)$, where \mathbf{X}_n is a n-th primitive unit root and $(\mathbf{j}, \mathbf{n}) = 1$ denotes j is relatively prime to n, is called the n-th cyclotomic polynomial.

Lemma 1.1 Let p be a prime. Then $j(p^n) = p^n - p^{n-1}$, where n is a positive integer, j is the Euler function.

Lemma 1.2 Let $\Phi_n(x)$ be the n-th cyclotomic polynomial. Then $\Phi_n(x)$ is irreducible over GF(q) if and only if that q is a primitive root modulo n, that is the order of q modulo n is j (n).

Lemma 1.3 Let p be a prime and m a positive integer. Then $\Phi_{p^m}(x) = \Phi_p(x^{p^{m-1}})$.

Proof: Since p is a prime,

$$\Phi_{p}(y) = \mathbf{p}_{0 < j < p, (j,p)=1}(y - \mathbf{X}_{p}^{j}) = \mathbf{p}_{0 \le j < p}(y - \mathbf{X}_{p}^{j})/(y - \mathbf{X}_{p}^{0}) = \frac{y^{p} - 1}{y - 1} = 1 + y + y^{2} + \dots + y^{p-1}.$$

Note that
$$X_{p^m}^j = \exp(\frac{2pi}{p^m}j) = \exp(\frac{2pi}{p^{m-1}}k) = X_{p^{m-1}}^k$$
, where $j = pk$,

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$$\Phi_{p^m}(\mathbf{x}) = \frac{x^{p^m} - 1}{\mathbf{p}_{0 \le i \le p^m, p \mid i}(\mathbf{x} - \mathbf{X}_{p^m}^i)} = \frac{x^{p^m} - 1}{\mathbf{p}_{0 \le k \le p^{m-1}}(\mathbf{x} - \mathbf{X}_{p^{m-1}}^k)} = \frac{x^{p^m} - 1}{x^{p^{m-1}} - 1} = \Phi_p(\mathbf{x}^{p^{m-1}}). \quad \blacksquare$$

Lemma 1.4 Let p and q be prime numbers, m and n be positive integers. Let $\Phi_{p^m}(x)^{q^n}$ denote $[\Phi_{p^m}(x)]^{q^n}$. Then

$$\Phi_{p^m}(x)^{q^n} = \Phi_{p^m}(x^{q^n}) = \Phi_p(x^{q^n})^{p^{m-1}}$$
, where the operation is over GF(q).

Proof: Since the operation is over GF(q), so qy=0.

As q is prime, thus
$$q \begin{vmatrix} q \\ i \end{vmatrix}$$
, $0 < i < q$, $\therefore (a+b)^q = a^q + b^q$,

$$\therefore \Phi_{p}(y)^{q} = (1+y+y^{2}+\cdots+y^{p-1})^{q} = 1+y^{q}+y^{2q}+\cdots+y^{(p-1)q} = \Phi_{p}(y^{q}),$$

By analogy, $\Phi_p(y)^{q^n} = \Phi_p(y^{q^n})$.

By lemma 1.3,
$$\Phi_{p^m}(x)^{q^n} = \Phi_p(x^{p^{m-1}})^{q^n} = \Phi_p(x^{p^{m-1}q^n}) = \Phi_p[(x^{q^n})^{p^{m-1}}] = \Phi_{p^m}(x^{q^n}).$$

Lemma 1.5 Let p be a prime, q a prime and a primitive root modulo p^2 . Then q is a primitive root modulo p^n , $n \ge 1$, so $\Phi_{n^n}(x)$ is irreducible over GF(q).

Proof: We first prove that q is a primitive root modulo p.

Suppose that $q^{\frac{p-1}{m}} \equiv 1 \pmod{p}$, where m is a positive integer, then $q^{\frac{p-1}{m}} = 1 + kp$, where k is a positive integer,

$$\therefore q^{\frac{p(p-1)}{m}} = (q^{\frac{p-1}{m}})^p = (1+kp)^p \equiv 1 \pmod{p^2}.$$

Since q is a primitive root modulo p², so m must be 1, thus q is a primitive root modulo p.

Secondly, we prove that q is a primitive root modulo p^n , $n \ge 2$, by induction,

Suppose the claim is true for $n \ge 2$, we now consider the case n+1.

Obviously, $q^{p^{n-1}-p^{n-2}} = 1 + kp^{n-1}$, where k is a positive integer;

Since q is a primitive root modulo p^n , so k is not divisible by p.

$$q^{p^{n}-p^{n-1}} = (q^{p^{n-1}-p^{n-2}})^{p} = 1 + kp^{n} + rp^{n+1}$$
, where r is a positive integer,

$$\therefore$$
 q $p^{n-p^{n-1}} \neq 1+s$ p p^{n+1} , where s is a positive integer.

Suppose the order of q modulo p^{n+1} is t, then $q^t = 1 + m p^{n+1}$, where m is a positive integer, and $t | j (p^{n+1})$.

By lemma 1.1, $t|(p^{n+1}-p^n)$.

As
$$q^{t} = 1 + (m p)p^{n}$$
, so $(p^{n} - p^{n-1})|t$, thus $t = p^{n} - p^{n-1}$ or $p^{n+1} - p^{n}$.

Since $t = p^n - p^{n-1}$ contradicts the fact that $q^{p^n - p^{n-1}} \neq 1 + s p^{n+1}$, thus $t = p^{n+1} - p^n$.

Therefore, q is a primitive root modulo p^n , $n \ge 1$.

By lemma 1.2, $\Phi_{p^n}(x)$ is irreducible over GF(q).

2. Main theorems concerning algorithms

The following lemma and its proof is from [4].

Lemma 2.1 Let $a=(a_0,a_1,\cdots,a_{N-1})$ be a finite sequence over GF(q), where $N=p^m$, p is a prime and q is a primitive root modulo p^2 . Let us denote a(x) as the generated function of the finite sequence (a_0,a_1,\cdots,a_{N-1}) and $A_i=(a_{(i-1)p^{m-1}},a_{(i-1)p^{m-1}+1},\cdots,a_{ip^{m-1}-1})$, $i=1,2,\cdots,p$. Then $(\Phi_{p^m}(x),a(x))\neq 1$, that is $\Phi_{p^m}(x)|a(x)$ if and only if $A_1=A_2=\cdots=A_p$.

Proof: Let $A_i(x)$ denote the generated function of A_i , $i=1,2,\dots,p$. Then,

$$a(x)=A_1(x)+x^{p^{m-1}}A_2(x)+\cdots+x^{(p-1)p^{m-1}}A(x)_p$$
.

We first show the necessity.

As $\Phi_{p^m}(x)|a(x)$, let $a(x)=t(x)\Phi_{p^m}(x)$, where t(x) is a polynomial over GF(q).

From lemma 1.3, $\Phi_{p^m}(x)=1+x^{p^{m-1}}+\cdots+x^{(p-1)p^{m-1}}$

$$\therefore \ \, A_{1}(x)\text{-}t(x)+x^{p^{m-1}}(A_{2}(x)\text{-}t(x))+\cdots+\,x^{(p-1)p^{m-1}}(A(x)_{p}\text{-}t(x))=0.$$

Since
$$\deg(a(x)) < p^m$$
 and $\deg(\Phi_{p^m}(x)) = (p-1) p^{m-1}$, so $\deg(t(x)) < p^m - (p-1) p^{m-1} = p^{m-1}$;

As $\deg(A_i(x)) < p^{m-1}$, so $\deg(A_i(x)-t(x)) < p^{m-1}$, thus $A_i(x)-t(x) = 0$, $i = 1, 2, \dots, p$, hence $A_1 = A_2 = \dots = A_p$.

Now come to the sufficiency.

As
$$A_1 = A_2 = \cdots = A_p$$
, so $a(x) = A_1(x)(1 + x^{p^{m-1}} + \cdots + x^{(p-1)p^{m-1}}) = A_1(x) \Phi_{p^m}(x)$, hence $\Phi_{p^m}(x) | a(x)$.

Theorem 2.1. Let $a=(a_0,a_1,\cdots,a_{N-1})$ be a finite sequence over GF(q), where $N=q^np^m$, p and q are prime numbers, q is a primitive root modulo p^2 . Let us denote a(x) as the generated function of the finite sequence (a_0,a_1,\cdots,a_{N-1}) , $M=q^{n-1}p^{m-1}$, $A_i=(a_{(i-1)M},a_{(i-1)M+1},\cdots,a_{iM-1})$, $A_i(x)$ be the generated function of A_i , $i=1,2,\cdots,qp$. Then,

(i).
$$\Phi_{p^m}(x)^{q^{n-1}}|a(x) \text{ if and only if } A_1 + A_{p+1} + \cdots + A_{(q-1)p+1} = A_2 + A_{p+2} + \cdots + A_{(q-1)p+2} = \cdots = A_p + A_{2p} + \cdots + A_{qp};$$

(ii). If
$$\Phi_{p^m}(x)^{q^{n-1}}|a(x)$$
, then $gcd(a(x), \Phi_{p^m}(x))^{q^n}) = \Phi_{p^m}(x)^{q^{n-1}}gcd(a'(x), \Phi_{p^m}(x))^{(q-1)q^{n-1}})$, where $a'(x) = A_1(x) + A_2(x) x^M + \cdots + A_{qp}(x) x^{(qp-1)M}$, $A_1 = A_1$, $A_2 = -A_1 + A_2$, \cdots , $A_{qp} = -A_{p-1} - A_{2p-1} - \cdots - A_{qp-1} + A_p + A_{2p} + \cdots + A_{qp}$;

(iii). If
$$\Phi_{p^m}(x)^{q^{n-1}}|a(x)$$
 not true, then $gcd(a(x), \Phi_{p^m}(x)^{q^n}) = gcd(a(x), \Phi_{p^m}(x)^{q^{n-1}}) = gcd(\sum_{i=1}^p [A_i(x) + A_{p+i}(x) + \cdots + A_{(q-1)p+i}(x)]x^{(i-1)M}, \Phi_{p^m}(x)^{q^{n-1}})$

(iv).
$$\gcd(a(x), 1-x^{qM}) = \gcd(\sum_{i=1}^{q} [A_i(x) + A_{q+i}(x) + \dots + A_{(p-1)q+i}(x)]x^{(i-1)M}, 1-x^{qM})$$

(v).
$$\gcd(a(x), 1-x^N) = \gcd(a(x), 1-x^{qM}) \gcd(a(x), \Phi_{n^m}(x)^{q^n})$$

Proof: (i) As q is a primitive root modulo p^2 , we know that $\Phi_{n^m}(x)$ is irreducible over GF(q).

From lemma 1.4, $\Phi_{n^m}(x)^{q^{n-1}} = 1 + x^M + \dots + x^{(p-1)M}$, let $b = \Phi_{p^m}(x)^{q^{n-1}}$. Then,

$$a(x) = A_1(x) + A_2(x) x^M + \cdots + A_{qp}(x) x^{(qp-1)M}$$

$$= A_1(x)b + A_2(x)bx^M + \cdots + A_p(x)bx^{(p-1)M}$$

$$+ A_{p+1}(x)bx^{pM} + A_{p+2}(x)bx^{(p+1)M} + \cdots + A_{2p}(x)bx^{(2p-1)M}$$

+••••

$$+ A_{(q-1)p+1}(x)bx^{(q-1)pM} + A_{(q-1)p+2}(x)bx^{[(q-1)p+1]M} + \cdots + A_{qp}(x)bx^{(qp-1)M}$$

$$-[A_{p}(x) + A_{2p}(x) + \cdots + A_{qp}(x)]bx^{qpM}$$

$$+\{[A_{1}(x)+A_{p+1}(x)+\cdots+A_{(q-1)p+1}(x)]+[A_{2}(x)+A_{p+2}(x)+\cdots+A_{(q-1)p+2}(x)]x^{M}+\cdots+[A_{p}(x)+A_{2p}(x)+\cdots+A_{qp}(x)]x^{(p-1)M}\}x^{qpM}$$
(1)

where,
$$A_1 = A_1$$
, $A_2 = -A_1 + A_2$, ..., $A_n = -A_{n-1} + A_n$,

$$A_{p+1}^{'} = -A_p + A_1 + A_{p+1}, A_{p+2}^{'} = -A_1 - A_{p+1} + A_2 + A_{p+2}, \cdots, A_{2p}^{'} = -A_{p-1} - A_{2p-1} + A_p + A_{2p}, \cdots$$

•••••,

$$A'_{(q-1)p+1} = -A_p - A_{2p} - \cdots - A_{(q-1)p} + A_1 + A_{p+1} + \cdots + A_{(q-1)p+1}$$

$$A'_{(q-1)p+2} = -A_1 - A_{p+1} - \cdots - A_{(q-1)p+1} + A_2 + A_{p+2} + \cdots + A_{(q-1)p+2}, \cdots,$$

$$A_{qp}^{'} = -A_{p-1} - A_{2p-1} - \cdots - A_{qp-1} + A_p + A_{2p} + \cdots + A_{qp}$$

To understand equality (1), we can add all the items of the right side concerning one polynomial, such as $A_2(x)$.

$$x^{M} \left[A_{2}(x)b - A_{2}(x)bx^{M} + A_{2}(x)bx^{pM} - A_{2}(x)bx^{(p+1)M} + \dots + A_{2}(x)bx^{(q-1)pM} - A_{2}(x)bx^{[(q-1)p+1]M} + A_{2}(x)x^{qpM} \right]$$

$$= x^{M} A_{2}(x) \left[b(1+x^{pM} + \cdots + x^{(q-1)pM}) - bx^{M} (1+x^{pM} + \cdots + x^{(q-1)pM}) + x^{qpM} \right]$$

$$= x^{M} A_{2}(x) [b(1-x^{M})(1+x^{pM}+\cdots+x^{(q-1)pM})+x^{qpM}]$$

$$= x^{M} A_{2}(x) [(1-x^{pM})(1+x^{pM}+\cdots+x^{(q-1)pM})+x^{qpM}] = A_{2}(x)x^{M} (1-x^{qpM}+x^{qpM}) = x^{M} A_{2}(x).$$

In the case of $A_{qp}(x)$,

$$x^{(qp-1)M} (A_{qp}(x)b-A_{qp}(x)bx^{M}+A_{qp}(x)x^{pM})=A_{qp}(x)x^{(qp-1)M} [b(1-x^{M})+x^{pM}]=A_{qp}(x)x^{(qp-1)M}$$

By analogy, we can verify other items of equality (1).

Since
$$(\Phi_{n^m}(x)^{q^{n-1}}, x^{qpM})=1$$
, so $\Phi_{n^m}(x)^{q^{n-1}}|a(x) \Leftrightarrow$

$$\Phi_{p^m}(\mathbf{x}^{-1})^{q^{n-1}} \left[\left\{ [\mathbf{A}_1(\mathbf{x}) + \mathbf{A}_{p+1}(\mathbf{x}) + \cdots + \mathbf{A}_{(q-1)p+1}(\mathbf{x})] + [\mathbf{A}_2(\mathbf{x}) + \mathbf{A}_{p+2}(\mathbf{x}) + \cdots + \mathbf{A}_{(q-1)p+2}(\mathbf{x})] \mathbf{x}^{-M} + \cdots + [\mathbf{A}_p(\mathbf{x}) + \mathbf{A}_{2p}(\mathbf{x}) + \cdots + \mathbf{A}_{qp}(\mathbf{x})] \mathbf{x}^{-(p-1)M} \right\}.$$

From lemma 2.1, we have $\Phi_{p^m}(x)^{q^{n-1}}|a(x) \Leftrightarrow$

$$A_{1}(x) + A_{p+1}(x) + \cdots + A_{(q-1)p+1}(x) = A_{2}(x) + A_{p+2}(x) + \cdots + A_{(q-1)p+2}(x) = \cdots = A_{p}(x) + A_{2p}(x) + \cdots + A_{qp}(x) + \cdots + A$$

$$\Leftrightarrow$$
 $A_1 + A_{p+1} + \cdots + A_{(q-1)p+1} = A_2 + A_{p+2} + \cdots + A_{(q-1)p+2} = \cdots = A_p + A_{2p} + \cdots + A_{qp}$.

(ii) If
$$\Phi_{n^m}(x)^{q^{n-1}}|a(x)$$
, then $A_1 + A_{p+1} + \cdots + A_{(q-1)p+1} = A_2 + A_{p+2} + \cdots + A_{(q-1)p+2} = \cdots = A_p + A_{2p} + \cdots + A_{qp}$.

Let a'(x)=a(x)/
$$\Phi_{p^m}(x)^{q^{n-1}}$$
. From equality (1), a'(x)= $A_1(x)+A_2(x)x^M+\cdots+A_{qp}(x)x^{(qp-1)M}$.

$$\text{Thus, } \gcd(a(x), \Phi_{p^m}\left(x\right.)^{q^n}) = \ \Phi_{p^m}\left(x\right.)^{q^{n-1}} \gcd(a'(x), \Phi_{p^m}\left(x\right.)^{(q-1)q^{n-1}}).$$

(iii) If
$$\Phi_{p^m}(x)^{q^{n-1}}|a(x)$$
 not true, then $gcd(a(x),\Phi_{p^m}(x)^{q^n})=gcd(a(x),\Phi_{p^m}(x)^{q^{n-1}})$.

From equality (1),
$$\gcd(\mathbf{a}(\mathbf{x}), \Phi_{p^m}(\mathbf{x})^{q^n}) = \gcd(\sum_{i=1}^p [\mathbf{A}_i(\mathbf{x}) + \mathbf{A}_{p+i}(\mathbf{x}) + \cdots + \mathbf{A}_{(q-1)p+i}(\mathbf{x})]\mathbf{x}^{(i-1)M}, \Phi_{p^m}(\mathbf{x})^{q^{n-1}}).$$

$$\begin{split} (\text{iv}) \ & \ a(x) = A_1(x) + A_2(x) \ x^M + \dots + A_{qp}(x) x^{(qp-1)M} \\ & = \{ \ A_1(x) + [\ A_1(x) + A_{q+1}(x)] x^{qM} + \dots + [\ A_1(x) + A_{q+1}(x) + \dots + A_{qp-2q+1}(x)] x^{(qp-2q)M} \ \} \\ & + [A_1(x) + A_{q+1}(x) + \dots + A_{qp-2q+1}(x) + A_{qp-q+1}(x)] x^{(qp-q)M} \end{split}$$

$$\{ A_{q}(x) + [A_{q}(x) + A_{2q}(x)]x^{qM} + \dots + [A_{q}(x) + A_{2q}(x) + \dots + A_{qp-q}(x)]x^{(qp-2q)M} \}x^{(q-1)M} (1-x^{qM}) + [A_{q}(x) + A_{2q}(x) + \dots + A_{qp-q}(x)]x^{(qp-1)M},$$

Since $gcd(x^{(qp-q)M}, 1-x^{qM})=1$, thus,

$$\gcd(a(x), 1-x \stackrel{qM}{=}) = \gcd([A_1(x) + A_{q+1}(x) + \cdots + A_{qp-2q+1}(x) + A_{qp-q+1}(x)] + \cdots + [A_q(x) + A_{2q}(x) + \cdots + A_{qp-q}(x)] + \cdots + [A_q(x) + A_{2q}(x) + \cdots + A_{qp-q}(x)] + \cdots + [A_q(x) + A_{2q}(x) + \cdots + A_{2q}(x)] + \cdots + [A_q(x) + A_{2$$

(v) Since
$$\frac{1-x^{N}}{1-x^{qM}} = 1+x^{qM} + \dots + x^{q(p-1)M} = \Phi_{p^{m}}(x^{q})^{q^{n-1}} = \Phi_{p^{m}}(x)^{q^{n}}$$
 and $gcd(1-x^{qM}, \Phi_{p^{m}}(x)^{q^{n}}) = 1$, thus,

$$\gcd(\mathbf{a}(\mathbf{x}),\mathbf{1}\mathbf{-x}^{^{N}})=\gcd(\mathbf{a}(\mathbf{x}),\mathbf{1}\mathbf{-x}^{^{qM}})\ \gcd(\mathbf{a}(\mathbf{x}),\ \Phi_{_{n^{^{m}}}}(\mathbf{x})^{^{q^{^{n}}}}).\ \blacksquare$$

Theorem 2.2. Let s be a sequence over GF(q) with period N and $a=(a_0,a_1,\cdots,a_{N-1})$ the first period, where $N=q^np^m$, p and q are prime numbers, q is a primitive root modulo p^2 . Let us denote a(x) as the generated function of the finite sequence (a_0,a_1,\cdots,a_{N-1}) , $M=q^{n-1}p^{m-1}$, $A_i=(a_{(i-1)M},a_{(i-1)M+1},\cdots,a_{iM-1})$, $A_i(x)$ be the generated function of A_i , $i=1,2,\cdots,qp$. Then $f_s(x)=f_{(b)}(x) \cdot \Phi_{n^m}(x)^z$, hence $c(s)=c((b))+(p-1)p^{m-1}z$,

where b=($A_1 + A_{q+1} + \cdots + A_{qp-q+1}$, \cdots , $A_q + A_{2q} + \cdots + A_{qp}$), (b) denotes the sequence with the first period b; $\deg(\Phi_{p^m}(x)) = j(p^m) = (p-1)p^{m-1}$, $\Phi_{p^m}(x)^z = \Phi_{p^m}(x)^{q^n}/\gcd(\Phi_{p^m}(x)^{q^n},a(x))$, hence $z=q^n$ -t, where t is the power exponent of $\Phi_{p^m}(x)$ in $\gcd(\Phi_{p^m}(x)^{q^n},a(x))$.

Proof: From (iv) and (v) of theorem 2.1,

$$\begin{split} & \text{f}_{s}(\mathbf{x}) \!\!=\!\! (1\!-\!\mathbf{x}^{\,N})/\gcd(\mathbf{a}(\mathbf{x}),\!1\!-\!\mathbf{x}^{\,N}) \!\!=\!\! [(1\!-\!\mathbf{x}^{\,qM})/\gcd(\mathbf{a}(\mathbf{x}),\!1\!-\!\mathbf{x}^{\,qM})] \! \bullet \! [\Phi_{p^{m}}(\mathbf{x})^{\,q^{n}}/\gcd(\mathbf{a}(\mathbf{x}), \Phi_{p^{m}}(\mathbf{x})^{\,q^{n}})] \\ & = \!\! [(1\!-\!\mathbf{x}^{\,qM})/\gcd([\mathbf{A}_{1}(\mathbf{x})\!+ \mathbf{A}_{q+1}(\mathbf{x}) + \cdots +\! \mathbf{A}_{qp-2q+1}(\mathbf{x}) +\! \mathbf{A}_{qp-q+1}(\mathbf{x})] \!\!+\! \cdots +\! [\mathbf{A}_{q}(\mathbf{x})\!+ \mathbf{A}_{2q}(\mathbf{x}) + \cdots +\! \mathbf{A}_{qp-q}(\mathbf{x}) \\ & +\! \mathbf{A}_{qp}(\mathbf{x})]\mathbf{x}^{\,(q-1)M},\! 1\!-\! \mathbf{x}^{\,qM})] \! \bullet \! [\Phi_{p^{m}}(\mathbf{x})^{\,q^{n}}/\gcd(\mathbf{a}(\mathbf{x}), \Phi_{p^{m}}(\mathbf{x})^{\,q^{n}})] \end{split}$$

= $f_{(b)}(x) \cdot \Phi_{n^m}(x)^z$, where (b) with the period qM.

It is easy to show that $c(s)=c((b))+(p-1)p^{m-1}z$.

3. Algorithms to compute the linear complexity of sequences over GF(q) with period q^np^m

The following algorithm was presented in [4] as algorithm 1.

Algorithm 3.1 Let s be a sequence over GF(q) with period N=pⁿ and the first period be denoted as $a=(a_0, a_1, \dots, a_{N-1})$, where q is a primitive root modulo p^2 .

Initial values: $a=(a_0, a_1, \dots, a_{N-1})$ is the first period of s, $k=p^n$, c=0, f=1.

- (i) If $a=(0, \dots, 0)$, then end; if k=1, then c=c+1, f=(1-x)f, end
- (ii) k=k/p, let $A_i = (a_{(i-1)k}, a_{(i-1)k+1}, \dots, a_{ik-1}), i=1,2,\dots,p;$
- (iii) If $A_1 = A_2 = \cdots = A_p$, then $a = A_1$; if $A_1 = A_2 = \cdots = A_p$ not true, then $a = A_1 + A_2 + \cdots + A_p$, c = c + (p-1)k, $f = f \Phi_{pk}(x)$;
- (iv) go to (i).
- (v) The final. The linear complexity of s: c(s)=c; the minimal polynomial of s: $f_s(x)=f$.

From lemma 2.1, it is easy to show the correctness of the algorithm and it computes the minimal polynomial in (n+1) loops at most. The reader is referred to [4] for a detailed proof.

Lemma 3.1 Let s be a sequence over GF(q) with period N= q^n and $a=(a_0,a_1,\cdots,a_{N-1})$ be the first period, where q is a prime. Let us denote a(x) as the generated function of the finite sequence a, M= q^{n-1} , $A_i = (a_{(i-1)M}, a_{(i-1)M+1}, \cdots, a_{iM-1})$, $A_i(x)$ be the generated function of A_i , $i=1,2,\cdots,q$. Then,

- (i) $(1-x^{M})|a(x)$ if and only if $\sum_{i=1}^{q} A_{i}=0$;
- (ii) if $(1-x^M)|a(x)$, then $\frac{a(x)}{1-x^M} = A_1(x) + [A_1(x) + A_2(x)]x^M + \dots + [A_1(x) + A_2(x) + \dots + A_q(x)]x^{(q-1)M}$;
- (iii) if $(1-x^M)|a(x)$ not true, then $gcd(a(x),1-x^N) = gcd(A_1(x) + A_2(x) + \cdots + A_n(x), 1-x^M)$
- (iv) if $(1-x^M)|a(x)$, then $gcd(a(x),1-x^N) = (1-x^M)gcd(A_1(x)+[A_1(x)+A_2(x)]x^M+\cdots+[A_1(x)+A_2(x)+\cdots+A_q(x)]x^M$, $(1-x^M)^{q-1}$.

Proof: From the following equality, we have (i) and (ii),

$$a(x) = A_1(x) + A_2(x)x^M + \cdots + A_q(x)x^{(q-1)M}$$

$$= (1 - x^{-M}) \{A_1(x) + [A_1(x) + A_2(x)]x^{-M} + \dots + [A_1(x) + A_2(x) + \dots + A_{-q}(x)]x^{-(q-1)M} \} + [A_1(x) + A_2(x) + \dots + A_{-q}(x)]x^{-qM} .$$

Note that the operation is over GF(q) and q is a prime,

$$(1-x)^q = 1-x^q$$
,

$$(1-x)^{q^2} = (1-x^q)^q = 1-x^{q^2}$$

By analogy, $(1-x)^N = (1-x)^{q^n} = 1-x^{q^n} = 1-x^N$,

$$\therefore 1-x^N = (1-x^M)^q$$

Therefore, we have (iii) and (iv). ■

From lemma 3.1, it is easy to show the correctness of the following algorithm.

Algorithm 3.2 Let s be a sequence over GF(q) with period $N = q^n$ and $a = (a_0, a_1, \dots, a_{N-1})$ be the first period, where q is a prime.

Initial values: $a = s^N$, $l = q^n$, c = 0, f = 1.

- (i) If l=1, then {If a=(0), then end; else c=c+1, f=(1-x)f, end.}
- (ii) If $l \neq 1$, then l = l/q, denote M = l, $A_i = (a_{(i-1)M}, a_{(i-1)M+1}, \dots, a_{iM-1})$, $i = 1, 2, \dots, q$. Set count=0.
- (iii) If $A_1 + A_2 + \cdots + A_q = 0$, then {count= count+1, if count<q, then set $A_i = A_i + A_{i-1}$ (i=2,...,q, sequentially), repeat (iii); if count=q, then end.}
- (iv) If $A_1 + A_2 + \cdots + A_q \neq 0$, then $a = A_1 + A_2 + \cdots + A_q$, c = c + (q count 1)M, $f = f(1 x^M)^{q count 1}$, go to (i).
- (v) The final. The linear complexity of s: c(s)=c; the minimal polynomial of s: f(x)=f.

The algorithm computes the minimal polynomial in n(q-1)+1 loops at most.

Let s be a sequence over GF(q) with period N and a=(a₀,a₁,···,a_{N-1}) the first period, where N= $q^n p^m$, p is a prime, q is a prime and a primitive root modulo p^2 . From theorem 2.2, the computation of $f_s(x)$ is equivalent to that of $f_{(b)}(x)$ and $\Phi_{p^m}(x)^z$, so we first introduce an algorithm to compute $\Phi_{p^m}(x)^z$.

Algorithm 3.3. Initial values: $a=(a_0, a_1, \dots, a_{N-1})$ is the first period of s, $l=q^n$, c=0, f=1, we denote $k=p^{m-1}$.

- (i) If $a=(0, \dots, 0)$, then end ;if l=1, then{let $A_i=(a_{(i-1)k}, a_{(i-1)k+1}, \dots, a_{ik-1}), i=1,2,\dots,p.$ If $A_1=A_2=\dots=A_p$, then end; else, c=c+(p-1)k, $f=f\Phi_{pk}(x)$, end.}
- (iii) If $A_1 + A_{p+1} + \cdots + A_{(q-1)p+1} = A_2 + A_{p+2} + \cdots + A_{(q-1)p+2} = \cdots = A_p + A_{2p} + \cdots + A_{qp}$, then {count = count + 1, if count < q, set A_1 , A_2 , A_2 , A_2 , A_2 , A_3 , repeat (iii); if count = q, end. }
- (iv) If $A_1 + A_{p+1} + \cdots + A_{(q-1)p+1} = A_2 + A_{p+2} + \cdots + A_{(q-1)p+2} = \cdots = A_p + A_{2p} + \cdots + A_{qp}$ not true, $a = (A_1 + A_{p+1} + \cdots + A_{(q-1)p+1}, A_2 + A_{p+2} + \cdots + A_{(q-1)p+2}, \cdots, A_p + A_{2p} + \cdots + A_{qp})$, c = c + (q count 1)(p 1)M, $f = f \Phi_{pk}(x)^{(q count 1)l}$, go to (i)
- (v) The final. f is $\Phi_{n^m}(x)^z$, c is $(p-1)p^{m-1}z$.

From theorem 2.1 and theorem 2.2, algorithm 3.3 immediately follows. It computes $\Phi_{p^m}(x)^z$ in n(q-1)+1 loops at most.

Now come to our main result, an efficient algorithm for computing the linear complexity and the minimal polynomial of sequences over GF(q).

Algorithm 3.4. Initial values: $a=(a_0, a_1, \dots, a_{N-1})$ is the first period of s, $l=q^n$, $k=p^m$, c=0, f=1.

- (i) If $a=(0, \dots, 0)$, then end; if k>1, then k=k/p and go to (iv).
- (ii) If l=1, then c=c+1, f=(1-x)f, end; else if $l\neq 1$, then l=l/q, denote M=l, $A_i=(a_{(i-1)M}, a_{(i-1)M+1}, \cdots, a_{iM-1})$, $i=1,2,\cdots,q$. Set count=0.
- (iii) If $A_1 + A_2 + \cdots + A_q = 0$, then {count= count+1, if count<q, then set $A_i = A_i + A_{i-1}$ (i=2,...,q, sequentially), repeat (iii); if count=q, then end.} else if $A_1 + A_2 + \cdots + A_q \neq 0$, then $a = A_1 + A_2 + \cdots + A_q$, c = c + (q count-1)M, $f = f(1 x^M)^{q count-1}$, go to (i).
- (iv) If l = 1, then{let $A_i = (a_{(i-1)k}, a_{(i-1)k+1}, \dots, a_{ik-1})$, $i=1,2,\dots,p$. If $A_1 = A_2 = \dots = A_p$, then $a = A_1$, go to (i); else $a = A_1 + A_2 + \dots + A_p$, c = c + (p-1)k, $f = f \Phi_{pk}(x)$, go to (i).}
- (v) If $l \neq 1$, then l = l/q, let M = l k, $A_i = (a_{(i-1)M}, a_{(i-1)M+1}, \cdots, a_{iM-1})$, $i = 1, 2, \cdots, qp$. $b = (A_1 + A_{q+1} + \cdots + A_{qp-2q+1} + A_{qp-q+1}, \cdots, A_q + A_{2q} + \cdots + A_{qp-q} + A_{qp})$. Set count=0.
- (vi) If $A_1 + A_{p+1} + \cdots + A_{(q-1)p+1} = A_2 + A_{p+2} + \cdots + A_{(q-1)p+2} = \cdots = A_p + A_{2p} + \cdots + A_{qp}$, count= count+1, if count<q, set A_1 , A_2 , \cdots , A_{qp} according to the definition in theorem 2.1, set $A_1 = A_1$, $A_2 = A_2$, \cdots , $A_{qp} = A_{qp}$, repeat (vi); if count=q, then a=b, $l = q^n$, go to (i).
- $\text{(vii)} \qquad \text{If} \quad \text{A}_1 + \text{A}_{p+1} + \cdots + \text{A}_{(q-1)\,p+1} = \text{A}_2 + \text{A}_{p+2} + \cdots + \text{A}_{(q-1)\,p+2} = \cdots = \text{A}_p + \text{A}_{2\,p} + \cdots + \text{A}_{qp} \quad \text{not true, } \text{a=(} \quad \text{A}_1 + \text{A}_{p+1} + \cdots + \text{A}_{(q-1)\,p+1}, \\ + \text{A}_{(q-1)\,p+1}, \quad \text{A}_2 + \text{A}_{p+2} + \cdots + \text{A}_{(q-1)\,p+2}, \cdots, \text{A}_p + \text{A}_{2\,p} + \cdots + \text{A}_{qp}), \text{c=c+(q-count-1)(p-1)M, f=f} \\ \Phi_{pk}(\textbf{x})^{(q-count-1)l}.$
- (viii) If l=1, then{let $A_i = (a_{(i-1)k}, a_{(i-1)k+1}, \dots, a_{ik-1})$, $i=1,2,\dots,p$. If $A_1 = A_2 = \dots = A_p$, then a=b, $l=q^n$, go to (i); else, c=c+(p-1)k, $f=f\Phi_{nk}(x)$, a=b, $l=q^n$, go to (i).}
- (ix) If $l \neq 1$, then l = l/q, let M = l k, $A_i = (a_{(i-1)M}, a_{(i-1)M+1}, \dots, a_{iM-1})$, $i = 1, 2, \dots, qp$. Set count=0, go to (vi).
- (x) The final. The linear complexity of s: c(s)=c; the minimal polynomial of s: f(x)=c.

From theorem 2.1, theorem 2.2, algorithm 3.1, algorithm 3.2 and algorithm 3.3, we know that algorithm 3.4 is correct. With a similar argument as that of algorithm 3.2 in [5], it is easy to show that it computes the minimal polynomial in

[n(q-1)+1](m+1) loops at most.

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