

A Compositional Query Algebra for Second-Order Logic and Uncertain Databases

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Abstract

World-set algebra is a variable-free query language for uncertain databases. It constitutes the core of the query language implemented in MayBMS, an uncertain database system. This paper shows that world-set algebra captures exactly second-order logic over finite structures, or equivalently, the polynomial hierarchy. The proofs also imply that world-set algebra is closed under composition, a previously open problem.

1 Introduction

Developing suitable query languages for uncertain databases is a substantial research challenge that is only currently starting to get addressed. In previous work [3], we have developed a query language in the spirit of relational algebra for processing uncertain data – *world-set algebra* (WSA). WSA consists of the operations of relational algebra plus two further operations, one to introduce uncertainty and one to compute possible tuples across groups of possible worlds. WSA is implemented in the MayBMS system [3, 2, 10, 9].

It remains to obtain an understanding of the complexity and expressive power of world-set algebra. The main result of this paper is a proof that world-set algebra over uncertain databases consisting of finite sets of possible worlds (each one a relational database) precisely captures second-order logic (SO) over finite structures, or equivalently, the polynomial hierarchy. This seems to be a somewhat surprising coincidence, since the language was not designed with this result as a goal but by abstraction from a set of use cases from the contexts of hypothetical (“what-if”) queries, decision support queries, and data cleaning. Viewed differently, WSA is a natural variable-free language equivalent to SO; it is to SO what relational algebra is to first-order logic. To the best of the author’s knowledge, no other such language is known.

The fact that WSA exactly captures second-order logic is a strong argument to justify it as a query language for uncertain data. Second-order logic is a natural yardstick for languages for querying possible worlds. Indeed, second-order quantifiers are the essence of what-if reasoning about databases. World-set algebra seems to be a strong candidate for a core algebra for forming query plans and optimizing and executing them in uncertain database management systems.

It was left open in previous work whether world-set algebra is closed under composition, or in other words, whether definitions are adding to the expressive power of the language. Compositionality is a desirable and rather commonplace property of query algebras, but

in the case of WSA it seems rather unlikely to hold. The reason for this is that the algebra contains an uncertainty-introduction operation that on the level of possible worlds is nondeterministic. First materializing a view and subsequently using it multiple times in the query is semantically quite different from composing the query with the view and thus obtaining several copies of the view definition that can now independently make their non-deterministic choices. In the paper, evidence is given that seems to suggest that definitions are essential for the expressive power of WSA.

The paper nevertheless gives a proof that definitions do not add to the power of the language, and WSA is indeed compositional. In fact, there is even a (nontrivial) practical linear-time translation from SO to WSA without definitions. This result, and the techniques for proving it, may also be relevant in other contexts. For example, it is shown that self-joins essentially can always be eliminated from classical relational algebra at the cost of introducing difference operators.

The proofs also imply that WSA is complete for the polynomial hierarchy with respect to data complexity and PSPACE-complete with respect to combined complexity [15, 14].

For use as a query language for probabilistic databases, WSA has been extended very slightly by a tuple confidence computation operation (see e.g. [9]). The focus of this paper is on the nonprobabilistic language of [3]. For the efficient processing of queries of this language, the confidence operation is naturally orthogonal to the remaining operations [2, 10, 9]. The expressiveness and complexity results obtained in the present paper constitute lower bounds for the probabilistic version of the language. But the non-probabilistic language is interesting and important in its own right: Many interesting queries can be phrased in terms of the alternatives possible in a data management scenario with uncertainty, without reference to the relative (probability) weights of these alternatives.

The structure of this paper is as follows. Section 2 establishes the connection between second-order logic and uncertain databases. Section 3 introduces world-set algebra and gives formal definitions of syntax and semantics. Section 4 proves that WSA exactly captures the expressive power of second-order logic over finite structures. These proofs assume the availability of a construct for making definitions (materializing views). Section 5 discusses the importance of being able to compose these definitions with the language, and shows why it should seem rather surprising that definitions are not needed for capturing second-order logic. Section 6 finally proves that definitions can indeed be eliminated without loss of expressive power, and a construction for composition is given. We obtain from these results that WSA with or without definitions is complete for the polynomial hierarchy with respect to data complexity and PSPACE-complete with respect to combined complexity. We discuss related work in Section 7 and conclude in Section 8.

2 Uncertain Databases

The schema of a relational database is a set of relation names together with a function sch that maps each relation name to a tuple of attribute names. We use calligraphic symbols such as \mathcal{A} for relational databases. The arity $|sch(R)|$ of a relation R is denoted by $ar(R)$.

We will use the standard syntax of second-order logic (SO) (see e.g. [11]). Its semantics is defined using the satisfaction relation \models , as usual. Throughout this paper, we will only use second-order logic *relativized* to some finite set of domain elements (say, D), as is common in finite model theory (cf. [11]). That is, first-order quantifiers $\exists x \phi$ are to be read as $\exists x D(x) \wedge \phi$ and second-order quantifiers $\exists R \phi$ are to be interpreted as $\exists R R \subseteq D^{ar(R)} \wedge \phi$.

An *uncertain database* over a given schema represents a finite set $W = \{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ of relational databases of that schema, called the *possible worlds*. One world among these is the true world, but we do not know which one.

A *representation* for a finite set of possible worlds W over schema (R_1, \dots, R_k) is a pair of a relational database schema and a formula ω over that database schema with free second-order variables R_1, \dots, R_k and without free first-order variables such that ω is true on exactly those structures that are in W :

$$(R_1, \dots, R_k) \models \omega \Leftrightarrow (R_1, \dots, R_k) \in W.$$

Example 2.1 (Standard Representation) Consider a representation of an uncertain database by relations that associate with each tuple a local condition in the form of a conjunction of propositional literals. A possible world is identified by a truth assignment for the propositional variables used, and a tuple is in a possible world if the world's truth assignment makes the tuple's clause true.

A representation database consists of a set V of propositional variables, a relation L such that $L(c, p, 1)$ is true iff variable p occurs positively in conjunction c and $L(c, p, 0)$ is true iff variable p occurs negated in c , and a representation relation R'_i for each schema relation R_i which extends the schema of R_i by a column to associate each tuple with a conjunction.

Possible worlds are identified by subsets $P \subseteq V$ of variables that are true. A tuple \vec{t} is in relation R_i in possible world P if $R'_i(\vec{t}, c)$ is true and conjunction c is true for the variable assignment that makes the variables in P true and the others false.

The representation formula $\omega(R_1, \dots, R_k)$ is

$$\exists P \ P \subseteq V \wedge \bigwedge_{i=1}^k \forall \vec{t} \ R_i(\vec{t}) \Leftrightarrow \exists c \ R'_i(\vec{t}, c) \wedge \forall p \ (L(c, p, 0) \Rightarrow \neg P(p)) \wedge (L(c, p, 1) \Rightarrow P(p)).$$

This is the representation system that is essentially used in MystiQ [5], Trio [4], and MayBMS [2]. It is a special case of c-tables [7] in which local conditions are in DNF, there is no global condition, and no variables occur in the data tuples themselves (just in the local conditions associated with the data tuples). Note that it is complete in the sense that it can represent any nonempty finite set of possible worlds. Moreover, it is succinct, i.e., the cardinality of the represented set of possible worlds is in general exponential in the size of the representation database. \square

It is now easy to use second-order logic for expressing queries on uncertain databases encoded by a representation. For instance, query ϕ is possible if $\exists R_1 \dots R_k \ \omega \wedge \phi$ and certain if $\forall R_1 \dots R_k \ \omega \Rightarrow \phi$. Second-order logic allows us to use succinct representations, but also yields very powerful hypothetical queries that can ask questions about possible choices of *sets* of tuples. Such a choice of sets could be e.g. clusters of tuples in record matching (also known as deduplication and under many other names).

3 The Algebra

3.1 Syntax and Semantics

World-set algebra (WSA) consists of the operations of relational algebra (selection σ , projection π , renaming ρ , product \times , union \cup , and difference $-$), two additional operations

repair-key and possible $_{\vec{A}}$, and definitions “let $R := Q$ in Q' ” where R is a new relation symbol that may be used in Q' . *WSA without definitions* is the set of WSA queries in which no let-expressions occur.

Conceptually all operations are evaluated in each possible world individually. The operations of relational algebra are evaluated within possible world \mathcal{A} in the normal way. Given input relation R , repair-key $_{\vec{A}}(R)$ nondeterministically chooses a maximal repair of the functional dependency $\vec{A} \rightarrow sch(R)$ on R , that is, it returns a subset R' of R in which \vec{A} is a (super)key such that there is no superset of R' which is a subset of R and in which \vec{A} is a (super)key. The operation possible $_{\vec{A}}(Q)$ is the only operation that can look into alternative possible worlds. It computes, for the current possible world given by \mathcal{A} , the set of possible tuples occurring in the results of Q across the group of possible worlds that agree with \mathcal{A} on $\pi_{\vec{A}}(Q)$. Definitions (statements “let $R := Q$ in Q' ”) extend \mathcal{A} by a named relation R defined by query Q . Since Q is nondeterministic in general, the overall set of possible worlds on which Q' runs (which is relevant for computing possible $_{\vec{A}}$) may increase.

Formally, the semantics of world-set algebra is defined using a translation $\llbracket \cdot \rrbracket_W^A$ such that for a context of a set of possible worlds W and a world $\mathcal{A} \in W$, R is a possible result of world-set algebra query Q iff $R \in \llbracket Q \rrbracket_W^A$:

$$\begin{aligned}
\llbracket \{\vec{t}\} \rrbracket_W^A &:= \{\{\vec{t}\}\} \\
&\dots \quad \vec{t} \text{ constant tuple} \\
\llbracket R \rrbracket_W^A &:= \{R^{\mathcal{A}}\} \\
\llbracket \theta(Q) \rrbracket_W^A &:= \{\theta(R) \mid R \in \llbracket Q \rrbracket_W^A\} \\
&\dots \quad \theta \in \{\sigma_\phi, \pi_{\vec{A}}, \rho_{A \rightarrow B}\} \\
\llbracket Q_1 \theta Q_2 \rrbracket_W^A &:= \{R_1 \theta R_2 \mid R_1 \in \llbracket Q_1 \rrbracket_W^A, R_2 \in \llbracket Q_2 \rrbracket_W^A\} \\
&\dots \quad \theta \in \{\times, \cup, -\} \\
\llbracket \text{repair-key}_{\vec{A}}(Q) \rrbracket_W^A &:= \{R' \mid R' \subseteq R \in \llbracket Q \rrbracket_W^A, \pi_{\vec{A}}(R) = \pi_{\vec{A}}(R'), \\
&\quad \vec{A} \text{ is a key for } R_0\} \\
\llbracket \text{possible}_{\vec{A}}(Q) \rrbracket_W^A &:= \left\{ \bigcup \{R' \mid \mathcal{B} \in W, R' \in \llbracket Q \rrbracket_{\mathcal{B}}^{\mathcal{B}}, \right. \\
&\quad \left. \pi_{\vec{A}}(R) = \pi_{\vec{A}}(R') \} \mid R \in \llbracket Q \rrbracket_W^A \right\} \\
\llbracket \text{let } R := Q \text{ in } Q' \rrbracket_W^A &:= \{\llbracket Q' \rrbracket_{W'}^{(\mathcal{A}, R)} \mid R \in \llbracket Q \rrbracket_W^A\} \\
\text{where } W' &= \{(\mathcal{B}, R') \mid \mathcal{B} \in W, R' \in \llbracket Q \rrbracket_{\mathcal{B}}^{\mathcal{B}}\}.
\end{aligned}$$

Queries are run against an uncertain database W , and $\llbracket Q \rrbracket_W^A$ gives the result of Q seen in possible world \mathcal{A} of W . Using possible $_{\emptyset}$, we can close the possible worlds semantics and ask for possible (or, using difference, certain) tuples. For such queries \mathcal{A} can be chosen arbitrarily (and the semantics function can be considered to be of the form $\llbracket Q \rrbracket_W$).

Definitions in subexpression are unaffected by the operations higher up in the expression tree and can be pulled to the top of the expression without modification. This is a direct consequence of the following fact, where we assume that θ may be any of the WSA operations. (Thus $0 \leq k \leq 2$ and for possible $_{\vec{A}}$, $k = 2$.)

Proposition 3.1 *For arbitrary WSA queries $Q, \theta(Q_1, \dots, Q_k)$, if V occurs only in Q_i ,*

$$\theta(Q_1, \dots, Q_{i-1}, (\text{let } V := Q \text{ in } Q_i), Q_{i+1}, \dots, Q_k) = (\text{let } V := Q \text{ in } \theta(Q_1, \dots, Q_k)).$$

Proof. It can be shown by an easy induction that for any Q , $\llbracket Q \rrbracket_W^{(\mathcal{A}, V)} = \llbracket Q \rrbracket_{W'}^{\mathcal{A}}$, where $W' = \{\mathcal{A} \mid (\mathcal{A}, V) \in W\}$ if relation name V does not appear in Q . This is immediate for all operations other than $\text{possible}_{\vec{A}}$. Let $Q = \text{possible}_{\vec{A}}(Q')$ and let the induction hypothesis hold for Q' , i.e., $\llbracket Q' \rrbracket_W^{(\mathcal{A}, V)} = \llbracket Q' \rrbracket_{W'}^{\mathcal{A}}$. Then

$$\begin{aligned} \llbracket \text{possible}_{\vec{A}}(Q') \rrbracket_W^{(\mathcal{A}, V)} &= \left\{ \bigcup \{R' \mid (\mathcal{B}, V') \in W, R' \in \llbracket Q' \rrbracket_W^{(\mathcal{B}, V')}, \right. \\ &\quad \left. \pi_{\vec{A}}(R) = \pi_{\vec{A}}(R') \} \mid R \in \llbracket Q' \rrbracket_W^{(\mathcal{A}, V)} \right\} \\ &= \left\{ \bigcup \{R' \mid V' \in W', R' \in \llbracket Q' \rrbracket_{W'}^{\mathcal{B}}, \pi_{\vec{A}}(R) = \pi_{\vec{A}}(R') \} \mid R \in \llbracket Q' \rrbracket_{W'}^{\mathcal{A}} \right\} \\ &= \llbracket \text{possible}_{\vec{A}}(Q') \rrbracket_{W'}^{\mathcal{A}}. \end{aligned}$$

Now we apply the fact just proven to the subqueries Q_j for $j \neq i$. By definition,

$$\llbracket \text{let } V := Q \text{ in } \theta(Q_1, \dots, Q_k) \rrbracket_{W'}^{\mathcal{A}} = \{ \llbracket \theta(Q_1, \dots, Q_k) \rrbracket_W^{(\mathcal{A}, V)} \mid V \in \llbracket Q \rrbracket_{W'}^{\mathcal{A}} \}.$$

We distinguish between the various operations θ . For relational algebra,

$$\begin{aligned} \llbracket \theta(Q_1, \dots, Q_k) \rrbracket_W^{(\mathcal{A}, V)} &= \left\{ \theta(R_1, \dots, R_k) \mid \bigwedge_j R_j \in \llbracket Q_j \rrbracket_W^{(\mathcal{A}, V)} \right\} \\ &= \left\{ \theta(R_1, \dots, R_k) \mid R_i \in \llbracket Q_i \rrbracket_W^{(\mathcal{A}, V)}, \bigwedge_{j \neq i} R_j \in \llbracket Q_j \rrbracket_{W'}^{\mathcal{A}} \right\} \end{aligned}$$

because V only occurs in Q_i and $\llbracket Q_j \rrbracket_W^{(\mathcal{A}, V)} = \llbracket Q_j \rrbracket_{W'}^{\mathcal{A}}$ for $j \neq i$. Thus

$$\begin{aligned} \llbracket \text{let } V := Q \text{ in } \theta(Q_1, \dots, Q_k) \rrbracket_{W'}^{\mathcal{A}} &= \left\{ \theta(R_1, \dots, R_k) \mid \underbrace{R_i \in \llbracket Q_i \rrbracket_W^{(\mathcal{A}, V)}, V \in \llbracket Q \rrbracket_{W'}^{\mathcal{A}}}_{R_i \in \llbracket \text{let } V := Q \text{ in } Q_i \rrbracket_{W'}^{\mathcal{A}}} \right. \\ &\quad \left. \bigwedge_{j \neq i} R_j \in \llbracket Q_j \rrbracket_{W'}^{\mathcal{A}} \right\} \\ &= \llbracket \theta(Q_1, \dots, Q_{i-1}, (\text{let } V := Q \text{ in } Q_i), Q_{i+1}, \dots, Q_k) \rrbracket_{W'}^{\mathcal{A}} \end{aligned}$$

The proof for the remaining operations proceeds similarly. \square

In other words, they can be considered “global”. That is, without loss of generality we could assume that each WSA query is of the form

$$\text{let } V_1 := Q_1 \text{ in } (\dots (\text{let } V_k := Q_k \text{ in } Q) \dots)$$

where Q does not contain definitions.

Observe that in the case of binary relational algebra operations θ , the set of possible worlds $\llbracket Q_1 \theta Q_2 \rrbracket_W^{\mathcal{A}}$ is obtained by pairing relations in the results of $\llbracket Q_1 \rrbracket_W^{\mathcal{A}}$ and $\llbracket Q_2 \rrbracket_W^{\mathcal{A}}$. This is consistent with the intuition that θ is applied to possible worlds \mathcal{B} that contain two relations $R_1^{\mathcal{B}}$ and $R_2^{\mathcal{B}}$ and the result in \mathcal{B} is $R_1^{\mathcal{B}} \theta R_2^{\mathcal{B}}$: Proposition 3.1 implies that

$$\theta(Q_1, \dots, Q_k) = (\text{let } V_1 := Q_1, \dots, V_k := Q_k \text{ in } \theta(V_1, \dots, V_k)).$$

As a convention, we use $\{\langle \rangle\}$ to represent truth and \emptyset to represent falsity, over a nullary relation schema.

Example 3.2 Given a relational database with relations $V(V)$ and $E(From, To)$ representing a graph (directed, or undirected if E is symmetric). Then the following WSA query Q returns true iff the graph is 3-colorable:

$$\text{let } R := \text{repair-key}_{sch(V)}(V \times \rho_C(\{r\} \cup \{g\} \cup \{b\})) \text{ in} \\ \text{possible}_\emptyset(\{\langle \rangle\} - \pi_\emptyset(\sigma_{1.V=2.From \wedge 2.To=3.V \wedge 1.C=3.C}(R \times E \times R))).$$

The possible relations R are all the functions $V \rightarrow \{r, g, b\}$, and Q simply asks whether there is such a function R such that there do not exist two adjacent nodes of the same color.

The corresponding SO sentence is

$$\exists R \phi_{R:V \rightarrow \{r,g,b\}} \wedge \neg \exists u, v, c R(u, c) \wedge E(u, v) \wedge R(v, c)$$

where $\phi_{R:V \rightarrow \{r,g,b\}}$ is a first-order sentence that states that R is a relation $\subset V \times \{r, g, b\}$ that satisfies the functional dependency $R : V \rightarrow \{r, g, b\}$. \square

3.2 Derived Operations: Syntactic Sugar

We will also consider the following operations, which are definable in the base language:

$$\begin{aligned} \llbracket \text{subset}(Q) \rrbracket_W^A &:= \{R' \mid R' \subseteq R \in \llbracket Q \rrbracket_W^A\} \\ \llbracket \text{choice-of}_{\vec{A}}(Q) \rrbracket_W^A &:= \{\pi_{\vec{A}=\vec{a}}(R) \mid R \in \llbracket Q \rrbracket_W^A, \vec{a} \in \pi_{\vec{A}}(R)\} \\ \llbracket \text{certain}_{\vec{A}}(Q) \rrbracket_W^A &:= \left\{ \bigcap \{R' \mid \mathcal{B} \in W, R' \in \llbracket Q \rrbracket_W^{\mathcal{B}}, \right. \\ &\quad \left. \pi_{\vec{A}}(R) = \pi_{\vec{A}}(R') \} \mid R \in \llbracket Q \rrbracket_W^A \right\} \\ \llbracket \text{possible}(Q) \rrbracket_W^A &:= \left\{ \bigcup \{R \mid \mathcal{B} \in W, R \in \llbracket Q \rrbracket_W^{\mathcal{B}} \} \right\} \\ \llbracket \text{certain}(Q) \rrbracket_W^A &:= \left\{ \bigcap \{R \mid \mathcal{B} \in W, R \in \llbracket Q \rrbracket_W^{\mathcal{B}} \} \right\} \end{aligned}$$

The operation `subset` nondeterministically chooses an arbitrary subset of its input relation. The operation `choice-of \vec{A}` (R) nondeterministically chooses an $\vec{a} \in \pi_{\vec{A}}(R)$ and selects those tuples \vec{t} of R for which $\vec{t}.\vec{A} = \vec{a}$. Conceptually, the operations `subset` and `repair-key` cause an exponential blowup of the possible worlds under consideration: for instance, on a certain database (i.e., consisting of a single possible world) `subset`(R) creates the powerset of relation R as the new set of possible worlds. The operation `certain \vec{A}` is the dual of `possible \vec{A}` and computes those tuples common to all the worlds that agree on $\pi_{\vec{A}}$. The operations `possible` and `certain` compute the possible respectively certain tuples across *all* possible worlds.

Proposition 3.3 *The operations `subset` and `possible` are expressible in WSA without definitions. The operations `choice-of \vec{A}` , `certain \vec{A}` , and `certain` are definable in WSA with definitions.*

Proof Sketch. The result is an immediate consequence of the following equivalences.

$$\begin{aligned} \text{choice-of}_{\vec{A}}(R) &= R \bowtie \text{repair-key}_\emptyset(\pi_{\vec{A}}(R)). \\ \text{certain}_{\vec{A}}(Q) &= Q - \text{possible}_{\vec{A}}(\text{possible}_{\vec{A}}(Q) - Q) \\ \text{subset}(R) &= \pi_{sch(R)}(\sigma_{A=1}(\text{repair-key}_{sch(R)}(R \times \rho_A(\{0, 1\})))) \\ &\quad (\text{w.l.o.g., } A \notin sch(R)). \\ \text{possible}(Q) &= \text{possible}_\emptyset(Q) \\ \text{certain}(Q) &= \text{certain}_\emptyset(Q) \end{aligned}$$

Company_Emp			C	E	Emp_Skills			E	S
			c_1	e_{11}				e_{11}	s_1
			c_1	e_{12}				e_{12}	s_1
			c_2	e_{21}				e_{21}	s_2
			c_2	e_{22}				e_{21}	s_1
			c_2	e_{23}				e_{22}	s_3
								e_{23}	s_2

(a)

U_1	C	E	U_2	C	E	U_3	C	E	U_4	C	E	U_5	C	E
	c_1	e_{11}		c_1	e_{12}		c_2	e_{21}		c_2	e_{22}		c_2	e_{23}

(b)

V_1	C	E	V_2	C	E	V_3	C	E	V_4	C	E	V_5	C	E
	c_1	e_{12}		c_1	e_{11}		c_2	e_{22}		c_2	e_{21}		c_2	e_{21}
							c_2	e_{23}		c_2	e_{23}		c_2	e_{22}

(c)

W_1	C	S	W_2	C	S	W_3	C	S	W_4	C	S	W_5	C	S
	c_1	s_1		c_1	s_1		c_2	s_2		c_2	s_2		c_2	s_2

(d)

Figure 1: Database (a) and intermediate query results (b-d) of Example 3.5.

The expression $\text{possible}_\emptyset(Q)$ computes the possible tuples of those worlds in which the result of Q is nonempty. But, obviously, in the remaining worlds there are no tuples to collect. By the definition of certain_Q in terms of possible_Q , the definition of certain is correct too. \square

Remark 3.4 The operation repair-key is also definable using the base operations without repair-key plus subset ; however, such a definition seems to need let -statements, while the definition of subset using repair-key does not.

In [3], it was shown that the fragment obtained from WSA by replacing repair-key by choice-of is a conservative extension of first-order logic. That is, every query of that language that maps from a single possible world to a single possible world is equivalent to a first-order query. It is not surprising that this is not true for full WSA.

3.3 A Hypothetical Query Processing Example

Example 3.5 Consider the relational database of Figure 1(a) which represents employees working in companies and their skills. The query, a simplified decision support problem, will be stated in four steps.

1. Suppose I choose to buy exactly one company and, as a consequence, exactly one (key) employee leaves that company.

$$U := \text{choice_of}_{C,E}(\text{Company_Emp})$$

(This nondeterministically chooses a tuple from Company_Emp .)

2. Who are the remaining employees?

$$V := \pi_{1.C, 2.E}(U \bowtie_{1.C=2.C \wedge 1.E \neq 2.E} \text{Company_Emp})$$

3. If I acquire that company, which skills can I obtain *for certain*?

$$W := \text{certain}_C(\pi_{C,S}(V \bowtie \text{Emp_Skills}))$$

(This query computes the tuples of $V \bowtie \text{Emp_Skills}$ that are certain assuming that the company was chosen correctly – i.e., certain in the set of possible worlds that agree with this world on the C column.)

4. Now list the *possible* acquisition targets if the gain of the skill s_1 shall be guaranteed by the acquisition.

$$\text{possible}(\pi_C(\sigma_{S=s_1}(W)))$$

Figure 1(b-d) shows the development of the uncertain database through steps 1 to 3. The first step creates five possible worlds corresponding to the five possible choices of company and renegade employee from relation `Company_Emp`. Steps two to four further process the query, and the overall result, which is the same in all five possible worlds, is

Result	C
	c_1

□

4 WSA with Definitions Captures SO Logic

In this section, it is shown that WSA with definitions has exactly the same expressive power as second-order logic over finite structures.

Theorem 4.1 *For every SO query, there is an equivalent WSA query with definitions.*

Proof. We may assume without loss of generality that the SO query is a first-order query prefixed by a sequence of second-order quantifiers. The proposition follows from induction.

Induction start: FO queries can be translated to relational algebra by a well-known translation known in the database context as one direction of Codd's Theorem (cf. [1]).

Induction step (second-order existential quantification, $\exists R_{k+1}(\subseteq D^l) \phi$): Let ϕ be an SO formula with free second-order variables R_1, \dots, R_{k+1} and free first-order variables \vec{x} where R_{k+1} has arity l . Let Q_ϕ be an equivalent WSA expression. Without loss of generality, we may assume that the relations R_1, \dots, R_k, Q_ϕ have disjoint schemas. Let

$$Q := (\text{let } R_{k+1} := \text{subset}(D^l) \text{ in } \pi_{\text{sch}(Q)}(\text{possible}_{\text{sch}(R_1) \dots \text{sch}(R_k)}(1_{R_1} \times \dots \times 1_{R_k} \times Q_\phi))).$$

where $1_{R_i} = R_i \times \{1\} \cup (D^{\text{ar}(R_i)} - R_i) \times \{0\}$. (Note that the relations 1_{R_i} will play a prominent role in later parts of this paper.) We prove that

$$(R_1, \dots, R_k, \vec{x}) \models \exists R_{k+1}(\subseteq D^l) \phi \Leftrightarrow \vec{x} \in R_Q$$

where $\{R_Q\} = \llbracket Q \rrbracket_W^{(R_1, \dots, R_k)}$. By definition of $\llbracket \cdot \rrbracket$,

$$\llbracket Q \rrbracket_W^{(R_1, \dots, R_k)} = \{ \pi_{\text{sch}(Q_\phi)}(\llbracket Q' \rrbracket_{W'}^{(R_1, \dots, R_{k+1})}) \mid R_{k+1} \subseteq D^l \}$$

where $W' = \{(R_1, \dots, R_{k+1}) \mid (R_1, \dots, R_k) \in W, R_{k+1} \subseteq D^l\}$ and Q' is a shortcut for $\text{possible}_{sch(R_1) \dots sch(R_k)}(1_{R_1} \times \dots \times 1_{R_k} \times Q_\phi)$.

We may assume a nonempty domain D , so the result of $1_{R_1} \times \dots \times 1_{R_k}$ is never empty, the mapping $(R_1, \dots, R_k) \mapsto 1_{R_1} \times \dots \times 1_{R_k}$ is injective, and Q will therefore group the possible outcomes of Q_ϕ for the various choices of R_{k+1} by R_1, \dots, R_k .

Formally, by definition of $\llbracket \cdot \rrbracket$,

$$\begin{aligned} \llbracket Q' \rrbracket_{W'}^{(R_1, \dots, R_{k+1})} &= \left\{ \bigcup \{ 1_{R_1} \times \dots \times 1_{R_k} \times \llbracket Q_\phi \rrbracket_{W'}^{(R_1, \dots, R_k, R'_{k+1})} \mid \right. \\ &\quad \left. (R_1, \dots, R_k, R'_{k+1}) \in W' \} \mid (R_1, \dots, R_{k+1}) \in W' \right\} \\ &= \left\{ 1_{R_1} \times \dots \times 1_{R_k} \times \bigcup \{ \llbracket Q_\phi \rrbracket_{W'}^{(R_1, \dots, R_k, R'_{k+1})} \mid R'_{k+1} \subseteq D^l \} \right\}. \end{aligned}$$

Thus, in a given world (R_1, \dots, R_k) , Q produces exactly one world as the result,

$$\llbracket Q \rrbracket_W^{(R_1, \dots, R_k)} = \left\{ \bigcup \{ \llbracket Q_\phi \rrbracket_{W'}^{(R_1, \dots, R_k, R'_{k+1})} \mid R'_{k+1} \subseteq D^l \} \right\} = \{R_Q\}$$

and this captures exactly second-order existential quantification.

The WSA expression for universal second-order quantifiers $\forall R_{k+1} (\subseteq D^l) \phi$ is similar. Alternatively, $\forall R_{k+1} \phi$ can also be taken as $\neg \exists R_{k+1} \neg \phi$, where complementation with respect to D is straightforward using the difference operation. \square

Example 4.2 Σ_2 -QBF is the following Σ_2^P -complete decision problem. Given two disjoint sets of propositional variables V_1 and V_2 and a DNF formula ϕ over the variables of V_1 and V_2 , does there exist a truth assignment for the variables V_1 such that ϕ is true for all truth assignments for the variables V_2 ?

Instances of this problem shall be represented by sets V_1 and V_2 , a set C of ids of clauses in ϕ , and a ternary relation $L(C, P, S)$ such that $\langle c, p, 1 \rangle \in L$ (resp., $\langle c, p, 0 \rangle \in L$) iff propositional variable p occurs positively (resp., negatively) in clause c of ϕ , i.e.,

$$\phi = \bigvee_{c \in C} \bigwedge_{\langle c, p, 1 \rangle \in L} p \wedge \bigwedge_{\langle c, p, 0 \rangle \in L} \neg p.$$

The QBF is true iff second-order sentence

$$\exists P_1 (P_1 \subseteq V_1) \wedge \forall P_2 (P_2 \subseteq V_2) \Rightarrow \psi$$

is true, where ψ is the first-order sentence

$$\exists c \neg \exists p (L(c, p, 0) \wedge (P_1(p) \vee P_2(p))) \vee (L(c, p, 1) \wedge \neg (P_1(p) \vee P_2(p))).$$

which asserts the truth of ϕ : that there is a clause c in ϕ of which no literal is inconsistent with the truth assignment $p \mapsto (p \in P_1 \cup P_2)$. By Theorem 4.1, this can be expressed as the Boolean WSA query

$$\begin{aligned} \text{let } P_1 &:= \text{subset}(V_1) \text{ in possible}(\{\langle \rangle\}) \\ &- \text{let } P_2 := \text{subset}(V_2) \text{ in possible}_{sch(P_1)}(1_{P_1} \times (\{\langle \rangle\} - Q)) \end{aligned}$$

where

$$Q = \pi_\emptyset(C - \pi_C((\sigma_{S=0}(L) \bowtie (P_1 \cup P_2)) \cup (\sigma_{S=1}(L) \bowtie ((V_1 \cup V_2) - (P_1 \cup P_2))))))$$

is relational algebra for ψ . \square

For the converse result, we must first make precise how second-order logic will be compared to WSA, since second-order logic queries are usually not “run” on uncertain databases. We will consider WSA queries that are evaluated against a (single-world) relational database \mathcal{A} representing an uncertain database (e.g., using the standard representation of Example 2.1). We already know that arbitrary uncertain databases (that is, nonempty finite sets of possible worlds) can be so represented, and this assumption means no loss of generality. The query constructs the uncertain database from the representation and is always evaluated as $\llbracket Q \rrbracket_{\{\mathcal{A}\}}^{\mathcal{A}}$, precisely as sketched at the end of Section 2.

Theorem 4.3 *For every WSA query, there is an equivalent second-order logic query.*

Proof Sketch. The proof revolves around the definition of a function $\llbracket \cdot \rrbracket_{so}$ that maps each WSA expression Q to an SO formula $\llbracket Q \rrbracket_{so}$ with free second-order variables \vec{R} and R_Q and without free first-order variables such that $\llbracket Q \rrbracket_{so}$ and Q are equivalent in the sense that $\llbracket Q \rrbracket_{so}$ is true on structure $(\mathcal{A}, \vec{R}, R_Q)$ iff R_Q is among the possible results of Q starting from possible world (\mathcal{A}, \vec{R}) . We can state this notion of correctness, which is the hypothesis of the following induction along the structure of the WSA expression, formally as

$$(\mathcal{A}, \vec{R}, R_Q) \models \llbracket Q \rrbracket_{so} \Leftrightarrow R_Q \in \llbracket Q \rrbracket_W^{(\mathcal{A}, \vec{R})}$$

for

$$W = \left\{ (\mathcal{A}, \vec{R}) \mid (\mathcal{A}, \vec{R}) \models \bigwedge_{V \text{ in } \vec{R}} \psi_V \right\}.$$

Here the free second-order variables \vec{R} are also the names of the views defined (using let-expressions) along the path from the root of the parse tree of the query to the subexpression Q . A formula ψ_V is identified by the name of the view relation V , assuming without loss of generality that each view name is introduced only once by a let expression across the entire query. The formulae ψ_V will be defined below.

For the operations θ of relational algebra,

$$\begin{aligned} \llbracket \theta(Q_1, \dots, Q_{ar(\theta)}) \rrbracket_{so}(\vec{R}, R_Q) &:= \exists R_{Q_1} \cdots R_{Q_{ar(\theta)}} \left(\bigwedge_{i=1}^{ar(\theta)} \llbracket Q_i \rrbracket_{so}(\vec{R}, R_{Q_i}) \right) \\ &\quad \wedge \forall \vec{x} R_Q(\vec{x}) \Leftrightarrow \phi_{\theta(Q_1, \dots, Q_{ar(\theta)})}(\vec{x}) \end{aligned}$$

where $0 \leq ar(\theta) \leq 2$ and $\phi_S(\vec{x}) := S(\vec{x})$, where S is either a relation from \mathcal{A} or a second-order variable from \vec{R} , $\phi_{\{\vec{t}\}}(\vec{x}) := \vec{x} = \vec{t}$, $\phi_{Q_1 \cup Q_2}(\vec{x}) := R_{Q_1}(\vec{x}) \vee R_{Q_2}(\vec{x})$, $\phi_{Q_1 - Q_2}(\vec{x}) := R_{Q_1}(\vec{x}) \wedge \neg R_{Q_2}(\vec{x})$, $\phi_{Q_1 \times Q_2}(\vec{x}, \vec{y}) := R_{Q_1}(\vec{x}) \wedge R_{Q_2}(\vec{y})$, $\phi_{\sigma_\gamma(Q)}(\vec{x}) := R_Q(\vec{x}) \wedge \gamma$, $\phi_{\pi_{\vec{x}}(Q)}(\vec{x}) := \exists \vec{y} R_Q(\vec{x}, \vec{y})$, and $\phi_{\rho_{\vec{x} \rightarrow \vec{y}}(Q)}(\vec{y}) := \exists \vec{x} R_Q(\vec{x}) \wedge \vec{x} = \vec{y}$. It is easy to verify that for any tuple \vec{x} and relational algebra operation θ , $(\mathcal{A}, R_{Q_1}, \dots, R_{Q_{ar(\theta)}}) \models \phi_{\theta(Q_1, \dots, Q_{ar(\theta)})}(\vec{x})$ if and only if \vec{x} is a result tuple of relational algebra query $\theta(R_{Q_1}, \dots, R_{Q_{ar(\theta)}})$. Assume that the induction hypothesis holds for the subqueries $Q_1, \dots, Q_{ar(\theta)}$, i.e., $(\mathcal{A}, \vec{R}, R_{Q_i}) \models \llbracket Q_i \rrbracket_{so}$ if and only if $R_{Q_i} \in \llbracket Q_i \rrbracket_W^{(\mathcal{A}, \vec{R})}$ for $1 \leq i \leq ar(\theta)$. The formula $\llbracket \theta(Q_1, \dots, Q_{ar(\theta)}) \rrbracket_{so}$ just states that R_Q is a relation consisting of exactly those tuples \vec{x} that satisfy $\phi_{\theta(Q_1, \dots, Q_{ar(\theta)})}(\vec{x})$ for a choice of possible results $R_{Q_i} \in \llbracket Q_i \rrbracket_W^{(\mathcal{A}, \vec{R})}$ of the subqueries Q_i , for $1 \leq i \leq ar(\theta)$. But this is exactly the definition of $\llbracket \theta(Q_1, \dots, Q_{ar(\theta)}) \rrbracket_W^{(\mathcal{A}, \vec{R})}$.

This in particular covers the nullary operations of relational algebra ($\{\vec{t}\}$ and R), which form the induction start.

The remaining operations are those special to WSA (with definitions):

$$\begin{aligned}
\llbracket \text{subset}(Q_1) \rrbracket_{so}(\vec{R}, R_Q) &:= \exists R_{Q_1} \llbracket Q_1 \rrbracket_{so}(\vec{R}, R_{Q_1}) \wedge R_Q \subseteq R_{Q_1} \\
\llbracket \text{repair-key}_{\vec{A}}(Q_1) \rrbracket_{so}(\vec{R}, R_Q) &:= \exists R_{Q_1} \llbracket Q_1 \rrbracket_{so}(\vec{R}, R_{Q_1}) \wedge R_Q \subseteq R_{Q_1} \\
&\quad \wedge \quad \vec{A} \text{ is a key for } R_Q \\
&\quad \wedge \quad \neg \exists R'_Q R_Q \subset R'_Q \subseteq R_{Q_1} \wedge \vec{A} \text{ is a key for } R'_Q \\
\llbracket \text{let } V := Q_1 \text{ in } Q_2 \rrbracket_{so}(\vec{R}, R_Q) &:= \exists V \psi_V \wedge \llbracket Q_2 \rrbracket_{so}(\vec{R}, V, R_Q) \\
&\quad \text{and define } \psi_V := \llbracket Q_1 \rrbracket_{so}(\vec{R}, V) \\
\llbracket \text{possible}_{\vec{A}}(Q_1) \rrbracket_{so}(\vec{R}, R_Q) &:= \exists R_{Q_1} \llbracket Q_1 \rrbracket_{so}(\vec{R}, R_{Q_1}) \wedge \forall \vec{x} R_Q(\vec{x}) \Leftrightarrow \\
&\quad \exists \vec{R} \left(\left(\bigwedge_{V \text{ in } \vec{R}} \psi_V \right) \right. \\
&\quad \wedge \exists R'_{Q_1} \llbracket Q_1 \rrbracket_{so}(\vec{R}, R'_{Q_1}) \\
&\quad \left. \wedge \pi_A(R_{Q_1}) = \pi_A(R'_{Q_1}) \wedge R'_{Q_1}(\vec{x}) \right)
\end{aligned}$$

where “ \vec{A} is a key for R ” and $\pi_{\vec{A}}(\cdot) = \pi_A(\cdot)$ are easily expressible in FO.

It is straightforward to verify the correctness of $\llbracket \cdot \rrbracket_{so}$ for subset and repair-key: The definitions of $\llbracket \cdot \rrbracket_{so}$ and $\llbracket \cdot \rrbracket$ essentially coincide.

Similarly, the correctness of the definition of $\llbracket \cdot \rrbracket_{so}$ for let is easy to verify. Here we also define the formulae ψ_V .

Finally, $\llbracket \text{possible}_{\vec{A}}(Q_1) \rrbracket_{so}$ makes reference to world-set W and for that purpose uses the formulae ψ_V : Indeed, the worlds in W are exactly those structures that satisfy all the ψ_V for relations V defined by let expressions on the path from the root of the query to the current subexpression $\text{possible}_{\vec{A}}(Q_1)$. The definition $\llbracket \text{possible}_{\vec{A}}(Q_1) \rrbracket_{so}$ is again very close to the definition of $\llbracket \text{possible}_{\vec{A}}(Q_1) \rrbracket$, and its correctness is straightforward to verify.

Note that by eliminating the definitions ψ_V we in general obtain an exponential-size formula. \square

5 Intermezzo: Why we are not done

The proof that WSA with definitions can express any SO query may seem to settle the expressiveness question for our language. However, understanding WSA without definitions is also important, for two reasons. First, it is a commonplace and desirable property of query algebras that they be compositional, i.e., that the power to define views is not needed for the expressive power, and all views can be eliminated by composing the query. Second, if this property does not hold, it means that in general we have to precompute and materialize views. And indeed, superficially we would expect that WSA is not compositional in that respect: it supports nondeterministic operations (repair-key and/or subset). If a view definition V contains such a nondeterministic operation and a query uses V at least twice, replacing each occurrence with the definition will not be equivalent because the two copies of the definition of V will produce different relations in some worlds. For example, $(\text{let } V := \text{subset}(U) \text{ in } V \bowtie V)$ is not at all equivalent to $\text{subset}(U) \bowtie \text{subset}(U)$.

The question remains whether for each WSA query there is an equivalent query in WSA without definitions via a less direct rewriting. The answer to this question is less obvious. Our language definition has assumed repair-key to be the base operation and subset definable using WSA with repair-key. Indeed, in WSA with definitions, either one can be defined using the other. However, it can be shown that repair-key cannot be expressed using subset without using definitions even though subset can guess subsets and appears comparable in expressiveness to repair-key.

Consider possible worlds databases in which each relation is independent from the other relations, i.e., the world set is of the form

$$\{(R_1, \dots, R_k) \mid R_1 \in W_1, \dots, R_k \in W_k\}.$$

WSA without definitions on such *relation-independent databases* gives rise to a much simpler and more intuitive semantics definition than the one of Section 3, via the following function $\llbracket \cdot \rrbracket_{ndef}$.

$$\begin{aligned} \llbracket \theta \rrbracket_{ndef}(W_1, \dots, W_{ar(\theta)}) &:= \{ \theta(R_1, \dots, R_{ar(\theta)}) \mid R_1 \in W_1, \dots, R_{ar(\theta)} \in W_{ar(\theta)} \} \\ &\quad \dots \text{ where } \theta \text{ is an operation of relational algebra} \\ \llbracket \text{repair-key}_{\vec{A}} \rrbracket_{ndef}(W) &:= \{ R \mid R \subseteq R' \in W, \pi_A(R) = \pi_A(R'), \vec{A} \text{ is a key for } R \} \\ \llbracket \text{subset} \rrbracket_{ndef}(W) &:= \{ R \mid R \subseteq R' \in W \} \\ \llbracket \text{possible}_{\vec{A}} \rrbracket_{ndef}(W) &:= \left\{ \bigcup \{ R' \in W \mid \pi_{\vec{A}}(R) = \pi_{\vec{A}}(R') \} \mid R \in W \right\} \end{aligned}$$

The correctness of this alternative semantics definition, stated next, is easy to verify.

Proposition 5.1 *For relation-independent databases and WSA without definitions, $\llbracket \cdot \rrbracket_{ndef}$ is equivalent to $\llbracket \cdot \rrbracket$ in the sense that for any operation θ ,*

$$\{ \llbracket \theta(Q_1, \dots, Q_{ar(\theta)}) \rrbracket_W^A \mid A \in W \} = \llbracket \theta \rrbracket_{ndef}(W_1, \dots, W_{ar(\theta)})$$

where $W_i = \bigcup \{ \llbracket Q_i \rrbracket_W^A \mid A \in W \}$ for all $1 \leq i \leq ar(\theta)$.

The following result asserts that adding subset to relational algebra yields little expressive power. By the existence of supremum of a set of worlds W , we assert the existence of an element $(\bigcup W) \in W$, denoted $\sup(W)$. An infimum is a set $\inf(W) := (\bigcap W) \in W$.

Theorem 5.2 *Any world-set computable using relational algebra extended by the operation subset has a supremum and an infimum.*

Proof. The nullary relational algebra expressions $(\{\vec{t}\})$ and R yield just a singleton world-set, and the single world is both the supremum and the infimum. Given a world-set W , $\sup(\llbracket \text{subset} \rrbracket_{ndef}(W)) := \sup(W)$ and $\inf(\llbracket \text{subset} \rrbracket_{ndef}(W)) := \emptyset$. For a positive relational algebra expression θ , $\sup(\llbracket \theta \rrbracket_{ndef}(W_1, \dots, W_k)) := \theta(\sup(W_1), \dots, \sup(W_k))$ and $\inf(\llbracket \theta \rrbracket_{ndef}(W_1, \dots, W_k)) := \theta(\inf(W_1), \dots, \inf(W_k))$. For relational difference, it can be verified that $\sup(\llbracket - \rrbracket_{ndef}(W_1, W_2)) := \sup(W_1) - \inf(W_2)$ and $\inf(\llbracket - \rrbracket_{ndef}(W_1, W_2)) := \inf(W_1) - \sup(W_2)$. It is easy to verify the correctness of these definitions, and together they yield the theorem. \square

Thus, not even $\text{repair-key}_{\emptyset}(\{0, 1\}) = \{ \{0\}, \{1\} \}$ can be defined.

Corollary 5.3 *The set of worlds $\{\{0\}, \{1\}\}$ is not definable in relational algebra extended by subset.*

In contrast, $\text{repair-key}_{\text{sch}(U)}(U \times \{0, 1\})$ can be defined as follows in the language fragment of relational algebra plus subset if definitions are available:

$$\text{let } R := \text{subset}(U) \text{ in } (R \times \{1\} \cup (U - R) \times \{0\}).$$

Thus, removing definitions seems to cause a substantial reduction of expressive power. In the remainder of this paper, we study whether $\text{possible}_{\bar{A}}$ and repair-key can offset this.

Before we move on, another simple result shall be stated that gives an intuition for the apparent weakness of WSA without definitions. If a view is defined by a query that involves one of the nondeterministic operations ($\text{possible}_{\bar{A}}$ or repair-key), then this view can only be used at one place in the query if the query is to be composed with the view. However, subsequent relational algebra operations will be monotonic with respect to that view.

Proposition 5.4 *Let Q be a nonmonotonic relational algebra query that is built using a relation R and constant relations. Then R occurs at least twice in Q .*

Proof. Assume a relational algebra query tree exists that expresses Q and in which R only occurs as a single leaf. Then the path from that leaf towards the root operation consists of unary operations and operations $Q_1 \theta Q_2$ where Q_1 contains R and Q_2 has only constant relations as leaves: Q_2 is constant. So $Q_1 \theta Q_2$ can be thought of as a unary operation. But all unary operations θ are monotonic, i.e., if $X \subseteq Y$, then $\theta(X) \supseteq \theta(Y)$ for the family of operations $(C - X)_{C \text{ const.}, \text{sch}(C) = \text{sch}(X)}$ and $\theta(X) \subseteq \theta(Y)$ for all other operations. It follows that Q , a sequence of such operations, is also monotonic. \square

6 WSA without Definitions Expresses all of SO Logic

As the main technical result of the paper, we now show that WSA without definitions (but using repair-key as in our language definition), captures all of SO. It follows that definitions, despite our nondeterministic operations, do not add power to the language. This is surprising given Theorem 5.2.

6.1 Indicator Relations

Let U be a *nonempty* relation (the *universe*) and let $R \subseteq U$. Then the indicator function $1_R : U \rightarrow \{0, 1\}$ is defined as

$$1_R : x \mapsto \begin{cases} 1 & \dots & x \in R \\ 0 & \dots & x \notin R \end{cases}$$

The corresponding indicator relation is just the relation $\{\langle x, 1_R(x) \rangle \mid x \in U\}$ which, obviously, has functional dependency $U \rightarrow \{0, 1\}$. Subsequently, we will always use indicator relations rather than indicator functions and will denote them by 1_R as well. By our assumption that $U \neq \emptyset$, indicator relations are always nonempty.

Given relations R and U with $R \subseteq U \neq \emptyset$, the indicator relation 1_R w.r.t. universe U can be constructed in relational algebra as

$$\text{ind}(R, U) := (R \times \{1\}) \cup ((U - R) \times \{0\}).$$

The expression $\text{repair-key}_{sch(U)}(U \times \{0, 1\})$ is equivalent to

$$\text{let } R := \text{subset}(U) \text{ in } \text{ind}(R, U)$$

and yields an indicator relation in each possible world.

Indicator relations have the nice property that their complement can be computed using a conjunctive query (with an inequality),

$$1_{U-R} = (U \times \{0, 1\}) - 1_R := \pi_{1,2}(\sigma_{1=3 \wedge 2 \neq 4}(U \times \{0, 1\} \times 1_R)).$$

Let \overline{R} denote the complement of relation R and let $U_i = R_i \cup \overline{R_i}$, called the *universe* of R_i . Note that

$$\overline{R_1 \times \cdots \times R_k} = \bigcup_{i=1}^k U_1 \times \cdots \times U_{i-1} \times \overline{R_i} \times U_{i+1} \times \cdots \times U_k.$$

The complement of a product $\vec{1} := 1_{R_1} \times \cdots \times 1_{R_k}$ can be obtained as

$$\begin{aligned} \text{compl}_{U_1, \dots, U_k}(\vec{1}) &= (U_1 \times \{0, 1\} \times \cdots \times U_k \times \{0, 1\}) - \vec{1} \\ &= \pi_{A_1, B_1, \dots, A_k, B_k}(\sigma_{\bigvee_i (A_i = A'_i \wedge B_i \neq B'_i)}(\rho_{A'_1 B'_1 \dots A'_k B'_k}(\vec{1}) \times \\ &\quad \rho_{A_1 B_1 \dots A_k B_k}(U_1 \times \{0, 1\} \times \cdots \times U_k \times \{0, 1\}))). \end{aligned}$$

if, for each $1 \leq i \leq k$, U_i is the universe of R_i . Moreover,

Lemma 6.1 *The k -times product of 1_R , denoted by $(1_R)_U^k := \overbrace{1_R \times \cdots \times 1_R}^{k \text{ times}}$, can be expressed as a relational algebra expression in which 1_R only occurs once.*

Proof. Let U be the universe of R .

$$\begin{aligned} (1_R)_U^k &= \rho_{A_1 B_1 \dots A_k B_k}((U \times \{0, 1\})^k) - \text{compl}_{U^k}(1_R^k) \\ &= \rho_{A_1 B_1 \dots A_k B_k}((U \times \{0, 1\})^k) \\ &\quad - \pi_{A_1, B_1, \dots, A_k, B_k}(\sigma_{\bigvee_{1 \leq i \leq k} (A_i = A'_i \wedge B_i \neq B'_i)}(\rho_{A'_1 B'_1 \dots A'_k B'_k}(1_R) \times \\ &\quad \rho_{A_1 B_1 \dots A_k B_k}((U \times \{0, 1\})^k) \times \rho_{A'_1 B'_1}(1_R))). \end{aligned}$$

□

As a convention, let $S^0 = \{\langle \rangle\}$ for nonempty relations S . In particular, $(1_R)_U^0 = \{\langle \rangle\}$.

6.2 The Quantifier-Free Case

By quantifier-free formulae we will denote formulae of predicate logic that have neither first- nor second-order quantifiers.

Lemma 6.2 *Let ϕ be a quantifier-free formula with relations \vec{R} . Then ϕ can be translated in linear time into a formula $\exists \vec{x} \alpha \wedge \beta$, where α is a Boolean combination of equalities and β is a conjunction of relational literals, which is equivalent to ϕ on structures in which each relation of \vec{R} and its complement are nonempty.*

Proof Sketch. Let R_1, \dots, R_s the set of distinct predicates (relation names) occurring in ϕ . First push negations in ϕ down to the atomic formulae using De Morgan's laws and the elimination of double negation and replace relational atomic formulae $\neg R_j(\vec{t})$, where \vec{t} is a tuple of variables and constants, by $\overline{R_j}(\vec{t})$.

Now apply the following translation inductively bottom-up. The translation is the identity on inequality literals. Rewrite atomic formulae $R_j(\vec{t})$ into $\exists \vec{v}_{j1} \vec{v}_{j1} = \vec{t} \wedge R_j(\vec{v}_{j1})$ and atoms $\overline{R_j}(\vec{t})$ into $\exists \vec{w}_{j1} \vec{w}_{j1} = \vec{t} \wedge \overline{R_j}(\vec{w}_{j1})$. Let

$$\gamma_{j,m,m'} = \bigwedge_{k=1}^m R_j(\vec{v}_{jk}) \wedge \bigwedge_{k=1}^{m'} \overline{R_j}(\vec{w}_{jk}).$$

A subformula $\psi_1 \vee \psi_2$ (resp., $\psi_1 \wedge \psi_2$) with

$$\psi_i = \exists \vec{v} \vec{w} \ \alpha_i \wedge \bigwedge_{j=1}^s \gamma_{j,n_{ij},n'_{ij}}$$

is turned into

$$\exists \vec{v} \vec{w} \ \alpha \wedge \bigwedge_{j=1}^s \gamma_{j,m_j,m'_j}$$

where $m_j = \max(n_{1j}, n_{2j})$, $m'_j = \max(n'_{1j}, n'_{2j})$ and $\alpha = \alpha_1 \vee \alpha_2$ (resp., $m_j = n_{1j} + n_{2j}$, $m'_j = n'_{1j} + n'_{2j}$, $\alpha = \alpha_1 \wedge \alpha'_2$, and α'_2 is obtained from α_2 by replacing each variable v_{jkl} by $v_{j(k+n_{1j})l}$ and each variable w_{jkl} by $w_{j(k+n_{1j})l}$).

For the equivalence of the rewritten formula to ϕ , it is only necessary to point out that since all the relations R_j and $\overline{R_j}$ are nonempty, ψ_i is equivalent to

$$\exists \vec{v} \vec{w} \ \alpha_i \wedge \bigwedge_{j=1}^s \gamma_{j,m_j,m'_j}.$$

It is not hard to verify that the translation can indeed be implemented to run in linear time and that the rewritten formula is of the form claimed in the lemma. \square

Theorem 6.3 *For any quantifier-free formula there is an equivalent expression in WSA over universe relations and indicator relations in which each indicator relation only occurs once.*

Proof Sketch. Assume R_1, \dots, R_s are all the predicates occurring in the formula. By Lemma 6.2, we only need to consider formulae of syntax

$$\phi = \exists \vec{v} \vec{w} \ \alpha \wedge \bigwedge_{j=1}^s \bigwedge_{k=1}^{m_j} R_j(\vec{v}_{jk}) \wedge \bigwedge_{k=1}^{m'_j} \neg R_j(\vec{w}_{jk})$$

where α does not contain relational atoms if each relation R_j is nonempty and different from U_j . Such a formula ϕ is equivalent to

$$\exists \vec{v} \vec{w} \vec{t} \vec{t}' \ \alpha' \wedge \bigwedge_{j=1}^s \bigwedge_{k=1}^{m_j} 1_{R_j}(\vec{v}_{jk}, t_{jk}) \wedge \bigwedge_{k=1}^{m'_j} 1_{R_j}(\vec{w}_{jk}, t'_{jk})$$

with

$$\alpha' = \alpha \wedge \bigwedge_{j=1}^s \left(\bigwedge_{k=1}^{m_j} t_{jk} = 1 \wedge \bigwedge_{k=1}^{m'_j} t'_{jk} = 0 \right).$$

This is true because $R_j(\vec{v}_{jk})$ is equivalent to $1_{R_j}(\vec{v}_{jk}, 1)$ and $\neg R_j(\vec{w}_{jk})$ is equivalent to $1_{R_j}(\vec{w}_{jk}, 0)$. Obtaining formulae of this form is indeed feasible because $1_{R_j} \neq \emptyset$ and $1_{R_j} \neq U_j$.

Let \vec{x} be the free variables of the formula. The WSA expression is

$$\pi_{\vec{x}}(\sigma_{\alpha'}(B_1 \times \cdots \times B_s))$$

with

$$B_j := \rho_{\vec{v}_{j1}t_{j1} \dots \vec{v}_{jm_j}t_{jm_j} \vec{w}_{j1}t'_{j1} \dots \vec{w}_{jm'_j}t'_{jm'_j}} ((1_{R_j})_{U_j}^{m_j+m'_j}).$$

Each B_j computes an $(m_j + m'_j)$ -times product of 1_{R_j} using the technique of Lemma 6.1 which just uses one occurrence of 1_{R_j} . All the relations 1_{R_j} only occur once. This proves the theorem. \square

Example 6.4 Consider an alternative encoding of 3-colorability in WSA which is based on guessing a subset of relation $U = V \times \rho_C(\{r, g, b\})$. Then 3-colorability is the problem of deciding the SO sentence $\exists C(\subseteq U) \neg \exists v, w, c, c' \phi_1 \vee \phi_2 \vee \phi_3$ with $\phi_1 = E(v, w) \wedge C(v, c) \wedge C(w, c)$, $\phi_2 = C(v, c) \wedge C(v, c') \wedge c \neq c'$, and $\phi_3 = \neg C(v, r) \wedge \neg C(v, g) \wedge \neg C(v, b)$, i.e., ϕ_1 asserts that two neighboring nodes have the same color, ϕ_2 that a node has simultaneously two colors, and ϕ_3 that a node has not been assigned any color at all. If neither is the case, we have a 3-coloring of the graph. Using Theorem 6.3, $\phi_1 \vee \phi_2 \vee \phi_3$ becomes

$$\pi = (\psi_1 \vee \psi_2 \vee \psi_3) \wedge 1_C(u_1, c_1, t_1) \wedge 1_C(u_2, c_2, t_2) \wedge 1_C(u_3, c_3, t_3) \wedge 1_E(v, w, t_4)$$

where

$$\begin{aligned} \psi_1 &= u_1 = v \wedge u_2 = w \wedge c_1 = c_2 \wedge t_1 = t_2 = t_4 = 1 \\ \psi_2 &= u_1 = u_2 \wedge c_1 \neq c_2 \wedge t_1 = t_2 = 1 \\ \psi_3 &= u_1 = u_2 = u_3 \wedge c_1 = r \wedge c_2 = g \wedge c_3 = b \wedge t_1 = t_2 = t_3 = 0; \end{aligned}$$

Following Theorem 6.3, formula π can be turned into WSA as

$$Q_\pi := \sigma_{\psi_1 \vee \psi_2 \vee \psi_3}(\rho_{u_1 c_1 t_1 u_2 c_2 t_2 u_3 c_3 t_3} ((1_C)_{V \times \{r, g, b\}}^3) \times \rho_{v w t_4}(E))$$

where $(1_C)_{V \times \{r, g, b\}}^3$ denotes the WSA expression for $1_C \times 1_C \times 1_C$ from Lemma 6.1.

The complete SO sentence can be stated as

$$\exists 1_C (1_C : V \times \{r, g, b\} \rightarrow \{0, 1\}) \wedge \neg \exists u_1 c_1 t_1 u_2 c_2 t_2 u_3 c_3 t_3 v w t_4 \pi.$$

If 1_C in Q_π is replaced by repair-key $_{V, C}(V \times \rho_C(\{r, g, b\}) \times \rho_T(\{0, 1\}))$, this sentence can be turned into WSA without definitions as possible $(\{\langle \rangle\} - \pi_\emptyset(Q_\pi))$. \square

6.3 Quantification and Alternation

Conceptually, in SO, there is no difference in the treatment of second-order variables and relations coming from the input structure; an existential second-order quantifier extends the structure over which the formula is evaluated. In our algebra, however, we have to construct the possible alternative relations for a second-order variable R at the beginning of the bottom-up evaluation of the algebra expression using repair-key and have to later test the existential quantifier $\exists R$ using the possible operation grouping the possible worlds that agree on R . For that we have to keep R around during the evaluation of the algebra expression. Selections also must not actually remove tuples because this would mean that the information about which world the tuple is missing from would be lost. For example, the algebra expression corresponding to a Boolean formula must not return false, but in some form must compute the pair $\langle R, \text{false} \rangle$.

Let ϕ be an SO formula with free second-order variables R_1, \dots, R_k and free first-order variables x_1, \dots, x_l . Conceptually, our proofs will produce a WSA expression for ϕ that computes, in each possible world identified by choices of relations R_1, \dots, R_k for the free second-order variables, the relation

$$R_1 \times \dots \times R_k \times \Theta$$

where Θ is a representation of a mapping

$$\vec{a} \mapsto \text{truth value of } \phi[\vec{x} \text{ replaced by } \vec{a}].$$

Truth and falsity cannot be just represented by 1 and 0, respectively, because an existential first-order quantifier will effect a projection on Θ whose result may contain both truth values 1 and 0 for a variable assignment \vec{a} . Thus, projection may map environments for which ϕ is true together with environments for which ϕ is false. In that case we would like to remove the tuples for which the truth value encoding is 0. Unfortunately, the function

$$F : \begin{cases} \{0\} \mapsto \{0\} \\ \{1\} \mapsto \{1\} \\ \{0, 1\} \mapsto \{1\} \end{cases}$$

is nonmonotonic, and by Proposition 5.4 cannot be expressed in relational algebra if the input relation is to occur in the query only once. Fortunately, we do not need such a function F .

Definition 6.5 *A PBIT (protected bit) is either $\{\perp\}$ (denoting 0) or $\{\perp, 1\}$ (denoting 1).*

Given a Boolean query Q (i.e., Q returns either $\{\langle \rangle\}$ or \emptyset),

$$PBIT(Q) := (Q \times \{1\}) \cup \{\perp\}.$$

The negation of PBIT B is obtained by $\{\perp, 1\} - (B \cap \{1\})$. The set union on PBITs effects a logical OR, thus a relation $\subseteq R \times PBIT$ for which $\langle \vec{a}, 1 \rangle \in R$ implies $\langle \vec{a}, \perp \rangle \in R$ guarantees that projecting away a column other than the rightmost corresponds to existential quantification.

For an SO formula ϕ with free second-order variables R_1, \dots, R_k and free first-order variables x_1, \dots, x_l , we will define a WSA expression that computes the relation

$$TT(\phi) := 1_{R_1} \times \dots \times 1_{R_k} \times \Theta$$

such that $\Theta = (D^l \times \{\perp\}) \cup \{\langle \vec{a}, 1 \rangle \mid \phi[\vec{x} \text{ replaced by } \vec{a}] \text{ is true}\}$ and D is a domain relation containing the possible values for the first-order variables. (So Θ can be thought of as a mapping $D^l \rightarrow PBIT$.) The complement of such a relation Θ is

$$\text{compl}_{D^l}(\Theta) := D^l \times \{\perp, 1\} - \sigma_{T=1}(\Theta).$$

Next we obtain an auxiliary construction for complementing a Θ relation while passing on the second-order relation. This will be the essential tool for alternation.

Lemma 6.6 *Let $P = 1_{R_1} \times \cdots \times 1_{R_k} \times \Theta$ where $\Theta \subseteq D_1 \times \cdots \times D_l \times PBIT$. There is a WSA expression without definitions for*

$$\text{compl}_{U_1, \dots, U_k; \vec{D}, T}(S) := 1_{R_1} \times \cdots \times 1_{R_k} \times \text{compl}(\Theta)$$

in which P only occurs once.

Proof. Let $\text{sch}(U_i) = A_i$ and $\text{sch}(1_{R_i}) = A_i B_i$. We write $\vec{1}$ for $1_{R_1} \times \cdots \times 1_{R_k}$ and \vec{U}^+ for $U_1 \times \cdots \times U_k \times \rho_{B_1 \dots B_k}(\{0, 1\}^k)$. A definition of $\text{compl}_{U_1, \dots, U_k}(\vec{1})$ was given in Section 6.1.

$$\begin{aligned} \text{compl}_{U_1, \dots, U_k; \vec{D}, T}(\vec{1} \times \Theta) &= \vec{1} \times (D^l \times \{\perp, 1\} - \sigma_{T=1}(\Theta)) \\ &= (\vec{U}^+ \times D^l \times \rho_T(\{\perp, 1\})) \\ &\quad - \text{compl}_{U_1, \dots, U_k}(\vec{1}) \times D^l \times \rho_T(\{\perp, 1\}) \\ &\quad - \vec{U}^+ \times \sigma_{T=1}(\Theta) \\ &= (\vec{U}^+ \times D^l \times \rho_T(\{\perp, 1\})) \\ &\quad - \pi_{A_1, B_1, \dots, A_k, B_k, T}(\sigma_{\bigvee_i (A_i = A'_i \wedge B_i \neq B'_i) \vee T' = T=1} (\\ &\quad \vec{U}^+ \times \rho_{A'_1 B'_1 \dots A'_k B'_k T'}(\underbrace{\vec{1} \times \Theta}_P \times \rho_T(\{\perp, 1\}))). \end{aligned}$$

The final WSA expression is in the desired form. \square

Now we are ready to prove the main result of this section.

Theorem 6.7 *Given a formula in second-order logic, an equivalent WSA expression without definitions can be computed in linear time in the size of the formula.*

Proof Sketch. The proof is by induction. Given second-order formula ϕ with free first-order variables \vec{x} and zero or more free second-order variables.

Induction start: Assume that ϕ is quantifier-free. Consider the quantifier-free formula

$$\psi(\vec{x}, \vec{y}, t) := \left(\bigwedge_{j: R_j \text{ is an SO var.}} R_j(\vec{y}_j) \right) \wedge (\phi \vee t = \perp),$$

where the variables \vec{y} and t are new and do not occur in ϕ . It is easy to verify that ψ defines the relation $TT(\phi)$. Specifically, the projection down to columns \vec{y}_j represents the free second-order variable R_j , the projection down to columns \vec{x} specifies all the possible assignments to the first-order variables, and t is a PBIT for the truth value of ϕ for a given assignment to the first- and second-order variables. The corresponding WSA expression without definitions is obtained using Theorem 6.3.

Induction step (ϕ has quantifiers): We assume that universal quantifiers $\forall \cdot$ have been replaced by $\neg \exists \cdot \neg$. Let P be the WSA expression for ψ claimed by the theorem.

- First-order existential quantification: If $\phi = \exists x_l \psi$, the corresponding WSA expression is $\pi_{sch(P)-x_l}(P)$. It is easy to verify that the projection has exactly the effect of existential first-order quantification, $TT(\exists x_l \psi) = \pi_{sch(P)-x_l}(TT(\psi))$.
- Second-order existential quantification: Let R_1, \dots, R_s be the free second-order variables in ψ . We may assume w.l.o.g. that these have disjoint schemas. If $\phi = \exists R_s \psi$, the corresponding WSA expression is

$$\pi_{sch(P)-sch(R_j)}(\text{possible}_{sch(R_1), \dots, sch(R_{s-1})}(P)).$$

Again, the correctness is straightforward, $TT(\exists R_s \psi) = \pi_{sch(P)-sch(R_j)}(TT(\psi))$.

- Negation: By Lemma 6.6, the WSA expression $\text{compl}_{U_1 \dots U_s, \vec{D}, T}(P)$ is equivalent to $\phi = \neg \psi$.

All that is left to be done is to provide WSA expressions for the indicator relations 1_{R_j} . For database relations R_j , the algebra expression is $\text{ind}(R_j, U_j)$. For second-order variables R_j , it is $\text{repair-key}_{sch(U_j)}(U \times \{0, 1\})$.

For an SO sentence ϕ (i.e., without free variables), the algebra expression computes a PBIT $TT(\phi)$ and its truth value is obtained as $\pi_{\emptyset}(\sigma_{T=1}(\cdot))$. \square

Example 6.8 We continue Example 4.2. Let

$$\phi = (L(c, p, 0) \wedge (P_1(p) \vee P_2(p))) \vee (L(c, p, 1) \wedge \neg(P_1(p) \vee P_2(p))).$$

Then Σ_2 -QBF can be expressed by the SO sentence

$$\exists P_1(\subseteq V_1) \neg \exists P_2(\subseteq V_2) \neg \exists c(c \in C) \neg \exists p \phi.$$

We can turn

$$(\phi \vee t = \perp) \wedge P_1(p_{12}) \wedge P_2(p_{22})$$

into WSA over indicator relations as

$$Q = \sigma_{\psi}(\rho_{cpst_L}(1_L) \times \rho_{p_{11}t_{11}p_{12}t_{12}}((1_{P_1})_{V_1}^2) \times \rho_{p_{21}t_{21}p_{22}t_{22}}((1_{P_2})_{V_2}^2) \times \rho_t(\{\perp, 1\}))$$

where $\psi = (t = \perp \vee (t_L = 1 \wedge p = p_{11} = p_{21} \wedge ((s = 0 \wedge (t_{11} = 1 \vee t_{21} = 1)) \vee (s = 1 \wedge t_{11} \neq 1 \wedge t_{21} \neq 1))))$. Note that we have simplified the expression of the proof somewhat by inlining the auxiliary variables \vec{v} and \vec{w} .

The complete WSA expression for the SO sentence is

$$\underbrace{\pi_{\emptyset} \circ \sigma_{t=1}}_{PBIT \text{ to bool}} \circ \underbrace{\pi_t \circ \text{possible}}_{\exists P_1} \circ \underbrace{\text{compl}_{V_1; T}}_{\neg} \circ \underbrace{\pi_{p_{12}t_{12}t}}_{\exists P_2} \circ \underbrace{\text{possible}_{p_{12}t_{12}}}_{\neg} \circ \underbrace{\text{compl}_{V_1, V_2; T}}_{\neg} \circ \underbrace{\pi_{p_{12}t_{12}p_{22}t_{22}t}}_{\exists c} \circ \underbrace{\text{compl}_{V_1, V_2; C, T}}_{\neg} \circ \underbrace{\pi_{p_{12}t_{12}p_{22}t_{22}ct}}_{\exists p} \circ \underbrace{Q}_{\phi}.$$

We replace 1_L by $\text{ind}(L, \cdot)$ and 1_{P_i} by $\text{repair-key}_p(\rho_{pt}(V_i \times \{0, 1\}))$. \square

Thus, definitions add no power to WSA.

Corollary 6.9 *WSA without definitions captures WSA.*

The data complexity of a query language refers to the problem of evaluating queries on databases assuming the queries fixed and only the database part of the input, while combined complexity assumes that both the query and the database are part of the input [15]. Since SO logic is complete for the polynomial hierarchy (PHIER) with respect to data complexity and PSPACE-complete with respect to combined complexity [14], a generalization of Fagin’s Theorem [6] (see also [11]),

Corollary 6.10 1. *WSA with or without definitions is PHIER-complete with respect to data complexity,*
 2. *WSA with definitions is PSPACE-hard with respect to combined complexity, and*
 3. *WSA without definitions is PSPACE-complete with respect to combined complexity.*

We cannot directly conclude an upper bound on the combined complexity of WSA with definitions from the reduction of Theorem 4.3 because it was exponential-time: In the case that WSA definitions are used, several copies of formulae ψ_V may be used in the SO formula constructed in the proof, and that recursively. However, we can think of the proof construction as a linear-time mapping from WSA with definitions to second-order logic with definitions. But the standard PSPACE algorithm for second-order logic extends directly to second-order logic with definitions: Of the formula, we only have to maintain a current path in its parse tree, which is clearly of polynomial size. It follows that

Proposition 6.11 *WSA with definitions is PSPACE-complete with respect to combined complexity.*

7 Related Work

In an early piece of related work, Libkin and Wong [12] define a query algebra for handling both nested data types and uncertainty. Their notion of uncertainty called *or-sets* (as a generalization of the or-sets of [8]) is treated as a special collection type that can syntactically be thought of as a set of data and is only interpreted as uncertainty on an additional “conceptual level”. The result is a very elegant and clean algebra that nicely combines complex objects with uncertainty. While their language is stronger and can manage nested data, there is nevertheless a close connection to WSA, which can be thought of as a flat relational version of their language. Indeed, the or-set language contains an operator α that is essentially equivalent to the repair-key operator of WSA.

TriQL, the query language of the Trio project [16], subsumes the power of relational algebra and supports an operation “groupalts” which expresses the repair-key operation of WSA applied to a certain relation. There are many more operations in TriQL, but it is hard to tell whether possible _{\bar{A}} is expressible in TriQL since no formal semantics of the language is available. Moreover, TriQL contains a number of representation-dependent (non-generic [1]) operations which may return semantically different results for different semantically equivalent representations of a probabilistic database. This makes TriQL hard to study and compare with WSA. However, it seems that WSA is a good candidate for a clean core to TriQL, and the results of the present paper provide additional evidence that it is highly expressive.

The probabilistic databases definable using repair-key from certain relations are also exactly the block independent-disjoint (BID) tables of Ré and Suciu. In their paper [13],

they study the related representability problems for BID tables. Their results suggest that BID tables are more powerful than tuple-independent tables, which correspond to uncertain tables definable using the subset operation. This is in line with observations made in Section 5 of the present paper.

The algebra defined in our own earlier work [3] is exactly the one described in the present paper, modulo the following details. Most importantly, while repair-key is introduced there as part of the algebra, most of the paper focuses on the fragment that is obtained by replacing repair-key by choice-of. Moreover, the syntax of possible _{\bar{A}} allows for the grouping of worlds by a query Q that can be given as a parameter; the syntax is possible _{Q} (Q'). An operation possible _{\bar{A}} in the syntax of the present paper corresponds to an operation possible _{$\pi_{\bar{A}}$} in the syntax of [3]. The results of this paper imply that allowing general queries Q for grouping adds no power, so we are indeed studying the same language. The paper [3] also gives an SQL-like syntax for WSA, in which the intuition of possible _{\bar{A}} is made explicit by the syntax “select possible ... group worlds by ...”.

In recent work [2, 10, 9], we have developed efficient techniques for processing a large part of WSA. The only operations that currently defy good solutions are possible _{\bar{A}} (i.e., with grouping, not possible _{\emptyset}) and, to a lesser extent, relational difference. Indeed, the repair-key operator on the standard representations described in Example 2.1 can be implemented efficiently, even though semantically it generally causes an exponential blowup in the size of the set of possible worlds. Thus, it is natural to ask for the expressive power of WSA with possible _{\bar{A}} replaced by possible. The construction of the proof of Theorem 4.1 can map any SO formula of the form $\exists R \phi$ or $\forall R \phi$ where ϕ is FO to WSA. It is not hard to see that despite the restriction to a *single* second-order quantifier, this fragment of WSA (with definitions) can express all of $\text{NP} \cup \text{co-NP}$. For an upper bound, it seems that all such restricted WSA queries have data complexity in Δ_2^P (i.e., P^{NP}).

8 Conclusions

The main contribution of this paper is to give the apparently first compositional algebra that exactly captures second-order logic over finite structures, a logic of wide interest.

Second-order logic is a natural yardstick for the expressiveness of query languages for uncertain databases. It is an elegant and well-studied formalism that naturally captures what-if queries. It can be argued that second-order logic takes the same role in uncertain databases that first-order logic and relational algebra take in classical relational databases. In that sense, the expressiveness result of this paper, $WSA = SO$, is an uncertain databases analog of Codd’s Theorem.

Finding the right query algebra for uncertain databases is important because efficient query processing techniques are easier to obtain for algebraic languages without variables or quantifiers, and algebraic operators are natural building blocks for database query plans. Of course, the expressiveness result of this paper also implies that WSA has high complexity and thus this paper can only be an initial call for the search for more efficiently processible fragments of WSA that retain some of its flavor of simplicity and cleanliness.

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