An Application of Proof-Theory in Answer Set Programming

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Abstract

Using a characterization of stable models of logic programs P as satisfying valuations of a suitably chosen propositional theory, called the set of *reduced defining equations* $r\Phi_P$, we show that the finitary character of that theory $r\Phi_P$ is equivalent to a certain continuity property of the Gelfond-Lifschitz operator GL_P associated with the program P.
We discuss possible extensions of techniques proposed in this paper to the context of cardinality constraints.

1 Introduction

The use of proof theory in logic based formalisms for constraint solving is pervasive. For example, in Satisfiability (SAT), proof theoretic methods are used to find lower bounds on complexity of various SAT algorithms. However, proof-theoretic methods have not played as prominent role in Answer Set Programming (ASP) formalisms. This is not to say that there were no attempts to apply proof-theoretic methods in ASP. To give a few examples, Marek and Truszczynski in [MT93] used the proof-theoretic methods to characterize Reiter's extensions in

Default Logic (and thus stable semantics of logic programs). Bonatti [Bo04] and separately Milnikel [Mi05] devised non-monotonic proof systems to study skeptical consequences of programs and default theories. Lifschitz [Li96] used proof-theoretic methods to approximate well-founded semantics of logic programs. Bondarenko et.al. [BTK93] studied an approach to stable semantics using methods with a clear proof-theoretic flavor. Marek, Nerode, and Remmel in a series of papers, [MNR90a, MNR90b, MNR91, MNR92, MNR94a, MNR94b], developed proof theoretic methods to study what they termed *non-monotonic rule systems* which have as special cases almost all ASP formalisms that have been seriously studied in the literature. Recently the area of proof systems for ASP (and more generally, nonmonotonic logics) received a lot of attention [GS07, JO07]. It is clear that the community feels that an additional research of this area is necessary. Nevertheless, there is no clear classification of proof systems for nonmonotonic reasoning analogous to that present in classical logic, and SAT in particular.
In this paper, we define a notion of P-proof schemes, which is a kind of a proof system that was previously used by Marek, Nerode, and Remmel to study complexity issues for stable semantics of logic programs [MNR94a]. This proof system abstracts of M-proofs of [MT93] and produces Hilbert-style proofs. The nonmonotonic character of our P-proofs is provided by the presence of guards, called the *support* of the proof scheme, to insure context-dependence. A different but equivalent, presentation of proof schemes, using a guarded resolution is also possible [MR09].
We shall show that we can use P-proof schemes to find a characterization of stable models via *reduced defining equations*. While in general these defining equations may be infinite, we study the case of programs for which all these equations are finite. This resulting class of programs, called FSP-programs, turn out to be characterized by a form of continuity of the Gelfond-Lifschitz operator.

1.1 Contributions of the paper

The contributions of this paper consist, primarily of investigations that elucidate the proof-theoretical character of the stable semantics for logic programs, an area with 20 years history [GL88]. The principal results of this paper are:

1. We show that the Gelfond-Lifschitz operator GL_P is, in fact a proof-theoretical construct (Proposition 4.3)

2. As a result of the analysis of the Gelfond-Lifschitz operator we are able to show that the upper-half continuity of that operator is equivalent to finiteness of (propositional) formulas in a certain class associated with the program P (Proposition 4.6)

We also discuss possible extension of these results to the case of programs with cardinality constraints.

2 Preliminaries

Let At be a countably infinite set of atoms. We will study programs consisting of clauses built of the atoms from At. A *program clause* C is a string of the form

$$p \leftarrow q_1, \ldots, q_m, \neg r_1, \ldots, \neg r_n \tag{1}$$

The integers m or n or both can be 0. The atom p will be called the head of C and denoted $head(C)$. We let $posBody(C)$ denote the set $\{q_1, \ldots, q_m\}$ and $negBody(C)$ denote the set $\{r_1, \ldots, r_n\}$. For any set of atoms X, we let $\neg X$ denote the conjunction of negations of atoms from X. Thus, we can write clause (1) as

$$head(C) \leftarrow posBody(C), \neg negBody(C).$$

Let us stress that the set $negBody(C)$ is a set of atoms, not a set of negated atoms as is sometimes used in the literature. A normal propositional program is a set P of such clauses. For any $M \subseteq At$, we say that M is model of C if whenever $q_1, \ldots, q_m \in M$ and $\{r_1, \ldots, r_n\} \cap M = \emptyset$, then $p \in M$. We say that M is a model of a program P if M is a model of each clause $C \in P$. Horn clauses are clauses with no negated literals, i.e. clauses of the form (1) where $n = 0$. We will denote by $Horn(P)$ the part of the program P consisting of its Horn clauses. Horn programs are logic programs P consisting entirely of Horn clauses. Thus for a Horn program P, $P = Horn(P)$.

Each Horn program P has a least model over the Herbrand base and the least model of P is the least fixed point of a continuous operator T_P representing one-step Horn clause logic deduction ([L89]). That is, for any set $I \subseteq At$, we let $T_P(I)$ equal the set of all $p \in At$ such that there is a clause $C = p \leftarrow q_1, \ldots, q_m$ in P and $q_1, \ldots, q_m \in I$. Then T_P has a least fixed point F_P which is obtained by iterating T_P starting at the empty set for ω steps, i.e., $F_P = \bigcup_{n \in \omega} T_P^n(\emptyset)$ where for any $I \subseteq At$, $T_P^0(I) = I$ and $T_P^{n+1}(I) = T_P(T_P^n(I))$. Then F_P is the least model of P.

The semantics of interest for us is the *stable semantics* of normal programs, although we will discuss some extensions in Section **??**. The stable models of a program P are defined as fixed points of the operator $T_{P,M}$. This operator is defined on the set of all subsets of At, $\mathcal{P}(At)$. If P is a program and $M \subseteq At$ is a subset of the Herbrand base, define operator $T_{P,M} \colon \mathcal{P}(At) \to \mathcal{P}(At)$ as follows:

$$\begin{aligned} T_{P,M}(I) = \{p \colon & \text{ there exist a clause } C = p \leftarrow q_1, \ldots, q_m, \neg r_1, \ldots, \neg r_n \\ & \text{in } P \text{ such that } q_1 \in I, \ldots, q_m \in I, r_1 \notin M, \ldots, r_n \notin M\} \end{aligned}$$

The following is immediate, see [Ap90] for unexplained notions.

Proposition 2.1 *For every program P and every set M of atoms the operator $T_{P,M}$ is monotone and continuous.*

Thus the operator $T_{P,M}$ like all monotonic continuous operators, possesses a least fixed point $F_{P,M}$.
Given program P and $M \subseteq At$, we define the *Gelfond-Lifschitz reduct* of P, P_M, as follows. For every clause $C = p \leftarrow q_1, \ldots, q_m, \neg r_1, \ldots, \neg r_n$ of P, execute the following operations.
(1) If some atom r_i, $1 \leq i \leq n$, belongs to M, then eliminate C altogether.
(2) In the remaining clauses that have not been eliminated by operation (1), eliminate all the negated atoms.
The resulting program P_M is a Horn propositional program. The program P_M possesses a least Herbrand model. If that least model of P_M coincides with M, then M is called a *stable model* for P. This gives rise to an operator GL_P which associates to each $M \subseteq At$, the least fixed point of $T_{P,M}$. We will discuss the operator GL_P and its proof-theoretic connections in section 4.2.

3 Proof schemes and reduced defining equations

In this section we recall the notion of a *proof scheme* as defined in [MNR90a, MT93] and introduce a related notion of *defining equations*.
Given a propositional logic program P, a proof scheme is defined by induction on its length. Specifically, a proof scheme w.r.t. P (in short P-proof scheme) is a sequence $S = \langle\langle C_1, p_1\rangle, \ldots, \langle C_n, p_n\rangle, U\rangle$ subject to the following conditions:
(I) when $n = 1$, $\langle\langle C_1, p_1\rangle, U\rangle$ is a P-proof scheme if $C_1 \in P$, $p_1 = head(C_1)$, $posBody(C_1) = \emptyset$, and $U = negBody(C_1)$ and
(II) when $\langle\langle C_1, p_1\rangle, \ldots, \langle C_n, p_n\rangle, U\rangle$ is a P-proof scheme,
$C = p \leftarrow posBody(C), \neg negBody(C)$ is a clause in the program P, and $posBody(C) \subseteq \{p_1, \ldots, p_n\}$, then

$$\langle\langle C_1, p_1\rangle, \ldots, \langle C_n, p_n\rangle, \langle C, p\rangle, U \cup negBody(C)\rangle$$

is a P-proof scheme.
When $S = \langle\langle C_1, p_1\rangle, \ldots, \langle C_n, p_n\rangle, U\rangle$ is a P-proof scheme, then we call (i) the integer n – the *length* of S, (ii) the set U – the *support* of S, and (iii) the atom p_n – the *conclusion* of S. We denote U by $supp(S)$.

Example 3.1 Let P be a program consisting of four clauses: $C_1 = p \leftarrow$, $C_2 = q \leftarrow p, \neg r$, $C_3 = r \leftarrow \neg q$, and $C_4 = s \leftarrow \neg t$. Then we have the following examples of P-proof schemes:

(a) $\langle\langle C_1, p\rangle, \emptyset\rangle$ is a P-proof scheme of length 1 with conclusion p and empty support.

(b) $\langle\langle C_1, p\rangle, \langle C_2, q\rangle, \{r\}\rangle$ is a P-proof scheme of length 2 with conclusion q and support $\{r\}$.

(c) $\langle\langle C_1, p\rangle, \langle C_3, r\rangle, \{q\}\rangle$ is a P-proof scheme of length 2 with conclusion r and support $\{q\}$.

(d) $\langle\langle C_1, p\rangle, \langle C_2, q\rangle, \langle C_3, r\rangle, \{q, r\}\rangle$ is a P-proof scheme of length 3 with conclusion r and support $\{q, r\}$.

Proof scheme in (c) is an example of a proof scheme with unnecessary items (the first term). Proof scheme (d) is an example of a proof scheme which is not internally consistent in that r is in the support of its proof scheme and is also its conclusion. $\Box$

A P-proof scheme carries within itself its own applicability condition. In effect, a P-proof scheme is a *conditional* proof of its conclusion. It becomes applicable when all the constraints collected in the support are satisfied. Formally, for any set of atoms M, we say that a P-proof scheme S is M*-applicable* if $M \cap supp(S) = \emptyset$. We also say that M *admits* S if S is M-applicable.
The fundamental connection between proof schemes and stable models [MNR90a, MT93] is given by the following proposition.

Proposition 3.1 *For every normal propositional program P and every set M of atoms, M is a stable model of P if and only if the following conditions hold.*

(i) For every $p \in M$, there is a P-proof scheme S with conclusion p such that M admits S.

(ii) For every $p \notin M$, there is no P-proof scheme S with conclusion p such that M admits S.

Proposition 3.1 says that the presence and absence of the atom p in a stable model depends *only* on the supports of proof schemes. This fact naturally leads to a characterization of stable models in terms of propositional satisfiability. Given $p \in At$, the *defining equation* for p w.r.t. P is the following propositional formula:

$$p \Leftrightarrow (\neg U_1 \vee \neg U_2 \vee \ldots) \tag{2}$$

where $\langle U_1, U_2, \ldots\rangle$ is the list of all supports of P-proof schemes. Here for any finite set $S = \{s_1, \ldots, s_n\}$ of atoms, $\neg S = \neg s_1 \wedge \cdots \wedge \neg s_n$. If p is not the conclusion of any proof scheme, then we set the defining equation of p to be $p \Leftrightarrow \bot$. In the case, where all the supports of proof schemes of p are empty, we set the defining equation of p to be $p \Leftrightarrow \top$. Up to a total ordering of the finite sets of atoms such a formula is unique. For example, suppose we fix a total order on At, $p_1 < p_2 < \cdots$. Then given two sets of atoms, $U = \{u_1 < \cdots < u_m\}$ and $V = \{v_1 < \cdots < v_n\}$, we say that $U \prec V$, if either (i) $u_m < v_n$, (ii) $u_m = v_n$ and $m < n$, or (iii) $u_m = v_n$, $n = m$, and $(u_1, \ldots, u_n)$ is lexicographically less than $(v_1, \ldots, v_n)$. We say that (2) is the *defining equation* for p relative to P if $U_1 \prec U_2 \prec \cdots$. We will denote the defining equation for p with respect to P by Eq_p^P.

For example, if P is a Horn program, then for every atom p, either the support of all its proof schemes are empty or p is not the conclusion of any proof scheme. The first of these alternatives occurs when p belongs to the least model of P, $lm(P)$. The second alternative occurs when $p \notin lm(P)$. The defining equations are $p \Leftrightarrow \top$ (that is p) when $p \in lm(P)$ and $p \Leftrightarrow \bot$ (that is $\neg p$) when $p \notin lm(P)$. When P is a stratified program the defining equations are more complex, but the resulting theory is logically equivalent to

$$\{p : p \in Perf_P\} \cup \{\neg p : p \notin Perf_P\}$$

where $Perf_P$ is the unique stable model of P.

Let Φ_P be the set $\{Eq_p^P : p \in At\}$. We then have the following consequence of Proposition 3.1.

Proposition 3.2 *Let P be a normal propositional program. Then stable models of P are precisely the propositional models of the theory Φ_P.*

When P is *purely negative*, i.e. all clauses C of P have $PosBody(C) = \emptyset$, the stable and supported models of P coincide [DK89] and the defining equations reduce to Clark's completion [Cl78] of P.

Let us observe that in general the propositional formulas on the right-hand-side of the defining equations may be infinite.

Example 3.2 Let P be an infinite program consisting of clauses $p \leftarrow \neg p_i$, for all $i \in n$. In this case, the defining equation for p in P is infinite. That is, it is

$$p \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \neg p_3 \vee \ldots)$$

$\square$

The following observation is quite useful. If U_1, U_2 are two finite sets of propositional atoms then

$$U_1 \subseteq U_2 \text{ if and only if } \neg U_2 \models \neg U_1$$

Here $\models$ is the propositional consequence relation. The effect of this observation is that not all the supports of proof schemes are important, only the inclusion-minimal ones.

Example 3.3 Let P be an infinite program consisting of clauses $p \leftarrow \neg p_1, \ldots, \neg p_i$, for all $i \in N$. The defining equation for p in P is

$$p \Leftrightarrow [\neg p_1 \vee (\neg p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee \ldots]$$

which is infinite. But our observation above implies that this formula is *equivalent* to the formula

$$p \Leftrightarrow \neg p_1$$

$\square$

Motivated by the Example 3.3, we define the *reduced defining equation* for p relative to P to be the formula

$$p \Leftrightarrow (\neg U_1 \vee \neg U_2 \vee \ldots) \tag{3}$$

where U_i range over *inclusion-minimal* supports of P-proof schemes for the atom p and $U_1 \prec U_2 \prec \cdots$. Again, if p is not the conclusion of any proof scheme, then we set the defining equation of p to be $p \Leftrightarrow \perp$. In the case, where there is a proof scheme of p with empty support, then we set the defining equation of p to be $p \Leftrightarrow \top$. We denote this formula as rEq_p^P, and define $r\Phi_P$ to be the theory consisting of rEq_p^P for all $p \in At$. We then have the following strengthening of Proposition 3.2.

Proposition 3.3 *Let P be a normal propositional program. Then stable models of P are precisely the propositional models of the theory $r\Phi_P$.*

In our example 3.3, the theory Φ_P involved formulas with infinite disjunctions, but the theory $r\Phi_P$ contains only normal finite propositions.

Given a normal propositional program P, we say that P is a *finite support program* (FSP-program) if all the reduced defining equations for atoms with respect to P are finite propositional formulas. Equivalently, a program P is an *FSP*-program if for every atom p there is only finitely many inclusion-minimal supports of P-proof schemes for p.

4 Continuity properties of operators and proof schemes

In this section we investigate continuity properties of operators and we will see that one of those properties characterizes the class of FSP programs.

4.1 Continuity properties of monotone and antimonotone operators

Let us recall that $\mathcal{P}(At)$ denotes the set of all subsets of At. We say that any function $O : \mathcal{P}(At) \rightarrow \mathcal{P}(At)$ is an operator on the set At of propositional atoms. An operator O is *monotone* if for all sets $X, Y \subseteq At$, $X \subseteq Y$ implies $O(X) \subseteq O(Y)$. Likewise an operator O is *antimonotone* if for all sets $X, Y \subseteq At$, $X \subseteq Y$ implies $O(Y) \subseteq O(X)$. For a sequence $\langle X_n \rangle_{n \in N}$ of sets of atoms, we say that $\langle X_n \rangle_{n \in N}$ is *monotonically increasing* if for all $i, j \in N$, $i \leq j$ implies $X_i \subseteq X_j$ and we say that $\langle X_n \rangle_{n \in N}$ is *monotonically decreasing* if for all $i, j \in N$, $i \leq j$ implies $X_j \subseteq X_i$.
There are four distinct classes of operators that we shall consider in this paper. First, we shall consider two types of monotone operators, upper-half continuous monotone operators and lower-half continuous monotone operators. That is, we say that a monotone operator O is *upper-half continuous* if for every monotonically increasing sequence $\langle X_n \rangle_{n \in N}$, $O(\bigcup_{n \in N} X_n) = \bigcup_{n \in N} O(X_n)$. We say that a monotone operator O is *lower-half continuous* if for every monotonically decreasing sequence $\langle X_n \rangle_{n \in N}$, $O(\bigcap_{n \in N} X_n) = \bigcap_{n \in N} O(X_n)$. In the Logic Programming literature the first of these properties is called *continuity*. The classic result due to van Emden and Kowalski is the following.

Proposition 4.1 *For every Horn program P, the operator T_P is upper-half continuous.*

In general, the operator T_P for Horn programs is *not* lower-half continuous. For example, let P be the program consisting of the clauses $p \leftarrow p_i$ for $i \in N$. Then the operator T_P is not lower-half continuous. That is, if $X_i = \{p_i, p_{i+1}, \ldots\}$, then clearly $p \in T_P(X_i)$ for all i. However, $\bigcap_i X_i = \emptyset$ and $p \notin T_P(\emptyset)$.
Lower-half continuous monotone operators have appeared in the Logic Programming literature [Do94]. Even more generally, for a monotone operator O, let us define its *dual* operator O^d as follows:

$$O^d(X) = At \setminus O(At \setminus X).$$

Then an operator O is upper-half continuous if and only if O^d is lower-half continuous [JT51]. Therefore, for any Horn program P, the operator T_P^d is lower-half continuous.
In case of antimonotone operators, we have two additional notions of continuity. We say an antimonotone operator O is *upper-half* continuous if for every monotonically increasing sequence $\langle X_n \rangle_{n \in N}$, $O(\bigcup_{n \in N} X_n) = \bigcap_{n \in N} O(X_n)$. Similarly, we say an antimonotone operator O is *lower-half* continuous if for every monotonically decreasing sequence $\langle X_n \rangle_{n \in N}$, $O(\bigcap_{n \in N} X_n) = \bigcup_{n \in N} O(X_n)$.

4.2 Gelfond-Lifschitz operator GL_P and proof-schemes

For the completeness sake, let us recall that the Gelfond-Lifschitz operator for a program P which we denote GL_P, assigns to a set of atoms M the least fixpoint of of the operator $T_{P,M}$ or, equivalently, the least model N_M of the program P_M which is the Gelfond-Lifschitz reduct of P via M [GL88]. The following fact is crucial.

Proposition 4.2 ([GL88]) *The operator GL is antimonotone.*

Here is a useful proof-theoretic characterization of the operator GL_P.

Proposition 4.3 *Let P be a normal propositional program and M be a set of atoms. Then*

$$GL_P(M) = \{p : \textit{ there exists a } P\textit{-proof scheme } S \textit{ such that } M \textit{ admits } S, \\ \textit{and } p \textit{ is the conclusion of } S\}$$

Proof: Let us assume that $p \in GL_P(M)$ that is $p \in N_M$. As N_M is the least model of the Horn program P_M, $N_M = \bigcup_{n \in N} T^n_{P_M}(\emptyset)$. Then it is easy to prove by induction on n, that if $p \in T^n_{P_M}(\emptyset)$, then there is a P-proof scheme S_p such that p is the conclusion of S_p and S_p is admitted by M. Conversely, we can show, by induction on the length of the P-proof schemes, that whenever such P-proof scheme S is admitted by M, then p belongs to $GL_P(M)$. $\Box$

4.3 Continuity properties of the operator GL_P

This section will be devoted to proving results on the continuity properties of the operator GL_P. First, we prove that for every program P, the operator GL_P is lower-half continuous. We then show that if f is a lower-half continuous anti-monotone operator, then $f = GL_P$ for a suitably chosen program P. Finally, we show that the operator GL_P is upper-half continuous if and only if P is an *FSP*-program. That is, GL_P is upper-half continuous if for all atoms p the reduced defining equation for any p (w.r.t. P) is finite.

Proposition 4.4 *For every normal program P, the operator GL_P is lower-half continuous.*

Proof: We need to prove that for every program P and every monotonically decreasing sequence $\langle X_n \rangle_{n \in N}$,

$$GL_P(\bigcap_{n \in N} X_n) = \bigcup_{n \in N} GL_P(X_n).$$

Our goal is to prove two inclusions: $\subseteq$, and $\supseteq$.
We first show $\supseteq$. Since

$$\bigcap_{j \in N} X_j \subseteq X_n$$

for every $n \in N$, by antimonotonicity of GL_P we have

$$GL_P(X_n) \subseteq GL_P(\bigcap_{j \in N} X_j).$$

As n is arbitrary,

$$\bigcup_{n \in N} GL_P(X_n) \subseteq GL_P(\bigcap_{j \in N} X_j).$$

Thus the inclusion $\supseteq$ holds.
Conversely, let $p \in GL_P(\bigcap_{n \in N} X_n)$. Then, by Proposition 4.3, there must be a proof scheme S with support support U and conclusion p such that

$$U \cap \bigcap_{n \in N} X_n = \emptyset.$$

But the family $\langle X_n \rangle_{n \in n}$ is monotonically descending and the set U is finite. Thus there is an integer n_0 so that

$$U \cap X_{n_0} = \emptyset.$$

This, however, implies that $p \in GL_P(X_{n_0})$, and thus

$$p \in \bigcup_{n \in N} GL_P(X_n).$$

As p is arbitrary, the inclusion $\subseteq$ holds. Thus $GL_P(\bigcap_{n \in N} X_n) = \bigcup_{n \in N} GL_P(X_n)$.
$\Box$

The lower-half continuity of antimonotone operators is closely related to programs, as shown in the following result.

Proposition 4.5 *Let At be a denumerable set of atoms. Let f be an antimonotone and lower-half continuous operator on $\mathcal{P}(At)$. Then there exists a normal logic program P such that $f = GL_P$.*

Proof.
We define the program $P = P_f$ as follows:

$$P = \{p \leftarrow \neg q_1, \ldots, \neg q_i : p \in f(At \setminus \{q_1, \ldots, q_i\})\}.$$

We claim that $f = GL_P$, that is, for all X, $f(X) = GL_P(X)$.

Let $X \subseteq At$ be given. We consider two cases.
Case 1: X is cofinite, $X = At \setminus \{q_1, \ldots, q_i\}$. We need to prove two inclusions, (a) $f(X) \subseteq GL_P(X)$ and (b) $GL_P(X) \subseteq f(X)$.
For (a), note that if $p \in f(X)$, then the clause $p \leftarrow \neg q_1, \ldots, \neg q_i$ belongs to P. Hence $p \leftarrow$ belongs to P_X and $p \in GL_P(X)$.
For (b), note that if $p \in GL_P(X)$, then given the form of the clauses in P, there must be some clause $p \leftarrow \neg q_{i_1}, \ldots, \neg q_{i_j}$ in P where $\{q_{i_1}, \ldots, q_{i_j}\} \subseteq \{q_1, \ldots, q_i\}$. But this means that $p \in f(At \setminus \{q_{i_1}, \ldots, q_{i_j}\})$. Since f is antimonote and $At \setminus \{q_1, \ldots, q_i\} \subseteq At \setminus \{q_{i_1}, \ldots, q_{i_j}\}$, we must have

$$f(At \setminus \{q_{i_1}, \ldots, q_{i_j}\}) \subseteq f(At \setminus \{q_1, \ldots, q_i\}) = f(X)$$

and, hence, $p \in f(X)$. Thus $GL_P(X) \subseteq f(X)$.

Case 2: X is not cofinite. Let $\{q_0, q_1, \ldots\}$ be an enumeration of $At \setminus X$. Let $Y_i = At \setminus \{q_0, \ldots, q_i\}$. Then, clearly, $X \subseteq Y_i$ for all $i \in N$. Moreover the sequence $\langle Y_i \rangle_{i \in N}$ is monotonically decreasing and $\bigcap_{i \in N} Y_i = X$. Therefore, by our assumptions on the operator f,

$$f(X) = \bigcup_{i \in N} f(Y_i).$$

Again, we need to prove two inclusions, (a) $f(X) \subseteq GL_P(X)$ and (b) $GL_P(X) \subseteq f(X)$. For (a), note that if $p \in f(X)$, then for some $i \in N$, $p \in F(Y_i)$. Therefore, for that i, $p \leftarrow \neg q_0, \ldots, \neg q_i$ is a clause in P. But then $X \cap \{q_0, \ldots, q_i\} = \emptyset$ so that the clause $p \leftarrow$ is in P_X and $p \in GL_P(X)$.

For the proof of (b), note that if $p \in GL_P(X)$, then because of the syntactic form of the clauses in our program there are atoms $r_0, \ldots, r_k$ so that the clause $p \leftarrow \neg r_0, \ldots, \neg r_k$ belongs to the program P, and $r_0, \ldots, r_k \notin X$. Thus $\{r_0, \ldots, r_k\} \subseteq \{q_0, q_1, \ldots\}$ and, hence, for some $i \in N$, $\{r_0, \ldots, r_k\} \subseteq \{q_0, \ldots, q_i\}$. Now, consider such a Y_i. Since Y_i is cofinite, it follows from Case 1 that $f(Y_i) = GL_P(Y_i)$. Since $X \subseteq Y_i$, $f(Y_i) \subseteq f(X)$ by the antimonotonicity of f. But $p \in GL_P(Y_i)$ because $r_0, \ldots, r_k \notin Y_i$ and, hence, $p \in f(Y_i)$. But since $f(Y_i) \subseteq f(X)$, $p \in f(X)$ as desired. $\Box$
We are now ready to prove the next result of this paper.

Proposition 4.6 *Let P be a normal propositional program. The following are equivalent:*

(a) *P is an* FSP-*program.*

(b) *The operator* GL_P *is upper-half continuous, i.e.*

$$GL_P(\bigcup_{n\in N} X_n) = \bigcap_{n\in N} GL_P(X_n)$$

for every monotonically increasing sequence $\langle X_n\rangle_{n\in N}$.

Proof: Two implications need to be proved: $(a) \Rightarrow (b)$, and $(b) \Rightarrow (a)$.
Proof of the implication $(a) \Rightarrow (b)$. Here, assuming (a), we need to prove two inclusions:
(i) $GL_P(\bigcup_{n\in N} X_n) \subseteq \bigcap_{n\in N} GL_P(X_n)$, and
(ii) $\bigcap_{n\in N} GL_P(X_n) \subseteq GL_P(\bigcup_{n\in N} X_n)$.
To prove (i), note that since $X_n \subseteq \bigcup_{j\in N} X_j$, we have

$$GL_P(\bigcup_{j\in N} X_j) \subseteq GL_P(X_n).$$

As n is arbitrary,

$$GL_P(\bigcup_{j\in N} X_j) \subseteq \bigcap_{n\in N} GL_P(X_n).$$

This proves (i).
To prove (ii), let $p \in \bigcap_{n\in N} GL_P(X_n)$. Then, for every $n \in N$, $p \in GL_P(X_n)$ and so, for every $n \in N$, there is an inclusion-minimal support U for p such that

$$U \cap X_n = \emptyset.$$

But by (a) there are only finitely many inclusion-minimal supports for P-proof schemes for p. Therefore there is a support of an inclusion minimal support of a proof scheme of p, U_0, such that for infinitely many n's

$$U_0 \cap X_n = \emptyset.$$

But the sequence $\langle X_n\rangle_{n\in N}$ is monotonically increasing. Therefore for *all* $n \in N$, $U_0 \cap X_n = \emptyset$. But then

$$U_0 \cap \bigcup_{n\in N} X_n = \emptyset,$$

so that $p \in GL_P(\bigcup_{n\in N} X_n)$. Thus (ii) holds and the implication $(a) \Rightarrow (b)$ follows.

To prove that $(b) \Rightarrow (a)$, assume that the operator GL_P is upper-half continuous. We need to show that for every p, the reduced defining equation for p is finite. So let us assume that rEq_p^P is not finite. This means that there is an infinite set $\mathcal{X} = \{U_1, U_2, \ldots\}$, where $U_1 \prec U_2 \prec \cdots$, such that

1. each U_i is finite,
2. the elements of $\mathcal{X}$ are pairwise inclusion-incompatible, and
3. for every set of atoms M, $p \in GL_P(M)$ if and only if for some $U_i \in \mathcal{X}$, $U_i \cap M = \emptyset$.

We will now define two sequences:

1. a sequence $\langle K_n \rangle_{n \in N}$ of infinite sets of integers and
2. a sequence $\langle p_n \rangle_{n \in N \setminus \{0\}}$ of atoms.

We define $K_0 = N$, and we define p_1 as the first element of U_1 such that

$$\{j : p \notin U_j\}$$

is infinite. Clearly, K_0 is well-defined. We need to show that p_1 is well-defined. If p_1 is not well-defined, then for every $p \in U_1$ there is an integer i_p such that for all $m > i_p$, $p \in U_m$. But U_1 is finite so taking $n = \max_{p \in U_1} i_p$, we find that for *all* $m > n$, $U_1 \subseteq U_m$ - which contradicts the fact that the sets in $\mathcal{X}$ are pairwise inclusion-incompatible. Thus p_1 is well-defined. We now set

$$K_1 = \{n \in K_0 : p_1 \notin U_n\} = \{n \in K_0 : \{p_1\} \cap U_n = \emptyset\}.$$

Clearly. K_1 is infinite.
Now, let us assume that we already defined p_l and K_l so that $K_l = \{n : U_n \cap \{p_1, \ldots, p_l\} = \emptyset\}$ is an infinite subset of N. We select p_{l+1} as the first element $p \in U_{l+1}$ so that

$$\{j : j \in K_l \text{ and } p \notin U_j\}$$

is infinite. Clearly, by an argument as above, there is such p, and so p_{l+1} is well-defined. We then set

$$K_{l+1} = \{j \in K_l : p_{l+1} \notin U_j\}.$$

Since $\{p_1, \ldots, p_l\} \cap U_j = \emptyset$ for all $j \in K_l$, $\{p_1, \ldots, p_{l+1}\} \cap U_j = \emptyset$ for all $j \in K_{l+1}$. By construction, the set K_{l+1} is infinite.
Now, we complete the argument as follows. We set $X_n = \{p_1, \ldots, p_n\}$. The sequence $\langle X_n \rangle_{n \in N}$ is monotonically increasing. For each n there is j (in fact infinitely many j's) so that $X_n \cap U_j = \emptyset$. Therefore, for each n, $p \in GL_P(X_n)$. Hence $p \in \bigcap_{n \in N} GL_P(X_n)$.
On the other hand, let $X = \bigcup_{n \in N} X_n$. Then

$$X = \{p_1, p_2, \ldots\}.$$

By our construction, $p_n \in U_n$, and so $U_n \cap X \neq \emptyset$. Therefore X does not admit *any* P-proof scheme for p. Thus $p \notin GL_P(X) = GL_P(\bigcup_{n \in N} X_n)$. But this would contradict our assumption that GL_P is upper-half continuous. Thus there can be no such p and hence P must be a *FSP*-program. □

5 Extensions to CC-programs

In [SNS02] Niemelä and coauthors defined a significant extension of logic programming with stable semantics which allows for programming with cardinality constraints, and, more generally, with weight constraints. This extension has been further studied in [MR04, MNT07]. To keep things simple, we will limit our discussion to cardinality constraints only, although it is possible to extend our arguments to any class of convex constraints [LT05]. *Cardinality constraints* are expressions of the form lXu, where $l, u \in N$, $l \leq u$ and X is a finite set of atoms. The semantics of an atom lXu is that a set of atoms M satisfies kXl if and only if $k \leq |M \cap X|$. When $l = 0$, we do not write it, and, likewise, when $u \geq |X|$, we omit it, too. Thus an atom p has the same meaning as $1\{p\}$ while $\neg p$ has the same meaning as $\{p\}0$.
The stable semantics for CC-programs is defined via fixpoints of an analogue of the Gelfond-Lifschitz operator GL_P; see the details in [SNS02] and [MR04]. The operator in question is neither monotone nor antimonotone. But when we limit our attention to the programs P where clauses have the property that the head consists of a single atom (i.e. are of the form $1\{p\}$), then one can define an operator $CCGL_P$ which is antimonotone and whose fixpoints are stable models of P. This is done as follows.

Given a clause C

$$p \leftarrow l_1X_1u_1, \ldots, l_mX_mu_m,$$

we transform it into the clause

$$p \leftarrow l_1X_1, \ldots, l_mX_m, X_1u_1, \ldots, X_mu_m \tag{4}$$

[MNT07]. We say that a clause C of the form (4) is a CC-Horn clause if it is of the form

$$p \leftarrow l_1X_1, \ldots, l_mX_m. \tag{5}$$

A CC-Horn program is a CC-program all of whose clauses are of the form (5). If P is a CC-Horn program, we can define the analogue of the one step provability operator T_P by defining that for a set of atom M,

$$T_P(M) = \{p : (\exists C = p \leftarrow l_1X_1, \ldots, l_mX_m)(\forall i \in \{1, \ldots m\})(|X_i \cap M| \geq l_i)\} \tag{6}$$

It is easy to see that T_P is monotone operator that the least fixed point of T_P is given by

$$lfp(T_P) = \bigcup_{n \geq 0} T_P^n(\emptyset). \tag{7}$$

We can define the analogue of the Gelfond-Lifschitz reduct of a CC-program, which we call the NSS-reduct of P, as follows. Let $\bar{P}$ denote the set of all transformed clauses derived from P. Given a set of atoms M, we eliminate from $\bar{P}$ those clauses where some upper-constraint ($X_i u_i$) is not satisfied by M, i.e. $|M \cap X_i| > u_i$. In the remaining clauses, the constraints of the form $X_i u_i$ are eliminated altogether. This leaves us with a CC-Horn program P_M. We then define $CCGL_P(M)$ to be the least fixed point of T_{P_M} and say that M is a CC-stable model if $M = CCGL_P(M)$. The equivalence of this construction and the original construction in [SNS02] for normal CC-programs is shown in [MNT07].

Next we define the analogues of P-proof schemes for normal CC-programs, i.e. programs which consists entirely of clauses of the form (4). This is done by induction as follows. When

$$C = p \leftarrow X_1 u_1, \ldots, X_k u_k$$

is a normal CC-clause without the cardinality-constraints of the form $l_i X_i$ then

$$\langle \langle C, p \rangle, \{X_1 u_1, \ldots, X_k u_k\} \rangle$$

is a P-CC-proof scheme with support $\{X_1 u_1, \ldots, X_k u_k\}$. Likewise, when

$$S = \langle \langle C_1, p_1 \rangle, \ldots, \langle C_n, p_n \rangle, U \rangle$$

is a P-CC-proof scheme,

$$p \leftarrow l_1 X_1, \ldots, l_m X_m, X_1 u_1, \ldots, X_m u_m$$

is a clause in P, and $|X_1 \cap \{p_1, \ldots, p_n\}| \geq l_1$, …, $|X_m \cap \{p_1, \ldots, p_n\}| \geq l_m$, then

$$\langle \langle C_1, p_1 \rangle, \ldots, \langle C_n, p_n \rangle, \langle C, p \rangle, U \cup \{X_1 u_1, \ldots, X_m u_m\} \rangle$$

is a P-CC-proof scheme with support $U \cup \{X_1 u_1, \ldots X_m u_m\}$. The notion of admittance of a P-CC-proof scheme is similar to the notion of admittance of P-proof scheme for normal programs P. That is, if $\mathcal{S} = \langle \langle C_1, p_1 \rangle, \ldots, \langle C_n, p_n \rangle, \langle C, p \rangle, U \rangle$ is a CC-proof scheme with support $U = \{X_1 u_1, \ldots X_n u_n\}$, then $\mathcal{S}$ is admitted by M if for every $X_i u_i \in U$, $M \models X_i u_i$, i.e. $|M \cap X_i| \leq u_i$.

Similarly, we can associate a propositional formula ϕ_U so that M admits $\mathcal{S}$ if and only if $M \models \phi_U$ as follows:

$$\phi_U = \bigwedge_{i=1}^{n} \bigvee_{W \subseteq X_i, |W| = |X_i| - u_i} \neg W. \tag{8}$$

Then we can define a partial ordering on the set of possible supports of proof scheme by defining $U_1 \preceq U_2 \iff \phi_{U_2} \models \phi_{U_1}$. For example if $U_1 = \langle\{1,2,3\}2,$ $\{4,5,6\}2\rangle$ and $U_2 = \langle\{1,2,3,4,5,6\},4\rangle$, then

$$\begin{aligned} \phi_{U_1} &= (\neg 1 \vee \neg 2 \vee \neg 3) \wedge (\neg 4 \vee \neg 5 \vee \neg 6) \\ \phi_{U_2} &= \bigvee_{1 \leq i < j \leq 6} (\neg i \wedge \neg j). \end{aligned}$$

Then clearly $\phi_{U_1} \models \phi_{U_2}$ so that $U_2 \preceq U_1$. We then define a normal propositional CC-program to be *FPS CC-program* if for each $p \in At$, there are finitely many $\preceq$-minimal supports of P-CC-proof schemes with conclusion p.
We can also define analogue of the defining equation $CCEq_p^P$ of p relative to a normal CC-program P as

$$p \Leftrightarrow (\phi_{U_1} \vee \phi_{U_2} \vee \cdots) \tag{9}$$

where $\langle U_1, U_2, \ldots\rangle$ is a list of supports of all P-CC-proofs schemes with conclusion p. Again up to a total ordering of possible finite supports, this formula is unique. Let Φ_P be the set $\{CCEq_p^P : p \in At\}$. Similarly, we define the *reduced defining equation* for p relative to P to be the formula

$$p \Leftrightarrow (\neg\phi_{U_1} \vee \neg\phi_{U_2} \vee \ldots) \tag{10}$$

where U_i range over $\preceq$-*minimal* supports of P-CC-proof schemes for the atom p. Then we have the following analogues of Propositions 3.1 and 3.2.

Proposition 5.1 *For every normal propositional CC-program P and every set M of atoms, M is a CC-stable model of P if and only if the following two conditions hold:*

(i) for every $p \in M$, there is a P-CC-proof scheme S with conclusion p such that M admits S and

(ii) for every $p \notin M$, there is no P-CC-proof scheme S with conclusion p such that M admits S.

Proposition 5.2 *Let P be a normal propositional CC-program. Then CC-stable models of P are precisely the propositional models of the theory Φ_P.*

We also can prove the analogues of Propositions 4.2 and 4.3.

Proposition 5.3 *For any CC-program P, the operator $CCGL_P$ is antimonotone.*

Proof: It is easy to see that if $M_1 \subseteq M_2$, then for any clause

$$C = p \to l_1 X_1, \ldots, l_m X_m, X_1 u_1, \ldots X_m l_m,$$

$M_2 \models X_i u_i$ implies $M_1 \models X_i u_i$. Thus it follows that $P_{M_2} \subseteq P_{M_1}$ and hence $\mathit{lfp}(T_{P_{M_2}}) \subseteq \mathit{lfp}(T_{P_{M_1}})$. $\Box$

Proposition 5.4 *Let P be a normal propositional CC-program and M be a set of atoms. Then*

$$CCGL_P(M) = \{p : \textit{ there exists a } P\textit{-proof scheme } S \textit{ such that } M \textit{ admits } S, \\ \textit{and } p \textit{ is the conclusion of } S\}$$

Proof: Let us assume that $p \in CCGL_P(M)$, i.e. $p \in \mathit{lfp}(T_{P_M})$. Since $\mathit{lfp}(T_{P_M}) = \bigcup_{n \geq 1} T^n_{P_M}(\emptyset)$, we can easily show by induction on n that if $p \in T^n_{P_M}(\emptyset)$, then there is a P-CC-proof scheme S_p such p is the conclusion of S_p and S_p is admitted by M.

Conversely, we can show, by induction on the length of the P-CC-proof schemes, that whenever there is P-CC-proof scheme S admitted by M, then p belongs to $\mathit{lfp}(T_{P_M})$. $\Box$

Next we prove that analogue of Proposition 4.4.

Proposition 5.5 *For every normal CC-program P, the operator $CCGL_P$ is lower-half continuous.*

Proof: We need to prove that for every normal CC-program P and every monotonically decreasing sequence $\langle X_n \rangle_{n \in N}$

$$CCGL_P(\bigcap_{n \in N} X_n) = \bigcup_{n \in N} CCGL_P(X_n).$$

We need to prove two inclusions: $\subseteq$, and $\supseteq$.

We first show $\supseteq$. Since

$$\bigcap_{j \in N} X_j \subseteq X_n$$

for every $n \in N$, it follows from the antimonotonicity of $CCGL_P$ that we have

$$CCGL_P(X_n) \subseteq GL_P(\bigcap_{j \in N} X_j).$$

As n is arbitrary,

$$\bigcup_{n \in N} CCGL_P(X_n) \subseteq CCGL_P(\bigcap_{j \in N} X_j).$$

Thus the inclusion $\supseteq$ holds.
Conversely, let $p \in CCGL_P(\bigcap_{n\in N} X_n)$. Then, by Proposition 5.4, there must be a CC-proof scheme S with support support $U = \{Y_1 u_1, \ldots, Y_n u_n\}$ and conclusion p such that

$$|Y_i \cap \bigcap_{n\in N} X_n| \leq u_i \text{ for } i = 1, \ldots, n.$$

Since the family $\langle X_n \rangle_{n\in n}$ is monotonically descending, it follows that

$$Y_i \cap X_1 \supseteq Y_i \cap X_2 \supseteq \cdots .$$

Since Y_i is finite, it is the case that if $|Y_i \cap \bigcap_{n\in N} X_n| \leq u_i$, then there is some m_i such that $|Y_i \cap X_{m_i}| \leq u_i$. Hence if $m = \max(m_1, \ldots, m_n)$, then

$$|Y_i \cap X_m| \leq u_i \text{ for } i = 1, \ldots, n.$$

This, however, implies that $p \in CCGL_P(X_m)$, and thus

$$p \in \bigcup_{n\in N} CCGL_P(X_n).$$

As p is arbitrary, the inclusion $\subseteq$ holds. Thus $CCGL_P(\bigcap_{n\in N} X_n) = \bigcup_{n\in N} CCGL_P(X_n)$.
$\Box$
Next we can prove the analogue of the first half of Proposition 4.6.

Proposition 5.6 *Let P be a normal propositional CC-program. Then if P is an* FSP-*program, the operator $CCGL_P$ is upper-half continuous, i.e.*

$$CCGL_P(\bigcup_{n\in N} X_n) = \bigcap_{n\in N} CCGL_P(X_n)$$

for every monotonically increasing sequence $\langle X_n \rangle_{n\in N}$.

Proof: Two implications need to be proved: $(a) \Rightarrow (b)$, and $(b) \Rightarrow (a)$.
Proof of the implication $(a) \Rightarrow (b)$. Here, assuming (a) we need to prove two inclusions:
(i) $GL_P(\bigcup_{n\in N} X_n) \subseteq \bigcap_{n\in N} GL_P(X_n)$, and
(ii) $\bigcap_{n\in N} GL_P(X_n) \subseteq GL_P(\bigcup_{n\in N} X_n)$.
To prove (i), note that since $X_n \subseteq \bigcup_{j\in N} X_j$, we have

$$CCGL_P(\bigcup_{j\in N} X_j) \subseteq CCGL_P(X_n).$$

As n is arbitrary,

$$CCGL_P(\bigcup_{j\in N} X_j) \subseteq \bigcap_{n\in N} CCGL_P(X_n).$$

This proves (i).
To prove (ii), let $p \in \bigcap_{n\in N} CCGL_P(X_n)$. Then, for every $n \in N$, $p \in CCGL_P(X_n)$ and so, for every $n \in N$, there is a minimal support $U_n = \{Y_1^{(n)} u_1^{(n)}, \ldots, Y_{m_n}^{(n)} u_{M_n}^{(n)}\}$ for p such that

$$|Y_i^{(n)} \cap X_n| \leq u_i^{(n)} \text{ for } i = 1, \ldots, m_n.$$

But there are only finitely many $\preceq$-minimal supports for P-CC-proof schemes for p. Therefore there is a support $U_0 = \{Z_1 w_1, \ldots, Z_t w_t\}$ for a P-CC-proof scheme with conclusion p such that for infinitely many n's

$$|Z_i \cap X_n| \leq w_i \text{ for } i = 1, \ldots, t.$$

But the sequence $\langle X_n \rangle_{n\in N}$ is monotonically increasing. Therefore for *all* $n \in N$,

$$|Z_i \cap X_n| \leq w_i \text{ for } i = 1, \ldots, t.$$

But since each Z_i is finite, then it must be the case that

$$|Z_i \cap \bigcup_{ninN} X_n| \leq w_i \text{ for } i = 1, \ldots, t.$$

so that $p \in CCGL_P(\bigcup_{n\in N} X_n)$. □

We note that, alternatively, one can easily give a direct reduction of our CC-programs to normal logic programs using the methods of [FL05] and the distributivity result of [LTT99]. Such reduction, of course, lead to an exponential blow up in the size of the representation.

6 Conclusions

We note that investigations of proof systems in a related area, SAT, play a key role in establishing lower bounds on the complexity of algorithms for finding the models. We wonder if there are analogous results in ASP. For achieving such a goal, we need to find and investigate proof systems for ASP. One candidate for such a proof system is provided in this paper by using P-proof schemes. We wonder if such a proof system can be used to develop a deeper understanding of the complexity issues related to finding stable models.

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References

[Ap90] Apt, K.. Logic programming, In: J. van Leeuven, ed, *Handbook of Theoretical Computer Science*, pages 493–574, MIT Press, 1990.

[Bo04] Bonatti, P.A. Reasoning with infinite stable models. *Artificial Intelligence 156*:75–111, 2004.

[BTK93] Bondarenko, A., Toni, F. and Kowalski, R.A., An Assumption-Based Framework for Non-Monotonic Reasoning. *Proceedings of LPNMR-93*, MIT Press, pages 171–189, 1993.

[Cl78] Clark, K. Negation as failure. In *Logic and data bases*, H. Gallaire and J. Minker, Eds. Plenum Press, pages 293–322, 1978.

[DP92] Davey, B.A., and Priestley, H.A., *Introduction to Lattices and Order*, Cambridge University Press, 1992.

[DK89] Dung, P.M. and Kanchanasut, K., On the generalized predicate completion of non-Horn programs, *Logic programming. Proceedings of the North American Conference*, 1989.

[Do94] Doets, K., *From Logic to Logic Programming*, MIT Press, 1994.

[FL05] Ferraris, P., and Lifschitz, V., Weight constraints as nested expressions, *Theory and Practice of Logic Programming*, *5*:45-74, 2005.

[FLL06] Ferraris, P., Lee, J. and Lifschitz, V. A generalization of Lin-Zhao theorem. *Annals of Mathematics and Artificial Intelligence 47*:79–101, 2006.

[GS07] Gebser, M. and Schaub, T., Generic Tableaux for Answer Set Programming, *Proceedings of International Conference on Logic Programming, 2007*, pages 119–133, 2007.

[GL88] Gelfond, M. and Lifschitz, V. The stable model semantics for logic programming. In *Proceedings. of the International Joint Conference and Symposium on Logic Programming*, pages 1070–1080, 1988.

[JT51] Jonsson, B. and Tarski, A. Boolean Algebras with Operators. *American Journal of Mathematics 73*:891–939, 1951.

[JO07] Järvisalo, M. and Oikarinen, E., Extended ASP Tableaux and Rule Redundancy in Normal Logic Programs, *Proceedings of International Conference on Logic Programming, 2007* pages 134–148, 2007.

[LT05] Liu, L. and Truszczyński, M., Properties of programs with monotone and convex constraints, *Proceedings of the 20th National Conference on Artificial Intelligence*, pages 701-706, 2005.

[Li96] Lifschitz, V., Foundations of logic programming, in *Principles of Knowledge Representation*, CSLI Publications, pages 69-127, 1996.

[LR06] V. Lifschitz and A. Razborov. Why are there so many loop formulas. *Annals of Mathematics and Artificial Intelligence 7*:261–268, 2006.

[LTT99] V. Lifschitz, L. R. Tang and H. Turner. Nested expressions in logic programs, *Annals of Mathematics and Artificial Intelligence, 25*:369-389, 1999.

[LZ02] F. Lin and Y. Zhao. ASSAT: Computing answer sets of a logic program by SAT solvers. *Proceedings of AAAI 2002*, pages 112–117. 2002

[L89] J. Lloyd, *Foundations of Logic Programming*, Springer-Verlag, 1989.

[MNR90a] Marek, W., Nerode, A., and Remmel, J.B., Nonmonotonic Rule Systems I. *Annals of Mathematics and Artificial Intelligence, 1*:241–273, 1990.

[MNR90b] Marek, W., Nerode, A., and Remmel, J.B., Nonmonotonic Rule Systems II. *Annals of Mathematics and Artificial Intelligence, 5*:229-264, 1992.

[MNR91] Marek, W., Nerode, A., and Remmel, J.B., A Context for Belief Revision: Normal Logic Programs (Extended Abstract) *Proceedings, Workshop on Defeasible Reasoning and Constraint Solving*, International Logic Programming Symposium, San Diego, CA., 1991.

[MNR92] Marek, W., Nerode, A., and Remmel, J.B., How Complicated is the Set of Stable Models of a Logic Program? *Annals of Pure and Applied Logic, 56*:119-136, 1992.

[MNR94a] Marek, W., Nerode, A., and Remmel, J.B., The stable models of predicate logic programs. *Journal of Logic Programming 21*:129-154, 1994.

[MNR94b] Marek, W., Nerode, A., and Remmel, J.B., Context for belief revision: Forward chaining-normal nonmonotonic rule systems, *Annals of Pure and Applied Logic 67*:269-324, 1994.

[MNR94] Marek, W., Nerode, A., and Remmel, J.B., The stable models of predicate logic programs. *Journal of Logic Programming 21*:129-154, 1994.

[MNT07] Marek, V.W., Niemelä, I. and Truszczynski, M. Logic programs with monotone abstract constraint atoms, *Theory and Practice of Logic Programming*, 8:167–199, 2008.

[MR04] Marek, V.W. and Remmel, J.B. Set Constraints in Logic Programming. In *Logic Programming and Nonmonotonic Reasoning, Proceedings of the 7th International Conference (LPNMR-04)*. *LNAI 2923*, pages 154–167, Springer-Verlag, 2004.

[MR09] Marek, V.W. and Remmel, J.B. Guarded resolution and Answer Set Programming, *Unpublished manuscript*, 2009.

[MT93] Marek, W. and Truszczyński, M. *Nonmonotonic Logic*, Springer-Verlag, Berlin, 1993.

[Mi05] Milnikel, R.S., Sequent Calculi for Skeptical Reasoning in Predicate Default Logic and Other Nonmonotonic Systems, *Annals of Mathematics and Artificial Intelligence 44*:1-34, 2005.

[SNS02] Simons, P., Niemelä, I., and Soininen, T. 2002. Extending and implementing the stable model semantics. *Artificial Intelligence 138*:181–234, 2002.