A method for hedging in Continuous Time

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1 Introduction

This article gives an analysis of the NormalHedge algorithm in continuous time. The NormalHedge algorithm is described and analyzed in discrete time in [CFH]. The continuous time analysis is mathematically cleaner, simpler and tighter than the discrete time analysis.

To motivate the continuous time framework consider the problem of portfolio management. Suppose we are managing N different financial instruments allowed to define a desired distribution of our wealth among the instruments. We ignore the details of the buy and sell orders that have to be placed in order to reach the desired distribution, we also ignore issues that have to do with transaction costs, buy-sell spreads and the like. We assume that at each moment the buy and sell prices for a unit of a particular instrument are the same and that there are no transaction costs.

Our goal is to find an algorithm for managing the portfolio distribution. In other words, we are looking for a mapping from past prices to a distribution over the instruments. As we are considering continuous time, the past can be arbitrarily close to the present. Formally speaking, we say that the portfolio distribution is "causal" or "unanticipating" to remove the possibly of defining a portfolio which is a function of the future gains as his would clearly be a cheat. We are interested in considering continuous time, because instrument prices can fluctuate very rapidly.

To model this very rapid fluctuation we use a type of stochastic process called an Itô process to model the log of the price as a function of time. Intuitively, an Itô process is a linear combination of a differentiable process and white noise. A more formal definition is given below. To read more about Itô processes see [Osk03].

Our algorithm and its analysis do not make *any* additional assumption on the price movement of the instruments. Of course, with no additional assumption we cannot have any guarantees regarding our future wealth. For example, if the price of all of the instruments decreases at a particular moment by 10%, our wealth will necessary decrease by 10%, regardless of our wealth distribution. However, surprisingly enough, we *can* give a guarantee on the *regret* associated with our method without any additional assumptions. Regret quantifies the difference between our wealth at time t and the wealth we would have had if we invested all of our money in the best one of the N instruments. Specifically, denote the log price of instrument i at time t by X_t^i and assume that the initial unit price for all instruments is one, i.e. $X_0^i = \log(1) = 0$. Let G_t be the log of our wealth at time t. We define our regret at time t as

$$R_t = \max_{i=1,\ldots,N} X_t^i - G_t \ .$$

Intuitively, the regret is large if by investing all of our money in a particular instrument (whose identity is known only in hind-sight) we would have made much more money than what we actually have. The main result of this paper is an algorithm for which the regret is bounded by $\sqrt{2ct(\ln N + 1)}$ where c is the amount of random fluctuations (white noise) in the instrument prices. It stands to reason that the bound depends on the number of instruments N, because the larger the potential impact of price fluctuations. If the instrument prices are are all simple random walks (brownian motion in continuous time) the expected price of the *best instrument* is proportional to $\sqrt{ct \ln N}$.

We can get potentially tighter bounds if we consider the *set* of best instruments. Suppose we sort X_t^i for a particular time t from the largest to the smallest. We say that the $\epsilon - quantile$ of the prices is the value x such that $\lfloor \epsilon N \rfloor$ of X_t^i are larger than x. We prove a bound of $\sqrt{2ct(\ln(1/\epsilon)+1)}$ on the regret of our algorithm relative to the ϵ-quantile for any $\epsilon > 0$. This generalized bound can be used when we are hedging over an infinite, even uncountably infinite set of instruments.

Case in point. Suppose that the instruments that we are combining are themselves portfolios. The set of fixed portfolios over $N > 1$ instruments consists of the $N-1$ dimensional simplex, which is an uncountably infinite set. Cover's universal porfolios algorithm [Cov91, CO96] uses this set as the set of portfolios to be combined. We can apply our algorithm to this set and guarantee that our regret relative to the top ϵ-quantile of fixed rebalanced portfolios is small.

2 Hedging in Continuous time

The portfolio management problem is defined as follows. Let X_t^i for $i = 1, \ldots, N$ define the log-prices of N instrumentss as a function of time. The initial unit price is one, thus $X_0^i = 0$. X_t^i is an Itô process. More specifically, let dW_t denote an N dimensional Wiener process where each coordinate is an independent process with unit variance. Then the differential of the total gain corresponding to the ith instrument is:

$$dX_t^i = \hat{a}^i(t)dt + \sum_{j=1}^{N} \hat{b}^{i,j}(t) dW_t^j \tag{1}$$

Where $\hat{a}^i(t), \hat{b}^{i,j}(t)$ are adapted (non-anticipatory) stochastic processes that are Itô-integrable with respect to W_t^j. In mathematical finance terms $\hat{a}_i$ correspond to *price drift* and $\hat{b}^{i,j}$ correspond to diffusion or *price volatility*.

The volatility of the ith instrument at time t is defined as

$$\hat{V}^i(t) = \sum_{j=1}^{N} \left(\hat{b}^{i,j}(t)\right)^2 \tag{2}$$

and the maximal volitility at time t is defined as

$$V^M(t) = \max_i \hat{V}^i(t) \tag{3}$$

An "aggregating strategy" is an portfolio management policy that defines how to distribute the wealth among the N instruments as a function of their past performance. Mathematically speaking, a policy is a stochastic process that is an adapted (non-anticipative) function of the past prices $X_t^i{}_{i=1}^N$. Given an instantiation of the stochastic processes $X_t^i{}_{i=1}^N$ the aggregating strategy defines N stochastic processes P_t^i such that for all t $P_t^i \geq 0$ and $\sum_i P_t^i = 1$. The cumulative gain of the master algorithm is defined to be 0 at $t = 0$, for $t \geq 0$ it is defined by the differential

$$dG_t = \sum_{j=1}^{N} P_t^i dX_t^j$$

The regret of the master algorithm relative to the ith instrument is defined to be zero at $t = 0$ and is otherwise defined by the differential

$$dR_t^i = dX_t^i - dG_t$$

And we can combine the last two equations to get:

$$\sum_{i=1}^{N} P_t^i dR_t^i = 0 \tag{4}$$

As R_t^i is a linear combination of X_t^i it is also an Itô process and can be expressed as

$$dR_t^i = a^i(t)dt + \sum_{j=1}^{N} b^{i,j}(t)dW_t^j \tag{5}$$

where

$$a^i(t) = \hat{a}^i(t) - \sum_{k=1}^{N} P_t^k \hat{a}^k(t) \tag{6}$$

and

$$b^{i,j}(t) = \hat{b}^{i,j}(t) - \sum_{k=1}^{N} P_t^k \hat{b}^{k,j}(t) \tag{7}$$

Similarly to X_t^i we define the diffusion rate of R_t^i to be

$$V^i(t) = \sum_{j=1}^{N} \left(b^{i,j}(t)\right)^2 \tag{8}$$

We prove an upper bound on $V^i(t)$

Lemma 1

$$\forall t, V^i(t) \leq 2V^M(t)$$

Proof: We use $\mathbf{b}^j$ and $\hat{\mathbf{b}}^j$ the N dimensional vectors $\left\langle b^{1,j}, \ldots, b^{N,j} \right\rangle$ and $\left\langle \hat{b}^{1,j}, \ldots, \hat{b}^{N,j} \right\rangle$ respectively.Using this notation we rewrite Equations (7) and (8) as

$$\mathbf{b}^i(t) = \hat{\mathbf{b}}^i(t) - \sum_{k=1}^{N} P_t^k \hat{\mathbf{b}}^k(t); \;\; V^i(t) = \|\mathbf{b}^i(t)\|_2^2$$

Equations (3) and (8) imply that $\|\hat{\mathbf{b}}^i(t)\|_2^2 \leq V^M(t)$ for all i. It follows that the norm of the convex combination is also bounded:

$$\left\|\sum_{k=1}^{N} P_t^k \hat{\mathbf{b}}^k(t)\right\|_2^2 = \sum_{k=1}^{N} P_t^k \left\|\hat{\mathbf{b}}^k(t)\right\|_2^2 \leq V^M(t)$$

From which it follows that

$$\left\|\hat{\mathbf{b}}^i(t) - \sum_{k=1}^{N} P_t^k \hat{\mathbf{b}}^k(t)\right\|_2^2 \leq 2V^M(t)$$

■

3 Normalhedge

NormalHedge is a particular aggregating strategy which is defined as follows.

We define a potential function that depends on two variables, x and c:

$$\phi(x,c) = \begin{cases} \exp\left(\frac{x^2}{2c}\right) & (x > 0) \\ 1 & (x \leq 0) \end{cases}$$

We will use the following partial derivatives of $\phi(x,c)$:

$$\phi'(x,c) \doteq \frac{\partial}{\partial x}\phi(x,c) = \begin{cases} \frac{x}{c}\exp\left(\frac{x^2}{2c}\right) & (x>0) \\ 0 & (x \leq 0) \end{cases} \qquad \phi''(x,c) \doteq \frac{\partial^2}{\partial x^2}\phi(x,c) = \begin{cases} \left(\frac{1}{c}+\frac{x^2}{c^2}\right)\exp\left(\frac{x^2}{2c}\right) & (x>0) \\ 0 & (x<0) \end{cases}$$

and

$$\phi^c(x,c) \doteq \frac{\partial}{\partial c}\phi(x,c) = \begin{cases} -\frac{x^2}{c^2}\exp\left(\frac{x^2}{2c}\right) & (x>0) \\ 0 & (x \leq 0) \end{cases}$$

The NormalHedge strategy is defined by the following conditions that should hold for every $t \geq 0$. If $R_t^i \leq 0$ for all $1 \leq i \leq N$ then $P_t^i = 1/N$. Otherwise P_t^i and $c(t)$ are defined by the following equations.

$$\frac{1}{N}\sum_{i=1}^{N}\phi(R_t^i, c(t)) = e \tag{9}$$

$$P_t^i = \frac{\phi'(R_t^i, c(t))}{\sum_{j=1}^{N}\phi'(R_t^j, c(t))} \tag{10}$$

4 Analysis

We introduce a new notion of regret. For a given time t we order the cumulative gains X_t^i for $i=1,\ldots,N$ from highest to lowest and define the *regret of the agregating strategy to the top ϵ-quantile* to be the difference between $G(t)$ and the $\lfloor \epsilon N \rfloor$-th element in the sorted list.

Lemma 2 *At any time t, the regret to the best instrument can be bounded as:*

$$\max_i R_{i,t} \leq \sqrt{2c(t)(\ln N + 1)}$$

Moreover, for any $0 \leq \epsilon \leq 1$ and any t, the regret to the top ϵ-quantile of instruments is at most

$$\sqrt{2c(t)(\ln(1/\epsilon)+1)}.$$

Proof: The first part of the lemma follows from the fact that, for any $i \in E_t$,

$$\exp\left(\frac{(R_{i,t})^2}{2c(t)}\right) = \exp\left(\frac{([R_{i,t}]_+)^2}{2c(t)}\right) \leq \sum_{i'=1}^{N}\exp\left(\frac{([R_{i',t}]_+)^2}{2c(t)}\right) \leq Ne$$

which implies $R_{i,t} \leq \sqrt{2c(t)(\ln N + 1)}$.

For the second part of the lemma, let $R_{i,t}$ denote the regret of our algorithm to the instrument with the ϵN-th highest price at time t. Then, the total potential of instruments with regrets greater than or equal to $R_{i,t}$ is at least:

$$\epsilon N \exp\left(\frac{([R_{i,t}]_+)^2}{2c(t)}\right) \leq Ne$$

from which the second part of the lemma follows. ■

We quote Itô's formula, as stated in [Osk03] (Theorem 4.2.1)

Theorem 3 (Itô) *Let*

$$dX(t) = udt + vdB(t)$$

be an n-dimensional Itô process. Let $g(t,x) = (g_1(t,x),\ldots,g_p(t,x))$ be a C^2 map from $[0,\infty)\times R^n$ into R^p. The the process

$$Y(t,\omega) = g(t, X(t))$$

is again an Itô process, whose component number k, Y_k, is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j$$

where $dB_i dB_j = \delta_{i,j}dt, dB_i dt = dtdBi = 0$.

We now give the main theorem, which characterizes the rate of increase of $c(t)$.

Theorem 4 *With probability one with respect to the Weiner process*

$$\forall t, \quad \frac{dc(t)}{dt} \leq 6V^M(t)$$

Proof: We denote the potential corresponding to the ith instrument by potential by Φ_t^i, i.e.

$$\Phi_t^i = \phi(R_t^i, c(t))$$

Using Itô's formula we can derive an equation for the differential $d\Phi_t^i$:

$$\begin{aligned} d\Phi_t^i &= \left[\frac{dc(t)}{dt}\phi^c(R_t^i, c(t)) + a^i(t)\phi'(R_t^i, c(t)) + \frac{1}{2}\left(\sum_{j=1}^N (b^{i,j}(t))^2\right)\phi''(R_t^i, c(t))\right]dt + \sum_{j=1}^N b^{i,j}(t)\phi'(R_t^i, c(t))dW_t^j \\ &= \left[\frac{dc(t)}{dt}\phi^c(R_t^i, c(t)) + \frac{1}{2}\left(\sum_{j=1}^N (b^{i,j}(t))^2\right)\phi''(R_t^i, c(t))\right]dt + dR_t^i\phi'(R_t^i, c(t)) \qquad (11) \end{aligned}$$

We sum Equation (11) over all instruments. As $c(t)$ is chosen so that the average potential is constant, the differential of the average potential is zero. We thus get:

$$\begin{aligned} 0 &= \sum_{i=1}^N d\Phi_t^i \qquad (12) \\ &= \sum_{i=1}^N \left[\frac{dc(t)}{dt}\phi^c(R_t^i, c(t)) + \frac{1}{2}\left(\sum_{j=1}^N (b^{i,j}(t))^2\right)\phi''(R_t^i, c(t))\right]dt + \sum_{i=1}^N dR_t^i\phi'(R_t^i, c(t)) \qquad (13) \end{aligned}$$

From Equation (4) we know that the last term is equal to zero. Removing this term and reorganizing the equation we arrive at an expression for the rate of change of $c(t)$:

$$\frac{dc(t)}{dt} = -\frac{\sum_{i=1}^N \left(\sum_{j=1}^N (b^{i,j}(t))^2\right)\phi''(R_t^i, c(t))}{2\sum_{i=1}^N \phi^c(R_t^i, c(t))}$$

we plug in the definitions of $V^i(t)$,ϕ^c and ϕ' to get:

$$\frac{dc(t)}{dt} = \frac{\sum_{i;R_t^i>0} V^i(t)\left(\frac{1}{c(t)} + \frac{(R_t^i)^2}{c(t)^2}\right)\exp\left(\frac{(R_t^i)^2}{2c(t)}\right)}{2\sum_{i;R_t^i>0}\frac{(R_t^i)^2}{c(t)^2}\exp\left(\frac{(R_t^i)^2}{2c(t)}\right)}$$

Multiplying the enumerator and denominator by $c(t)$, using the bound $V^i(t) \leq V^M$ and denoting $x_i \doteq R_t^i/\sqrt{c(t)}$ we get the inequality

$$\frac{dc(t)}{dt} \leq V^M(t)\frac{\sum_{i;x_i>0}(1+x_i^2)e^{x_i^2/2}}{\sum_{i;x_i>0} x_i^2 e^{x_i^2/2}} \qquad (14)$$

The maximum of the ratio on the right hand side under the constraint $(1/N)\sum_{i;x_i>0} e^{x_i^2/2} = e$ is achieved when $x_i = \sqrt{2}$ for all i. Plugging this value back into equation (refeqn:final) yields the statement of the theorem. ■

5 references

There are many good sources for stochastic differential equations and the Itô calculus. One which I found particularly appealing is a set of lecture notes for a course on "Stochastic Calculus, Filtering, and Stochastic Control" by Ramon van Handel, available from the web here:

`http://www.princeton.edu/~rvan/acm217/ACM217.pdf`

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