On the relationship between fuzzy logic and four-valued relevance logic

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In fuzzy propositional logic, to a proposition a partial truth in $[0,1]$ is assigned. It is well known that under certain circumstances, fuzzy logic collapses to classical logic. In this paper, we will show that under dual conditions, fuzzy logic collapses to four-valued (relevance) logic, where propositions have truth-value `true`,`false`, `unknown`, or `contradiction`. As a consequence, fuzzy entailment may be considered as "in between" four-valued (relevance) entailment and classical entailment.

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1. INTRODUCTION

Since the introduction of fuzzy sets by Zadeh [1965], an impressive work has been carried out around them, not least the numerous studies on fuzzy logic. In classical set theory, membership of a subset S of the universe of objects U, is often viewed as a (crisp) characteristic function μ_S from U to $\{0,1\}$ (called, *valuation set*) such that

$$\mu_S(u) = \begin{cases} 1 & \text{iff } u \in S \\ 0 & \text{iff } u \notin S. \end{cases}$$

In fuzzy set theory, the valuation set is allowed to be the real interval $[0,1]$ and $\mu_S(u)$ is called the *grade of membership*. The closer the value to 1, the more u belongs to S. Of course, S is a subset of U that has no sharp boundary.

When we switch to fuzzy propositional logic, the notion of grade of membership of an element u in an universe U with respect to a fuzzy subset S over U is regarded as the *truth-value* of the proposition *"u is S"*.

In this paper we will consider a fuzzy propositional logic in which expressions are boolean combinations of simpler expressions of type $\langle A{\geq}n\rangle$ and $\langle A{\leq}n\rangle$, where A is a propositional statement having a truth-value in $[0,1]$. Both express a constraint on the truth-value of A, i.e. a lower bound and an upper bound, respectively (see, e.g. [Chen and Kundu 1996; Straccia 2000]). While it is well-known that the fuzzy entailment relation, $\approx$, is bounded *upward* by classical entailment, $\models_2$, i.e. there cannot be fuzzy entailment without classical entailment, in this paper we will

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establish that the fuzzy entailment relation is bounded *below* by the four-valued (relevance) entailment relation described in [Anderson and Belnap 1975; Belnap 1977; Dunn 1986; Levesque 1984; Straccia 1997], in which propositions have a truth-value `true,false, unknown`, or `contradiction`. As a consequence, fuzzy entailment is "in between" the four-valued logical entailment relation $\models_4$ and the classical two-valued logic entailment relation $\models_2$.

We proceed as follows. In the next section we introduce syntax, semantics of the fuzzy propositional logic considered, give main definitions, describe some basic properties and a decision procedure. In Section 3 we will present the four-valued propositional logic considered in this paper, describe its properties and present a decision procedure. Section 4 is the main part of this paper where the relations among fuzzy logic, four-valued logic and classical two-valued logic are described. Section 5 concludes.

2. A FUZZY PROPOSITIONAL LOGIC

2.1 Syntax and semantics

Our logical language has two parts. At the *objective level*, let $\mathcal{L}$ be the language of propositional logic, with connectives $\wedge, \vee$, $\neg$, and the logical constants $\perp$ (false) and $\top$ (true). We will use metavariables $A, B, C, \ldots$ and $p, q, r, \ldots$ for propositions and propositional letters, respectively.[1] $\perp, \top$, letters and their negations are called *literals* (denoted l). As we will see below, propositions will have a truth-value in $[0, 1]$.

At the *meta level*, let $\mathcal{L}^f$ be the language of *meta propositions* (denoted by ψ). $\mathcal{L}^f$ consists of *meta atoms*, i.e. expressions of type $\langle A{\geq}n\rangle$ and $\langle A{\leq}n\rangle$, where A is a proposition in $\mathcal{L}$ and $n \in [0, 1]$, the connectives $\wedge, \vee$, $\neg$ and the logical constants $\perp$ and $\top$. Essentially, a meta-atom $\langle A{\leq}n\rangle$ constrains the truth-value of A to be less or equal to n (similarly for $\geq$). But, unlike [Pavelka 1979] where the truth-value of $\langle A{\leq}n\rangle$ can be any number in $[0, 1]$, in our case $\langle A{\geq}n\rangle$ and $\langle A{\leq}n\rangle$ will have the truth-value 0 or 1. Furthermore, a *meta letter* is a meta atom of the form $\langle p{\geq}n\rangle$ and $\langle p{\leq}n\rangle$, where p is a propositional letter. $\perp, \top$, meta letters and their negations are called *meta literals*. A *meta proposition* is then any $\wedge, \vee$, $\neg$ combination of meta atoms. For instance, $(\neg\langle r \wedge s{\leq}0.6\rangle \vee \langle p \vee q{\geq}0.2\rangle) \wedge \langle r \wedge s{\leq}0.6\rangle$ is a meta proposition, while $(\langle p{\leq}0.3\rangle \geq 0.4)$ is not. We will use $\langle A{<}n\rangle$ as a short form of $\neg\langle A{\geq}n\rangle$ and similarly for $\langle A{>}n\rangle$; likewise, $\langle A{=}n\rangle$ is a short form for $\langle A{\leq}n\rangle\wedge\langle A{\geq}n\rangle$. The meta letter $\langle p{\geq}n\rangle$ is *non-trivial* if $n > 0$, and similarly for $\langle p{\leq}n\rangle$. The meta letter $\langle p{\geq}1\rangle$ corresponds to the classical letter p (p is true), and $\langle p{\leq}0\rangle$ corresponds to the classical literal $\neg p$ (p is false). Therefore, $\mathcal{L}^f$ contains $\mathcal{L}$.

The classical definitions of *Negation Normal Form* (NNF), *Conjunctive Normal Form* (CNF) and *Disjunctive Normal Form* (DNF) are easily extended to our context. For instance, a meta proposition ψ in *negation normal form* is a combination of meta literals, using the connectives $\wedge$ and $\vee$; a meta proposition ψ in *conjunctive normal form* is a conjunction of disjunction of meta literals. Similarly for the DNF case.

From a semantics point of view, an interpretation $\mathcal{I}$ is a mapping $(\cdot)^{\mathcal{I}}$ from

[1] In the following, all metavariables could have an optional subscript and superscript.

propositional letters into $[0,1]$. We extend $\mathcal{I}$ to propositions via the usual min, max and 1-complement functions: $\top^{\mathcal{I}} = 1$, $\bot^{\mathcal{I}} = 0$, $(\neg A)^{\mathcal{I}} = 1 - A^{\mathcal{I}}$, $(A \wedge B)^{\mathcal{I}} = \min\{A^{\mathcal{I}}, B^{\mathcal{I}}\}$, $(A \vee B)^{\mathcal{I}} = \max\{A^{\mathcal{I}}, B^{\mathcal{I}}\}$.

Given an interpretation $\mathcal{I}$ we will assign a boolean truth-value in $\{0,1\}$ to each meta atom in the obvious way: namely,

$$\begin{array}{l} \langle A{\geq}n\rangle^{\mathcal{I}} = 1 \quad \text{iff} \quad A^{\mathcal{I}} \geq n, \text{ and} \\ \langle A{\leq}n\rangle^{\mathcal{I}} = 1 \quad \text{iff} \quad A^{\mathcal{I}} \leq n. \end{array}$$

Finally, we assign a boolean truth-value to each meta proposition like $\langle A{\geq}n_1\rangle \vee \langle B{\leq}n_2\rangle$ using the classical method of combining truth-values and we say that an interpretation $\mathcal{I}$ *satisfies* a meta proposition ψ iff $\psi^{\mathcal{I}} = 1$; in that case, we will say that $\mathcal{I}$ is a *model* of ψ.

A *meta theory* (denoted by Σ) is a finite set of meta propositions. Given an interpretation $\mathcal{I}$ and a meta theory Σ, we say that $\mathcal{I}$ *satisfies* Σ if $\mathcal{I}$ satisfies each $\psi \in \Sigma$; in that case we say that $\mathcal{I}$ is a *model* of Σ. We say that a meta theory Σ *entails* a meta proposition ψ if every model of Σ is a model of ψ; this is denoted by $\Sigma {\approx\!\!\!\!\mid}\, \psi$. A meta proposition ψ is *valid* if it is entailed by the empty meta theory, i.e. $\emptyset {\approx\!\!\!\!\mid}\, \psi$. An example of valid meta proposition is $\langle p \vee \neg p{\geq}0.5\rangle$. Two propositions A and B are said to be *equivalent* (denoted by $A \equiv B$) if $A^{\mathcal{I}} = B^{\mathcal{I}}$, for each interpretation $\mathcal{I}$. For example, $\neg(A \wedge \neg B)$ is equivalent to $\neg A \vee B$. The equivalence of two meta propositions, $\psi \cong \psi'$, is defined similarly.

Given a meta theory Σ and a proposition A, it is of interest to compute A's best lower and upper truth-value bounds [Straccia 2000]. To this end we define the *least upper bound* and the *greatest lower bound* of A with respect to Σ (written $lub(\Sigma, A)$ and $glb(\Sigma, A)$, respectively) as $lub(\Sigma, A) = \inf\{n : \Sigma {\approx\!\!\!\!\mid}\, \langle A{\leq}n\rangle\}$ and $glb(\Sigma, A) = \sup\{n : \Sigma {\approx\!\!\!\!\mid}\, \langle A{\geq}n\rangle\}$.

2.2 Some basic properties

In order to make our paper self-contained, we recall some properties of the logic $\mathcal{L}^f$, which will be of use (see [Straccia 2000]). The first ones are straightforward: for any proposition A, B and C, $\neg\top \equiv \bot$, $A \wedge \top \equiv A$, $A \vee \top \equiv \top$, $A \wedge \bot \equiv \bot$, $A \vee \bot \equiv A$, $\neg\neg A \equiv A$, $\neg(A \wedge B) \equiv \neg A \vee \neg B$, $\neg(A \vee B) \equiv \neg A \wedge \neg B$, $(A \wedge (B \vee C)) \equiv (A \wedge B) \vee (A \wedge C)$ and $(A \vee (B \wedge C)) \equiv (A \vee B) \wedge (A \vee C)$. It can be verified that each proposition may easily be transformed, by preserving equivalence, into either $\top$, $\bot$ or a proposition in NNF, CNF and DNF in which neither $\top$ nor $\bot$ occur. Please note, we do not have $A \wedge \neg A \equiv \bot$. In general we can only say that $(A \wedge \neg A)^{\mathcal{I}} \leq 0.5$, for any interpretation $\mathcal{I}$ and similarly $(A \vee \neg A)^{\mathcal{I}} \geq 0.5$.

Concerning meta propositions, as meta propositions have a boolean truth-value, we have the equivalencies of classical propositional logic, e.g. $\psi \wedge \top \cong \psi$, $\neg\psi \wedge \psi \cong \bot$, as well as

$$\begin{array}{rcl} \langle \top{\geq}n\rangle & \cong & \top \\ \langle \top{\leq}n\rangle & \cong & \begin{cases} \top & \text{if } n = 1 \\ \bot & \text{otherwise} \end{cases} \\ \langle p{\geq}0\rangle & \cong & \top \end{array}$$

$$\begin{aligned} \langle p \leq 1 \rangle &\cong \top \\ \langle \neg A \geq n \rangle &\cong \langle A \leq 1-n \rangle \\ \langle A \wedge B \geq n \rangle &\cong \langle A \geq n \rangle \wedge \langle B \geq n \rangle \\ \langle A \vee B \geq n \rangle &\cong \langle A \geq n \rangle \vee \langle B \geq n \rangle \end{aligned} \tag{1}$$

and likewise for the cases $\leq, <$ and $>$. Therefore, each meta proposition may easily be transformed by, preserving equivalence, into $\top$, $\perp$ or into a meta proposition in NNF, CNF and DNF in which neither $\top$ nor $\perp$ occur. Since $\Sigma \approx \top, \Sigma \not\approx \perp$ (unless Σ is unsatisfiable), $glb(\Sigma, \top) = 1, glb(\Sigma, \perp) = 0, lub(\Sigma, \top) = 1, lub(\Sigma, \perp) = 0$, $\Sigma \cup \{\top\}$ and Σ share the same set of models and $\Sigma \cup \{\perp\}$ is unsatisfiable, for the rest of the paper, if not stated otherwise, *we will always assume that meta propositions are always in NNF in which neither trivial meta letters nor* $\top$ *nor* $\perp$ *occur*.

As showed in [Straccia 2000], there is a strict relation between meta propositions and classical propositions. Let us consider the following transformation $\sharp(\cdot)$ of meta propositions into propositions, where $\sharp(\cdot)$ takes the "crisp" propositional part of a meta proposition:

$$\begin{aligned} \sharp(\langle p \geq n \rangle) &\mapsto p \\ \sharp(\langle p \leq n \rangle) &\mapsto \neg p \\ \sharp(\neg \psi) &\mapsto \neg \sharp(\psi) \\ \sharp(\psi_1 \wedge \psi_2) &\mapsto \sharp(\psi_1) \wedge \sharp(\psi_2) \\ \sharp(\psi_1 \vee \psi_2) &\mapsto \sharp(\psi_1) \vee \sharp(\psi_2). \end{aligned}$$

Further, for a meta theory Σ, $\sharp(\Sigma) = \{\sharp(\psi) : \psi \in \Sigma\}$. Then the following proposition holds.

Proposition 2.1 [Straccia 2000]. *Let* Σ *be a meta theory and let* ψ *be a meta proposition:*

(*1*) *if* Σ *is unsatisfiable then* $\sharp(\Sigma)$ *is classically unsatisfiable;*
(*2*) *if* $\Sigma \approx \psi$ *then* $\sharp(\Sigma) \models_2 \sharp(\psi)$*, where* $\models_2$ *is classical entailment.*

Proposition 2.1 states that *there cannot be entailment without classical entailment*. In this sense $\approx$ is correct with respect to $\models_2$.

Example 2.2. Let Σ be the set $\Sigma = \{\langle p \geq 0.8 \rangle \vee \langle q \leq 0.3 \rangle, \langle p \leq 0.3 \rangle\}$. Let ψ be $\langle q \leq 0.6 \rangle$. It follows that $\sharp(\Sigma) = \{p \vee \neg q, \neg p\}$. It is easily verified that $\Sigma \approx \langle q \leq 0.6 \rangle$ and that $\sharp(\Sigma) \models_2 \neg q$, thereby confirming Proposition 2.1.

The converse of Proposition 2.1 does not hold in the general case.

Example 2.3. Let Σ be the set $\Sigma = \{\langle p \leq 0.5 \rangle \vee \langle q \geq 0.6 \rangle, \langle p \geq 0.3 \rangle\}$. It follows that $\sharp(\Sigma) = \{\neg p \vee q, p\}$. It is easily verified that $\sharp(\Sigma) \models_2 q$, but $\Sigma \not\approx \langle q \geq n \rangle$, for all $n > 0$.

The following result establishes the converse of Proposition 2.1. It directly relates to a similar result described in [Lee 1972]. We say that a meta proposition ψ is *normalised* iff for each meta literal ψ' occurring in ψ,

(1) if ψ' is $\langle p \geq n \rangle$ then $n > 0.5$;
(2) if ψ' is $\langle p \leq n \rangle$ then $n < 0.5$;

(3) if ψ' is $\langle p{>}n\rangle$ then $n \geq 0.5$;

(4) if ψ' is $\langle p{<}n\rangle$ then $n \leq 0.5$.

PROPOSITION 2.4 [STRACCIA 2000]. *Let Σ be a meta theory and let ψ be a meta proposition. Furthermore, we assume that each $\psi' \in \Sigma$ is normalised as well as is an equivalent NNF of $\neg\psi$. Then*

(1) Σ is satisfiable iff $\sharp(\Sigma)$ is classically satisfiable;

(2) $\Sigma{\approx}\!\!\!\!\mid\,\psi$ iff $\sharp(\Sigma) \models_2 \sharp(\psi)$, where $\models_2$ is classical entailment.

Example 2.5. Consider Example 2.2. An equivalent NNF of $\neg\psi$ is $\psi' = \langle q{>}0.6\rangle$. It is easily verified that both Σ and ψ are normalised. Indeed, both $\Sigma{\approx}\!\!\!\!\mid\,\psi$ and $\sharp(\Sigma) \models_2 \sharp(\psi)$ hold. On the other hand, in Example 2.3, Σ is not normalised, e.g. for $\langle p{\geq}0.3\rangle$ we have $0.3 < 0.5$.

Dually to normalisation, we say that a meta proposition ψ is *sub-normalised* iff for each meta literal ψ' occurring in ψ,

(1) if ψ' is $\langle p{\geq}n\rangle$ then $n \leq 0.5$;

(2) if ψ' is $\langle p{\leq}n\rangle$ then $n \geq 0.5$;

(3) if ψ' is $\langle p{>}n\rangle$ then $n < 0.5$;

(4) if ψ' is $\langle p{<}n\rangle$ then $n > 0.5$.

Furthermore, for any letter p and meta proposition ψ, let $(\max\emptyset = 0, \min\emptyset = 1)$:

$$
\begin{aligned}
p_\psi^{\geq} &= \max\{n : \langle p{\geq}n\rangle \text{ occurs in } \psi\} && (2)\\
p_\psi^{>} &= \max\{n : \langle p{>}n\rangle \text{ occurs in } \psi\} && (3)\\
p_\psi^{\leq} &= \min\{n : \langle p{\leq}n\rangle \text{ occurs in } \psi\} && (4)\\
p_\psi^{<} &= \min\{n : \langle p{<}n\rangle \text{ occurs in } \psi\}. && (5)
\end{aligned}
$$

For any p and ψ, $p_\psi^{\geq}, p_\psi^{>}$ and $p_\psi^{\leq}, p_\psi^{<}$, determine the greatest lower bound and the least upper bound which p's truth value has to satisfy, respectively. We extend the above definition to the case of meta theories as follows:

$$
\begin{aligned}
p_\Sigma^{\geq} &= \max\{p_\psi^{\geq} : \psi \in \Sigma\} && (6)\\
p_\Sigma^{>} &= \max\{p_\psi^{>} : \psi \in \Sigma\} && (7)\\
p_\Sigma^{\leq} &= \min\{p_\psi^{\leq} : \psi \in \Sigma\} && (8)\\
p_\Sigma^{<} &= \min\{p_\psi^{<} : \psi \in \Sigma\}. && (9)
\end{aligned}
$$

The following proposition holds.

PROPOSITION 2.6. *Let ψ be a sub-normalised meta proposition in NNF. Then ψ is satisfiable.*

PROOF. For any letter p consider $p_\psi^{\geq}, p_\psi^{>}, p_\psi^{\leq}$ and $p_\psi^{<}$. Since ψ is sub-normalised it follows that for each letter p, there is $\epsilon_p \geq 0$ such that

$$\underline{p} = \max\{p^{\geq}_{\psi}, p^{>}_{\psi} + \epsilon_p\} \leq \min\{p^{\leq}_{\psi}, p^{<}_{\psi} - \epsilon_p\} = \overline{p} \tag{10}$$

i.e., for each p, its greatest lower bound constraint is less or equal than its least upper bound constraint. Now, let $\mathcal{I}$ be an interpretation such that

(1) $\top^{\mathcal{I}} = 1$ and $\bot^{\mathcal{I}} = 0$;

(2) $p^{\mathcal{I}} = \underline{p}$, for all letters p .

We will show on induction on the number of connectives of ψ that $\mathcal{I}$ is an interpretation satisfying ψ.

ψ is a meta letter.

(1) Suppose ψ is a meta letter $\langle p{\geq}n\rangle$. By definition, $n \leq \underline{p}$ and, thus, $\mathcal{I}$ satisfies $\langle p{\geq}n\rangle$.

(2) Suppose ψ is a meta letter $\langle p{\leq}n\rangle$. By definition, $n \geq \overline{p} \geq \underline{p}$ and, thus, $\mathcal{I}$ satisfies $\langle p{\leq}n\rangle$.

Induction step.

(1) Suppose ψ is a meta proposition $\psi_1 \wedge \psi_2$. By induction on ψ_1 and ψ_2, $\mathcal{I}$ satisfies ψ_1 and ψ_2 and, thus, $\mathcal{I}$ satisfies ψ.

(2) The cases $\vee$ is similar.

$\Box$

The above property is easily generalised to sub-normalised meta theories. We say that a meta theory Σ is *sub-normalised* iff each element of it is.

COROLLARY 2.7. *Let Σ be a sub-normalised meta theory in NNF. Then Σ is satisfiable.*

PROOF. Similarly to Proposition 2.6, for any letter p, consider $p^{\geq}_{\Sigma}, p^{>}_{\Sigma}, p^{\leq}_{\Sigma}$ and $p^{<}_{\Sigma}$. Since Σ is sub-normalised, it follows that for each letter p, there is $\epsilon_p \geq 0$ such that $\underline{p} = \max\{p^{\geq}, p^{>} + \epsilon_p\} \leq \min\{p^{\leq}, p^{<} - \epsilon_p\} = \overline{p}$. Now, let $\mathcal{I}$ be an interpretation such that

(1) $\top^{\mathcal{I}} = 1$ and $\bot^{\mathcal{I}} = 0$;

(2) $p^{\mathcal{I}} = \underline{p}$, for all letters p .

It is easily verified that $\mathcal{I}$ satisfies Σ. $\Box$

As it happens for classical entailment, entailment in $\mathcal{L}^f$ can be reduced to satisfiability checking: indeed, for a meta theory Σ and a meta proposition ψ

$$\Sigma {\approx\!\!\!|}\, \psi \ \text{ iff } \ \Sigma \cup \{\neg\psi\} \text{ is unsatisfiable.} \tag{11}$$

We conclude this section by showing that the computation of the least upper bound can be reduced to the computation of the greatest lower bound. Let Σ be a meta theory and let A be a proposition. By (1), $\langle A{\leq}n\rangle \equiv \langle \neg A{\geq}1-n\rangle$ holds and, thus, $\Sigma {\approx\!\!\!|} \langle A{\leq}n\rangle$ iff $\Sigma {\approx\!\!\!|} \langle \neg A{\geq}1-n\rangle$ holds. Therefore,

$$\begin{aligned}
1 - lub(\Sigma, A) &= 1 - \inf\{n : \Sigma {\approx\!\!\!|} \langle A{\leq}n\rangle\} \\
&= \sup\{1 - n : \Sigma {\approx\!\!\!|} \langle A{\leq}n\rangle\} \\
&= \sup\{n : \Sigma {\approx\!\!\!|} \langle A{\leq}1-n\rangle\} \\
&= \sup\{n : \Sigma {\approx\!\!\!|} \langle \neg A{\geq}n\rangle\} \\
&= glb(\Sigma, \neg A)
\end{aligned}$$

and, thus,

$$lub(\Sigma, A) \; = \; 1 - glb(\Sigma, \neg A), \tag{12}$$

i.e. the lub can be determined through the glb (and vice-versa).
In [Straccia 2000] a simple method has been developed in order to compute the glb. The method is based on the fact that from Σ it is possible to determine a finite set $N^{\Sigma} \subset [0,1]$, where $|N^{\Sigma}|$ is $O(|\Sigma|)$, such that $glb(\Sigma, A) \in N^{\Sigma}$. Therefore, $glb(\Sigma, A)$ can be determined by computing the greatest value $n \in N^{\Sigma}$ such that $\Sigma \models\!\approx \langle A \geq n \rangle\}$. An easy way to search for this n is to order the elements of N^{Σ} and then to perform a binary search among these values.

Proposition 2.8 [Straccia 2000]. *Let Σ be a meta theory. Then $glb(\Sigma, A) \in N^{\Sigma}$, where*

$$\begin{array}{rcl} N^{\Sigma} & = & \{0, 0.5, 1\} \cup \\ & & \{n : \langle p \geq n \rangle \; or \; \langle p > n \rangle \; occurs \; in \; \Sigma\} \cup \\ & & \{1 - n : \langle p \leq n \rangle \; or \; \langle p < n \rangle \; occurs \; in \; \Sigma\}. \end{array} \tag{13}$$

For instance, for the meta theory in Example 2.2, N^{Σ} is given by $\{0, 0.5, 1\} \cup \{0.8, 0.7\}$. The value of $glb(\Sigma, A)$ can, thus, be determined in $O(\log |N^{\Sigma}|)$ entailment tests.

Note that, since for a proposition A, $\langle A < 0.5 \rangle$ is normalised, it follows from Proposition 2.4 that $\models_2 A$ iff $\models\!\approx \langle A \geq 0.5 \rangle$, i.e. the truth-value of a classical propositional tautology is greater or equal than 0.5. But, by Proposition 2.8, $glb(\emptyset, A) \in \{0, 0.5\}$ and, thus, $\models_2 A$ iff $glb(\emptyset, A) = 0.5$, i.e. a classical tautology has 0.5 as its greatest truth-value lover bound.

2.3 Decision procedure in $\mathcal{L}^f$

In this section we will present a procedure for deciding the main problem within $\mathcal{L}^f$: deciding whether a meta theory Σ is satisfiable or not (by (11), the entailment problem is solved too). We call it the *fuzzy SAT problem* in order to distinguish it from the classical SAT problem.

We recall here a simplified version of the decision procedure proposed in [Straccia 2000]. Given two meta propositions ψ_1 and ψ_2 we say that (i) ψ_1 *subsumes* ψ_2 (denoted by $subs(\psi_1, \psi_2)$) iff $\psi_1 \models\!\approx \psi_2$; and that (ii) ψ_1 and ψ_2 are *pairwise contradictory* (denoted by $ctd(\psi_1, \psi_2)$) iff $\psi_1 \models\!\approx \neg \psi_2$. For instance, $\langle p \geq 0.3 \rangle \vee \langle q \leq 0.6 \rangle$ subsumes $\langle p \geq 0.2 \rangle \vee \langle q \leq 0.9 \rangle$, while $\langle p \geq 0.3 \rangle \vee \langle q \leq 0.6 \rangle$ and $\langle p \leq 0.2 \rangle \wedge \langle q \geq 0.7 \rangle$ are pairwise contradictory. Since $\psi_1 \models\!\approx \neg \psi_2$ iff $\psi_2 \models\!\approx \neg \psi_1$ it follows that $ctd(\cdot, \cdot)$ is symmetric. By definition, $ctd(\psi_1, \psi_2)$ iff $subs(\psi_1, \neg \psi_2)$ holds which relates $ctd(\cdot, \cdot)$ to $subs(\cdot, \cdot)$. If ψ_1 and ψ_2 are two meta literals, it is quite easy to check whether $subs(\psi_1, \psi_2)$ holds, as shown in Table I, on the left. Each entry in the table specifies the condition under which ψ_1 subsumes ψ_2. We are now ready to specify the calculus. The calculus is based on the following set of rules, $\mathcal{R}^T = \{(\perp), (\wedge), (\vee)\}$, described in Table II. As usual, a deduction is represented as a tree, called *deduction tree*. A branch ϕ in a deduction tree is closed iff it contains $\perp$. A deduction tree is *closed* iff each branch in it is closed. With ϕ^M we indicate the set of meta

Table I. On the left: ψ_1 subsumes ψ_2. On the right: ψ_1 and ψ_2 pairwise contradictory.

ψ_1	ψ_2			
	$\langle p \geq m \rangle$	$\langle p > m \rangle$	$\langle p \leq m \rangle$	$\langle p < m \rangle$
$\langle p \geq n \rangle$	$n \geq m$	$n > m$	$\times$	$\times$
$\langle p > n \rangle$	$n \geq m$	$n \geq m$	$\times$	$\times$
$\langle p \leq n \rangle$	$\times$	$\times$	$n \leq m$	$n < m$
$\langle p < n \rangle$	$\times$	$\times$	$n \leq m$	$n \leq m$

ψ_1	ψ_2	
	$\langle p \geq m \rangle$	$\langle p > m \rangle$
$\langle p \leq n \rangle$	$n < m$	$n \leq m$
$\langle p < n \rangle$	$n \leq m$	$n \leq m$

Table II. Simple Tableaux inference rules for $\mathcal{L}^f$.

$$(\bot) \quad \frac{\psi, \psi'}{\bot} \qquad \text{where } \psi, \psi' \text{ are meta literals and } ctd(\psi, \psi')$$

$$(\wedge) \quad \frac{\psi_1 \wedge \psi_2}{\psi_1, \psi_2}$$

$$(\vee) \quad \frac{\psi_1 \vee \psi_2}{\psi_1 \mid \psi_2}$$

procedure $\mathsf{SAT}(\Sigma)$
Convert each $\psi \in \Sigma$ into an equivalent NNF. $\mathsf{SAT}(\Sigma)$ starts from the root labelled Σ. So, we initialise Φ with $\Phi = \{\phi\}$, where $\phi^M = \Sigma$. Φ is managed as a multiset, i.e. there could be elements in Φ which are replicated.

(1) if $\Phi = \emptyset$ then return `false` and exit;
    `/* all branches are closed and, thus, Σ is unsatisfiable */`
(2) otherwise, select a branch $\phi \in \Phi$ and remove it from Φ, i.e. $\Phi \leftarrow \Phi \setminus \{\phi\}$;
(3) try to apply a rule to ϕ with the following priority among the rules: $(\bot) \succ (\wedge) \succ (\vee)$:
  (a) if the $(\bot)$ rule is applicable to ϕ then go to step 1.
  (b) if the $(\wedge)$ rule is applicable to ϕ then expand ϕ by the application of the $(\wedge)$ rule. Let ϕ' be the resulting branch. If ϕ' is not closed then add it to Φ, i.e. $\Phi \leftarrow \Phi \cup \{\phi'\}$; Go to step 1.
  (c) if the $(\vee)$ rule is applicable to ϕ then expand ϕ by the application of the $(\vee)$ rule. Let ϕ_1 and ϕ_2 be the resulting branches. For each $\phi_i, i = 1, 2$, if ϕ_i is not closed then add it to Φ, i.e. $\Phi \leftarrow \Phi \cup \{\phi_i\}$. Go to step 1.
  (d) otherwise, if no rule is applicable to ϕ, then return `true` and exit.
    `/* φ is completed and, thus, Σ is satisfiable */`

end SAT

Fig. 1. The procedure SAT.

propositions occurring in ϕ. A meta theory Σ has a *refutation* iff each deduction tree is closed. A branch ϕ is *completed* iff it is not closed and no rule can be further applied to it. A branch ϕ is *open* iff it is not closed and not completed.

Given a meta theory Σ, the procedure $\mathsf{SAT}(\Sigma)$ described in Figure 1 determines whether Σ is satisfiable or not. $\mathsf{SAT}(\Sigma)$ starts from the root labelled Σ and applies the rules until the resulting tree is either closed or there is a completed branch. If the tree is closed, $\mathsf{SAT}(\Sigma)$ returns false, otherwise true and from the completed branch a model of Σ can be build. The set of not closed branches ϕ which may be expanded during the deduction is hold by Φ.

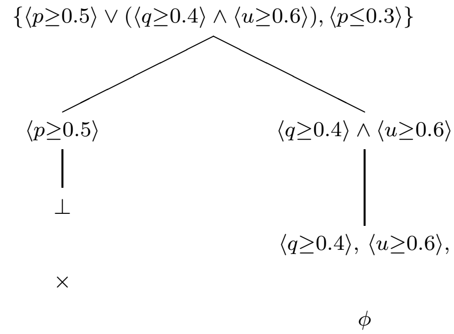

Fig. 2. Deduction tree for Σ.

Example 2.9. Let Σ be the set

$$\Sigma \;=\; \{\langle p{\geq}0.5\rangle \vee (\langle q{\geq}0.4\rangle \wedge \langle u{\geq}0.6\rangle), \langle p{\leq}0.3\rangle\}$$

Figure 2 shows a deduction tree produced by $\mathsf{SAT}(\Sigma)$. The branch on the left is closed, while the branch ϕ on the right is completed. Consider $\phi'^M \subseteq \phi^M$ where ϕ'^M contains all the meta literals occurring in ϕ^M, i.e.

$$\phi'^M \;=\; \{\langle p{\leq}0.3\rangle, \langle q{\geq}0.4\rangle, \langle u{\geq}0.6\rangle\}.$$

From ϕ'^M a model $\mathcal{I}$ of Σ can easily be build as follows: $p^{\mathcal{I}} = 0.3, q^{\mathcal{I}} = 0.4$ and $u^{\mathcal{I}} = 0.6$.

The following proposition establishing correctness and completeness of the SAT procedure.

Proposition 2.10 [Straccia 2000]. *Let Σ be a meta theory. Then* $\mathsf{SAT}(\Sigma)$ *iff Σ is satisfiable.*

3. FOUR-VALUED PROPOSITIONAL LOGIC

The four-valued propositional logic we will rely on can be found in [Anderson and Belnap 1975; Belnap 1977; Dunn 1986; Levesque 1984; Straccia 1997; Straccia 1999]. In the following we will describe briefly syntax, semantics, basic properties and a decision procedure for the four-valued entailment problem.

Expressions in four-valued propositional logic are propositions in which no $\top$ and $\perp$ appear. A theory is a set of propositions. From a semantics point of view, a *four-valued interpretation* $\mathcal{I}$ maps a proposition into an element of $2^{\{t,f\}} = \{\emptyset, \{t\}, \{f\}, \{t, f\}\}$. The four truth-values, $\emptyset, \{t\}, \{f\}, \{t, f\}$ stand for `unknown`, `true`, `false` and `contradiction`, respectively. Furthermore, $\mathcal{I}$ has to satisfy the following equations:

$$\begin{array}{lll}
t \in (A \wedge B)^{\mathcal{I}} & \text{iff} & t \in A^{\mathcal{I}} \text{ and } t \in B^{\mathcal{I}};\\
f \in (A \wedge B)^{\mathcal{I}} & \text{iff} & f \in A^{\mathcal{I}} \text{ or } f \in B^{\mathcal{I}};\\
t \in (A \vee B)^{\mathcal{I}} & \text{iff} & t \in A^{\mathcal{I}} \text{ or } t \in B^{\mathcal{I}};\\
f \in (A \vee B)^{\mathcal{I}} & \text{iff} & f \in A^{\mathcal{I}} \text{ and } f \in B^{\mathcal{I}};\\
t \in (\neg A)^{\mathcal{I}} & \text{iff} & f \in A^{\mathcal{I}}\\
f \in (\neg A)^{\mathcal{I}} & \text{iff} & t \in A^{\mathcal{I}}.
\end{array}$$

It is worth noting that a two-valued interpretation is just a four-valued interpretation $\mathcal{I}$ such that $p^{\mathcal{I}} \in \{\{t\},\{f\}\}$, for each letter p. We might characterise the distinction between two-valued and four-valued semantics as the distinction between *implicit* and *explicit* falsehood: in a two-valued logic a formula is (implicitly) false in an interpretation iff it is not true, while in a four-valued logic this need not be the case. Our truth conditions are always given in terms of belongings $\in$ (and never in terms of non belongings $\notin$) of truth-values to interpretations. Let $\mathcal{I}$ be a four-valued interpretation, let A, B be two propositions and let Σ be a theory: $\mathcal{I}$ *satisfies* (*is a model of*) A iff $t \in A^{\mathcal{I}}$; $\mathcal{I}$ *satisfies* (*is a model of*) Σ iff $\mathcal{I}$ is a model of each element of Σ; A and B are *equivalent* (written $A \equiv_4 B$) iff they have the same models; Σ *entails* B (written $\Sigma \models_4 B$) iff all models of Σ are models of B. Without loss of generality, we can restrict our attention to propositions in NNF only, as $\neg\neg A \equiv_4 A$, $\neg(A \wedge B) \equiv_4 \neg A \vee \neg B$ and $\neg(A \vee B) \equiv_4 \neg A \wedge \neg B$ hold. For easy of notation, we will write $A \models_4 B$ in place of $\{A\} \models_4 B$.

The following relations can easily be verified:

$$\begin{array}{l}
A \wedge B \models_4 A\\
A_1 \models_4 A_2 \text{ and } A_2 \models_4 A_3 \text{ implies } A_1 \models_4 A_3\\
A \models_4 A \vee B\\
A \wedge (\neg A \vee B) \not\models_4 B\\
A \models_4 B \text{ implies } \neg B \models_4 \neg A\\
A \models_4 B \text{ implies } A \models_2 B\ .
\end{array}$$

Note that there are no tautologies, i.e. there is no A such that $\models_4 A$, e.g. $\not\models_4 p \vee \neg p$ (consider $\mathcal{I}$ such that $p^{\mathcal{I}} = \emptyset$). Moreover, every theory is satisfiable. Hence, $p \wedge \neg p \not\models_4 q$, as there is a model $\mathcal{I}$ ($p^{\mathcal{I}} = \{t, f\}$, $q^{\mathcal{I}} = \emptyset$) of $p \wedge \neg p$ not satisfying q. Moreover, $\models_4$ is a subset of classical entailment $\models_2$, i.e. $\models_4$ is sound w.r.t. classical entailment.

In [Straccia 1997] a simple procedure, deciding whether $\Sigma \models_4 A$ holds, has been presented. The calculus, a tableaux, is based on *signed propositions of type* α ("conjunctive propositions") and of *type* β ("disjunctive propositions") and on their *components* which are defined as usual [Smullyan 1968]:[2]

α	α_1	α_2
$\mathtt{T}A \wedge B$	$\mathtt{T}A$	$\mathtt{T}B$
$\mathtt{NT}A \vee B$	$\mathtt{NT}A$	$\mathtt{NT}B$

β	β_1	β_2
$\mathtt{T}A \vee B$	$\mathtt{T}A$	$\mathtt{T}B$
$\mathtt{NT}A \wedge B$	$\mathtt{NT}A$	$\mathtt{NT}B$

$\mathtt{T}A$ and $\mathtt{NT}A$ are called *conjugated signed propositions*. An interpretation $\mathcal{I}$ *satisfies* $\mathtt{T}A$ iff $\mathcal{I}$ satisfies A, whereas $\mathcal{I}$ *satisfies* $\mathtt{NT}A$ iff $\mathcal{I}$ does not satisfy A. A set of signed propositions is *satisfiable* iff each element of it is satisfiable. Therefore,

[2] T and NT play the role of "True" and "Not True", respectively. In classical calculi NT may be replaced with F ("False").

Table III. Simple Tableaux inference rules for four-valued $\mathcal{L}$.

$$(\bot^4) \quad \frac{\mathtt{T}p, \mathtt{NT}p}{\bot}$$

$$(\wedge^4) \quad \frac{\alpha}{\alpha_1, \alpha_2}$$

$$(\vee^4) \quad \frac{\beta}{\beta_1 \mid \beta_2}$$

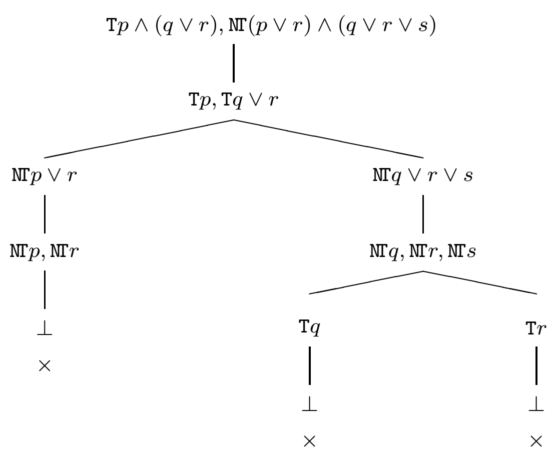

Fig. 3. Deduction tree for $p \wedge (q \vee r) \models_4 (p \vee r) \wedge (q \vee r \vee s)$.

$$\Sigma \models_4 A \text{ iff } \mathtt{T}\Sigma \cup \{\mathtt{NT}A\} \text{ is not satisfiable,} \tag{14}$$

where $\mathtt{T}\Sigma = \{\mathtt{T}A : A \in \Sigma\}$.

We present here a simplified version of the calculus for signed propositions in NNF, which is based on the set of rules, $\mathcal{R}_4^T = \{(\bot^4), (\wedge^4), (\vee^4)\}$, described in Table III. With SAT_4 we indicate the decision procedure that decides whether a set of signed propositions is (four-valued) satisfiable or not: SAT_4 derives directly from SAT in Table 1, where the deduction rules $\mathcal{R}^T$ for $\mathcal{L}^f$ have been replaced with the set of rules $\mathcal{R}_4^T$ for four-valued propositional logic.

It has been shown in [Straccia 1997] that $\Sigma \models_4 A$ iff $\mathsf{SAT}_4(\mathtt{T}\Sigma \cup \{\mathtt{NT}A\})$ returns false. For instance, Figure 3 is a closed deduction tree for $p \wedge (q \vee r) \models_4 (p \vee r) \wedge (q \vee r \vee s)$. Note that, if we switch to the classical two-valued setting, soundness and completeness is obtained by extending signed propositions as usual: just consider additionally the following signed propositions of type α.

α	α_1	α_2
$\mathtt{T}\neg A$	$\mathtt{NT}A$	$\mathtt{NT}A$
$\mathtt{NT}\neg A$	$\mathtt{T}A$	$\mathtt{T}A$

Therefore, in the general case the only difference between four-valued and two-valued semantics relies on the negation connective. This is not a surprise as we already said that the semantics for the negation is constructive, i.e. expressed in terms of $\in$ rather than on $\notin$.

The following proposition establishes correctness and completeness of the SAT_4 procedure.

Proposition 3.1 [Straccia 1997]. *Let S be a set of signed propositions. Then $\mathsf{SAT}_4(S)$ iff S is four-valued satisfiable.*

4. RELATIONS AMONG FUZZY ENTAILMENT AND FOUR-VALUED ENTAILMENT

The objective of this section is to establish some relationships between fuzzy and four-valued propositional logic.

At first, we show that

Proposition 4.1. *Let A and B be two propositions. If (i) $A \models_4 B$ or (ii) $\models_2 \neg A \wedge B$ then for all $n > 0$, $\langle A \geq n \rangle \approx \langle B \geq n \rangle$.*

Proof. (i) Assume $A \models_4 B$ and suppose to the contrary that there is an $n' > 0$ such that $\langle A \geq n' \rangle \not\approx \langle B \geq n' \rangle$. Therefore, there is a fuzzy interpretation $\mathcal{I}'$ such that $A^{\mathcal{I}'} \geq n'$ and $B^{\mathcal{I}'} < n'$. Let $\mathcal{I}$ be the following four-valued interpretation:

$$\begin{array}{l} t \in p^{\mathcal{I}} \text{ iff } p^{\mathcal{I}'} \geq n' \\ f \in p^{\mathcal{I}} \text{ iff } 1 - p^{\mathcal{I}'} \geq n'. \end{array}$$

We show on induction on the structure of a proposition C that $\mathcal{I}'$ satisfies $\langle C \geq n' \rangle$ iff $t \in C^{\mathcal{I}}$.

Case letter p. If $\mathcal{I}'$ satisfies $\langle p \geq n' \rangle$ then $p^{\mathcal{I}'} \geq n'$. By definition, $t \in p^{\mathcal{I}}$ follows. If $\mathcal{I}'$ does not satisfy $\langle p \geq n' \rangle$ then $p^{\mathcal{I}'} < n'$. By definition, $t \notin p^{\mathcal{I}}$ follows.

Case literal $\neg p$. If $\mathcal{I}'$ satisfies $\langle \neg p \geq n' \rangle$ then $1 - p^{\mathcal{I}'} \geq n'$. By definition, $f \in p^{\mathcal{I}}$ follows and, thus, $t \in \neg p^{\mathcal{I}}$. If $\mathcal{I}'$ does not satisfy $\langle \neg p \geq n' \rangle$ then $1 - p^{\mathcal{I}'} < n'$. By definition, $f \notin p^{\mathcal{I}}$ follows and, thus, $t \notin \neg p^{\mathcal{I}}$.

Case $A_1 \wedge A_2$. If $\mathcal{I}'$ satisfies $\langle A_1 \wedge A_2 \geq n' \rangle$ then ${A_1}^{\mathcal{I}'} \geq n'$ and ${A_2}^{\mathcal{I}'} \geq n'$. By induction on A_1 and A_2, both $t \in {A_1}^{\mathcal{I}}$ and $t \in {A_2}^{\mathcal{I}}$ hold and, thus, $t \in (A_1 \wedge A_2)^{\mathcal{I}}$. The case $A_1 \vee A_2$ is similar.

As a consequence, since $\mathcal{I}'$ satisfies $\langle A \geq n' \rangle$ but not $\langle B \geq n' \rangle$, it follows that $t \in A^{\mathcal{I}}$ and $t \notin B^{\mathcal{I}}$, which is contrary to the assumption $A \models_4 B$.
(ii) Assume that $\models_2 \neg A \wedge B$ holds, i.e. $\models_2 B$ and $\models_2 \neg A$. Consider $n \in (0, 1]$. Then either $n \leq 0.5$ or $n > 0.5$. From $\models_2 B$ it follows that $glb(\emptyset, B) = 0.5$, i.e. $\approx \langle B \geq n \rangle$. Therefore, for $n \leq 0.5$, $\langle A \geq n \rangle \approx \langle B \geq n \rangle$ follows. From $\models_2 \neg A$, $glb(\emptyset, \neg A) = 0.5$ follows, i.e. $lub(\emptyset, A) = 0.5$. As a consequence, for $n > 0.5$, $\langle A \geq n \rangle$ is unsatisfiable and, thus, $\langle A \geq n \rangle \approx \langle B \geq n \rangle$ holds. Therefore, for all $n > 0$ $\langle A \geq n \rangle \approx \langle B \geq n \rangle$ holds. $\square$

Proposition 4.2. *Let A and B be two propositions. If $\not\models_2 B$ and $A \not\models_4 B$ then there is a four-valued interpretation $\mathcal{I}$ such that $t \in A^{\mathcal{I}}$, $t \notin B^{\mathcal{I}}$ and for no letter p, $p^{\mathcal{I}} = \emptyset$.*

PROOF. Since $A \not\models_4 B$, $\mathsf{SAT}_4(\{\mathtt{T}A, \mathtt{NT}B\})$ returns true. Therefore, there is a completed branch ϕ. Suppose that for each completed branch ϕ_i, $1 \le i \le b$, there is a letter p_i occurring in B such that both $\mathtt{NT}p_i \in \phi_i^M$ and $\mathtt{NT}\neg p_i \in \phi_i^M$. As a consequence, collecting all the NT expressions in branches ϕ_i, informally $\mathtt{NT}B$ is equivalent to

$$\begin{aligned} \mathtt{NT}B &\equiv \textstyle\bigvee_{i=1}^{b}(\mathtt{NT}p_i \wedge \mathtt{NT}\neg p_i \wedge \mathtt{NT}F_i) \\ &\equiv \textstyle\bigvee_{i=1}^{b}(\mathtt{NT}(p_i \vee \neg p_i \vee F_i) \\ &\equiv \mathtt{NT}\textstyle\bigwedge_{i=1}^{b}(p_i \vee \neg p_i \vee F_i) \end{aligned}$$

and, thus, B is classically equivalent to $\bigwedge_i (p_i \vee \neg p_i \vee F_i)$. It follows that $\models_2 B$, contrary to our assumption. Therefore, there is a *completed* branch ϕ such that for any letter p occurring in B,

(1) if $\mathtt{NT}p \in \phi^M$ then $\mathtt{NT}\neg p \notin \phi^M$ and $\mathtt{T}p \notin \phi^M$;
(2) if $\mathtt{NT}\neg p \in \phi^M$ then $\mathtt{NT}p \notin \phi^M$ and $\mathtt{T}\neg p \notin \phi^M$.

Let $\mathcal{I}$ be the following four-valued interpretation:

$$\begin{array}{ll} t \in p^{\mathcal{I}} & \text{iff } \mathtt{T}p \in \phi^M \\ f \in p^{\mathcal{I}} & \text{iff } \mathtt{T}\neg p \in \phi^M \\ p^{\mathcal{I}} = \{f\} & \text{iff } \mathtt{NT}p \in \phi^M \\ p^{\mathcal{I}} = \{t\} & \text{iff } \mathtt{NT}\neg p \in \phi^M. \end{array}$$

It follows that for any letter p, $p^{\mathcal{I}} \neq \emptyset$. Furthermore, it can easily be shown on induction on the structure of A and B that $\mathcal{I}$ satisfies both $\mathtt{T}A$ and $\mathtt{NT}B$. Therefore, $t \in A^{\mathcal{I}}$ and $t \notin B^{\mathcal{I}}$. □

PROPOSITION 4.3. *Let A and B be two propositions and consider $0 < n \le 0.5$. $\langle A \ge n \rangle \approx \langle B \ge n \rangle$ iff $\models_2 B$ or $A \models_4 B$.*

PROOF.
$\Rightarrow$.) Assume $0 < n \le 0.5$ and $\langle A \ge n \rangle \approx \langle B \ge n \rangle$. If $\approx \langle B \ge n \rangle$ then $\models_2 B$, by Proposition 2.1. Otherwise, $\not\approx \langle B \ge n \rangle$ implies $\not\models_2 B$ (as a NNF of $\neg\langle B \ge n \rangle$ is normalised and by Proposition 2.4). So, let us show that $A \models_4 B$. Suppose to the contrary that $A \not\models_4 B$. From Proposition 4.2, there is an interpretation $\mathcal{I}$ such that $t \in A^{\mathcal{I}}$, $t \notin B^{\mathcal{I}}$ and for no letter p, $p^{\mathcal{I}} = \emptyset$. Consider the following fuzzy interpretation $\mathcal{I}'$:

(1) if $p^{\mathcal{I}} = \{t\}$ then $p^{\mathcal{I}'} = 1$;
(2) if $p^{\mathcal{I}} = \{f\}$ then $p^{\mathcal{I}'} = 0$;
(3) if $p^{\mathcal{I}} = \{t, f\}$ then $p^{\mathcal{I}'} = 0.5$.

Let us show on induction of the structure of any proposition C and any $0 < n \le 0.5$ that $t \in C^{\mathcal{I}}$ iff $C^{\mathcal{I}'} \ge n$ holds.

Case letter p. By definition, $t \in p^{\mathcal{I}}$ implies $p^{\mathcal{I}'} = 1$ and, thus, $p^{\mathcal{I}'} \ge n$. On the other hand, $t \notin p^{\mathcal{I}}$ implies $p^{\mathcal{I}} = \{f\}$ and, thus, $p^{\mathcal{I}'} = 0$. As a consequence, $p^{\mathcal{I}'} < n$;

Case literal $\neg p$. $t \in (\neg p)^{\mathcal{I}}$ implies $f \in p^{\mathcal{I}}$. Therefore, either $p^{\mathcal{I}'} = 0$ or $p^{\mathcal{I}'} = 0.5$. As a consequence, $(\neg p)^{\mathcal{I}'} = 1 - p^{\mathcal{I}'} \ge n$ ($n \le 0.5$). On the other hand, $t \notin (\neg p)^{\mathcal{I}}$ implies $f \notin p^{\mathcal{I}}$. Therefore, $p^{\mathcal{I}} = \{t\}$ and, by definition, $p^{\mathcal{I}'} = 1$ follows. As a consequence, $(\neg p)^{\mathcal{I}'} = 1 - p^{\mathcal{I}'} = 0 < n$;

Cases $A_1 \wedge A_2$ *and* $A_1 \vee A_2$. Straightforward.

Therefore, from $t \in A^{\mathcal{I}}$ it follows that $\mathcal{I}'$ satisfies $\langle A{\geq}n\rangle$. From $t \notin B^{\mathcal{I}}$ it follows that $\mathcal{I}'$ does not satisfy $\langle B{\geq}n\rangle$, contrary to the assumption that $\langle A{\geq}n\rangle{\approx}\langle B{\geq}n\rangle$ holds.

$\Leftarrow$.) From $A \models_4 B$ and from Proposition 4.1 for all $n \in (0,1]$, $\langle A{\geq}n\rangle{\approx}\langle B{\geq}n\rangle$ follows. Otherwise, if $\models_2 B$ then ${\approx}\langle B{\geq}0.5\rangle$ and, thus, for any $0 < n \leq 0.5$ $\langle A{\geq}n\rangle{\approx}\langle B{\geq}n\rangle$. □

Proposition 4.3 can be generalised as follows. At first, we show that

PROPOSITION 4.4. *Let ψ_1 and ψ_2 be two meta propositions such that ψ_1 is subnormalised and let $n \in (0, 0.5]$. If $\psi_1{\approx}\psi_2$ then $\langle\sharp(\psi_1){\geq}n\rangle{\approx}\langle\sharp(\psi_2){\geq}n\rangle$.*

PROOF. Assume $\psi_1{\approx}\psi_2$. Mark all meta-literals in ψ_2 with *. Consider a deduction of $\mathsf{SAT}(\{\psi_1, \neg\psi_2\})$, which returns false, and let T be the deduction tree. As a consequence, all branches ϕ in T are closed.

Let us consider the following substitution, $\overline{(\cdot)}$, in each branch ϕ. For each meta literal ψ occurring in ϕ^M, (i) if $\psi = \langle p\ r\ m\rangle$ is not marked with * then for $r \in \{\geq, >\}$ replace ψ with $\langle p\ r\ n\rangle$ and for $r \in \{\leq, <\}$ replace ψ with $\langle p\ r\ 1-n\rangle$; and (ii) if $\psi = \langle p\ r\ m\rangle^*$ is marked with * then for $r \in \{\geq, >\}$ replace ψ with $\langle p{>}1-n\rangle$ and for $r \in \{\leq, <\}$ replace ψ with $\langle p{<}n\rangle$. $\perp$ is mapped into it. Let $\overline{\psi}$ and $\overline{\phi}$ be the result of this substitution, for each meta proposition ψ and for each (closed) branch ϕ, respectively.

We show on induction of the depth d of each branch ϕ in the deduction tree T, that $\overline{\phi}$ is a branch in a deduction tree of $\mathsf{SAT}(\{\langle\sharp(\psi_1){\geq}n\rangle, \langle\sharp(\psi_2){<}n\rangle\})$ and, thus, the tree $\overline{T}$ formed out by the (closed) branches $\overline{\phi}$, for ϕ branch in T, is a closed deduction tree for $\mathsf{SAT}(\{\langle\sharp(\psi_1){\geq}n\rangle, \langle\sharp(\psi_2){<}n\rangle\})$. Therefore, $\langle\sharp(\psi_1){\geq}n\rangle{\approx}\langle\sharp(\psi_2){\geq}n\rangle$.

Case $d = 1$. Therefore, there is an unique closed branch ϕ in T as the result of the application of the $(\perp)$ rule, i.e. $\phi^M = \{\psi_1, \neg\psi_2, \perp\}$. Since ϕ is closed, $ctd(\psi_1, \neg\psi_2)$. There are eight possible cases for $r, r' \in \{\geq, >, \leq, <\}$ such that $\psi_1 = \langle p\ r\ k\rangle$, $\neg\psi_2 \cong \langle p\ r'\ m\rangle^*$ and $ctd(\psi_1, \neg\psi_2)$. Let us consider the cases (a) $\psi_1 = \langle p{\geq}k\rangle$, $\neg\psi_2 \cong \langle p{\leq}m\rangle^*$. By definition, $\overline{\phi}^M$ is $\{\langle p{\geq}n\rangle, \langle p{<}n\rangle^*, \perp\}$. Therefore, $\overline{\phi}$ is a closed branch of a deduction tree for $\mathsf{SAT}(\{\langle\sharp(\psi_1){\geq}n\rangle, \langle\sharp(\psi_2){<}n\rangle\})$ $= \mathsf{SAT}(\{\langle p{\geq}n\rangle, \langle p{<}n\rangle\})$; (b) $\psi_1 = \langle p{\leq}k\rangle$, $\neg\psi_2 \cong \langle p{\geq}m\rangle^*$. By definition, $\overline{\phi}^M$ is $\{\langle p{\leq}1-n\rangle, \langle p{>}1-n\rangle^*, \perp\}$. Therefore, $\overline{\phi}$ is a closed branch of a deduction tree for $\mathsf{SAT}(\{\langle\sharp(\psi_1){\geq}n\rangle, \langle\sharp(\psi_2){<}n\rangle\}) = \mathsf{SAT}(\{\langle\neg p{\geq}n\rangle, \langle\neg p{<}n\rangle\})$. The other cases can be shown similarly.

Case $d > 1$. Consider a branch ϕ of depth $d > 1$. ϕ is the result of the application of one of the rules in $\mathcal{R}^T$ to a branch ϕ' of depth $d-1$. On induction on ϕ', $\overline{\phi'}$ is a branch in a deduction tree of $\mathsf{SAT}(\{\langle\sharp(\psi_1){\geq}n\rangle, \langle\sharp(\psi_2){<}n\rangle\})$. Let us show that $\overline{\phi}$ is still a branch in a deduction tree of $\mathsf{SAT}(\{\langle\sharp(\psi_1){\geq}n\rangle, \langle\sharp(\psi_2){<}n\rangle\})$. (1) Suppose that rule $(\wedge)$ has been applied to $\psi \wedge \psi' \in \phi'^M$ and, thus, $\psi, \psi' \in \phi^M$. By definition of $\overline{(\cdot)}$, $\overline{(\psi \wedge \psi')}$ is in $\overline{\phi'}^M$, i.e. $\overline{\psi} \wedge \overline{\psi'}$ is in $\overline{\phi'}^M$. As a consequence, the $(\wedge)$ rule can be applied to it and, thus, $\overline{\psi}$ and $\overline{\psi'}$ are in $\overline{\phi'}^M$. (2) the case of rule $(\vee)$ is similar. Finally, (3) suppose that rule $(\perp)$ has been applied to literals $\psi, \psi' \in \phi'^M$ such that $ctd(\psi, \psi')$ and $\perp \in \phi^M$. By definition of $\overline{(\cdot)}$, $\overline{\psi}$ and $\overline{\psi'}$ are in $\overline{\phi'}^M$. Now,

we proceed similarly to the case $d = 1$. As ψ_1 is sub-normalised, either ψ or ψ' has to be marked with *. Without loss of generality, we can distinguish two cases (i) only ψ' is marked with *; and (ii) both ψ and ψ' are marked with *. Let us consider case (i). There are eight possible cases for $r, r' \in \{\geq, >, \leq, <\}$ such that $\psi = \langle p\ r\ n\rangle$, $\psi' = \langle p\ r'\ m\rangle^*$ and $ctd(\psi, \psi')$. Let us consider the case (a) $\psi = \langle p{\geq}k\rangle$, $\psi' = \langle p{\leq}m\rangle^*$. By definition, $\overline{\phi'}^M$ contains both $\langle p{\geq}n\rangle$ and $\langle p{<}n\rangle^*$, which are pairwise contradictory. Therefore, rule $(\perp)$ can be applied to $\overline{\phi'}$ and $\overline{\phi}^M$ contains $\perp$ and, thus, ϕ is closed; (b) $\psi = \langle p{\leq}k\rangle$, $\psi' = \langle p{\geq}m\rangle^*$. By definition, $\overline{\phi'}^M$ contains both $\langle p{\leq}1-n\rangle$ and $\langle p{>}1-n\rangle^*$, which are pairwise contradictory. Therefore, rule $(\perp)$ can be applied to $\overline{\phi'}$ and $\overline{\phi}^M$ contains $\perp$ and, thus, $\overline{\phi}$ is closed. The other cases are similar. Finally, consider the case (ii) both ψ and ψ' are marked with *. Without loss of generality, there are four possible cases for $r, r' \in \{\geq, >, \leq, <\}$ such that $\psi = \langle p\ r\ n\rangle^*$, $\psi' = \langle p\ r'\ m\rangle^*$ and $ctd(\psi, \psi')$. Let us consider the case $\psi = \langle p{\geq}k\rangle^*$, $\psi' = \langle p{\leq}m\rangle^*$. By definition, $\overline{\phi'}^M$ contains both $\langle p{>}1-n\rangle$ and $\langle p{<}n\rangle^*$, which are pairwise contradictory, for $n \in (0, 0.5]$. Then proceed similarly as above. The other cases are similar.
□

Note that the converse of the above proposition does not hold. For instance, given $n \in (0, 0.5]$, $\langle p{\geq}n\rangle {\approx\!\!\!\!\models} \langle p \vee q{\geq}n\rangle$, but $\langle p{\geq}0.2\rangle \not\approx\!\!\!\!\models \langle p{\geq}0.3\rangle \vee \langle p{\geq}0.1\rangle$.

Proposition 4.5. *Let ψ_1 and ψ_2 be two meta propositions such that ψ_1 is sub-normalised. If $\psi_1 {\approx\!\!\!\!\models} \psi_2$ then either $\models_2 \sharp(\psi_2)$ or $\sharp(\psi_1) \models_4 \sharp(\psi_2)$.*

Proof. From hypothesis, from Proposition 4.3 and Proposition 4.4 it follows immediately that either $\models_2 \sharp(\psi_2)$ or $\sharp(\psi_1) \models_4 \sharp(\psi_2)$ holds. □

As a meta theory is equivalent to a conjunction of meta propositions, we have immediately,

Proposition 4.6. *Let Σ be a sub-normalised meta theory and let ψ be a meta proposition. If $\Sigma {\approx\!\!\!\!\models} \psi$ then either $\models_2 \sharp(\psi)$ or $\sharp(\Sigma) \models_4 \sharp(\psi)$.*

The converse of the above propositions does not hold. Indeed, $p \models_4 p \vee q$, but $\langle p{\geq}0.2\rangle \not\approx\!\!\!\!\models \langle p{\geq}0.3\rangle \vee \langle p{\geq}0.1\rangle$.

Dually to Proposition 4.3 we have

Proposition 4.7. *Let A and B be two propositions and consider $0.5 < n \leq 1$. $\langle A{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$ iff for a DNF $A_1 \vee \ldots \vee A_l$ of A and for each $j = 1, \ldots l$, either (i) $\models_2 \neg A_j$ or (ii) $A_j \models_4 B$.*

Proof.

$\Rightarrow$.) Assume $n > 0.5$ and $\langle A{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$. Consider a DNF $A_1 \vee \ldots \vee A_l$ of A. From $\langle A{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$ it follows that $\bigvee_{j=1}^{l} \langle A_j{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$ and, thus, for each $j = 1, \ldots l$, $\langle A_j{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$, i.e. $S_j = \{\langle A_j{\geq}n\rangle, \langle B{<}n\rangle\}$ is unsatisfiable. Mark all the meta literals in a NNF of B with *. Let $\phi_1, \ldots, \phi_h$ be all the branches of a deduction of $\mathsf{SAT}(S_j)$. Consider a branch ϕ_i. Obviously, ϕ_i is closed. Therefore, there are meta literals $\psi_i, \psi'_i \in \phi_i^M$ such that $ctd(\psi_i, \psi'_i)$. Consider the four pairs for ψ_i and ψ'_i, respectively:

$$\begin{aligned} \langle p{\geq}n\rangle &\ , \ \langle p{\leq}1-n\rangle & (15)\\ \langle p{\geq}n\rangle &\ , \ \langle p{<}n\rangle^* & (16)\\ \langle p{\leq}1-n\rangle &\ , \ \langle p{>}1-n\rangle^* & (17)\\ \langle p{<}n\rangle^* &\ , \ \langle p{>}1-n\rangle^* & (18)\end{aligned}$$

At first, if (15) is the case ($n > 0.5$) then A_j is unsatisfiable, i.e. $\models_2 \neg A_j$ and, thus, condition (i) is satisfied. Second, (18) cannot be the case as $n > 0.5$. So, for the other cases, we can assume that A_j is satisfiable.

Let us consider the following transformation $\mathtt{S}\sharp(\cdot)$ for each branch ϕ_i. For each meta literal ψ occurring in ϕ^M, (i) if $\psi = \langle p{\geq}n\rangle$ is not marked with * then $\psi \mapsto \mathtt{T}p$; (ii) if $\psi = \langle p{\leq}1-n\rangle$ is not marked with * then $\psi \mapsto \mathtt{T}\neg p$; (iii) if $\psi = \langle p{<}n\rangle^*$ is marked with * then $\psi \mapsto \mathtt{NT}p$; and (iv) if $\psi = \langle p{>}1-n\rangle^*$ is marked with * then $\psi \mapsto \mathtt{NT}\neg p$. Let $\mathtt{S}\sharp(\psi)$, $\mathtt{S}\sharp(\phi)$ and $\mathtt{S}\sharp(S)$ be the result of this transformation, for each meta proposition ψ, for each branch ϕ_i and for each set of meta propositions S, respectively.

Similarly to Proposition 4.4, it can be shown on induction of the depth d of each branch ϕ_i of a deduction $\mathsf{SAT}(S_j)$, that the branch $\mathtt{S}\sharp(\phi_i)$ is a closed branch of a four-valued deduction $\mathsf{SAT}_4(\mathtt{S}\sharp(S_j))$. But, $\mathtt{S}\sharp(S_j)$ is $\{\mathtt{T}A_j, \mathtt{NT}B\}$ and, thus, $A_j \models_4 B$. In the induction proof, it suffices to show that if $ctd(\psi_i, \psi_i')$ then $ctd(\mathtt{S}\sharp(\psi_i), \mathtt{S}\sharp(\psi_i'))$, i.e. if the $(\perp)$ rule is applicable to ϕ_i then the $(\perp)^4$ rule is applicable to $\mathtt{S}\sharp(\phi_i)$. For the other rules the proof is immediate. So, as we have seen above, either case (16) or case (17) holds. If (16) is the case then $\mathtt{S}\sharp(\psi) = \mathtt{T}p$ and $\mathtt{S}\sharp(\psi') = \mathtt{NT}p$ and, thus, $ctd(\mathtt{S}\sharp(\psi_i), \mathtt{S}\sharp(\psi_i'))$. Otherwise, if (17) is the case then $\mathtt{S}\sharp(\psi) = \mathtt{T}\neg p$ and $\mathtt{S}\sharp(\psi') = \mathtt{NT}\neg p$ and, thus, $ctd(\mathtt{S}\sharp(\psi_i), \mathtt{S}\sharp(\psi_i'))$, which completes $\Rightarrow$.).

$\Leftarrow$.) Consider $n > 0.5$. It suffices to show that for each $j = 1, \ldots l$, if either $\models_2 \neg A_j$ or $A_j \models_4 B$ then $\langle A_j{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$. If $\models_2 \neg A_j$ then $\langle A_j{\geq}n\rangle$ is unsatisfiable, as $n > 0.5$ and, thus $\langle A_j{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$. Otherwise, if $A_j \models_4 B$ then, by Proposition 4.1, it follows that $\langle A_j{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$. $\square$

We conclude with

PROPOSITION 4.8. *Let A and B be two propositions, $n_1 \leq 0.5$ and $n_2 > 0.5$. It follows that, for both $n \in \{n_1, n_2\}$, $\langle A{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$ iff either (i) $A \models_4 B$; or (ii) $\models_2 \neg A \wedge B$ holds.*

PROOF. $\Rightarrow$.) Assume that for both $n \in \{n_1, n_2\}$, $\langle A{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$ holds. If $A \models_4 B$ then condition (i) is trivially satisfied. Otherwise, assume $A \not\models_4 B$. From Proposition 4.3, $\models_2 B$ follows. But then, we know that $glb(\emptyset, B) = 0.5$. As a consequence, for $n = n_2 > 0.5$ no interpretation satisfies $\langle B{\geq}n\rangle$. Therefore, since by hypothesis $\langle A{\geq}n\rangle {\approx\!\!\!\!\models} \langle B{\geq}n\rangle$ holds for $n > 0.5$, it follows that for $n > 0.5$, $\langle A{\geq}n\rangle$ has to be unsatisfiable, i.e. ${\approx\!\!\!\!\models} \langle A{<}n\rangle$ and, thus, ${\approx\!\!\!\!\models} \langle \neg A{>}1-n\rangle$, for $n > 0.5$. As a NNF of $\neg\langle \neg A{>}1-n\rangle$ is normalised, from Proposition 2.4 it follows that $\models_2 \neg A$. Therefore, condition (ii) is satisfied.[3]

[3]An example of case (ii) is the following: for $n = n_1, n_2$, where $n_1 \leq 0.5$ and $n_2 > 0.5$, $\langle p \wedge \neg p{\geq}n\rangle {\approx\!\!\!\!\models} \langle q \vee \neg q{\geq}n\rangle$.

$\Leftarrow$.) From Proposition 4.1 it follows directly that for all $n > 0$, $\langle A{\geq}n\rangle \approx\!\!\!\!\models \langle B{\geq}n\rangle$. In particular, $\langle A{\geq}n\rangle \approx\!\!\!\!\models \langle B{\geq}n\rangle$ holds for $n = n_1, n_2$, where $n_1 \leq 0.5$ and $n_2 > 0.5$. $\Box$

An interesting application of the above proposition is the following. Consider the quite natural and common fuzzy entailment relation, $\models\!\!\!\approx_r$, among propositions, defined as follows (see, e.g. [Xiachun et al. 1995; Yager 1985]):

$$A \models\!\!\!\approx_r B \text{ iff } \text{ for all fuzzy interpretations } \mathcal{I},\ A^{\mathcal{I}} \leq B^{\mathcal{I}}.$$

Now, it is quite easy to show that

Proposition 4.9. *Let A and B be two propositions. It follows that $A \models\!\!\!\approx_r B$ iff for all $n > 0$, $\langle A{\geq}n\rangle \models\!\!\!\approx_r \langle B{\geq}n\rangle$.*

Proof. $\Rightarrow$.) Assume that $A \models\!\!\!\approx_r B$. Suppose to the contrary that $\exists n > 0$ such that $\langle A{\geq}n\rangle \not\models\!\!\!\approx \langle B{\geq}n\rangle$. Therefore, there is a fuzzy interpretation $\mathcal{I}$ such that $A^{\mathcal{I}} \geq n$ and $B^{\mathcal{I}} < n$. But, from the hypothesis $n \leq A^{\mathcal{I}} \leq B^{\mathcal{I}} < n$ follows. Absurd.

$\Leftarrow$.) Assume that for all $n > 0$, $\langle A{\geq}n\rangle \models\!\!\!\approx \langle B{\geq}n\rangle$. Suppose to the contrary that $A \not\models\!\!\!\approx_r B$. Therefore, there is a fuzzy interpretation $\mathcal{I}$ such that $A^{\mathcal{I}} > B^{\mathcal{I}}$. Consider $\overline{n} = A^{\mathcal{I}}$. Of course, $\mathcal{I}$ satisfies $\langle A{\geq}\overline{n}\rangle$. Therefore, from the hypothesis it follows that $\mathcal{I}$ satisfies $\langle B{\geq}\overline{n}\rangle$, i.e. $B^{\mathcal{I}} \geq \overline{n} = A^{\mathcal{I}} > B^{\mathcal{I}}$. Absurd. $\Box$

Finally, we can apply Proposition 4.8 and Proposition 4.1 and obtain

Corollary 4.10. *Let A and B be two propositions. It follows that $A \models\!\!\!\approx_r B$ iff either (i) $A \models_4 B$; or (ii) $\models_2 \neg A \wedge B$ holds.*

Essentially, Corollary 4.10 establishes that *for all interesting cases, i.e. the theory A is classically satisfiable and the conclusion B is not a classical tautology, fuzzy entailment $\models\!\!\!\approx_r$ is equivalent to four-valued entailment* $\models_4$. In particular, Yager [1985] further restricts $\models\!\!\!\approx_r$ to the case where the premise should be classically satisfiable and, thus, from Corollary 4.10 it follows that $A \models\!\!\!\approx_r B$ iff $A \models_4 B$. In fact, a closer look to the axiomatization provided by Yager reveals that it is a (not minimal) axiomatization for four-valued logic.

Finally, some alternative, still popular, definitions of fuzzy entailment are (see, e.g. [Xiachun et al. 1995]):

(1) $A \models\!\!\!\approx_a B$ iff for all fuzzy interpretations $\mathcal{I}$, $\max\{1 - A^{\mathcal{I}}, B^{\mathcal{I}}\} \geq 0.5$;

(2) $A \models\!\!\!\approx_b B$ iff for all fuzzy interpretations $\mathcal{I}$, $A^{\mathcal{I}} \geq 0.5$ implies $B^{\mathcal{I}} \geq 0.5$;

(3) $A \models\!\!\!\approx_c B$ iff for all fuzzy interpretations $\mathcal{I}$, $A^{\mathcal{I}} > 0.5$ implies $B^{\mathcal{I}} > 0.5$.

The following relations are easily verified.

(1) $A \models\!\!\!\approx_a B$ iff $\models\!\!\!\approx \langle \neg A \vee B{\geq}0.5\rangle$. As $\langle \neg A \vee B{\geq}0.5\rangle$ is normalised, we already know that this is equivalent to $\models_2 \neg A \vee B$, i.e. $A \models_2 B$. Therefore, $A \models\!\!\!\approx_a B$ iff $A \models_2 B$. This result already has been proven differently in [Lee 1972];

(2) $A \models\!\!\!\approx_b B$ iff $\langle A{\geq}0.5\rangle \models\!\!\!\approx \langle B{\geq}0.5\rangle$. From Proposition 4.1 and Proposition 4.3 it follows that $A \models\!\!\!\approx_b B$ iff either $\models_2 B$ or $A \models_4 B$;

(3) $A \models\!\!\!\approx_c B$ iff $\langle A{>}0.5\rangle \models\!\!\!\approx \langle B{>}0.5\rangle$. According to Proposition 4.7 $A \models\!\!\!\approx_c B$ iff for a DNF $A_1 \vee \ldots \vee A_l$ of A and for each $j = 1, \ldots l$, either $\models_2 \neg A_j$ or $A_j \models_4 B$.

5. CONCLUSIONS

In this paper we have shown that there is a strict relation between various common definitions of fuzzy entailment ($\approx_{(\cdot)}$), four-valued entailment ($\models_4$) and two-valued entailment ($\models_2$). While the presented results allow to describe *qualitatively* what is inferable according to $\approx_{(\cdot)}$, neither $\models_4$ nor $\models_2$ can solve the *quantitative* aspect, e.g. the computation of the greatest lower bound, $glb(\Sigma, A)$.

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