

# TOWARDS AN INFORMATION-THEORETICALLY SAFE CRYPTOGRAPHIC PROTOCOL

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ABSTRACT. We introduce what –if some kind of group action exists– is a truly (information-theoretically) safe cryptographic communication system: a protocol which provides *zero* information to any passive adversary having full access to the channel.

## 1. THE FALSE ALGORITHM, SIMPLE VERSION

Assume Alice wants to share a secret  $s$ , which we assume for simplicity<sup>1</sup> is a non-zero rational number  $s = p/q \in \mathbb{Q}^*$ . For example,  $s$  could be the key of a symmetric key protocol, a password or even a complete message such as a pair of coordinates in a map or a time.

Alice picks another random rational  $t$  and calls  $v = (s, t)$  to the corresponding point in  $\mathbb{Q}^2$ .

She chooses a random transformation  $A \in GL_2(\mathbb{Q})$  in the linear group of  $\mathbb{Q}^2$  and computes  $v_1 = v \cdot A$ . Alice sends  $v_1$  to Bob.

Bob picks another random transformation  $B \in GL_2(\mathbb{Q})$  and computes  $v_2 = v_1 \cdot B$ , and sends  $v_2$  back to Alice. Notice that  $v_1$  gives no information to Bob or an eavesdropper (Eve) about  $s$ , because  $t$  is random and  $v_1$  can be *any* point in  $\mathbb{Q}^2$ , depending on  $t$  and  $A$ , which are both unknown to both Bob and Eve. For a similar reason, the knowledge of  $v_1$  and  $v_2$  gives no useful information about  $B$ .

Alice now computes  $v_3 = v_2 \cdot A^{-1}$  and sends  $v_3$  back to Bob. Again, the knowledge of  $v_1$ ,  $v_2$  and  $v_3$  is useless in order to retrieve the original  $v$ .

Finally, Bob computes  $v_4 = v_3 \cdot B^{-1}$ .

If only  $v_4 = v$ ...!

## 2. THE PROTOCOL “WOULD BE” SAFE

Let us assume the above algorithm ends up with  $v_4 = v$  and let us prove its safeness under this condition.

**Theorem 1.** *The above method of communication is information-theoretically safe, assuming  $v$ ,  $A$  and  $B$  (and their inverses, obviously) are kept secret. That is, the knowledge of the whole communication gives no information on the message.*

*Proof.* We only need to show that an eavesdropper which knows all the communication has no clue about what  $s$  may be. In other words, it is

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<sup>1</sup>This assumption might be relaxed, using an infinite set is for exposition reasons, see section 3.

enough to show that for any rational  $s'$ , there exist another rational number  $t'$  and matrices  $A', B'$  such that the communication between Alice and Bob is the same (i.e.  $v_1, v_2$  and  $v_3$ ). But this is trivial.  $\square$

**Remark:** The algorithm described above obviously does not work because  $GL_2(\mathbb{Q})$  is non-commutative (in general, the linear group is noncommutative for dimension greater than 1).

### 3. WHAT IS NEEDED?

A natural question comes to mind: what are the necessary conditions for a group action on a set for the above algorithm to provide a valid system? What we used above is:

- (1) A set  $S$  (either finite or infinite) (the rational plane in the example).
- (2) An action  $G \times S^2 \rightarrow S^2$  of a *commutative* group  $G$  on  $S^2$  (the group of movements of the plane in the example, which is *not* commutative). This condition means that after the above protocol is carried out completely, one always gets the original message.
- (3) Conditions on the action. At least the following ones, but more might be needed:
  - Given  $(s, t) \in S^2$  and  $g \in G$ , for any  $s' \in S$  there are  $t' \in S$  and  $g' \in G$  such that  $g \cdot (s, t) = g' \cdot (s', t')$ .
  - For any  $(s, t) \in S^2$  and  $A, B \in G$ , there are  $(s', t')$  and  $A', B' \in G$  for which the sequences in the above algorithm are the same:

$$[(s, t) \cdot A, (s, t) \cdot A \cdot B, (s, t) \cdot A \cdot B \cdot A^{-1}] = [(s', t') \cdot A', (s', t') \cdot A' \cdot B', (s', t') \cdot A' \cdot B' \cdot (A')^{-1}].$$

In fact, we do not need exactly an action of  $G$  on  $S^2$ .

**Definition 1.** Let  $G$  be a (not necessarily commutative) group acting on a set  $T$ . We say that  $t \in T$  is *comm-fixed* if  $g \cdot t = t$  for any  $g \in \text{Comm}(G)$  (the commutator of  $G$ ). A subset  $S \subset T$  is *comm-fixed* if any  $s \in S$  is comm-fixed.

It is clear that a subset  $S \subset T$  is comm-fixed if and only if, for any  $s \in S$  and any  $g, h \in G$ , one has  $s = h^{-1}g^{-1}hgs$ . From this, it follows that we do not need exactly an action of a commutative group on  $S^2$  but an action of a (not necessarily commutative) group on a set  $X \supset S^2$  for which  $S^2$  is comm-fixed and which satisfies, at least, condition (3) above.

We would like to prove two results; the first one seems relatively easy, while we have no clue (but are somewhat pessimistic) about the second one:

**Conjecture 1.** With the above conditions on  $X$ ,  $S^2$  and  $G$ , the protocol described in section 1 is information-theoretically safe.

**Question 1.** Do there exist  $X, S$  and a group  $G$  acting on  $X$  for which  $S^2 \subset X$  is comm-fixed and such that the stated conditions hold?

**Remark:** it is obvious that  $S^2$  can be changed by any set of the same cardinal.

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