

# Coupled quasi-harmonic bases

A. Kovnatsky<sup>1</sup>, M. M. Bronstein<sup>1</sup>, A. M. Bronstein<sup>2</sup>,  
 K. Glashoff<sup>3</sup>, and R. Kimmel<sup>4</sup>

<sup>1</sup>Faculty of Informatics, Università della Svizzera Italiana

<sup>2</sup>School of Electrical Engineering, Tel Aviv University

<sup>3</sup>Department of Mathematics, Hamburg University

<sup>4</sup>Department of Computer Science, Technion

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## Abstract

The use of Laplacian eigenbases has been shown to be fruitful in many computer graphics applications. Today, state-of-the-art approaches to shape analysis, synthesis, and correspondence rely on these natural harmonic bases that allow using classical tools from harmonic analysis on manifolds. However, many applications involving multiple shapes are obstructed by the fact that Laplacian eigenbases computed independently on different shapes are often incompatible with each other. In this paper, we propose the construction of common approximate eigenbases for multiple shapes using approximate joint diagonalization algorithms. We illustrate the benefits of the proposed approach on tasks from shape editing, pose transfer, correspondence, and similarity.

## 1 Introduction

It is well-established that the eigenfunctions of the Laplace-Beltrami operator (*manifold harmonics*) of a 3D shape modeled as a 2-manifold play the role of the Fourier basis in the Euclidean space [Tau95, Lév06]. Methods based on the Laplace-Beltrami operator have been used in a wide range of applications, including remeshing [Kob97, NISA06], parametrization [FH05], compression [KG00], recognition [RWP05, Rus07], and clustering. Many methods in computer graphics and geometry processing draw inspiration from the world of physics, finding analogies between physical processes such as heat diffusion [CL06] or wave propagation and the geometric properties of the shape [SOG09]. Several papers have studied consistent discretizations of the Laplace-Beltrami operator [PP93, MDSB03, WMKG08].

The influential paper of Taubin [Tau95] drew the analogy between the classical signal processing theory and manifold harmonics, showing that standard tools in signal processing such as analysis and synthesis of signals can be carried

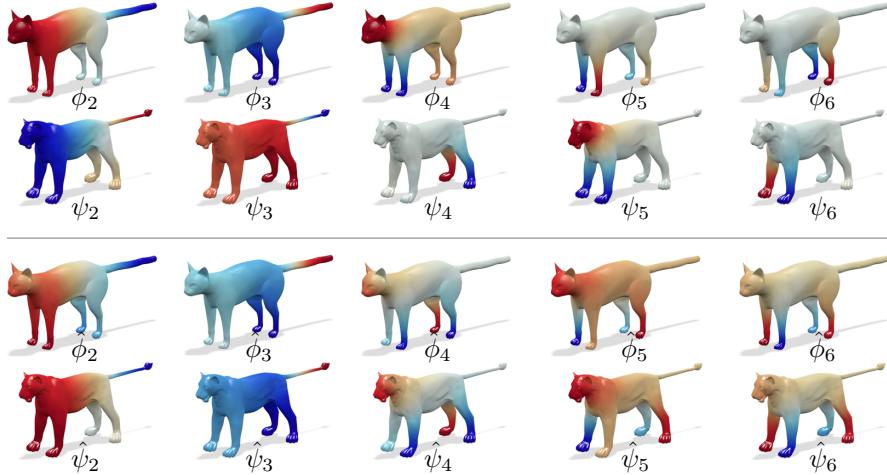


Figure 1: Top: Laplace-Beltrami eigenfunctions  $\{\phi_i\}$  and  $\{\psi_i\}$  of cat and lion shapes: besides trivial sign flips in eigenfunctions 2 and 3, higher frequency eigenfunctions (4-6) manifest quite different behaviors. Bottom: coupled basis function  $\{\hat{\phi}_i\}, \{\hat{\psi}_i\}$  obtained using approximate joint diagonalization procedure proposed in this paper behave consistently. Hot and cold colors represent positive and negative values, respectively.

out on manifolds. This idea was extended in [KR05] and later in [Lév06, LZ09], who showed a practical framework for shape filtering and editing using the manifold harmonics transform. In [OBCS<sup>+</sup>12], the authors proposed a novel representation of correspondences between shapes as linear maps between functional spaces on manifolds. In this representation, the Laplace-Beltrami eigenbases of the shapes play a crucial role, as they allow to parametrize the linear map as a matrix mapping the Fourier coefficients from one shape to another.

Applications involving multiple shapes rely on the fact that the harmonic bases computed on each shape independently are *compatible* with each other. However, this assumption is often unrealistic. First, eigenfunctions are only defined up to sign flips for shapes having simple spectra (i.e., no multiplicity of eigenvalues). In this case every vector in the eigenspace is called an eigenvector. In the more general case, eigenfunctions corresponding to an eigenvalue with non-trivial multiplicity are not defined at all, and one can select an arbitrary orthonormal basis spanning each such an eigenspace. Second, due to numerical instabilities, the ordering of the eigenfunctions, especially those representing higher frequencies, is not repeatable across shapes. Finally, harmonic bases computed independently on different shapes can be expected to be reasonably compatible only when the shapes are approximately isometric, since isometries preserve the eigenfunctions of the Laplace-Beltrami operator. When this assumption is violated, it is generally impossible to expect that the  $n$ -th harmonic of one shape will correspond to the  $n$ -th harmonic of another shape.

These drawbacks limit the use of harmonic bases in simultaneous shape analysis and processing to approximately isometric shapes, they do not allow to use high frequencies, and usually require some intervention to order the eigenfunctions or solve sign ambiguities.

**Contributions.** In this paper, we propose a general framework allowing to extend the notion of harmonic bases by finding a common (approximate) eigenbasis of multiple Laplacians. Numerically, this problem is posed as *approximate joint diagonalization* of several matrices. Such methods have received limited attention in the numerical mathematics community [BGBM93] and have been employed mostly in blind source separation applications [CS93, CS96, Yer02, Zie05]. Philosophically similar ideas have been used in the machine learning community for multi-view spectral clustering [KRDI11]. To the best of our knowledge, this is the first time they are applied to problems in computer graphics. We show a few example of applications of such coupled quasi-harmonic bases in Section 5.

## 2 Background

Let us be given a shape modeled as a compact two-dimensional manifold  $X$ . Given a smooth scalar field  $f$  on  $X$ , the negative divergence of the gradient of a scalar field,  $\Delta f = -\operatorname{div} \operatorname{grad} f$ , is called the *Laplace-Beltrami operator* of  $f$  and can be considered as a generalization of the standard notion of the Laplace operator to manifolds [Tau95, LZ09].

Since the Laplace-Beltrami operator is a positive self-adjoint operator, it admits an eigendecomposition with non-negative eigenvalues  $\lambda$  and corresponding orthonormal eigenfunctions  $\phi$ ,

$$\Delta\phi = \lambda\phi \tag{1}$$

where orthonormality is understood in the sense of the local inner product induced by the Riemannian metric on the manifold. Furthermore, due to the assumption that our manifold is compact, the spectrum is discrete,  $0 = \lambda_1 < \lambda_2 < \dots$ . In physics, (1) is known as the *Helmholtz equation* representing the spatial component of the wave equation. Thinking of our shape as of a vibrating membrane, the eigenfunctions  $\phi_i$  can be interpreted as natural vibration modes of the membrane, while the  $\lambda_i$ 's assume the meaning of the corresponding vibration frequencies. The eigenbasis of the Laplace-Beltrami operator is frequently referred to as the *harmonic basis* of the manifold, and the functions  $\phi_i$  as *manifold harmonics*.

In the discrete setting, we represent the manifold  $X$  as a triangular mesh built upon the vertex set  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ . A function  $f$  on the manifold is represented by the vector  $\mathbf{f} = (f(\mathbf{x}_1), \dots, f(\mathbf{x}_n))^T$  of its samples. A common approach to discretizing manifold harmonics is by first constructing a discrete Laplace-Beltrami operator on the mesh, represented as an  $n \times n$  matrix, followed by its eigendecomposition. A popular discretization is the cotangent scheme [MDSB03], resulting in the generalized eigenvalue problem  $\mathbf{W}\Phi = \mathbf{D}\Phi\Lambda$ ,

or equivalently, the eigenvalue problem  $\mathbf{L}\Phi = \Phi\Lambda$ , where  $\mathbf{L} = \mathbf{D}^{-1}\mathbf{W}$ ,

$$\mathbf{D} = \text{diag}(s_1, \dots, s_n)/3, \quad (2)$$

$$w_{ij} = \begin{cases} (\cot(\alpha_{ij}) + \cot(\beta_{ij}))/2 & i \neq j; \\ \sum_{k \neq i} w_{ik} & i = j, \end{cases} \quad (3)$$

$s_i > 0$  denotes the sum of the areas of all triangles sharing the vertex  $i$ , and  $\alpha_{ij}, \beta_{ij}$  are the two angles opposite to the edge between vertices  $i$  and  $j$  in the two triangles sharing the edge [WBH<sup>+</sup>07, RWP06]. Here,  $\Phi = (\phi_1, \dots, \phi_n)$  is the matrix of eigenvectors arranged as columns, discretizing the eigenfunctions of the Laplace-Beltrami operator at the sampled points. Numerically, the aforementioned eigendecomposition problem can be posed as the minimization

$$\min_{\Phi} \text{off}(\Phi^T \mathbf{W}_X \Phi) \text{ s.t. } \Phi^T \mathbf{D}_X \Phi = \mathbf{I}, \quad (4)$$

of the sum of squared off-diagonal elements,  $\text{off}(\mathbf{A}) = \sum_{i \neq j} a_{ij}^2$  [CS96]. Several classical algorithms for finding eigenvectors such as the Jacobi method in fact try to reduce the off-diagonal values in an iterative way.

Laplace-Beltrami eigenbases are equivalent to Fourier bases on Euclidean domains, and allow to represent square-integrable functions on the manifold as linear combinations of eigenfunctions, akin to Fourier analysis. In particular, solutions of PDEs on non-Euclidean domains can be expressed in the Laplacian eigenbasis, giving rise to numerous efficient methods for computing e.g. local descriptors based on fundamental solutions of heat and wave equations [SOG09, ASC11], isometric embeddings of shapes [BN03, Rus07], diffusion metrics [CL06], shape correspondence and similarity [RWP05, BBK<sup>+</sup>10, OMMG10, DK10].

Unlike the Euclidean plane where the Fourier basis is fixed, in non-Euclidean spaces, the harmonic basis depends on the domain on which it is defined. Many applications working with several shapes (such as shape matching or pose transfer) rely on the fact that harmonic basis defined on two or more different shapes are consistent and behave in a similar way [Lév06]. While experimentally it is known that often low-frequency harmonics have similar behavior (finding protrusions in shapes, a fact often employed for shape segmentation [Reu10]), there is no theoretical guarantee whatsoever of such behavior. Theoretically, consistent behavior of eigenfunctions can be guaranteed only in very restrictive settings of isometric shapes with simple spectrum [OBCS<sup>+</sup>12]. As an illustration, we give here two examples of applications in which the assumption of consistent behavior of basis functions is especially crucial. Additional applications are discussed in the Section 5.

**Functional correspondence.** Ovsjanikov et al. [OBCS<sup>+</sup>12] proposed an elegant way to avoid direct representation of correspondences as maps between shapes using a functional representation. The authors noted that when two shapes  $X$  and  $Y$  are related by a bijective correspondence  $t : X \rightarrow Y$ , then for any real function  $f : X \rightarrow \mathbb{R}$ , one can construct a corresponding function  $g : Y \rightarrow \mathbb{R}$  as  $g = f \circ t^{-1}$ . In other words, the correspondence  $t$  uniquely defines a mapping between two function spaces  $T : \mathcal{F}(X, \mathbb{R}) \rightarrow \mathcal{F}(Y, \mathbb{R})$ , where  $\mathcal{F}(X, \mathbb{R})$  denotes the space of real functions on  $X$ . Furthermore, such a mapping is *linear*.

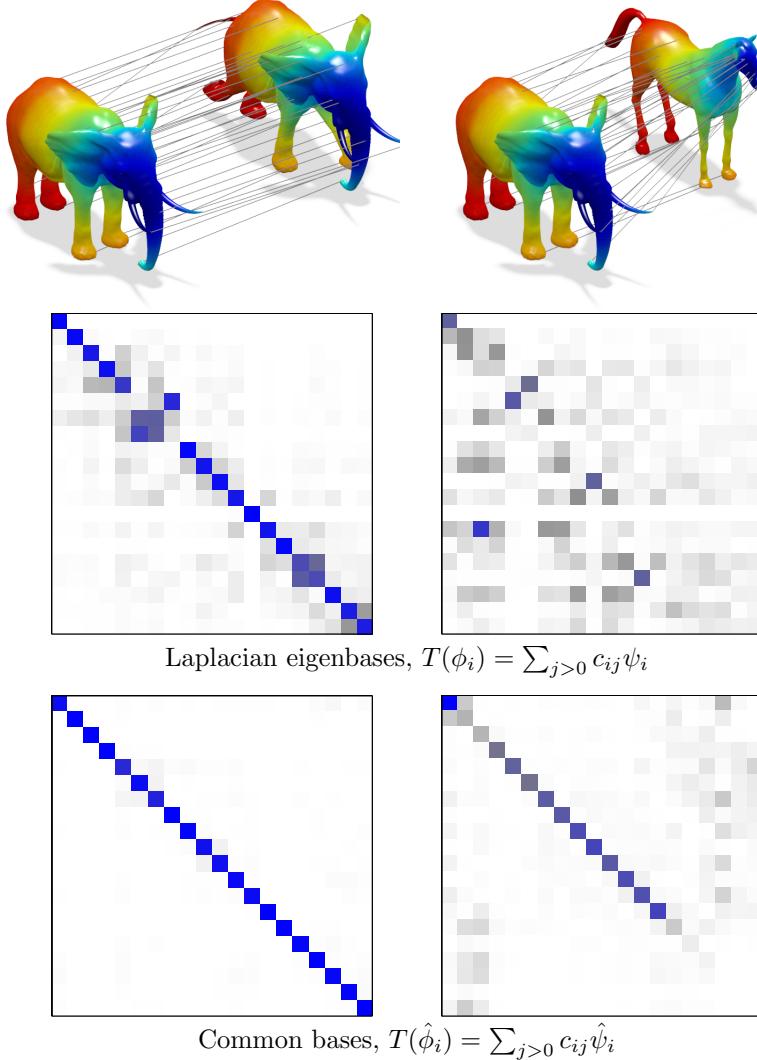


Figure 2: Matrix  $\mathbf{C}$  of coefficients expressing a given correspondence between two poses of an elephant (left) and elephant and horse (right) in functional representation of [OBCS<sup>+</sup>12]. The shapes are shown in the first row, with similar colors representing corresponding points. Second and third rows: coefficients defined using Laplacian and our coupled bases,  $T(\hat{\phi}_i) = \sum_{j>0} c_{ij} \hat{\psi}_j$

Equipping  $X$  and  $Y$  with harmonic bases,  $\{\phi_i\}_{i \geq 1}$  and  $\{\psi_j\}_{j \geq 1}$ , respectively, one can represent a function  $f : X \rightarrow \mathbb{R}$  using the set of (generalized) Fourier coefficients  $\{a_i\}_{i \geq 1}$  as  $f = \sum_{i \geq 1} a_i \phi_i$ . Then, translating the representation into the other harmonic basis, one obtains a simple representation of the correspon-

dence between the shapes

$$T(f) = \sum_{i,j \geq 1} a_i c_{ij} \psi_j, \quad (5)$$

where  $c_{ij}$  are Fourier coefficients of the basis functions of  $X$  expressed in the basis of  $Y$ , defined as  $T(\phi_i) = \sum_{j \geq 1} c_{ij} \psi_j$ . The correspondence can be thus by approximated using  $k$  basis functions and encoded by a  $k \times k$  matrix  $\mathbf{C} = (c_{ij})$  of these coefficients, referred to as the *functional matrix* in [OBCS<sup>+</sup>12]. In this representation, the computation of the shape correspondence  $t : X \rightarrow Y$  is translated into a simpler task of finding  $\mathbf{C}$  from a set of correspondence constraints. This matrix has a diagonal structure if the harmonic bases are compatible, an assumption crucial for the efficient computation of the correspondence. However, the authors report that in practice the elements of  $\mathbf{C}$  spread off the diagonal with the increase of the frequency due to the lack of perfect compatibility of the harmonic bases.

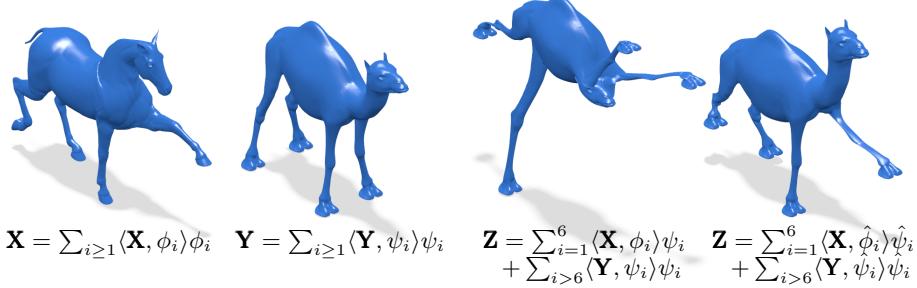


Figure 3: Pose transfer from horse (leftmost) to camel shape (second from left) by substituting the first 6 Fourier coefficients in the decomposition of extrinsic coordinates of the shape in the Laplacian eigenbasis as done in [Lév06] (third from left) and joint approximate eigenbasis (rightmost). Coupling between the basis function is crucial for this approach to work.

**Pose transfer.** Lévy [Lév06] proposed a pose transfer approach based on the Fourier decomposition of the manifold embedding coordinates. Given two shapes  $X$  and  $Y$  embedded in  $\mathbb{R}^3$  with the corresponding harmonic bases  $\{\phi_i\}_{i \geq 1}$  and  $\{\psi_i\}_{i \geq 1}$ , respectively, one first obtains the Fourier decompositions of the embeddings

$$\mathbf{X} = \sum_{i \geq 1} \mathbf{a}_i \phi_i, \quad \mathbf{Y} = \sum_{i \geq 1} \mathbf{b}_i \psi_i \quad (6)$$

(we denote by  $\mathbf{X}$  and  $\mathbf{Y}$  the Euclidean embeddings of manifolds  $X$  and  $Y$ , and by  $\mathbf{a}_i, \mathbf{b}_i$  the three-dimensional vectors of the Fourier coefficients corresponding to each embedding coordinate). Next, a new shape  $Z$  is composed according to

$$\mathbf{Z} = \sum_{j=1}^n \mathbf{a}_j \psi_j + \sum_{i>n} \mathbf{b}_i \psi_i, \quad (7)$$

with the first  $n$  low frequency coefficients taken from  $X$ , and higher frequencies taken from  $Y$ . This transfers the “layout” (pose) of the shape  $X$  to the shape  $Y$  while preserving the geometric details of  $Y$ . This method works when the  $\mathbf{a}_i$ ’s and the  $\mathbf{b}_i$ ’s are expressed in the same “language”, i.e., when the Laplacian eigenfunctions behave consistently in  $X$  and  $Y$ .

### 3 Coupled quasi-harmonic bases

Let us be given two shapes  $X, Y$  with the corresponding Laplacians  $\Delta_X, \Delta_Y$ .<sup>1</sup> If  $X$  and  $Y$  are related by an isometry  $t : X \rightarrow Y$  and have simple Laplacian spectrum (no eigenvalues with multiplicity greater than 1), the eigenfunctions are defined up to a sign flip,  $\psi_i = \pm \phi_i \circ t^{-1}$ . If some eigenvalue  $\lambda_i = \dots = \lambda_{i+p}$  has multiplicity  $p+1$ , the individual eigenvectors are not defined, but rather the subspaces they span:  $\text{span}\{\psi_i, \dots, \psi_{i+p}\} = \text{span}\{\phi_i, \dots, \phi_{i+p}\} \circ t^{-1}$ . More generally, if the shapes are not isometric, the behavior of their eigenfunctions can differ dramatically (Figure 1, top).

This poses severe limitations on applications we mentioned in the previous section. Figure 2 (middle) shows the coefficient matrix  $\mathbf{C}$  defined in (5), representing the functional correspondence between two shapes. For near-isometric shapes (two deformations of an elephant, Figure 2, middle left), since  $\psi_i \approx \pm \phi_i \circ t^{-1}$ , the coefficients  $c_{ij} \approx \pm \delta_{ij}$ , and thus the matrix  $\mathbf{C}$  is nearly diagonal. However, when trying to express correspondence between non-isometric shapes (elephant and horse, Figure 2, middle right), the Laplacian eigenfunctions manifest a very different behavior breaking this diagonality.

The same problem is observed when we try to use the technique of Lévy [Lév06] for pose transfer by expressing the embedding coordinates of the shape in the respective Laplacian eigenbasis and substituting the low-frequency coefficients from another shape. Lévy disclaims that his method works “provided that the eigenfunctions that correspond to the lower frequencies match” [Lév06]. However, such a consistent behavior is not guaranteed at all; Figure 3 (third from left) shows how the pose transfer breaks when the eigenfunction are inconsistent.

The main idea of this paper is to try to find bases  $\hat{\phi}_i, \hat{\psi}_i$  that approximately diagonalize the respective Laplacians ( $\Delta_X \hat{\phi}_i \approx \lambda_X \hat{\phi}_i$ ,  $\Delta_Y \hat{\psi}_i \approx \lambda_Y \hat{\psi}_i$ ) and are coupled ( $\hat{\psi}_i \approx \hat{\phi}_i \circ t^{-1}$ ). Such new *coupled bases*, while being nearly harmonic, make the basis functions consistent across shapes and cure the problems we outlined above (Figure 1, bottom). Figure 2 (bottom) shows the functional correspondence represented in the coupled bases  $\{\hat{\phi}_i\}_{i \geq 1}, \{\hat{\psi}_i\}_{i \geq 1}$ . Due to the coupling, the basis functions behave consistently resulting in almost perfectly diagonal matrices  $\mathbf{C}$  even when the shapes are highly non-isometric (bottom right). Likewise, Figure 3 (rightmost) shows that pose transfer using Fourier coefficients in the coupled bases works correctly for shapes with inconsistent Laplacian eigenfunctions.

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<sup>1</sup>We consider the case of a pair of shapes for the mere sake of simplicity. Extension to a collection of more shapes is straightforward.

**Approximate joint diagonalization.** We assume that the shapes are sampled at  $n_X, n_Y$  points, and their Laplacians are discretized as matrices  $\mathbf{W}_X, \mathbf{W}_Y$  and  $\mathbf{D}_X, \mathbf{D}_Y$  of size  $n_X \times n_X$  and  $n_Y \times n_Y$  respectively, as defined in (3). We further assume that we know the correspondence between  $l$  points  $x_{j_1}, \dots, x_{j_l}$  and  $y_{j'_1}, \dots, y_{j'_l}$  (as we show in Section 5, very few and very rough correspondences are required). We represent these correspondences by an  $l \times n_X$  matrix  $\mathbf{P}$  with elements  $p_{i,j_i} = 1$  and zero elsewhere; the  $l \times n_Y$  matrix  $\mathbf{Q}$  is defined accordingly as  $q_{i,j'_i} = 1$  zero elsewhere.

The problem of *joint approximate diagonalization* (JD) can be formulated as the coupling of two problems (4),

$$\begin{aligned} \min_{\hat{\Phi}, \hat{\Psi}} \quad & \text{off}(\hat{\Phi}^T \mathbf{W}_X \hat{\Phi}) + \text{off}(\hat{\Psi}^T \mathbf{W}_Y \hat{\Psi}) + \mu \|\mathbf{P}\hat{\Phi} - \mathbf{Q}\hat{\Psi}\|_F^2 \\ \text{s.t. } & \hat{\Phi}^T \mathbf{D}_X \hat{\Phi} = \mathbf{I}, \quad \hat{\Psi}^T \mathbf{D}_Y \hat{\Psi} = \mathbf{I} \end{aligned} \quad (8)$$

where off denotes some off-diagonality penalty, e.g., the sum of the squared off-diagonal elements as defined in Section 2. The parameter  $\mu$  determines the coupling strength. For  $\mu = 0$ , the problem becomes uncoupled and boils down to individual diagonalization (4) of the two Laplacians. The joint eigenvectors in this setting coincide with the eigenvectors of the Laplacians:  $\hat{\Phi} = \Phi$  and  $\hat{\Psi} = \Psi$ .

We can parametrize the joint basis functions as linear combinations of the Laplacian eigenvectors,  $\hat{\Phi} = \Phi \mathbf{A}$  and  $\hat{\Psi} = \Psi \mathbf{B}$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are matrices of combination coefficients of size  $n_X \times n_X$  and  $n_Y \times n_Y$ , respectively. Noticing that  $\hat{\Phi}^T \mathbf{W}_X \hat{\Phi} = \mathbf{A}^T \Phi^T \mathbf{W}_X \Phi \mathbf{A} = \mathbf{A}^T \Lambda_X \mathbf{A}$ , and same way,  $\hat{\Psi}^T \mathbf{W}_Y \hat{\Psi} = \mathbf{B}^T \Psi^T \mathbf{W}_Y \Psi \mathbf{B} = \mathbf{B}^T \Lambda_Y \mathbf{B}$ , we transform problem (8) into

$$\begin{aligned} \min_{\mathbf{A}, \mathbf{B}} \quad & \text{off}(\mathbf{A}^T \Lambda_X \mathbf{A}) + \text{off}(\mathbf{B}^T \Lambda_Y \mathbf{B}) + \mu \|\mathbf{P}\Phi\mathbf{A} - \mathbf{Q}\Psi\mathbf{B}\|_F^2 \\ \text{s.t. } & \mathbf{A}^T \mathbf{A} = \mathbf{I}, \quad \mathbf{B}^T \mathbf{B} = \mathbf{I} \end{aligned} \quad (9)$$

Since  $\mathbf{A}$  and  $\mathbf{B}$  are orthonormal, they act as isometries in the respective eigenspaces of the Laplacians of  $X$  and  $Y$ . We can thus think geometrically of problem (9) as an attempt to rotate and reflect the eigenbases  $\hat{\Phi}$  and  $\hat{\Psi}$  such that they align in the best way (in the least squares sense) at corresponding points, while still approximately diagonalizing the Laplacians.

Since in many applications we are not interested in the entire eigenbasis but in the first  $k$  eigenvectors, this formulation is especially convenient, as it allows us to express the first  $k$  joint eigenvectors as a linear combination of  $k'$  eigenvectors (we provide a justification of this assumption in Appendix A), thus having the matrices  $\mathbf{A}$  and  $\mathbf{B}$  of size  $k' \times k$ :

$$\begin{aligned} \min_{\mathbf{A}, \mathbf{B}} \quad & \text{off}(\mathbf{A}^T \bar{\Lambda}_X \mathbf{A}) + \text{off}(\mathbf{B}^T \bar{\Lambda}_Y \mathbf{B}) + \mu \|\mathbf{P}\bar{\Phi}\mathbf{A} - \mathbf{Q}\bar{\Psi}\mathbf{B}\|_F^2 \\ \text{s.t. } & \mathbf{A}^T \mathbf{A} = \mathbf{I}, \quad \mathbf{B}^T \mathbf{B} = \mathbf{I}, \end{aligned} \quad (10)$$

where  $\bar{\Phi} = (\phi_1, \dots, \phi_{k'})$  and  $\bar{\Lambda}_X = \text{diag}(\lambda_k^X, \dots, \lambda_{k'}^X)$ ; matrices  $\bar{\Psi}, \bar{\Lambda}_Y$  are defined accordingly. Typically,  $k, k' \ll n_X, n_Y$ , and thus the problem is much

smaller than the full joint diagonalization (8). Note that the coupling term provides  $lk$  constraints, so in order not to over-determine the problem, we should have  $2k' > l$ . Typical values used in our experiments were  $k' \sim 30$ ,  $l \sim 15$ , and  $k \sim 20$ .

**Off-diagonality penalty.** It is important to note that in problem (10) the use of the sum of squared off-diagonal elements as the off-penalty does not produce an ordered set of approximate joint eigenvectors. This issue can be solved by using an alternative penalty,

$$\|\hat{\Phi}^T \mathbf{L}_X \hat{\Phi} - \bar{\Lambda}_X\|_F^2 = \|\mathbf{A}^T \bar{\Lambda}_X \mathbf{A} - \bar{\Lambda}_X\|_F^2, \quad (11)$$

which is similar to the sum of squared off-diagonal elements but also includes the difference of the diagonal elements. The penalty for the shape  $Y$  is defined in the same way.

**Coupling term.** It is possible to change the coupling strength at different frequencies, by using a more generic form of the coupling term  $\|(\mathbf{P}\bar{\Phi}\mathbf{A} - \mathbf{Q}\bar{\Psi}\mathbf{B})\mathbf{V}\|_F^2$ , where  $\mathbf{V} = \text{diag}(v_1, \dots, v_k)$  is a diagonal matrix of weights  $v_i$  decreasing with frequency. Such frequency-dependent coupling can be useful in applications like pose transfer or functional correspondence we are interested in strong coupling at low frequencies and can afford weaker coupling of higher ones.

**Procrustes problem.** Interestingly, in the limit case  $\mu \rightarrow \infty$  where we can ignore the off-diagonal penalties, problem (10) becomes

$$\min_{\mathbf{A}, \mathbf{B}} \|\mathbf{P}\bar{\Phi}\mathbf{A} - \mathbf{Q}\bar{\Psi}\mathbf{B}\|_F^2 \quad \text{s.t.} \quad \mathbf{A}^T \mathbf{A} = \mathbf{I}, \quad \mathbf{B}^T \mathbf{B} = \mathbf{I}, \quad (12)$$

Using the invariance of the Frobenius norm under orthogonal transformation, we can rewrite problem (12) as an *orthogonal Procrustes problem*

$$\min_{\Omega} \|\mathbf{P}\bar{\Phi} - \mathbf{Q}\bar{\Psi}\Omega\|_F^2 \quad \text{s.t.} \quad \Omega^T \Omega = \mathbf{I}, \quad (13)$$

where  $\Omega = \mathbf{B}\mathbf{A}^T$ . The problem has an analytic solution  $\Omega = \mathbf{S}\mathbf{R}^T$ , where  $\bar{\Phi}^T \mathbf{P}^T \mathbf{Q}\bar{\Psi} = \mathbf{S}\Sigma\mathbf{R}^T$  is the singular value decomposition of the matrix  $\bar{\Phi}^T \mathbf{P}^T \mathbf{Q}\bar{\Psi}$  with left- and right singular vectors  $\mathbf{S}, \mathbf{R}$  [Sch66]. Then,  $\mathbf{A} = \mathbf{S}$  and  $\mathbf{B} = \mathbf{R}$ .

## 4 Numerical computation

Problem (10) is a non-linear optimization problem with orthogonality constraints. In our experiments, we used the first-order constrained minimization algorithm implemented in MATLAB Optimization Toolbox. We provide below the gradients of our cost function.

**Gradient of the off-diagonality penalty** is given by

$$\nabla_{\mathbf{A}} \sum_i (\mathbf{A}^T \bar{\Lambda}_X \mathbf{A})_{ii}^2 = 4(\bar{\Lambda}_X \mathbf{A} \mathbf{A}^T \bar{\Lambda}_X \mathbf{A} - \mathbf{O} \circ \mathbf{A} \bar{\Lambda}_X), \quad (14)$$

where  $\mathbf{O}$  is a matrix of equal columns  $\mathbf{o} = \text{diag}(\mathbf{A}^T \bar{\Lambda}_X \mathbf{A})$  and  $\circ$  denotes element-wise product of matrices (see derivation in Appendix B). Gradient of the alternative penalty (11) is derived similarly as

$$\nabla_{\mathbf{A}} \|\mathbf{A}^T \bar{\Lambda}_X \mathbf{A} - \bar{\Lambda}_X\|_F^2 = 4(\bar{\Lambda}_X \mathbf{A} \mathbf{A}^T \bar{\Lambda}_X \mathbf{A} - \bar{\Lambda}_X \mathbf{A} \bar{\Lambda}_X). \quad (15)$$

**Gradient of the coupling term** w.r.t. to  $\mathbf{A}$  is given by

$$\nabla_{\mathbf{A}} \|\mathbf{P} \bar{\Phi} \mathbf{A} - \mathbf{Q} \bar{\Psi} \mathbf{B}\|_F^2 = 2\bar{\Phi}^T \mathbf{P}^T (\mathbf{P} \bar{\Phi} \mathbf{A} - \mathbf{Q} \bar{\Psi} \mathbf{B}); \quad (16)$$

the gradient w.r.t.  $\mathbf{B}$  is obtained in the same way.

**Initialization.** Assuming the Laplacians are nearly jointly diagonalizable and have simple spectrum, their joint eigenvectors will be equal to the harmonic basis functions up to sign flips. Thus, in this case  $\mathbf{A}$  and  $\mathbf{B}$  are diagonal matrices of  $\pm 1$ . This was found to be a reasonable initialization to the joint diagonalization procedure. We set  $\mathbf{A} = \mathbf{I}$ , and then solve sign flip by setting the elements of  $\mathbf{B}$  to

$$b_{ij} = \begin{cases} +1 & i = j, \quad \|\mathbf{P}\phi_i - \mathbf{Q}\psi_i\| \leq \|\mathbf{P}\phi_i + \mathbf{Q}\psi_i\|; \\ -1 & i = j, \quad \|\mathbf{P}\phi_i - \mathbf{Q}\psi_i\| > \|\mathbf{P}\phi_i + \mathbf{Q}\psi_i\|; \\ 0 & \text{else.} \end{cases} \quad (17)$$

Initialized this way, the problem starts with decoupled but diagonalizing bases, and the optimization tries to improve the coupling.

**Band-wise computation.** In applications requiring the computation of many approximate joint eigenvectors ( $k \gg 1$ ), rather than solving problem (10) for  $k \times k'$  matrices  $\mathbf{A}, \mathbf{B}$ , we can split the eigenvectors into non-overlapping bands of size  $k''$ , and solve  $k/k''$  problems (10) for  $k'' \times k''$  matrices  $\mathbf{A}, \mathbf{B}$  (here we assume that  $k$  is a multiple of  $k''$  and that  $k'' > l$ ). For the  $i$ th band, we use  $\Phi = (\phi_{(i-1)k''}, \dots, \phi_{ik''})$  and  $\bar{\Lambda}_X = \text{diag}(\lambda_{(i-1)k''}^X, \dots, \lambda_{ik''}^X)$  and  $\Psi, \bar{\Lambda}_Y$  defined accordingly in (10).

## 5 Results and Applications

In this section, we show additional examples of coupled bases construction, as well as some potential applications of the proposed approach. We used shapes from publicly available datasets [BBK08, SP04, SMKF04]. Mesh sizes varied widely between 600 - 25K vertices. Discretization of the Laplace-Beltrami operator was done using the cotangent formula [MDSB03]. In all our examples, we constructed coupled bases solving the JD problem (10) with off-diagonality penalty (11), as described in Section 4. Typical time to compute 15 joint eigenvectors was about 1 minute.

Figure 4 (top) shows examples of joint diagonalization of Laplacians of different shapes: near isometric (two poses of an elephant) and non-isometric (elephant and horse). We computed the first  $k = 20$  joint approximate eigenvectors. The Laplacians are almost perfectly diagonalized by the obtained coupled bases

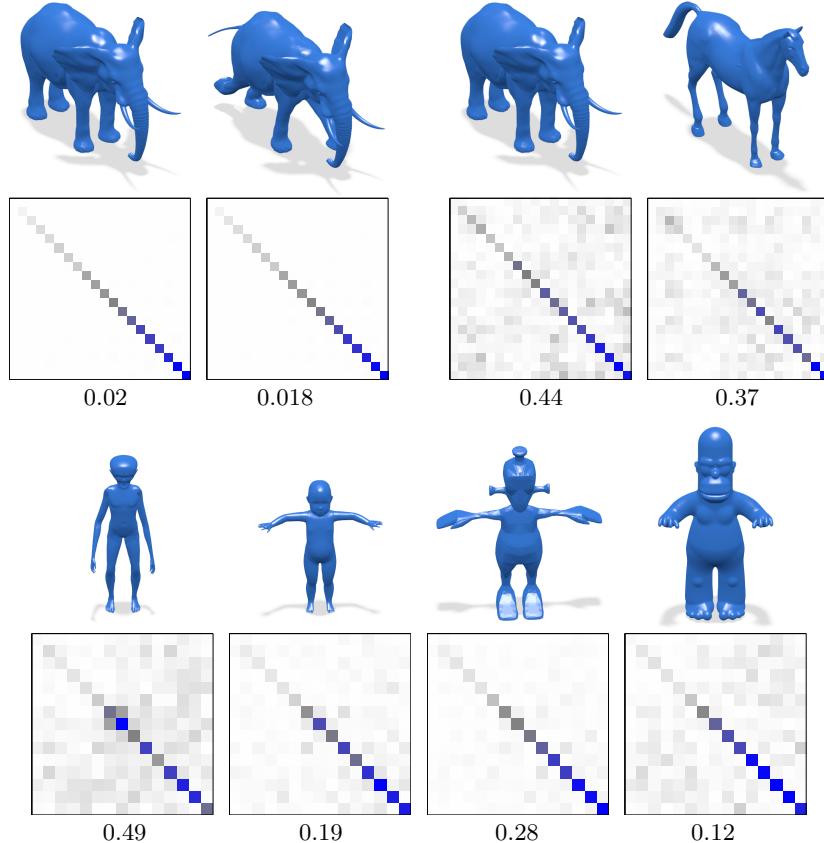


Figure 4: Examples of joint diagonalization of Laplacians of near-isometric shapes (two poses of an elephant, top left) and non-isometric shapes (elephant and horse, top right; four humanoids, bottom). Second and fourth rows show the approximate diagonalization of the respective Laplacians. Coupling was done using 40 points for the elephant and horse shapes and 25 for the humanoid shapes. Numbers show the ratio of the norm of the diagonal and off-diagonal values.

in the case of near-isometric shapes; for non-isometric shapes, off-diagonal elements are more prominent. Nevertheless, a clear diagonally-dominant structure is present. Figure 4 (bottom) shows an additional example of joint diagonalization of the Laplacian matrices of four humanoid shapes by first 15 elements of the coupled bases.

**Sensitivity to correspondence error.** Figure 5 shows the sensitivity of the presented procedure to errors in correspondence used for coupling. In this experiment, we used coupling at 10 points that deviated from groundtruth correspondence by up to 15% of the geodesic diameter of the shape. Figure 6 shows examples of the first elements of the coupled bases obtained in the lat-

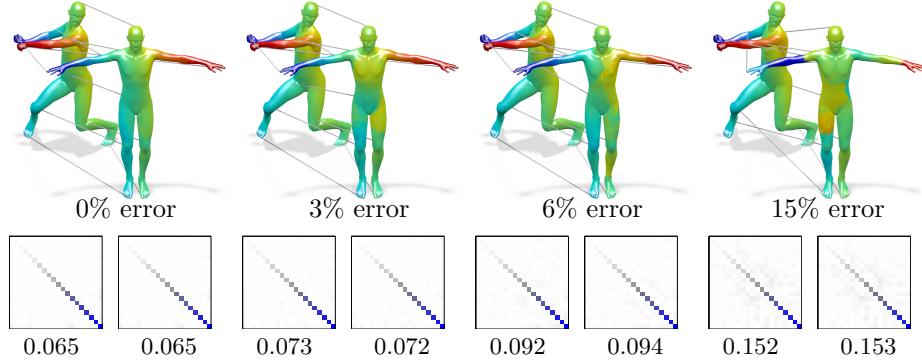


Figure 5: Sensitivity of joint diagonalization to errors in correspondence (in % of geodesic diameter of the shape) used in the coupling term. Shapes are shown with similar colors representing corresponding points. Correspondences between 10 points used for coupling are shown with lines. Numbers show the ratio of the norm of the diagonal and off-diagonal values.

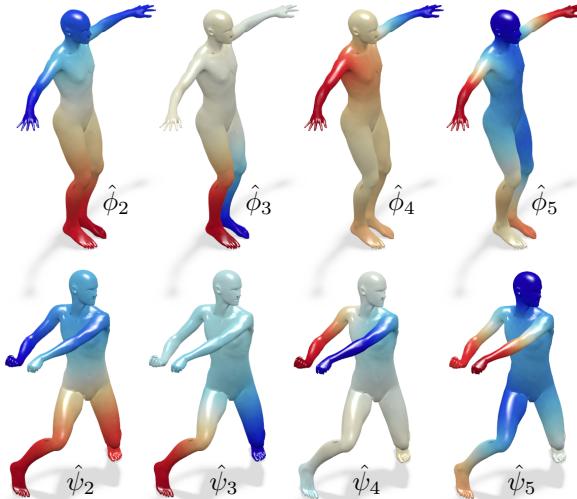


Figure 6: Coupled bases elements of the human shapes from Figure 5 obtained using 10 points with inaccurate correspondence (error of 15% geodesic diameter) for coupling.

ter setting. This experiment illustrates that very few roughly corresponding points are required for the coupling term in our problem, and that the proposed procedure is robust to correspondence noise.

**Shape correspondence.** Ovsjanikov et al. [OBCS<sup>+</sup>12] computed correspondence between near-isometric shapes from a set of constraints on the

$k \times k$  functional matrix  $\mathbf{C}$  (encoding the correspondence represented using the first  $k$  harmonic functions as described in Section 2). Given a set of functions  $f_1, \dots, f_p$  on  $X$  and corresponding functions  $g_1, \dots, g_q$  on  $Y$  (for example, indicator functions of stable regions) represented as  $n_X \times p$  and  $n_Y \times p$  matrices  $\mathbf{F} = (\mathbf{f}_1, \dots, \mathbf{f}_p)$  and  $\mathbf{G} = (\mathbf{g}_1, \dots, \mathbf{g}_p)$ , respectively,  $\mathbf{C}$  is recovered from a system of  $pk$  equations with  $k^2$  variables,

$$\mathbf{F}^T \boldsymbol{\Phi} = \mathbf{G}^T \boldsymbol{\Psi} \mathbf{C}. \quad (18)$$

Additional constraints stemming from the properties of the matrix  $\mathbf{C}$  are also added [OBCS<sup>+</sup>12].

The use of our coupled bases in place of standard Laplace-Beltrami eigenbases allows to exploit the sparse structure of  $\mathbf{C}$ , which is mentioned by Ovsjanikov et al., but never used explicitly in their paper. Approximating  $\mathbf{C} \approx \text{diag}(c_{11}, \dots, c_{kk})$ , we can rewrite (18) as a system of  $pk$  equations with only  $k$  variables corresponding to the diagonal of  $\mathbf{C}$ ,

$$\begin{bmatrix} \text{diag}(\mathbf{g}_1^T \hat{\boldsymbol{\Psi}}) \\ \vdots \\ \text{diag}(\mathbf{g}_p^T \hat{\boldsymbol{\Psi}}) \end{bmatrix} \begin{bmatrix} c_{11} \\ \vdots \\ c_{kk} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}^T \mathbf{f}_1 \\ \vdots \\ \hat{\boldsymbol{\Phi}}^T \mathbf{f}_p \end{bmatrix}. \quad (19)$$

Equation (19) allows to use significantly less data to fully determine the correspondence, and is also more computationally efficient. The method can be applied iteratively, by first establishing some (rough) correspondence to produce coupled bases, then finding  $\mathbf{C}$  by solving (19) and using the computed correspondence to refine the coupling and recompute the bases.

Figure 7 shows an example of finding functional correspondence between non-isometric shapes of human and gorilla. As functions  $\{f_i\}_{i=1}^p$  and  $\{g_i\}_{i=1}^q$  we used binary indicator functions of  $p = 25$  regions detected using the MSER algorithm [LBB11]. We compared the method described in [OBCS<sup>+</sup>12] for computing  $\mathbf{C}$  by solving the system (18) in the Laplace-Beltrami eigenbases (Figure 7, middle) and the diagonal-only approximation (19) in the coupled bases. In both cases, we used  $k = 20$  first basis vectors. Then, point-wise correspondence was obtained from  $\mathbf{C}$  using the ICP-like approach proposed in [OBCS<sup>+</sup>12].

**Simultaneous mesh editing.** Rong et al. [RCG08] proposed an approach for mesh editing based on elastic energy minimization. Given a shape with embedding coordinates  $\mathbf{X}$ , the method attempts to find a deformation field  $\mathbf{d}$  producing a new shape  $\mathbf{X}' = \mathbf{X} + \mathbf{d}$ , providing a set of user-defined  $n'$  anchor points for which the displacement is known (w.l.o.g. assuming to be the first  $n'$  points,  $\mathbf{d}_i = \mathbf{d}'_i$  for  $i = 1, \dots, n'$ ), as a solution of the system of equations

$$\begin{bmatrix} k_b \mathbf{L}_X^2 - k_c \mathbf{L}_X & \mathbf{M} \\ \mathbf{M}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d}' \end{bmatrix}, \quad (20)$$

where  $\mathbf{M} = (\mathbf{I}, \mathbf{0})^T$  is an  $n \times n'$  identity matrix,  $\boldsymbol{\gamma}$  are unknown Lagrange multipliers corresponding to the constraints on anchor points, and  $k_b, k_c$  are parameters

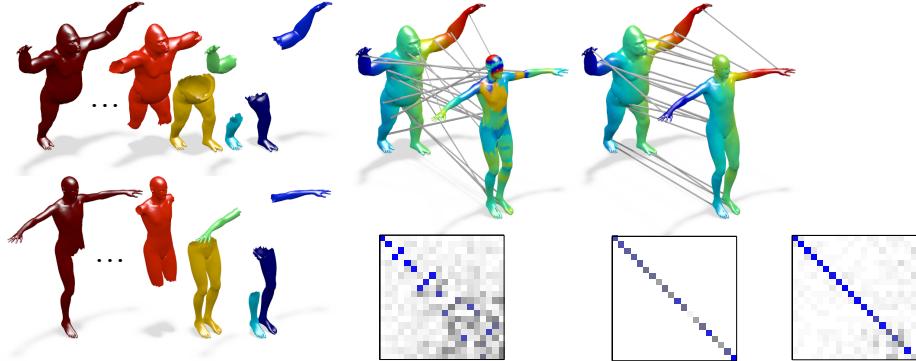


Figure 7: Functional correspondence computation. Left: some of the MSER regions. Second and third columns: correspondence and matrix  $\mathbf{C}$  computed using the method of [OBCS<sup>+</sup>12] in Laplace-Beltrami eigenbases and diagonal-only approach in coupled bases, respectively. Fourth column: groundtruth correspondence matrix  $\mathbf{C}$  represented in coupled bases.

trading off between resistance to bending and stretching, respectively [RCG08]. The system of equations can be expressed in the frequency domain using  $k \ll n$  first harmonic basis functions,

$$\begin{bmatrix} \bar{\Phi}^T(k_b \mathbf{L}_X^2 - k_c \mathbf{L}_X) \bar{\Phi} & \bar{\Phi}^T \mathbf{M} \\ \mathbf{M}^T \bar{\Phi} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{d}' \end{bmatrix} \quad (21)$$

where  $\boldsymbol{\alpha} = \bar{\Phi}^T \mathbf{d}$  are the  $k$  Fourier coefficients. The desired deformation field is obtained by solving the system of equations for  $\boldsymbol{\alpha}$  and transforming it to the spatial domain  $\mathbf{d} = \bar{\Phi} \boldsymbol{\alpha}$  (for details, the reader is referred to [RCG08]).

Using coupled bases, it is possible to easily extend this approach to simultaneous editing of multiple shapes, solving the system (21) with the coupled basis  $\hat{\Phi}$  in place of  $\bar{\Phi}$ , and applying the deformation to the second mesh using  $\mathbf{d} = \hat{\Psi} \boldsymbol{\alpha}$ . Figure 8 exemplifies this idea, showing how a deformation of the cat shape is automatically transferred to the lion shape, which accurately and naturally repeats the cat pose.

**Shape similarity.** The diagonalization quality of the Laplacians can be used as a criterion for shape similarity, with isometric shapes resulting in ideal diagonalization. With this approach, it is possible to compare two shapes from a small number of inaccurate correspondences provided for coupling in the joint diagonalization problem. It is possible to compare more than two shapes at once, with the diagonalization quality acting like some “variance” of the collection of shapes.

Figure 9 shows the similarity matrix between 25 shapes belonging to 8 different classes. Each shape is present with 3-4 near-isometric deformation. We used 25 points for coupling; dissimilarity of a pair of shapes was computed by

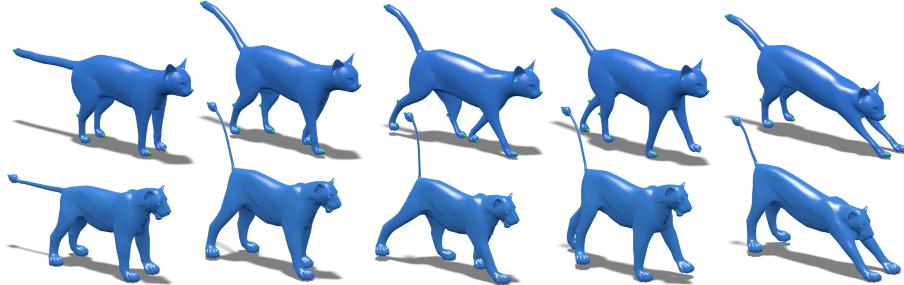


Figure 8: Simultaneous shape editing in the frequency domain using the approach of [RCG08]. Top: editing the cat shape (green points denote the anchor vertices used as constraints in problem (20)). Bottom: the same pose is automatically transferred to the lion shape using coupled basis.

jointly diagonalizing the respective Laplacians and then computing the average ratio of the norms of the diagonal and off-diagonal elements of both matrices.

## 6 Conclusions

We showed several formulations of numerically efficient joint approximate diagonalization algorithms for the construction of coupled bases of the Laplacians of multiple shapes. Such quasi-harmonic bases allow to extend many shape analysis and synthesis tasks to cases where the standard harmonic bases computed on each shape separately cease being compatible. The proposed construction can be used as an alternative to the standard harmonic bases. A particularly promising direction is non-rigid shape matching in the functional correspondence representation. The sparse structure of the functional matrix  $\mathbf{C}$  has not been yet taken advantage of for the computation of correspondence in a proper way, and naturally calls for sparse modeling methods successfully used in signal processing. However, the correctness of this model largely depends on the basis in which  $\mathbf{C}$  is represented, and standard Laplace-Beltrami eigenbasis performs quite poorly when one deviates from the isometry assumptions. In follow-up studies, we intend to explore the relation between sparse models and joint approximate diagonalization in shape correspondence problems.

## Appendix A - Perturbation analysis of JD

We analyze here the solution of the basic problem (8). For simplicity of analysis, we assume that  $n_X = n_Y = l$ , the points are ordered as  $i_k = j_k = k$ , and  $\mu \rightarrow \infty$ . This setting of the joint diagonalization problem has been considered in numerical mathematics [BGBM93] and blind source separation [CS96] literature. In this case, the matrices are of equal size, have row- and column-wise correspondence, and a single basis ( $\hat{\Phi} = \hat{\Psi}$ ) is searched.

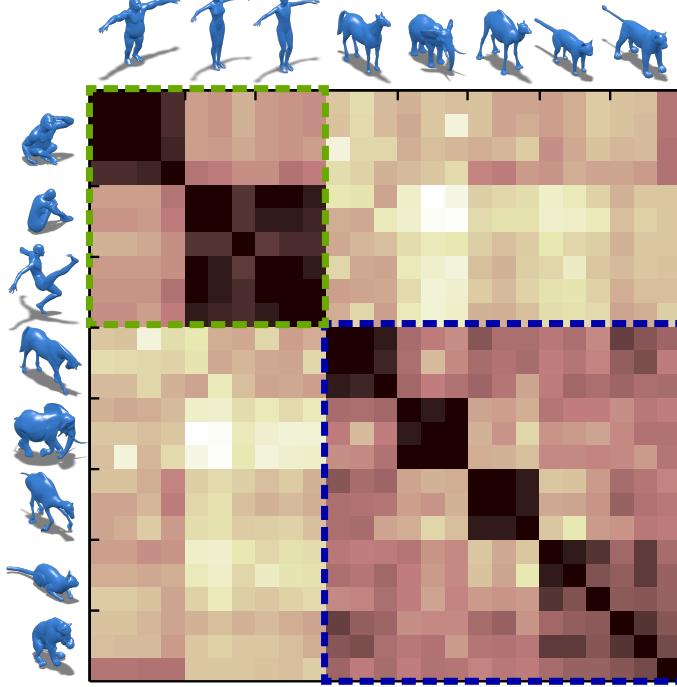


Figure 9: Shape similarity using joint diagonalization. Darker colors represent more similar shapes. One can clearly distinguish blocks of isometric shapes. Also, two classes of two- and four-legged shapes (marked with green and blue) are visible. Small figures show representative shapes from each class.

We further assume that  $\mathbf{L}_X = \Phi\Lambda\Phi^T$  has a simple  $\tau$ -separated spectrum (i.e.,  $|\lambda_i - \lambda_j| \geq \tau$ ). If  $Y$  is a near-isometric deformation of  $X$ , its Laplacian can be described as a perturbation  $\mathbf{L}_Y = \Phi\Lambda\Phi^T + \epsilon\mathbf{R}$  of  $\mathbf{L}_X$ . Ignoring permutation of eigenfunctions and sign flips, the joint approximate eigenbasis can be written as the first-order perturbation [Car95]

$$\hat{\phi}_i \approx \phi_i + \epsilon \sum_{j \neq i} \alpha_{ij} \phi_j, \quad (22)$$

where  $\alpha_{ij} = \phi_i^T \mathbf{R} \phi_j / 2(\lambda_j - \lambda_i)$ . We can bound these coefficients by the spectral norm  $|\alpha_{ij}| \leq \|\mathbf{R}\|_2 / 2\tau = \alpha_{\max}$ . We can conclude that the first  $k$  joint approximate eigenvectors  $\hat{\phi}_1, \dots, \hat{\phi}_k$  can be well represented as linear combinations of  $\phi_1, \dots, \phi_{k'}$ , with square error bounded by  $\epsilon \sum_{j > k'} |\alpha_{ij}|^2 \leq \epsilon(n_X - k' - 1)\alpha_{\max}$ . This result also justifies the use of band-wise computation discussed in Section 4.

To relate this bound to the geometry of the shape, let us assume that  $X$  and  $Y$  have the same connectivity and that the angles  $\beta, \bar{\beta}$  of triangles in the

two meshes satisfy  $\theta_0 \leq \beta, \bar{\beta} \leq \pi - \theta_0$  (all triangles are at least  $\theta_0$  fat), the area elements are at least  $s, \bar{s} \geq s_0$ , and each vertex is connected to at most  $\nu$  vertices. We assume that the mesh  $Y$  is obtained as a deformation of mesh  $X$  changing the angles by  $|\bar{\beta} - \beta| \leq \Delta\theta$  and the area elements by  $1 - \delta \leq \bar{s}/s \leq 1 + \delta$ . We are interested in a bound on the spectral norm  $\|\mathbf{L}_X - \mathbf{L}_Y\|_2 = \epsilon \|\mathbf{R}\|_2$  expressed in terms of parameters  $\Delta\theta, \delta$  (strength of non-isometric deformation) and constants  $\theta_0, s_0$ . From trigonometric identities, we have for  $i \neq j$

$$|\bar{w}_{ij} - w_{ij}| \leq \left| \frac{\sin(\bar{\beta} - \beta)}{\sin \beta \sin \bar{\beta}} \right| \leq \frac{\sin \Delta\theta}{\sin^2 \theta_0} \leq \frac{\Delta\theta}{\sin^2 \theta_0}, \quad (23)$$

and  $|\bar{w}_{ii} - w_{ii}| \leq \frac{\nu \Delta\theta}{\sin^2 \theta_0}$ . Applying the triangle inequality, for  $i \neq j$  we get  $|\bar{l}_{ij} - l_{ij}| = 3|\bar{w}_{ij}\bar{s}_i^{-1} - w_{ij}s_i^{-1}| \leq 3|\bar{w}_{ij} - w_{ij}|s_i^{-1} + 3|\bar{w}_{ij}||\bar{s}_i^{-1} - s_i^{-1}|$ . Plugging in the bounds on  $|\bar{w}_{ij} - w_{ij}|$  and  $\bar{s}_i$ , we get

$$\begin{aligned} |\bar{l}_{ij} - l_{ij}| &\leq \frac{3}{s_0} \left( \frac{\Delta\theta}{\sin^2 \theta_0} + \cot(\theta_0)\delta \right) \leq \frac{3}{s_0} \left( \frac{\Delta\theta}{\sin^2 \theta_0} + \frac{\delta}{\theta_0} \right) \\ &\leq \frac{3(\Delta\theta + \delta)}{s_0 \sin^2 \theta_0}, \end{aligned}$$

from which it follows by norm inequality

$$\|\mathbf{L}_X - \mathbf{L}_Y\|_2 \leq n_X^{1/2} \|\mathbf{L}_X - \mathbf{L}_Y\|_1 \leq \frac{6\nu n_x^{3/2}}{s_0 \sin^2 \theta_0} (\Delta\theta + \delta),$$

where  $\epsilon = \Delta\theta + \delta$  is the degree of shape deformation (“lack of isometry”).

## Appendix B - Off-diagonality penalty gradient

Let us rewrite the penalty as  $\text{off}(\mathbf{A}) = \|\mathbf{A}^T \bar{\Lambda}_X \mathbf{A}\|_F^2 - \sum_i (\mathbf{A}^T \bar{\Lambda}_X \mathbf{A})_{ii}^2$ . The first term is bi-quadratic and its gradient derivation is trivial, so here we derive the gradient of the second term only. We have  $\sum_i (\mathbf{A}^T \bar{\Lambda}_X \mathbf{A})_{ii}^2 = \sum_i (\sum_k a_{ki}^2 \lambda_k)^2$ ; differentiating in a coordinate-wise manner,

$$\begin{aligned} \frac{\partial}{\partial a_{pq}} \sum_i \left( \sum_k a_{ki}^2 \lambda_k \right)^2 &= 2 \sum_i \left( \sum_k a_{ki}^2 \lambda_k \right) 2a_{pq} \lambda_p \delta_{iq} \\ &= 4 \sum_k a_{kq}^2 \lambda_k a_{pq} \lambda_p. \end{aligned} \quad (24)$$

Observing that  $\mathbf{A}\Lambda = (a_{pq}\lambda_p)$  and denoting by  $o_{pq} = (\sum_k a_{kq}^2 \lambda_k)$  the elements of the equal-columns matrix  $\mathbf{O}$ , we get the expression  $4\mathbf{O} \circ \mathbf{A}\Lambda$ .

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