On the digital homology groups of digital images

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Abstract : In this article we study the digital homology groups of digital images which are based on the singular homology groups of topological spaces in algebraic topology. Specifically, we define a digitally standard n-simplex, a digitally singular n-simplex, and the digital homology groups of digital images with k-adjacency relations. We then construct a covariant functor from a category of digital images and digitally continuous functions to the one of abelian groups and group homomorphisms, and investigate some fundamental and interesting properties of digital homology groups of digital images, such as the digital version of the dimension axiom which is one of the Eilenberg-Steenrod axioms.

Key words and phrases. Digitally continuous function, digital homeomorphism, digital homotopy, digital fundamental group, digitally standard n-simplex, digitally singular n-simplex, digitally singular n-chains, digital homology group.

1 Introduction

The digital fundamental groups [8, 3] of pointed digital images may be thought of as the tool in order to characterize properties of pointed digital images in a fashion analogous to that of classical fundamental groups [9] of topological spaces. They are basically derived from a classical notion of homotopy classes of based loops in the pointed homotopy category of pointed topological spaces or pointed CW-spaces.

Even though the digital fundamental group is a nice tool to classify the digital images with k-adjacency relations, it does not yields information at all in a large

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class of obvious problems. This is hardly surprising when we recall that the digital fundamental group of a digital image completely depends on the k-adjacency and the digital homotopy type of digital images, and even is difficult or fails to distinguish higher dimensions of pointed digital images. For example, as in the case of algebraic topology, S^2 and S^3 can not be classified by using the classical fundamental groups. But it is very easy to say that they do not have the same homotopy type when we use the singular (or simplicial) homology groups or 2 (or 3)-dimensional homotopy groups. Motivated from the above statements, we need to set up a new algebraic device which is called the digital homology groups in order to classify the various digital images with k-adjacency relations.

The homology groups and higher (or stable) homotopy groups are useful algebraic tools in a large number of topological problems, and are in practice the standard tools of algebraic topology. In the same lode, the digital homology group can be an important gadget to classify digital images from the point of view for the digital version of the homotopy type, mathematical morphology and image synthesis.

This paper is concerned with setting up more algebraic invariants for a digital image with k-adjacency. The paper is organized as follows: In Section 2 we introduce the general notions of digital images with k-adjacency relations. In Section 3 we define a digitally standard n-simplex, a digitally singular n-simplex, the digitally singular n-chains, and the digital homology groups of digital images. We then construct a covariant functor from a category of digital images and digitally continuous functions to the one of abelian groups and group homomorphisms, and investigate some fundamental and interesting properties of digital homology groups of digital images. Moreover, we show that the digital version of the dimension axiom, one of the Eilenberg-Steenrod axioms in algebraic topology, is satisfied.

2 Preliminaries

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of all integers and real numbers, respectively. Let $\mathbb{Z}^n$ be the set of lattice points in the Euclidean n-dimensional space $\mathbb{R}^n$. A *(binary) digital image* is a pair (X, k), where X is a finite subset of $\mathbb{Z}^n$ and k indicates some adjacency relation for the members of X. The k-adjacency relations are used in the study of digital images in $\mathbb{Z}^n$. For a positive integer u with $1 \leq u \leq n$, we define an adjacency relation of a digital image in $\mathbb{Z}^n$ as follows. Two distinct points $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in $\mathbb{Z}^n$ are $k(u, n)$*-adjacent* [6] if

- there are at most u distinct indices i such that $|p_i - q_i| = 1$; and

- for all indices j, if $|p_i - q_i| \neq 1$, then $p_j = q_j$.

A $k(u,n)$-adjacency relation on $\mathbb{Z}^n$ may be denoted by the number of points that are $k(u,n)$-adjacent to a point $p \in \mathbb{Z}^n$. Moreover,

- the $k(1,1)$-adjacent points of $\mathbb{Z}$ are called 2-adjacent;
- the $k(1,2)$-adjacent points of $\mathbb{Z}^2$ are called 4-adjacent, and the $k(2,2)$-adjacent points in $\mathbb{Z}^2$ are called 8-adjacent;
- the $k(1,3)$-adjacent points of $\mathbb{Z}^3$ are called 6-adjacent, the $k(2,3)$-adjacent points of $\mathbb{Z}^3$ are called 18-adjacent, and the $k(3,3)$-adjacent points of $\mathbb{Z}^3$ are called 26-adjacent;
- the $k(1,4)$-, $k(2,4)$-, $k(3,4)$-, and $k(4,4)$-adjacent points of $\mathbb{Z}^4$ are called 8-adjacent, 32-adjacent, 64-adjacent, and 80-adjacent, respectively; and so forth.

We note that the above number $k(u,n), 1 \leq u \leq n$ is just the cardinality of the set of lattice points which have the $k(u,n)$-adjacency relations centered at p in $\mathbb{Z}^n$. We sometimes denote the $k(u,n)$-adjacency by k-adjacency for short if there is no chance of ambiguity.

Definition 2.1 A *k-neighbor* of a lattice point $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is k-adjacent to p.

Definition 2.2 ([3]) Let k be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is said to be *k-connected* if and only if for every pair of points $\{x, y\} \subset X$ with $x \neq y$, there exists a set $P = \{x_0, x_1, \ldots, x_s\} \subset X$ of $s+1$ distinct points such that $x = x_0, x_s = y$, and x_i and x_{i+1} are k-adjacent for $i = 0, 1, \ldots, s-1$. The *length* of the set P is the number s.

The following generalizes an earlier definition of digital continuity given in [10].

Definition 2.3 Let $X \subset \mathbb{Z}^{n_1}$ and $Y \subset \mathbb{Z}^{n_2}$ be the digital images with k_1-adjacent and k_2-adjacent relations, respectively. A function $f : X \to Y$ is said to be a (k_1, k_2)-*continuous function* if the image under f of every k_1-connected subset of the digital image X is a k_2-connected subset of Y.

The following is an easy consequence of the above definition: Let X and Y be digital images with k_1-adjacency and k_2-adjacency, respectively. Then the function $f : X \to Y$ is a (k_1, k_2)-continuous function if and only if for every $\{x_1, x_2\} \subset X$ such that x_1 and x_2 are k_1-adjacent in X, either $f(x_1) = f(x_2)$ or $f(x_1)$ and $f(x_2)$ are k_2-adjacent in Y.

We note that if $f : X_1 \to X_2$ is (k_1, k_2)-continuous and if $g : X_2 \to X_3$ is (k_2, k_3)-continuous, then the composite $g \circ f : x_1 \to X_3$ is (k_1, k_3)-continuous.

Definition 2.4 ([2]) Two digital images (X, k_1) and (Y, k_2) with adjacency relations k_1 and k_2, respectively, are (k_1, k_2)-*homeomorphic* if there is a bijective function $f : X \to Y$ that is (k_1, k_2)-continuous such that the inverse function $f^{-1} : Y \to X$ is (k_2, k_1)-continuous. In this case, we call the function $f : X \to Y$ a *digital* (k_1, k_2)-*homeomorphism*, and we denote it by $X \approx_{(k_1,k_2)} Y$.

Definition 2.5 ([4]) Let $a, b \in \mathbb{Z}, a < b$. A *digital interval* is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

in which 2-adjacency is assumed. A *digital* k*-path* in a digital image X is a $(2,k)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$. If $f(0) = f(m)$, we call f a *digital* k*-loop*. If f is a constant function, it is called a *trivial loop*.

Definition 2.6 ([5]) Let X and Y be digital images with k_1-adjacent and k_2-adjacent relations, respectively, and let $f, g : X \to Y$ be the (k_1, k_2)-continuous functions. Suppose that there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is $(2, k_2)$-continuous; and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_1, k_2)-continuous.

Then F is called a *digital* (k_1, k_2)*-homotopy* between f and g, written $F : f \simeq_{(k_1,k_2)} g$, and f and g are said to be *digitally* (k_1, k_2)*-homotopic* in Y.

We use $[f]$ to denote the digital homotopy class of a (k_1, k_2)-continuous function $f : X \to Y$, i.e.,

$$[f] = \{g : X \to Y \mid g \text{ is } (k_1, k_2) - \text{continuous, and } f \simeq_{(k_1,k_2)} g\}.$$

Similarly, we denote by $[f]$ the k-loop class of a digital k-loop $f : [0, m]_{\mathbb{Z}} \to X$ in a digital image X with k-adjacency.

A *pointed digital image* is a pair (X, x_0), where X is a digital image and $x_0 \in X$; x_0 is called the *base point* of (X, x_0). A *pointed digitally continuous function* $f : (X, x_0) \to (Y, y_0)$ is a digitally continuous function from X to Y such that $f(x_0) = y_0$. A digital homotopy $F : X \times [0, m]_{\mathbb{Z}} \to Y$ between f and g is said to be *pointed digital homotopy* between f and g if for all $t \in [0, m]_{\mathbb{Z}}$, $F(x_0, t) = y_0$. If a pointed digital homotopy between f and g exists, we say f and g belong to the same pointed digital homotopy class. It is not difficult to see that the (pointed) digital homotopy is an equivalence relation among the (pointed) digital homotopy classes of digitally continuous functions.

We consider the digital version of products just as in the case of products of paths (or loops) of homotopy classes in homotopy theory. If $f : [0, m_1]_{\mathbb{Z}} \to X$ and $g : [0, m_2]_{\mathbb{Z}} \to X$ are digital k-paths in the digital image X with $f(m_1) = g(0)$, the *product* $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ (see [3] and [5]) of f and g is the digital k-path in X defined by

$$(f * g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathbb{Z}}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

The following result shows that the '$*$' product operation of digital loop classes is well-defined.

Proposition 2.7 ([7]) *Suppose f_1, f_2, g_1 and g_2 are digital loops in a pointed digital image (X, x_0) with $f_2 \in [f_1]$ and $g_2 \in [g_1]$. Then $f_2 * g_2 \in [f_1 * g_1]$.*

We now describe the notion of trivial extension [6] which is used to allow a loop to stretch and remain in the same pointed homotopy class.

Definition 2.8 Let f and g be digital k-loops in a pointed digital image (X, x_0). We say that g is a *trivial extension* of f if there are sets of k-paths $\{f_1, f_2, \ldots, f_s\}$ and $\{G_1, G_2, \ldots, G_t\}$ in X such that

1. $s \leq t$;
2. $f = f_1 * f_2 * \cdots * f_s$;
3. $g = G_1 * G_2 * \cdots * G_t$; and
4. there are indices $1 \leq i_1 < i_2 < \cdots < i_s \leq t$ such that
   - $G_{i_j} = f_j, 1 \leq j \leq s$; and
   - $i \notin \{i_1, i_2, \ldots, i_s\}$ implies G_i is a trivial loop.

Two digital loops f and g with the same base point $x_0 \in X$ belong to the same digital loop class $[f]$ (see [4]) if they have trivial extensions that can be joined by a homotopy that holds the endpoints fixed.

We end this section with the digital k-fundamental group originally derived from a classical notion of homotopy theory (see [11, 12]). Let (X, x_0) be a pointed digital image with k-adjacency. Consider the set $\pi_1^k(X, x_0)$ of digital k-loop classes $[f]$ in (X, x_0) with base point x_0. By Proposition 2.7, the product operation

$$[f] + [g] = [f * g]$$

is well-defined on $\pi_1^k(X, x_0)$. One can see that $\pi_1^k(X, x_0)$ becomes a group under the '$*$' product operation which is called the *digital k-fundamental group* of (X, x_0). As in the case of basic notions in algebraic topology, it is well known in [3, Theorem 4.14] that π_1^k is a covariant functor from the category of pointed digital images and pointed digitally continuous functions to the category of groups and group homomorphisms.

3 Digital homology groups

We now consider the digital version of singular homology groups as follows: For $i = 0, 1, \ldots, n$, let e_i denote the point in $\mathbb{Z}^{n+1}$ having coordinates all zeros except for 1 in the $(i+1)$st position, i.e., $e_0 = (1, 0, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 0, 1)$.

Definition 3.1 A *digitally convex combination* of points $e_0, e_1, \ldots, e_n$ in $\mathbb{Z}^{n+1}$ is a point x with

$$x = t_0 e_0 + t_1 e_1 + \cdots + t_n e_n,$$

where $\sum_{i=0}^{n} t_i = 1$, and $t_i = 0$ or 1. The entries $(t_0, t_1, \ldots, t_n)$ of $x = t_0 e_0 + t_1 e_1 + \cdots + t_n e_n$ are called the *digitally barycentric coordinates* of x.

We denote by Δ^n the set of all digitally convex combinations of points $e_0, e_1, \ldots, e_n$ in $\mathbb{Z}^{n+1}$, that is, $\Delta^n = \{e_0, e_1, \ldots, e_n\}$. If we consider Δ^n as the digital image with $k(2, n+1)$-adjacency, then it is $k(2, n+1)$-connected.

Definition 3.2 By using the $k(2, n+1)$-adjacent relation in the digital image $\Delta^n = \{e_0, e_1, \ldots, e_n\}$, we define an *orientation* of Δ^n by a linear ordering of its vertices, and call it a *digitally standard n-simplex* with a linear order for its orientation.

Definition 3.3 Let (X, k) be a digital image with k-adjacency. A *digitally singular n-simplex* in (X, k) is a $(k(2, n+1), k)$-continuous function

$$\sigma : \Delta^n \to (X, k),$$

where Δ^n is the digitally standard n-simplex.

Definition 3.4 Let (X, k) be a digital image with k-adjacency. For each $n \geq 0$, define $\check{S}_n(X, k)$ as the free abelian group with basis all digitally singular n-simplexes in (X, k), and define $\check{S}_{-1}(X, k) = 0$. The elements of $\check{S}_n(X, k)$ are called *digitally singular n-chains* in (X, k).

The oriented boundary of a digitally singular n-simplex $\sigma : \Delta^n \to (X, k)$ have to be $\sum_{i=0}^{n}(-1)^i(\sigma|_{\{e_0,\ldots,\hat{e}_i,\ldots,e_n\}})$, where the symbol $\hat{e}_i$ means that the vertex e_i is to be deleted from the array in the digitally standard n-simplex Δ^n. Technically, we prefer that this is a digitally singular $(n-1)$-chain in (X, k).

Definition 3.5 For each n and i, we define the *ith face function*

$$\epsilon_i = \epsilon_i^n : \Delta^{n-1} \to \Delta^n$$

to be the function sending the ordered vertices $\{e_0, \ldots, e_{n-1}\}$ to the ordered vertices $\{e_0, \ldots, \hat{e}_i, \ldots, e_n\}$ preserving the displayed orderings as follows:

- $\epsilon_0^n : (t_0, t_1, \ldots, t_{n-1}) \mapsto (0, t_0, t_1, \ldots, t_{n-1})$; and
- $\epsilon_i^n : (t_0, t_1, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$ for $i \geq 1$.

For example, there are three face functions $\epsilon_i^2 : \Delta^1 \to \Delta^2$ such as $\epsilon_0^2 : \{e_0, e_1\} \to \{e_1, e_2\}$; $\epsilon_1^2 : \{e_0, e_1\} \to \{e_0, e_2\}$; and $\epsilon_2^2 : \{e_0, e_1\} \to \{e_0, e_1\}$.

Definition 3.6 Let (X, k) be a digital image with k-adjacency. If $\sigma : \Delta^n \to (X, k)$ is a digitally singular n-simplex, then the function $\partial_n : \check{S}_n(X, k) \to \check{S}_{n-1}(X, k)$ defined by

$$\partial_n \sigma = \begin{cases} \displaystyle\sum_{i=0}^{n}(-1)^i \sigma \circ \epsilon_i^n & \text{if} \quad n \geq 1; \\ 0 & \text{if} \quad n = 0; \end{cases}$$

is called the *digitally boundary operator* of the digital image (X, k).

We note that $\partial_n : \check{S}_n(X, k) \to \check{S}_{n-1}(X, k)$ is a homomorphism. We thus extend the above definition by linearity to the digitally singular n-chains. In particular, if $X = \Delta^n$ and $I : \Delta^n \to \Delta^n$ is the identity, then

$$\partial_n(I) = \sum_{i=0}^{n}(-1)^i \epsilon_i^n.$$

Lemma 3.7 *If $k < j$, then*

$$\epsilon_j^{n+1} \circ \epsilon_k^n = \epsilon_k^{n+1} \circ \epsilon_{j-1}^n : \Delta^{n-1} \to \Delta^{n+1}.$$

Proof We first consider

$$\begin{aligned} \epsilon_j^{n+1} \circ \epsilon_k^n \quad & (t_0, \ldots, t_k, \ldots, t_j, \ldots, t_{n-1}) \\ & = \epsilon_j^{n+1}(t_0, \ldots, t_{k-1}, 0, t_k, \ldots, t_j, \ldots, t_{n-1}) \\ & = (t_0, \ldots, t_{k-1}, 0, t_k, \ldots, t_{j-2}, 0, t_{j-1}, \ldots, t_{n-1}). \end{aligned}$$

Secondly,

$$\begin{aligned}\epsilon_k^{n+1} \circ \epsilon_{j-1}^n \quad & (t_0, \dots, t_k, \dots, t_j, \dots, t_{n-1}) \\ & = \epsilon_k^{n+1}(t_0, \dots, t_k, \dots, t_{j-2}, 0, t_{j-1}, \dots, t_{n-1}) \\ & = (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_{j-2}, 0, t_{j-1}, \dots, t_{n-1}),\end{aligned}$$

as required. □

Theorem 3.8 *For all $n \geq 0$, we have $\partial_n \circ \partial_{n+1} = 0$.*

Proof It suffices to show that $\partial_n \circ \partial_{n+1}(\sigma) = 0$ for each digitally singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to (X, k)$. By Lemma 3.7, we have

$$\begin{aligned}\partial_n \circ \partial_{n+1}(\sigma) \quad & = \partial_n\Big(\sum_{j=0}^{n+1} (-1)^j \sigma \circ \epsilon_j^{n+1}\Big) \\ & = \partial_n(\sigma \circ \epsilon_0^{n+1} - \sigma \circ \epsilon_1^{n+1} + \sigma \circ \epsilon_2^{n+1} + \cdots + (-1)^{n+1} \sigma \circ \epsilon_{n+1}^{n+1}) \\ & = \sum_{k=0}^{n} (-1)^k \sigma \circ \epsilon_0^{n+1} \circ \epsilon_k^n - \sum_{k=0}^{n} (-1)^k \sigma \circ \epsilon_1^{n+1} \circ \epsilon_k^n + \\ & \qquad\qquad \cdots + (-1)^{n+1} \sum_{k=0}^{n} (-1)^k \sigma \circ \epsilon_{n+1}^{n+1} \circ \epsilon_k^n \\ & = \sum_{j=0}^{n+1} (-1)^j \Big[\sum_{k=0}^{n} (-1)^k \sigma \circ \epsilon_j^{n+1} \circ \epsilon_k^n\Big] \\ & = \sum_{j=0}^{n+1} \sum_{k=0}^{n} (-1)^{j+k} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_k^n \\ & = \sum_{j \leq k} (-1)^{j+k} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_k^n + \sum_{k<j} (-1)^{j+k} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_k^n \\ & = \sum_{j \leq k} (-1)^{j+k} \sigma \circ \epsilon_j^{n+1} \circ \epsilon_k^n + \sum_{k<j} (-1)^{j+k} \sigma \circ \epsilon_k^{n+1} \circ \epsilon_{j-1}^n. \qquad (\star)\end{aligned}$$

The left-hand term and the right-hand term in $(\star)$ denote the upper triangular matrix and the lower triangular matrix in the $(n+2) \times (n+1)$-matrix. These terms cancel in pairs, and thus $\partial_n \circ \partial_{n+1}(\sigma) = 0$. □

Definition 3.9 The kernel of $\partial_n : \check{S}_n(X,k) \to \check{S}_{n-1}(X,k)$ is called the *group of digitally singular n-cycles* in (X,k) and denoted by $\check{Z}_n(X,k)$. The image of $\partial_{n+1} : \check{S}_{n+1}(X,k) \to \check{S}_n(X,k)$ is called the *group of digitally singular n-boundaries* in (X,k) and denoted by $\check{B}_n(X,k)$.

By Theorem 3.8, each digitally singular n-boundary of digitally singular $(n+1)$-chains is automatically a digitally singular n-cycle, that is, $\check{B}_n(X,k)$ is a normal subgroup of $\check{Z}_n(X,k)$ for each $n \geq 0$. Thus we can define

Definition 3.10 For each $n \geq 0$, the *nth digital homology group* of a digital image (X, k) with k-adjacency is defined by

$$\breve{H}_n(X, k) = \breve{Z}_n(X, k)/\breve{B}_n(X, k).$$

The coset $[z_n] = z_n + \breve{B}_n(X, k)$ is called the *digital homology class* of z_n, where z_n is a digitally singular n-cycle.

We note that if $f : (X, k_1) \to (Y, k_2)$ is a (k_1, k_2)-continuous function and if $\sigma : \Delta^n \to (X, k_1)$ is a digitally singular n-simplex in (X, k_1), then $f \circ \sigma : \Delta^n \to (Y, k_2)$ is a digitally singular n-simplex in (Y, k_2). By extending by linearity, we have a homomorphism

$$f_\sharp : \breve{S}_n(X, k_1) \to \breve{S}_n(Y, k_2)$$

defined by

$$f_\sharp(\Sigma m_\sigma \sigma) = \Sigma m_\sigma (f \circ \sigma),$$

where $m_\sigma \in \mathbb{Z}$.

Lemma 3.11 *If $f : (X, k_1) \to (Y, k_2)$ is a (k_1, k_2)-continuous function, then for every $n \geq 0$ there is a commutative diagram*

$$\begin{tikzcd}
\breve{S}_n(X, k_1) \arrow[r, "f_\sharp"] \arrow[d, "\partial_n"] & \breve{S}_n(Y, k_2) \arrow[d, "\partial'_n"] \\
\breve{S}_{n-1}(X, k_1) \arrow[r, "f_\sharp"] & \breve{S}_{n-1}(Y, k_2).
\end{tikzcd}$$

Proof It suffices to evaluate each composite on a digital generator $\sigma : \Delta^n \to (X, k_1)$, a digitally singular n-simplex, as follows.

$$\begin{aligned}
f_\sharp \circ \partial_n(\sigma) &= f_\sharp(\sum_{j=0}^{n} (-1)^j \sigma \circ \epsilon_j^n) \\
&= \sum_{j=0}^{n} (-1)^j f_\sharp(\sigma \circ \epsilon_j^n) \\
&= \sum_{j=0}^{n} (-1)^j (f \circ \sigma) \circ \epsilon_j^n \\
&= \partial'_n(f \circ \sigma) \\
&= \partial'_n \circ f_\sharp(\sigma),
\end{aligned}$$

as required. $\square$

Let $\mathcal{D}$ be the category of digital images and digitally continuous functions, and let $\mathcal{G}$ be the category of abelian groups and group homomorphisms. Then we have

Theorem 3.12 *For each $n \geq 0$, $\check{H}_n : \mathcal{D} \to \mathcal{G}$ is a covariant functor.*

Proof If $f : (X, k_1) \to (Y, k_2)$ is a digitally (k_1, k_2)-continuous function, then we can define

$$f_* = \check{H}_n(f) : \check{H}_n(X, k_1) \to \check{H}_n(Y, k_2)$$

by $\check{H}_n([z_n]) = f_\sharp(z_n) + \check{B}_n(Y, k_2)$, where $z_n \in \check{Z}_n(X, k_1)$. We note that if $z_n \in \check{Z}_n(X, k_1)$, then $\partial_n(z_n) = 0$ and thus

$$\partial'_n \circ f_\sharp(z_n) = f_\sharp \circ \partial_n(z_n) = 0.$$

In other words, $f_\sharp(z_n) \in \ker \partial'_n = \check{Z}_n(Y, k_2)$. Moreover, if $b_n \in \check{B}_n(X, k_1)$, then

$$\partial_{n+1}(b_{n+1}) = b_n$$

for some $b_{n+1} \in \check{S}_{n+1}(X, k_1)$, and

$$f_\sharp(b_n) = f_\sharp \circ \partial_{n+1}(b_{n+1}) = \partial'_{n+1} \circ f_\sharp(b_{n+1}) \in \check{B}_n(Y, k_2).$$

If $b_n \in \check{B}_n(X, k_1)$, then

$$f_\sharp(z_n + b_n) + \check{B}_n(Y, k_2) = f_\sharp(z_n) + \check{B}_n(Y, k_2),$$

that is, the definition of $\check{H}_n(f)$ is independent of the choice of representative.

If $1_X : (X, k_1) \to (X, k_1)$ is the identity and $\sigma : \Delta^n \to (X, k_1)$ is a digitally singular n-simplex, then

$$1_{X\sharp}(\sigma) = 1_X \circ \sigma = \sigma,$$

that is, $\check{H}_n(1_X) = 1_{\check{H}_n(X,k_1)}$, the identity automorphism on $\check{H}_n(X, k_1)$.

If $f : (X, k_1) \to (Y, k_2)$ is digitally (k_1, k_2)-continuous and $g : (Y, k_2) \to (Z, k_3)$ is a digitally (k_2, k_3)-continuous function, then

$$(g \circ f)_\sharp(\sigma) = (g \circ f) \circ (\sigma) = g \circ (f \circ \sigma) = g_\sharp(f_\sharp(\sigma)).$$

Thus $\check{H}_n(g \circ f) = \check{H}_n(g) \circ \check{H}_n(f)$. $\square$

The following shows that the digital homology groups are digitally invariant.

Corollary 3.13 *If $f : (X, k_1) \to (Y, k_2)$ is a digitally (k_1, k_2)-homeomorphism, then $f_* : \check{H}_n(X, k_1) \to \check{H}_n(Y, k_2)$ is an isomorphism.*

Proof Let $g : (Y, k_2) \to (X, k_1)$ be the digitally (k_2, k_1)-continuous function such that $f \circ g = 1_{(Y,k_2)}$ and $g \circ f = 1_{(X,k_1)}$. Then we get

$$\check{H}_n(f) \circ \check{H}_n(g) = \check{H}_n(f \circ g) = 1_{\check{H}_n(Y,k_2)}$$

and similarly,

$$\check{H}_n(g) \circ \check{H}_n(f) = \check{H}_n(g \circ f) = 1_{\check{H}_n(X,k_1)}.$$

Therefore, $f_* = \check{H}_n(f)$ is an isomorphism, as required. $\square$

The following is the digital version of the dimension axiom which is one of the Eilenberg-Steenrod axioms in algebraic topology.

Theorem 3.14 *If (X,k) is a one-point digital image with k-adjacency, then*

$$\check{H}_n(X,k) = \begin{cases} 0 & \textit{for all} \quad n \geq 1; \\ \mathbb{Z} & \textit{for} \quad n = 0. \end{cases}$$

Proof For each $n \geq 0$, we have

$$\check{S}_n(X,k) \ \cong \ \mathbb{Z} \ \cong \ < \sigma_n >$$

generated by the digitally singular n-simplex $\sigma_n : \Delta^n \to (X,k)$ which is a constant function in this case. From the definition of the boundary operator $\partial_n : \check{S}_n(X,k) \to \check{S}_{n-1}(X,k)$, we have

$$\partial_n(\sigma_n) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \sigma_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Thus, ∂_n is the trivial homomorphism if n is odd, and it is an isomorphism if n is even.

Assume that $n \geq 1$, and consider the sequence

$$\cdots \longrightarrow \check{S}_{n+1}(X,k) \xrightarrow{\partial_{n+1}} \check{S}_n(X,k) \xrightarrow{\partial_n} \check{S}_{n-1}(X,k) \longrightarrow \cdots.$$

(1) If $n \geq 1$ is odd, then ∂_n is the trivial homomorphism and ∂_{n+1} is an isomorphism, and thus $\check{S}_n(X,k) = \ker\, \partial_n = \check{Z}_n(X,k)$ and $\check{S}_n(X,k) = \mathrm{im}\, \partial_{n+1} = \check{B}_n(X,k)$. Therefore

$$\check{H}_n(X,k) = \check{Z}_n(X,k)/\check{B}_n(X,k) = \check{S}_n(X,k)/\check{S}_n(X,k) = 0.$$

(2) Similarly, if $n \geq 2$ is even, then ∂_n is an isomorphism, and thus $\check{Z}_n(X,k) = \ker\, \partial_n = 0$ so that

$$\check{H}_n(X,k) = \check{Z}_n(X,k)/\check{B}_n(X,k) = 0/\check{B}_n(X,k) = 0.$$

On the other hand, from the sequence

$$\cdots \longrightarrow \check{S}_1(X,k) \xrightarrow{\partial_1} \check{S}_0(X,k) \xrightarrow{\partial_0} 0,$$

we have

$$\begin{cases} \ker\, \partial_0 = \check{Z}_0(X,k) \cong \mathbb{Z}; & \text{and} \\ \mathrm{im}\, \partial_1 = \check{B}_0(X,k) \cong 0; \end{cases}$$

so that

$$\check{H}_0(X,k) = \check{Z}_0(X,k)/\check{B}_0(X,k) \cong \mathbb{Z},$$

as required. $\square$

Theorem 3.15 *Let $f, g : (X, k_1) \to (Y, k_2)$ be the digitally (k_1, k_2)-continuous functions. Assume that there are homomorphisms $\varphi_n : \check{S}_n(X, k_1) \to \check{S}_{n+1}(Y, k_2)$ such that*

$$f_\sharp - g_\sharp = \partial'_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \partial_n,$$

where $\partial_n : \check{S}_n(X, k_1) \to \check{S}_{n-1}(X, k_1)$ and $\partial'_{n+1} : \check{S}_{n+1}(Y, k_2) \to \check{S}_n(Y, k_2)$ are digital boundary operators of (X, k_1) and (Y, k_2), respectively. Then $f_* = g_* : \check{H}_n(X, k_1) \to \check{H}_n(Y, k_2)$ as group homomorphisms.

Proof Since

$$\check{H}_n(f)([z_n]) = \check{H}_n(f)(z_n + \check{B}_n(X, k_1)) = f_\sharp(z_n) + \check{B}_n(Y, k_2),$$

where $z_n \in \check{Z}_n(X, k_1)$, we have

$$\begin{aligned} (f_\sharp - g_\sharp)(z_n) &= (\partial'_{n+1} \circ \varphi_n + \varphi_{n-1} \circ \partial_n)(z_n) \\ &= \partial'_{n+1} \circ \varphi_n(z_n) + 0 \in \check{B}_n(Y, k_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \check{H}_n(f)([z_n]) &= f_\sharp(z_n) + \check{B}_n(Y, k_2) \\ &= g_\sharp(z_n) + \check{B}_n(Y, k_2) \\ &= \check{H}_n(g)([z_n]), \end{aligned}$$

as required. □

In graph theory, a graph product is a certain kind of binary operation on graphs, such as the cartesian product, tensor product, lexicographical product, normal product, conormal product and rooted product. We recall that the *cartesian product* [1] of simple graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$.

Let X be a digital image with k_1-adjacency, and let $X \times [0, m]_{\mathbb{Z}}$ be the digital image with the cartesian product whose adjacency is denoted by k_c. Then the map $\psi_i : X \to X \times [0, m]_{\mathbb{Z}}$ defined by $\psi_i(x) = (x, i)$ for $i = 0, m$ is (k_1, k_c)-continuous. Moreover, we have

Theorem 3.16 *If $\check{H}_n(\psi_0) = \check{H}_n(\psi_m) : \check{H}_n(X) \to \check{H}_n(X \times [0, m])$ as group homomorphisms, and if $f \simeq_{(k_1, k_2)} g : (X, k_1) \to (Y, k_2)$, then $\check{H}_n(f) = \check{H}_n(g) : \check{H}_n(X, k_1) \to \check{H}_n(Y, k_2)$.*

Proof Let $F : X \times [0, m]_{\mathbb{Z}} \to Y$ be a digital (k_1, k_2)-homotopy between f and g. Then we have

$$f = F \circ \psi_0 \text{ and } g = F \circ \psi_m.$$

Since $\check{H}_n : \mathcal{D} \to \mathcal{G}$ is a covariant functor, we get

$$\begin{aligned} \check{H}_n(f) &= \check{H}_n(F \circ \psi_0) = \check{H}_n(F) \circ \check{H}_n(\psi_0) \\ &= \check{H}_n(F) \circ \check{H}_n(\psi_m) = \check{H}_n(F \circ \psi_m) \\ &= \check{H}_n(g), \end{aligned}$$

as required. $\square$

Theorem 3.17 *Let (A,k) be a nonempty subset of digital image (X,k) with k-adjacency, and let $i : A \hookrightarrow X$ be the inclusion. Then $i_\sharp : \check{S}_n(A,k) \to \check{S}_n(X,k)$ is a monomorphism for every $n \geq 0$.*

Proof Let $\sigma_j : \Delta^n \to (A,k)$ be the digitally singular n-simplex in (A,k). Then

$$\gamma = \sum_j m_j \sigma_j \in \check{S}_n(A,k).$$

If $\gamma \in \ker(i_\sharp)$, then

$$0 = i_\sharp\Big(\sum_j m_j \sigma_j\Big) = \sum_j m_j (i \circ \sigma_j).$$

Since $i \circ \sigma_j : \Delta^n \to (X,k)$ differs from $\sigma_j : \Delta^n \to (A,k)$ only in having its target enlarged from (A,k) to (X,k), and $\check{S}_n(X,k)$ is a free abelian group with basis all digitally singular n-simplexes in (X,k), it follows that every $m_j = 0$ and thus $\gamma = 0$. Therefore $\ker(i_\sharp) = 0$ which means $i_\sharp$ is a monomorphism. $\square$

Let $D_1^k(X,x_0)$ be the set of all digital k-loops in a pointed digital image (X,x_0) with k-adjacency. Then, by using the equivalence relation, namely '$\simeq_{(2,k)}$' as previously described in Section 2, we have the digital k-fundamental group of (X,x_0),

$$\pi_1^k(X,x_0) = D_1^k(X,x_0)/\simeq_{(2,k)}$$

with the '$*$' product operation.

One can construct a relation between the homotopy groups of a pointed topological space and the singular homology groups with integer (or rational) coefficients by using the Hurewicz homomorphism in the sense of [11, pp. 388-390] which plays an important role in algebraic topology. We end this paper with a matter for the following possibility of the digital Hurewicz homomorphism.

Remark 3.18 Let $\eta : \Delta^1 \to [0,2]_{\mathbb{Z}}$ be the function defined by $\eta(e_0) = 0$ and $\eta(e_1) = 1$. Then η is a digitally $(k(2,2),2)$-continuous function. Let (X,k) be a digital image with k-adjacency, and let $f : [0,m]_{\mathbb{Z}} \to X, m \geq 3$ be a digital k-loop in (X,k). Then we can think of a well-defined function

$$h : D_1^k(X,x_0) \to \check{S}_1(X,k)$$

given by $h(f) = f \circ i \circ \eta$, where $i : [0,2]_{\mathbb{Z}} \hookrightarrow [0,m]_{\mathbb{Z}}$ is the inclusion. In general, even though the digital image (X,k) is k-connected, there is no guarantee to say that the function

$$H : \pi_1^k(X, x_0) \to \check{S}_1(X,k)$$

defined by $H([f]) = f \circ i \circ \eta$ is well-defined. Here is an example. Let $X = \{x_0, x_1, x_2, x_3\}$ be a digital image in $\mathbb{Z}^2$ with 4-adjacency, where $x_0 = (0,0), x_1 = (1,0), x_2 = (1,1)$, and $x_3 = (0,1)$. Let $f : [0,4]_{\mathbb{Z}} \to X$ be the digital 4-loop in the pointed digital image (X, x_0) defined by

$$\begin{cases} f(0) = f(4) = x_0; \\ f(i) = x_i \text{ for } i = 1,2,3, \end{cases}$$

and let $g : [0,5]_{\mathbb{Z}} \to X$ be the digital 4-loop in (X, x_0) defined by

$$\begin{cases} g(0) = g(1) = g(5) = x_0; \\ g(i) = x_{i-1} \text{ for } i = 2,3,4. \end{cases}$$

Then $e_{x_0} * f \simeq_{(2,4)} g$ (the equality holds strictly in this case), where $e_{x_0} : [0,1]_{\mathbb{Z}} \to X$ is a constant function at x_0, that is, $e_{x_0}([0,1]_{\mathbb{Z}}) = \{x_0\}$. By using the trivial extensions, we have $[f] = [g]$ in $\pi_1^k(X, x_0)$. However,

$$H([f]) = f \circ i \circ \eta \neq g \circ i \circ \eta = H([g]).$$

It raises the following fundamental questions:

(1) Can we define the digital version of the higher homotopy groups of digital images?

(2) Under what conditions can we construct a digital version of the Hurewicz homomorphism?

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