SOME RESULTS ON $\mathbb{R}$-COMPUTABLE STRUCTURES

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1. Introduction

The theory of effectiveness properties on countable structures whose atomic diagrams are Turing computable is well-studied (see, for instance, [1, 14]). Typical results describe which structures in various classes are computable (or have isomorphic copies that are) [18], or the potential degree of unsolvability of various definable subsets of the structure [15]. The goal of the present paper is to survey some initial results investigating similar concerns on structures which are effective in a different sense.

A rather severe limitation of the Turing model of computability is its traditional restriction to the countable. Of course, many successful generalizations have been made (see, for instance, [27, 11, 12, 22, 23, 25] and the other papers in the present volume). The generalization that will be treated here is based on the observation that while there is obviously no Turing machine for addition and multiplication of real numbers, there is strong intuition that these operations are "computable." The BSS model of computation, first introduced in [4], approximately takes this to be the definition of computation on a given ring (a more formal definition is

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forthcoming). This allows several problems of computation in numerical analysis and continuous geometry to be treated rigorously. The monograph [3] gives the examples of the "decision problem" of the points for which Newton's method will converge to a root, and determining whether a given point is in the Mandelbrot set.

1.1. **Basic Definitions.** The definition of a BSS machine comes from [3]. Such a machine should be thought of as the analogue of a Turing machine (indeed, the two notions coincide where $R = \mathbb{Z}$). Let R be a ring with 1. Let R^∞ be the set of finite sequences of elements from R, and R_∞ the bi-infinite direct sum

$$\bigoplus_{i\in\mathbb{Z}} R.$$

Definition 1.1. A machine M over R is a finite connected directed graph, containing five types of nodes: input, computation, branch, shift, and output, with the following properties:

(1) The unique input node has no incoming edges and only one outgoing edge.
(2) Each computation and shift node has exactly one output edge and possibly several input branches.
(3) Each output node has no output edges and possibly several input edges.
(4) Each branch node η has exactly two output edges (labeled 0_η and 1_η) and possibly several input edges.
(5) Associated with the input node is a linear map $g_I : R^\infty \to R_\infty$.
(6) Associated with each computation node η is a rational function $g_\eta : R_\infty \to R_\infty$.
(7) Associated with each branch node η is a polynomial function $h_\eta : R_\infty \to R$.
(8) Associated with each shift node is a map $\sigma_\eta \in \{\sigma_l, \sigma_r\}$, where $\sigma_l(x)_i = x_{i+1}$ and $\sigma_r(x)_i = x_{i-1}$.
(9) Associated with each output node η is a linear map $O_\eta : R_\infty \to R^\infty$.

A machine may be understood to compute a function in the following way:

Definition 1.2. Let M be a machine over R.

(1) A *path* through M is a sequence of nodes $(\eta_i)_{i=0}^n$ where η_0 is the input node, η_n is an output node, and for each i, we have an edge from η_i to η_{i+1}.
(2) A *computation* on M is a sequence of pairs $((\eta_i, x_i))_{i=0}^n$ with a number x_{n+1}, where $(\eta_i)_{i=0}^n$ is a path through M, where $x_0 \in R^\infty$, and where, for each i, the following hold:
  (a) If η_i is an input node, $x_{i+1} = g_I(x_i)$.
  (b) If η_i is a computation node, $x_{i+1} = g_{\eta_i}(x_i)$.
  (c) If η_i is a branch node, $x_{i+1} = x_i$ and η_{i+1} determined by h_{η_i} so that if $h_{\eta_i}(x_i) \geq 0$, then η_{i+1} is connected to η_i by 1_{η_i} and if $h_{\eta_i}(x_i) < 0$, then η_{i+1} is connected to η_i by 0_{η_i}. (Note that in all other cases, η_{i+1} is uniquely determined by the definition of path.)
  (d) If η_i is a shift node, $x_{i+1} = \sigma_{\eta_i}(x_i)$
  (e) If η_i is an output node, $x_{i+1} = O_{\eta_i}(x_i)$.

The proof of the following lemma is an obvious from the definitions.

Lemma 1.3. *Given a machine M and an element $z \in R^\infty$, there is at most one computation on M with $x_0 = z$.*

Definition 1.4. The function $\varphi_M : R^\infty \to R^\infty$ is defined in the following way: For each $z \in R^\infty$, let $\varphi_M(z)$ be x_{n+1}, where $(((\eta_i, x_i))_{i=0}^n, x_{n+1})$ is the unique computation, if any, where $x_0 = z$. If there is no such computation, then φ_M is undefined on z.

Since a machine is a finite object, involving finitely many real numbers as parameters, it may be coded by a member of R^∞.

Definition 1.5. If σ is a code for M, we define $\varphi_\sigma = \varphi_M$.

We can now say that a set is computable if and only if its characteristic function is φ_M for some M.

Example 1.6. Let $R = \mathbb{Z}$. Now the R-computable functions are exactly the classical Turing-computable functions.

Example 1.7. Let $R = \mathbb{R}$. Then the Mandelbrot set is not R-computable (see Chapter 2 of [3]).

Definition 1.8. A *machine over R with oracle X* is exactly like a machine over R, except that it has an additional type of nodes, the oracle nodes. Each oracle node is exactly like a computation node, except that $g_\eta = \chi_X$. Computations in oracle machines are defined in the obvious way.

We say that a set S is decidable (respectively, X-decidable) over R if and only if S is both the halting set of an R-machine (respectively, with oracle X) and the complement of the halting set of an R-machine (respectively, with oracle X). We also say that S is semi-decidable if and only if S is the domain of an R-computable function (if R is a real closed field, it is equivalent to say that S is the range of an R-computable function [3]). Ziegler [29] gives a specialized but recent survey of results on ℝ-computation. The following result, first presented by Michaux, but proved in detail in [9], is useful in characterizing the decidable and semi-decidable sets:

Proposition 1.9. *Let $S \subseteq \mathbb{R}^\infty$. Then S is semi-decidable if and only if S is the union of a countable family of semialgebraic sets defined over a single finitely generated extension of $\mathbb{Q}$.*

The "only if" part of this statement is the upshot of an earlier theorem described in [3], called the Path Decomposition Theorem. We can now proceed to define computable structures.

Definition 1.10. Let $\mathcal{L} = (\{P_i\}_{i \in I_P}, \{f_i\}_{i \in I_f}, \{c_i\}_{i \in I_C})$ be a language with relation symbols $\{P_i\}_{i \in I_P}$, function symbols $\{f_i\}_{i \in I_f}$, and constant symbols $\{c_i\}_{i \in I_C}$. Let $\mathcal{A}$ be an $\mathcal{L}$-structure with universe $A \subseteq R^\infty$.

(1) We say that $\mathcal{L}$ is R-computable if the sets of relations, functions, and constants are each decidable over R, and if, in addition, there are R-machines which will tell, given P_i (respectively, f_i), the arity of P_i (respectively, f_i).
(2) We identify $\mathcal{A}$ with its atomic diagram; in particular,
(3) We say that $\mathcal{A}$ is computable if and only if the atomic diagram of $\mathcal{A}$ is decidable.

The obstructions to a direct parallel between the theory of ℝ-comptuable structures and that of Turing computable structures which we have encountered so far are two in number (one for the parsimonious):

(1) The real numbers do not admit an ω-like well-ordering to facilitate searching or priority constructions, and in particular
(2) There exist $\mathbb{R}$-computable injective functions whose inverses are not $\mathbb{R}$-computable.

1.2. **Plan of the Paper.** In the present paper, we will survey recent work on the theory of $\mathbb{R}$-computable structures. In Section 2, we give some basic calculations, showing some parallels with the classical theory, including computable ordinals (Section 2.1), satisfaction of computable infinitary formulas (Section 2.2), and the use of forcing to carry out a simple priority construction (Section 2.3). In Section 3, we explore effective categoricity, using vector spaces as an example. In Section 4, we describe some recent results in effective geometry and topology from the perspective of $\mathbb{R}$-computation. In Section 5 we address the relationship of $\mathbb{R}$-computation with other models of effective mathematics for uncountable structures. In Section 6 we summarize the state of $\mathbb{R}$-computable model theory and describe some directions for future research.

2. Basic Results

2.1. **$\mathbb{R}$-computable Ordinals.** The Turing computable ordinals constitute a proper initial segment of the countable ordinals [28, 19]. This initial segment includes, for instance, the ordinal $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$. In the present section, we will establish the following theorem:

Theorem 2.1. *A well-ordering $(L, <)$ has an isomorphic copy which is $\mathbb{R}$-computable if and only if L is countable.*

Proposition 2.2. *Every countable well-ordering $(M, \prec)$ has an isomorphic copy $(L, <)$ which is $\mathbb{R}$-computable.*

Proof. Since $(M, \prec)$ is countable, it has an isomorphic copy with universe ω. Now $D(M) = \{(a, b) \in M^2 | a \prec b\}$ is a subset of ω^2. Now we define a real number ℓ in the following way:

$$\ell = \sum_{i \in \omega} 10^{-i} \chi_{D(M)}(i).$$

There is a $\mathbb{R}$ machine which, given a pair $(a, b) \in \omega^2$ will return the $10^{-\langle a,b \rangle}$ place of ℓ if that place is 1 and will diverge if that place is 0. This shows that $D(M)$ is the halting set of a $\mathbb{R}$-computable function, as required. $\square$

Proposition 2.3. *Suppose $(L, <)$ is a $\mathbb{R}$-computable well-ordering. Then $|L| \leq \aleph_0$.*

Proof. Since $(L, <)$ is $\mathbb{R}$-computable, the set $L_< := \{(a, b) \subseteq L^2 : a < b\}$ is the halting set of a $\mathbb{R}$-machine. By Path Decomposition, it must be a disjoint union of semialgebraic sets, and consequently Borel. By the Kunen-Martin Theorem (Theorem 31.5 of [17]), analytic (and hence Borel) well-orderings are countable. $\square$

A rather different proof of Proposition 2.3, using Fubini's Theorem, is possible and enlightening.

Proof. Since $L_<$ is uncountable and Borel, $|L_<| = 2^{\aleph_0}$. This implies L is Borel with $|L| = 2^{\aleph_0}$. Without loss of generality, we suppose that L is order isomorphic to the cardinal $2^{\aleph_o}$; otherwise, an initial segment of L which was isomorphic to this cardinal would also be $\mathbb{R}$-computable. In particular, L contains a Cantor set C. Fix a Borel measure μ on C such that $\mu(C) = 1$ and extend μ to L by setting $\mu(L \setminus C) = 0$.

We define two auxiliary sets:

$$\begin{aligned} L_x &= \{b \in L : (x, b) \in L_<\} \\ L^y &= \{a \in L : (a, y) \in L_<\} \end{aligned}$$

Each of these is a Borel set. For any y, we have $|L^y| < 2^{\aleph_0}$, since $2^{\aleph_0}$ is a cardinal and L^y is isomorphic to an ordinal less than $2^{\aleph_0}$. Since L^y is Borel, we have $|L^y| = \aleph_0$. This implies the set L_y is co-countable for any y.

Since $L_<$ is Borel, we can apply Fubini's theorem to calculate $\int_{L_<} 1 d\lambda$, where λ is the product measure $\mu \times \mu$. On the one hand,

$$\int_{L_<} 1 d\lambda = \int_L \int_{L_x} 1(d\mu)(d\mu) = \int_L \mu(L_x) d\mu = \int_L 1 d\mu = \mu(L) = 1$$

since L_x is co-countable for each x, and thus of full measure. On the other hand, since L^y is countable for each y, we have $\int_{L_y} 1 dy = 0$. Hence

$$\int_{L_<} 1 d\lambda = \int_L \int_{L^y} 1(d\mu)(d\mu) = \int_L \mu(L^y) d\mu = \int_L 0 d\mu = 0,$$

which is a contradiction. □

2.2. **The Complexity of Satisfaction.** We define the class of $\mathbb{R}$-computable infinitary formulas. The definition is by analogy with the (Turing) computable infinitary formulas already in broad usage, described in [1]. The choice of computable infinitary formulas is nontrivial, since there are uncountably many $\mathbb{R}$-machines. One natural approach, not pursued here, would be to work in the $\mathbb{R}$-computable fragment of $\mathcal{L}_{(2^{\aleph_0})^+,\omega}$. This would certainly be an interesting logic to understand, but the present authors found it more desirable at first to understand the more familiar $\mathbb{R}$-computable fragment of $\mathcal{L}_{\omega_1\omega}$. At issue is which conjunctions and disjunctions are allowed in a "computable" formula. The logic $\mathcal{L}_{\omega_1\omega}$ allows countable conjunctions and disjunctions, while $\mathcal{L}_{(2^{\aleph_0})^+,\omega}$ allows any of size at most $2^{\aleph_0}$. However, the difficulty of describing what is meant by, for instance, an interval of formulas is a motivation (beyond the avoidance of set-theoretic independence) to consider first the countably long formulas.

Definition 2.4. Let $\mathcal{L}$ be an R-computable language.

(1) The Σ_0 formulas of $\mathcal{L}$ are exactly the finitary quantifier-free formulas. The Π_0 formulas are the same.
(2) For any ordinal $\alpha = \beta + 1$, the Σ^0_α formulas are those of the form
$$\bigvee\!\!\!\bigvee_{i \in S} \exists \bar{y} [\varphi_i(\bar{x}\bar{y})]$$
where S is countable and is the halting set of an R-machine, and there is a finitely generated field $F \subset \mathbb{Q}$ such that all parameters in ϕ_i are in F.
(3) For any ordinal $\alpha = \beta + 1$, the Π^0_α formulas are those of the form
$$\bigwedge\!\!\!\bigwedge_{i \in S} \forall \bar{y} [\varphi_i(\bar{x}\bar{y})]$$

where S is countable and is the halting set of an R-machine, and there is a finitely generated field $F \subset \mathbb{Q}$ such that all parameters in ϕ_i are in F.

(4) Suppose $\alpha = \lim_n \beta_n$ where β_n is a bounded R-computable sequence of ordinals, and there is a finitely generated field $F \subset \mathbb{Q}$ such that all parameters in ϕ_i are in F.
   (a) The Σ_α formulas are those of the form

$$\bigvee\!\!\!\bigvee_{n \in S} \varphi_n,$$

   where for each n the formula φ_n is a Σ_{β_n} formula and S is countable and is the halting set of an R-machine.
   (b) The Π_α formulas are those of the form

$$\bigwedge\!\!\!\bigwedge_{n \in S} \varphi_n,$$

   where for each n the formula φ_n is a Π_{β_n} formula and S is countable and is the halting set of an R-machine.

The $\mathbb{R}$-computable infinitary formulas will be exactly the formulas which belong to either Σ_α or Π_α for some countable (i.e. $\mathbb{R}$-computable) α. Ash showed that Turing computable Σ_α formulas defined sets which were Σ^0_α [1].

We will say that a set is *semantically* $\mathbb{R}$-Σ_α if and only if it is the set of solutions to an $\mathbb{R}$-computable Σ_α formula, and similarly for Π_α. We will say that a set is *topologically* Σ^0_α if it is of that level in the standard Borel hierarchy using the order topology on $\mathbb{R}$.

Theorem 2.5. *We characterize the topological structure of sets in the semantic hierarchy:*

(1) *The semantically* $\mathbb{R}$-Σ_0 *sets are topologically* Δ^0_2.
(2) *If* $0 < \alpha < \omega$, *then the semantically* $\mathbb{R}$-Σ_α *sets are included among the topologically* $\Sigma^0_{\alpha+1}$ *sets.*
(3) *If* $\alpha \geq \omega$, *then the semantically* $\mathbb{R}$-Σ_α *sets are included among the topologically* Σ^0_α *sets.*

Proof. Since $\mathcal{A}$ is a $\mathbb{R}$-computable structure, the semantically $\mathbb{R}$-Σ_0 sets are all countable unions of semialgebraic sets, and the completes of semantically $\mathbb{R}$-Σ_0 are all countable unions of semialgebraic sets. Since all semialgebraic sets are topologically Δ^0_2 (that is, both topologically Σ^0_2 and Π^0_2), the countable unions of them are all topologically Σ^0_2. Now if the statement holds for $n \leq k$, it clearly holds for $n = k + 1$ by the definitions of the various classes involved.

Toward the final statement, notice that the semantically $\mathbb{R}$-Σ_ω sets are countable unions of sets at lower levels, and are all topologically Σ_ω. Above that level, the induction follows exactly as before. □

At the finite levels, Cucker proved [9] that the union of all the semantically $\mathbb{R}$-Σ_n for $n < \omega$ is the class of Borel sets of finite order. Cucker [9] defined another arithmetical hierarchy: we call a set *computationally* $\Sigma_{\alpha+1}$ if it can be enumerated by a real machine with a computationally Σ_α oracle. In particular, the semi-decidable sets are the computationally Σ_1 sets. Cucker proved that for all $k < \omega$, the computationally Σ_k sets are exactly the semantically $\mathbb{R}$-Σ_k sets. It seems likely that this result could be generalized for transfinite α, but we do not have a proof of this.

2.3. **Forcing as a Construction Technique.** Aside from the lack of inverse functions, the most difficult part of classical computability theory to get by without is the priority construction. Unless this niche can be filled, we are not optimistic concerning the parallel between Turing-computable structures and $\mathbb{R}$-computable structures. Consequently, although there are ad-hoc methods to construct $\mathbb{R}$-incomparable sets [21], we give an example in this section that is potentially more easily generalized.

Proposition 2.6. *There exist sets A_o and A_1 such that neither is computable by a $\mathbb{R}$-machine using the other as an oracle.*

Proof. The proof will closely follow the second proof given for the classical case by Lerman [20]. Let $(F, \leq)$ be the set of pairs of partial functions from $\mathbb{R}$ to 2 whose complement contains an interval, partially ordered by extension in the sense that $(p_0, p_1) \leq (q_0, q_1)$ if and only if p_i extends q_i for each i. It suffices to satisfy the following requirements for every $e \in \mathbb{R}^\infty$:

$$P_{e,i} : M_e^{A_i} \neq A_{1-i}.$$

We say that $(p_0, p_1) \Vdash P_{e,i}$ if there is some x such that either $M_e^{p_i}(x) \downarrow \neq p_{1-i}(x)$ and the latter is defined, or for any p extending P_i, we have $M_e^p \uparrow$.

Lemma 2.7 (Density Lemma)**.** *For any pair (e, i), the set $\{p \in F : p \Vdash P_{e,i}\}$ is dense.*

Proof. Let $q = (q_0, q_1) \in F$. We will show that there is some $p \leq q$ such that $p \Vdash P_{e,i}$. Let x be outside the domain of q_{1-i}. If there is no pair s, r such that r extends q_i and $M_{e,s}^r(x) \downarrow$, then $q \Vdash P_{e,i}$, so assume that such an s exists. Now we extend q_{1-i} by setting $p_{1-i} = q_{1-i} \cup \{(x, 1 - M_e^{q_i}(x))\}$, and set $p_i = q_i$. $\square$

The following Lemma is the only part of the construction which becomes genuinely more difficult in the uncountable case.

Lemma 2.8 (Existence of a Generic)**.** *Let $\mathcal{C}$ be the collection of all sets of the form $\{p \in F : p \Vdash P_{e,i}\}$. There exists a $\mathcal{C}$-generic set; that is, a pair of functions $G = (G_0, G_1)$ where $G_i : \mathbb{R} \to 2$ such that for each pair (e, i), the function G extends some element of F which forces $P_{e,i}$.*

Proof. Let $G^0 := (\emptyset, \emptyset)$, and well-order the requirements. We define $G^{\alpha+1}$ to be the extension of G^α which forces the αth requirement. For limit ordinals γ, we define $G^\gamma := \bigcup_{\beta<\gamma} G^\beta$. The union of all the G^α is a $\mathcal{C}$-generic. $\square$

Of course, we may take G to be total, by setting all undefined values to 0. Now we take A_0 to be the set whose characteristic function is G_0 and A_1 the set with characteristic function G_1. $\square$

3. Effective Categoricity

It is often possible to produce two classically computable structures which are isomorphic, but for which the isomorphism is not witnessed by a computable function. Any theory of effective mathematics must take account of this phenomenon.

Definition 3.1. A computable structure $\mathcal{M}$ is said to be *computably categorical* if and only if for any computable structure $\mathcal{N} \simeq \mathcal{M}$ there is a computable function $f : \mathcal{N} \overset{\simeq}{\to} \mathcal{M}$. The number of equivalence classes under computable isomorphism contained in an isomorphism type is called its *computable dimension*.

In the present section, we describe progress toward a parallel to the following classical result:

Theorem 3.2 (see [26], although it was almost certainly known earlier)**.** *If V is a countable vector space over $\mathbb{Q}$,*

(1) *There is a Turing-computable copy of V,*
(2) *The categoricity properties are as follows:*
   (a) *If $dim(V)$ is finite, then V is computably categorical, and*
   (b) *If $dim(V) = \omega$, then the computable dimension of V is ω.*

The existence part of the theorem is still true without serious modification.

Proposition 3.3. *Let $n \in \aleph_0 \cup \{\aleph_0, 2^{\aleph_0}\}$. Then there is a $\mathbb{R}$-computable vector space V^n of dimension n. Further, V^n has a $\mathbb{R}$-computable basis.*

Proof. Consider the language of real vector spaces (addition, plus one scaling operation for each element of $\mathbb{R}$). Let $\{b_i : i \in I\}$ be a $\mathbb{R}$-computable set of constants, where $|I| = n$. The set of closed terms with constants from $\{b_i : i \in I\}$, modulo provable equivalence (in the theory of vector spaces) is a model of the theory of vector spaces, and has dimension n. □

Of course, the categoricity result highlights an additional concern with $\mathbb{R}$-computation: It may happen that there is a $\mathbb{R}$-computable isomorphism with no $\mathbb{R}$-computable inverse. Thus, while the following result establishes, according to Definition 3.1, something very close to part 2a of Theorem 3.2, it falls short of full analogy.

Proposition 3.4. *Let $n < \aleph_0$. Then for any real vector space W of dimension n, there is a computable isomorphism $f : V^n \to W$.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a basis of W. Each member of V^n is a $\mathbb{R}$-linear combination $\sum_{i=1}^{n} \lambda_i b_i$. We map $\sum_{i=1}^{n} \lambda_i b_i$ to $\sum_{i=1}^{n} \lambda_i a_i$. □

The classical way to prove part 2b of Theorem 3.2 is to produce a computable vector space with a computable basis, and an isomorphic (i.e. same dimension) vector space with no computable basis. Without recourse to priority constructions, this strategy seems, for the present, very difficult in the $\mathbb{R}$-computable context.

4. Geometry and Topology

In the talk by the first author at EMU 2008, an early slide asked for a context in which one could formulate effectiveness questions for results like Thom's Theorem on cobordism or the classification of compact 2-manifolds. Some work in the intervening months, which began at that meeting, has yielded interesting results in $\mathbb{R}$-computable topology.

An n-manifold is a topological space which is locally homeomorphic to $\mathbb{R}^n$, satisfying some fairly obvious regularity conditions on the intersections of the neighborhoods on which homeomorphism holds. The following definition is given in [8].

Definition 4.1. A *real-computable d-manifold M* consists of real-computable i, j, j', k, the *inclusion functions*, satisfying the following conditions for all $m, n \in \omega$.

- If $i(m,n) \downarrow= 1$, then $\phi_{j(m,n)}$ is a total real-computable homeomorphism from $\mathbb{R}^d$ into $\mathbb{R}^d$, and $\phi_{j'(m,n)} = \phi^{-1}_{j(m,n)}$, and $k(m,n) \downarrow= k(n,m) \downarrow= m$.
- If $i(m,n) \downarrow= 0$, then $k(m,n) \downarrow= k(n,m) \downarrow\in \omega$ with $i(k(m,n),m) = i(k(m,n),n) = 1$ and for all $p \in \omega$, if $i(p,m) = i(p,n) = 1$, then $i(p,k(m,n)) = 1$, and for all $q \in \omega$, if $i(m,q) = i(n,q) = 1$, then $i(k(m,n),q) = 1$ with
$$\mathrm{range}(\phi_{j(m,q)}) \cap \mathrm{range}(\phi_{j(n,q)}) = \mathrm{range}(\phi_{j(k(m,n),q)}).$$
- If $i(m,n) \notin \{0,1\}$, then $i(m,n) \downarrow= i(n,m) \downarrow= -1$, and
$$(\forall p \in \omega)[i(p,m) \neq 1 \text{ or } i(p,n) \neq 1],$$
and for all $q \in \omega$, if $i(m,q)$ and $i(n,q)$ both lie in $\{0,1\}$, then
$$\mathrm{range}(\phi_{j(k(m,q),q)}) \cap \mathrm{range}(\phi_{j(k(n,q),q)}) = \emptyset.$$
- For all $q \in \omega$, if $i(m,n) = i(n,q) = 1$, then $i(m,q) = 1$ and
$$\phi_{j(n,q)} \circ \phi_{j(m,n)} = \phi_{j(m,q)}.$$

In essence, each natural number m represents a chart U_m. The functions $i(m,n)$ tell whether U_m is a subset of U_n and whether U_n is a subset of U_m. The function $j(m,n)$ is the index for a computable map giving the inclusion of U_m in U_n.

4.1. **Classifying Compact 2-Manifolds.**

Classification of n-manifolds up to homeomorphism in general is quite difficult. However, a well-known theory of disputed priority offers the following classification of compact connected 2-manifolds.

Theorem 4.2. *Let X be a compact connected 2-manifold. Then X is homeomorphic to a connected sum of 2-spheres, copies of $\mathbb{RP}^2$, copies of $S^1 \times S^1$, and copies of the Klein Bottle.*

Unpublished work by the first author and Montalban, inspired in part by discussions with R. Miller, gives an effective version of this result.

Theorem 4.3 (Calvert–Montalban)**.** *Let M and N be $\mathbb{R}$-computable compact 2-manifolds. Then there is a $\mathbb{R}$-computable homeomorphism $f : M \to N$.*

Proof outline. We can triangulate each of M and N to form a finite simplicial complex. The function f consists of a mapping on the complexes, with a smoothing effect. □

Corollary 4.4. *Let X be a compact connected $\mathbb{R}$-computable 2-manifold. Then X is homeomorphic by a $\mathbb{R}$-computable function to a connected sum of 2-spheres, copies of $\mathbb{RP}^2$, copies of $S^1 \times S^1$, and copies of the Klein Bottle.*

4.2. **Computing Homotopy Groups.**

One standard set of topological invariants for a manifold M is the sequence of groups $(\pi_n(M))_{n\in\omega}$, where $\pi_n(M)$ is the group of continuous mappings from S^n to M, up to homotopy equivalence. Under the classical model of computation, manifolds are often represented by simplicial complexes in order to discuss the possibility of computing various topological invariants. Brown showed [5] that there is a procedure which will, given a finite simplicial complex M, compute a set of generators and relations for each of the groups $\pi_n(M)$. It is natural to ask, now that we have a notion of computation that gives us algorithmic access to the manifolds themselves, whether this can be computed directly from the manifolds. We restrict attention here to the case of π_1, studied in detail in [8], although it is likely that similar results could be established for π_n.

Lemma 4.5 (Calvert–Miller [8]). *Every loop f in a computable manifold M is homotopic to a computable loop in M whose only real parameters are the base point and the inclusion functions necessary to define M.*

Nevertheless, the answer to the question of computing a fundamental group from a manifold is largely negative:

Theorem 4.6 (Calvert–Miller [8]). *Let M be a $\mathbb{R}$-computable manifold which is connected but not simply connected. Then there is no algorithm to decide whether a given loop is nullhomotopic.*

Theorem 4.7 (Calvert–Miller [8]). *There is no $\mathbb{R}$-computable function which will decide, given a $\mathbb{R}$-computable manifold, whether that manifold is simply connected.*

Nevertheless, there is a canonical family of loops, sufficient to represent the whole (but not recoverable by a uniform procedure) from which we could make the necessary computations for a fundamental group.

Lemma 4.8 (Calvert–Miller [8]). *Let M be a $\mathbb{R}$-computable manifold. Then there is a $\mathbb{R}$-computable function S_M, defined on the naturals, such that the set $S_M(n)$ consists of a set of indices for loops and contains exactly one representative from each homotopy equivalence type.*

While we cannot effectively pass from an index for M to an index for S_M, this step includes all of the difficulty in computing $\pi_1(M)$:

Theorem 4.9 (Calvert–Miller [8]). *Let M be a $\mathbb{R}$-computable manifold. Then there is a uniform procedure to pass from an index for S_M to an index for a real-computable presentation of the group $\pi_1(M)$.*

5. Relations with Other Models

5.1. **Local Computability.** Let T be a $\forall$-axiomatizable theory in a language with n symbols.

Definition 5.1. A *simple cover* of $\mathcal{S}$ is a (finite or countable) collection $\mathfrak{U}$ of finitely generated models $\mathcal{A}_0, \mathcal{A}_1, ...$ of T, such that:

- every finitely generated substructure of $\mathcal{S}$ is isomorphic to some $\mathcal{A}_i \in \mathfrak{U}$; and
- every $\mathcal{A}_i \in \mathfrak{U}$ embeds isomorphically into $\mathcal{S}$.

A simple cover $\mathfrak{U}$ is *computable* if every $\mathcal{A}_i \in \mathfrak{U}$ is a computable structure whose domain is an initial segment of ω. $\mathfrak{U}$ is *uniformly computable* if the sequence $\langle(\mathcal{A}_i, \overline{a}_i)\rangle_{i\in\omega}$ can be given uniformly: there must exist a computable function which, on input i, outputs a tuple of elements $\langle e_1, ..., e_n, \langle a_0, ..., a_k\rangle\rangle \in \omega^n \times \mathcal{A}_i^{<\omega}$ such that $\{a_0, ..., a_{k_i}\}$ generates $\mathcal{A}_i$ and ϕ_{e_j} computes the j-th function, relation, or constant in $\mathcal{A}_i$.

Definition 5.2. An embedding $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ *lifts* to the inclusion $\mathcal{B} \subset \mathcal{C}$, via isomorphisms $\beta : \mathcal{A}_i \twoheadrightarrow \mathcal{B}$ and $\gamma : \mathcal{A}_j \twoheadrightarrow \mathcal{C}$, if the diagram below commutes:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow[\subseteq]{} & \mathcal{C} \\ {\scriptstyle\beta}\Big\uparrow{\scriptstyle\cong} & & {\scriptstyle\gamma}\Big\uparrow{\scriptstyle\cong} \\ \mathcal{A}_i & \xrightarrow{f} & \mathcal{A}_j \end{array} \qquad \text{with } \gamma \circ f = \beta$$

A *cover* of $\mathcal{S}$ consists of a simple cover $\mathfrak{U} = \{\mathcal{A}_0, \mathcal{A}_1, ...\}$ of $\mathcal{S}$, along with sets $I_{ij}^{\mathfrak{U}}$ (for all $\mathcal{A}_i, \mathcal{A}_j \in \mathfrak{U}$) of injective homomorphisms $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$, such that:

(1) for all finitely generated substructures $\mathcal{B} \subseteq \mathcal{C}$ of $\mathcal{S}$, there exists $i, j \in \omega$ and an $f \in I_{ij}^{\mathfrak{U}}$ which lifts to $\mathcal{B} \subseteq \mathcal{C}$ via some isomorphisms $\beta : \mathcal{A}_i \twoheadrightarrow \mathcal{B}$ and $\gamma : \mathcal{A}_j \twoheadrightarrow \mathcal{C}$; and
(2) for every i and j, every $f \in I_{ij}^{\mathfrak{U}}$ lifts to an inclusion $\mathcal{B} \subseteq \mathcal{C}$ in $\mathcal{S}$ via some isomorphism β and γ.

This cover is *uniformly computable* if $\mathfrak{U}$ is a uniformly computable simple cover of $\mathcal{S}$ and there exists a c.e. set W such that for all $i, j \in \omega$

$$I_{ij}^{\mathfrak{U}} = \{\phi_e \upharpoonright \mathcal{A}_i : \langle i, j, e\rangle \in W\}.$$

A structure $\mathcal{B}$ is *locally computable* if it has a uniformly computable cover.

Proposition 5.3 ([22])**.** *A structure $\mathcal{S}$ is* locally computable *if and only if it has a uniformly computable simple cover.*

Proposition 5.4 ([22])**.** *The ordered field of real numbers is not locally computable.*

However, the ordered field of real numbers is trivially $\mathbb{R}$-computable. It appears at first that the ordering might be essential in escaping local computability.

Definition 5.5. A $\mathbb{R}$-machine is said to be *equational* if and only if each branch node is decided by a polynomial *equation*. We call a structure *equationally* $\mathbb{R}$-*computable* if its diagram is computable by an equational $\mathbb{R}$-machine.

Lemma 5.6 (Path Decomposition for Equational Machines)**.** *Let M be an equational $\mathbb{R}$-machine. Then the halting set of M is a countable disjoint union of algebraic sets.*

Proof. The proof is exactly the same as for normal $\mathbb{R}$-machines. □

Corollary 5.7. *The ordered field of real numbers is not equationally $\mathbb{R}$-computable.*

Theorem 5.8. *There is an equationally $\mathbb{R}$-computable structure which is not locally computable.*

Proof. Let S be a noncomputable set of natural numbers, and denote by C_n a cyclic graph on n vertices (i.e. an n-gon). Now let $\mathcal{G}$ be the structure given by

$$\left(\dot{\bigcup_{n\in S}} C_{2n} \right) \dot{\bigcup} \left(\dot{\bigcup_{n\notin S}} C_{2n+1} \right).$$

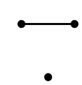
To show that $\mathcal{G}$ is equationally $\mathbb{R}$-computable, we observe that the disjoint union of two $\mathbb{R}$-computable structures is $\mathbb{R}$-computable (since the same is true of the cardinal sum). However, each of the graphs C_k has a $\mathbb{R}$-computable copy by Lagrange interpolation.

Suppose f is a uniform computable enumeration of the finitely generated substructures of $\mathcal{G}$. Then we could compute whether $n \in S$ by searching the structures indexed by $f(t)$ for successive t until we see a substructure of type C_{2n} or of type C_{2n+1}. Since S is noncomputable, no such f can exist, so that $\mathcal{G}$ is not locally computable. □

Theorem 5.9. *There is a locally computable structure which is not* $\mathbb{R}$*-computable.*

Proof. Let X be the set of all countable graphs with universe ω, and let E be the isomorphism relation on X. Now E is complete analytic [13], so $F = E^c$ is complete co-analytic. Now for any $x \in X$, we have $\neg xFx$, and for any $x, y \in X$ we have xFy if and only if yFx. Thus, F defines the adjacency relation of a graph on X. Let $\mathcal{X}$ denote the graph (X, F).

Now $\mathcal{X}$ is not real-computable, since its diagram is complete co-analytic (contradicting path decomposition). We will show that $\mathcal{X}$ is locally computable. Now the finitely generated substructures of $\mathcal{X}$ are all finite graphs, and it only remains to determine which finite graphs are included. Let T be the following graph:



Let $\mathcal{G}$ be a finite T-free graph. We will show that $\mathcal{G}$ embeds in $\mathcal{X}$. Let $\mathcal{G} = (\{0, \dots, n\}, G)$. We will define an equivalence structure R with universe $N = \{0, \dots, n\}$. For $x, y \in N$, we say that xRy if and only if $\neg xGy$. This relation R will be reflexive and symmetric. Since G is T-free, R will also be transitive. Now since the isomorphism relation is Borel complete [13], there is a function $f : N \to X$ such that xRy if and only if $f(x)Ef(y)$. This function can also be required to be injective [16].

Now let Φ be a computable Friedberg enumeration of finite graphs up to isomorphism (i.e. a total computable function whose range consists of an index for exactly one representative from each isomorphism class of finite graphs). Such an enumeration was given in [7]. We will define a Friedberg enumeration Ψ of finite T-free graphs up to isomorphism as follows: $\Psi(x)$ will be $\Phi(x')$ for the least x' such that $\Phi(x')$ is T-free and $\Phi(x') \notin ran(\Psi|_x)$. Since all of the graphs are finite, we can effectively check whether each is T-free, so that Ψ is computable. Now Ψ provides a uniform simple computable cover for $\mathcal{X}$. □

Corollary 5.10. *There is another structure* $\tilde{\mathcal{X}}$ *with the same uniform simple computable cover as* $\mathcal{X}$*, such that* $\tilde{\mathcal{X}}$ *is* $\mathbb{R}$*-computable.*

Proof. Let $\tilde{\mathcal{X}}$ be the disjoint union

$$\dot{\bigcup_{x \in \omega}} \Psi(x).$$

Now $\tilde{\mathcal{X}}$ is countable, and so is trivially $\mathbb{R}$-computable. □

One can say more about the structure described in Theorem 5.9. The structure satisfies a stronger condition called *perfectly local computability.* We recall the definition of perfectly locally computable and leave the details to the reader.

Definition 5.11. Let $\mathfrak{U}$ be a uniformly computable cover for a structure $\mathcal{S}$. A Set M is a *correspondence system* for $\mathfrak{U}$ and $\mathcal{S}$ if it satisfies all of the following:

(1) Each element of M is an embedding of some $\mathcal{A}_i \in \mathfrak{U}$ into $\mathcal{S}$; and
(2) Every $\mathcal{A}_i \in \mathfrak{U}$ is the domain of some $\beta \in M$; and
(3) Every generated $\mathcal{B} \subset \mathcal{S}$ is the image of some $\beta \in M$; and
(4) For every i and j and every $\beta \in M$ with domain $\mathcal{A}_i$, every $f \in I_{ij}^{\mathfrak{U}}$ lifts to an inclusion $\beta(\mathcal{A}_i) \subset \gamma(\mathcal{A}_i)$ via β and some $\gamma \in M$; and
(5) For every i, every $\beta \in M$ with domain $\mathcal{A}_i$, and every finitely generated $\mathcal{C} \subset \mathcal{S}$ containing $\beta(\mathcal{A}_i)$, there exist a j and an $f \in I_{ij}^{\mathfrak{U}}$ which lifts to $\beta(\mathcal{A}_i) \subset \mathcal{C}$ via β and some $\gamma : \mathcal{A}_j \twoheadrightarrow \mathcal{C} \in M$.

The correspondence system is *perfect* if it also satisfies

6. For every finitely generated $\mathcal{B} \subset \mathcal{S}$, if $\beta : \mathcal{A}_i \twoheadrightarrow \mathcal{B}$ and $\gamma : \mathcal{A}_j \twoheadrightarrow \mathcal{B}$ both lie in M and have image $\mathcal{B}$, then $\gamma^{-1} \circ \beta \in I_{ij}^{\mathfrak{U}}$.

If a perfect correspondence system exists, then its elements are called *perfect matches* between their domains and their images. $\mathcal{S}$ is then said to be *perfectly locally computable* with *perfect cover* $\mathfrak{U}$.

5.2. **Σ-Definability.** The following definition is standard, and appears in equivalent forms in [2] and [10].

Definition 5.12. Given a structure $\mathcal{M}$ with universe M, we define a new structure $HF(\mathcal{M})$ as follows.

(1) The universe of $HF(\mathcal{M})$ is the union of the chain $HF_n(M)$ defined as follows:
  (a) $HF_0(M) = M$
  (b) $HF_{n+1}(M) = \mathcal{P}^{<\omega}(M \cup HF_n(M))$, where $\mathcal{P}^{<\omega}(S)$ is the set of all finite subsets of S
(2) The language for $HF(\mathcal{M})$ consists of a unary predicate U for $HF_0(M)$, as well as a predicate $\in$ interpreted as membership, plus a symbol σ^* for each symbol σ of the language of $\mathcal{M}$, given the interpretation of σ on $M = HF_0(M)$.

Ershov gave a definition [10] of a notion generalizing computability to structures other than $\mathfrak{N}$. We will first give Barwise's definition [2] of the class of Σ-formulas.

Definition 5.13. The class of Σ-formulas are defined by induction.

(1) Each Δ_0 formula is a Σ-formula.
(2) If Φ and Ψ are Σ-formulas, then so are $(\Phi \wedge \Psi)$ and $(\Phi \vee \Psi)$.
(3) For each variable x and each term t, if Φ is a Σ-formula, then the following are also Σ-formulas:
  (a) $\exists(x \in t) \;\; \Phi$
  (b) $\forall(x \in t) \;\; \Phi$, and
  (c) $\exists x \Phi$.

A predicate S is called a *Δ-predicate* if both S and its complement are defined by Σ-formulas.

Definition 5.14. Let $\mathcal{M}$ and $\mathcal{N} = (N, P_0, P_1, \dots)$ be structures. We say that $\mathcal{N}$ is Σ-definable in $HF(\mathcal{M})$ if and only if there are Σ-formulas $\Psi_0, \Psi_1, \Psi_1^*, \Phi_0, \Phi_0^*, \Phi_1, \Phi_1^*, \dots$ such that

(1) $\Psi_0^{HF(\mathcal{M})} \subseteq HF(\mathcal{M})$ is nonempty,
(2) Ψ_1 defines a congruence relation on $\left(\Psi_0^{HF(\mathcal{M})}, \Phi_0^{HF(\mathcal{M})}, \Phi_1^{HF(\mathcal{M})}, \dots\right)$,
(3) $(\Psi_1^*)^{HF(\mathcal{M})}$ is the relative complement in $(\Psi_0^{HF(\mathcal{M})})^2$ of $\Psi_1^{HF(\mathcal{M})}$,
(4) For each i, the set $(\Phi_i^*)^{HF(\mathcal{M})}$ is the relative complement in $\Psi_0^{HF(\mathcal{M})}$ of $\Phi_i^{HF(\mathcal{M})}$, and
(5) $\mathcal{N} \simeq \left(\Psi_0^{HF(\mathcal{M})}, \Phi_0^{HF(\mathcal{M})}, \Phi_1^{HF(\mathcal{M})}, \dots\right) /_{\Psi_1^{HF(\mathcal{M})}}$.

Theorem 5.15 (Calvert [6]). *The structures which have isomorphic copies Σ-definable over $HF(\mathbb{R})$ are exactly the ones which have isomorphic copies which are $\mathbb{R}$-computable.*

An interesting consequense of this (an immediate corollary of Theorem 5.15 and a result of Morozov and Korovina [23]) gives a sense in which some $\mathbb{R}$-computable structures can be approximated by classically computable structures.

Definition 5.16. Let $\mathcal{A}$ and $\mathcal{B}$ be structures in a common signature. We write that $\mathcal{A} \leq_1 \mathcal{B}$ if $\mathcal{A}$ is a substructure of $\mathcal{B}$, and for all existential formulas $\varphi(\bar{x})$ and for all tuples $\bar{a} \subseteq \mathcal{A}$, we have

$$\mathcal{B} \models (\varphi(\bar{a}) \Rightarrow \mathcal{A} \models \varphi(\bar{a}).$$

Corollary 5.17. *For any $\mathbb{R}$-computable structure $\mathcal{M}$ whose defining machine involves only algebraic reals as parameters, there is a computable structure $\mathcal{M}^*$ such that $\mathcal{M}^* \leq_1 \mathcal{M}$.*

5.3. **F-Parameterizability.** Morozov introduced a concept that he called *F-parameterizability* in order to understand the elementary substructure relation on both automorphism groups and the structure of hereditarily finite sets over a given structure [25]. In a talk at Stanford University, though, he identified this notion as one "which generalizes the notion of computable" [24].

Definition 5.18 ([25]). Let $\mathcal{M}$ be a structure in a finite relational language $\left(P_n^{k_n}\right)_{n \leq k}$. We say that $\mathcal{M}$ is F-parameterizable if and only if there is an injection $\xi : \mathcal{M} \to \omega^\omega$ with the following properties:
(1) The image of ξ is analytic in the Baire space, and
(2) For each n, the set $\left\{(\xi(a_i))_{i \leq k_n} : \mathcal{M} \models P_n(\bar{a})\right\}$ is analytic.

The function ξ is called an F-parameterization of $\mathcal{M}$. Morozov also introduced the following stronger condition, essentially requiring that $\mathcal{M}$ be able to define its own F-parameterization.

Definition 5.19 ([25]). Let $\mathcal{M}$ be an F-parameterizable structure. We say that $\mathcal{M}$ is weakly selfparameterizable if and only if there are functions $\Xi, p : \mathcal{M} \times \omega \to \omega$, both definable without parameters in $HF(\mathcal{M})$, with the following properties:
(1) For all $x \in \mathcal{M}$ and all $m \in \omega$, we have $\Xi(x, m) = \xi(x)[m]$, and
(2) For all $f \in \omega^\omega$ there is some $x \in \mathcal{M}$ such that for all $n \in \omega$ we have $p(x, n) = f(n)$.

In making sense of effectiveness on uncountable structures, a major motivation is to describe a sense in which real number arithmetic — an operation that, while not Turing computable, does not seem horribly ineffective — can be considered to be effective.

Proposition 5.20 (Morozov [25])**.** *The real field is weakly F-selfparameterizable.*

Outline of proof. Define a function $\xi : \mathbb{R} \to \omega^\omega$ maps x to its decimal expansion. This function is definable without parameters in $HF(\mathbb{R})$, in the sense required by Definition 5.19. □

Theorem 5.21 (Calvert [6])**.** *Every $\mathbb{R}$-computable structure is F-parameterizable. On the other hand, the structure $(\mathbb{R}, +, \cdot, 0, 1, e^x)$ is weakly F-selfparameterizable but not $\mathbb{R}$-computable.*

6. Conclusion

We state here some open problems arising from issues discussed in the present paper. The first is perhaps the most vital.

Problem 6.1. Develop a substitute for the priority method which is capable of handling constructions with injury.

Question 6.2. Is it true that for any $\mathbb{R}$-computable finite dimensional $\mathbb{R}$-vector spaces M and N with the same dimension, there is a $\mathbb{R}$-computable isomorphism from M to N?

Conjecture 6.3. *A $\mathbb{R}$-computable $\mathbb{R}$-vector space of dimension greater than $\aleph_0$ is not $\mathbb{R}$-computably categorical.*

We would also like to know about the categoricity of vector spaces of dimension $\aleph_0$, but are not ready to hazard a conjecture at this time.

Question 6.4. Does there exist a $\mathbb{R}$-computable Banach space of infinite dimension in the language of vector spaces, augmented by a sort for $\mathbb{R}$ and a function interpreted as the norm?

Question 6.5. Does there exist a $\mathbb{R}$-computable Hilbert space of infinite dimension in the language of vector spaces, augmented by a sort for $\mathbb{R}$ and a binary function interpreted as the inner product?

On each of the previous two questions, the authors had difficulty guaranteeing completeness.

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