REINFORCEMENT LEARNING WITH LINEAR FUNCTION APPROXIMATION AND LQ CONTROL CONVERGES*

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ABSTRACT. Reinforcement learning is commonly used with function approximation. However, very few positive results are known about the convergence of function approximation based RL control algorithms. In this paper we show that TD(0) and Sarsa(0) with linear function approximation is convergent for a simple class of problems, where the system is linear and the costs are quadratic (the LQ control problem). Furthermore, we show that for systems with Gaussian noise and non-completely observable states (the LQG problem), the mentioned RL algorithms are still convergent, if they are combined with Kalman filtering.

1. Introduction

Reinforcement learning is commonly used with function approximation. However, the technique has little theoretical performance guarantees: for example, it has been shown that even linear function approximators (LFA) can diverge with such often used algorithms as Q-learning or value iteration [1, 8]. There are positive results as well: it has been shown [10, 7, 9] that $TD(\lambda)$, Sarsa, importance-sampled Q-learning are convergent with LFA, if the policy remains constant (policy evaluation). However, to the best of our knowledge, the only result about the control problem (when we try to find the optimal policy) is the one of Gordon's [4], who proved that TD(0) and Sarsa(0) can not diverge (although they may oscillate around the optimum, as shown in [3])¹.

In this paper, we show that RL control with linear function approximation can be convergent when it is applied to a linear system, with quadratic cost functions (known as the LQ control problem). Using the techniques of Gordon [4], we were prove that under appropriate conditions, TD(0) and Sarsa(0) converge to the optimal value function. As a consequence, Kalman filtering with RL is convergent for observable systems, too.

Although the LQ control task may seem simple, and there are numerous other methods solving it, we think that this Technical Report has some significance: (i) To our best knowledge, this is the first paper showing the convergence of an RL control algorithm using LFA. (ii) Many problems can be translated into LQ form [2].

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¹These are results for policy iteration (e.g. [5]). However, by construction, policy iteration could be very slow in practice.

2. THE LQ CONTROL PROBLEM

Consider a linear dynamical system with state $\mathbf{x}_t \in \mathbb{R}^n$, control $\mathbf{u}_t \in \mathbb{R}^m$, in discrete time t:

$$\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t.$$

Executing control step \mathbf{u}_t in \mathbf{x}_t costs

(2)
$$c(\mathbf{x}_t, \mathbf{u}_t) := \mathbf{x}_t^T Q \mathbf{x}_t + \mathbf{u}_t^T R \mathbf{u}_t,$$

and after the N^{th} step the controller halts and receives a final cost of $\mathbf{x}_N^T Q_N \mathbf{x}_N$. The task is to find a control sequence with minimum total cost.

First of all, we slightly modify the problem: the run time of the controller will not be a fixed number N. Instead, after each time step, the process will be stopped with some fixed probability p (and then the controller incurs the final cost $c_f(\mathbf{x}_f) := \mathbf{x}_f^T Q^f \mathbf{x}_f$). This modification is commonly used in the RL literature; it makes the problem more amenable to mathematical treatments.

2.1. The cost-to-go function. Let $V_t^*(\mathbf{x})$ be the optimal cost-to-go function at time step t, i.e.

(3)
$$V_t^*(\mathbf{x}) := \inf_{\mathbf{u}_t, \mathbf{u}_{t+1}, \dots} E[c(\mathbf{x}_t, \mathbf{u}_t) + c(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) + \dots + c_f(\mathbf{x}_f) \big| \mathbf{x}_t = \mathbf{x}].$$

Considering that the controller is stopped with probability p, Eq. 3 assumes the following form

(4)
$$V_t^*(\mathbf{x}) = p \cdot c_f(\mathbf{x}) + (1-p) \inf_{\mathbf{u}} \left(c(\mathbf{x}, \mathbf{u}) + V_{t+1}^*(F\mathbf{x} + G\mathbf{u}) \right)$$

for any state \mathbf{x} . It is an easy matter to show that the optimal cost-to-go function is time-independent and it is a quadratic function of \mathbf{x} . That is, the optimal cost-to-go action-value function assumes the form

$$V^*(\mathbf{x}) = \mathbf{x}^T \Pi^* \mathbf{x}.$$

Our task is to estimate the optimal value functions (i.e., parameter matrix Π^*) on-line. This can be done by the method of temporal differences.

We start with an arbitrary initial cost-to-go function $V_0(\mathbf{x}) = \mathbf{x}^T \Pi_0 \mathbf{x}$. After this,

- (1) control actions are selected according to the current value function estimate
- (2) the value function is updated according to the experience, and
- (3) these two steps are iterated.

The t^{th} estimate of V^* is $V_t(\mathbf{x}) = \mathbf{x}^T \Pi_t \mathbf{x}$. The greedy control action according to this is given by

(6)
$$\mathbf{u}_{t} = \arg\min_{\mathbf{u}} \left(c(\mathbf{x}_{t}, \mathbf{u}) + V_{t}(F\mathbf{x}_{t} + G\mathbf{u}) \right)$$
$$= \arg\min_{\mathbf{u}} \left(\mathbf{u}^{T} R\mathbf{u} + (F\mathbf{x}_{t} + G\mathbf{u})^{T} \Pi_{t}(F\mathbf{x}_{t} + G\mathbf{u}) \right)$$
$$= -(R + G^{T} \Pi_{t} G)^{-1} (G^{T} \Pi_{t} F) \mathbf{x}_{t}.$$

The 1-step TD error is

(7)
$$\delta_t = \begin{cases} c_f(\mathbf{x}_t) - V_t(\mathbf{x}_t) & \text{if } t = t_{STOP}, \\ \left(c(\mathbf{x}_t, \mathbf{u}_t) + V_t(\mathbf{x}_{t+1}) \right) - V_t(\mathbf{x}_t), & \text{otherwise.} \end{cases}$$

Initialize
$$\mathbf{x}_0$$
, \mathbf{u}_0 , Π_0
repeat
$$\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t$$

$$\nu_{t+1} := \text{random noise}$$

$$\mathbf{u}_{t+1} = -(R + G^T\Pi_{t+1}G)^{-1}(G^T\Pi_{t+1}F)\mathbf{x}_{t+1} + \nu_{t+1}$$
with probability p ,
$$\delta_t = \mathbf{x}_t^T Q^f \mathbf{x}_t - \mathbf{x}_t^T \Pi_t \mathbf{x}_t$$
STOP
else
$$\delta_t = \mathbf{u}_t^T R \mathbf{u}_t + \mathbf{x}_{t+1}^T \Pi_t \mathbf{x}_{t+1} - \mathbf{x}_t^T \Pi_t \mathbf{x}_t$$

$$\Pi_{t+1} = \Pi_t + \alpha_t \delta_t \mathbf{x}_t \mathbf{x}_t^T$$

$$t = t+1$$
end

FIGURE 1. TD(0) with linear function approximation for LQ control

and the update rule for the parameter matrix Π_t is

(8)
$$\Pi_{t+1} = \Pi_t + \alpha_t \cdot \delta_t \cdot \nabla_{\Pi_t} V_t(\mathbf{x}_t)$$

$$= \Pi_t + \alpha_t \cdot \delta_t \cdot \mathbf{x}_t \mathbf{x}_t^T,$$

where α_t is the learning rate.

The algorithm is summarized in Fig. 1.

2.2. Sarsa. The cost-to-go function is used to select control actions, so the action-value function $Q_t^*(\mathbf{x}, \mathbf{u})$ is more appropriate for this purpose. The action-value function is defined as

$$Q_t^*(\mathbf{x}, \mathbf{u}) := \inf_{\mathbf{u}_{t+1}, \mathbf{u}_{t+2}, \dots} E[c(\mathbf{x}_t, \mathbf{u}_t) + c(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) + \dots + c_f(\mathbf{x}_f) | \mathbf{x}_t = \mathbf{x}, \mathbf{u}_t = \mathbf{u}],$$

and analogously to V_t^* , it can be shown that it is time independent and can be written in the form

(9)
$$Q^*(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} \mathbf{x}^T & \mathbf{u}^T \end{pmatrix} \begin{pmatrix} \Theta_{11}^* & \Theta_{12}^* \\ \Theta_{21}^* & \Theta_{22}^* \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{x}^T & \mathbf{u}^T \end{pmatrix} \Theta^* \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}.$$

Note that Π^* can be expressed by Θ^* using the relationship $V(\mathbf{x}) = \min_{\mathbf{u}} Q(\mathbf{x}, \mathbf{u})$:

(10)
$$\Pi^* = \Theta_{11}^* - \Theta_{12}^* (\Theta_{22}^*)^{-1} \Theta_{21}^*$$

If the t^{th} estimate of Q^* is $Q_t(\mathbf{x}, \mathbf{u}) = [\mathbf{x}^T, \mathbf{u}^T]^T \Theta_t[\mathbf{x}^T, \mathbf{u}^T]$, then the greedy control action is given as

(11)
$$\mathbf{u}_{t} = \arg\min_{\mathbf{u}} Q_{t}(\mathbf{x}, \mathbf{u}) = -\Theta_{22}^{-1} \frac{\Theta_{21}^{T} + \Theta_{21}}{2} \mathbf{x}_{t} = -\Theta_{22}^{-1} \Theta_{21} \mathbf{x}_{t},$$

where subscript t of Θ has been omitted to improve readability.

The estimation error and the weight update are similar to the state-value case:

(12)
$$\delta_t = \begin{cases} c_f(\mathbf{x}_t) - Q_t(\mathbf{x}_t, \mathbf{u}_t) & \text{if } t = t_{STOP}, \\ \left(c(\mathbf{x}_t, \mathbf{u}_t) + Q_t(\mathbf{x}_{t+1}, \mathbf{u}_{t+1}) \right) - Q_t(\mathbf{x}_t, \mathbf{u}_t), & \text{otherwise,} \end{cases}$$

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Initialize \mathbf{x}_0, \mathbf{u}_0, \Theta_0

\mathbf{z}_0 = (\mathbf{x}_0^T \mathbf{u}_0^T)^T

repeat
 \begin{aligned} \mathbf{x}_{t+1} &= F \mathbf{x}_t + G \mathbf{u}_t \\ \nu_{t+1} &:= \text{random noise} \\ \mathbf{u}_{t+1} &= -(\Theta_t)_{22}(\Theta_t)_{21} \mathbf{x}_{t+1} + \nu_{t+1} \\ \mathbf{z}_{t+1} &= (\mathbf{x}_{t+1}^T \mathbf{u}_{t+1}^T)^T \\ \text{with probability } p, \\ \delta_t &= \mathbf{x}_t^T Q^f \mathbf{x}_t - \mathbf{z}_t^T \Theta_t \mathbf{z}_t \\ \text{STOP} \end{aligned} 
else
 \delta_t &= \mathbf{u}_t^T R \mathbf{u}_t + \mathbf{z}_{t+1}^T \Theta_t \mathbf{z}_{t+1} - \mathbf{z}_t^T \Theta_t \mathbf{z}_t \\ \Theta_{t+1} &= \Theta_t + \alpha_t \delta_t \mathbf{z}_t \mathbf{z}_t^T \end{aligned} 
 t = t+1
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FIGURE 2. Sarsa(0) with linear function approximation for LQ control

(13)
$$\Theta_{t+1} = \Theta_t + \alpha_t \cdot \delta_t \cdot \nabla_{\Theta_t} Q_t(\mathbf{x}_t, \mathbf{u}_t)$$
$$= \Theta_t + \alpha_t \cdot \delta_t \cdot \begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{pmatrix} \begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{pmatrix}^T.$$

The algorithm is summarized in Fig. 2.

3. Convergence

Theorem 3.1. If $\Pi_0 \geq \Pi^*$, there exists an L such that $||F + GL|| \leq 1/\sqrt{1-p}$, then there exists a series of learning rates α_t such that $0 < \alpha_t \leq 1/||\mathbf{x}_t||^4$, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, and it can be computed online. For all sequences of learning rates satisfying these requirements, Algorithm 1 converges to the optimal policy.

The proof of the theorem can be found in Appendix B.

The same line of thought can be carried over for the action-value function $Q(\mathbf{x}, \mathbf{u}) = (\mathbf{x}^T \mathbf{u}^T)^T \Theta(\mathbf{x}^T \mathbf{u}^T)$, which we do not detail here, giving only the result:

Theorem 3.2. If $\Theta_0 \geq \Theta^*$, there exists an L such that $||F + GL|| \leq 1/\sqrt{1-p}$, then there exists a series of learning rates α_t such that $0 < \alpha_t \leq 1/||\mathbf{x}_t||^4$, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, and it can be computed online. For all sequences of learning rates satisfying these requirements, Sarsa(0) with LFA (Fig. 2) converges to the optimal policy.

4. Kalman filter LQ control

Now let us examine the case when we do not know the exact states, but we have to estimate them from noisy observations. Consider a linear dynamical system with state $\mathbf{x}_t \in \mathbb{R}^n$, control $\mathbf{u}_t \in \mathbb{R}^m$, observation $\mathbf{y}_t \in \mathbb{R}^k$, noises $\xi_t \in \mathbb{R}^n$ and $\zeta_t \in \mathbb{R}^k$ (which are assumed to be uncorrelated Gaussians with covariance matrix Ω^{ξ} and

 Ω^{ζ} , respectively), in discrete time t:

$$\mathbf{x}_{t+1} = F\mathbf{x}_t + G\mathbf{u}_t + \xi_t$$

$$\mathbf{y}_t = H\mathbf{x}_t + \zeta_t.$$

Assume that the initial state has mean $\hat{\mathbf{x}}_1$, and covariance Σ_1 . Furthermore, assume that executing control step \mathbf{u}_t in \mathbf{x}_t costs

(16)
$$c(\mathbf{x}_t, \mathbf{u}_t) := \mathbf{x}_t^T Q \mathbf{x}_t + \mathbf{u}_t^T R \mathbf{u}_t,$$

After each time step, the process will be stopped with some fixed probability p, and then the controller incurs the final cost $c_f(\mathbf{x}_f) := \mathbf{x}_f^T Q^f \mathbf{x}_f$.

We will show that the separation principle holds for our problem, i.e. the control law and the state filtering can be computed independently from each other. On one hand, state estimation is independent of the control selection method (in fact, the control could be anything, because it does not affect the estimation error), i.e. we can estimate the state of the system by the standard Kalman filtering equations:

$$\hat{\mathbf{x}}_{t+1} = F\hat{\mathbf{x}}_t + G\mathbf{u}_t + K_t(\mathbf{y}_t - H\hat{\mathbf{x}}_t)$$

(18)
$$K_t = F\Sigma_t H^T (H\Sigma_t H^T + \Omega^e)^{-1}$$

(19)
$$\Sigma_{t+1} = \Omega^w + F \Sigma_t F^T - K_t H \Sigma_t F^T.$$

On the other hand, it is easy to show that the optimal control can be expressed as the function of $\hat{\mathbf{x}}_t$. The proof (similarly to the proof of the original separation principle) is based on the fact that the noise and error terms appearing in the expressions are either linear and have zero mean or quadratic and independent of \mathbf{u} . In both cases they can be omitted. More precisely, let W_t denote the sequence $\mathbf{y}_1, \ldots, \mathbf{y}_t, \mathbf{u}_1, \ldots, \mathbf{u}_{t-1}$, and let $\mathbf{e}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$. Equation (6) for the filtered case can be formulated as

(20)
$$\mathbf{u}_{t} = \arg\min_{\mathbf{u}} E\left(c(\mathbf{x}_{t}, \mathbf{u}) + V_{t}(F\mathbf{x}_{t} + G\mathbf{u} + \xi_{t})\middle|W_{t}\right)$$
$$= \arg\min_{\mathbf{u}} E\left(\mathbf{x}_{t}^{T}Q\mathbf{x}_{t} + \mathbf{u}^{T}R\mathbf{u} + (F\mathbf{x}_{t} + G\mathbf{u} + \xi_{t})\middle|W_{t}\right).$$

Using the fact that $E(\mathbf{x}_t^T Q \mathbf{x}_t | W_t)$ and $E(\xi_t^T \Pi_t \xi_t | W_t)$ are independent of \mathbf{u} and that $E((F\mathbf{x}_t + G\mathbf{u})^T \Pi_t \xi_t | W_t) = 0$, furthermore that $\mathbf{x}_t = \hat{\mathbf{x}}_t + \mathbf{e}_t$, we get

$$\mathbf{u}_t = \arg\min_{\mathbf{u}} E\left(\mathbf{u}^T R \mathbf{u} + (F\hat{\mathbf{x}}_t + F\mathbf{e}_t + G\mathbf{u})^T \Pi_t (F\hat{\mathbf{x}}_t + F\mathbf{e}_t + G\mathbf{u}) \middle| W_t\right)$$

Finally, we know that $E(\mathbf{e}_t|W_t) = 0$, because the Kalman filter is an unbiased estimator, furthermore $E(\mathbf{e}_t^T \Pi_t \mathbf{e}_t | W_t)$ is independent of \mathbf{u} , which yields

$$\mathbf{u}_{t} = \arg\min_{\mathbf{u}} E\left(\mathbf{u}^{T} R \mathbf{u} + (F \hat{\mathbf{x}}_{t} + G \mathbf{u})^{T} \Pi_{t} (F \hat{\mathbf{x}}_{t} + G \mathbf{u}) \middle| W_{t}\right)$$
$$= -(R + G^{T} \Pi_{t} G)^{-1} (G^{T} \Pi_{t} F) \hat{\mathbf{x}}_{t},$$

i.e. for the computation of the greedy control action according to V_t we can use the estimated state instead of the exact one. The proof of the separation principle for SARSA(0) is quite similar and therefore is omitted here.

The resulting algorithm using TD(0) is summarized in Fig. 3. The algorithm using Sarsa can be derived in a similar manner.

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Initialize \mathbf{x}_0, \hat{\mathbf{x}}_0, \mathbf{u}_0, \Pi_0, \Sigma_0 repeat \begin{aligned} \mathbf{x}_{t+1} &= F\mathbf{x}_t + G\mathbf{u}_t + \xi_t \\ \mathbf{y}_t &= H\mathbf{x}_t + \zeta_t \\ \Sigma_{t+1} &= \Omega^\xi + F\Sigma_t F^T - K_t H\Sigma_t F^T \\ K_t &= F\Sigma_t H^T (H\Sigma_t H^T + \Omega^\zeta)^{-1} \\ \hat{\mathbf{x}}_{t+1} &= F\hat{\mathbf{x}}_t + G\mathbf{u}_t + K_t (\mathbf{y}_t - H\hat{\mathbf{x}}_t) \\ \nu_{t+1} &:= \text{random noise} \\ \mathbf{u}_{t+1} &= -(R + G^T \Pi_{t+1} G)^{-1} (G^T \Pi_{t+1} F) \hat{\mathbf{x}}_{t+1} + \nu_{t+1} \\ \text{with probability } p, \\ \delta_t &= \hat{\mathbf{x}}_t^T Q^f \hat{\mathbf{x}}_t - \hat{\mathbf{x}}_t^T \Pi_t \hat{\mathbf{x}}_t \\ \text{STOP} \end{aligned}else \delta_t = \mathbf{u}_t^T R \mathbf{u}_t + \hat{\mathbf{x}}_{t+1}^T \Pi_t \hat{\mathbf{x}}_{t+1} - \hat{\mathbf{x}}_t^T \Pi_t \hat{\mathbf{x}}_t \\ \Pi_{t+1} &= \Pi_t + \alpha_t \delta_t \hat{\mathbf{x}}_t \hat{\mathbf{x}}_t^T \\ t &= t+1 \end{aligned}end
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FIGURE 3. Kalman filtering with TD control

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Appendix A. The boundedness of
$$\|\mathbf{x}_t\|$$

We need several technical lemmas to show that $\|\mathbf{x}_t\|$ remains bounded for the linear-quadratic case, and also, $E(\|\mathbf{x}_t\|)$ remains bounded for the Kalman filter case. The latter result implies that for the KF case, $\|\mathbf{x}_t\|$ remains bounded with high probability.

For any positive semidefinite matrix Π and any state \mathbf{x} , we can define the action vector which minimizes the one-step-ahead value function:

$$\mathbf{u}_{greedy} := \arg\min_{\mathbf{u}} \left(\mathbf{u}^T R \mathbf{u} + (F \mathbf{x} + G \mathbf{u})^T \Pi (F \mathbf{x} + G \mathbf{u}) \right)$$
$$= -(R + G^T \Pi G)^{-1} (G^T \Pi F) \mathbf{x}.$$

Let

$$L_{\Pi} := -(R + G^T \Pi G)^{-1} (G^T \Pi F)$$

denote the greedy control for matrix Π , and let

$$L^* = -(R + G^T \Pi^* G)^{-1} (G^T \Pi^* F)$$

be the optimal policy, furthermore, let $q := 1/\sqrt{1-p}$.

Lemma A.1. If there exists an L such that ||F + GL|| < q, then $||F + GL^*|| < q$ as well.

Proof. Indirectly, suppose that $||F + GL^*|| \ge q$. Then for a fixed \mathbf{x}_0 , let \mathbf{x}_t be the optimal trajectory

$$\mathbf{x}_{t+1} = (F + GL^*)\mathbf{x}_t.$$

Then

$$V^*(\mathbf{x}_0) = p \ c_f(\mathbf{x}_0) + (1-p)c(\mathbf{x}_0, L^*\mathbf{x}_0)$$
+ $(1-p)p \ c_f(\mathbf{x}_1) + (1-p)^2c(\mathbf{x}_1, L^*\mathbf{x}_1)$
+ $(1-p)^2p \ c_f(\mathbf{x}_2) + (1-p)^3c(\mathbf{x}_2, L^*\mathbf{x}_2)$
+ ...,

$$V^{*}(\mathbf{x}_{0}) \geq p(c_{f}(\mathbf{x}_{0}) + (1-p)c_{f}(\mathbf{x}_{1}) + (1-p)^{2}c_{f}(\mathbf{x}_{2}) + \ldots)$$

= $p\sum_{f}(1-p)^{k}\mathbf{x}_{0}^{T}(F+GL^{*})^{k}^{T}Q^{f}(F+GL^{*})^{k}\mathbf{x}_{0}.$

We know that Q^f is positive definite, so there exists an ϵ such that $\mathbf{x}^T Q^f \mathbf{x} \ge \epsilon \|\mathbf{x}\|^2$, therefore

$$V^*(\mathbf{x}_0) \geq \epsilon p \sum (1-p)^k \left\| (F+GL^*)^k \mathbf{x}_0 \right\|^2.$$

If \mathbf{x}_0 is the eigenvector corresponding to the maximal eigenvalue of $F + GL^*$, then $(F + GL^*)\mathbf{x}_0 = ||F + GL^*||\mathbf{x}_0$, and so $(F + GL^*)^k\mathbf{x}_0 = ||F + GL^*||^k\mathbf{x}_0$. Consequently,

$$V^{*}(\mathbf{x}_{0}) \geq \epsilon p \sum_{k} (1-p)^{k} \|F + GL^{*}\|^{2k} \|\mathbf{x}_{0}\|^{2}$$
$$\geq \epsilon p \sum_{k} (1-p)^{k} \frac{1}{(1-p)^{k}} \|\mathbf{x}_{0}\|^{2} = \infty.$$

On the other hand, because of ||F + GL|| < q, it is easy to see that the value of following the control law L from \mathbf{x}_0 is finite, therefore we get $V^L(\mathbf{x}_0) < V^*(\mathbf{x}_0)$, which is a contradiction.

Lemma A.2. For positive definite matrices A and B, if $A \ge B$ then $||A^{-1}B|| \le 1$.

Proof. Indirectly, suppose that $||A^{-1}B|| > 1$. Let λ_{max} be the maximal eigenvalue of $A^{-1}B$, and \mathbf{v} be a corresponding eigenvector.

$$A^{-1}B\mathbf{v} = \lambda_{max}\mathbf{v},$$

and according to the indirect assumption,

$$\lambda_{max} = \left\| A^{-1}B \right\| > 1.$$

 $A \geq B$ means that for each \mathbf{x} , $\mathbf{x}^T A \mathbf{x} \geq \mathbf{x}^T B \mathbf{x}$, so this holds specifically for $\mathbf{x} = A^{-1} B \mathbf{v} = \lambda_{max} \mathbf{v}$, too. So, on one hand,

$$(\lambda_{max}\mathbf{v})^T B(\lambda_{max}\mathbf{v}) = \lambda_{max}^2 \mathbf{v}^T B \mathbf{v} > \mathbf{v}^T B \mathbf{v},$$

and on the other hand.

$$(\lambda_{max}\mathbf{v})^T A(\lambda_{max}\mathbf{v}) = (A^{-1}B\mathbf{v})^T A(A^{-1}B\mathbf{v}) = \mathbf{v}^T (BA^{-1}B)\mathbf{v},$$

so,

$$\mathbf{v}^T (BA^{-1}B)\mathbf{v} > \mathbf{v}^T B\mathbf{v},$$

However, from $A \geq B$, $A^{-1} \leq B^{-1}$. Multiplying this with B from both sides, we get $BA^{-1}B \leq B$, which is a contradiction.

Lemma A.3. If there exists an L such that ||F + GL|| < q then for any Π such that $\Pi \ge \Pi^*$, $||F + GL_{\Pi}|| < q$, too.

Proof. We will apply the Woodbury identity [6], stating that for positive definite matrices R and Π ,

$$(R + G^T \Pi G)^{-1} G^T \Pi = R^{-1} G^T (GR^{-1} G^T + \Pi^{-1})^{-1}$$

Consequently,

$$F + GL_{\Pi} = F - G(R + G^{T}\Pi G)^{-1}(G^{T}\Pi F)$$

= $F - \left(GR^{-1}G^{T}\right)\left((GR^{-1}G^{T} + \Pi^{-1})^{-1}\right)F.$

Let

$$U_{\Pi} := I - \left(GR^{-1}G^{T}\right)\left(GR^{-1}G^{T} + \Pi^{-1}\right)^{-1}$$
$$= \Pi^{-1}\left(GR^{-1}G^{T} + \Pi^{-1}\right)^{-1}$$

and

$$U^* := I - \left(GR^{-1}G^T\right) \left(GR^{-1}G^T + (\Pi^*)^{-1}\right)^{-1}$$
$$= (\Pi^*)^{-1} \left(GR^{-1}G^T + (\Pi^*)^{-1}\right)^{-1}$$

Both matrices are positive definite, because they are the product of positive definite matrices. With these notations, $F + GL_{\Pi} = U_{\Pi}F$ and $F + GL^* = U^*F$.

It is easy to show that $U_{\Pi} \leq U^*$ exploiting the fact that $\Pi \geq \Pi^*$ and several well-known properties of matrix inequalities: if $A \geq B$ and C is positive semidefinite, then $-A \leq -B$, $A^{-1} \leq B^{-1}$, $A + C \geq B + C$, $A \cdot C \geq B \cdot C$.

From Lemma A.1 we know that $||U^*F|| = ||F + GL^*|| < q$, and from the previous lemma we know that $||U_{\pi}(U^*)^{-1}|| \le 1$, so

$$||F + GL_{\Pi}|| = ||U_{\Pi}F|| = ||U_{\Pi}(U^*)^{-1}U^*F|| \le ||U_{\Pi}(U^*)^{-1}|| ||U^*F|| \le 1 \cdot q$$

Corollary A.4. If there exists an L such that $||F + GL|| \le q$, then the state sequence generated by the noise-free LQ equations is bounded, i.e., there exists $M \in \mathbb{R}$ such that $||\mathbf{x}_t|| \le M$.

Proof. This is a simple corollary of the previous lemma: in each step we use a greedy control law L_t , so

$$\|\mathbf{x}_{t+1}\| = \|(F + GL_t)\mathbf{x}_t\| \le q \|\mathbf{x}_t\|$$

Corollary A.5. If there exists an L such that $||F + GL|| \le q$, then the state sequence generated by the Kalman-filter equations is bounded with high probability, i.e., for any e > 0, there exists $M \in \mathbb{R}$ such that $||\mathbf{x}_t|| \le M$ with probability $1 - \epsilon$.

Proof.

$$E \|\mathbf{x}_{t+1}\| = E \|(F + GL_t)\mathbf{x}_t + \xi_t\| \le \sqrt{E \|(F + GL_t)\mathbf{x}_t\| + \Omega_{\xi}}$$

$$\le \sqrt{qE \|\mathbf{x}_t\| + \Omega_{\xi}},$$

so there exists a bound M' such that $E \|\mathbf{x}_t\| \leq M'$. From Markov's inequality,

$$\Pr(\|\mathbf{x}_t\| > M'/e) < e,$$

therefore, M = M'/e satisfies our requirements.

APPENDIX B. THE PROOF OF THE MAIN THEOREM

We will use the following lemma:

Lemma B.1. Let J be a differentiable function, bounded from below by J^* , and let ∇J be Lipschitz-continuous. Suppose the weight sequence w_t satisfies

$$w_{t+1} = w_t + \alpha_t b_t$$

for random vectors b_t independent of w_{t+1}, w_{t+2}, \ldots , and b_t is a descent direction for J, i.e. $E(b_t|w_t)^T \nabla J(w_t) \leq -\delta(\epsilon) < 0$ whenever $J(w_t) > J^* + \epsilon$. Suppose also that

$$E(\|b_t\|^2|w_t) \le K_1 J(w_t) + K_2 E(b_t|w_t)^T \nabla J(w_t) + K_3$$

and finally that the constants α_t satisfy $\alpha_t > 0$, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$. Then $J(w_t) \to J^*$ with probability 1.

In our case, the weight vectors are $n \times n$ dimensional, with $w_{n \cdot i+j} := \Pi_{ij}$. For the sake of simplicity, we denote this by $w_{(ij)}$. Let w^* be the weight vector corresponding to the optimal value function, and let

$$J(w) = \frac{1}{2} \|w - w^*\|^2.$$

Theorem B.2 (Theorem 3.1). If $\Pi_0 \geq \Pi^*$, there exists an L such that $||F + GL|| \leq q$, then there exists a series of learning rates α_t such that $0 < \alpha_t \leq 1/||\mathbf{x}_t||^4$, $\sum_t \alpha_t = \infty$, $\sum_t \alpha_t^2 < \infty$, and it can be computed online. For all sequences of learning rates satisfying these requirements, Algorithm 1 converges to the optimal policy.

Proof. First of all, we prove the existence of a suitable learning rate sequence. Let α_t' be a sequence of learning rates that satisfy two of the requirements, $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$. Fix a probability 0 < e < 1. By the previous lemma, there exists a bound M such that $\|\mathbf{x}_t\| \leq M$ with probability 1 - e. The learning rates

$$\alpha_t := \min\{\alpha_t', 1/\|\mathbf{x}_t\|^4\}$$

will be satisfactory, and can be computed on the fly. The first and third requirements are trivially satisfied, so we only have to show that $\sum_t \alpha_t = \infty$. Consider the index set $H = \{t : \alpha_t' \leq 1/M^4\} \cup \{t : \alpha_t' \leq 1/\|\mathbf{x}_t\|^4\}$. By the first condition only finitely many indices are excluded. The second condition excludes indices with $1/M^4 < \alpha_t' < 1/\|\mathbf{x}_t\|^4$, which happens at most with probability e. However,

$$\sum_{t} \alpha_t \ge \sum_{t \in H} \alpha_t = \sum_{t \in H} \alpha_t' = \infty.$$

The last equality holds, because if we take a divergent sum of nonnegative terms, and exclude finitely many terms or an index set with density less than 1, then the remaining subseries will remain divergent.

An update step of the algorithm is $\alpha_t \delta_t \mathbf{x}_t \mathbf{x}_t^T$. To make the proof simpler, we decompose this into a step size α_t' and a direction vector $(\alpha_t/\alpha_t')\delta_t \mathbf{x}_t \mathbf{x}_t^T$. Denote the scaling factor by

$$A_t := \alpha_t / \alpha_t' = \min\{1, 1/(\alpha_t' \|\mathbf{x}_t\|^4)\}$$

Clearly, $A_t \leq 1$. In fact, it will be one most of the time, and will damp only the samples that are too big.

We will show that

$$b_t = A_t \delta_t \mathbf{x}_t \mathbf{x}_t^T$$

is a descent direction for every t.

$$E(b_t|w_t)^T \nabla J(w_t) = A_t E(\delta_t|w_t) \mathbf{x}_t \mathbf{x}_t^T (w_t - w^*)$$

$$= A_t E(\delta_t|w_t) \mathbf{x}_t^T (\Pi_t - \Pi^*) \mathbf{x}_t$$

$$= A_t E(\delta_t|w_t) (V_t(\mathbf{x}_t) - V^*(\mathbf{x}_t)).$$

For the sake of simplicity, from now on we do not note the dependence on w_t explicitly.

We will show that for all t, $E(\Pi_t) > 0$, $E(\Pi_{t-1}) > E(\Pi_t)$ and $E(\delta_t) \leq -p\mathbf{x}_t^T(\Pi_t - \Pi^*)\mathbf{x}_t$. We proceed by induction.

- $\bullet t = 0$. $\Pi_0 > \Pi^*$ holds by assumption.
- Induction step part 1: $E(\delta_t) \leq -p \mathbf{x}_t^T (\Pi_t \Pi^*) \mathbf{x}_t$. Recall that

(21)
$$\mathbf{u}_{t} = \arg\min_{\mathbf{u}} \left(c(\mathbf{x}_{t}, \mathbf{u}) + V_{t}(F\mathbf{x}_{t} + G\mathbf{u}) \right)$$
$$= L_{t}\mathbf{x}_{t},$$

where

$$L_t = -(R + G^T \Pi_t G)^{-1} (G^T \Pi_t F)$$

is the greedy control law with respect to V_t . Clearly, by the definition of L_t ,

$$c(\mathbf{x}_t, L_t \mathbf{x}_t) + V_t(F\mathbf{x}_t + GL_t \mathbf{x}_t) \le c(\mathbf{x}_t, L^* \mathbf{x}_t) + V_t(F\mathbf{x}_t + GL^* \mathbf{x}_t).$$

This yields

$$(22) E(\delta_t) = p c_f(\mathbf{x}_t) + (1-p) \left(c(\mathbf{x}_t, L_t \mathbf{x}_t) + V_t (F \mathbf{x}_t + G L_t \mathbf{x}_t) \right) - V_t(\mathbf{x}_t)$$

$$\leq p c_f(\mathbf{x}_t) + (1-p) \left(c(\mathbf{x}_t, L^* \mathbf{x}_t) + V_t (F \mathbf{x}_t + G L^* \mathbf{x}_t) \right) - V_t(\mathbf{x}_t).$$

We know that the optimal value function satisfies the fixed-point equation

(23)
$$0 = \left(p \ c_f(\mathbf{x}_t) + (1-p)\left(c(\mathbf{x}_t, L^*\mathbf{x}_t) + V^*(F\mathbf{x}_t + GL^*\mathbf{x}_t)\right)\right) - V^*(\mathbf{x}_t).$$

Subtracting this from Eq. (22), we get

(24)
$$E(\delta_t) \leq (1-p) \left(V_t (F \mathbf{x}_t + G L^* \mathbf{x}_t) - V^* (F \mathbf{x}_t + G L^* \mathbf{x}_t) \right) - (V_t (\mathbf{x}_t) - V^* (\mathbf{x}_t)).$$

(25)
$$= (1-p)\mathbf{x}_{t}^{T}(F+GL^{*})^{T}(\Pi_{t}-\Pi^{*})(F+GL^{*})\mathbf{x}_{t}$$

$$-\mathbf{x}_t^T (\Pi_t - \Pi^*) \mathbf{x}_t.$$

Let $\epsilon_1 = \epsilon_1(p) := 1/(1-p) - ||F + GL^*||^2 > 0$. Inequality (24) implies

$$(27) E(\delta_t) \leq (1-p)(\frac{1}{1-p} - \epsilon_1(p))\mathbf{x}_t^T(\Pi_t - \Pi^*)\mathbf{x}_t - \mathbf{x}_t^T(\Pi_t - \Pi^*)\mathbf{x}_t$$

$$= -(1-p)\epsilon_1(p)\mathbf{x}_t^T(\Pi_t - \Pi^*)\mathbf{x}_t.$$

$$= -\epsilon_2(p)\mathbf{x}_t^T(\Pi_t - \Pi^*)\mathbf{x}_t,$$

where we defined $\epsilon_2(p) = (1-p)\epsilon_1(p)$

• Induction step part 2: $E(\Pi_{t+1}) > \Pi^*$.

$$(30) E(\delta_t) = p c_f(\mathbf{x}_t) + (1-p) \left(c(\mathbf{x}_t, L_t \mathbf{x}_t) + V_t(F\mathbf{x}_t + GL_t \mathbf{x}_t) \right) - V_t(\mathbf{x}_t)$$

$$\geq p c_f(\mathbf{x}_t) + (1-p) \left(c(\mathbf{x}_t, L_t \mathbf{x}_t) + V^*(F\mathbf{x}_t + GL_t \mathbf{x}_t) \right) - V_t(\mathbf{x}_t).$$

Subtracting eq. 23, we get

$$(31) E(\delta_t) \geq (1-p)\Big(\Big(c(\mathbf{x}_t, L_t\mathbf{x}_t) + V^*(F\mathbf{x}_t + GL_t\mathbf{x}_t)\Big)$$

$$-\Big(c(\mathbf{x}_t, L^*\mathbf{x}_t) + V^*(F\mathbf{x}_t + GL^*\mathbf{x}_t)\Big)\Big) + V^*(\mathbf{x}_t) - V_t(\mathbf{x}_t)$$

$$\geq V^*(\mathbf{x}_t) - V_t(\mathbf{x}_t) \geq -\|\Pi_t - \Pi^*\| \|\mathbf{x}_t\|^2.$$

Therefore

(32)
$$E(\Pi_{t+1}) - \Pi^* \geq \Pi_t + \alpha'_t A_t E(\delta_t) \mathbf{x}_t \mathbf{x}_t^T - \Pi^*$$

$$(33) \geq (\Pi_t - \Pi^*) - \alpha_t \|\mathbf{x}_t\|^4 \|\Pi_t - \Pi^*\| I$$

$$(34) \qquad \qquad > (\Pi_t - \Pi^*) - \|\Pi_t - \Pi^*\| I > 0.$$

• Induction step part 3: $\Pi_t > E(\Pi_{t+1})$.

$$(35) \quad \Pi_t - E(\Pi_{t+1}) = -\alpha_t' A_t E(\delta_t) \mathbf{x}_t \mathbf{x}_t^T \ge \alpha_t \epsilon_2(p) \mathbf{x}_t^T (\Pi_t - \Pi^*) \mathbf{x}_t \cdot \mathbf{x}_t \mathbf{x}_t^T,$$

but $\alpha_t' \epsilon_2(p) > 0$, $\mathbf{x}_t^T (\Pi_t - \Pi^*) \mathbf{x}_t > 0$ and $\mathbf{x}_t \mathbf{x}_t^T > 0$, so their product is positive as well.

The induction is therefore complete.

We finish the proof by showing that the assumptions of Lemma B.1 hold:

 $\frac{b_t \text{ is a descent direction.}}{\Pi_t - \Pi^* \text{ is positive definite, so } \Pi_t - \Pi^* \geq \epsilon_3(\epsilon), \text{ but } \Pi_t - \Pi^* \text{ is positive definite, so } \Pi_t - \Pi^* \geq \epsilon_3(\epsilon)I.$

$$E(b_t|w_t)^T \nabla J(w_t) = A_t E(\delta_t|w_t) (V_t(x_t) - V^*(x_t))$$

$$\leq -\epsilon_2(p) A_t \mathbf{x}_t^T (\Pi_t - \Pi^*) \mathbf{x}_t \cdot \mathbf{x}_t^T (\Pi_t - \Pi^*) \mathbf{x}_t$$

$$\leq -\epsilon_2 \epsilon_3^2 A_t \|\mathbf{x}_t\|^4$$

$$\leq -\epsilon_2 \epsilon_3^2 \min\{\|\mathbf{x}_t\|^4, 1/\alpha_t'\}$$

$$\frac{E(\|b_{t}\|^{2} | w_{t}) \text{ is bounded.}}{E(\|b_{t}\|^{2} | w_{t})} |E(\delta_{t})| \leq |\mathbf{x}_{t}^{T} (\Pi_{t} - \Pi^{*}) \mathbf{x}_{t}|. \text{ Therefore}$$

$$E(\|b_{t}\|^{2} | w_{t}) \leq |A_{t}|^{2} |E(\delta_{t})|^{2} \|\mathbf{x}_{t}\|^{2}$$

$$\leq \|\Pi_{t} - \Pi^{*}\|^{2} \cdot \min\{1, 1/(\alpha_{t}^{\prime 2} \|\mathbf{x}_{t}\|^{8})\} \cdot \|\mathbf{x}_{t}\|^{6}$$

$$\leq \|\Pi_{t} - \Pi^{*}\|^{2} \cdot \min\{\|\mathbf{x}_{t}\|^{6}, 1/(\alpha_{t}^{\prime 2} \|\mathbf{x}_{t}\|^{2})\}$$

$$\leq K \cdot J(w_{t}).$$

Consequently, The assumptions of lemma B.1 hold, so the algorithm converges to the optimal value function with probability 1. \Box

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