Lecture 20

Error Bounds & Higher Taylor Methods

Example 20.1. We computed approximate solution at $t = 0.04$ of the following problem

$$y' = y, \quad y(0) = 1 \tag{20.1}$$

which are

$$\begin{aligned} w_1 &= 1 + 0.01 = 1.01 \\ w_2 &= 1.01 + 10.1(1.01) = 1.0201 \\ w_3 &= 1.0201 + 0.01(1.0201) = 1.030301 \\ w_4 &= 1.03301 + 0.1(1.030301) = 1.040606 \end{aligned} \tag{20.2}$$

The exact solution is $y = e^x$,

$$\begin{aligned} y(0.01) &= 1.010050167 \\ y(0.02) &= 1.020201340 \\ y(0.03) &= 1.030454534 \\ y(0.04) &= 1.040810774 \end{aligned} \tag{20.3}$$

Observe that

$$|y''| = e^x \leq e^{0.04} := M \tag{20.4}$$

and

$$L = \frac{\partial f}{\partial y} = 1 \tag{20.5}$$

Then we have the actual error and the upper bound of the error from the above formula are as follow

$$0.000050167 = |y(0.01) - w_1| \leq \frac{0.01 \times e^{0.04}}{2} \left(e^{(0.01-0)} - 1\right) = 0.0000523 \tag{20.6}$$

$$0.000101340 = |y(0.02) - w_2| \leq \frac{0.01 \times e^{0.04}}{2} \left(e^{(0.02-0)} - 1\right) = 0.000105 \tag{20.7}$$

$$0.000153534 = |y(0.03) - w_3| \leq \frac{0.01 \times e^{0.04}}{2} \left(e^{(0.03-0)} - 1\right) = 0.0001584 \tag{20.8}$$

$$0.000204774 = |y(0.04) - w_4| \leq \frac{0.01 \times e^{0.04}}{2} \left(e^{(0.04-0)} - 1\right) = 0.000212 \tag{20.9}$$

Remark 20.2. *The Euler's method works best on problems whose solutions have relatively small second derivatives. If M is large, it can produce large error even though h is small.*

Example 20.3. Consider the IVP

$$y' = ky, \quad y(0) = 1 \tag{20.10}$$

The exact solution of this equation is $y = e^{kt}$. Now observe that for $k < 0$ and $|k|$ very large, the slope of the curve is decreasing very fast as t increases. But the Euler's method is the slope of tangent line at t_i and is constant on interval $[t_i, t_{i+1}]$. Therefore, this produce a big error even if we make h small.

20.0.1 Round-off Error

The approximate solution of Euler method is

$$w_{i+1} = w_i + hf(t, w_i), \qquad w_0 = y_0 \tag{20.11}$$

for $i = 0, \ldots, n-1$. Since the round off error is inevitable in any numerical computation with calculator and/or computer, we assume δ_i for $i = 0, \ldots, n$ are round-off error. We consider the following equation

$$u_{i+1} = u_i + hf(t, u_i) + \delta_{i+1}, \qquad u_0 = y_0 + \delta_0 \tag{20.12}$$

for $i = 0, \ldots, n-1$. Then we have the following theorem.

Theorem 20.1: Round-off Error Bound

If f is continuous and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) : t \in [a, b], y \in \mathbb{R}\} \tag{20.13}$$

and $|y''| \leq M$ for some $M > 0$ where $y(t)$ is a solution of the IVP

$$\begin{cases} y'(t) = f(t, y(t)) & a \leq t \leq b \\ y(a) = y_0 \end{cases} \tag{20.14}$$

and u_i, for $i = 0, \ldots, n$ be approximate solution of (20.12). If $|\delta_i| < \delta$ for each $i = 0, \ldots, n$, then

$$|y(t_i) - u_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_i - a)} - 1\right) + |\delta_0| e^{L(t_i - a)} \tag{20.15}$$

for $i = 0, \ldots, n$.

20.1 Trapezoid Method

Since the Euler method has global truncation error of $O(h)$, it is slow and nor very accurate. One can improve it with slightly modifying the method. Consider the IVP

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(a) = y_0 \end{cases} \tag{20.16}$$

in $t \in [a, b]$. Assume f only depends on t. Then we can integrate both side of this equation on $[t_i, t_{i+1}]$

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} y'(t)\, dt = \int_{t_i}^{t_{i+1}} f(t)\, dt \tag{20.17}$$

If we use the Trapezoidal rule of integration we have

$$\begin{aligned}
y(t_{i+1}) - y(t_i) &= \frac{h}{2}\left(f(t_{i+1}) + f(t_i)\right) - \frac{h^3}{12} f''(\xi) \\
&= \frac{h}{2}\left(f(t_i + h) + f(t_i)\right) - \frac{h^3}{12} f''(\xi)
\end{aligned} \tag{20.18}$$

Removing the error term we will have the Trapezoid Method approximation, for $i = 0, \ldots, n-1$, is

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i + h) + f(t_i)\right], \qquad w_0 = y_0. \tag{20.19}$$

In general, if f depends on t and y, we have

The Trapezoid Method Approximation, for $i = 0, \ldots, n-1$, is

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f\left(t_i + h, w_i + hf(t_i, w_i)\right)\right], \qquad w_0 = y_0. \tag{20.20}$$

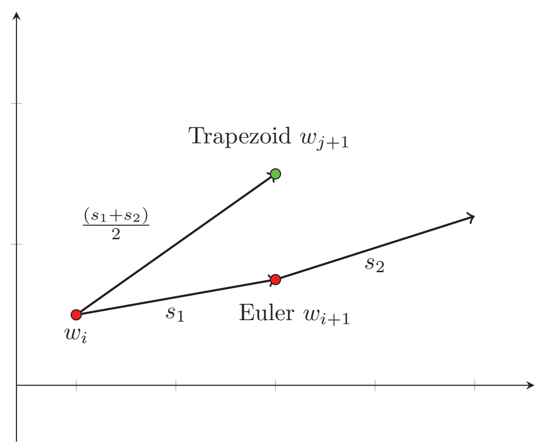

Figure 20.1: The slope of trapezoid line is average of slope of Euler's line.

Remark 20.4 (Geometric Interpretation of Trapezoid Method)**.** *Consider two points w_i and w_{i+1} of Euler's method, the trapezoid method is the average of slope of tangent lines at w_i and w_{i+1}. See Figure 19.1.*

Example 20.5. Consider the IVP

$$y' = y, \quad y(0) = 1 \tag{20.21}$$

We want to find the approximate at $t = 0.04$. The Trapezoid method is

$$w_{i+1} = w_i + \frac{h}{2}\left(w_i + w_i + hw_i\right) = \left(1 + h + \frac{h^2}{2}\right) w_i \tag{20.22}$$

with $h = 0.01$. Then we have

$$\begin{aligned}
w_1 &= 1 + 0.01 + 0.00005 = 1.01005 \\
w_2 &= (1 + 0.1 + 0.00005)\, 1.01005 = 1.0202010025 \\
w_3 &= (1 + 0.1 + 0.00005)\, 1.0202010025 = 1.0304540226 \\
w_4 &= (1 + 0.1 + 0.00005)\, 1.0304540226 = 1.0408100855
\end{aligned} \tag{20.23}$$

The exact solution is $y = e^x$, which will be $y(0.04) = 1.0408107742$.

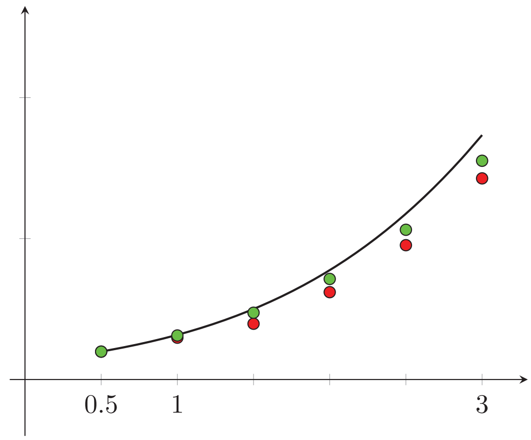

Figure 20.2: An approximate solution $y'(t) = t^3/12 + t/3 + 0.219$ with initial condition $y(0.5) = 0.39583$ from the Euler's method and Trapezoid method demonstrated by red and green points, respectively.

20.2 Higher Taylor Methods

Consider the IVP

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(a) = y_0 \end{cases} \tag{20.24}$$

in $t \in [a, b]$. Assume f is differentiable of order k, then consider the Taylor expansion at t_i is

$$\begin{aligned} y(t_i + h) &= y(t_i) + hy'(t_i) + \frac{h^2}{2!}y''(t_i) + \cdots + \frac{h^k}{k!}y^{(k)}(t_i) + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(\xi) \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2!}f'(t_i, y(t_i)) + \cdots + \frac{h^k}{k!}f^{(k)}(t_i, y(t_i)) + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(\xi_i) \end{aligned} \tag{20.25}$$

where ξ_i lies between $t_i + h$ and t_i and prime is the total derivative

$$f'(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + y'(t)\frac{\partial f}{\partial y}(t, y(t)) \tag{20.26}$$

The Taylor Method of order k Approximation, for $i = 0, \ldots, n-1$, is

$$w_{i+1} = w_i + hf(t_i, w_i) + \frac{h^2}{2!}f'(t_i, w_i) + \cdots + \frac{h^k}{k!}f^{(k)}(t_i, w_i), \qquad w_0 = y_0. \tag{20.27}$$

Example 20.6. Consider the IVP

$$y' = y, \quad y(0) = 1 \tag{20.28}$$

We want to find the approximate at $t = 0.04$ using the second taylor method with $h = 0.01$. Observe that

$$f(t, y) = y, \quad f'(t, y) = y, \quad f''(t, y) = y \tag{20.29}$$

$$w_{i+1} = w_i + hw_i + \frac{h^2}{2}w_i = \left(1 + h + \frac{h^2}{2}\right)w_i \tag{20.30}$$

This is same as the Trapezoid method for this example and in general will be different. Then we have

$$\begin{aligned} w_1 &= 1 + 0.01 + 0.00005 = 1.01005 \\ w_2 &= (1 + 0.1 + 0.00005)\,1.01005 = 1.0202010025 \\ w_3 &= (1 + 0.1 + 0.00005)\,1.0202010025 = 1.0304540226 \\ w_4 &= (1 + 0.1 + 0.00005)\,1.0304540226 = 1.0408100855 \end{aligned} \tag{20.31}$$

The exact solution is $y = e^x$, which will be $y(0.04) = 1.0408107742$.

Example 20.7. Determine the second-order Taylor method for the first order linear equation

$$\begin{cases} y' = ty + t^3 \\ y(0) = y_0 \end{cases} . \tag{20.32}$$

Since $f(t, y) = ty + t^3$, it follows that

$$f'(t, y) = f_t + f_y f = y + 3t^2 + t(ty + t^3) \tag{20.33}$$

This implies that

$$w_{i+1} = w_i + h(t_i w_i + t_i^3) + \frac{h^2}{2}\left(w_i + 3t_i^2 + t_i(t_i w_i + t_i^3)\right). \tag{20.34}$$