Non-conservative Hamiltonian Mechanics and its Applications

by

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Abstract

In classical mechanics, the principle of least action states that an object's motion minimizes the action, a fundamental scalar quantity related to its energy. Lagrangian mechanics, derived from the Hamiltonian principle, excels in predicting trajectories within conservative fields where forces depend solely on position. Nevertheless, traditional Lagrangian mechanics face limitations in handling nonconservative forces like friction or air drag. Recent breakthroughs in Dissipative Lagrangian mechanics have addressed this gap, especially in describing non-conservative forces.

Our research extends this innovative formalism to Hamiltonian mechanics, emphasizing dissipation in dynamic systems. The primary focus lies in applying non-conservative Hamiltonian mechanics to compute dissipation for physical quantities such as energy and momentum in non-conservative systems. The study explores consequential alterations in Liouville's theorem to describe the behavior of phase space volume under dissipation. In conventional Liouville's theorem, the phase space volume remains constant over time. The new dissipative Hamiltonian mechanics enable the description of the change in phase space volume under dissipation. The results are validated by computational simulations to compare with classical examples like the damped harmonic oscillator and a falling ball with drag.

Having established a non-conservative Hamiltonian formalism, our future goal aims to apply it to calculate dissipation resulting from gravitational waves in binary systems and address unresolved issues surrounding dissipation in physical systems.

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Chapter 1

Introduction

Newton's three laws of motion are the earliest formalism to describe the motion of objects in a macroscopic system. Newton introduced the concept of force to quantify the change of motion through a vector quantity. Newtonian Mechanics marked the birth of classical mechanics. Although Newtonian mechanics are applicable to low-velocity macroscopic systems, physicists later discovered the difficulty of systems with high complexity.

In the 18th century, physicists turned their focus to the conservation law of energy and the constraints imposed on systems. In a conservative field, the work done by an object is path-independent, which only depends on the position in the field. No energy is dissipated, and the motion of the object is the result of the conversion of kinetic and potential energy. Physicists can describe the scalar quantity of motions in the system. In addition, the concept of constraints restricts the degree of freedom of objects to allow the derivation of the equations of motion.

In 1760, based on the analytic mechanics approach above, Joseph-Louis Lagrange applied the calculus of variation to derive the stationary path for objects in a conservative field [4]. The path of the object is extremized to reduce the virtual displacement. Any system tends to reduce the action of the motion of objects. This approach by Lagrange is named Lagrangian Mechanics. Later, Hamilton summarized the phenomenon as Hamilton's principle [12].

"Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies."

The application of Hamilton's principle has proven instrumental in describing the dynamics of conservative systems, wherein the equations of motion are elegantly formulated through Lagrangian mechanics [12]. Furthermore, Hamilton introduced Hamiltonian mechanics through the canonical transformation of Lagrangian mechanics. Hamiltonian mechanics extend the physical concepts apart from the equations of motions to the phase space, providing a deeper connection between positions and momenta of objects.

The Lagrangian and Hamiltonian mechanics, leveraging the conservation of energy and the forces of constraint, provide a robust tool for handling intricate dynamic problems.

The application of Lagrangian and Hamiltonian mechanics is built on the fundamentals of a conservative field. Energy is conserved so that the state of the system is constant. Hamiltonian mechanics cannot be applied to non-conservative systems. The traditional approach encounters limitations when applied to irreversible processes in non-conservative systems [2]. Fortunately, in 2013, Dr. Galley introduced a double coordinate method, utilizing modified Lagrangian mechanics, to articulate the equations of motion for objects in such systems [6]. This breakthrough allows us to surpass the conventional constraints of calculus of variation

and articulate the impact of dissipation on object motion within a non-conservative field.

In my thesis, the objective is to extend the double coordinate method from modified Lagrangian mechanics to develop a new Hamiltonian formalism for describing non-conservative action. After Dr. Galley obtained the dissipative Lagrangian, he attempted to extend the Lagrangian mechanics to a Hamiltonian setting. However, Dr. Galley commented that the details of Hamiltonian formalism will be stated elsewhere [5]. However, in Galley's follow-up paper, the part of Hamiltonian formalism is not mentioned anymore [6]. In order to fill the knowledge gap, I developed the approach to represent non-conservative forces in Hamiltonian-related theorems and their underlying physical significance.

One of the foundational theorem, Liouville's theorem, shows that the density in phase space volume remains constant under time evolution. The introduction of dissipation suggests an additional impact on the phase-space distribution function over time. What patterns in area change correspond to the type and magnitude of energy dissipation? These questions, rooted in the double coordinate formalism of Hamiltonian mechanics, constitute the groundwork for exploring the undiscovered influence of non-conservative forces on fundamental theorems in classical mechanics.

I propose to construct illustrative examples under the double coordinates Hamiltonian, including a Maxwell element (a spring and a dashpot model in a series to demonstrate the viscoelasticity of a material, the damped harmonic oscillator and a ball with the drag force, and

Chapter 2

Conventional and Dissipative Lagrangian and related Theorems

2.1 Variational Principles and Lagrangian's Equations

The principle of least action is one of the most fundamental ideas in physics, stating that the motion of any object minimizes the action—a fundamental quantity related to its energy [11]. In 1834, Hamilton formulated the Hamiltonian principle, stating that among all possible paths within a dynamical system for a given time interval and under the same constraints, the actual path is the one that minimizes the action integral [8]. The action integral, denoted as S, represents the time integral of the Lagrangian, expressed as the difference between kinetic energy (T) and potential (U) energy:

$$L = T - U \tag{2.1}$$

$$S = \int_{t_i}^{t_f} L\, dt \tag{2.2}$$

$$\text{where} \quad \delta S = \delta \int_{t_i}^{t_f} L(q_1, \ldots, q_n, \dot{q_1}, \ldots, \dot{q_n}, t)\, dt = 0 \tag{2.3}$$

Here, S denotes the action of the system, with t_i and t_f denoting the initial and final times. The Lagrangian, L, is the difference between kinetic and potential energy ($L = T - V$), where q and $\dot{q}$ denote the generalized coordinates of displacement and velocity, respectively. The indices $\{1, \ldots, n\}$ represent the number of degrees of freedom in the system. The meaning of the Hamiltonian principle is that the action of an object has to be extremized. It allows us to distinguish the actual path from the sets of varied paths. [7].The equation of the varied path:

$$q_i(t, \epsilon) = q_i(t, 0) + \epsilon \eta_i(t) \tag{2.4}$$

$$\text{where} \quad \eta_i(t_i) = \eta_i(t_f) \tag{2.5}$$

where q_i is the i th generalized coordinate. $q_i(t, 0)$ represents the optimal path. ϵ is the infinitesimal parameter with auxiliary function $\eta(t)$. [7] The expression shows that the varied path $q_i(t, \epsilon)$ is the actual path $q_i(t, 0)$ with a small deviation $\epsilon \eta_i(t)$. The deviation vanishes at the fixed endpoints at t_i, t_f

Since the action is extremized, the actual path is a stationary path and its derivative is equal to zero.

$$\left(\frac{dS}{d\epsilon}\right)_{\epsilon=0} = \int_{t_i}^{t_f} \sum \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} d\epsilon + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon} d\epsilon\right) dt = 0 \tag{2.6}$$

By performing integration by part at the 2nd term on the integral. Then, the Euler-Lagrange equation (EL equation) is derived for each coordinate q_i:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = \text{number of degrees of freedom} \tag{2.7}$$

The derivation of the Euler-Lagranian equation is based on Goldstein's Classical Mechanics (3rd edition) [7]. The EL equation crucially relies on fixed boundary conditions at $q(t_1) = q_1$ and $q(t_2) = q_2$. However, this fixed endpoint assumption poses challenges in the context of non-conservative forces, where the dissipation results in an unknown endpoint. To address this issue, Galley introduces the double coordinate method and imposes four new boundary conditions, reconstructing a modified EL equation to describe the equations of motion in a non-conservative force field [5].

2.2 Review of Dissipated Double Coordinate Lagrangian

Inspired by the Feynman integral, Galley introduces a modification to traditional Lagrangian mechanics, creating a double coordinate Lagrangian that accounts for the dissipation in the new Euler-Lagrange (EL) equation. The construction of this double coordinate Lagrangian, based on the limitations of ordinary Lagrangian mechanics, unfolds as follows: [6]

1. **Introduction of Double Degrees of Freedom:**

   In the context of non-conservative systems, where only the initial condition is available, a double set of degrees of freedom q_1 and q_2 is introduced. The double coordinates allow a non conservative potential in the new Lagrangian (details in part 2). This captures the variation in paths due to the dissipation, representing possible different endpoints of the path, which, in physical situations, converge to the same endpoint. Notice that q_1, q_2 here are different from the q_i in ordinary Euler Lagrangian equations. $q_{1,2}$ in double coordinate Lagrangian represents the two corresponding paths q_1, q_2. The number of generalized coordinates is represented by I, The expression of the set of I generalized coordinates is q^I.

$$\mathcal{S}\left[q_1, q_2\right] = \int_{t_i}^{t_f} dt \left[L\left(q_1, \dot{q}_1, t\right) - L\left(q_2, \dot{q}_2, t\right)\right] \tag{2.8}$$

2. **Enhancing Lagrangian for dissipation:**

   Recognizing that the ordinary Lagrangian is inadequate for fully describing dissipation, the double set of degrees of freedom allows for more freedom in extremizing the non-conservative action S. This enhancement enables the Lagrangian to incorporate a new "potential" K corresponding to non-conservative forces. K possesses certain properties:
   (i) it vanishes in conservative fields, and
   (ii) it is anti-symmetric to allow interchange of q_1 and q_2.

$$\begin{aligned}\mathcal{S}\left[q_1, q_2\right] = \int_{t_i}^{t_f} dt \big[&L\left(q_1, \dot{q}_1, t\right) - L\left(q_2, \dot{q}_2, t\right) \\ &+ K\left(q_1, q_2, \dot{q}_1, \dot{q}_2, t\right)\Big]\end{aligned} \tag{2.9}$$

   We can denote the non conservative Lagrangian as:

$$\begin{aligned}&\Lambda(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = \\ &L\left(q_1, \dot{q}_1, t\right) - L\left(q_2, \dot{q}_2, t\right) + K\left(q_1, q_2, \dot{q}_1, \dot{q}_2, t\right)\end{aligned} \tag{2.10}$$

And define the double coordinate canonical momenta $\pi_I(q^I_{1,2}, \dot{q}^I_{1,2})$ in the double coordinate Lagrangian:

$$\pi_{1,2I}\left(q_{1,2}, \dot{q}_{1,2}\right) \equiv \frac{\partial \Lambda}{\partial \dot{q}^I_{1,2}(t)} = \frac{\partial L\left(q_{1,2}, \dot{q}_{1,2}\right)}{\partial \dot{q}^I_{1,2}(t)} + \frac{\partial K}{\partial \dot{q}^I_{1,2}(t)}, \tag{2.11}$$

It is also called the conjugate momenta for π_1, π_2 to show the momenta is related in two paths.

3. **Varied Path, Boundary Conditions and Constraints:**

The introduction of double coordinates in our system suggests the varied path is doubled:

$$q_{1,2}(t, \epsilon) = q_{1,2}(t, 0) + \epsilon \eta_{1,2}(t) \tag{2.12}$$

The two varied paths $q_1(t), q_2(t)$ have its own auxiliary function $\eta_1(t), \eta_2(t)$ to represent the individual virtual displacements.

The system now has two varied paths for the motion of objects and requires double the amount of boundary conditions. However, only the initial condition is available from a non conservative force field. Four Boundary Conditions are required to obtain the double coordinate EL equation (2 B.C. for 1 varied path). Except for the initial condition, Galley imposes his innovation of the equality conditions at the final endpoint. and a physical limit (PL) to constrain the degrees of freedom. Details are given below. Except for the initial condition, Galley imposes the his innovation of the equality conditions at the final endpoint. and a physical limit (PL) to constrain the degrees of freedom. Details are given below.

(a) Initial condition:

$$\dot{q}^I_1 = \dot{q}^I_2 \tag{2.13}$$

The initial condition is obtained at the initial endpoint (t_i). The two coordinates are the same at the initial time of the system.

(b) Equality condition at on η at t_f

$$\eta_1\left(t_f\right) = \eta_2\left(t_f\right) \tag{2.14}$$

The auxiliary function is equal at the final time. The variation vanishes at the final endpoint.

(c) Equality of final conjugate momenta ($\boldsymbol{\pi}$)

$$\boldsymbol{\pi}_1\left(t_f\right) = \boldsymbol{\pi}_2\left(t_f\right) \tag{2.15}$$

The conjugate momenta are equal at the final time. The two equality conditions Eqs. (2.14)-(2.15) shows that the displacement and moment of both paths are equal. The variation at the final vanishes to obtain the actual path.

(d) Physical limit (PL)

$$q^I_1 = q^I_2, \quad \dot{q}^I_1 = \dot{q}^I_2 \tag{2.16}$$

The physical limit shows the two paths are the same eventually. In a classical system, the object has only one actual path. After calculating the non conservative potential, the the physical limit converges into a single physical path.

4. **Reparametrization**

   A reparametrization of coordinates is applied to represent the average (q_+^I) and difference (q_-^I) of two paths

$$q_+^I \equiv \frac{q_1^I + q_2^I}{2} \tag{2.17}$$

$$q_-^I \equiv q_1^I - q_2^I \tag{2.18}$$

   The reparameterization connects the two coordinates q_1, q_2 as the average and difference of varied paths to allow a more convenient expression of the double coordinate Euler-Lagrangian Equations. The two parameterization represents the same physical facts while $q_\pm$ is more convenient. The physical limit in $q_\pm$ is

$$q_+^I = q^I, q_- = 0 \tag{2.19}$$

   where q^I is the actual coordinate of the path. After the physical limit, the average of the two varied paths is the actual path while the difference is zero.

5. **Hamiltonian Principle and Double Coordinate Euler-Lagrangian Equations**

   The Hamiltonian principle is then employed on the non-conservative action to obtain double coordinate EL equations. Notice that the Hamiltonian principle is still applied to the action of the object in a non conservative field. The object still follows the only path with extremized action. The issue in ordinary Lagrangian is that the Euler-Lagranian equation is not obtainable with non fixed final endpoint at t_f. In double coordinate Lagrangian, a double coordinate Euler-Lagrangian equations are derived. Starting from the Hamiltonian Principle:

$$0 = \left[\frac{dS}{d\epsilon}\right]_{\epsilon=0} \tag{2.20}$$

   From Eqs. (2.9) -(2.10), we can obtained the variation in the action:

$$\begin{aligned} 0 = \int_{t_i}^{t_f} dt \left\{ \eta_1^I \left[\frac{\partial \Lambda}{\partial q_1^I} - \frac{d\pi_{1I}}{dt}\right]_0 - \eta_2^I \left[\frac{\partial \Lambda}{\partial q_2^I} - \frac{d\pi_{2I}}{dt}\right]_0 \right\} \\ + \left[\eta_1^I \pi_{1I} - \eta_2^I \pi_{2I}\right]_{t=t_f}. \end{aligned} \tag{2.21}$$

   After we impose the boundary conditions stated in (3). We obtain the double coordinate Euler-Lagrangian equation as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^I} - \frac{\partial L}{\partial q^I} = \left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_-^I} - \frac{\partial L}{\partial q_-^I}\right]_p .l. = Q_I\left(q^I, \dot{q}^I, t\right) \tag{2.22}$$

   Q_I represents the non conservative force of the dynamics system. The LHS of the double coordinate Euler-Lagranian equation can be interpreted as the usual conservative term in an ordinary EL equation. In a conservative force field, the term is equal to zero. In the double coordinate EL equation, a non conservative potential K is introduced and the EL equation is no longer equal to zero. The middle term represents the variation on potential K and is represented as the generalized non conservative force Q_I. The physical meaning is the EL equation is not conserved with a dissipation term Q_I.

The entire process is visualized in the concept diagram in Galley's paper, as shown in Figure 2.1.

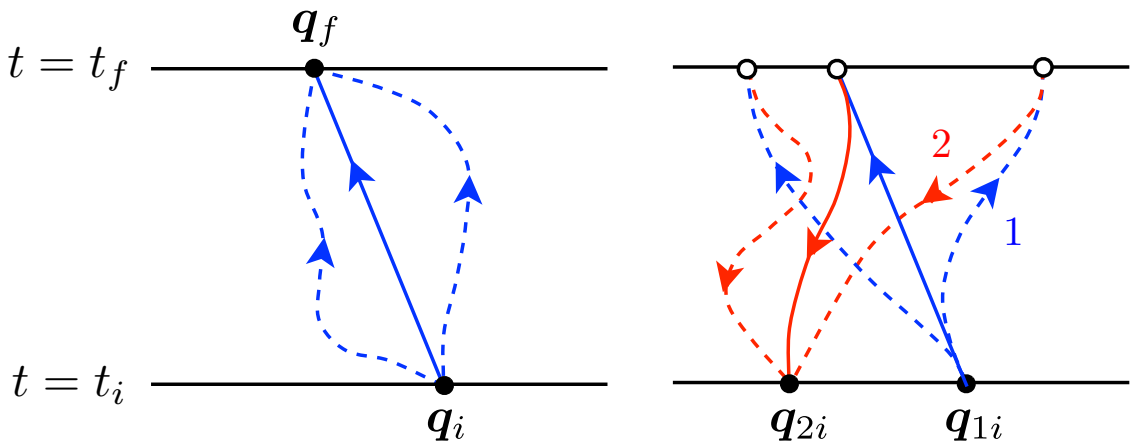

Figure 2.1: Double Varied Path Diagram
The diagram illustrates the differences in possible trajectories between conservative and non-conservative mechanics. Dashed lines represent possible particle paths, while the solid line represents the actual path. In conservative mechanics (left), both endpoints are fixed at q_i, q_f, and the actual path is obtained using the Hamiltonian principle. In non-conservative mechanics (right), only initial data q_{1i}, q_{2i} is available. After applying boundary conditions and double coordinate EL equations, the actual path is determined. The arrows indicate the direction of the integral, not the motion of particles. Figure 1 is cited from Galley's paper. [5].

2.3 Example : Maxwell Element

To illustrate the application of dissipative Lagrangian mechanics, Dr. Galley presented multiple examples of dissipative systems in discrete mechanics [6]. The following example of a Maxwell element is referred to in Galley's paper to demonstrate the method to obtain equations of motion in a multiple-degree-of-freedom system. The Maxwell element is a viscoelastic model used to demonstrate the dissipative behavior of the harmonic oscillator. The system consists of a massless spring with spring constant k attached to a mass m with displacement q, and a massless dashpot to produce damping with the displacement d from the mass.

The conservative Lagrangian of the Maxwell element is

$$L(q, \dot{q}, d, \dot{d}, t) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}k(q-d)^2. \tag{2.23}$$

While the nonconservative K potential is: (where $F(t)$ is the external forcing term, λ is the linear dissipative term)

$$K = q_- \cdot F(t) - \lambda d_- \cdot \dot{d}_+ \tag{2.24}$$

The equation of motion of $x(t)$ is obtained by applying Eq. (2.22).

$$m\ddot{q} + k(q-d) = F(t) \tag{2.25}$$

And, the equation of motion of $d(t)$ is also obtained in the same way

$$\lambda\dot{d} = k(q-d) \tag{2.26}$$

The example above illustrates how the equations of motion of a dissipative system can be obtained by applying the dissipative Lagrangian.

Chapter 3

Conventional Hamiltonian Mechanics and Related Theorems

3.1 Review of Hamiltonian Mechanics

Sir William Rowan Hamilton reformulated Lagrangian mechanics into Hamiltonian mechanics [9]. Both Hamiltonian and Lagrangian mechanics are developed under the Hamiltonian principle that the action of a system is minimized. The benefit of Hamiltonian mechanics is that it provides a different perspective on dynamic systems. Apart from the equations of motion, Hamiltonian mechanics allows for the exploration of Hamilton-Jacobi theory, perturbation approaches, and chaos [7]. Moreover, Hamiltonian mechanics is also well applied in quantum mechanics and statistical mechanics.

Hamiltonian mechanics introduces the generalized momentum (p) associated with the Lagrangian [7]. The definition of generalized momentum is

$$p = \frac{\partial L(q, \dot{q}, t)}{\partial q}, \tag{3.1}$$

where q and $\dot{q}$ are the sets of generalized coordinates and their corresponding velocities, respectively. Note that the variables in the subsequent chapters represent the sets of corresponding quantities with degrees of freedom. For example, if the system has n degrees of freedom, p represents the set of generalized momenta: $p = \{p_1, p_2, \ldots, p_n\}$.

Hamilton defined the Hamiltonian $H(q, p)$ as the Legendre transform of the Lagrangian $L(q, \dot{q})$.

$$H(q, p, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(q, \dot{q}, t) \tag{3.2}$$

The equation of motion of the dynamic system can be derived by comparing the differential form of the Legendre transformation and the total differential of the Hamiltonian.

$$dL = \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt. \tag{3.3}$$

From the differential form of the Lagrangian, the differential form of the Hamiltonian is obtained:

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt, \tag{3.4}$$

dH can also be written as the total differential form of (q, p), which is expressed as:

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt, \tag{3.5}$$

By the comparing the like terms, the Hamiltonian equations are obtained:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{3.6}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{3.7}$$

$$\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{3.8}$$

Finding the equations of motion of a dynamic system requires a similar amount of effort in Hamiltonian and Lagrangian mechanics. However, the Hamiltonian allows us to further explore the dynamics in the phase space (q, p), defined as the space consisting of all possible states of the dynamic system [13].

Apart from the Legendre transform, the Hamiltonian equations of motion can be derived from the Hamiltonian principle. In Hamiltonian mechanics, the generalized position (q) and the generalized momentum (p) are considered as $2n$ independent generalized coordinates in the $2n$ configuration space known as the phase space. Consider $L = p\dot{q} - H$, the Hamiltonian equations can be derived from the Hamiltonian principle [10].

$$0 = \delta S[q] = \int dt \delta L = \int dt \delta(p\dot{q} - H) \tag{3.9}$$

Then, variation is applied on the $L = p\dot{q} - H$,

$$0 = \delta L = \delta(p\dot{q} - H) = \dot{q}\delta p + p\delta\dot{q} - \delta H \tag{3.10}$$

The variation of the Hamiltonian $H(q, p)$ is the total derivative of $H(q, p)$

$$\delta H = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p \tag{3.11}$$

By the Hamiltonian principle, the variation is equation to zero.

$$0 = \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q + \frac{d}{dt}(p\delta q) \tag{3.12}$$

The coefficients of the variation on q and p need to be zero to zero out the variation of the action S. Hence, the Hamiltonian equations from Eqs. (3.6) and (3.7) are obtained. The aim of this paper is to convert Dr. Galley's double-coordinate Lagrangian to a dissipative Hamiltonian to describe the evolution of motion under dissipation through the variational principle.

3.2 Liouville's Theorem

The generalized momenta are defined by Eq. (3.1). Through the transformation from $(q, \dot{q}, t)$ to (q, p, t), it allows the exploration of the concept of phase space as both the possible positions and momenta of all particles [12]. One of the key differences between the Legendre transform from Lagrangian to Hamiltonian is that in the Lagrangian formulation, the state of space is represented by an n-dimensional configuration space [7]. $\dot{q}_i$ depends on generalized coordinates q_i. However, in Hamiltonian Mechanics, the number of independent variables extends to $2n$ as

a result of the generalized momenta (p_i). Therefore, the phase space describes all the possible positions and the momenta of the particles.

The Liouville's theorem describes the behavior of the trajectory of particles in phase space. In a conservative field, the trajectory of particles is fully determined by the initial conditions of the dynamic system. The phase space volume encloses a certain number of particles. Liouville's theorem states that the density of the phase space volume (V) enclosed by any closed surface is constant through time [14].

The following derivation of the traditional Liouville theorem is based on the *Introduction to Statistical Mechanics* by Peter Eastman [3]. Considering an infinitesimally small volume element in phase space, δV is defined by the product of differential changes in the generalized coordinates and their conjugate momenta, δq and δp, respectively. Mathematically,

$$\delta V = \delta q \delta p \tag{3.13}$$

.

The theorem states that the temporal evolution of the phase space volume element remains constant throughout the dynamical evolution of the system.

The derivation of Liouville's Theorem begins by considering the time derivative of the phase space volume:

$$\frac{dV}{dt} = \frac{d}{dt}(\delta q \delta p) \tag{3.14}$$

Applying the product rule, we obtain:

$$\frac{dV}{dt} = \delta q \frac{d(\delta p)}{dt} + \delta p \frac{d(\delta q)}{dt} \tag{3.15}$$

Expanding this derivative involves a linearization of the variables q and p, which leads to expressing the time derivatives as a sum of individual variations:

$$\frac{dV}{dt} = \delta q \left(\frac{d(p + \delta p)}{dt} - \frac{dp}{dt} \right) + \delta p \left(\frac{d(q + \delta q)}{dt} - \frac{dq}{dt} \right) \tag{3.16}$$

Taking the total derivative on $p + \delta p$ and $q + \delta q$ and using a linear approximation:

$$\frac{dV}{dt} = \delta q \left(\frac{dp}{dt} + \delta p \frac{\partial}{\partial p} \frac{dp}{dt} - \frac{dp}{dt} \right) + \delta p \left(\frac{dq}{dt} + \delta q \frac{\partial}{\partial q} \frac{dq}{dt} - \frac{dq}{dt} \right) \tag{3.17}$$

The remaining terms are:

$$\frac{dV}{dt} = \delta q \delta p \left(\frac{\partial}{\partial p} \frac{dp}{dt} + \frac{\partial}{\partial q} \frac{dq}{dt} \right) \tag{3.18}$$

By substituting the Hamiltonian equations from Eqs. (3.6)- (3.7) into the time derivative of q and p:

$$\frac{dV}{dt} = \delta q \delta p \left(\frac{\partial}{\partial p} \left(-\frac{\partial H}{\partial q} \right) + \frac{\partial}{\partial q} \left(\frac{\partial H}{\partial p} \right) \right) = \delta q \delta p \left(-\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial q \partial p} \right) = 0 \tag{3.19}$$

The time derivative of the phase-space volume concretely signifies that the phase-space volume, under the Hamiltonian flow, does not change with time. Liouville's Theorem is

a fundamental theory in statistical mechanics, showing the time evolution of particles in macroscopic dynamical systems.

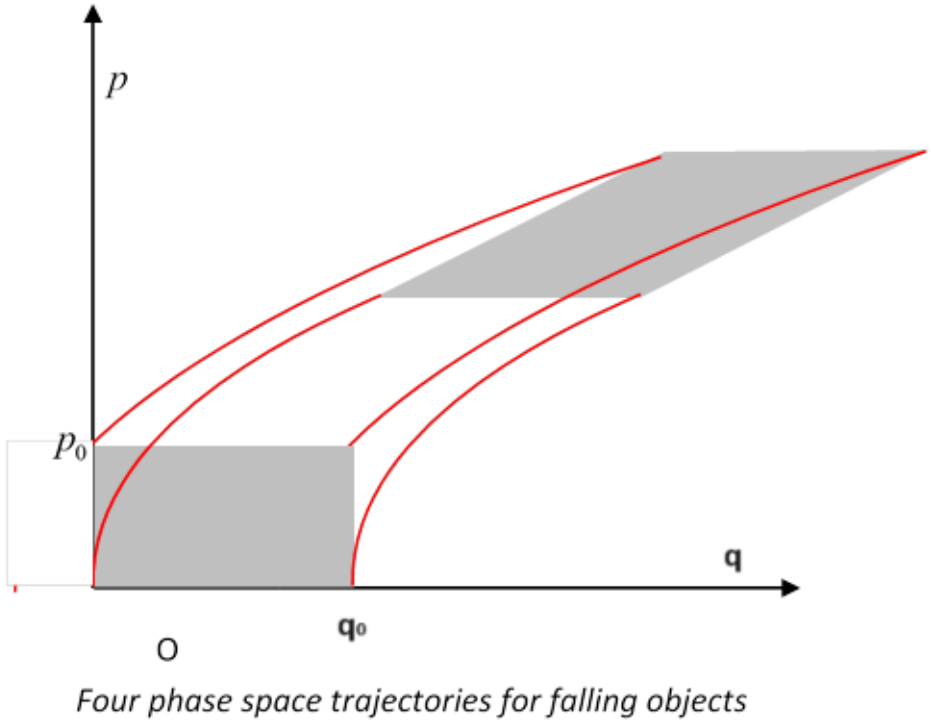

Figure 3.1: Phase Space Volume remains constant over time evolution
The picture shows the time evolution of the phase space enclosed by the four phase space trajectories. It illustrates that although the shape of the phase space volume changes, the phase space volume remains constant through time. The image is taken from the textbook *Graduate Classical Mechanics* by Michael Fowler [1].

3.3 Numerical Method to Estimate Phase Space Volume with Simple Harmonic Oscillator as An Example

In this session, I apply the computational simulation of the phase space diagram of a simple harmonic oscillator and a falling ball with linear drag. By imposing 25 x 25 initial conditions of the simple harmonic oscillator, we can numerically estimate the change in the phase space volume using the convex hull method. The Python code is implemented in a Jupyter Notebook. The animation of the simulation is attached in the QR code in the graph.

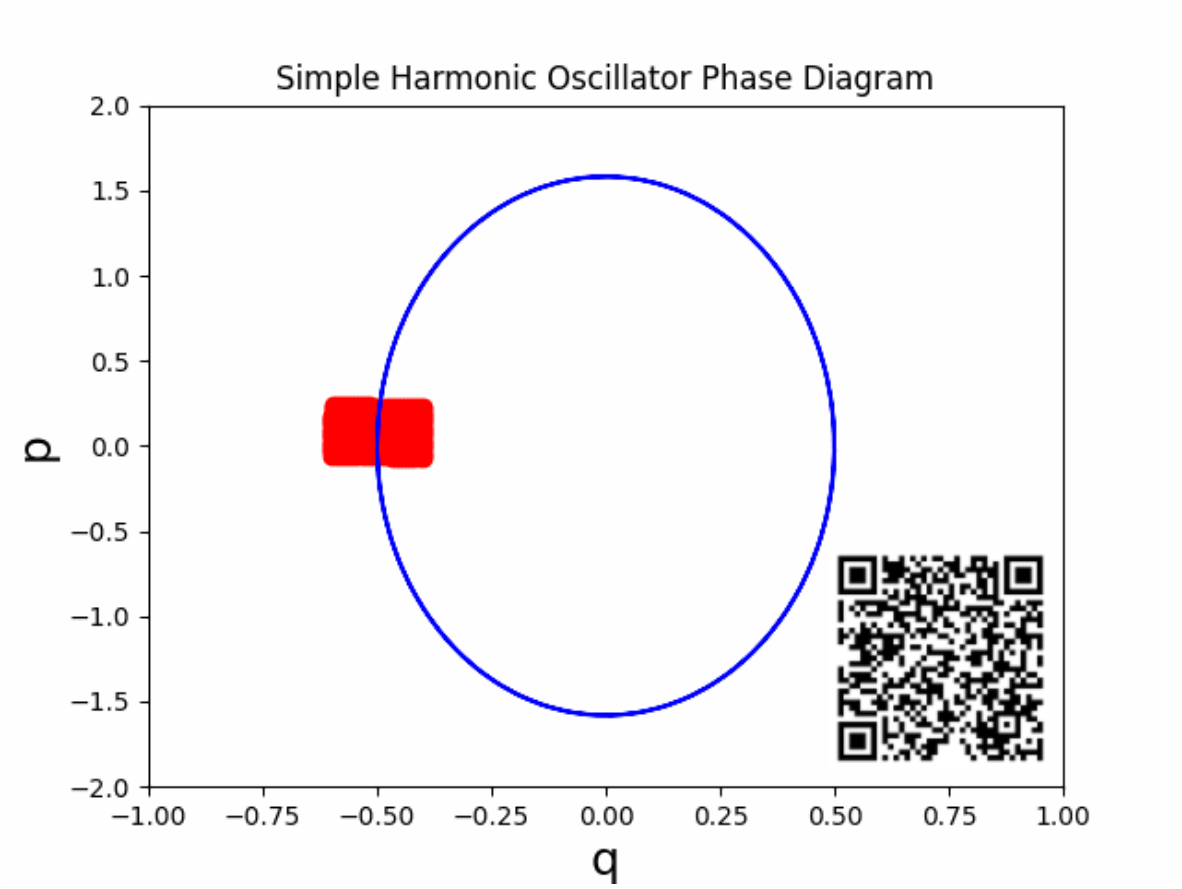

Figure 3.2: Phase Space Diagram of a simple Harmonic Oscillator with the 25 x 25 initial conditions

The numerical simulation of the phase space trajectory of the simple harmonic oscillator is depicted. The blue line represents the trajectory of one of the initial conditions of position (q) and momentum (p). The duration of the simulation is 10 seconds. The trajectory remains an ellipse for multiple cycles, demonstrating that the symmetry of the motion of the particle is preserved because no energy is dissipated to the surroundings. The red rectangle contains the 25 x 25 different initial conditions of (q, p). The numerical phase space volume is estimated using the convex hull method. The codes for both the animation (see Program 6.1) and the phase space volume estimation (see Program 6.2) are attached in the appendix. The QR code is the link to the corresponding animation.

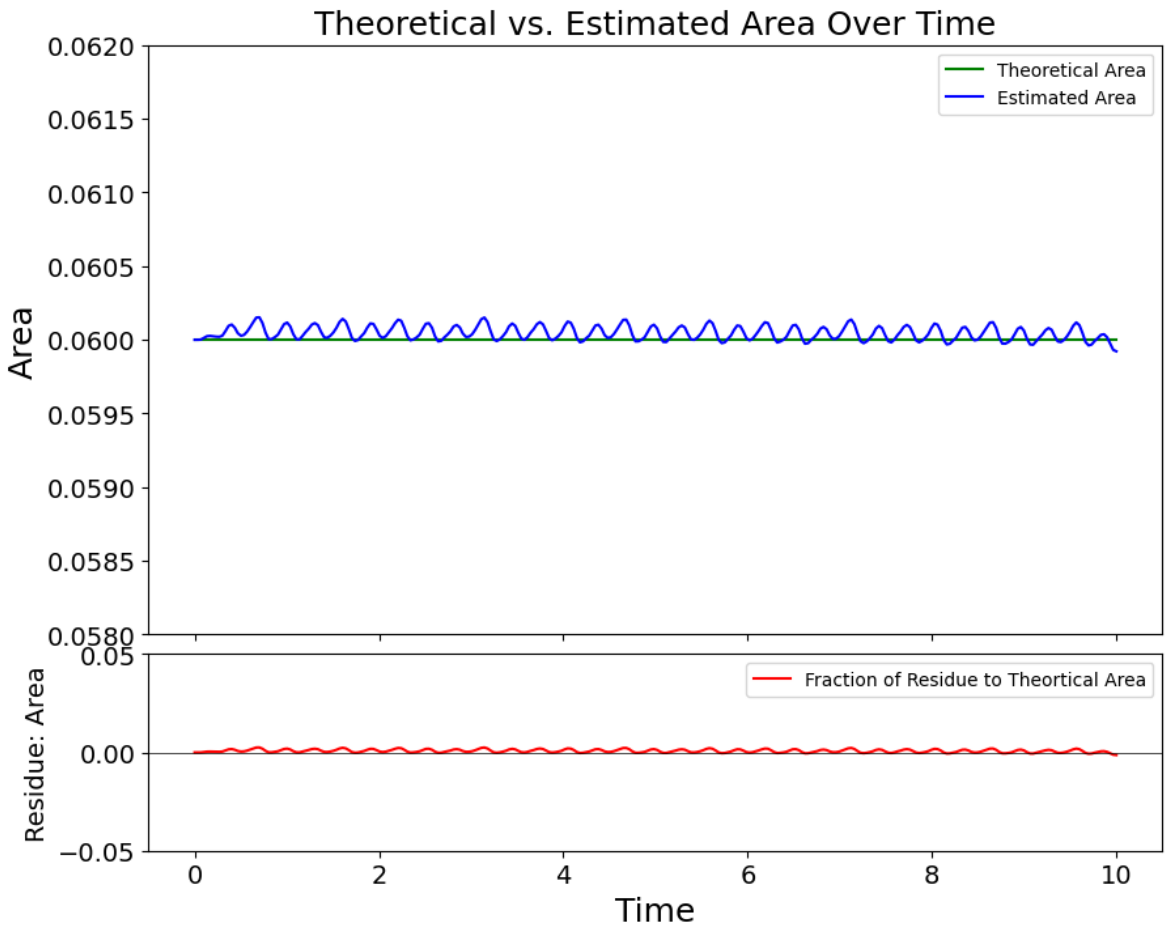

Figure 3.3: Area estimate of the Phase Space Volume and the theoretical curve for the simple harmonic oscillator

The graph shows the numerical results of the convex hull method. Overall, the theoretical value agrees with the estimated area from the convex hull method. The deviation is due to the convex hull method estimating the area by the smallest convex polygon enclosing all the points, which differs from the actual phase space volume. The error can be reduced by imposing more initial condition points.

Chapter 4

Dissipate Double Coordinate Hamiltonian Formalism

4.1 General Formalism

After obtaining the dissipative double coordinate Lagrangian, Dr. Galley attempted to extend Lagrangian mechanics to a Hamiltonian setting. However, Dr. Galley commented that the details of Hamiltonian formalism would be stated elsewhere [5]. Nevertheless, in Galley's follow-up paper, the part of Hamiltonian formalism is not mentioned anymore [6]. In order to fill the knowledge gap, I developed the dissipative Hamiltonian equation from the variational principle.

The Legendre Transform from the dissipative Lagrangian is formulated by:

$$\mathcal{H}(q_{1,2}, p_{1,2}, t) = p_1\dot{q}_1 - p_2\dot{q}_2 - L(q_1, \dot{q}_1, t) + L(q_2, \dot{q}_2, t) - K(q_{1,2}, p_{1,2}, t) \tag{4.1}$$

It is equivalent to formulate the dissipative Hamiltonian in the double varied path formalism by $H = p\dot{q} - L$

$$\mathcal{H}(q_{1,2}, p_{1,2}, t) = H(q_1, p_1, t) - H(q_2, p_2, t) - K(q_{1,2}, p_{1,2}, t) \tag{4.2}$$

After reparameterization from $(1, 2)$ to $(+, -)$, notice that the average in position and the difference in position are defined as following:

$$q_+ := \frac{q_1 + q_2}{2}, \quad q_- := q_1 - q_2 \tag{4.3}$$

The same applies to momenta:

$$p_+ := \frac{p_1 + p_2}{2}, \quad p_- := p_1 - p_2 \tag{4.4}$$

The varied paths (q_1, p_1) and (q_2, p_2) have no intrinsic difference. The physical meanings is attached when the varied paths $(1, 2)$ is reparameterized into $(+, -)$. The average$(+)$ represents the average of position and momentum, while the difference $(-)$ represents the dissipation quantity.

After the reparameterization from $(1, 2)$ to $(+, -)$. the $(1, 2)$ Hamiltonian at Eq. (4.1) changes to be:

$$\mathcal{H}(q_\pm, p_\pm, t) = p_+\dot{q}_- + p_-\dot{q}_+ - \Lambda(q_\pm, p_\pm, t) \tag{4.5}$$

I apply the Hamiltonian principle to the dissipative Hamiltonian to obtain the dissipative Hamiltonian equations.

$$S = \int_{t_i}^{t_f} \Lambda(q_{1,2}, p_{1,2}, t)\, dt = \int_{t_i}^{t_f} \left(p_1\dot{q}_1 - p_2\dot{q}_2 - \mathcal{H}(q_{1,2}, p_{1,2}, t)\right) dt \tag{4.6}$$

Then, apply the reparameterization from $(1,2)$ to $(+,-)$:

$$S = \int_{t_i}^{t_f} \left(p_+\dot{q}_- + p_-\dot{q}_+ - \mathcal{H}(q_\pm, p_\pm, t)\right) dt \tag{4.7}$$

Formulate the varied path of $q_\pm, p_\pm$ with the corresponding auxiliary functions $\eta_{p_\pm,q_\pm}(t)$ as follows:

$$p_\pm = p_{0,\pm} + \epsilon\eta_{p_\pm}(t), \quad q_\pm = q_{0,\pm} + \epsilon\eta_{q_\pm}(t) \tag{4.8}$$

where $p_{0,\pm}$ and $q_{0,\pm}$ are the optimal paths of the momenta and positions, respectively, and ϵ is an infinitesimal parameter. The $+$ corresponds to the average and $-$ corresponds to the difference of paths.

The variation of S is as followed:

$$\delta S = \frac{\partial S}{\partial \epsilon} = \int_{t_i}^{t_f} dt \frac{\partial p_+}{\partial \epsilon}\dot{q}_- + p_+\frac{\partial \dot{q}_-}{\partial \epsilon} + \frac{\partial p_-}{\partial \epsilon}\dot{q}_+ + p_-\frac{\partial \dot{q}_+}{\partial \epsilon} - \frac{\partial \mathcal{H}}{\pm}\frac{\partial q_\pm}{\partial \epsilon} + -\frac{\partial \mathcal{H}}{\pm}\frac{\partial p_\pm}{\partial \epsilon} \tag{4.9}$$

Based on the, the varied path from Eqs. (4.8), the variation of S is

$$\delta S = \int_{t_i}^{t_f} dt \quad \eta_{p_+}(t)\left(\dot{q}_- - \frac{\partial \mathcal{H}}{\partial p_+}\right) + \eta_{p_-}(t)\left(\dot{q}_+ - \frac{\partial \mathcal{H}}{\partial p_-}\right) - \eta_{q_-}(t)\frac{\partial \mathcal{H}}{\partial q_-} - \eta_{q_+}(t)\frac{\partial \mathcal{H}}{\partial q_+} + \dot{\eta}_{q_-}(t)p_+ + \dot{\eta}_{q_+}(t)p_- \tag{4.10}$$

Notice that the last two terms with $\dot{\eta}_{q_\pm}(t)$ are the first time derivative of of the auxiliary functions, we can handle them by integration by parts:

$$\int_{t_i}^{t_f} dt\dot{\eta}_{q_-}(t)p_+ = \eta_{q_-}(t)p_+|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\eta_{q_-}(t)\dot{p}_+ \tag{4.11}$$

Notice that the second term on the right has the variation $\eta_{q_-}(t)$ to compare with Eq. (4.1). The first term is the boundary term which will be discussed with the integration part of $q_+(t)$ below later.

$$\int_{t_i}^{t_f} dt\dot{\eta}_{q_+}(t)p_- = \eta_{q_+}(t)P_-|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt\eta_{q_+}(t)\dot{P}_- \tag{4.12}$$

Now, only the auxiliary functions without time derivative remains. Therefore, apply the Hamiltonian principle and compare the coefficients of the auxiliary functions:

$$\begin{aligned} 0 = \delta S = {} & \eta_{q_-}(t_f)P_+(t_f) - \eta_{q_-}(t_i)P_+(t_i) + \eta_{q_+}(t_f)P_-(t_f) - \eta_{q_+}(t_i)P_-(t_i) \\ & - \int_{t_i}^{t_f} dt\, \eta_{p_+}(t)\left(\dot{q}_- - \frac{\partial \mathcal{H}}{\partial p_+}\right) \\ & + \eta_{p_-}(t)\left(\dot{q}_+ - \frac{\partial \mathcal{H}}{\partial p_-}\right) \\ & - \eta_{q_+}(t)\left(\dot{p}_- + \frac{\partial \mathcal{H}}{\partial q_+}\right) \\ & - \eta_{q_-}(t)\left(\dot{p}_+ + \frac{\partial \mathcal{H}}{\partial q_-}\right) \end{aligned} \tag{4.13}$$

Now that $0 = \delta S$, the boundary terms are equal to zero (discussed later). Observe that the coefficients of each auxiliary function must be zero to result in zero variation in S. Thus, the Hamiltonian equations are reached:

$$\dot{q}_\pm = \frac{\partial \mathcal{H}}{\partial p_\mp}, \quad \dot{p}_\pm = -\frac{\partial \mathcal{H}}{\partial q_\mp} \tag{4.14}$$

For the boundary term in Eq. (4.13), at time $t = t_i$, the particle starts at the initial position, which is known, so there is no variation. Thus, $\eta_{q_-}(t_i) = 0$ and $\eta_{q_+}(t_i) = 0$. At the final time $t = t_f$, the differences in dissipative momentum $p_-(t_f)$ and variation of dissipative position $\eta_{q_-}(t_f)$ are equal to zero. Note that (q_-, p_-) coordinate is not the arbitary dissipative path that reduce to be zero at final time (t_f). Therefore, all the boundary terms are cancelled out, resulting in Eq. (4.14).

$$\eta_{q_-}(t_f)p_+(t_f) - \eta_{q_-}(t_i)p_+(t_i) + \eta_{q_+}(t_f)p_-(t_f) - \eta_{q_+}(t_i)p_-(t_i) \tag{4.15}$$

After deriving the Hamiltonian equations in the two varied path setting, physical limits (p.l.) are taken to constrain the two varied paths into a single one. A similar approach was also used in Galley's paper [5] to derive the dissipative Euler Lagrangian equation. The physical limits are as follows:

$$q_+ = q, \quad q_- = 0, \tag{4.16}$$

$$p_+ = p, \quad p_- = 0 \tag{4.17}$$

After taking the physical limits, the average of position q_+ and momentum p_+ of the two paths is constrained to the physical path q and p. The two varied paths are mathematical tools to include the dissipative K potential. In the physical sense, only one physical path is allowed by the Hamiltonian principle. Therefore, the average of paths (q_+, p_+) is constrained to (q, p), while the difference of paths $(q_-, p_-) = (0, 0)$. The dissipative Hamiltonian equation, after taking the physical limits, is as follows:

$$\dot{q} = [\dot{q}_+]_{p.l.} = \left[\frac{\partial \tilde{H}}{\partial p}\right]_{p.l.} \tag{4.18}$$

$$\dot{p} = [\dot{p}_+]_{p.l.} = -\left[\frac{\partial \tilde{H}}{\partial q}\right]_{p.l.}. \tag{4.19}$$

4.2 Application of Dissipative Hamiltonian Equations: Damped Harmonic Oscillator

In this section, the example of a damped harmonic oscillator is presented to show how the equations of motion can be obtained from Galley's dissipative Lagrangian [5] and the dissipative Hamiltonian from the variational principle.

From Galley's paper, the Lagrangian of the damped harmonic oscillator is given by the following. The usual conservative Lagrangian with mass (m), spring constant (k):

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{4.20}$$

While the dissipative K potential is as follows, where λ is the damping factor:

$$K(q_\pm, \dot{q}_\pm) = -\lambda q_- \cdot \dot{q}_+ \tag{4.21}$$

Translation from q_1, q_2 to q_+, q_- setting:

$$\Lambda(q_1, q_2, \dot{q}_1, \dot{q}_2, t) = \frac{1}{2}m\dot{q}_1^2 - \frac{1}{2}kq_1^2 - \lambda(q_1 - q_2)\left(\frac{\dot{q}_1 + \dot{q}_2}{2}\right) - \frac{1}{2}m\dot{q}_2^2 + \frac{1}{2}kq_2^2 \tag{4.22}$$

$$\Lambda(q_\pm, \dot{q}_\pm, t) = m\dot{q}_-\dot{q}_+ - kq_-q_+ - \lambda q_-\dot{q}_+ \tag{4.23}$$

By the Galley's Dissipative Euler-Lagrangian equations [5], the equation of motion can be obtained:

$$\left[\frac{d}{dt}\frac{\partial \Lambda}{\partial \dot{q}_-} - \frac{\partial \Lambda}{\partial q_-}\right]_{PL} = 0 \tag{4.24}$$

$$m\ddot{q}_+ + kq_+ + \lambda \dot{q}_+ = 0 \tag{4.25}$$

$$m\ddot{q} + kq + \lambda \dot{q} = 0 \quad \text{(apply physical limits)} \tag{4.26}$$

Conversion of the Lagrangian into Hamiltonian mechanics: The canonical momentum in double coordinate Lagrangian:

$$p_+ = \frac{\partial \Lambda}{\partial \dot{q}_-} = m\dot{q}_+ \tag{4.27}$$

$$p_- = \frac{\partial \Lambda}{\partial \dot{q}_+} = m\dot{q}_- \tag{4.28}$$

The dissipative Hamiltonian is derived from the Lagrangian in Eq. (4.23) as follows:

$$\mathcal{H}(q_\pm, p_\pm, t) = \frac{p_+ p_-}{m} - kq_- q_+ + \frac{\lambda q_- p_+}{m} \tag{4.29}$$

The dissipative Hamiltonian follows from Eq. (4.2) and also agrees with Eq. (4.29). Then, apply the dissipative Hamiltonian equations Eqs. (4.18) - (4.19).

$$\dot{q}_+ = \frac{p_+}{m} \tag{4.30}$$

$$\dot{p}_+ = -kq_+ - \frac{\lambda p_+}{m} \tag{4.31}$$

After applying physical limits on Eq. (4.31), the equation of motion $m\ddot{q} + kq + \lambda\dot{q} = 0$ is obtained.

The example of the damped harmonic oscillator demonstrates the method to obtain the equations of motion from the dissipative Hamiltonian.

4.3 Dissipative Liuoville's Theorem

The dissipative Hamiltonian has been developed in the above sections. In this session, I apply the new formalism to Liouville's theorem. In conventional Liouville's theorem, the phase space volume depends on the generalized coordinates (q, p) from conventional Hamiltonian mechanics. Since conventional Hamiltonian mechanics does not include dissipation, the impact of dissipation on the phase space diagram has not been fully explored. One of the goals of this thesis is to modify Liouville's theorem based on the dissipative Hamiltonian to describe how phase space volume changes in response to dissipation.

Notice that the generalized coordinates of $(q_\pm, p_\pm)$ are introduced, the phase space volume can be defined as following:

$$V_{++} = \delta q_+ \delta p_+ \tag{4.32}$$

$$V_{+-} = \delta q_+ \delta p_- \tag{4.33}$$

$$V_{-+} = \delta q_- \delta p_+ \tag{4.34}$$

$$V_{--} = \delta q_- \delta p_- \tag{4.35}$$

The physical phase space volume is defined by V_{++} because q_+ and p_+ are defined as the physical path after taking physical limits. To derive the dissipative Liouville's theorem, the conventional phase space volume $V_{++} = \delta q \delta p$ is replaced by V_{++}. The general derivative follows the same process as in section 3.2 until Eq. (3.18). The reason is that the new Hamiltonian equation is derived at Eqs. (4.18) - (4.19).

$$\frac{dV_{++}}{dt} = \delta q_+ \delta p_+ \left(\frac{\partial}{\partial p_+} \left(-\frac{dp_+}{dt} \right) + \frac{\partial}{\partial q_+} \frac{dq_+}{dt} \right) \tag{4.36}$$

By the dissipative Hamiltonian equations Eqs. (4.18) - (4.19),

$$\frac{dV_{++}}{dt} = \delta q_+ \delta p_+ \left[-\frac{\partial^2}{\partial p_+ \partial q_-} + \frac{\partial^2}{\partial q_+ \partial p_-} \right] \mathcal{H}\left(q_\pm, p_\pm, t\right) \tag{4.37}$$

Since the difference in conjugation between $(p_+ q_-)$ and $(q_+ p_-)$, the first time derivative of the dissipative phase space volume is allowed to be non-zero. Therefore, the phase space volume can vary across time.

The following variable is defined to be the exponential factor $D(q_\pm, p_\pm, t)$ of the phase space behavior:

$$D(q_\pm, p_\pm, t) = -\left[\frac{\partial^2}{\partial p_+ \partial q_-} - \frac{\partial^2}{\partial q_+ \partial p_-} \right] \mathcal{H}\left(q_\pm, p_\pm, t\right) \tag{4.38}$$

The ordinary differential equation from Eq. (4.36) can be expressed as:

$$\frac{dV_{++}}{dt} = \delta V_{++} \left(-D(q_\pm, p_\pm, t) \right) \tag{4.39}$$

Then, take the physical limit to obtain the physical exponential factor $D(q, p, t)$:

$$D(q, p, t) = \left[D(q_\pm, p_\pm, t) \right]_{p.l.} \tag{4.40}$$

After solving the ordinary differential equation at Eq. (4.37), the phase space behavior equation is obtained:

$$V(q, p, t) = V_0 e^{-\int_0^t dt D(q,p,t)} \tag{4.41}$$

where V_0 is the original phase space volume at $t = t_i$. The equation reveals the exponential behavior of the phase space diagram under dissipation.

The phase space behavior of the physical system has been discussed above. In this paragraph, some mathematical insights about the phase space volumes of V_{+-}, V_{-+}, and V_{--} are explored. For V_{+-} and V_{-+}, the dissipative factor $D(q_\pm, p_\pm, t)$ is zero because of the conjugation of the coordinates $(q_\pm, p_\mp)$ that cancels out the dissipation operator.

$$\frac{dV_{+-}}{dt} = \delta q_+ \delta p_- \left[-\frac{\partial^2}{\partial q_+ \partial q_-} + \frac{\partial^2}{\partial q_+ \partial p_-} \right] \mathcal{H}\left(q_\pm, p_\pm, t\right) = 0 \tag{4.42}$$

while

$$\frac{dV_{-+}}{dt} = \delta q_- \delta p_+ \left[-\frac{\partial^2}{\partial q_- \partial p_+} + \frac{\partial^2}{\partial q_- \partial p_+} \right] \mathcal{H}\left(q_\pm, p_\pm, t\right) = 0 \tag{4.43}$$

It is obvious that the derivative operators in V_{-+} and V_{+-} cancel out itself to result in the time derivative is equal to zero.

For V_{--}, the derivative operator is negative of V_{++}, apply the same approach as Eq. (4.36) to V_{--},

$$\frac{dV_{--}}{dt} = \delta q_- \delta p_- \left(\frac{\partial}{\partial p_-} \frac{dp_-}{dt} + \frac{\partial}{\partial q_-} \frac{dq_-}{dt} \right) \tag{4.44}$$

Apply Eq.(4.14) to the time derivative of p_- and q_-.

$$\frac{dV_{--}}{dt} = \delta q_- \delta p_- \left(\frac{\partial}{\partial p_-} \frac{dp_-}{dt} + \frac{\partial}{\partial q_-} \frac{dq_-}{dt} \right) \mathcal{H}\left(q_\pm, p_\pm, t\right) \tag{4.45}$$

The exponential factor $D_{--}(q_\pm, p_\pm, t)$ is

$$D_{--}(q_\pm, p_\pm, t) = \left[-\frac{\partial^2}{\partial p_- \partial q_+} + \frac{\partial^2}{\partial q_- \partial p_+} \right] \mathcal{H}\left(q_\pm, p_\pm, t\right) = D_{++}(q_\pm, p_\pm, t) \tag{4.46}$$

The exponential factor in V_{--} is the negative of that in the physical phase space V_{++}. Consequently, the phase space volume changes in opposite ways for V_{++} and V_{--}. While the phase space volume exhibits exponential decay in the physical phase space V_{++}, the phase space V_{--} experiences exponential growth with the same magnitude but opposite sign of $D(q, p, t)$. Although the mathematical behaviors of the arbitrary phase space volumes are intriguing, their physical meaning remains to be explored further.

For the subsequent discussion, the phase space volume V is considered as V_{++}, and the exponential factor is defined as $D(q, p, t) = \left[D_{++}(q_\pm, p_\pm, t)\right]_{p.l.}$.

4.4 Damped Harmonic Oscillator

Now, I apply the dissipative Liouville's theorem from section 4.3 to a damped harmonic oscillator. The result is compared to the numerical result developed from section 3.3. For the example of damped harmonic oscillator, substitute Eq. (4.29) into the Eq. (4.38) of exponential factor $D(q, p, t)$.

$$D(q_\pm, p_\pm, t) = \left[-\frac{\partial^2}{\partial p_+ \partial q_-} + \frac{\partial^2}{\partial q_+ \partial p_-} \right] \left(\frac{p_+ p_-}{m} - k q_- q_+ + \frac{\lambda q_- p_+}{m} \right) \tag{4.47}$$

The physical exponential factor is obtained after taking physical limit.

$$D(q_\pm, p_\pm, t) = -\frac{\lambda}{m} \tag{4.48}$$

Apply the result to Eq. (4.38), the phase space behaviour for the damped harmonic oscillator.

$$V(q, p, t) = V_0 e^{\int_0^t dt \frac{-\lambda}{m}} = V_0 e^{-\frac{\lambda}{m} t} \tag{4.49}$$

From the phase space behavior factor, we can observe that the phase space volume decreases exponentially as $e^{\frac{-\lambda}{m} t}$. The physical interpretation is to consider the phase space volume as a small area around a point (q, p) in phase space, representing an initial condition. This small enclosed area constitutes a set of different initial conditions around the point. Conventional Liouville's theorem states that the phase space volume remains constant over time in a system with conservative forces. The density of points remains constant, suggesting that the

trajectories of different initial conditions do not diverge over time, and no dissipation occurs.

For the dissipative Liouville's theorem, a dissipative term $-\lambda\dot{q}$ is introduced, causing all trajectories starting at their initial conditions to experience energy loss due to dissipation. The velocity decreases and the position of all particles tends to converge to the equilibrium position eventually. Consequently, all particles within the enclosed area shrink towards the same state, which is the equilibrium position at the end. Eq. (4.49) describes the exponential decay of phase space volume in a damped harmonic oscillator.

To validate this hypothesis and Eq. (4.49), I conduct a computational simulation with 625 points representing different initial conditions around the center point. The convex hull method is applied to validate the theoretical value.

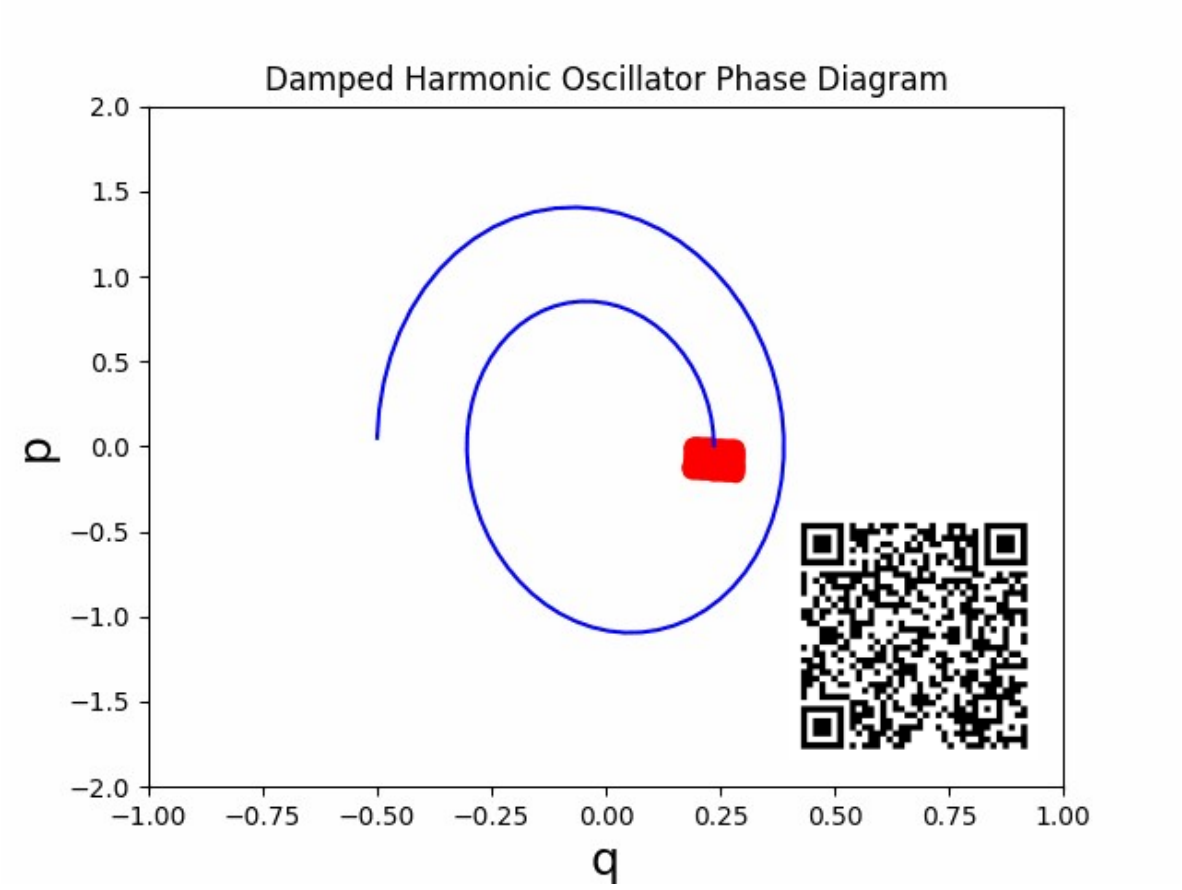

Figure 4.1: Phase Space Diagram of a damped Harmonic Oscillator with the 25 x 25 initial conditions

The parameters of the simulation are as follows: spring constant $k = 10\,\mathrm{N/m}$, mass $m = 1\,\mathrm{kg}$, and damping coefficient $\lambda = 0.5$. The range of position is from $-0.6\,\mathrm{m}$ to $-0.4\,\mathrm{m}$, and the range of velocities is from $-0.1\,\mathrm{m/s}$ to $0.2\,\mathrm{m/s}$. All the initial condition points are evenly distributed. A QR code is provided as the link to the animation of the phase diagram, which shows that the area of the phase space decreases exponentially and all the trajectories approach $(0, 0)$ as the equilibrium position.

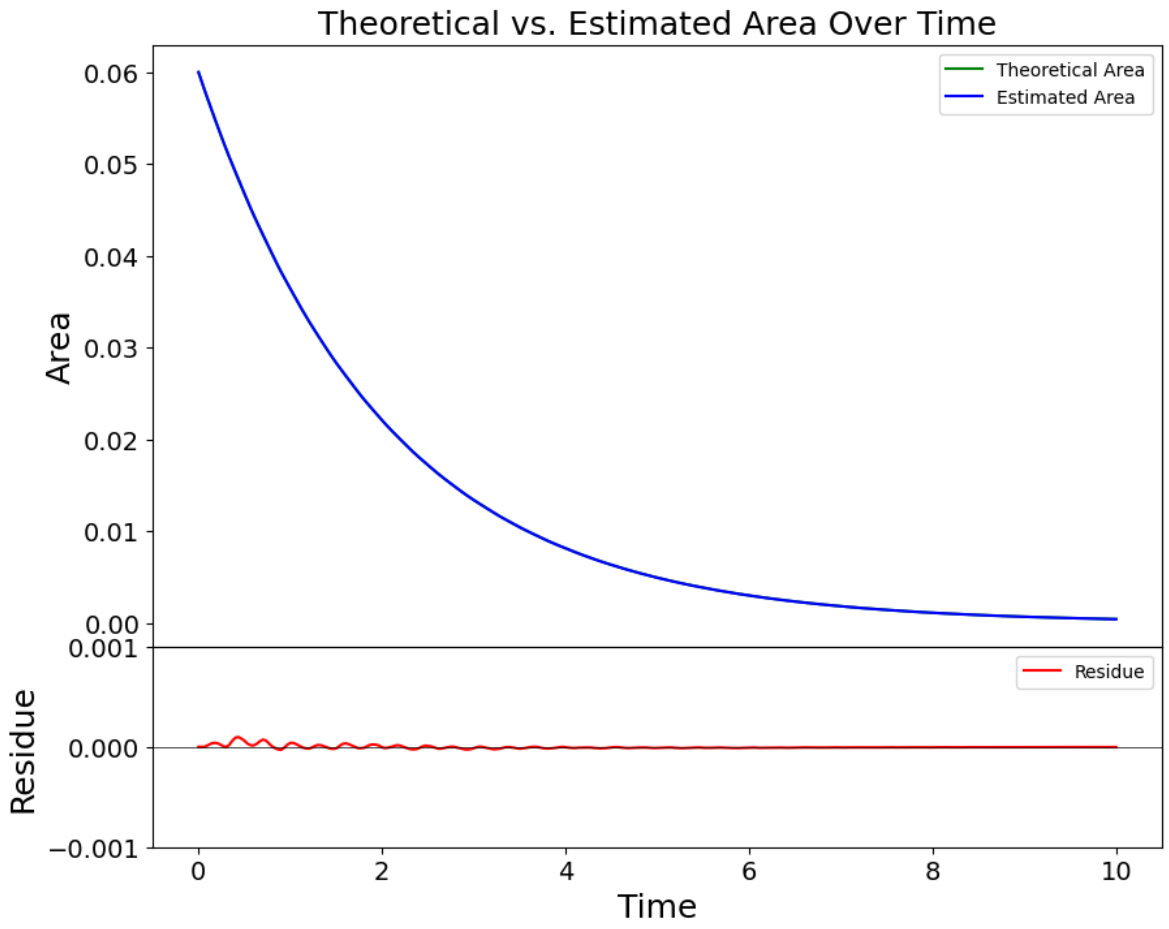

Figure 4.2: Area estimate of the Phase Space Volume and the theoretical curve for a damped harmonic oscillator

The graph shows the theoretical value agrees with the estimated area from the convex hull method. The two curve overlaps the residue is small. The convex hull validates our therotical equation Eq. (4.49)

4.5 Dropping Ball with Drag

The dropping ball with a linear drag $f = -b\dot{q}$ is another typical example of dissipative systems. A ball with mass m is falling under the gravitational acceleration (g). The conservative Hamiltonian is

$$H(q, \dot{q}, t) = \frac{p^2}{2m} + mgq \tag{4.50}$$

. The dissipative K potential is constructed as

$$K(q_\pm, p_\pm, t) = \frac{b}{m} q_- p_+ \tag{4.51}$$

The disspative Hamiltonian is (based from Eq. (4.2)):

$$\mathcal{H}(q_\pm, p_\pm, t) = \frac{p_+ p_-}{m} + mgq_- + \frac{bq_- p_+}{m} \tag{4.52}$$

While the exponential factor derived from Eq. (4.38)is

$$D(q_{\pm}, p_{\pm}, t) = -\frac{b}{m} \tag{4.53}$$

The resulting phase space volume equation is

$$V(q, p, t) = V_0 e^{-\frac{b}{m}t} \tag{4.54}$$

The numerical result is shown below.

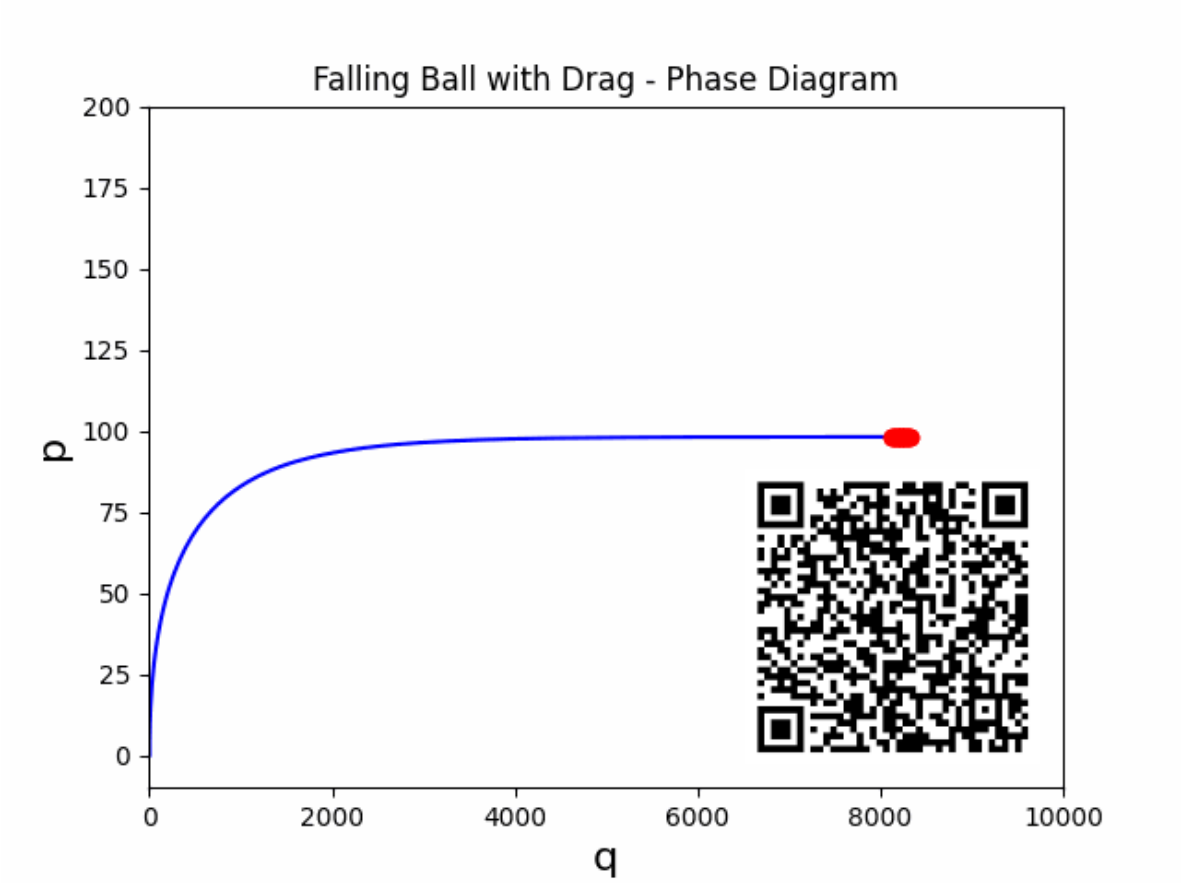

Figure 4.3: Phase Space Diagram of a falling ball with the linear drag with the 25 x 25 initial conditions
The parameters of the simulation are as follows: gravity $g = 9.81\,\mathrm{m/s^2}$, mass $m = 1\,\mathrm{kg}$, and linear drag coefficient $\lambda = 0.1$. The range of position is from 0 m to 20 m, and the range of velocities is from 0 m/s to 20 m/s. All the initial condition points are evenly distributed. A QR code provides the link to the animation of the phase diagram, which shows that the area of the phase space decreases and all the trajectories eventually reach the terminal velocity. The final phase space exhibits a line rather than a point, reflecting the diversity in initial velocities and positions.

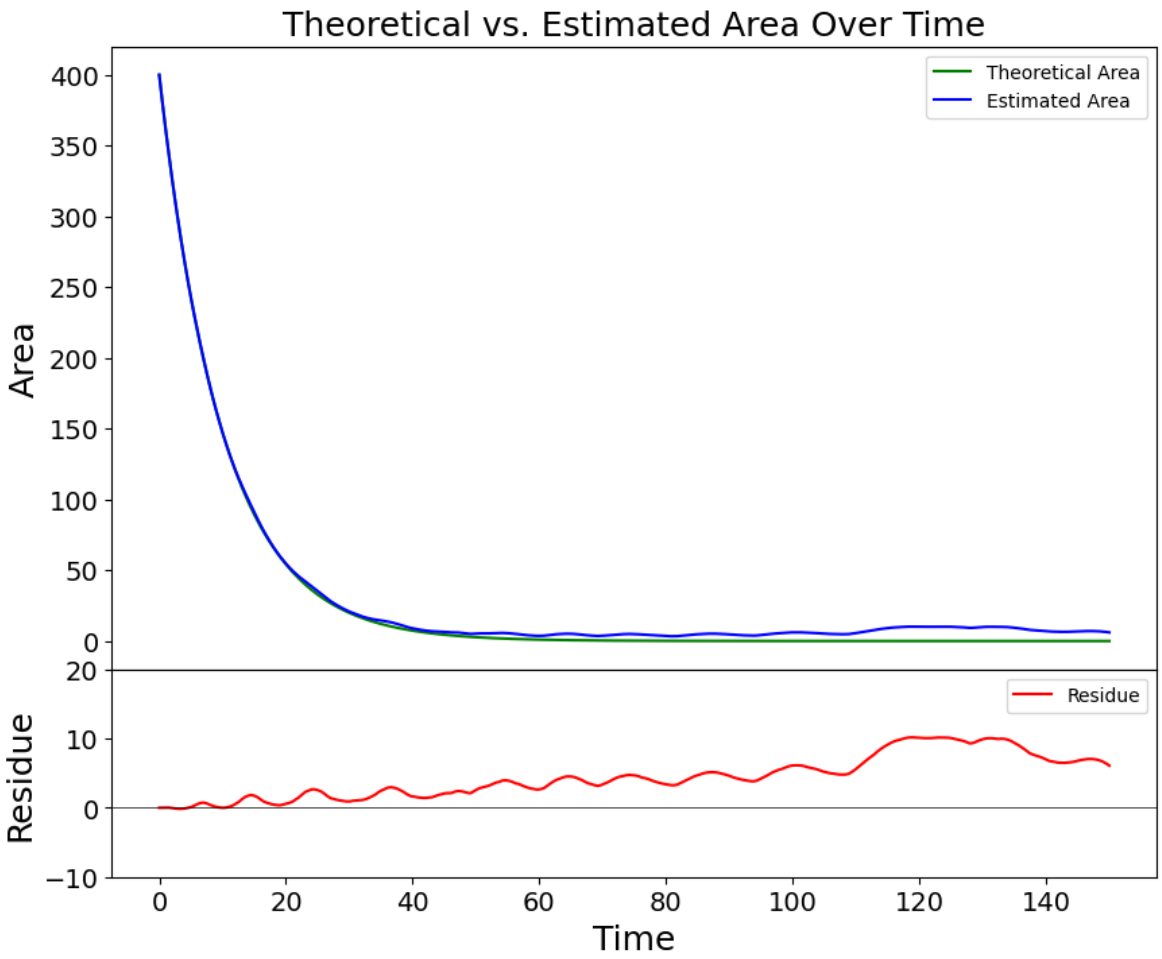

Figure 4.4: Area estimate of the Phase Space Volume and the theoretical curve of a falling ball with the linear drag

The graph shows the theoretical value agrees with the estimated area from the convex hull method at the early state. When the time increases, the deviation from theoretical value and the estimated area is due to the error in approximation the are of a line under convex hull method. Since the final phase space volume is a line with a theoretical volume approaches zero while the estimation of convex hull method results in a non zero volume of an area enclosed by the smallest polygon. It shows the limitation of computational stimulation. Another algorithm may be applied to handle the problem in the future

In my thesis, I purpose that all the linear dissipative term in the force equation can be describled by the K potential $K = bq_{-}\dot{q}_{+}$. The dissipative Liuoville's thereom can be applied to all physical systems only depend on the linear velocity dissipation. All the physical system will perturbation around the equilibrium can be modelled in the similar way with damped harmonic oscillator.

4.6 Future Goal

The next step of the project is to apply the dissipative Hamiltonian formalism to more complicated physical systems, like resonance capture of a planet. For the theoretical aspect, I plan to extend the dissipative Hamiltonian to Noether's Theorem to explore the impact of dissipation on the existing symmetry in the dynamics system. The final goal is to apply the dissipative Hamiltonian to describe the energy dissipation through gravitational waves in a binary system.

Chapter 5

Conclusion

In this thesis, I have derived dissipative Hamiltonian mechanics from Galley's Dissipative Lagrangian framework. This approach allowed us to apply the new formalism to Liouville's theorem, revealing the exponential decay behavior in the phase space distribution function—an insight that enhances our understanding of dissipative systems. Further, certain examples are provided to show the application of the Dissipative Hamiltonian to classical mechanics systems such as the damped harmonic oscillator, a falling ball subject to drag, and a damped single pendulum. The implications of the thesis maybe applied to describle a more complex dissipative system.

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Chapter 6

Appendix

Program 6.1 Python program that computes the phase space diagram of simple harmonic oscillator

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
import matplotlib.animation as animation
# Constants for the spring-mass system
k = 10  # Spring constant
m = 1   # Mass
b = 0 # Damping coefficient =0

# ODE for the spring-mass system with friction
def spring_mass_friction_ODE(t, y):
    return [y[1], -k * y[0] / m - b * y[1] / m]

positions = np.linspace(-0.6, -0.4, 25)
velocities = np.linspace(-0.1, 0.2, 25)
initial_conditions = [(x, v) for x in positions for v in velocities]

# Solve ODE for each set of initial conditions
time_span = [0, 10]
time_points = np.linspace(time_span[0], time_span[1], 300)  # More time points
solutions = [solve_ivp(spring_mass_friction_ODE, time_span, ic, t_eval=time_points) for ic i

# Set up the figure for animation
fig, ax = plt.subplots()
ax.set_xlim(-1, 1)
ax.set_ylim(-2, 2)
ax.set_xlabel('q', fontsize=18)
ax.set_ylabel('p',fontsize=18)
ax.set_title('Simple Harmonic Oscillator Phase Diagram')

# Initialize the animation with empty plots
dots = [ax.plot([], [], 'ro')[0] for _ in initial_conditions]
line, = ax.plot([], [], 'b-')  # Blue line for trajectory

def init():
    for dot in dots:
        dot.set_data([], [])
    line.set_data([], [])
    return dots + [line]

# Animation function to update the plots
def animate(i):
    for j, dot in enumerate(dots):
        dot.set_data(solutions[j].y[0][i], m * solutions[j].y[1][i])
    # Update the blue line for the middle trajectory
    if i == 0:
        line.set_data([], [])
    else:
        # Choosing the middle initial condition for the blue line trajectory
        middle_index = len(initial_conditions) // 2
        line.set_data(solutions[middle_index].y[0][:i], m * solutions[middle_index].y[1][:i]
    return dots + [line]
```

Program 6.2 Python program that computes the numerical value of the phase space volume

```
# Time span for the simulation
time_span = [0, 10]
time_points = np.linspace(time_span[0], time_span[1], 300)

# Solving the ODE for each set of initial conditions
solutions = [solve_ivp(spring_mass_friction_ODE, time_span, ic, t_eval=time_points) for ic i

# Theoretical volume (area) change function
def theoretical_area_change(t, V0, b, m):
    return V0 * np.exp(-b / m * t)

# Calculate the initial volume (area), here using the provided initial conditions
V0 = (positions[-1] - positions[0]) * (velocities[-1] - velocities[0])

# Calculate theoretical areas over time
theoretical_areas = theoretical_area_change(time_points, V0, b, m)

# Function to calculate the convex hull area at each time step
def calculate_area_over_time(solutions):
    areas = []
    for i in range(len(solutions[0].t)):
        current_points = np.array([(s.y[0][i], m * s.y[1][i]) for s in solutions])
        hull = ConvexHull(current_points)
        areas.append(hull.volume)  # In 2D, volume attribute gives the area
    return areas

# Calculate the areas over time using the convex hull method
areas_over_time = calculate_area_over_time(solutions)

# Calculate the residue (difference) between the estimated and theoretical areas
residue = np.array(areas_over_time) - np.array(theoretical_areas)

# Create a figure with subplots, the second subplot being smaller
fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(10, 8), sharex=True, gridspec_kw={'height_rati

# Plot the theoretical and estimated areas on the first subplot
ax1.plot(time_points, theoretical_areas, label='Theoretical Area', color='green')
ax1.plot(time_points, areas_over_time, label='Estimated Area', color='blue')
ax1.set_ylabel('Area', fontsize=18)
ax1.set_title('Theoretical vs. Estimated Area Over Time', fontsize=18)
ax1.tick_params(axis='both', which='major', labelsize=14)
ax1.legend()

# Plot the residue on the second subplot
ax2.plot(time_points, residue, label='Residue', color='red')
ax2.axhline(0, color='black', linewidth=0.5)  # Horizontal line at residue = 0
ax2.set_ylim(-0.001, 0.001)  # Set y-axis limits for residue
ax2.set_xlabel('Time', fontsize=18)
ax2.set_ylabel('Residue', fontsize=18)
ax2.tick_params(axis='both', which='major', labelsize=14)
ax2.legend()

# Adjust the layout so there is no space between subplots
```

Program 6.3 Python program that computes the phase space diagram of damped harmonic oscillator

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
import matplotlib.animation as animation

def spring_mass_friction_ODE(t, y):
    return (y[1],  - k * y[0] / m - b * y[1] / m)

# Assign constants numerical values
g = 9.81
k = 10
m = 1
b = 1  # Set to zero for no damping

# Generate 25 sets of initial conditions
positions = np.linspace(-0.5, -0.45, 5)
velocities = np.linspace(0.0, 0.1, 5)

initial_conditions = [(x, v) for x in positions for v in velocities]

# Solve ODE for each set of initial conditions
solutions = [solve_ivp(spring_mass_friction_ODE, [0, 10], ic,
t_eval=np.linspace(0, 10, 10 * 30)) for ic in initial_conditions]
Energy = [0.5 * m * (solutions[0].y[1])**2 + 0.5 * k * (solutions[0].y[0])**2]
# Animate the phase diagram for all 25 sets of initial conditions
fig, ax = plt.subplots()

phase_curve, = ax.plot([], [], 'b')
phase_dots = [ax.plot([], [], 'ro')[0] for _ in range(25)]

ax.set_title('Spring Mass System - Phase Diagram')
ax.set_xlim(-1, 1)
ax.set_ylim(-1.2, 1.5)
ax.set_xlabel(r'$q$ (m)')
ax.set_ylabel(r'$p$ (kg m/s)')
ax.grid()

def animate(i):
    phase_curve.set_data(solutions[0].y[0][:i], m*solutions[0].y[1][:i])
    for j in range(25):
        phase_dots[j].set_xdata(solutions[j].y[0][i])
        phase_dots[j].set_ydata(m*solutions[j].y[1][i])

ani = animation.FuncAnimation(fig, animate, frames=len(solutions[0].t), interval=50)
# Use the line below to save the animation as a GIF
ani.save('DHO2_0107.gif', writer='pillow')

# Uncomment the line below to save the animation as an MP4 file (requires ffmpeg)
# ani.save('phase_diagram_DHO2.mp4', writer='ffmpeg')

plt.show()
```

Program 6.4 Python program that computes the phase space diagram of a ball with drag

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from scipy.integrate import solve_ivp

# Constants
g = 9.81
m = 1
b = 0.1  # Linear drag coefficient

# Define the ODE function for a ball with linear drag
def linear_drag_ball_ODE(t, y):
    return [y[1], g - (b/m) * y[1]]

# Generate 25 sets of initial conditions
positions = np.linspace(0, 20, 25)
velocities = np.linspace(0, 20, 25)

initial_conditions = [(x, v) for x in positions for v in velocities]

# Solve ODE for each set of initial conditions
solutions = [solve_ivp(linear_drag_ball_ODE, t_span=(0, 150), y0=[x, v], t_eval=np.linspace(

# Animate the phase diagram for all 25 sets of initial conditions
fig, ax = plt.subplots()

phase_curve, = ax.plot([], [], 'b')
phase_dots = [ax.plot([], [], 'ro')[0] for _ in range(25)]

ax.set_title('Falling Ball with Drag - Phase Diagram')
ax.set_xlim(-10, 10000)
ax.set_ylim(-10, 200)
ax.set_xlabel('q',fontsize=16)
ax.set_ylabel('p',fontsize=16)
ax.grid(False)

# Calculate the interval for the desired duration
duration = 10  # seconds
num_frames = len(solutions[0].t)
interval = duration * 1000 / num_frames  # Convert to milliseconds

def animate(i):
    phase_curve.set_data(solutions[0].y[0][:i], solutions[0].y[1][:i])
    for j in range(25):
        phase_dots[j].set_xdata(solutions[j].y[0][i])
        phase_dots[j].set_ydata(solutions[j].y[1][i])

ani = animation.FuncAnimation(fig, animate, frames=len(solutions[0].t), interval=interval)
ani.save('fallingballwithdrag0417.gif', writer='pillow', fps=30)  # Set the frames per secon

# Uncomment the line below to save the animation as an MP4 file (requires ffmpeg)
# ani.save('falling_ball_with_drag.mp4', writer='ffmpeg')
plt.show()
```