Notes for Analytical Mechanics

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1 Mathematics

Theorem 1 (Chain rule)**.** *If $F(q_1, \cdots, q_n)$ is a function of n variables, then*

$$dF = \sum_i \frac{\partial F}{\partial q_i} dq_i.$$

In particular, if $F(q_1, \cdots, q_n, t)$ is a function of coordinates and time, then

$$\dot{F} = \frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial t}.$$

Definition 1 (Levi-Cevita symbol)**.** The Levi-Civita symbol $\epsilon_{i_1 \cdots i_n}$ is the sign of the permutation $(i_1 \cdots i_n)$. It has the following properties:

1. It is always 1, -1 or 0.
2. Swapping any two indexes inverts the Levi-Civita symbol.
3. $\epsilon_{i_1 \cdots i_n} = 0$ if $i_\alpha = i_\beta$ for any $\alpha \neq \beta$.
4. A cyclic permutation does not change the Levi-Civita symbol.

2 Newtonian mechanics

Theorem 2. *The work done by a force equals the change in kinetic energy:*

$$W_1 2 = \int_1^2 \mathbf{F} d\mathbf{s} = T_2 - T_1$$

Definition 2 (Conservative force)**.** A *conservative force* is the gradient of a scalar potential:

$$\mathbf{F} = -\nabla V(\mathbf{r})$$

Theorem 3. *A conservative force conserves energy:*

$$T_2 - T_1 = W_{12} = V_1 - V_2$$

Definition 3 (Lagrange multipliers)**.** content

3 Constraints, coordinates and displacements

Definition 4 (Holonomic constraint)**.** Consider a system of n particles with (actual, physical) coordinates $\mathbf{r_1}, \cdots \mathbf{r_n}$ and the time t. A *holonomic* constraint on the system can be expressed as a set of equations

$$f(\mathbf{r_1}, \cdots \mathbf{r_n}, t) = 0.$$

Remark 1. We generally limit ourselves to holonomic constraints, and at a microscopic scale, these are generally the only ones involved.

Definition 5 (Rheonomous and scleronomous constraints)**.** A constraint is called *rheonomous* if it contains the time as an explicit variable. If it does *not*, it is called *scleronomous*.

Definition 6 (Virtual displacement)**.** content

Postulate 1 (d'Alembert's principle)**.** *In the physical systems that we are interested in, reaction forces do no work under a virtual displacement.*

Definition 7 (Generalized force)**.**

$$Q_\sigma = \sum_{i=1}^{3N} F_i^{(a)} \frac{\partial x_i}{\partial q_\sigma}$$

4 Lagrangian mechanics

Definition 8 (Lagrangian)**.** The Lagrangian is the kinetic energy T minus the potential energy V:

$$L = T - V$$

Theorem 4 (Euler-Lagrange equation, without constraints)**.**

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

Definition 9 (Action)**.** The *action* S of a system evolving from t_1 to t_2 is the integral of the Lagrangian:

$$S = \int_{t_1}^{t_2} L(t)\ dt$$

Definition 10 (Variation)**.** Variation:

$$\delta S =$$

what is this really?

Postulate 2 (Hamilton's principle)**.** *A system will evolve in such a way that the action has a stationary value at the actual path of motion:*

$$\delta S = 0$$

5 Hamiltonian mechanics

Definition 11 (Canonical momentum from the Lagrangian)**.**

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Definition 12 (Hamiltonian)**.** The Hamiltonian (i running over all variables) is calculated from a Lagrangian $L(\{q_i, \dot{q}_i\}, t)$ and the canonical coordinates $\{q_i\}$ and momenta $\{p_i\}$ as

$$H = \sum_i p_i \dot{q}_i - L$$

Theorem 5 (Hamiltonian equations of motion)**.** *Let $H(\{q_i, p_i\}, t)$ be a Hamiltonian (derived from a Lagrangian L, or from another Hamiltonian by a canonical transformation). Then, for all i,*

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Furthermore, if H is derived from a Lagrangian L, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Definition 13 (Canonical transformation)**.** A transformation $(p_i, q_i) \to (P_i, Q_i)$ is *canonical* if it preserves the form of the Hamiltonian equations of motion, i.e.

$$\dot{Q} = \frac{\partial H}{\partial P}$$
$$\dot{P} = -\frac{\partial H}{\partial Q}$$

Definition 14 (Poisson bracket)**.** The Poisson bracket of two functions f and g, with respect to coordinates $\{q_i, p_i\}$, is defined as

$$\{f, g\}_{\{q_i, p_i\}} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Theorem 6. *A transformation is canonical iff it preserves Poisson brackets:*

$$\{f, g\}_{\{q_i, p_i\}} = \{f, g\}_{\{Q_i, P_i\}} \quad \textit{for all functions } f, g$$

Theorem 7 (Hamilton-Jacobi equation)**.** *Let H be a Hamiltonian function and S a generating function such that $H + \frac{\partial S}{\partial t} = 0$. Then*

$$H\left(q_1, \cdots, q_n, \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_1}, t\right) + \frac{\partial S}{\partial t} = 0$$

Not much of a theorem, really just by construction. But useful.

6 Central force problem

Theorem 8 (Virial theorem)**.**

$$\bar{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

7 Rotating coordinate systems

Theorem 9 (Effective force)**.** *If a particle is applied a force of* $\mathbf{F}$ *in a reference system, an observer in a coordinate system rotating with angular velocity* $\boldsymbol{\omega}$ *will see it as affected by a force*

$$\mathbf{F}_{\mathit{eff}} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v_r}) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{v_r}$ *is the velocity of the particle in the rotating coordinate system.*