Limit Cycles

Math 451S: Nonlinear Dynamics and Chaos

Spring 2025

Introduction

When analyzing non-linear planar systems in the xy-plane, we can find equilibrium points and analyze how the trajectories of the system look in the neighborhood of each equilibrium point i.e. solve for the x and y-nullclines and consider the linearization of the non-linear system, and compute the Jacobian at each equilibrium point to characterize each equilibrium point as a saddle point, nodal sink, nodal source, spiral sink or a spiral source.

Another important possibility which can influence how the trajectories look is if an isolated closed trajectory C is traced. Isolated means that neighboring trajectories are not closed. Such isolated closed trajectories are called *limit cycles*. If this happens, the associated solution $x(t)$ will be geometrically realized by a point which goes round the curve C with a certain period T.

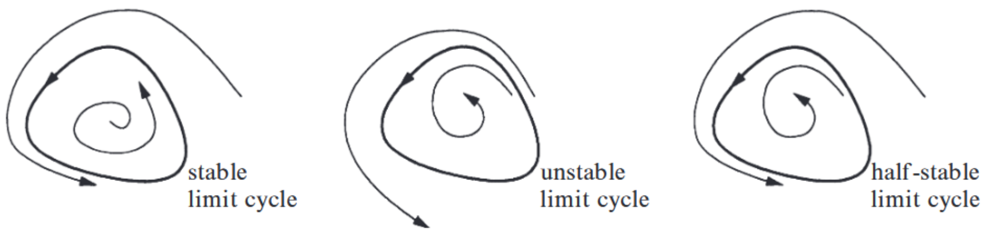
$$\mathbf{x}(t) \;=\; (x(t),\, y(t))$$

can be written as a pair of periodic functions with period T:

$$x(t+T) \;=\; x(t), \quad y(t+T) \;=\; y(t)$$

for all t.

It is worth noting that limit cycles are inherently nonlinear. If there is such a closed curve, the nearby trajectories must behave somewhat like C. The possibilities are illustrated below. The nearby trajectories can either spiral in toward C (attracting limit set) or they can spiral away from C (repelling limit set). Limit cycles are called stable, unstable, or half-stable according to whether the nearby curves spiral toward C, away from C, or both. Note that



Proving non-existence of limit cycles

By intuition or looking at numerical evidence, we can often suspect that a particular system has no periodic solutions. Here are three methods which can be used to show that a limit cycle does not exist.

1. **Gradient Systems**

   Suppose the system can be written in the form $\dot{\mathbf{x}} = \nabla V$ for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$

   Theorem: Closed orbits are impossible in gradient systems.

   Proof. Suppose there were a closed orbit. We obtain a contradiction by considering the change in V after one circuit. On the one hand, $\Delta V = 0$ since V is single-valued. But on the other hand,

$$\begin{aligned} \Delta V &= \int_0^T \tfrac{dV}{dt}\,dt \\ &= \int_0^T (\nabla V \cdot \dot{\mathbf{x}})\,dt \\ &= \text{-}\int_0^T \|\dot{\mathbf{x}}\|^2\,dt \\ &< 0 \end{aligned}$$

   (unless $\dot{\mathbf{x}} \equiv \mathbf{0}$, in which case the trajectory is a fixed point, not a closed orbit). This contradiction shows that closed orbits cannot exist in gradient systems.

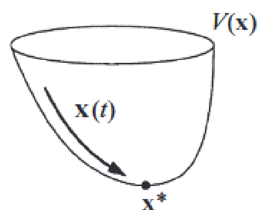
2. **Liapunov Functions**

   Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose that we can find a *Liapunov function*, i.e., a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

   (a) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite.*)

   (b) $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward x^*.)

   Then x^* is *globally asymptotically stable*: for all initial conditions, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In particular, the system has no closed orbits. A detailed proof is omitted here. The intuition is that all trajectories move monotonically down the graph of $V(\mathbf{x})$ toward $\mathbf{x}^*$.



   The solutions can't get stuck anywhere else because if they did, V would stop changing; but by assumption, $\dot{V} < 0$ everywhere except at $\mathbf{x}^*$.

3. **Dulac's Criterion / Bendixson's Criterion**

Proving existence of limit cycles

The main tool which is used to show that closed orbits exist in particular systems is the Poincare-Bendixson Theorem.

Poincare-Bendixson Theorem:.

Suppose that:

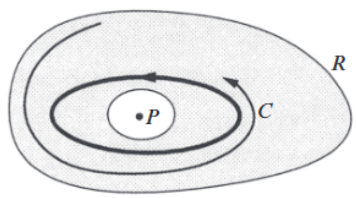
1. R is a closed, bounded subset of the plane;

2. $\dot{\mathbf{x}} = f(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

3. R does not contain any fixed points; and

4. There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time

Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R *contains a closed orbit* (shown as a heavy curve in the figure below).



The proof of this theorem is omitted here. In essence, if a trajectory is confined to a closed, bounded region that contains no fixed points, then the trajectory must eventually approach a closed orbit. This result depends crucially on the two-dimensionality of the plane. In higher-dimensional systems ($n > 3$), the Poincaré-Bendixson theorem no longer applies.

Lienard Systems

Lienard's equation is the second-order ODE:

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0,$$

We can rewrite this second-order equation as the first-order system:

$$\dot{x} = y, \dot{y} = -\,g(x) \;-\; f(x)\,y.$$

The following theorem states that this system has a unique, stable limit cycle under appropriate hypotheses on f and g.

Levinson-Smith Theorem. Suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

1. $f(x)$ and $g(x)$ are continuously differentiable for all x.
2. $g(-x) = -g(x)$ for all x (i.e. $g(x)$ is an odd function).
3. $g(x) > 0$ for all $x > 0$.
4. $f(-x) = f(x)$ for all x (i.e. $f(x)$ is an even function).
5. The odd function
$$F(x) = \int_0^x f(u)\,\mathrm{d}u$$
has exactly one positive zero at $x = a$, is negative for $0 < x < a$, is positive and nondecreasing for $x > a$, and $F(x) \to \infty$ as $x \to \infty$.

Then the system above has a unique, stable limit cycle surrounding the origin in the phase plane. Lienard's equation can be seen as a generalization of the van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + g(x) = 0$.

Relaxation Oscillations

Given that a closed orbit exists, what can we say about its shape and period? In general, such problems can't be solved exactly but we can still obtain useful approximations *if some parameter is large or small.*

In general, we can often observe limit cycles that consist of an extremely slow buildup followed by a sudden discharge, followed by another slow buildup, and so on. Such relaxation oscillations have two time scales that operate sequentially—a slow buildup is followed by a fast discharge. This is best illustrated with the example of the van der Pol equation.

Weakly Nonlinear Oscillators

Regular perturbation theory and its failure

We will now deal with equations of the form

$$\ddot{x} + \Omega_0^2 x = \epsilon h(x)$$

where $0 < \epsilon << 1$ and $h(x)$ is an arbitrary smooth, nonlinear function of x. Such equations represent small perturbations of the linear oscillator $\ddot{x} + \Omega_0^2 x = 0$. Therefore, these are called *weak nonlinear oscillators*. As an example, consider the simple pendulum, which obeys

$$\ddot{\theta} + \Omega_0^2 \sin\theta = 0$$

where $\Omega_0^2 = \frac{g}{l}$, with l the length of the pendulum. We may rewrite this equation as

$$\ddot{\theta} + \Omega_0^2\theta = \Omega_0^2(\theta - \sin\theta) = \frac{1}{6}\,\Omega_0^2\,\theta^3 - \frac{1}{120}\,\Omega_0^2\,\theta^5 + \ldots$$

The right-hand side above is a nonlinear function of θ. We can define this function to be $h(\theta)$, and take $\epsilon = 1$.

In order to predict the shape, period, radii etc. of such limit cycles, we will exploit our understanding of simple harmonic oscillators, which weak nonlinear oscillators are in "close proximity" to.

First, we will expand x as a power series in ϵ.

$$x = x_0 + \epsilon x_1 + \epsilon x_2 + ...$$

By Taylor's Theorem, we let $\eta = \epsilon x_1 + \epsilon^2 x^2 + ...$ have that

$$h(x_0 + \eta) = h(x_0) + h'(x_0)\eta + \frac{1}{2}h''(x_0)\eta^2 + ...$$

Working out the resulting expansion in powers of ϵ is tedious. One finds

$$h(x) = h(x_0) + \epsilon\, h'(x_0)\, x_1 + \epsilon^2 \Big[h'(x_0)\, x_2 + \frac{1}{2}\, h''(x_0)\, x_1^2\Big] + \ldots \tag{1}$$

Equating terms of the same order in ϵ, we obtain a hierarchical set of equations:

$$\begin{gathered}
\ddot{x}_0 + \Omega_0^2\, x_0 = 0, \\
\ddot{x}_1 + \Omega_0^2\, x_1 = h(x_0), \\
\ddot{x}_2 + \Omega_0^2\, x_2 = h'(x_0)\, x_1, \\
\ddot{x}_3 + \Omega_0^2\, x_3 = h'(x_0)\, x_2 + \tfrac{1}{2}\, h''(x_0)\, x_1^2 \\
\ldots
\end{gathered}$$

where the prime denotes differentiation with respect to the argument. The first of these is easily solved:

$$x_0(t) = A\cos(\Omega_0 t + \varphi),$$

where A and φ are constants. This solution is then plugged in at the next order to obtain an inhomogeneous equation for $x_1(t)$. Solve for $x_1(t)$ and insert into the following equation to solve for $x_2(t)$, and so on.

The problem is that *resonant forcing terms* generally appear on the right-hand side of each equation in our hierarchical set, except the first equation. Define $\theta \equiv \Omega_0 t + \varphi$. Then $x_0(\theta)$ is an even periodic function of θ with period 2π, hence so is $h(x_0)$. We may then expand $h(x_0(\theta))$ in a Fourier series

$$h(A\cos\theta) = \sum_{n=0}^{\infty} h_n(A)\ \cos(n\theta)$$

The $n = 1$ term leads to resonant forcing. Thus, the solution for $x_1(t)$ is

$$x_1(t) = \frac{1}{\Omega_0^2} \sum_{n=0;n\neq 1}^{\infty} \frac{h_n(A)}{1-n^2} \cos(n\,\Omega_0 t + n\,\varphi)\ +\ \frac{h_1(A)}{2\,\Omega_0}\, t\ \sin(\Omega_0 t + \varphi).$$

which increases linearly with time. As an example, consider a cubic nonlinearity with $h(x) = rx^3$, where r is a constant. Then, using

$$\cos^3\theta = \frac{3}{4}\ \cos\theta + \frac{1}{4}\ \cos(3\theta)$$

we have that $h_1 = \frac{3}{4} r A^3$ and $h_3 = \frac{1}{4} r A^3$.

We have a good approximation given small enough ϵ, but we are often interested in behavior of x for fixed ϵ, not fixed time t. In that case, we can only expect the perturbation approximation to work for $t << O(1/\epsilon)$.

Poincare-Lindstedt method

The problem here is that the nonlinear oscillator has a different frequency than its linear counterpart. These unbounded, secular terms indicate that the naive expansion is failing to capture the slight shift in the oscillator's natural frequency due to the perturbation. With this as motivation, if we assume the frequency Ω is a function of ϵ, with

$$\Omega(\epsilon) = \Omega_0 + \epsilon\, \Omega_1 + \epsilon^2\, \Omega_2 + \ldots,$$

then subtracting the unperturbed solution from the perturbed one and expanding in ϵ yields

$$\cos(\Omega t) - \cos(\Omega_0 t) = -\, \sin(\Omega_0 t)\, (\Omega - \Omega_0)\, t \;-\; \frac{1}{2} \cos(\Omega_0 t)\, (\Omega - \Omega_0)^2\, t^2 \;+\; \ldots$$

$$= -\,\epsilon\, \sin(\Omega_0 t)\, \Omega_1\, t \;-\; \epsilon^2 \Big(\sin(\Omega_0 t)\, \Omega_2\, t \;+\; \frac{1}{2}\, \cos(\Omega_0 t)\, \Omega_1^2\, t^2 \Big) + \mathcal{O}(\epsilon^3).$$

What perturbation theory can do for us is to provide a good solution up to a given time, provided that ϵ is sufficiently small. It will not give us a solution that is close to the true answer for *all* time. We see above that for each order, and to recover the shifted frequency $\Omega(\epsilon)$, we would have to resum perturbation theory to all orders.

The Poincare–Lindstedt method obviates this difficulty by assuming $\Omega = \Omega(\epsilon)$ from the outset. Define a dimensionless time $s = \Omega t$ and write our nonlinear DE can now be written as

$$\Omega^2 \frac{d^2 x}{ds^2} + \Omega_0^2 x = \epsilon h(x)$$

where

$$x = x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots, \qquad \Omega^2 = a_0 + \epsilon\, a_1 + \epsilon^2\, a_2 + \ldots.$$

We now plug the above expansions into the rewritten nonlinear DE:

$$(a_0 + \epsilon\, a_1 + \epsilon^2\, a_2 + \ldots) \left(\frac{d^2 x_0}{ds^2} + \epsilon\, \frac{d^2 x_1}{ds^2} + \epsilon^2\, \frac{d^2 x_2}{ds^2} + \ldots \right) + \Omega_0^2 (x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots) =$$

$$\epsilon\, h(x_0) + \epsilon^2\, h'(x_0)\, x_1 + \epsilon^3 \Big[h'(x_0)\, x_2 + \frac{1}{2}\, h''(x_0)\, x_1^2 \Big] + \ldots$$

Now let's write down equalities at each order in ϵ:

$$\begin{gathered}
\mathrm{a}_0 \, \tfrac{d^2 x_0}{ds^2} \; + \; \Omega_0^2 \, x_0 = 0, \\
\mathrm{a}_0 \, \tfrac{d^2 x_1}{ds^2} \; + \; \Omega_0^2 \, x_1 = h(x_0) \; - \; a_1 \, \tfrac{d^2 x_0}{ds^2}, \\
\mathrm{a}_0 \, \tfrac{d^2 x_2}{ds^2} \; + \; \Omega_0^2 \, x_2 = h'(x_0) \, x_1 \; - \; a_2 \, \tfrac{d^2 x_0}{ds^2} \; - \; a_1 \, \tfrac{d^2 x_1}{ds^2} \\
\ldots
\end{gathered}$$

The first equation of the hierarchy is immediately solved by

$$a_0 = \Omega_0^2, \quad x_0(s) = A \cos(s + \varphi).$$

At $O(\epsilon)$, we then have

$$\frac{d^2 x_1}{ds^2} + x_1 = \frac{1}{\Omega_0^2} \, h(A \cos(s + \varphi)) + \frac{1}{\Omega_0^2} \, a_1 A \cos(s + \varphi)$$

The left-hand side of the above equation has a natural frequency of 1 (in terms of the dimensionless time s). We expect $h(x_0)$ to contain resonant forcing terms as discussed earlier. However, we now have the freedom to adjust the undetermined coefficient a_1 to cancel any such resonant term. Clearly, we must choose

$$a_1 = \frac{h_1(A)}{A}$$

The solution for $x_1(s)$ is then

$$x_1(s) = \frac{1}{\Omega_0^2} \sum_{n=0; n \neq 1}^{\infty} \frac{h_n(A)}{1 - n^2} \, \cos(n \, s + n \, \varphi)$$

which is periodic and does not increase in magnitude without bound. The perturbed frequency is then obtained from

$$\Omega^2 = \Omega_0^2 \; - \; \frac{h_1(A)}{A} \, \epsilon \; + \; \mathcal{O}(\epsilon^2) \quad \Longrightarrow \quad \Omega(\epsilon) = \Omega_0 \; - \; \frac{h_1(A)}{2 \, \Omega_0 \, A} \, \epsilon \; + \; \mathcal{O}(\epsilon^2)$$

Note that Ω depends on the amplitude of the oscillations.

As an example, consider an oscillator with a quartic nonlinearity in the potential, i.e. $h(x) = r \, x^3$. Then

$$h(A \cos \theta) = r \, (A \cos \theta)^3 = \frac{3 \, r \, A^3}{4} \, \cos \theta \; + \; \frac{r \, A^3}{4} \, \cos(3\theta) \; + \; \ldots,$$

and we then obtain, setting $\epsilon = 1$ at the end of the calculation,

$$\Omega(\epsilon) = \Omega_0 \; + \; \frac{3 \, r \, A^2}{8 \, \Omega_0} \; + \; \ldots,$$

where the remainder is higher order in the amplitude A.

Multiple time-scale method

Note our previous framework of using perturbation theory, then eliminating secular terms (i.e. driving terms which oscillate at the resonant frequency of the unperturbed oscillator). Another method to do so is the *multiple time-scale method*. Apart from thinking from the point of view of eliminating secular terms, this method is motivated by the reasoning that we have a fast time-scale for sinusoidal oscillations, and a slow time-scale over which the amplitude and frequency gradually change.

If we simply expand quantities like $e^{\epsilon^2 t}$ in powers of ϵ, you only match the exact solution for small t. Once t grows large (on the order of $1/\epsilon$ or $1/\epsilon^2$, we need infinitely many terms to capture the slow decay or the slight frequency shift accurately. Truncating the series at any finite order causes the approximate solution to:

1. Keep amplitude nearly constant when it should be slowly decaying
2. Retain unperturbed frequency, which causes large phase error over time

To capture both the fast oscillation and the slow evolution of amplitude/frequency, we are motivated to explicitly build in the slow time-scale(s), avoiding the need for infinitely many terms to track behavior for large t.

Consider the equation

$$\ddot{x} + x = \epsilon\, h(x, \dot{x}),$$

where ϵ is presumed small, and $h(x, \dot{x})$ is a nonlinear function of position and/or velocity. We define a hierarchy of time-scales:

$$T_0 = t, \quad T_1 = \epsilon\, t, \quad T_2 = \epsilon^2\, t, \quad \ldots$$

Thus,

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\, \frac{\partial}{\partial T_1} + \epsilon^2\, \frac{\partial}{\partial T_2} + \ldots = \sum_{n=0}^{\infty} \epsilon^n \frac{\partial}{\partial T_n}.$$

Next, we expand

$$x(t) = \sum_{n=0}^{\infty} \epsilon^n\, x_n(T_0, T_1, T_2, \ldots).$$

Thus, we have

$$\left(\sum_{n=0}^{\infty}\epsilon^n\frac{\partial}{\partial T_n}\right)^2\left(\sum_{k=0}^{\infty}\epsilon^k x_k\right) \;+\; \sum_{k=0}^{\infty}\epsilon^k x_k \;=\; \epsilon\, h\Big(\sum_{k=0}^{\infty}\epsilon^k x_k,\, \sum_{n=0}^{\infty}\epsilon^n\frac{\partial}{\partial T_n}\Big(\sum_{k=0}^{\infty}\epsilon^k x_k\Big)\Big).$$

We now evaluate this order by order in ϵ:

$$\mathcal{O}(\epsilon^0): \quad \Big(\frac{\partial^2}{\partial T_0^2}+1\Big)x_0 = 0,$$

$$\mathcal{O}(\epsilon^1): \quad \Big(\frac{\partial^2}{\partial T_0^2}+1\Big)x_1 = -\,2\,\frac{\partial^2 x_0}{\partial T_0\,\partial T_1} \;+h(x_0,\frac{\partial x_0}{\partial T_0})$$

$$\mathcal{O}(\epsilon^2): \quad \Big(\frac{\partial^2}{\partial T_0^2}+1\Big)x_2 = -\,2\,\frac{\partial^2 x_1}{\partial T_0\,\partial T_1} - 2\,\frac{\partial^2 x_0}{\partial T_0\,\partial T_2} - \frac{\partial^2 x_0}{\partial T_1^2} + \frac{\partial h}{\partial x}\bigg|_{(x_0,\dot{x}_0)} x_1 + \frac{\partial h}{\partial \dot{x}}\bigg|_{(x_0,\dot{x}_0)}\Big(\frac{\partial x_1}{\partial T_0}+\frac{\partial x_0}{\partial T_1}\Big)$$

$$\ldots$$

Let's carry this procedure out to first order in ϵ. To order ϵ^0, we have

$$x_0 = A\cos(T_0+\phi)$$

where A and ϕ are arbitrary (at this point) functions of the slower time scales $\{T_1, T_2, \ldots\}$. Now we solve the next equation in the hierarchy, for x_1. Let $\theta \equiv T_0+\phi$. Then $\frac{\partial}{\partial T_0}=\frac{\partial}{\partial\theta}$, and we have

$$\Big(\frac{\partial^2}{\partial\theta^2}+1\Big)\,x_1 = 2\,\frac{\partial A}{\partial T_1}\,\sin\theta \;+\; 2\,A\,\frac{\partial\phi}{\partial T_1}\,\cos\theta \;+\; h(A\cos\theta,\,-A\sin\theta)$$

Since the arguments of h are periodic under $\theta \;\mapsto\; \theta+2\pi$, we may expand h in a Fourier series:

$$h(\theta) = h(A\cos\theta,\,-A\sin\theta) = \sum_{k=1}^{\infty}\alpha_k(A)\,\sin(k\theta) \;+\; \sum_{k=0}^{\infty}\beta_k(A)\,\cos(k\theta)$$

The inverse of this relation is

$$\alpha_k(A) = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\,\sin(k\theta)\,d\theta, \quad k > 0,$$

$$\beta_0(A) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta,$$

$$\beta_k(A) = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\,\cos(k\theta)\,d\theta, \quad k > 0.$$

We now demand that the secular terms on the right-hand side — those terms proportional to $\cos\theta$ and $\sin\theta$ — must vanish. This means

$$2\,\frac{\partial A}{\partial T_1} + \alpha_1(A) = 0,$$

$$2\,A\,\frac{\partial \phi}{\partial T_1} + \beta_1(A) = 0$$

These two first-order equations require two initial conditions, which is sensible since our initial equation $\ddot{x} + x = \epsilon\,h(x,\dot{x})$ is second order in time.

With the secular terms eliminated, we may solve for x_1:

$$x_1 = \sum_{k\neq 1}^{\infty}\left(\frac{\alpha_k(A)}{1-k^2}\,\sin(k\theta) \;+\; \frac{\beta_k(A)}{1-k^2}\,\cos(k\theta)\right) \;+\; C_0\cos\theta \;+\; D_0\sin\theta$$

where (i) the $k = 1$ terms are excluded from the sum, and (ii) an arbitrary solution to the homogeneous analog of the hierarchical DE involving x_1 is included. The constants C_0 and D_0 are arbitrary functions of T_1, $T_2, \ldots$.

The equations for A and ϕ are both first order in T_1. They will therefore involve two constants of integration - call them A_0 and ϕ_0. At second order, these constants are taken as dependent upon the superslow time scale T_2.

We will look at the example of the van der Pol oscillator.

Further reading

Recall the equation of motion for a forced, damped linear oscillator

$$\ddot{x} + 2\mu\dot{x} + x = f_0\cos\Omega t$$

We can consider adding a nonlinearity and study the equation

$$\ddot{x} + x = \epsilon h(x, \dot{x}) + \epsilon f_0 \cos(t + \epsilon \nu t)$$

We can also study relaxation oscillations, which is related to writing the original second-order DE (Lienard's equation) as two coupled first-order equations. If there is some parameter in the equation that is assumed to be very small / large, we can exploit this to observe that one equation evolves fast, while another evolves much more slowly.