Study Notes

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LIST OF DEFINITIONS

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Acronyms

Preface

Contents

0.1 Features of this template

> *TeX, stylized within the system as LaTeX, is a typesetting system which was designed and written by Donald Knuth and first released in 1978. TeX is a popular means of typesetting complex mathematical formulae; it has been noted as one of the most sophisticated digital typographical systems.*
>
> - Wikipedia

0.1.1 crossref

different styles of clickable definitions and theorems

- nameref: Gaussian distribution
- autoref: Definition A.1, **??**
- cref: Definition A.1,
- hyperref: Gaussian,

0.1.2 ToC (Table of Content)

- mini toc of sections at the beginning of each chapter
- list of theorems, definitions, figures
- the chapter titles are bi-directional linked

0.1.3 header and footer

fancyhdr

- right header: section name and link to the beginning of the section
- left header: chapter title and link to the beginning of the chapter
- footer: page number linked to ToC of the whole document

0.1.4 bib

- titles of reference is linked to the publisher webpage e.g., [Kit+02]
- backref (go to the page where the reference is cited) e.g., [Chi09]
- customized video entry in reference like in [Bab16]

0.1.5 preface, index, quote (epigraph) and appendix

index page at the end of this document...

Part I

Machine Learning

Chapter 1

Probability

Contents

1.1 Maximum Likelihood Estimation

$$\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)^{\mathrm{T}}, \boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})^{\mathrm{T}} \tag{1.1}$$

in which N is the number of samples, p is the number of features. The data is sampled from a distribution $p(\boldsymbol{x} \mid \theta)$, where θ is the parameter of the distribution.

For N i.i.d. samples, the likelihood function is $p(\boldsymbol{X} \mid \theta) = \prod_{i=1}^{N} p(\boldsymbol{x}_i \mid \theta))$

In order to get θ, we use Maximum Likelihood Estimation (MLE) to maximize the likelihood function.

$$\theta_{\texttt{MLE}} = \underset{\theta}{\operatorname{argmax}} \log p(\boldsymbol{X} \mid \theta) = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{N} \log p(\boldsymbol{x}_i \mid \theta) \tag{1.2}$$

1.2 Maximum A Posteriori Estimation

In Bayes' theorem, the θ is not a constant value, but $\theta \sim p(\theta)$. Hence,

$$p(\theta \mid \boldsymbol{X}) = \frac{p(\boldsymbol{X} \mid \theta)p(\theta)}{p(\boldsymbol{X})} = \frac{p(\boldsymbol{X} \mid \theta)p(\theta)}{\int_{\theta} p(\boldsymbol{X} \mid \theta)p(\theta)d\theta} \tag{1.3}$$

In order to get θ, we use Maximum A Posteriori Estimation (MAP) to maximize the posterior function.

$$\theta_{\texttt{MAP}} = \underset{\theta}{\operatorname{argmax}}\, p(\theta \mid \boldsymbol{X}) = \underset{\theta}{\operatorname{argmax}} \frac{p(\boldsymbol{X} \mid \theta)p(\theta)}{p(\boldsymbol{X})} \tag{1.4}$$

After θ is estimated, then calculating $\frac{p(\boldsymbol{X} \mid \theta) \cdot p(\theta)}{\int_\theta p(\boldsymbol{X} \mid \theta) p(\theta) d\theta}$ to get the posterior distribution. We can use the posterior distribution to predict the probability of a new sample $\boldsymbol{x}$.

$$p(x_{\texttt{new}} \mid \boldsymbol{X}) = \int_\theta p(x_{\texttt{new}} \mid \theta) \cdot p(\theta \mid \boldsymbol{X}) d\theta \tag{1.5}$$

1.3 Gaussian Distribution

Gaussian distribution is also called normal distribution.

$$\theta = (\mu, \sigma^2), \quad \mu = \frac{1}{N} \sum_{i=1}^{N} x_i, \quad \sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \tag{1.6}$$

For MLE,

$$\theta = (\mu, \Sigma) = (\mu, \sigma^2), \quad \theta_{\texttt{MLE}} = \operatorname*{argmax}_\theta \log p(\boldsymbol{X} \mid \theta) = \operatorname*{argmax}_\theta \sum_{i=1}^{N} \log p(\boldsymbol{x}_i \mid \theta) \tag{1.7}$$

Generally, the Probability Density Function (PDF) of a Gaussian distribution is:

$$p(x \mid \mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma)}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \mu)^{\mathrm{T}} \Sigma^{-1} (\boldsymbol{x} - \mu)\right) \tag{1.8}$$

in which μ is the mean vector, Σ is the covariance matrix, det is the determinant of matrix. det is the product of all eigenvalues of a matrix.

Hence,

$$\log p(\boldsymbol{X} \mid \theta) = \sum_{i=1}^{N} \log p(x_i \mid \theta) = \sum_{i=1}^{N} \log \frac{1}{\sqrt{(2\pi)^p \det(\Sigma)}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \mu)^{\mathrm{T}} \Sigma^{-1} (\boldsymbol{x} - \mu)\right) \tag{1.9}$$

Let's only consider 1 dimension case for brevity, then

$$\log p(\boldsymbol{X} \mid \theta) = \sum_{i=1}^{N} \log p(x_i \mid \theta) = \sum_{i=1}^{N} \log \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2}\right) \tag{1.10}$$

Let's get the optimal value for μ,

$$\mu_{\texttt{MLE}} = \operatorname*{argmax}_\mu \log p(\boldsymbol{X} \mid \theta) = \operatorname*{argmin}_\mu \sum_{i=1}^{N} \frac{1}{2} (x_i - \mu)^2 \tag{1.11}$$

So,

$$\frac{\partial \log p(\boldsymbol{X} \mid \theta)}{\partial \mu} = \sum_{i=1}^{N} (\mu - x_i) = 0 \rightarrow \mu_{\texttt{MLE}} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{1.12}$$

Let's get the optimal value for σ^2,

$$
\begin{aligned}
\sigma_{\texttt{MLE}} &= \underset{\sigma}{\operatorname{argmax}} \log p(\boldsymbol{X} \mid \theta) \\
&= \underset{\sigma}{\operatorname{argmax}} \sum_{i=1}^{N} \log \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) \\
&= \underset{\sigma}{\operatorname{argmax}} \sum_{i=1}^{N} \left[-\log\sqrt{2\pi\sigma^2} - \frac{(x-\mu)^2}{2\sigma^2}\right] \\
&= \underset{\sigma}{\operatorname{argmin}} \sum_{i=1}^{N} \left[\log\sigma + \frac{(x-\mu)^2}{2\sigma^2}\right]
\end{aligned}
$$

Hence,

$$
\frac{\partial}{\partial\sigma} \sum_{i=1}^{N} \left[\log\sigma + \frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0 \to \sigma^2_{\texttt{MLE}} = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 \tag{1.13}
$$

$\mathbb{E}_D\left[\mu_{\texttt{MLE}}\right]$ is unbaised.

$$
\mathbb{E}_D\left[\mu_{\texttt{MLE}}\right] = \mathbb{E}_D\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_D\left[x_i\right] = \frac{1}{N}\sum_{i=1}^{N}\mu = \mu \tag{1.14}
$$

However, $\mathbb{E}_D\left[\sigma^2_{\texttt{MLE}}\right]$ is biased.

$$
\begin{aligned}
\mathbb{E}_D\left[\sigma^2_{\texttt{MLE}}\right] &= \mathbb{E}_D\left[\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_{\texttt{MLE}})^2\right] && (1.15)\\
&= \mathbb{E}_D\left[\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu_{\texttt{MLE}})^2\right] && (1.16)\\
&= \mathbb{E}_D\left[\frac{1}{N}\sum_{i=1}^{N}\left(x_i^2-2x_i\mu_{\texttt{MLE}}+\mu^2_{\texttt{MLE}}\right)\right] = \mathbb{E}_D\left[\sum_{i=1}^{N}x_i^2 - 2\frac{1}{N}\sum_{i=1}^{N}x_i\mu_{\texttt{MLE}}+\mu^2_{\texttt{MLE}}\right] && (1.17)\\
&= \mathbb{E}_D\left[\frac{1}{N}\sum_{i=1}^{N}\left(x_i^2-\mu^2\right)+\mu^2-\mu^2_{\texttt{MLE}}\right] && (1.18)\\
&= \sigma^2 - \mathbb{E}_D\left[\mu^2_{\texttt{MLE}}-\mu^2\right] && (1.19)\\
&= \sigma^2 - \left(\mathbb{E}_D\left[\mu^2_{\texttt{MLE}}\right]-\mathbb{E}_D\left[\mu^2_{\texttt{MLE}}\right]\right) && (1.20)\\
&= \sigma^2 - \texttt{Var}\left[\mu_{\texttt{MLE}}\right] = \sigma^2 - \texttt{Var}\left[\frac{1}{N}\sum_{i=1}^{N}x_i\right] && (1.21)\\
&= \sigma^2 - \frac{1}{N^2}\sum_{i=1}^{N}\texttt{Var}\left[x_i\right] = \frac{N-1}{N}\sigma^2 && (1.22)\\
& && (1.23)
\end{aligned}
$$

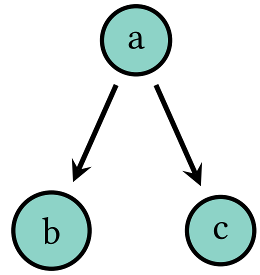

Figure 1.1: A simple Bayesian network.

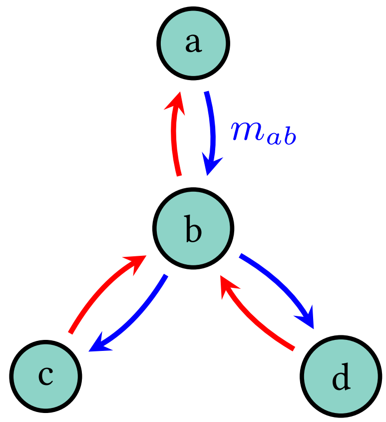

Figure 1.2: Belief propagation.

1.4 Bayesian Network

1.5 Probability Graph

this section is not finished yet. Need to be reviewed p54

1.5.1 Variables Elimination

1.5.2 Belief propagation

Belief propagation is mainly used for tree data structure, and equals Section 1.5.1 with caching.

1.5.3 Max-product Algorithm

1.5.4 Factor Graph

1.6 Expectation Maximum

$$\Theta^{(t+1)} = \underset{\Theta}{\text{argmax}} \int_Z \log P(x, z \mid \theta) \cdot P(z \mid x, \Theta^{(t)})\, dz$$

continue on p60

1.7 Hidden Markov Model

Part II

Algorithm and Data Structure

Chapter 2

Algorithm

Contents

2.1 Graph

Part III

Programming

Chapter 3

C++

Chapter 4

Rust

Part IV

Research

Chapter 5

Paper Reading

Appendix A

Formulas

A.1 Gaussian distribution

Definition A.1 (Gaussian distribution)**.** *Gaussian distribution*

Theorem A.1 (Central limit theorem)**.**

Bibliography

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