Instrumental variables: Experiments with non-compliance

INFO/STSCI/ILRST 3900: Causal Inference

21 Oct 2025

Logistics

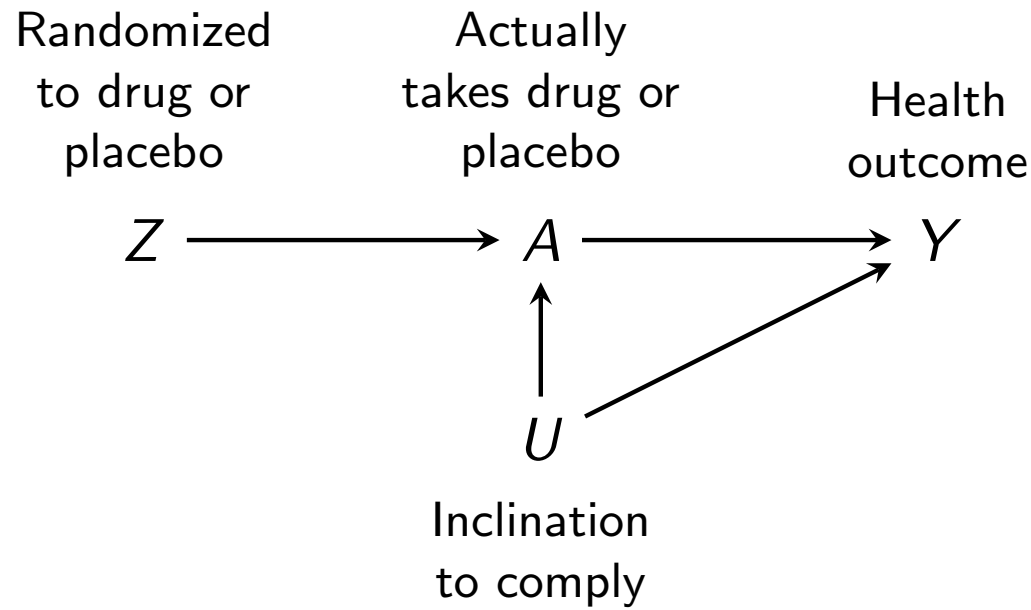
- PSET 4 posted (due Oct 27)
- Project Groups will be assigned this week
- Project Part 2 details will be posted

Learning goals for today

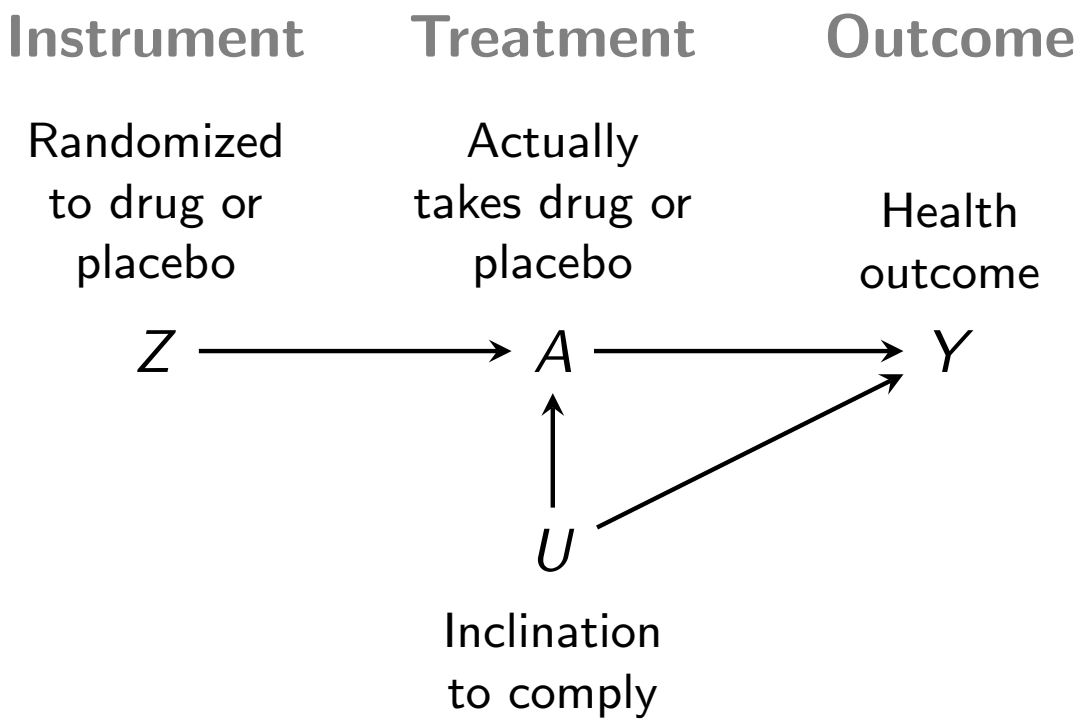
At the end of class, you will be able to:

1. Understand the logic of instrumental variables
2. Derive the average effect among compliers in experiments with noncompliance

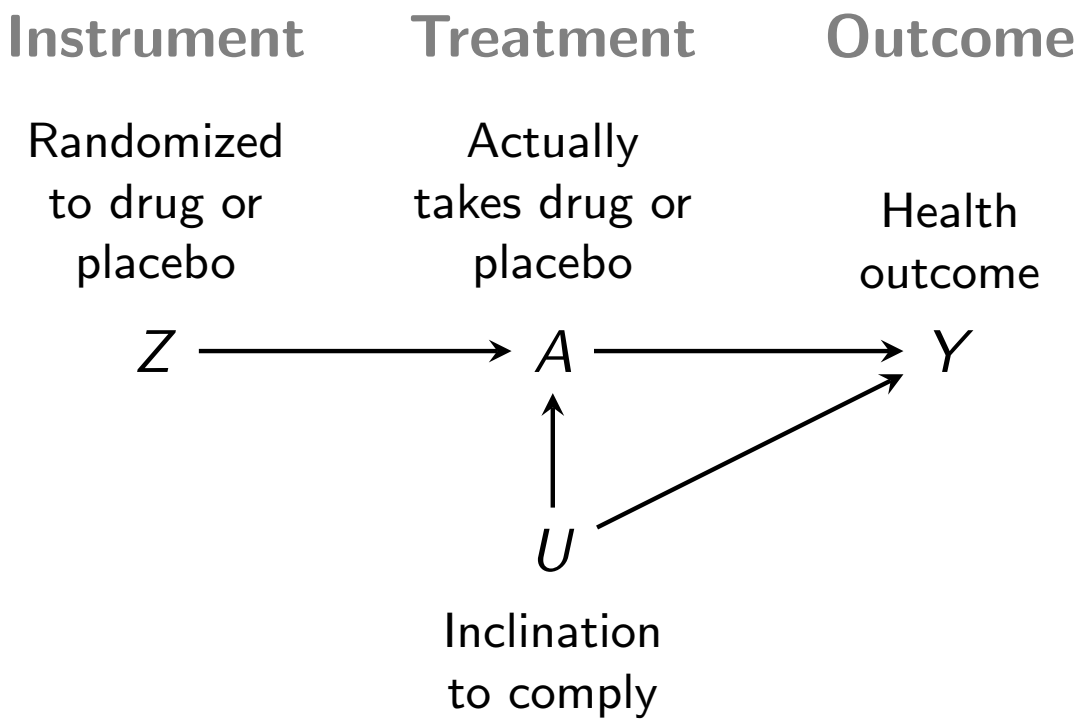
Instrumental variables: Experiment with noncompliance



Instrumental variables: Experiment with noncompliance



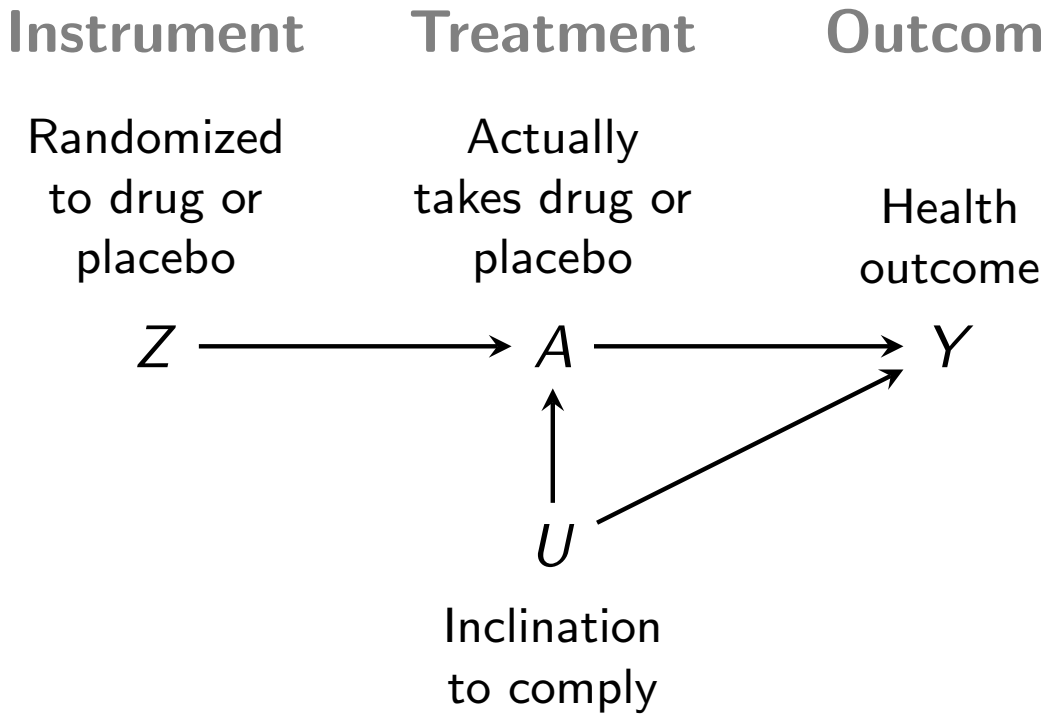
Instrumental variables: Experiment with noncompliance



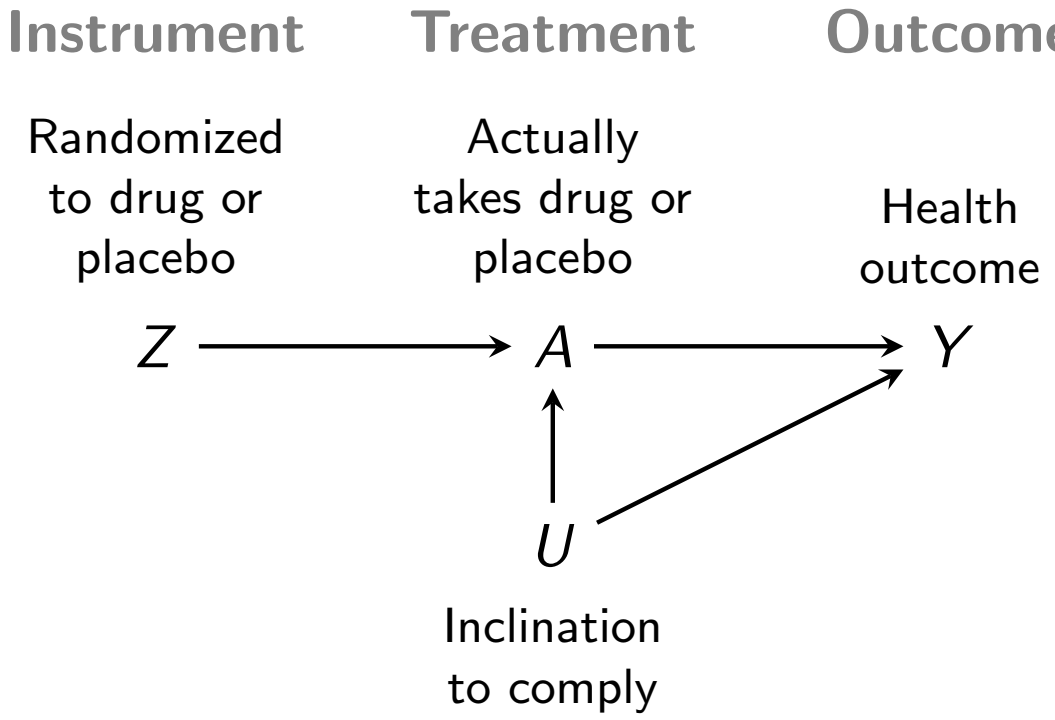
Two ideas

1. Intent to treat effect
2. Average effect among compliers

Instrumental variables: 1) Intent to treat effect

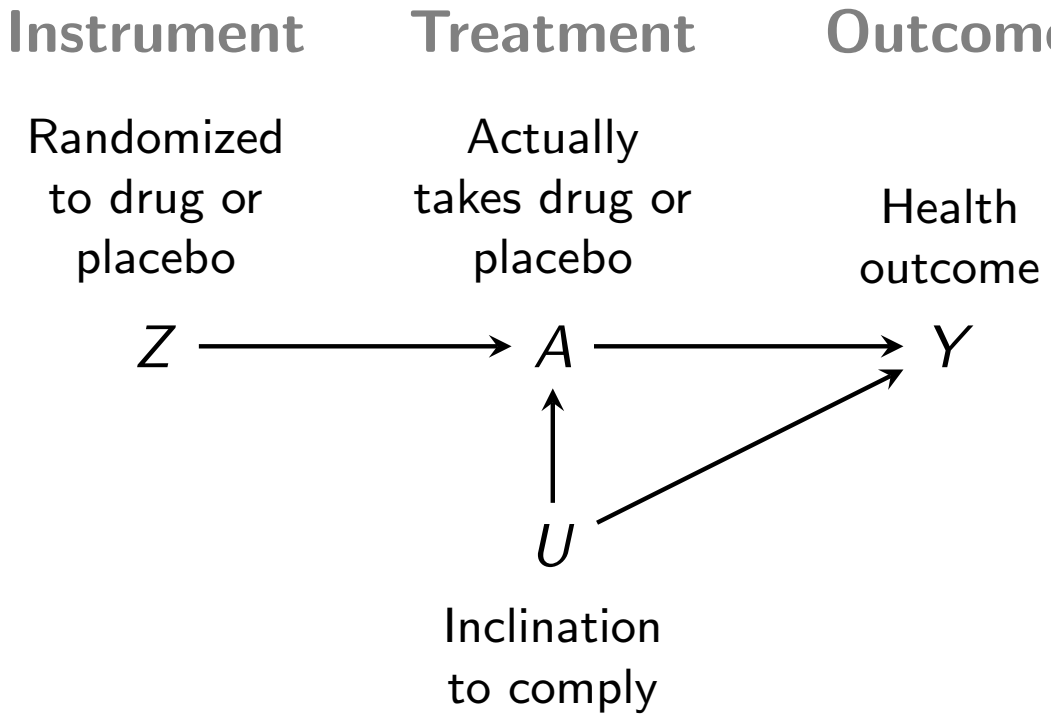


Instrumental variables: 1) Intent to treat effect



Ignore A. What is the effect of Z on Y?

Instrumental variables: 1) Intent to treat effect

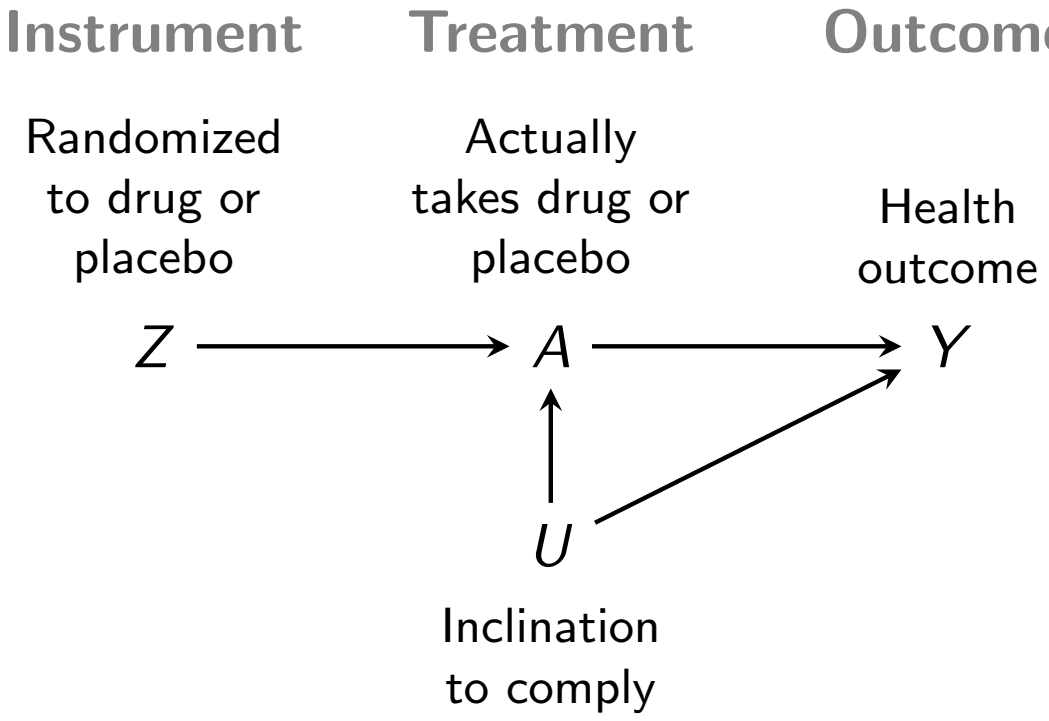


Ignore A. What is the effect of Z on Y?

$$\mathsf{E}(Y^{z=1} - Y^{z=0})$$

- $Y^{z=1}$: Outcome under $Z = 1$
- $Y^{z=0}$: Outcome under $Z = 0$

Instrumental variables: 1) Intent to treat effect

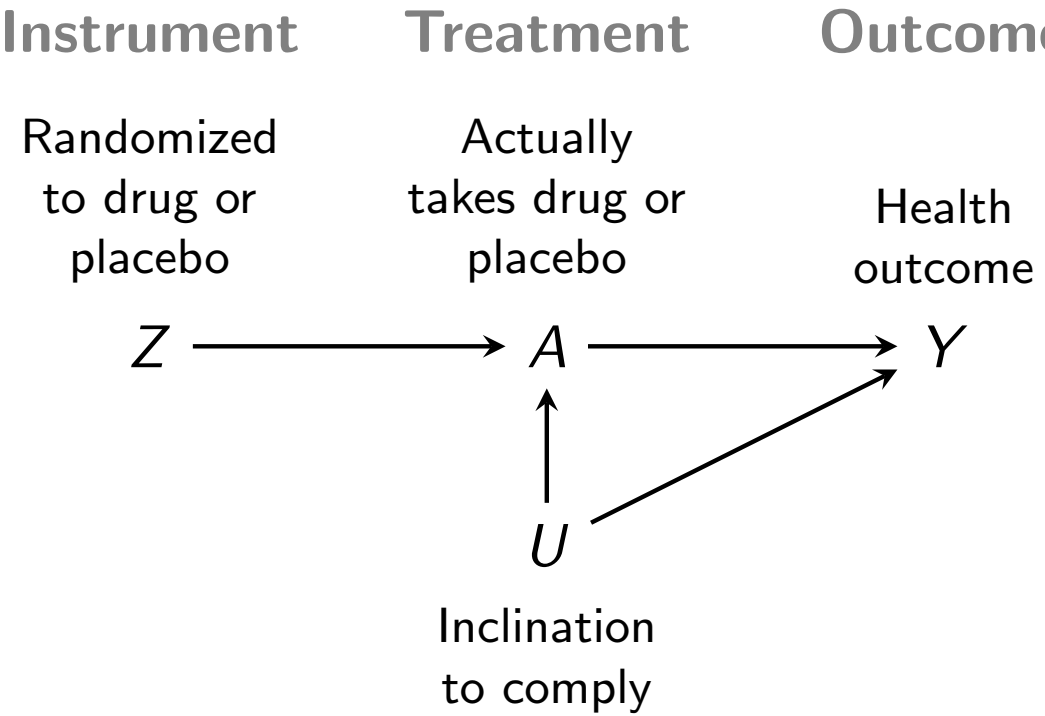


Ignore A. What is the effect of Z on Y?

$$\mathsf{E}(Y^{z=1} - Y^{z=0}) = \mathsf{E}(Y|Z=1) - \mathsf{E}(Y|Z=0)$$

- $Y^{z=1}$: Outcome under $Z = 1$
- $Y^{z=0}$: Outcome under $Z = 0$
- $=$: By Positivity, Consistency, Exchangeability for Z

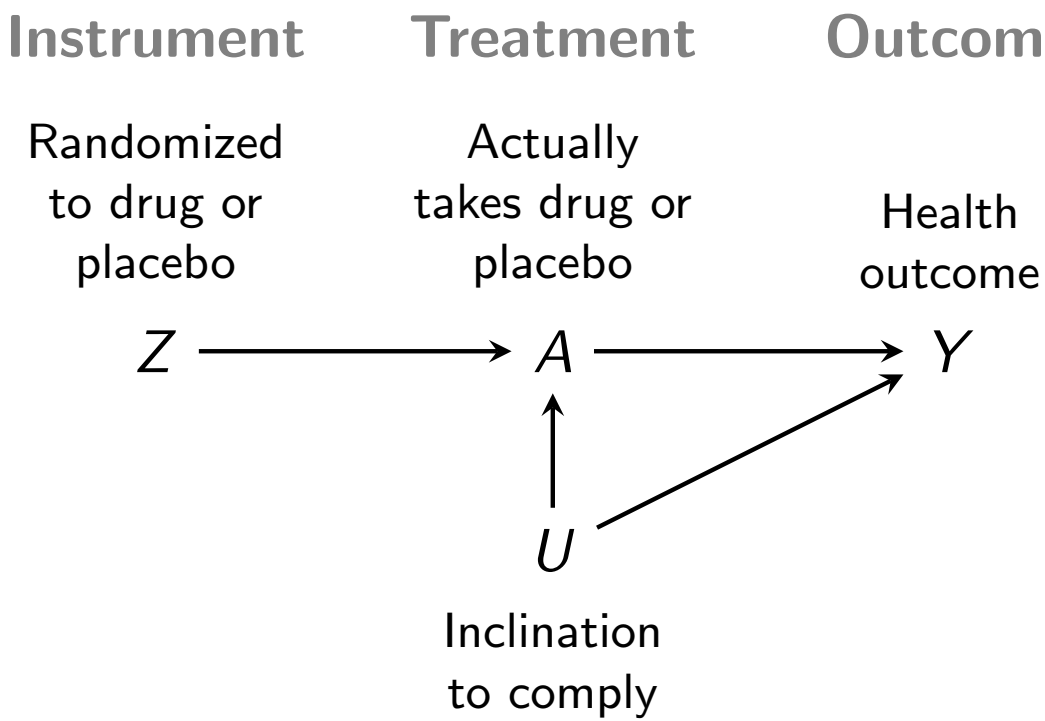
Instrumental variables: 1) Intent to treat effect



Ignore A. What is the effect of Z on Y?

$$\underbrace{\mathsf{E}(Y^{z=1} - Y^{z=0})}_{\substack{\nearrow \\ \text{Outcome under } Z=1 \\ \nwarrow \\ \text{Outcome under } Z=0}} \underbrace{=}_{\substack{\uparrow \\ \text{By Positivity Consistency Exchangeability for } Z}} \underbrace{\mathsf{E}(Y \mid Z=1) - \mathsf{E}(Y \mid Z=0)}_{\text{Mean difference in observable outcomes}}$$

Instrumental variables: 2) Average effect among compliers



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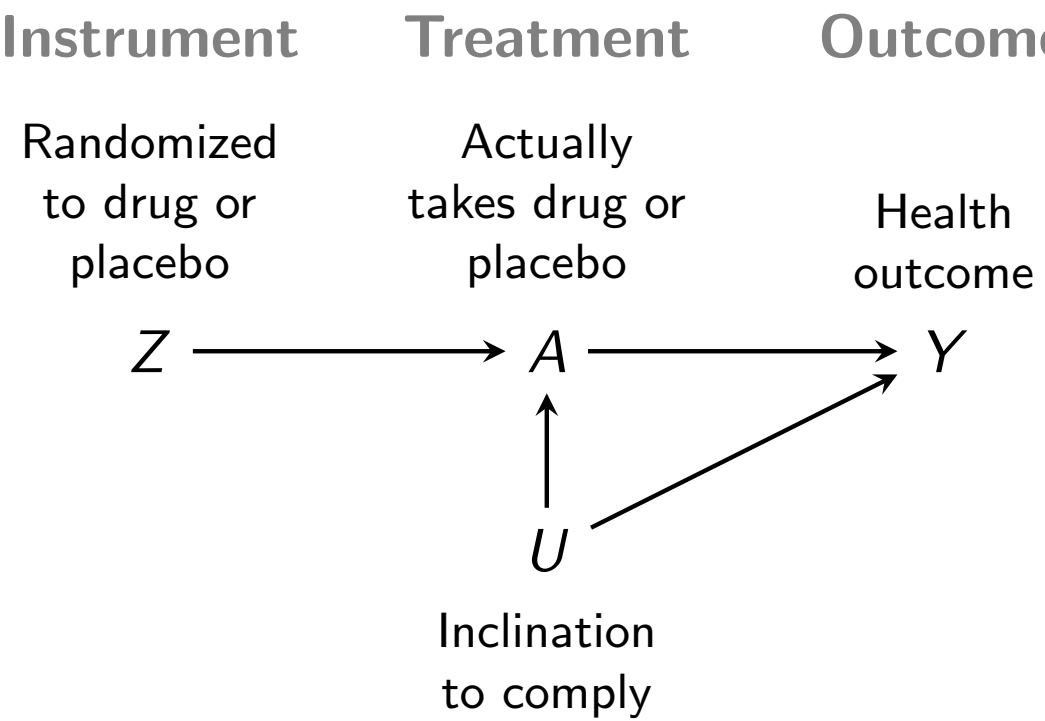
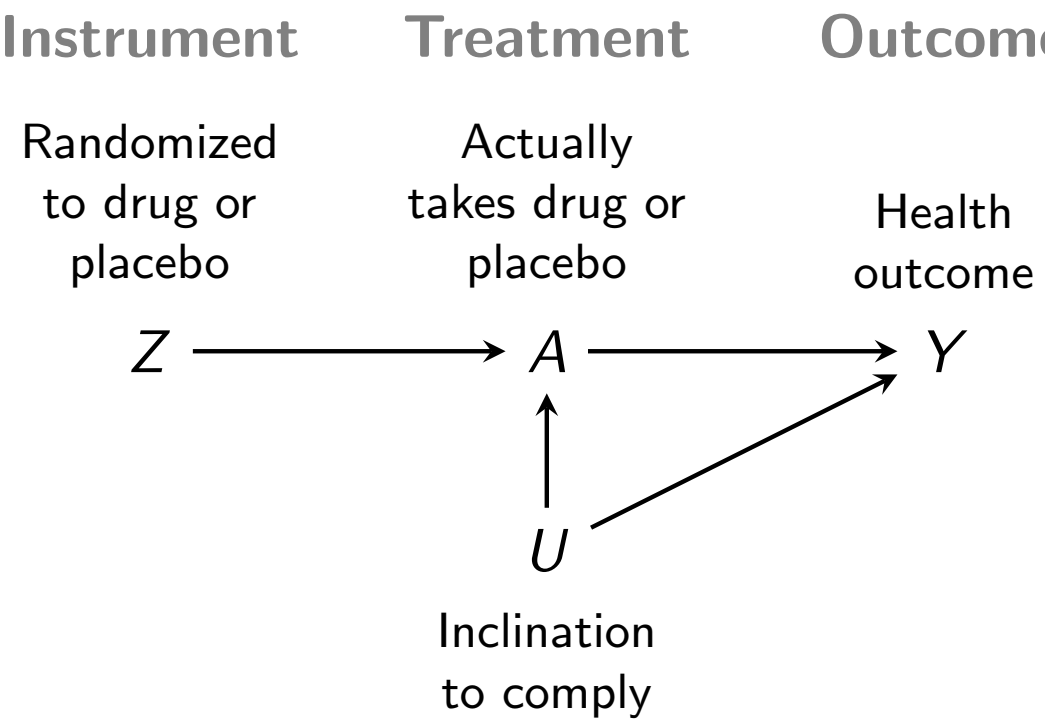


Key insight: The effect of Z on Y operates entirely through A

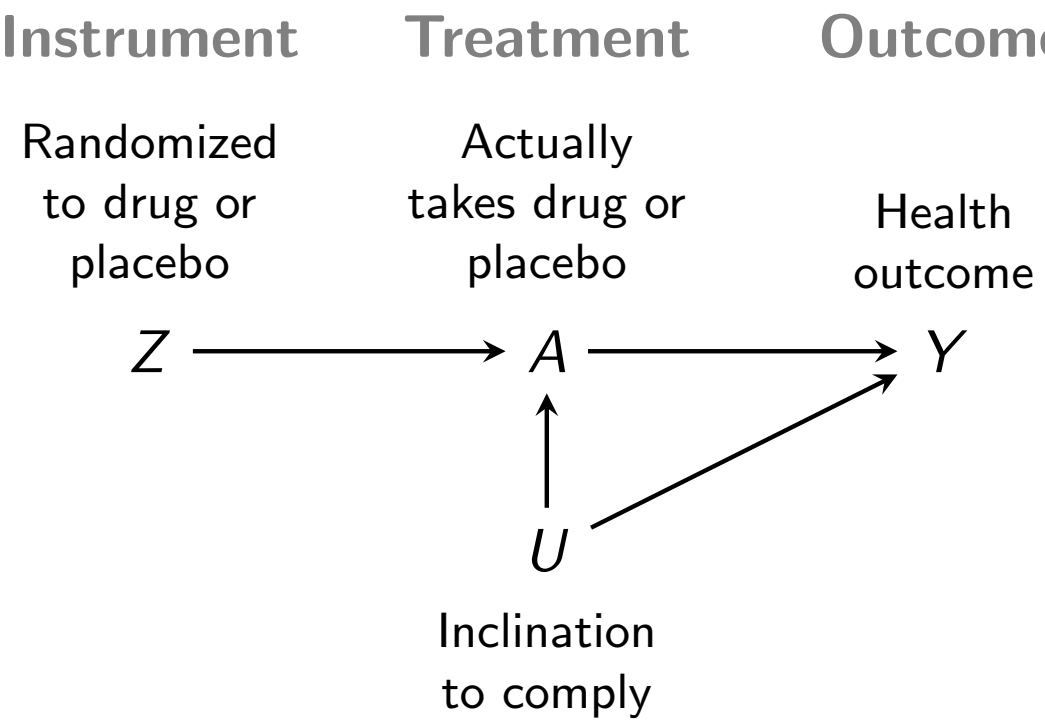
Instrumental variables: 2) Average effect among compliers



Key insight: The effect of Z on Y operates entirely through A

1. Study the effect of $Z \to Y$

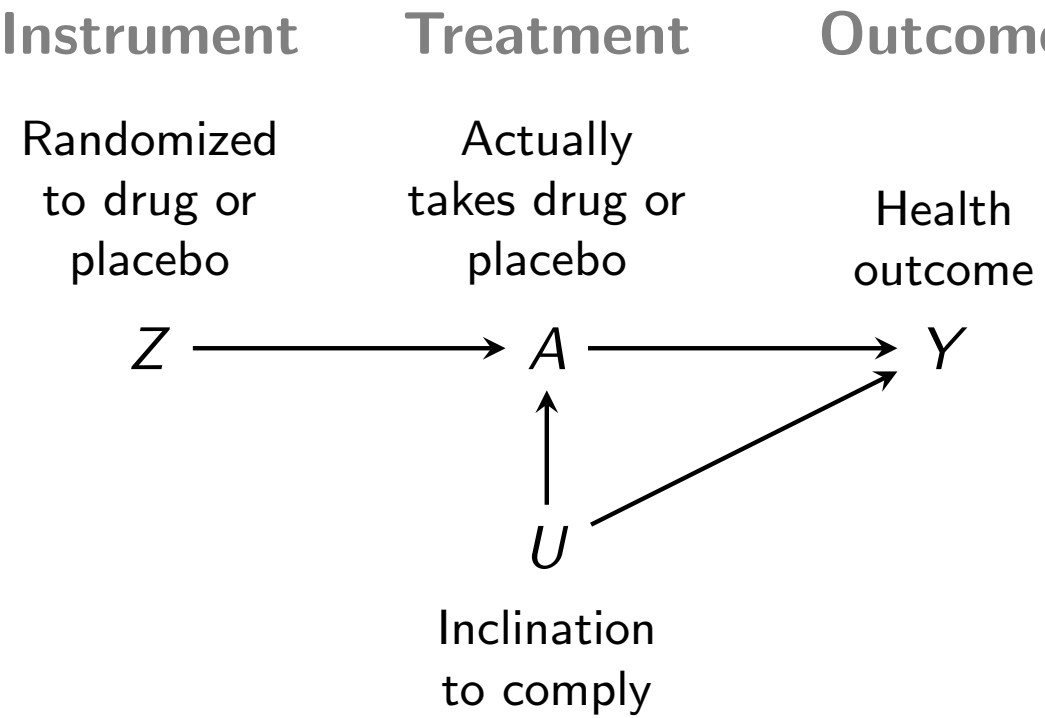
Instrumental variables: 2) Average effect among compliers



Key insight: The effect of Z on Y operates entirely through A

1. Study the effect of $Z \to Y$ (we just did)

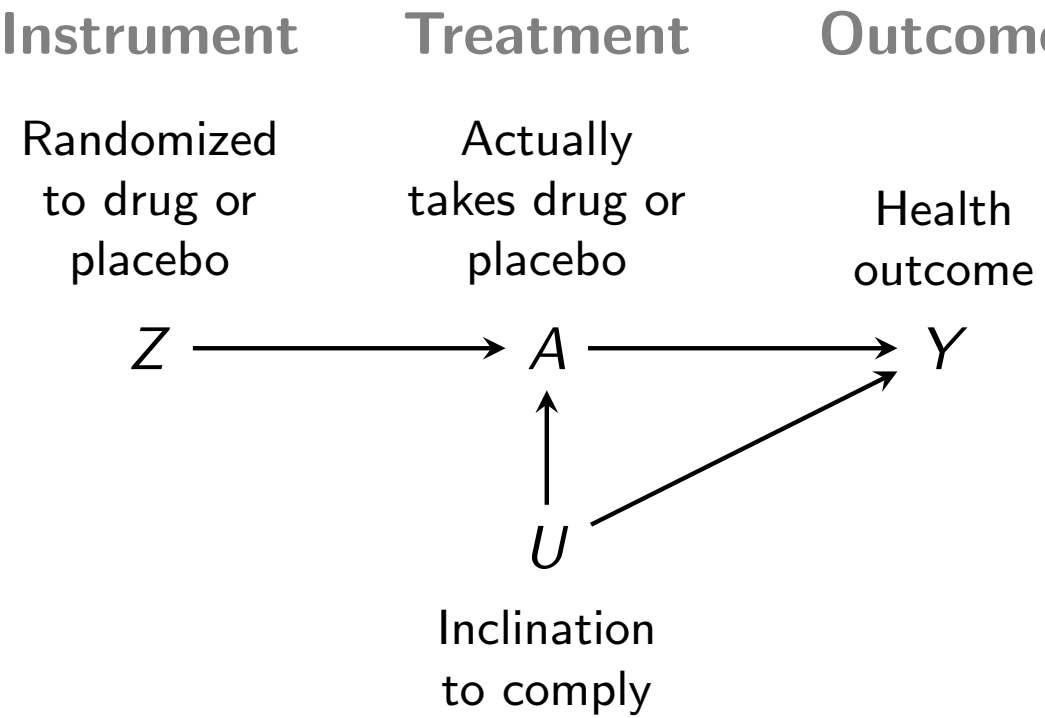
Instrumental variables: 2) Average effect among compliers



Key insight: The effect of Z on Y operates entirely through A

1. Study the effect of $Z \to Y$ (we just did)
2. Study the effect of $Z \to A$

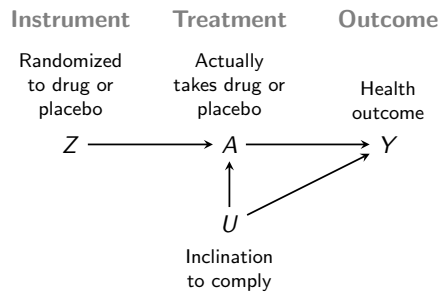
Instrumental variables: 2) Average effect among compliers



Key insight: The effect of Z on Y operates entirely through A

1. Study the effect of $Z \to Y$ (we just did)
2. Study the effect of $Z \to A$
3. Learn about $A \to Y$ since $Z \to Y$ is $Z \to A \to Y$

Instrumental variables: 2) Average effect among compliers



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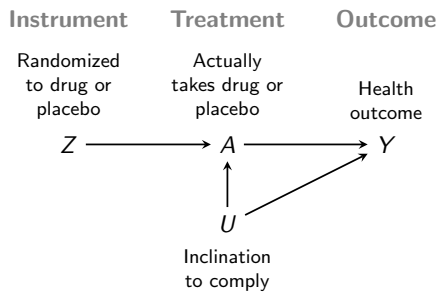
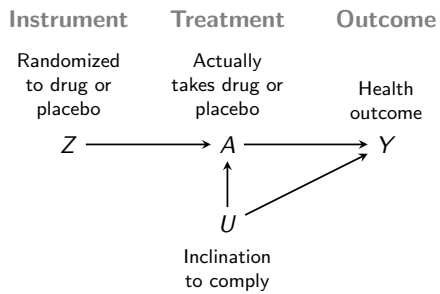


The effect $Z \rightarrow A$ has four **principal strata**:
latent sets of people who respond to Z a particular way

Instrumental variables: 2) Average effect among compliers



The effect $Z \to A$ has four **principal strata**:
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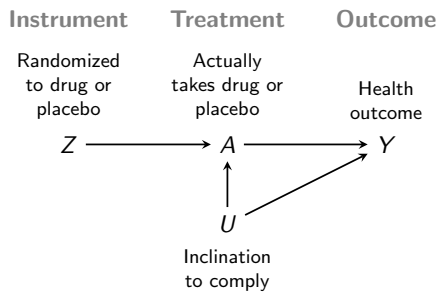
Compliers $\quad A^{Z=0} = 0 \quad A^{Z=1} = 1 \quad$ (follow assignment)

Instrumental variables: 2) Average effect among compliers



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Compliers	$A^{Z=0} = 0$	$A^{Z=1} = 1$	(follow assignment)
Always takers	$A^{Z=0} = 1$	$A^{Z=1} = 1$	(always take treatment)

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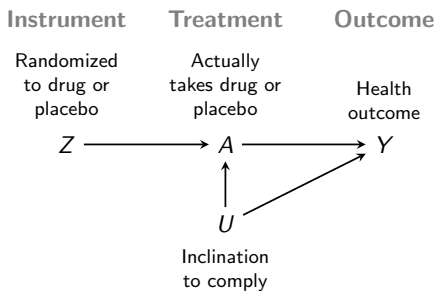

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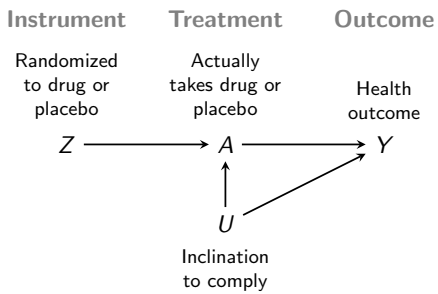

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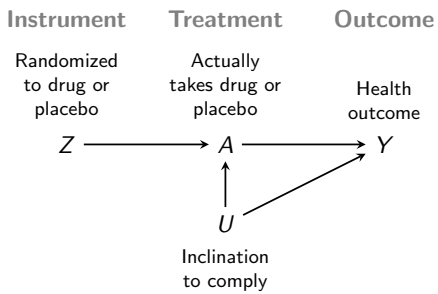
Instrumental variables: 2) Average effect among compliers



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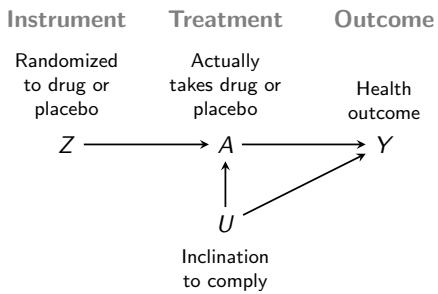
Fundamental problem of causal inference applies to $Z \to A$

Instrumental variables: 2) Average effect among compliers



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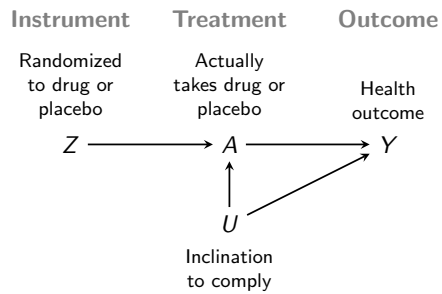
Fundamental problem of causal inference applies to $Z \to A$

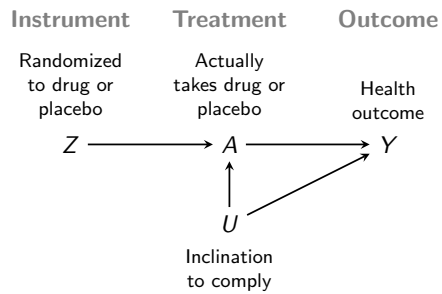
Discuss: In which strata is the effect $Z \to Y$ zero?

Instrumental variables: 2) Average effect among compliers



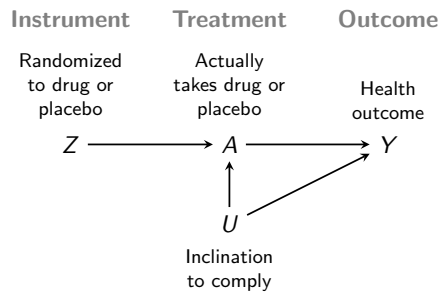
Among **always takers** and **never takers**,

Instrumental variables: 2) Average effect among compliers



Among **always takers** and **never takers**,
Z does not affect A

Instrumental variables: 2) Average effect among compliers



Among **always takers** and **never takers**,
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Z only affects Y through A

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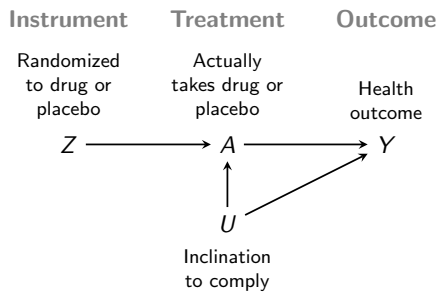

Among **always takers** and **never takers**,
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Z only affects Y through A

In these strata, Z does not affect Y

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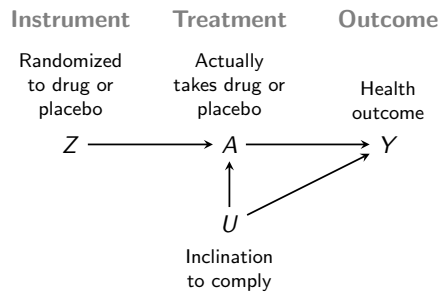
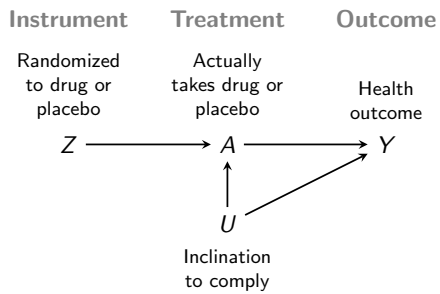


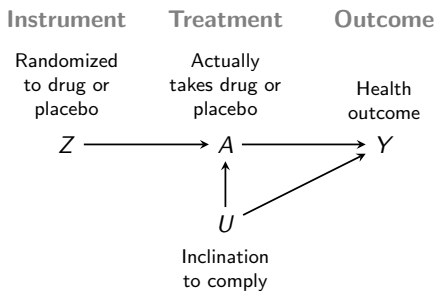
Among **compliers**,

Instrumental variables: 2) Average effect among compliers



Among **compliers**,
$Z = 1$ implies $A = 1$ and
$Z = 0$ implies $A = 0$

Instrumental variables: 2) Average effect among compliers



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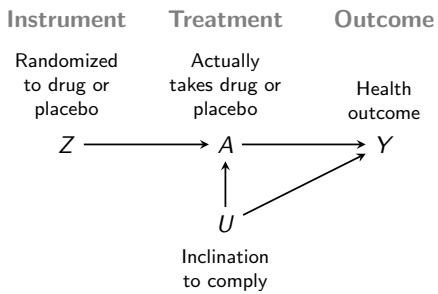

Among **compliers**,
$Z = 1$ implies $A = 1$ and
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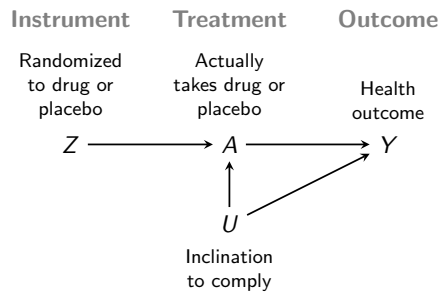
In these strata, $Z = A$

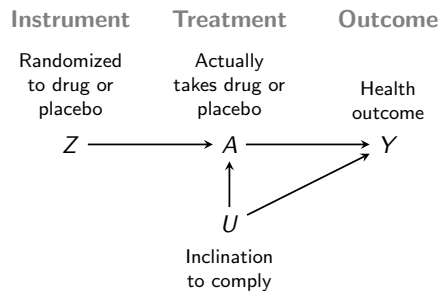
$Z \to Y$ and $A \to Y$ are the same

Instrumental variables: 2) Average effect among compliers



Among **defiers**,

Instrumental variables: 2) Average effect among compliers



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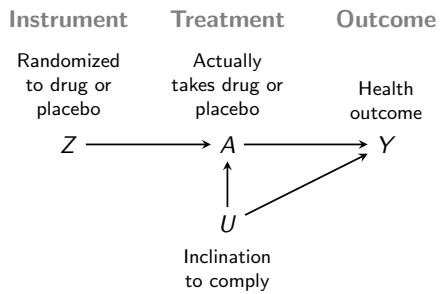

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$Z = 1$ implies $A = 0$ and
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In these strata, $Z = 1 - A$

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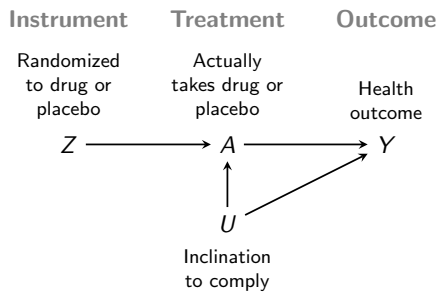

Among **defiers**,
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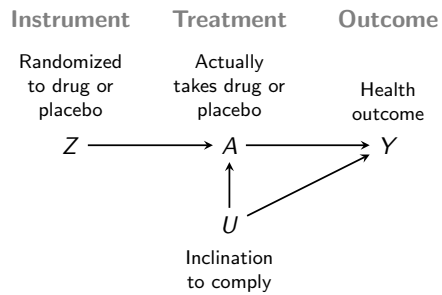
In these strata, $Z = 1 - A$

$Z \to Y$ and $A \to Y$ are the same magnitude
but have opposite signs

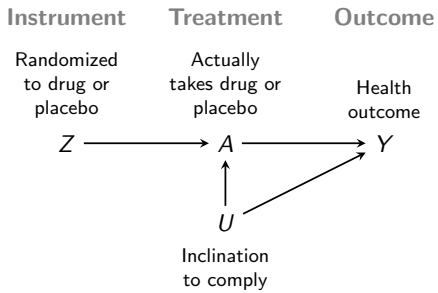
Instrumental variables: 2) Average effect among compliers



Four principal strata

Compliers	$(Z \to A) = +1$	$(Z \to Y) = (A \to Y)$
Always takers	$(Z \to A) = 0$	$(Z \to Y) = 0$
Never takers	$(Z \to A) = 0$	$(Z \to Y) = 0$
Defiers	$(Z \to A) = -1$	$(Z \to Y) = -(A \to Y)$

Instrumental variables: 2) Average effect among compliers



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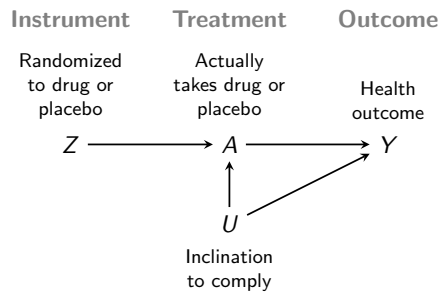
Assume **no defiers** in the population

Instrumental variables: 2) Average effect among compliers



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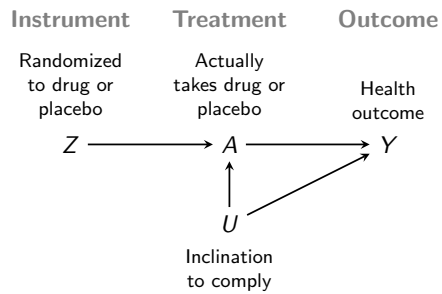

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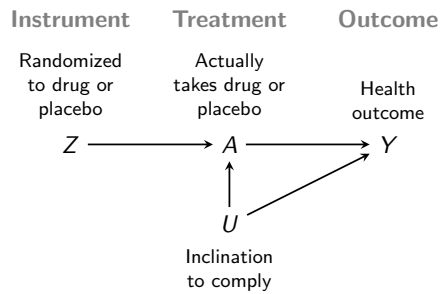
Discuss a hypothetical.

Instrumental variables: 2) Average effect among compliers



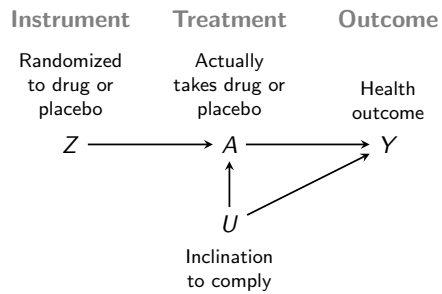
Four principal strata

Compliers	$(Z \to A) = +1$	$(Z \to Y) = (A \to Y)$
Always takers	$(Z \to A) = 0$	$(Z \to Y) = 0$
Never takers	$(Z \to A) = 0$	$(Z \to Y) = 0$

Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers

Instrumental variables: 2) Average effect among compliers



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Average effect of $Z \to Y$ among compliers is 0.6

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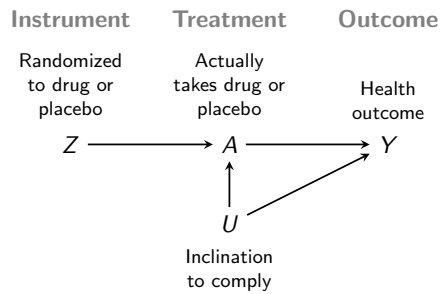

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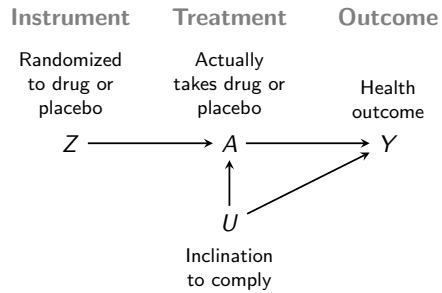

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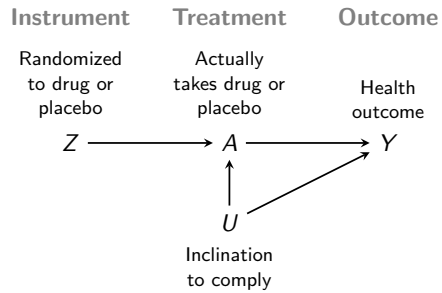

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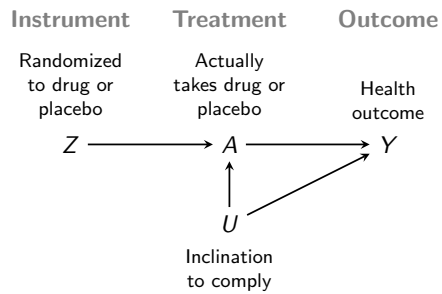
Assume **no defiers** in the population

Instrumental variables: 2) Average effect among compliers



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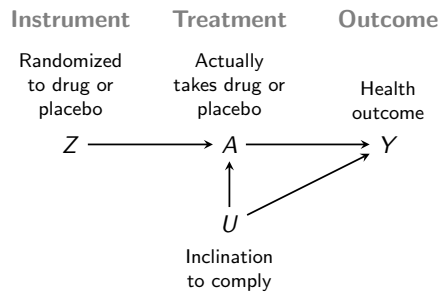

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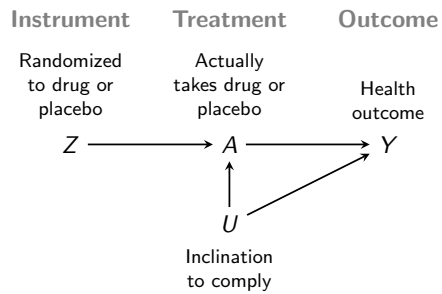

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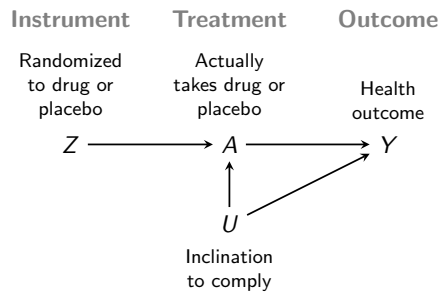

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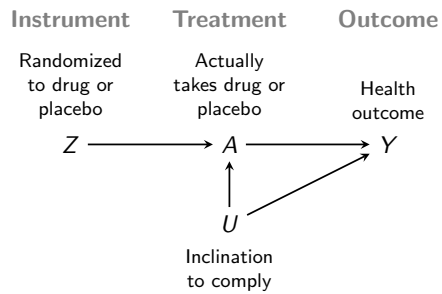

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Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers
Average effect of $Z \to Y$ among compliers is 0.6

What is the average effect of $Z \to Y$ in the population?
Could you calculate the proportion of compliers in the population?

Instrumental variables: 2) Average effect among compliers

Four principal strata

$$\begin{array}{rll} \text{Compliers} & (Z \to A) = +1 & (Z \to Y) = (A \to Y) \\ \text{Always takers} & (Z \to A) = 0 & (Z \to Y) = 0 \\ \text{Never takers} & (Z \to A) = 0 & (Z \to Y) = 0 \end{array} \tag{1}$$

Discuss a hypothetical.

Population is 50% compliers, 25% always takers, 25% never takers

Average effect of $Z \to Y$ among compliers is 0.6

What is the average effect of $Z \to Y$ in the population?

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\end{aligned}
\tag{2}
$$

Assuming no defiers,

$$\pi_{\text{Complier}} = \mathrm{E}(A \mid Z=1) - \mathrm{E}(A \mid Z=0)$$

Instrumental variables: 2) Average effect among compliers

Deriving the general case:

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$$\mathsf{E}(Y \mid Z = 1) - \mathsf{E}(Y \mid Z = 0) = \mathsf{E}(Y^{Z=1} - Y^{Z=0})$$

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$$

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Instrumental variables: 2) Average effect among compliers

$$\mathrm{E}(Y^{Z=1} - Y^{Z=0}) = \mathrm{E}(Y^{Z=1} - Y^{Z=0} \mid S = \text{Complier})\pi_{\text{Complier}}$$

(4)

Instrumental variables: 2) Average effect among compliers

Average effect of instrument = Average among compliers × Proportion who comply

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Among compliers, $(Z \to Y) = (A \to Y)$.

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Rearrange to get the complier average treatment effect

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Rearrange to get the complier average treatment effect

$$\mathrm{E}(Y^{A=1} - Y^{A=0} \mid S = \text{Complier}) = \frac{\mathrm{E}(Y^{Z=1} - Y^{Z=0})}{\pi_{\text{Complier}}} \tag{4}$$

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Average effect of instrument = Average among compliers × Proportion who comply

$$\mathrm{E}(Y^{Z=1} - Y^{Z=0}) = \mathrm{E}(Y^{Z=1} - Y^{Z=0} \mid S = \text{Complier})\pi_{\text{Complier}}$$

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$$\begin{aligned}\mathrm{E}(Y^{A=1} - Y^{A=0} \mid S = \text{Complier}) &= \frac{\mathrm{E}(Y^{Z=1} - Y^{Z=0})}{\pi_{\text{Complier}}} \\ &= \frac{\mathrm{E}(Y \mid Z = 1) - \mathrm{E}(Y \mid Z = 0)}{\mathrm{E}(A \mid Z = 1) - \mathrm{E}(A \mid Z = 0)} \quad (4)\end{aligned}$$

Learning goals for today

At the end of class, you will be able to:

1. Understand the logic of instrumental variables
2. Derive the average effect among compliers in experiments with noncompliance