

MA 323 Geometric Modelling

Course Notes: Day 09

Quintic Hermite Interpolation

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We want to explore some properties of higher degree polynomials, today. In general, higher degree polynomials allow more flexibility as more data is required to specify a higher degree polynomial. Higher degree polynomials also allow more complicated basic curves as curve elements. Quadratic polynomials are basically only parabolas (with the degenerate case of straight lines). Cubic polynomials allow a greater variety of curves, and allows self-intersection of the curves. Increasing the degree past cubic enlarges the basic shapes further. Higher order polynomials also allow easier construction of piecewise curves that have higher order degree smoothness. In particular, today, we want to extend the ideas involved in cubic Hermite interpolation to quintic Hermite interpolation.

9.1 A Review of Cubic Hermite Interpolation

To construct a cubic curve by Hermite interpolation, we provide two points that the curve must pass through and then the tangent vectors at these two points (the value of the first derivative (velocity) at these points). We note that this a symmetric way of providing data, each point is treated in exactly the same manner.

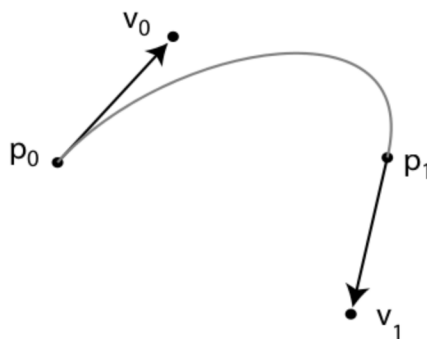


Figure 1: Cubic Hermite Curve Data and Cubic Hermite Curve

To create the cubic curve possessing this data, we look for a curve $\mathbf{c}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3$ that satisfies

$$\begin{array}{lcl} \mathbf{c}(0) = \mathbf{p}_0 & \text{and} & \mathbf{c}(1) = \mathbf{p}_1 \\ \mathbf{c}'(0) = \mathbf{v}_0 & & \mathbf{c}'(1) = \mathbf{v}_1 \end{array}$$

This means solving the system of equations

$$\begin{aligned}\mathbf{p}_0 &= \mathbf{a}_0 \\ \mathbf{v}_0 &= \mathbf{a}_1 \\ \mathbf{p}_1 &= \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\ \mathbf{v}_1 &= \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3\end{aligned}$$

which gives

$$\begin{aligned}\mathbf{a}_0 &= \mathbf{p}_0 \\ \mathbf{a}_1 &= \mathbf{v}_0 \\ \mathbf{a}_2 &= 3\mathbf{p}_1 - 3\mathbf{p}_0 - 2\mathbf{v}_0) - \mathbf{v}_1\mathbf{a}_3 = -2\mathbf{p}_1 + 2\mathbf{p}_0 + \mathbf{v}_0 + \mathbf{v}_1\end{aligned}$$

Rearranging terms, this yields the curve as

$$\mathbf{c}(t) = (1 - 3t^2 + 2t^3) \mathbf{p}_0 + (t - 2t^2 + t^3) \mathbf{v}_0 + (-t^2 + t^3) \mathbf{v}_1 + (3t^2 - 2t^3) \mathbf{p}_1$$

which may be written in the basis function form

$$\mathbf{c}(t) = H_0^3(t) \mathbf{p}_0 + H_1^3(t) \mathbf{v}_0 + H_2^3(t) \mathbf{v}_1 + H_3^3(t) \mathbf{p}_1$$

where

$$\begin{aligned}H_0^3(t) &= 1 - 3t^2 + 2t^3 \\ H_1^3(t) &= t - 2t^2 + t^3 \\ H_2^3(t) &= -t^2 + t^3 \\ H_3^3(t) &= 3t^2 - 2t^3\end{aligned}$$

The basis functions maybe obtained without using algebra to solve for the coefficients and then rearranging terms, but by specifying the properties desired of the basis functions.

Starting with the basis function representation of the curve

$$\mathbf{c}(t) = H_0^3(t) \mathbf{p}_0 + H_1^3(t) \mathbf{v}_0 + H_2^3(t) \mathbf{v}_1 + H_3^3(t) \mathbf{p}_1,$$

we list out the properties desired, that is

$$\begin{aligned}\mathbf{c}(0) &= \mathbf{p}_0 = H_0^3(0) \mathbf{p}_0 + H_1^3(0) \mathbf{v}_0 + H_2^3(0) \mathbf{v}_1 + H_3^3(0) \mathbf{p}_1 \\ \mathbf{c}'(0) &= \mathbf{v}_0 = (H_0^3)'(0) \mathbf{p}_0 + (H_1^3)'(0) \mathbf{v}_0 + (H_2^3)'(0) \mathbf{v}_1 + (H_3^3)'(0) \mathbf{p}_1 \\ \mathbf{c}(1) &= \mathbf{p}_1 = H_0^3(1) \mathbf{p}_0 + H_1^3(1) \mathbf{v}_0 + H_2^3(1) \mathbf{v}_1 + H_3^3(1) \mathbf{p}_1 \\ \mathbf{c}'(1) &= \mathbf{v}_1 = (H_0^3)'(1) \mathbf{p}_0 + (H_1^3)'(1) \mathbf{v}_0 + (H_2^3)'(1) \mathbf{v}_1 + (H_3^3)'(1) \mathbf{p}_1\end{aligned}$$

The independence of the data (that there is no relations between the data and we must obtain the same functions independent of the data provided) means that we must have in each of the equations above one value for a basis function 1 and the others 0. Thus, we need to find functions that satisfy

$$\begin{array}{cccc} H_0^3(0) = 1 & H_1^3(0) = 0 & H_2^3(0) = 0 & H_3^3(0) = 0 \\ (H_0^3)'(0) = 0 & (H_1^3)'(0) = 1 & (H_2^3)'(0) = 0 & (H_3^3)'(0) = 0 \\ H_0^3(1) = 0 & H_1^3(1) = 0 & H_2^3(1) = 0 & H_3^3(1) = 1 \\ (H_0^3)'(1) = 0 & (H_1^3)'(1) = 0 & (H_2^3)'(1) = 1 & (H_3^3)'(1) = 0 \end{array}, \text{ and }$$

Such functions are given by

$$H_0^3(t) = 1 - 3t^2 + 2t^3$$

$$H_1^3(t) = t - 2t^2 + t^3$$

$$H_2^3(t) = -t^2 + t^3$$

$$H_3^3(t) = 3t^2 - 2t^3$$

This is just an alternate method for finding the basis function, without recourse to solving a particular system of equations, but rather specifying what we desire to be the representation of the model and then determining the representation. In linear algebra, this is similar to the process of constructing a change of basis. This point of view is sometimes useful in geometric modelling, especially when converting between modelling methods.

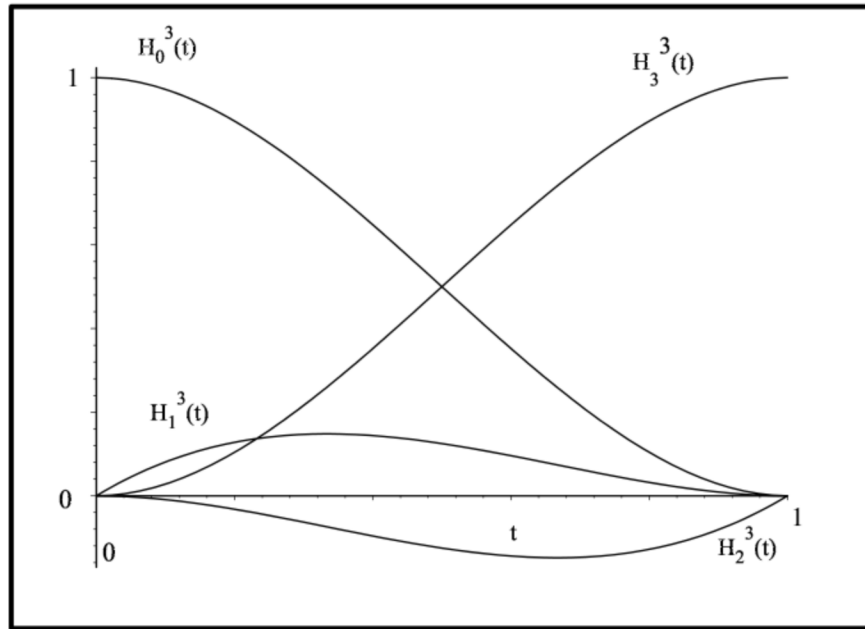


Figure 2: Cubic Hermite Basis Functions $H_i^3(t)$

9.2 Quintic Hermite Interpolation

As a direct generalization of cubic Hermite interpolation, we will consider a quintic Hermite interpolation. The quintic Hermite interpolation problem is to find a curve $\mathbf{c}(t)$ that has $\mathbf{c}(0) = \mathbf{p}_0$, $\mathbf{c}'(0) = \mathbf{v}_0$, $\mathbf{c}''(0) = \mathbf{a}_0$, $\mathbf{c}(1) = \mathbf{p}_1$, $\mathbf{c}'(1) = \mathbf{v}_1$ and $\mathbf{c}''(1) = \mathbf{a}_1$. The simplest solution involves a quintic (5th order) polynomial curve.

Solving this interpolation problem is accomplished in the same manner as for the cubic Hermite interpolation problem. We seek general quintic

$$\mathbf{c}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_5 t^5$$

and write the equations for the $\mathbf{c}(0) = \mathbf{p}_0$, $\mathbf{c}'(0) = \mathbf{v}_0$, $\mathbf{c}''(0) = \mathbf{a}_0$, $\mathbf{c}(1) = \mathbf{p}_1$, $\mathbf{c}'(1) = \mathbf{v}_1$

and $\mathbf{c}''(1) = \mathbf{a}_1$:

$$\begin{aligned}\mathbf{p}_0 &= \mathbf{b}_0 \\ \mathbf{v}_0 &= \mathbf{b}_1 \\ \mathbf{a}_0 &= \mathbf{b}_2 \\ \mathbf{p}_1 &= \mathbf{b}_0 + \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \mathbf{b}_4 + \mathbf{b}_5 \\ \mathbf{v}_1 &= \mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3 + 4\mathbf{b}_4 + 5\mathbf{b}_5 \\ \mathbf{a}_1 &= 2\mathbf{b}_2 + 6\mathbf{b}_3 + 12\mathbf{b}_4 + 20\mathbf{b}_5\end{aligned}$$

An alternate method is to use the basis function representation,

$$\mathbf{c}(t) = H_0^5(t) \mathbf{p}_0 + H_1^5(t) \mathbf{v}_0 + H_2^5(t) \mathbf{a}_0 + H_3^5(t) \mathbf{a}_1 + H_4^5(t) \mathbf{v}_1 + H_5^5(t) \mathbf{p}_1$$

and determine the basis functions $H_i^5(t)$. Symmetry considerations, that is the same curve geometrically should arise with the data \mathbf{p}_1 , $-\mathbf{v}_1$, \mathbf{a}_1 , \mathbf{p}_0 , $-\mathbf{v}_0$ and \mathbf{a}_0 , imply that $H_0(t) = H_5(1-t)$, $H_1(t) = -H_4(1-t)$, $H_2(t) = H_3(1-t)$. Therefore we need to find quintic polynomials satisfying

$$\begin{array}{lll} H_0^5(0) = 1 & H_1^5(0) = 0 & H_2^5(0) = 0 \\ (H_0^5)'(0) = 0 & (H_1^5)'(0) = 1 & (H_2^5)'(0) = 0 \\ (H_0^5)''(0) = 0 & (H_1^5)''(0) = 0 & (H_2^5)''(0) = 1 \\ H_0^5(1) = 0 & H_1^5(1) = 0 & H_2^5(1) = 0 \\ (H_0^5)'(1) = 0 & (H_1^5)'(1) = 0 & (H_2^5)'(1) = 0 \\ (H_0^5)''(1) = 0 & (H_1^5)''(1) = 0 & (H_2^5)''(1) = 0 \end{array}$$

Either of these two methods yields the basis function

$$\begin{aligned}H_0^5(t) &= 1 - 10t^3 + 15t^4 - 6t^5 \\ H_1^5(t) &= t - 6t^3 + 8t^4 - 3t^5 \\ H_2^5(t) &= \frac{1}{2}t^2 - \frac{3}{2}t^3 + \frac{3}{2}t^4 - \frac{1}{2}t^5 \\ H_3^5(t) &= \frac{1}{2}t^3 - t^4 + \frac{1}{2}t^5 \\ H_4^5(t) &= -4t^3 + 7t^4 - 3t^5 \\ H_5^5(t) &= 10t^3 - 15t^4 + 6t^5\end{aligned}$$

which have graphs on the following page

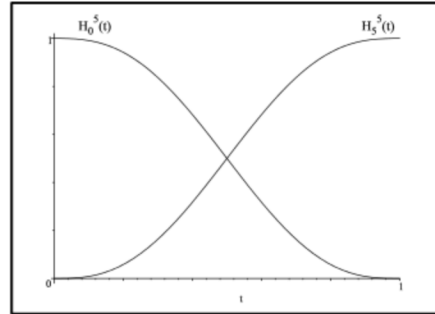


Figure 3: Quintic Hermite Basis Functions $H_0^5(t)$ and $H_5^5(t)$

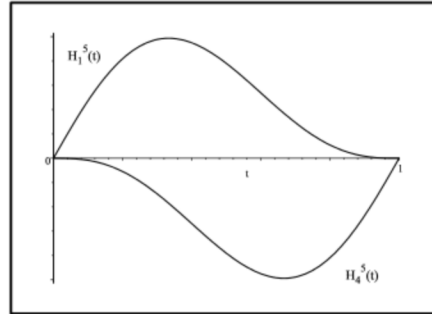


Figure 4: Quintic Hermite Basis Functions $H_1^5(t)$ and $H_4^5(t)$

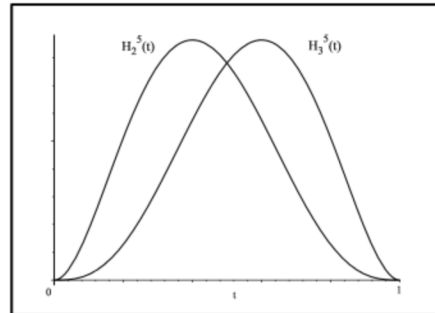


Figure 5: Quintic Hermite Basis Functions $H_2^5(t)$ and $H_3^5(t)$

Note that these basis function possess the same type of symmetries as the cubic Hermite basis functions $H_0^3(t)$, $H_1^3(t)$, $H_2^3(t)$, $H_3^3(t)$.

We note that creating a quintic Hermite curve does not admit the elegant geometrical solution that the cubic Hermite curve did. The reason for this is simple there is more information to supply at each point. We can still view the geometric construction as supplying six points $\{q_i\}$ with $i = 0, 1, 2, \dots, 5$. The first point q_0 defines the first point p_0 to be interpolated, the second point q_1 is used to define the tangent vector v_0 at p_0 as $v_0 = q_1 - q_0$, and the third point q_2 is used to define the acceleration vector a_0 as $a_0 = q_2 - q_0$. The next set of three points defines the corresponding information at the second point to be interpolated. The point q_3 defines the second point p_1 to be interpolated, the point q_4 is used to define the tangent vector v_1 at p_1 as $v_1 = q_4 - q_3$, and the last point q_5 is used to define the acceleration vector a_1 at p_1 as $a_1 = q_5 - q_3$. This is not as elegant a geometric solution as you end up supplying two vectors at each point, see diagrams below.

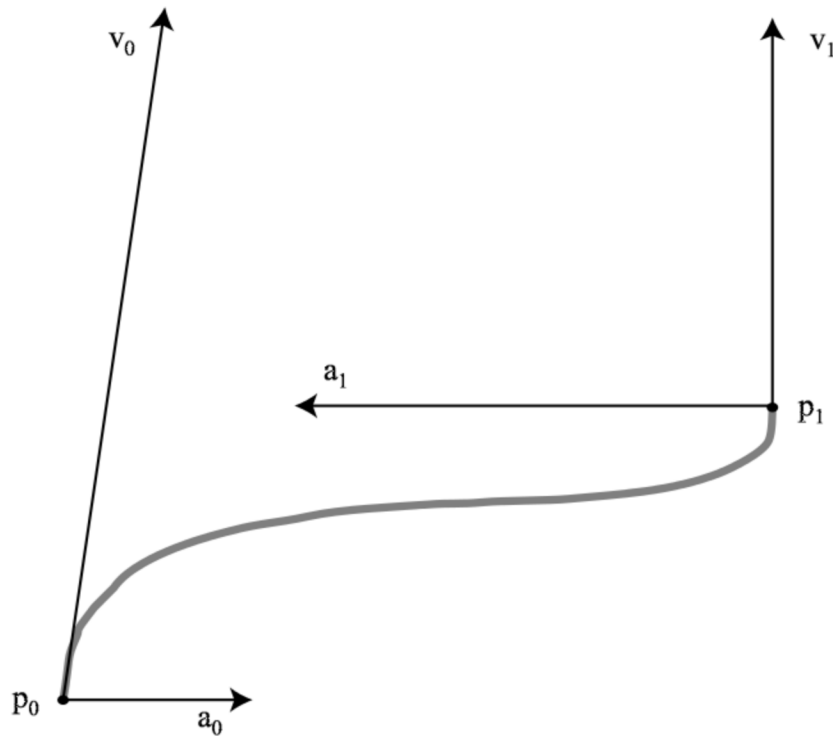


Figure 6:

Quintic Hermite interpolation could arise in motion planning problems where to control the motion one specifies the position, velocity and acceleration at several times. It is worth noting that typically there are other constraints in solving this type of problem. These constraints could restrict the area where the curve may lie, for instance in specifying the path needed for a robot to navigate a room there may be obstacles to avoid (see diagram below). This type of constraint restricts the possible points, velocity vectors and acceleration vectors that can be prescribed. Once, this constraint is satisfied typically one also constraints on the permissible velocity and acceleration, meaning that the magnitude of the velocity and acceleration may have restrictions.

This type of interpolation may also be used in animation where specifies the velocity and acceleration of the object in order to movement of an object. At each control point defining the object, one would describe a velocity and acceleration vector describing the movement of

that control point. The animation is accomplished through defining a sequence of position, velocity and acceleration vectors. For instance, the motion of the parabola below is given by the control points $\mathbf{p}_0(t)$, $\mathbf{p}_1(t)$, $\mathbf{p}_2(t)$ whose paths are specified by quintic Hermite curves.

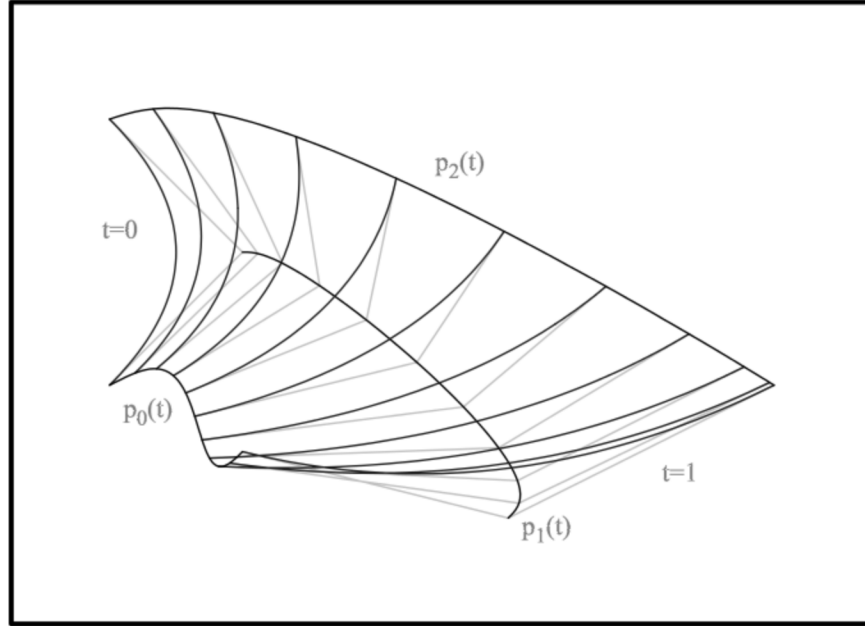


Figure 7: Animation of Parabola $c(s, t) = (1 - s)^2 \mathbf{p}_0(t) + 2s(1 - s) \mathbf{p}_1(t) + s^2 \mathbf{p}_2(t)$

We note like cubic Hermite curves the quintic Hermite curves can be used to construct smooth piecewise defined curves. The quintic construction will ensure the curves are parametrically smooth of order two C^2 , that is twice differentiable. This means that the acceleration vector is continuous. This is an advantage for motion planning and control movement as continuous acceleration is preferable to the highly *jerky* acceleration provided by noncontinuous acceleration. In fact, smooth acceleration is even more preferable, which requires even higher degree polynomials or more sophisticated methods.

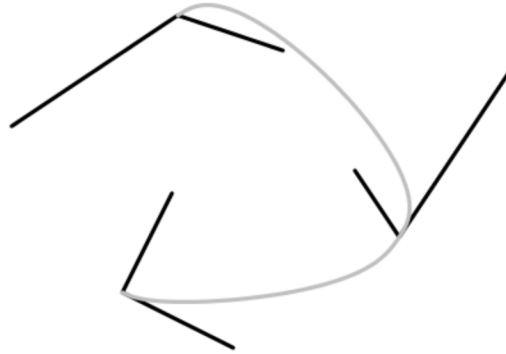


Figure 8: Piecewise Quintic Curve

9.3 Exercises

1. (Computational) Find the Hermite data for the quintic curve that passes through $p_0 = [1, 0]$, $p_1 = [2, 0]$, $p_2 = [2, 2]$, $p_3 = [3, 1]$, $p_4 = [3, 2]$, $p_5 = [4, 1]$ using equal length parameter spacing, that is $t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, \dots , $t_5 = 1.0$.
2. (Interactive) Complete the interactive exercises associated with the Quintic Hermite Polynomial Applet.
3. (Thought) How do the basis functions for cubic Hermite curves and quintic Hermite change if instead of the curve passing through desired points at times 0 and 1, we use arbitrary times $t = t_0$ and $t = t_1$? HINT: Use the chain rule to recast the problem in terms of $\tau = 0$ and $\tau = 1$ using the substitution $\tau = (t - t_0)/(t_1 - t_0)$.
4. (Computational and Thought) Sketch the piecewise quintic Hermite polynomial curve $c(t)$ that has given values for $c(t_i)$, $c'(t_i)$, $c''(t_i)$.

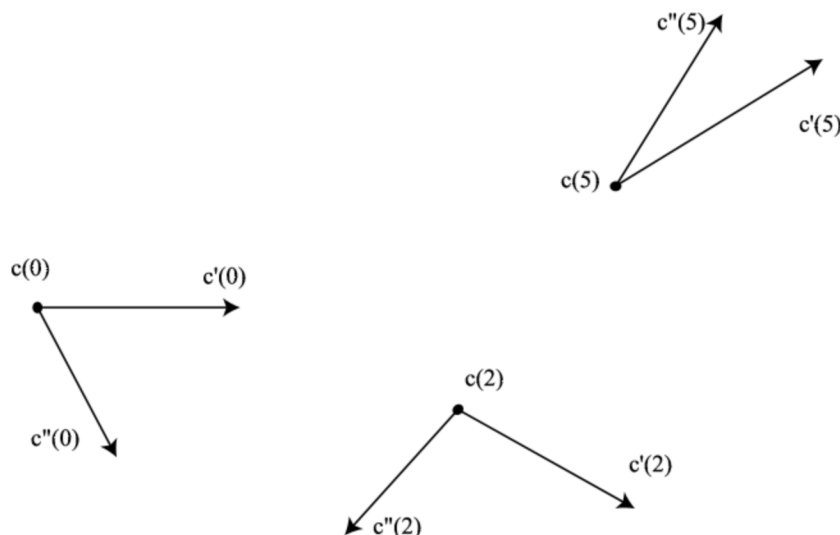


Figure 9: Sketch the piecewise quintic curve with the data

5. (Computational Challenge) Find the acceleration \mathbf{a} that minimizes the distance travelled for a particle following the quintic Hermite polynomial curve $c(t)$ with $c(0) = [1, 0]$, $c'(0) = [1, 1]$, $c''(0) = \mathbf{a}$, $c(1) = [2, 2]$, $c'(1) = [1, -1]$, $c''(1) = [0, 0]$.