LAW OF LARGE NUMBER AND CENTRAL LIMIT THEOREM

“

REFERENCES:

1) *Law of Large Numbers Chapter 8.2 of Grinstead*
2) *Central limit Chapter 9.1, 9.2 of Grinstead*

”

IID

- A sequence or a collection of random variables {Xi} is **independent and identically distributed (i.i.d. or iid or IID)** if
    - Each Xi has the same probability distribution as the others
    - All random variables are mutually independent.
- In general, if a sequence {Xi} satisfies:
  P(Xi | Xj) = P(Xi) (all i,j) — mutually independent
  Each Xi has the same CDF (could be N(0,1), exp(**λ**), Bernoulli(p), etc.)
  we say: The sequence {Xi} is iid with distribution N(0,1), exp(**λ**), Bernoulli(p), or etc.

IID

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    - Each Xi has the same probability distribution as the others
    - All random variables are mutually independent.
- eg:(discrete case)
  Xi is the ith coin toss and {Xi} for i = 1,2,3,..,n,… is a collection of r.v. mutually independent. Each of them follows a Bernoulli distribution with probability 1/2
- We say {Xi} is iid with distribution Bernoulli(1/2)

IID

- A sequence or a collection of random variables {Xi} is **independent and identically distributed (i.i.d. or iid or IID**) if
  - Each Xi has the same probability distribution as the others
  - All random variables are mutually independent.
- eg:(continuous case)
  Suppose we choose at random X1,…,Xn — n numbers from [0,1] with uniform distribution U[0,1]. {Xi} is a collection of r.v. mutually independent and each of them follows U[0,1]
- We say {Xi} is IID with distribution U[0,1]

LAW OF LARGE NUMBERS

Theorem 1. *Let $X_1, \cdots, X_n$ IID random variables with $E[X_i] = \mu$ and* $\mathrm{var}(X_i)$ *for all i. Then we have*

$$P\left(\left|\frac{X_1 + \cdots X_n}{n} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2}$$

In particular the right hand side goes to 0 has $n \to \infty$.

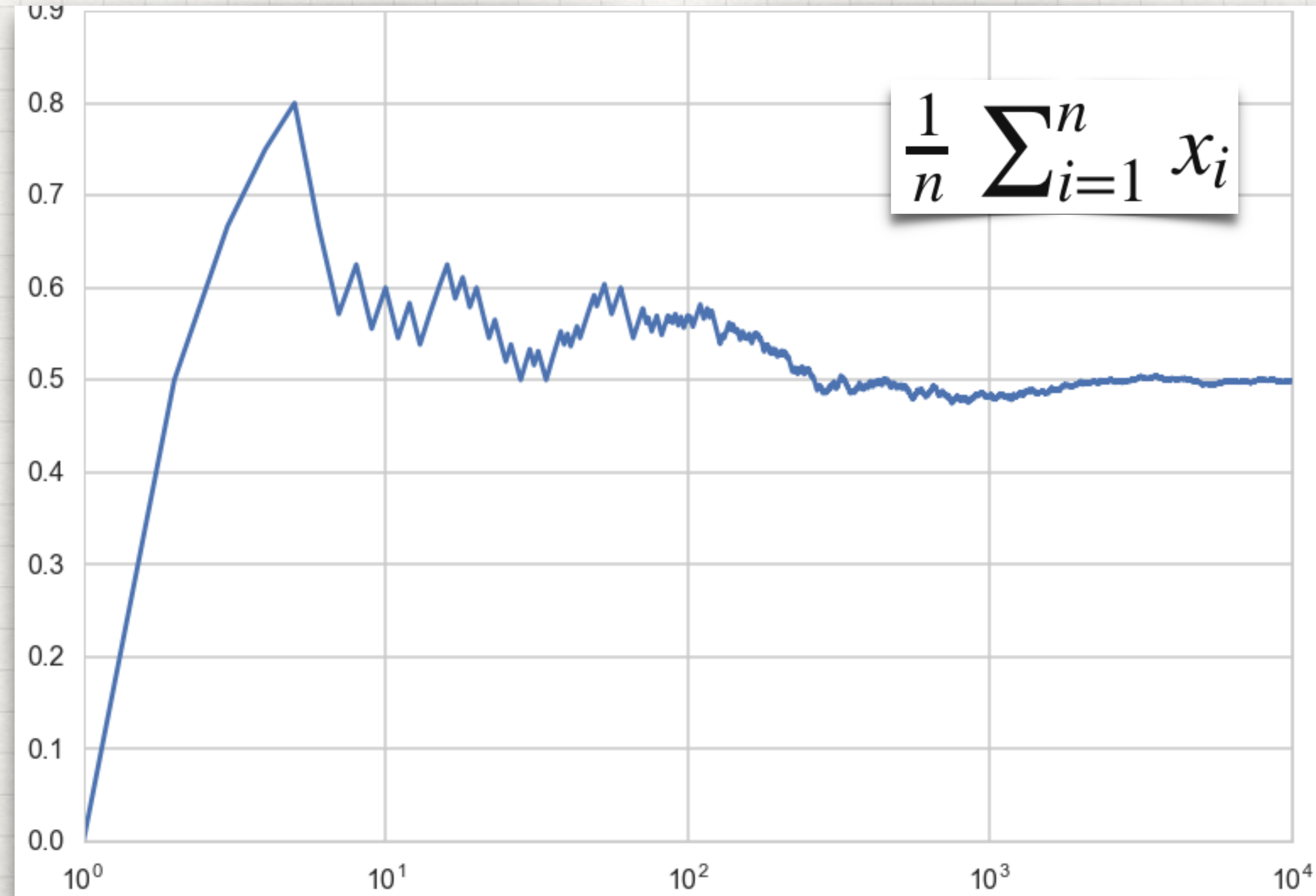
- Idea:
  When doing experiment large number of times (n goes to infinity), the average is very close to the expected value μ (that is E(Xi)) with very high probability (goes to 1)

FLIPPING COIN EXAMPLE – DISCRETE CASE

- Imagine a sequence of length n of coin flips. X1,.., Xn
- The ith coin flip Xi follows Bernoulli(0.5)
- The ith coin flip does not depend on other coin flips
- So {Xi} is IID with Bernoulli(0.5)
- Lets keep on flipping the coin
- Compute a running average of the coin-flip random variables

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

We plot this running mean
It converges to the mean of the distribution from which the random variables are plucked, i.e. the Bernoulli distribution with p=0.5.



CONTINUOUS CASE

- Suppose we choose at random X1,…,Xn — n numbers from [0,1] with uniform distribution U[0,1].
- Each draw does not depend on other draws and each draw follows U[0,1]
- So {Xi} is IID with U[0,1]
- What is the behavior for the average?(example 8.5 in Grinstead)

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

CENTRAL LIMIT THEOREM – STATISTICAL AND MATHEMATICAL

CENTRAL LIMIT THEOREM

Central Limit Theorem: Let $X_1, X_2, ..., X_n$ be a random sample from a population with mean μ and standard deviation σ. Let $\bar{X}$ be the sample average of $X_1, X_2, ..., X_n$. Then the distribution of $\bar{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$.

- Mathematically, {Xi} is a sequence of IID with CDF F(x), mean E(Xi) = μ, and standard deviation std(Xi) = **σ**

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

- The average follows a normal distribution of mean μ and standard deviation **σ**/sqrt(n)

WHY CENTRAL LIMIT THEOREM?

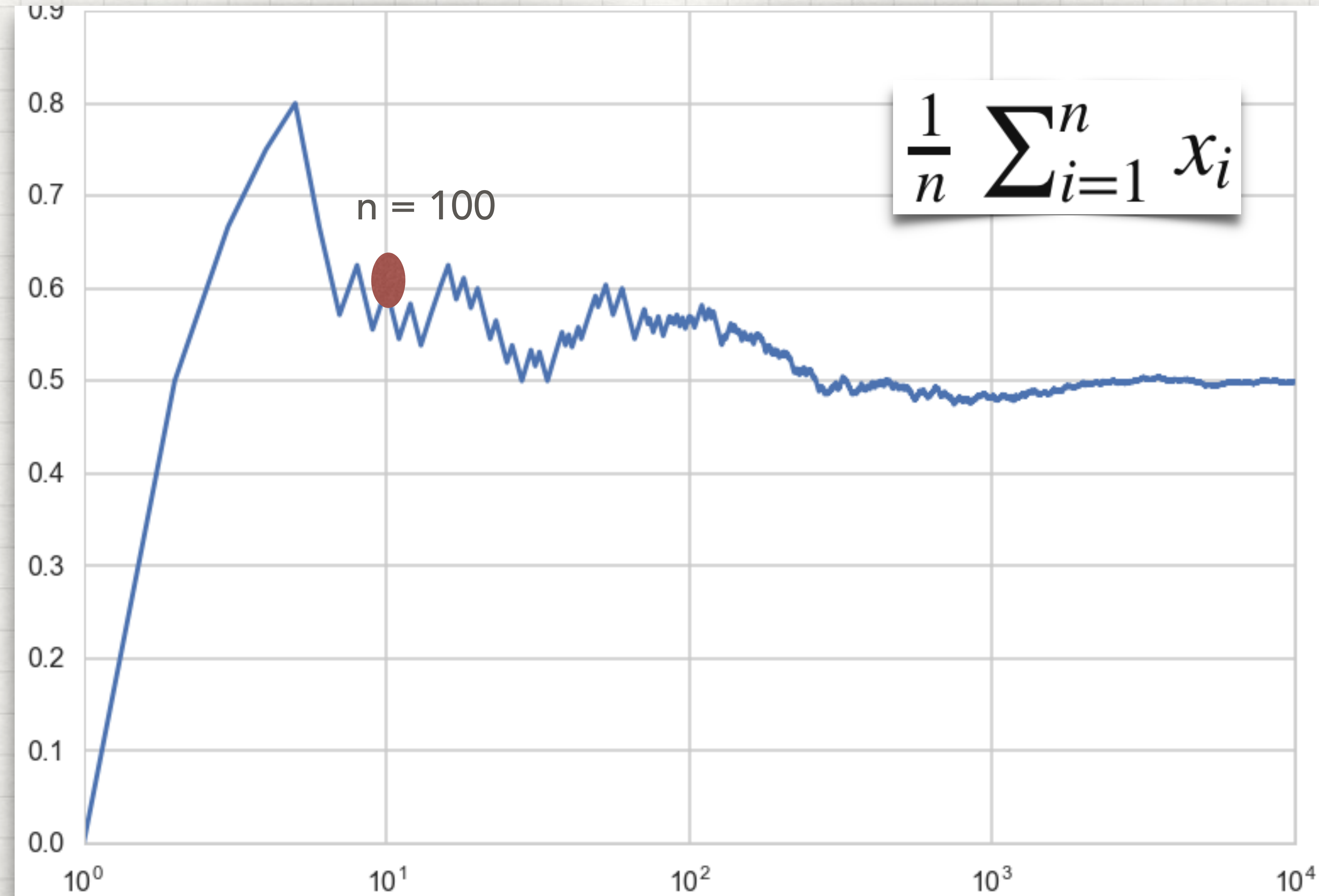
- Chebyshev inequality gives a very rough estimation
- Law large number — describes one experiment runs a large number of times and the average result turn out to be the "expected" result E(X)
- **Central limit Theorem gives a more detailed estimation using normal distribution**
- Central limit Theorem — distribution of N samples — each sample consists of an average n independent experiment
- Law of Large Number — one limit
  Central Limit Theorem — an estimate of distribution of averages

EXAMPLE OF COMPUTER SIMULATION…

- How many heads come up in 100 coin tosses?
- We can flip it 100 times and see what we get.

if we only flip one coin many times, keeps on calculating portion of heads we get.

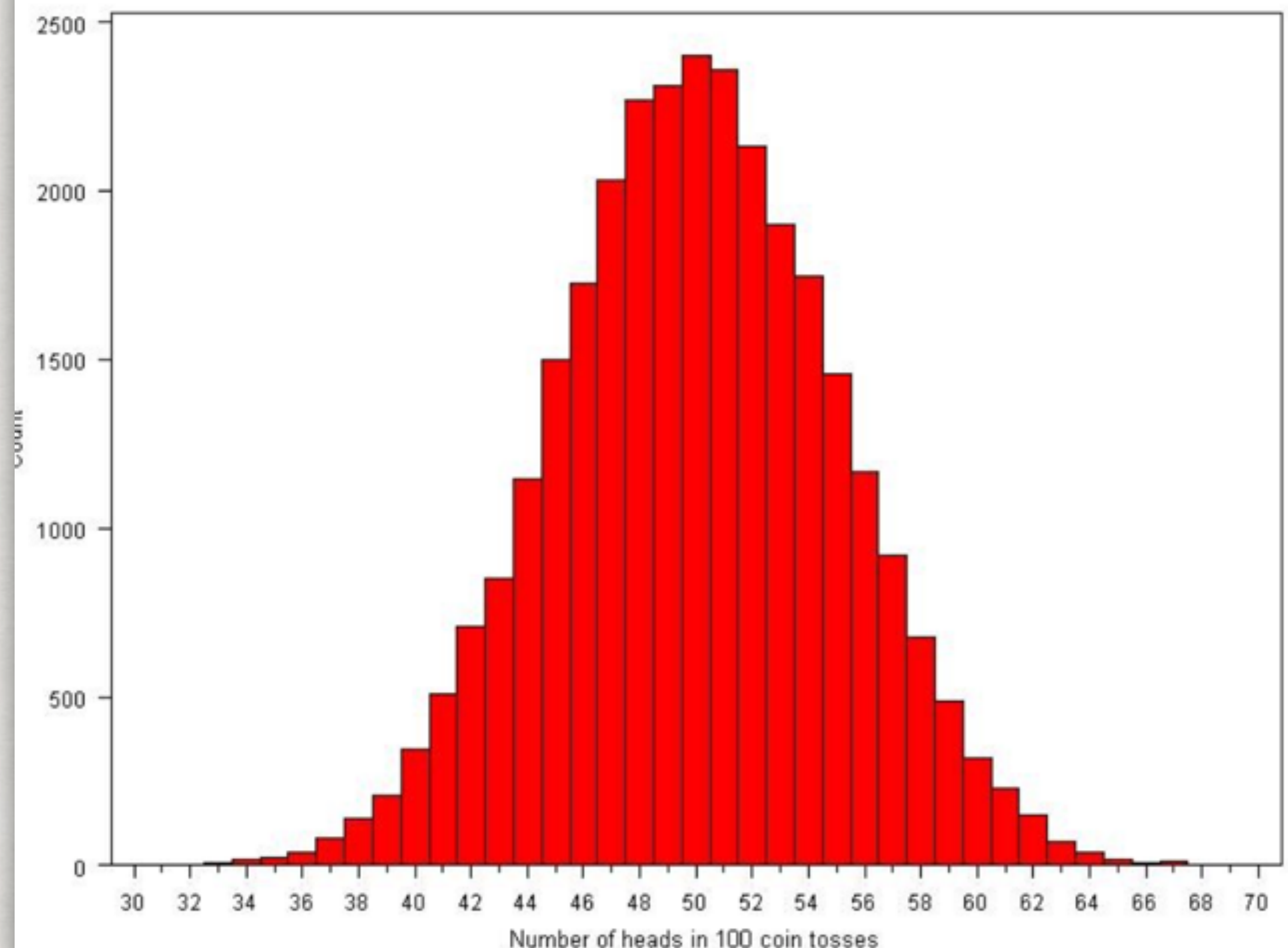
We will obtain a curve converges to a flat curve around E(Xi)



EXAMPLE OF COMPUTER SIMULATION…

- How many heads come up in 100 coin tosses?
- Flip coins virtually
  - Flip a coin 100 times; count the number of heads.
  - Repeat this over and over again a large number of times (how about 30,000 repeats?)
- Plot the 30,000 results.
- Note: we treat flip 100 coin tosses as ONE sample

HISTOGRAM RESULT

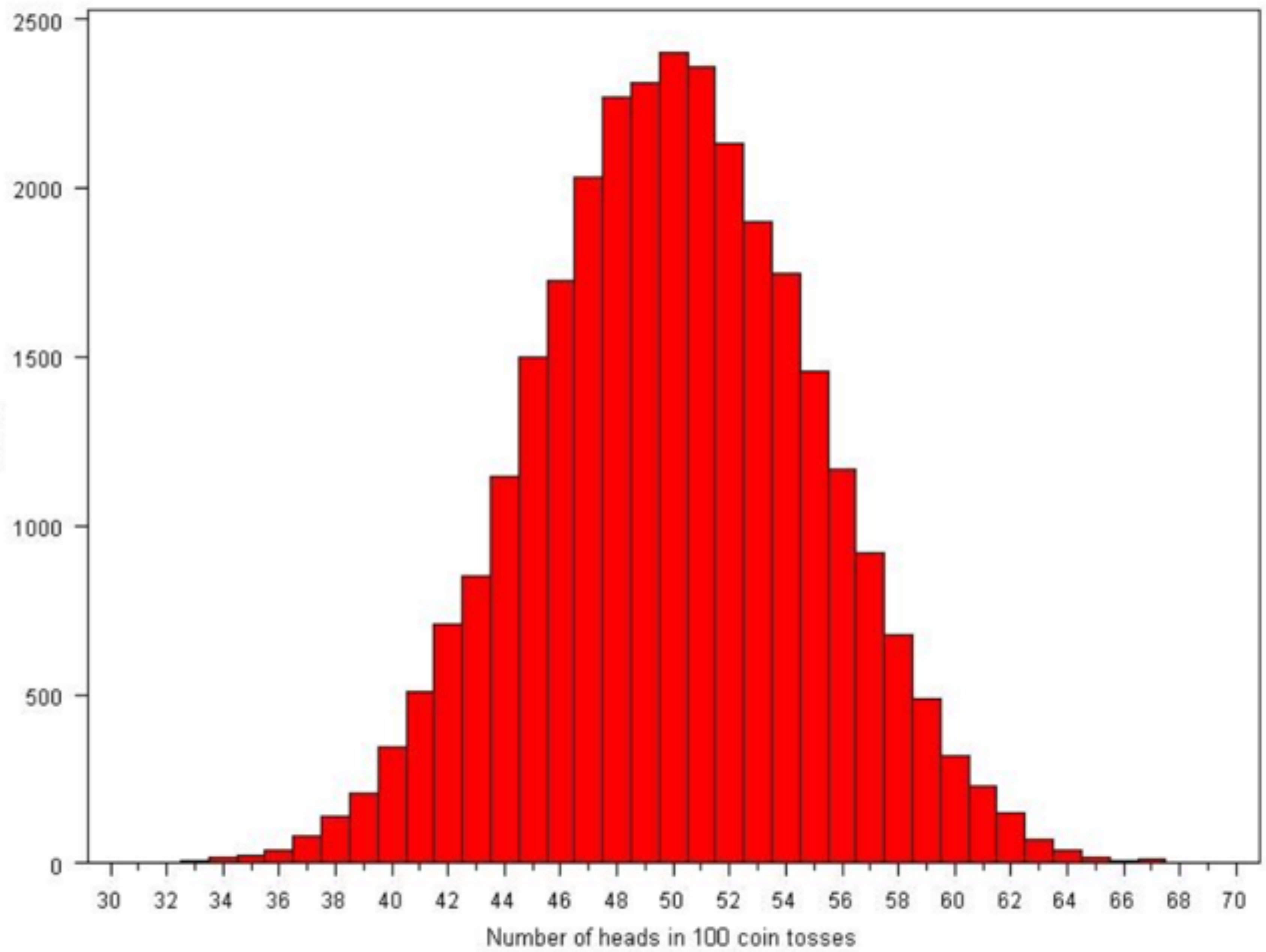


Conclusions:

We usually get between 40 and 60 heads when we flip a coin 100 times.

It's extremely unlikely that we will get 30 heads or 70 heads (didn't happen in 30,000 experiments!).

WHAT DOES THE EXPERIMENTAL RESULT SUGGEST?



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EXPERIMENTAL DESCRIPTION: CENTRAL LIMIT THEOREM!

- If all possible random samples, each of size n, are taken from any population with a mean μ and a standard deviation **σ**, the sampling distribution of the sample means (averages) will:

1. have mean: $\boxed{\mu_{\bar{x}} = \mu}$

2. have standard deviation: $\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}}$

3. be approximately normally distributed regardless of the shape of the parent population (normality improves with larger *n*). It all comes back to Z!

$\mu_{\overline{x}}$ **The mean of the sample means.**

$\sigma_{\overline{x}}$ **The standard deviation of the sample means. Also called "the standard error of the mean."**

CENTRAL LIMIT THEOREM

Central Limit Theorem: Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with mean μ and standard deviation σ. Let $\bar{X}$ be the sample average of $X_1, X_2, \ldots, X_n$. Then the distribution of $\bar{X}$ is approximately normal with mean μ and standard deviation $\sigma/\sqrt{n}$.

- Mathematically, {Xi} is a sequence of IID with CDF F(x), mean E(Xi) = μ, and standard deviation std(Xi) = **σ**

$$\frac{1}{n} \sum_{i=1}^{n} x_i$$

- The average follows a normal distribution of mean μ and standard deviation **σ**/sqrt(n)

BACK TO COIN TOSSES

Flip coins: I want to do 3 experiments:

A:

1) Flip a coin 10 times; count the number of heads; and dive the number of heads by 10

2) Repeat (1) 200 times — gives 200 samples

B:

1) Flip a coin 100 times; count the number of heads; and dive the number of heads by 100

2) Repeat (1) 200 times — gives 200 samples

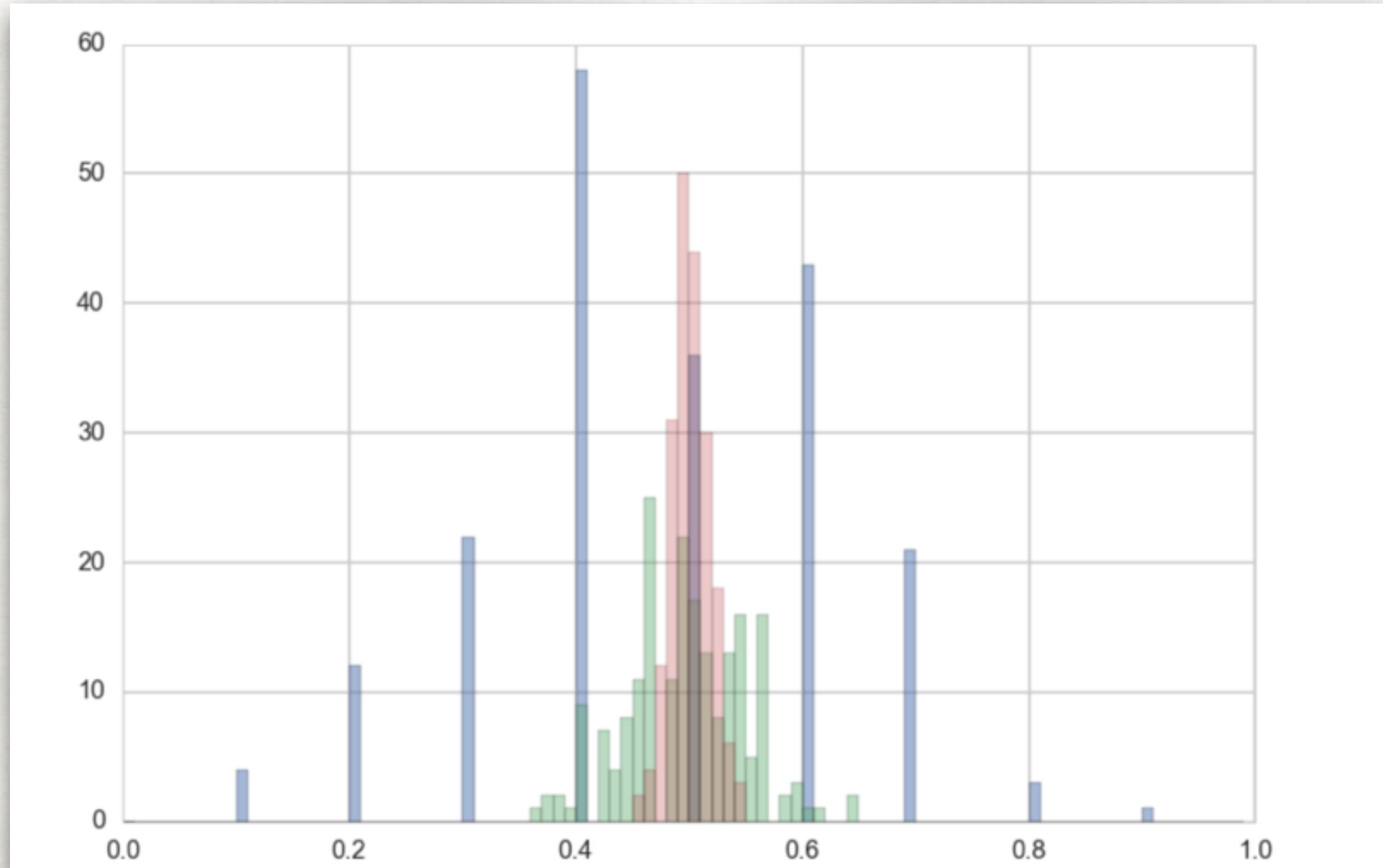
C:

1) Flip a coin 1000 times; count the number of heads; and dive the number of heads by 1000

2) Repeat (1) 200 times — gives 200 samples

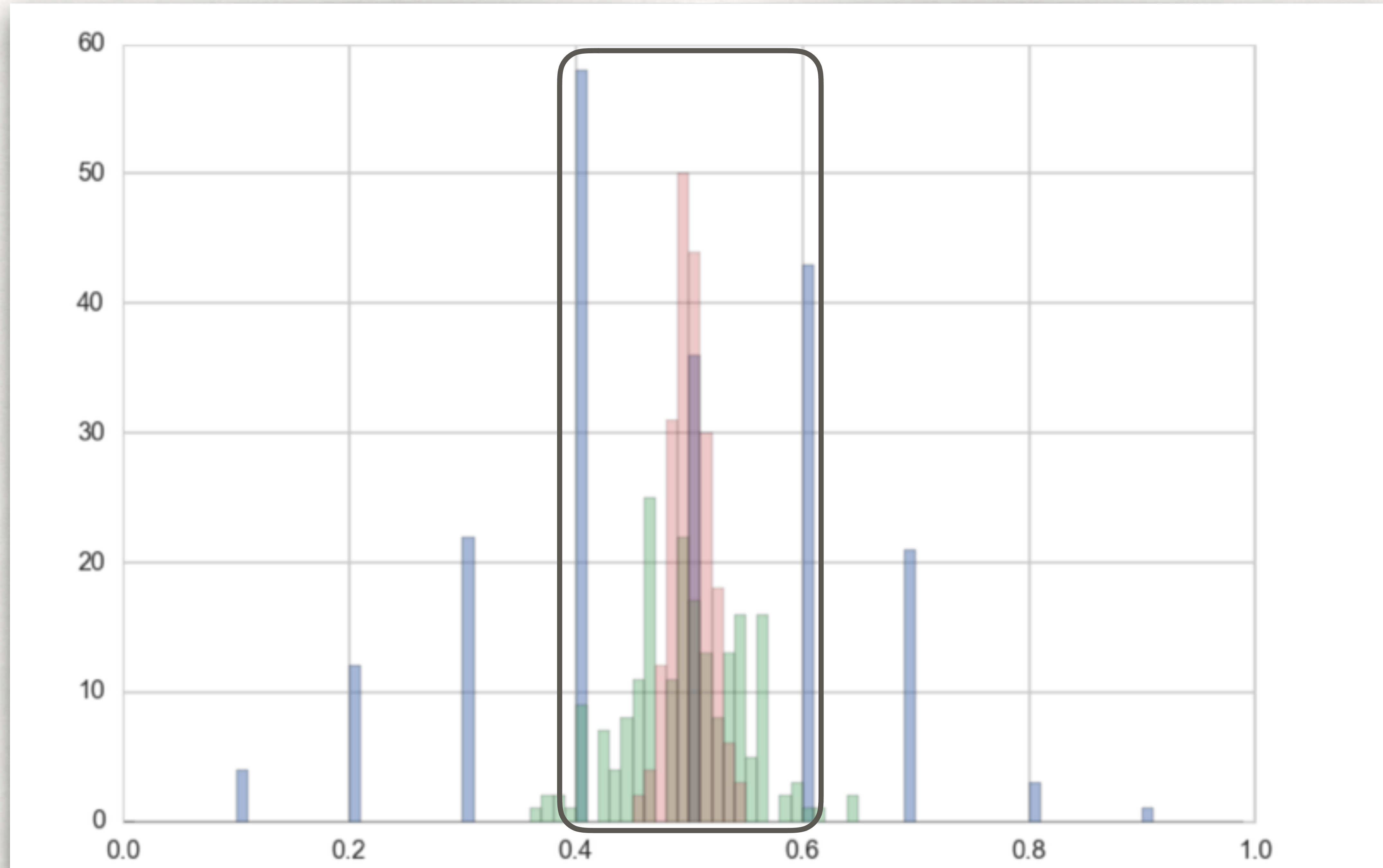
PLOT HISTOGRAM OF A,B,C

A: BLUE-10, B: GREEN-100, C: RED-1000

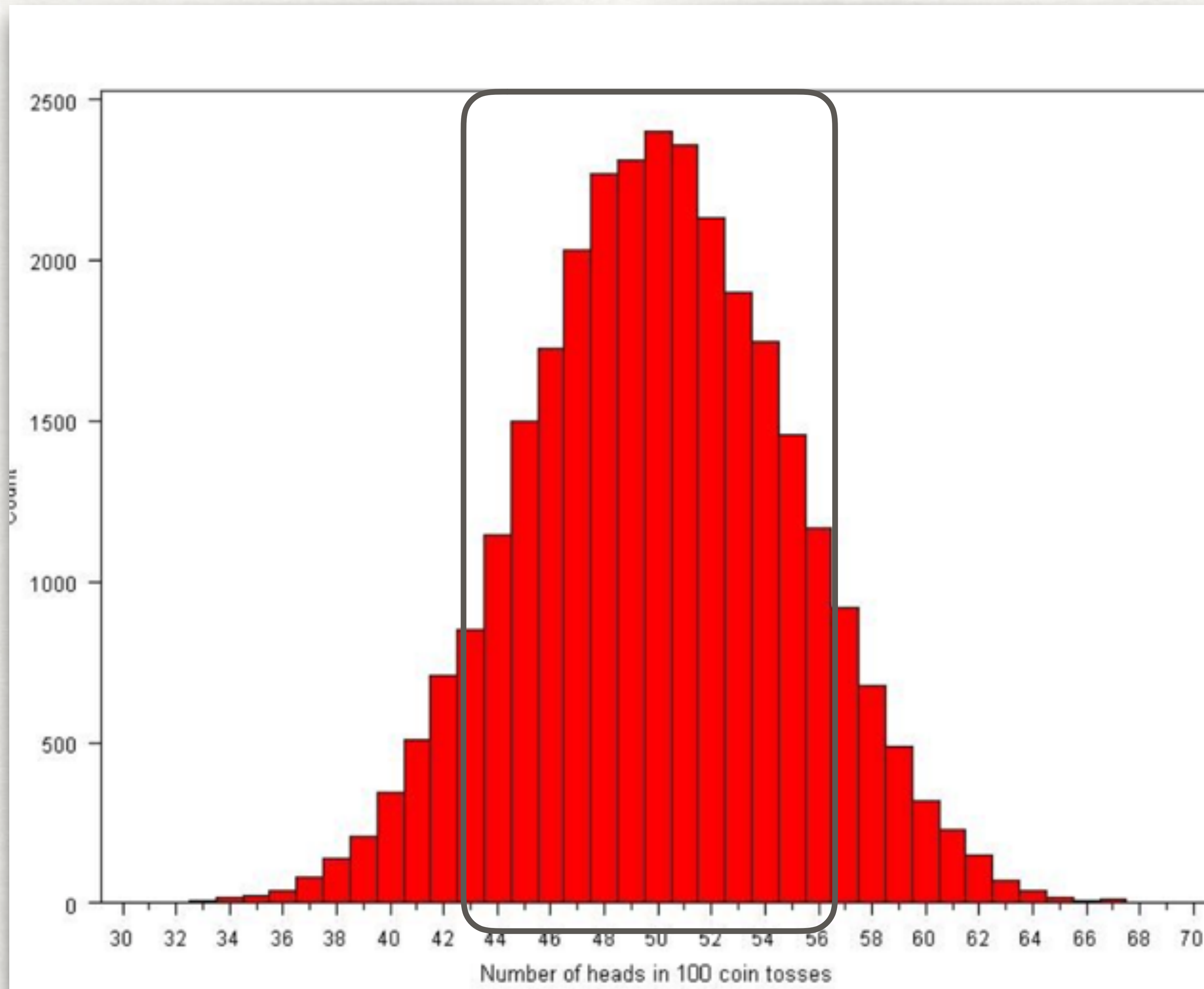


PLOT HISTOGRAM OF A,B,C

A: BLUE-10, B: GREEN-100, C: RED-1000

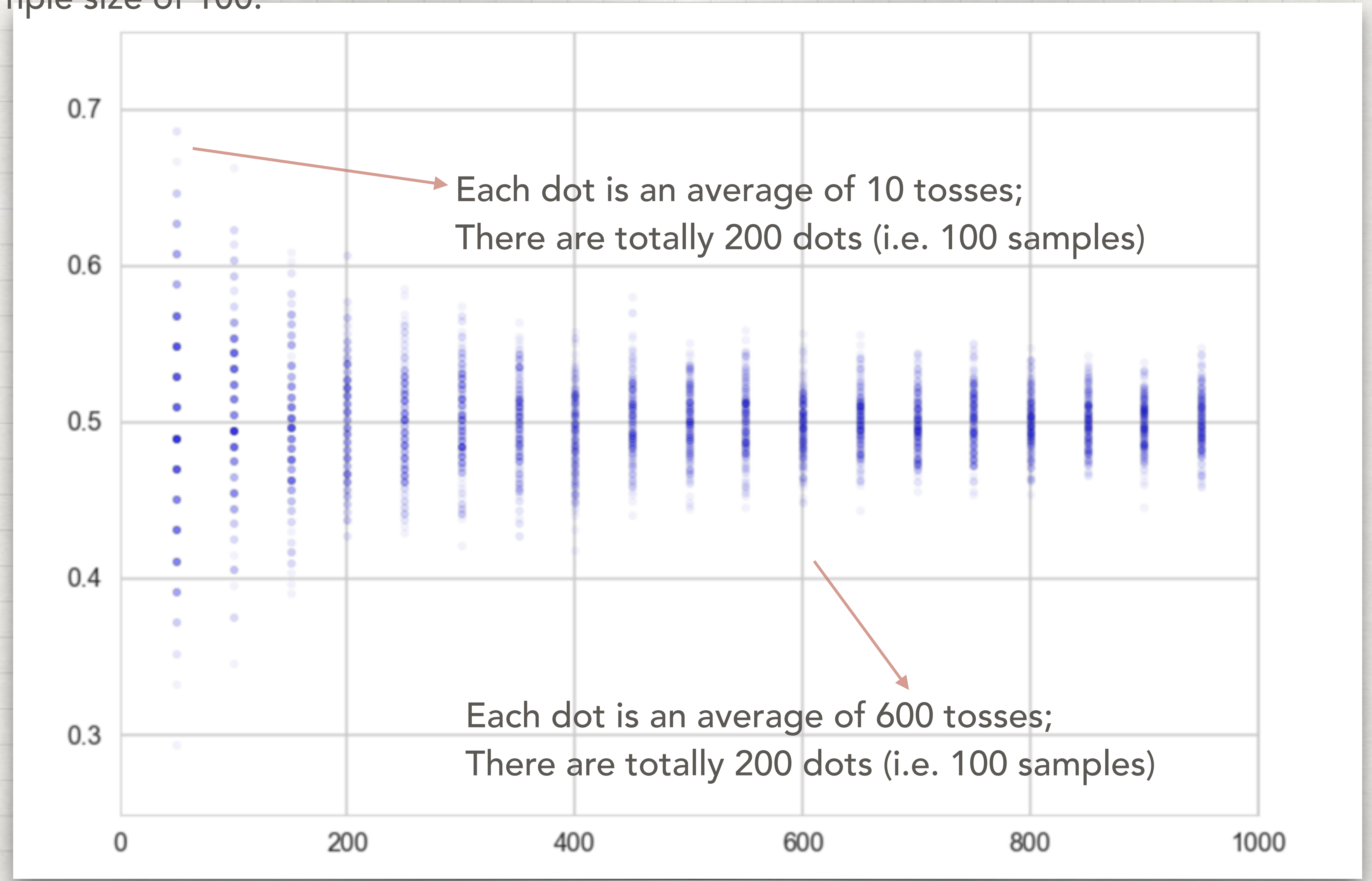


SAME AS WHAT WE HAVE BEFORE



The distribution is much tighter at large sample sizes, and that you can have way low and way large means at small sample sizes.

Indeed there are means as small as 0.1 at a sample size of 10, and as small as 0.3 at a sample size of 100.



EXAMPLE FROM LAST TIME

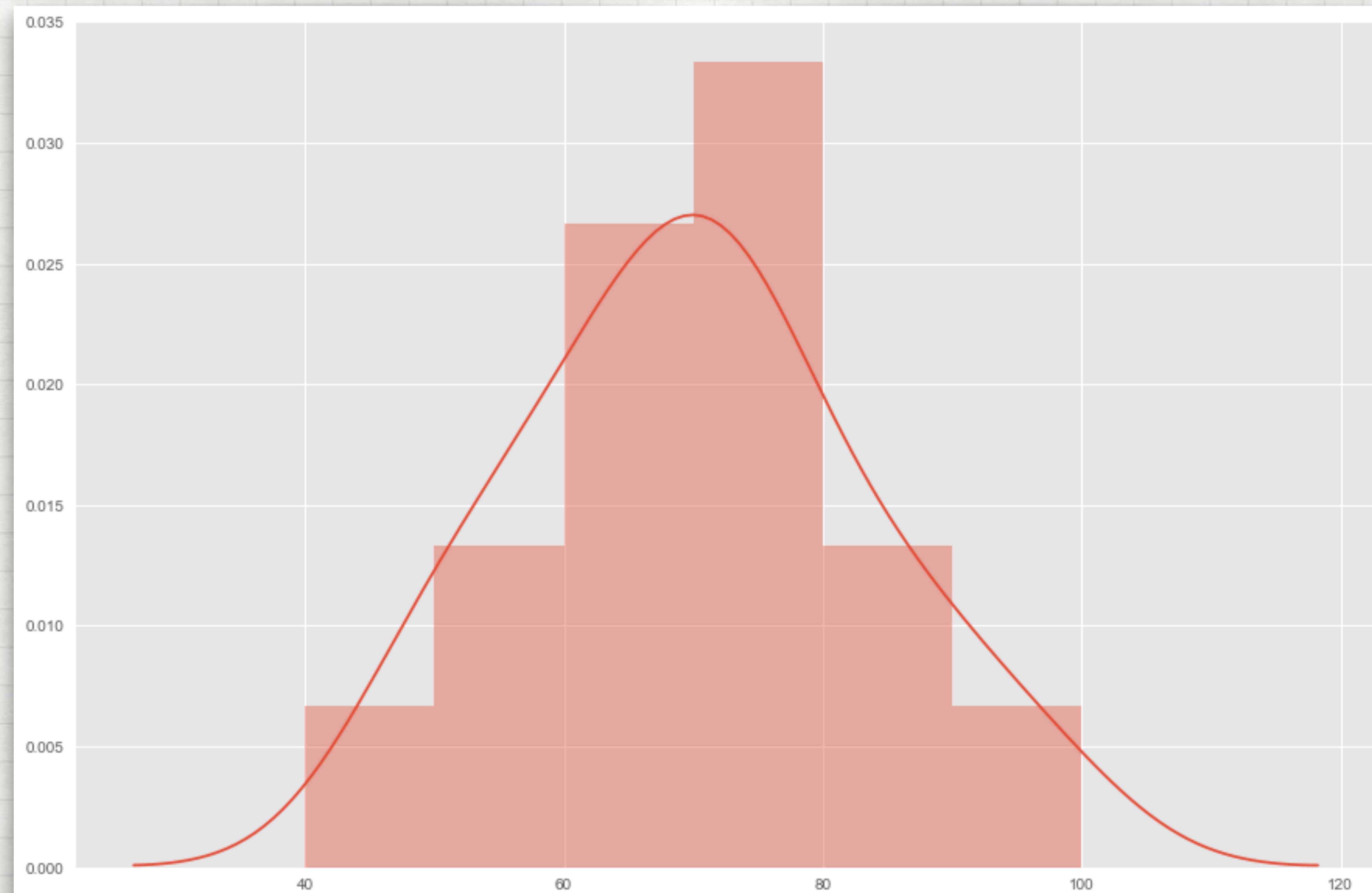
sum of relative sequence is 1 makes this a probability density function — in statistical sense

Step 1: Add a row at bottom of table. Put in total number of observations in the data set.

In this example, there are 15 (1+2+4+5+2+1= 15) total observations.

Score	Count (also called Frequency)	Relative Frequency
[40–50)	1	0.07
[50–60)	2	0.13
[60–70)	4	0.27
[70–80)	5	0.33
[80–90)	2	0.13
[90–100]	1	0.07
Total	15	

eg: In score [40 - 50), there are 1/15 = 0.07 much of data density estimate histogram — looks like normal distribution



NORMAL DISTRIBUTION

THE NORMAL DISTRIBUTION

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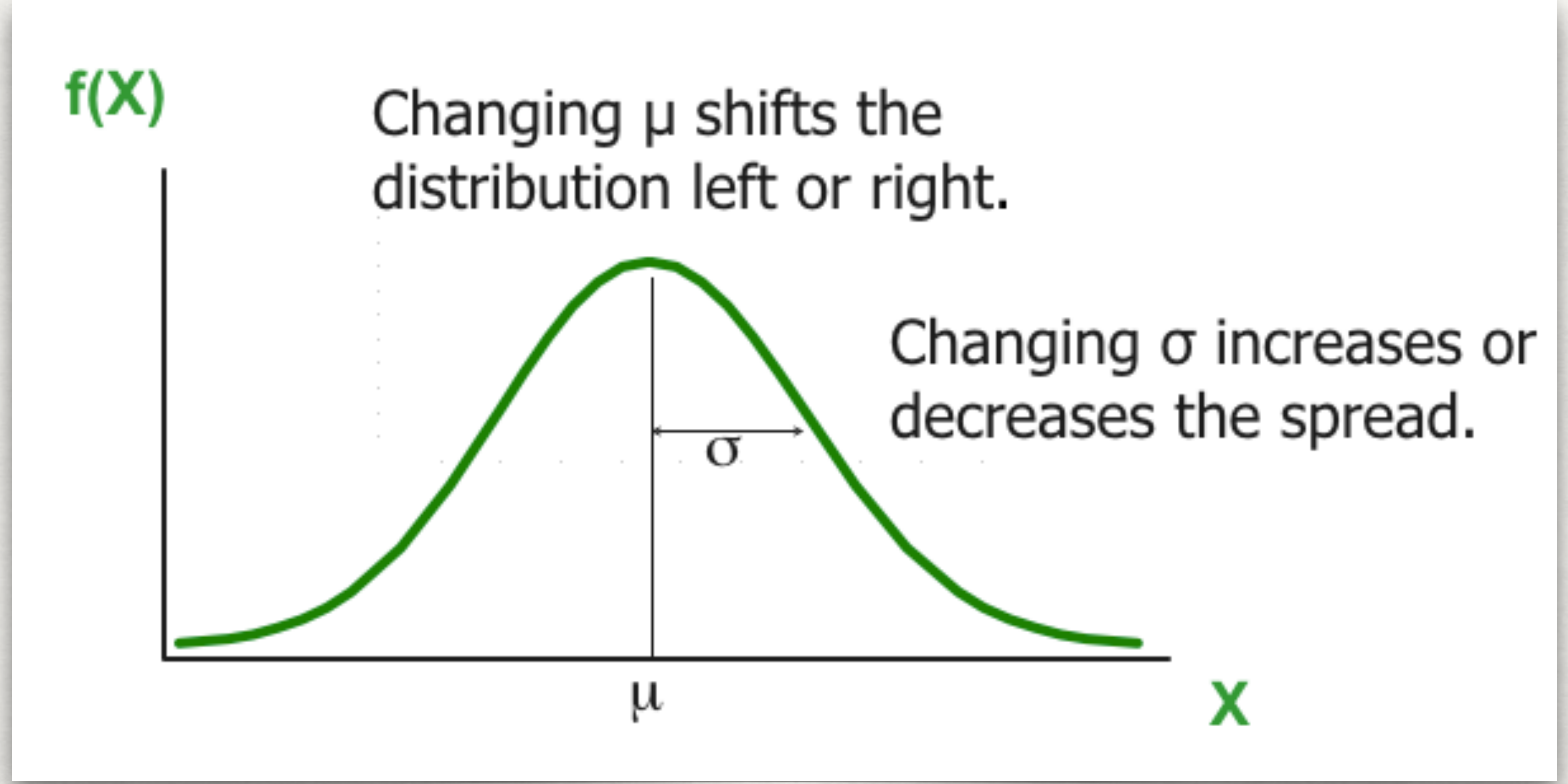
- Probability density function of a Normal or Gaussian distribution, written as N(μ,**σ**)



THE NORMAL DISTRIBUTION

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- The Normal Distribution:
  as mathematical function (pdf)

$$f(x) = \frac{1}{\boxed{\sigma}\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\boxed{\mu}}{\sigma}\right)^2}$$

This is a bell shaped curve with different centers and spreads depending on μ and σ

<u>Note constants:</u>
π=3.14159
e=2.71828

THE NORMAL DISTRIBUTION

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- It's a probability function, so no matter what the values of μ and **σ**, must integrate to 1!

$$E(X)=\mu = \boxed{\int_{-\infty}^{+\infty} x \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx}$$

$$Var(X)=\sigma^2 = \boxed{\int_{-\infty}^{+\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx) - \mu^2}$$

$$\text{Standard Deviation}(X)=\sigma$$

WHAT IS WEIRD?

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- **Normal curve:**

  No matter what μ and σ are,

  - the area between $\mu-\sigma$ and $\mu+\sigma$ is about 68%;
  - the area between $\mu-2\sigma$ and $\mu+2\sigma$ is about 95%;
  - and the area between $\mu-3\sigma$ and $\mu+3\sigma$ is about 99.7%.

- Almost all values fall within 3 standard deviations.

- 68-95-99.7 Rule

THE NORMAL DISTRIBUTION

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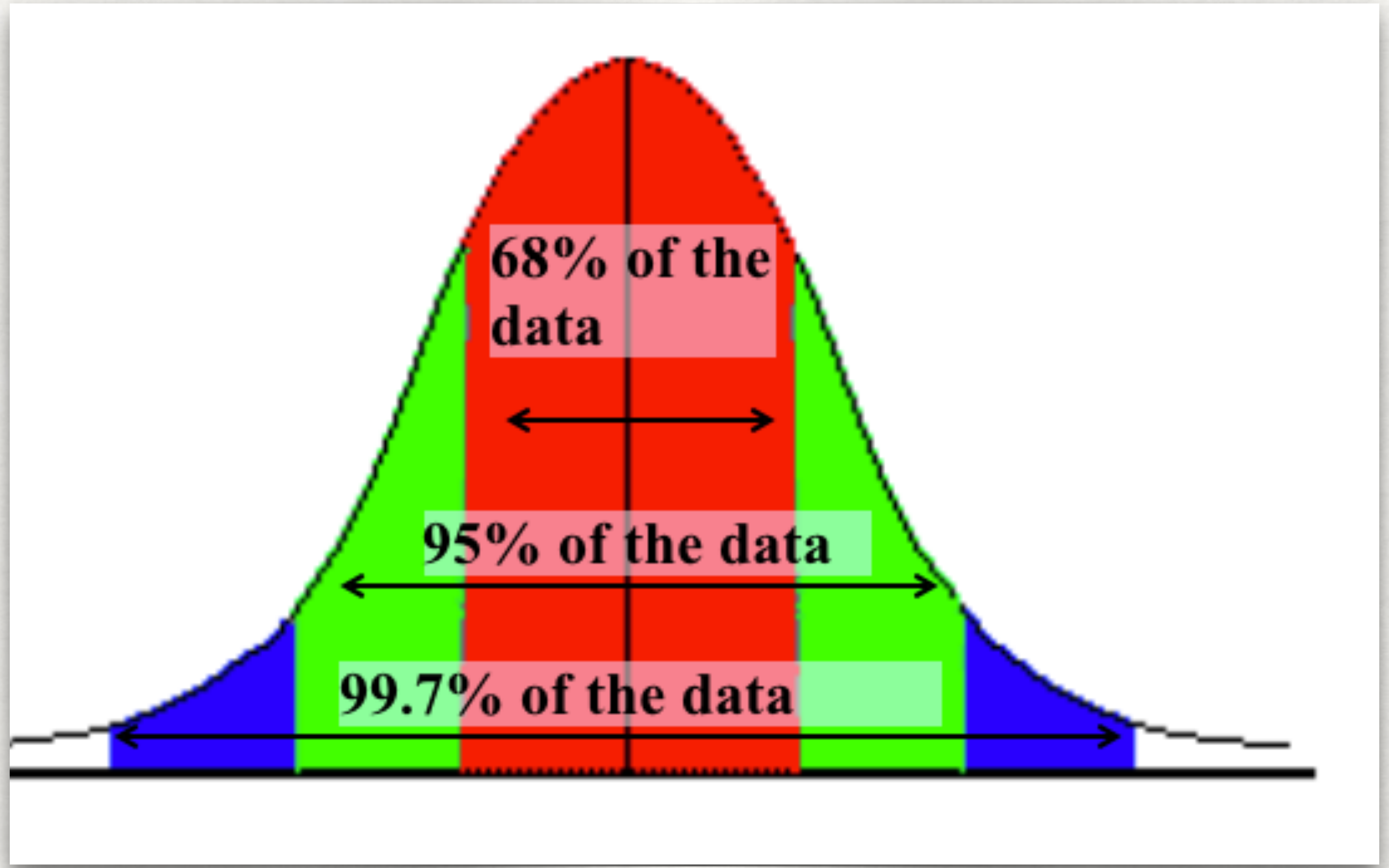

THE NORMAL DISTRIBUTION

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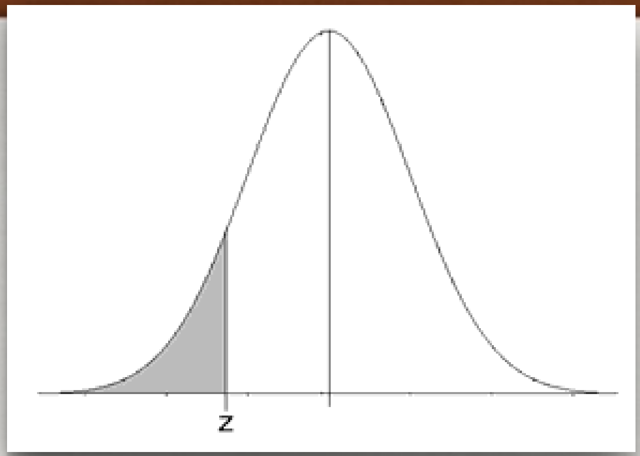
- 68-95-99.7 Rule (more data later)
  in Math terms…

$$\int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{\sigma\sqrt{2\pi}} \bullet e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = .68$$

$$\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \bullet e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = .95$$

$$\int_{\mu-3\sigma}^{\mu+3\sigma} \frac{1}{\sigma\sqrt{2\pi}} \bullet e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = .997$$

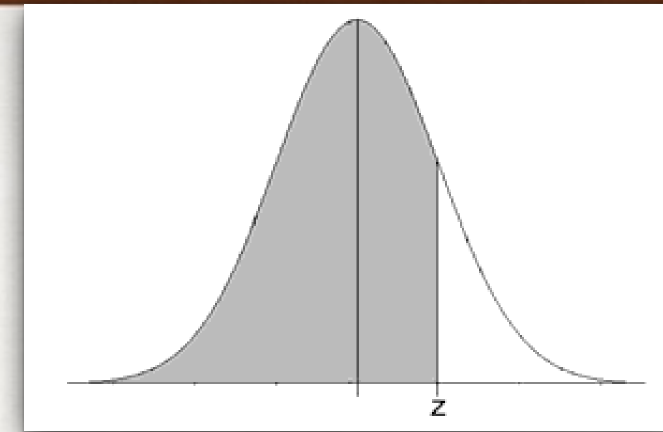
Standard Normal Cumulative Probability Table



Cumulative probabilities for NEGATIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890