Practice Midterm

Please show your work.

1. X is a random variable with mean 2 and variance 1/4. Find expectation $E(X/2)$ and standard deviation of $X - 1$.

2. $\{X_1, X_2, \ldots X_{10}\}$ is a sequence of IID follows a Bernoulli distribution with parameter 0.1. That is

$$p(X_i = 1) = 0.1, p(X_i = 0) = 0.9, E(X_i) = 0.1, Var(X_i) = 0.09$$

. Let $A_{10} = \frac{\sum_{i=1}^{10} X_i}{10}$ Find the following values

   (a) $E(A_{10})$

   (b) $Var(A_{10})$

   (c) $std(A_{10})$

   (d) Use the Central Limit Theorem to find a normal distribution $N(\mu, \sigma^2)$ (i.e. to find μ and σ) to approximate the distribution of A_{10}.

3. Suppose you choose a real number X from the interval [1, e] with a density function of the form:

$$f(x) = \frac{C}{x}$$

where C is a constant. Note that e is the Euler's constant 2.718281....

   (a) Find C

   (b) Find $P(E)$, where $E = [1, \frac{e}{2}]$ is a subinterval of [1 , e].

   (c) Find the cumulative density function of X

4. Let U, V be random numbers chosen independently from the interval $[0, 1]$ uniformly. Find the cumulative distribution and density for the random variable $Y = max(U, V)$

5. X follows a Normal Distribution with mean 5 and variance 4. Calculate the following values

   (a) $P(|X| > 3)$

   (b) $P(X < 9)$

   (c) $P(X > 5)$

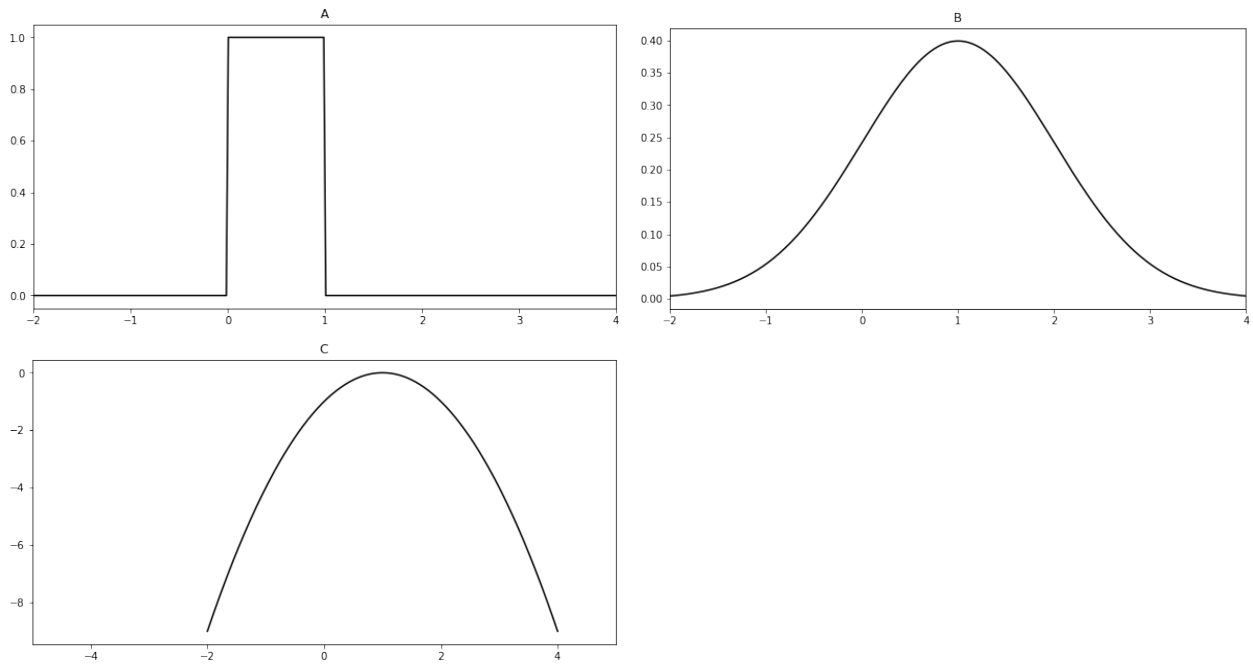
   (d) $P(e^X < 1)$

(e) $E(X^2)$

6. X follows a Normal Distribution with mean μ and standard deviation σ. Let Z denote the standard normal distribution. Is it true that $P(|X-\mu| > \sigma)$ is equal to $P(|Z| > 1)$? Justify your answer.

7. Let X be a random variable with $E(X) = 0$ and $Var(X) = 0.01$. What integer value k will assure us that $P(|X| \geq k) \leq 0.01$?

8. Which plot corresponds to a normal distribution?



9. The probability density function of X is $f(x) = \begin{cases} 3(x-1)^2, & \text{if } 2 \geq x \geq 1 \\ 0, & \text{otherwise} \end{cases}$

   Find the inverse Cumulative Density Function of X

10. Use Hypothesis Testing to answer the question. There are rumors that students at a certain liberal arts college are more inclined to use Gahoo email than U.S. college students in general. Suppose that in a simple random sample of 100 students from the college, 50 admitted to Gahoo email use. Do the data provide enough evidence to conclude that the proportion of Gahoo email users among the students in the college (p) is higher than the national average, which is 0.3?

    (a) Determine Null Hypothesis H_0

    (b) Determine Alternative Hypothesis H_a

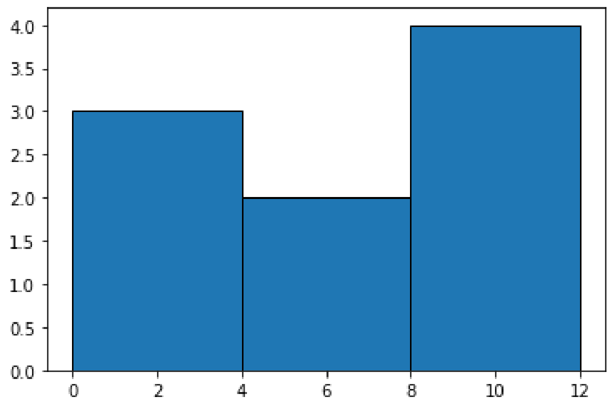
    (c) Determine z-score.

(d) Assume the confidence level is $\alpha = 5\%$. Do the data provide enough evidence to conclude that the proportion of Gahoo email users among the students in the college (p) is higher than the national proportion, which is 0.3?

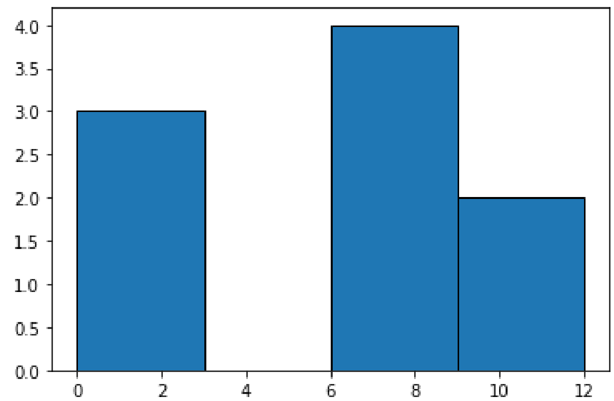
11. In python, if I run the following code, select the right output graph.

```python
import numpy as np
import matplotlib.pyplot as plt

data = [1,2,2,7,7.5,8,8.5,9,12]
plt.hist(data,bins = 3, range = (0,12),edgecolor='black', linewidth=1)
plt.show()

```

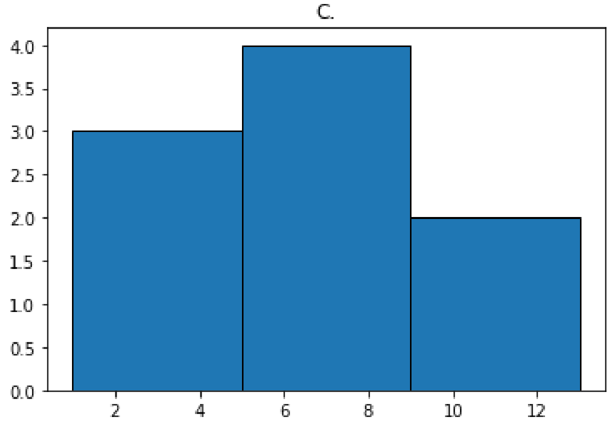
A.



B.



C.



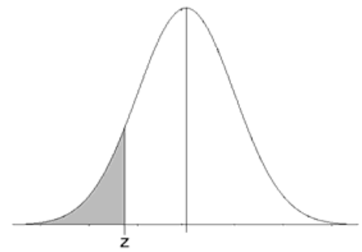
12. In python, if I run the following code, select the right output.

```python
arr = np.array([8,9,10,-1])
print(np.where(arr<0)[0])

```

A. [4]
B. [3]

C. [-1]
D. [10]
E. None of the above

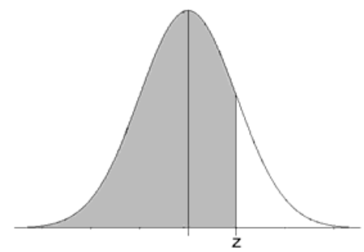
Standard Normal Cumulative Probability Table



Cumulative probabilities for NEGATIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Probability Cheat Sheet

Distributions

Unifrom Distribution

`notation` $U[a,b]$

`cdf` $\frac{x-a}{b-a}$ for $x \in [a,b]$

`pdf` $\frac{1}{b-a}$ for $x \in [a,b]$

`expectation` $\frac{1}{2}(a+b)$

`variance` $\frac{1}{12}(b-a)^2$

`mgf` $\frac{e^{tb}-e^{ta}}{t(b-a)}$

`story:` all intervals of the same length on the distribution's support are equally probable.

Gamma Distribution

`notation` $Gamma(k,\theta)$

`pdf` $\frac{\theta^k x^{k-1} e^{-\theta x}}{\Gamma(k)} \mathbb{I}_{x>0}$

$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$

`expectation` $k\theta$

`variance` $k\theta^2$

`mgf` $(1-\theta t)^{-k}$ for $t < \frac{1}{\theta}$

`ind. sum` $\sum_{i=1}^n X_i \sim Gamma\left(\sum_{i=1}^n k_i, \theta\right)$

`story:` the sum of k independent exponentially distributed random variables, each of which has a mean of θ (which is equivalent to a rate parameter of θ^{-1}).

Geometric Distribution

`notation` $G(p)$

`cdf` $1-(1-p)^k$ for $k \in \mathbb{N}$

`pmf` $(1-p)^{k-1} p$ for $k \in \mathbb{N}$

`expectation` $\frac{1}{p}$

`variance` $\frac{1-p}{p^2}$

`mgf` $\frac{pe^t}{1-(1-p)e^t}$

`story:` the number X of Bernoulli trials needed to get one success. Memoryless.

Poisson Distribution

`notation` $Poisson(\lambda)$

`cdf` $e^{-\lambda} \sum_{i=0}^k \frac{\lambda^i}{i!}$

`pmf` $\frac{\lambda^k}{k!} \cdot e^{-\lambda}$ for $k \in \mathbb{N}$

`expectation` λ

`variance` λ

`mgf` $\exp\left(\lambda\left(e^t-1\right)\right)$

`ind. sum` $\sum_{i=1}^n X_i \sim Poisson\left(\sum_{i=1}^n \lambda_i\right)$

`story:` the probability of a number of events occurring in a fixed period of time if these events occur with a known average rate and independently of the time since the last event.

Normal Distribution

`notation` $N\left(\mu,\sigma^2\right)$

`pdf` $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/\left(2\sigma^2\right)}$

`expectation` μ

`variance` σ^2

`mgf` $\exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$

`ind. sum` $\sum_{i=1}^n X_i \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$

`story:` describes data that cluster around the mean.

Standard Normal Distribution

`notation` $N(0,1)$

`cdf` $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$

`pdf` $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

`expectation` $\frac{1}{\lambda}$

`variance` $\frac{1}{\lambda^2}$

`mgf` $\exp\left(\frac{t^2}{2}\right)$

`story:` normal distribution with $\mu = 0$ and $\sigma = 1$.

Exponential Distribution

`notation` $exp(\lambda)$

`cdf` $1-e^{-\lambda x}$ for $x \geq 0$

`pdf` $\lambda e^{-\lambda x}$ for $x \geq 0$

`expectation` $\frac{1}{\lambda}$

`variance` $\frac{1}{\lambda^2}$

`mgf` $\frac{\lambda}{\lambda - t}$

`ind. sum` $\sum_{i=1}^k X_i \sim Gamma(k,\lambda)$

`minimum` $\sim exp\left(\sum_{i=1}^k \lambda_i\right)$

`story:` the amount of time until some specific event occurs, starting from now, being memoryless.

Binomial Distribution

`notation` $Bin(n,p)$

`cdf` $\sum_{i=0}^k \binom{n}{i} p^i (1-p)^{n-i}$

`pmf` $\binom{n}{i} p^i (1-p)^{n-i}$

`expectation` np

`variance` $np(1-p)$

`mgf` $\left(1-p+pe^t\right)^n$

`story:` the discrete probability distribution of the number of successes in a sequence of n independent yes/no experiments, each of which yields success with probability p.

Basics

Comulative Distribution Function

$F_X(x) = \mathbb{P}(X \leq x)$

Probability Density Function

$F_X(x) = \int_{-\infty}^\infty f_X(t)\, dt$

$\int_{-\infty}^\infty f_X(t)\, dt = 1$

$f_X(x) = \frac{d}{dx} F_X(x)$

Quantile Function

The function $X^* : [0,1] \to \mathbb{R}$ for which for any $p \in [0,1]$, $F_X\left(X^*(p)^-\right) \leq p \leq F_X\left(X^*(p)\right)$

$F_{X^*} = F_X$

$\mathbb{E}(X^*) = \mathbb{E}(X)$

Expectation

$\mathbb{E}(X) = \int_0^1 X^*(p) dp$

$\mathbb{E}(X) = \int_{-\infty}^0 F_X(t)\, dt + \int_0^\infty (1-F_X(t))\, dt$

$\mathbb{E}(X) = \int_{-\infty}^\infty x f_X x dx$

$\mathbb{E}(g(X)) = \int_{-\infty}^\infty g(x) f_X x dx$

$\mathbb{E}(aX+b) = a\mathbb{E}(X) + b$

Variance

$\mathrm{Var}(X) = \mathbb{E}\left(X^2\right) - (\mathbb{E}(X))^2$

$\mathrm{Var}(X) = \mathbb{E}\left((X-\mathbb{E}(X))^2\right)$

$\mathrm{Var}(aX+b) = a^2 \mathrm{Var}(X)$

Standard Deviation

$\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Covariance

$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

$\mathrm{Cov}(X,Y) = \mathbb{E}((X-\mathbb{E}(x))(Y-\mathbb{E}(Y)))$

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$

Correlation Coefficient

$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X, \sigma_Y}$

Moment Generating Function

$M_X(t) = \mathbb{E}\left(e^{tX}\right)$

$\mathbb{E}(X^n) = M_X^{(n)}(0)$

$M_{aX+b}(t) = e^{tb} M_{aX}(t)$

Joint Distribution

$\mathbb{P}_{X,Y}(B) = \mathbb{P}((X,Y) \in B)$
$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$

Joint Density

$\mathbb{P}_{X,Y}(B) = \iint_B f_{X,Y}(s,t)\,dsdt$

$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dtds$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dsdt = 1$

Marginal Distributions

$\mathbb{P}_X(B) = \mathbb{P}_{X,Y}(B \times \mathbb{R})$
$\mathbb{P}_Y(B) = \mathbb{P}_{X,Y}(\mathbb{R} \times Y)$

$F_X(a) = \int_{-\infty}^{a}\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dtds$

$F_Y(b) = \int_{-\infty}^{b}\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dsdt$

Marginal Densities

$f_X(s) = \int_{-\infty}^{\infty} f_{X,Y}(s,t)dt$

$f_Y(t) = \int_{-\infty}^{\infty} f_{X,Y}(s,t)ds$

Joint Expectation

$\mathbb{E}(\varphi(X,Y)) = \iint_{\mathbb{R}^2} \varphi(x,y)\, f_{X,Y}(x,y)\,dxdy$

Independent r.v.

$\mathbb{P}(X \le x, Y \le y) = \mathbb{P}(X \le x)\,\mathbb{P}(Y \le y)$
$F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$
$f_{X,Y}(s,t) = f_X(s)\,f_Y(t)$
$\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
Independent events:
$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

Conditional Probability

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

`bayes` $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

Conditional Density

$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

$f_{X|Y=n}(x) = \frac{f_X(x)\,\mathbb{P}(Y=n \mid X=x)}{\mathbb{P}(Y=n)}$

$F_{X|Y=y} = \int_{-\infty}^{x} f_{X|Y=y}(t)\,dt$

Conditional Expectation

$\mathbb{E}(X \mid Y=y) = \int_{-\infty}^{\infty} x f_{X|Y=y}(x)\,dx$
$\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X)$
$\mathbb{P}(Y=n) = \mathbb{E}(\mathbb{I}_{Y=n}) = \mathbb{E}(\mathbb{E}(\mathbb{I}_{Y=n} \mid X))$

Sequences and Limits

$\limsup A_n = \{A_n \text{ i.o.}\} = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n$

$\liminf A_n = \{A_n \text{ eventually}\} = \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} A_n$

$\liminf A_n \subseteq \limsup A_n$
$(\limsup A_n)^c = \liminf A_n^c$
$(\liminf A_n)^c = \limsup A_n^c$

$\mathbb{P}(\limsup A_n) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{n=m}^{\infty} A_n\right)$

$\mathbb{P}(\liminf A_n) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{n=m}^{\infty} A_n\right)$

Borel-Cantelli Lemma

$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(\limsup A_n) = 0$

And if A_n are independent:

$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}(\limsup A_n) = 1$

Convergence

Convergence in Probability

`notation` $X_n \xrightarrow{p} X$

`meaning` $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

Convergence in Distribution

`notation` $X_n \xrightarrow{D} X$

`meaning` $\lim_{n\to\infty} F_n(x) = F(x)$

Almost Sure Convergence

`notation` $X_n \xrightarrow{a.s.} X$

`meaning` $\mathbb{P}\left(\lim_{n\to\infty} X_n = X\right) = 1$

Criteria for a.s. Convergence

- $\forall\varepsilon\exists N\forall n > N : \mathbb{P}(|X_n - X| < \varepsilon) > 1 - \varepsilon$
- $\forall\varepsilon\mathbb{P}(\limsup(|X_n - X| > \varepsilon)) = 0$
- $\forall\varepsilon \sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon) < \infty$ (by B.C.)

Convergence in L_p

`notation` $X_n \xrightarrow{L_p} X$

`meaning` $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0$

Relationships

$$\xrightarrow{L_q} \quad \underset{q>p\ge1}{\Rightarrow} \quad \xrightarrow{L_p}$$
$$\Downarrow$$
$$\xrightarrow{a.s.} \quad \Rightarrow \quad \xrightarrow{p} \quad \Rightarrow \quad \xrightarrow{D}$$

If $X_n \xrightarrow{D} c$ then $X_n \xrightarrow{p} c$
If $X_n \xrightarrow{p} X$ then there exists a subsequence n_k s.t. $X_{n_k} \xrightarrow{a.s.} X$

Laws of Large Numbers

If X_i are i.i.d. r.v.,

`weak law` $\overline{X_n} \xrightarrow{p} \mathbb{E}(X_1)$

`strong law` $\overline{X_n} \xrightarrow{a.s.} \mathbb{E}(X_1)$

Central Limit Theorem

$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{D} N(0,1)$

If $t_n \to t$, then

$\mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le t_n\right) \to \Phi(t)$

Inequalities

Markov's inequality

$\mathbb{P}(|X| \ge t) \le \frac{\mathbb{E}(|X|)}{t}$

Chebyshev's inequality

$\mathbb{P}(|X - \mathbb{E}(X)| \ge \varepsilon) \le \frac{\mathrm{Var}(X)}{\varepsilon^2}$

Chernoff's inequality

Let $X \sim Bin(n,p)$; then:
$\mathbb{P}(X - \mathbb{E}(X) > t\sigma(X)) < e^{-t^2/2}$
Simpler result; for every X:
$\mathbb{P}(X \ge a) \le M_X(t)\,e^{-ta}$

Jensen's inequality

for φ a convex function, $\varphi(\mathbb{E}(X)) \le \mathbb{E}(\varphi(X))$

Miscellaneous

$\mathbb{E}(Y) < \infty \iff \sum_{n=0}^{\infty} \mathbb{P}(Y > n) < \infty \ (Y \ge 0)$

$\mathbb{E}(X) = \sum_{n=0}^{\infty} \mathbb{P}(X > n) \ (X \in \mathbb{N})$

$X \sim U(0,1) \iff -\ln X \sim exp(1)$

Convolution

For ind. X, Y, $Z = X + Y$:

$f_Z(z) = \int_{-\infty}^{\infty} f_X(s)\,f_Y(z-s)\,ds$

Kolmogorov's 0-1 Law

If A is in the tail σ-algebra $\mathcal{F}^t$, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$

Ugly Stuff

cdf of Gamma distribution:

$\int_0^t \frac{\theta^k x^{k-1} e^{-\theta k}}{(k-1)!}\,dx$

This cheatsheet was made by Peleg Michaeli in January 2010, using LaTeX.
version: 1.01
comments: peleg.michaeli@math.tau.ac.il