Cheat sheet and z-table will be provided as in the practice exam.

Notes, calculator or any other study aids are NOT permitted.

Solution 1

1.] In general, Given a random variable. X, a constant c:

$$E(cX) = cE(X), \quad E(c) = c, \quad E(X+c) = E(X)+c$$

If Y is another random Variable and, b = a constant.

$$E(cX+bY) = cE(X) + bE(Y).$$

$$Var(cX) = c^2 \, Var(X). \qquad Var(X+C) = Var(X),$$

$$\Downarrow$$

$$std(cX) = \sqrt{Var(cX)} = |c| \sqrt{Var(X)}$$

$$Var(X+Y) = Var(X) + Var(Y) + 2cov(X,Y)$$

$$Var(X) := E((X-E(X))^2) = E(X^2) - E(X)^2.$$

Note $E(g(x)) = \int_{-\infty}^{+\infty} g(x) I_{\{spx\}} f_X(x)\, dx$. where $f_X(x)$ is pdf of X if X is continuous

$E(g(x)) = \sum_{x \in \Omega} g(x) \cdot m(x) \leftarrow$ for X discrete r.v. with mass $m(x)$

$E(X) = 2.$ $\quad var(X) = \frac{1}{4}$

$$E(\tfrac{X}{2}) = \tfrac{1}{2}E(X) = \tfrac{1}{2} \cdot 2 = 1 \; ; \; Std(X-1) = \sqrt{Var(X-1)} = \sqrt{Var(X)} = \tfrac{1}{2}$$

②

2] $\{X_i\}$ IID — identically independent Distribution : - same mean & Variance
in/ distribution. - same distribution.

Central limit : $\frac{\sum X_i}{n} \approx N\left(E(X_i), \frac{Var(X_i)}{n}\right)$

Note : $\frac{(\sum X_i)}{n}$ is a r.v. with mean: $E(X_i)$
and variance $\frac{Var(X_i)}{n}$.

a) $E(A_{10}) = E(X_i) = 0.1$

b) $Var(A_{10}) = \frac{Var(X_i)}{10} = \frac{0.09}{10} = 0.009$

c) $Std(A_{10}) = \sqrt{Var(A_{10})} = \sqrt{0.009}$

d) $N\left(E(X_i), \cancel{\frac{Var(X_i)}{n}} \frac{Var(X_i)}{n}\right) = N(0.1, 0.009)$

i.e. $\mu = 0.1$. $\sigma = \sqrt{0.009}$.

3). pdf - probability density function $f(x)$ for a r.v. X must satisfy:

* 1): $f(x) \geq 0$ for all $x \in \mathbb{R}$

** 2): $\int_{\mathbb{R}} f(x)\,dx = 1$.

3) $\underbrace{P(\cancel{x} \in A)}_{\text{probability event A happens}} = \int_A f(x)\,dx$.

Since X is choosen from $[1, e]$, $f_X(x) = 0$ for all x outsider $[1, e]$.

@ c is a real number s.t. (*) & (**) are both true.

$\boxed{**}$ $\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow 1 = \int_{\mathbb{R}} \frac{c}{x} I_{\{x \in [1,e]\}}\,dx$

$= \int_1^e \frac{c}{x}\,dx = c \ln x \Big|_1^e = c(1-0) = c = 1$.

Note: $\boxed{*}$ for $c=1$. $f(x) = \frac{1}{x} > 0$. for all $x \in [1, e]$.

So $c=1$ makes $f(x) = \frac{1}{x}$ a pdf over $[1, e]$

b). $P(E)$ where $E = [1, \frac{e}{2}]$.

step 1: check ~~math~~ relation between E & $[1, e]$.

$[1, \frac{e}{2}] = E \subseteq [1, e]$

step2: $\int_E f(x)\,dx = \int_1^{\frac{e}{2}} \frac{1}{x}\,dx = \ln(\frac{e}{2}) - \ln(1)$

$= 1 - \ln 2$

eg: $A = [\frac{e}{2}, 10]$. find $P(A)$.

step 1: $A \nsubseteq [1, e]$, event: $A' = A \cap [1, e]$

$A \cap [1, e] = [\frac{e}{2}, 10] \cap [1, e] = [\frac{e}{2}, e]$

step2: $\int_A f(x)\,dx = \int_{A'} f(x)\,dx = \int_{\frac{e}{2}}^{e} \frac{1}{x}\,dx$

$= \ln(e) - \ln(\frac{e}{2}) = \ln 2.$

$P(W)$ where $W = [\frac{e}{4}, 7]$, since $[1, e] \subseteq W$, $P(W) = 1$.

c): CDF: $F_X(x) = P(X < x) = \int_{-\infty}^{x} f(x) I_{\{\text{supp}(x)\}}\,dx = \int_1^x \frac{1}{x}\,dx$

$= \ln x$. for $x \in [1, e]$. ★

for $x > e$. $F_X(x) = 1$, for $x < 1$, $F_X(x) = 0$

41: CDF: $P(\text{r.v.} < x)$.

$$P(\max(u,v) < x) = P(u < x \text{ and } v < x).$$

u, v – independent $\longrightarrow$ $= P(u < x) \cdot P(v < x)$

$$\boxed{\text{uniform CDF: } F(x) = \begin{cases} x & x \in [0,1] \\ 0 & x < 0 \\ 1 & x > 1 \end{cases}}$$

$$\begin{cases} x^2 & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1. \end{cases}$$

Extra: $u_1, u_2, \cdots, u_n$ — n IId on uniform $[0,1]$.

$$Y := \max(u_1, \cdots, u_n).$$

$$P(Y < x) = P(\max(u_1, \cdots, u_n) < x)$$

$$= P(u_1 < x)\, P(u_2 < x) \cdots P(u_n < x)$$

$$= \begin{cases} x^n & \text{if } x \in [0,1] \\ 0 & \text{if } x < 0 \\ 1 & \text{if } x > 1. \end{cases}$$

5) $X \sim N(\mu, \sigma^2)$. $\quad Z \sim N(0,1)$

then $X = \sigma Z + \mu$

$\mu = 5$. $\sigma^2 = 4$

$\Big\}\ \cancel{X} = 2Z + 5$.

(a) $P(|2Z+5|>3) = P(2Z+5>3) + P(2Z+5<-3)$

$= \underbrace{P(Z>-1)} + P(Z<-4)$.

$= \boxed{1-P(Z<-1)} + P(Z<-4)$

z-table $\approx 1 - 0.1587 + 0$

$= 0.8413$.

(b). $P(X<9) = P(2Z+5<9) = P(Z<2) = 0.9772$.

(c). $P(X>5) = 1-P(X<5) = 1-P(2Z+5<5) = 1-P(Z<0)$

$= \frac{1}{2}$

(d) $P(e^X<1) = P(X<\ln(1)) = P(X<0) = P(2Z+5<0)$

$= P(Z<-2.5) = 0.0062$.

5 (e): $E(X^2)$ ← use $Var(X) = E(X^2) - E(X)^2$

so $E(X^2) = Var(X) + E(X)^2 = 4 + 5^2 = 29$.

6. $X \sim N(\mu, \sigma^2)$ then $X = \sigma Z + \mu$, $|X - \mu| = \sigma |Z|$.

so $P(|X-\mu| > \sigma) = P(\sigma|Z| > \sigma) = P(|Z| > 1)$.

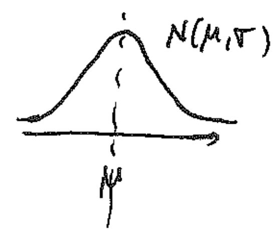
7: cheby: $P(|X - \mu| > \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$ where $\mu = E(X)$

$\sigma^2 = Var(X)$

$P(|X - 0| > \overset{k}{\varepsilon}) \le \frac{0.01}{\varepsilon^2} k$ $\quad k = 1$

8: normal bell shape ~~etc~~. . . . (~~...~~)

Symmetry about $x = \mu$.



9. pdf: $f(x) = \begin{cases} 3(x-1)^2, & 1 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$

CDF: $\int_{-\infty}^{x} f(t)\,dt = \int_1^x f(t)\,dt = (t-1)^3 \Big|_1^x = (x-1)^3$

Let $u = (t-1)^3 \Rightarrow t = \sqrt[3]{u} + 1$ ← Inverse CDF.

10| a) H_0 = the college is the same as national average

$P = P_0 = 0.3$

b) H_a = the college has higher average

$P > P_0 = 0.3$

c) $z\text{-score} = \dfrac{\hat{P} - P}{\sqrt{\dfrac{P(1-P)}{n}}} = \dfrac{\frac{50}{100} - 0.3}{\sqrt{\dfrac{0.3 \cdot 0.7}{100}}} \approx 4.364.$

D) P-value $\approx P(Z > 4.364)$ (since H_a : $P > 0.3$)

$\approx 0. < 0.05.$

so data provides enough evidence to reject. H_0 & accept H_a.

11| A

12| B.