Solutions to Chapter 2 of Sheldon M. Ross' *Probability Models For Computer Science*

1. Let $q = 1 - p$.

   (a) The vertex set $\{1, ..., k\}$ is a component of the graph if it is connected and there are no edges between nodes from the respective sets $\{1, ..., k\}$ and $\{k+1, ..., n\}$.

   $$P_k \cdot \left(q^{n-k}\right)^k$$

   (b) The probability that vertex 1 is a member of a component of size k is the sum of the probabilities of all subsets of size k containing vertex 1 is a component.

   $$\binom{n-1}{k-1} \cdot P_k \cdot q^{k(n-k)}$$

   (c) The graph of n nodes is connected if an arbitrary vertex, say vertex 1, belongs to a component of size n; that is, the vertex does not belong to a component of size other than n.

   $$P_n = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \cdot P_k \cdot q^{k(n-k)}$$

   (d)

   $$\begin{aligned}
   P_1 &= 1 \\
   P_2 &= 1 - q \\
   P_3 &= 1 - 3q^2 + 2q^3 \\
   P_4 &= 1 - 4q^3 - 3q^4 + 12q^5 - 6q^6 \\
   P_5 &= 1 - 5q^4 - 10q^6 + 20q^7 + 30q^8 - 60q^9 + 24q^{10} \\
   P_6 &= 1 - 6q^5 - 15q^8 + 20q^9 + 120q^{11} - 90q^{12} - 270q^{13} + 360q^{14} - 120q^{15}
   \end{aligned}$$

2. Upon choosing a pivot for a sequence of $n > 0$ distinct elements, $n - 1$ comparisons are made before recursively sorting two subsets. If the pivot element is the ith rank, the sizes of the two subsets are $i - 1$ and $n - i$.

   Hence, for $n > 0$, noting also that the pivot is selected uniformly at random among n

elements,

$$\begin{aligned}
M_n &= (n-1) + \sum_{i=1}^{n} \frac{1}{n} \left(M_{i-1} + M_{n-i} \right) \\
&= n - 1 + \frac{1}{n} \left(\sum_{i=1}^{n} M_{i-1} + \sum_{i=1}^{n} M_{n-i} \right) \\
&= n - 1 + \frac{2}{n} \sum_{j=0}^{n-1} M_j .
\end{aligned}$$

Using base case $M_0 = 0$,

i	1	2	3	4	5	6	7	8
M_i	0	1	$8/3$	$29/6$	$37/5$	$103/10$	$472/35$	$2369/140$

3. Let C denote the number of comparisons and let R denote the number of remaining values in the non-empty pile when the other is first emptied. Suppose n is positive.

   Every comparison corresponds to the removal of one value whilst both piles are non-empty. Given there are n values in total, and R remaining values upon the first emptying of a pile, there must have been $n - R$ comparisons.

   It is clear that the remaining values is some maximal suffix of the sorted sequence. Hence, R is the number of sequential values, in descending order from the largest value, that were allocated to the same pile. The largest value always remains in the non-empty pile, hence $R \geq 1$, and, noting that all values were equally likely to be have been initially placed in either pile,

$$P(R \geq i) = \left(\frac{1}{2} \right)^{i-1}$$

   for $1 \leq i \leq n$.

   Using $E[X] = \sum_{i=1}^{\infty} P(X \geq i)$ for a non-negative random variable X,

$$\begin{aligned}
E[C] &= E[n - R] \\
&= n - \sum_{i=1}^{\infty} P(R \geq i) \\
&= n - \sum_{i=1}^{n} \left(\frac{1}{2} \right)^{i-1} \\
&= n - 2 + \left(\frac{1}{2} \right)^{n-1} .
\end{aligned}$$

4. Define $Q_{i,j} = 1 - P_i - P_j$, the probability that neither e_i nor e_j are requested at a particular time.

   Let $I_{i,j}$ indicate the event that element e_i precedes e_j at time n. This event can occur if

   - e_i initially preceded e_j and neither e_i nor e_j were selected, which happens with probability $\frac{1}{2} \cdot Q_{i,j}^{n-1}$ since each element is equally likely to have preceded the other in the initial ordering; or

- e_j is not selected after the last time e_i is selected. Conditioning on the time of the last selection of e_i, this occurs with probability $\sum_{k=1}^{n-1} P_i \cdot Q_{i,j}^{n-k-1}$.

Let R_i denote the position of e_i at time n.

$$R_i = 1 + \sum_{j \neq i} I_{j,i}$$

Let R denote the position of the element requested at time n. Conditioning on the element requested at time n, its expected value is

$$\begin{aligned}
E[R] &= \sum_i P_i E[R_i] \\
&= \sum_i P_i \left(1 + \sum_{j \neq i} E[I_{j,i}] \right) \\
&= \sum_i P_i + \sum_i P_i \sum_{j \neq i} \left(\frac{1}{2} \cdot Q_{i,j}^{n-1} + \sum_{k=1}^{n-1} P_j \cdot Q_{i,j}^{n-k-1} \right) \\
&= 1 + \sum_i \sum_{j \neq i} \frac{1}{2} \cdot P_i \cdot Q_{i,j}^{n-1} + \sum_i \sum_{j \neq i} \sum_{k=1}^{n-1} P_i \cdot P_j \cdot Q_{i,j}^{n-k-1}.
\end{aligned}$$

5. Let $p_{i,j}$ denote the probability that e_i precedes e_j, and R the position of the requested element. Given the requested element, its position is 1 more the number of elements that precede it. Hence, conditioning on the requested element, the expected position of the requested element is

$$\begin{aligned}
E[R] &= 1 + \sum_i P_i \sum_{j \neq i} p_{j,i} \\
&= 1 + \sum_i \sum_{j<i} (p_{j,i} P_i + p_{i,j} P_j) \\
&= 1 + \sum_i \sum_{j<i} ((1 - p_{i,j}) P_i + p_{i,j} P_j) \\
&= 1 + \sum_i \sum_{j<i} (P_i + p_{i,j}(P_j - P_i)).
\end{aligned}$$

The expected position can be minimised by $p_{i,j} = 1$ where $P_i \geq P_j$; that is, ordering the elements in decreasing order of their probabilities of being selected.

6. Let F denote the number of fixed points and, for $1 \leq i \leq n$, let I_i indicate that position i is a fixed point of the random permutation. Then,

$$F = I_1 + ... + I_n.$$

The expected value of the Bernoulli random variable I_i is $\frac{1}{n}$, so the expected number of fixed points is

$$E[F] = \sum_{i=1}^{n} E[I_i] = n \cdot \frac{1}{n} = 1.$$

Let $p_{i,j}$ denote the joint probability mass function of I_i and I_j, for $i \neq j$. It is clear that the function is symmetric; $p_{i,j}(a, b) = p_{i,j}(b, a)$.

The covariance of I_i and I_j, for $i \neq j$, is

$$\begin{aligned}
\mathrm{Cov}(I_i, I_j) &= E[(I_i - E[I_i])(I_j - E[I_j])] \\
&= \sum_{x=0}^{1}\sum_{y=0}^{1} p_{i,j}(x, y) \cdot \left(x - \frac{1}{n}\right)\left(y - \frac{1}{n}\right) \\
&= p_{i,j}(0, 0) \cdot \frac{1}{n^2} + 2p_{i,j}(1, 0) \cdot \frac{1-n}{n^2} + p_{i,j}(1, 1) \cdot \frac{(n-1)^2}{n^2} \\
&= \left(\frac{1}{n} + \frac{n-2}{n}\frac{n-2}{n-1}\right) \cdot \frac{1}{n^2} + 2\left(\frac{1}{n}\frac{n-2}{n-1}\right) \cdot \frac{1-n}{n^2} + \left(\frac{1}{n}\frac{1}{n-1}\right) \cdot \frac{(n-1)^2}{n^2} \\
&= \frac{n^2 - 3n + 3}{n^3(n-1)} + \frac{-2n^2 + 6n - 4}{n^3(n-1)} + \frac{n^2 - 2n + 1}{n^3(n-1)} \\
&= \frac{1}{n^2(n-1)}.
\end{aligned}$$

Hence, given also that the variance of the Bernoulli random variable I_i is $\frac{1}{n}\left(1 - \frac{1}{n}\right) = \frac{n-1}{n^2}$, the variance of the number of fixed points is

$$\begin{aligned}
\mathrm{Var}(F) &= \sum_{i=1}^{n} \mathrm{Var}(I_i) + \sum_{i}\sum_{j \neq i} \mathrm{Cov}(I_i, I_j) \\
&= n \cdot \frac{n-1}{n^2} + n(n-1) \cdot \frac{1}{n^2(n-1)} \\
&= 1 - \frac{1}{n} + \frac{1}{n} \\
&= 1.
\end{aligned}$$

An alternative method of calculating the variance computes the expected value of F^2.

$$\begin{aligned}
E[F^2] &= E\left[\left(\sum_{i=1}^{n} I_i\right)^2\right] \\
&= E\left[\sum_{i=1}^{n} I_i^2\right] + E\left[\sum_{i=1}^{n}\sum_{j \neq i}^{n} I_i I_j\right] \\
&= \sum_{i=1}^{n} E[I_i^2] + \sum_{i=1}^{n}\sum_{j \neq i}^{n} E[I_i I_j] \\
&= \sum_{i=1}^{n} E[I_i] + \sum_{i=1}^{n}\sum_{j \neq i}^{n} p_{i,j}(1, 1) \\
&= n \cdot \frac{1}{n} + n(n-1) \cdot \left(\frac{1}{n}\frac{1}{n-1}\right) \\
&= 2
\end{aligned}$$

Then the variance is $\mathrm{Var}(F) = E[F^2] - E[F]^2 = 2 - 1^2 = 1$.

7. Let $j_1, ..., j_n$ be an arbitrary permutation of $1, ..., n$.
   Supposing X is a random permutation, the probability of
   $$X(j_1) = 1, ..., X(j_n) = n \tag{1}$$
   is $1/n!$. Equation (1) may be restated
   $$j_1 = X^{-1}(1), ..., j_n = X^{-1}(n)$$
   so
   $$P\left\{(X^{-1}(1), ..., X^{-1}(n)) = (j_1, ..., j_n)\right\} = \frac{1}{n!}.$$

8. Let P denote the number of matched pairs, and let I_i for $0 < i \leq n$ indicate that i belongs to a matched pair.
   $$P = \frac{I_1 + ... + I_n}{2}$$
   (a) If there is a single person, $n = 1$, there are no matched pairs, $P = 0$.
   Otherwise, supposing $n > 1$, the expected value of I_i is the probability that i belongs to a matched pair,
   $$E[I_i] = \frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n},$$
   and the expected value of P is
   $$\frac{1}{2}\sum_{i=1}^{n} I_i = \frac{1}{2}\left(n \cdot \frac{1}{n}\right) = \frac{1}{2}.$$
   Hence,
   $$E[P] = \begin{cases} 0 & \text{if } n = 1 \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$
   (b) For $i \neq j$, the probability i and j belong to the same matched pair (i.e., i chooses j's hat and j chooses i's hat) is
   $$P(I_i = 1, I_j = 1, i \text{ and } j \text{ matched pair}) = \frac{1}{n}\frac{1}{n-1}.$$
   The probability that both i and j belong to matched pairs, but not the same, depends on n. If $n < 4$, it is not possible for i and j to belong to distinct pairs. Otherwise, if $n \geq 4$, it is the probability that i matches with some other than j, with probability $\frac{n-2}{n}\frac{1}{n-1}$, and j belongs to a matched pair amongst the remaining, with probability $\frac{n-3}{n-2}\frac{1}{n-3}$. Putting that together,
   $$P(I_i = 1, I_j = 1, i \text{ and } j \text{ not matched pair}) = \begin{cases} 0 & \text{if } 1 < n < 4 \\ \frac{1}{n(n-1)} & \text{otherwise} \end{cases}.$$
   Hence, for $i \neq j$,
   $$P(I_i = 1, I_j = 1) = \begin{cases} \frac{1}{n(n-1)} & \text{if } 1 < n < 4 \\ \frac{2}{n(n-1)} & \text{otherwise} \end{cases}.$$

If there is a single person, $n=1$, there is P has zero variance since $P=0$. Otherwise, supposing $n>1$, the variance of P is

$$\begin{aligned} E[P^2]-E[P]^2 &= E\left[\left(\frac{I_1+...+I_n}{2}\right)^2\right]-E[P]^2 \\ &= \frac{1}{4}\left(\sum_{i=1}^{n} E[I_i^2]+\sum_i\sum_{j\neq i} E[I_iI_j]\right)-\left(\frac{1}{2}\right)^2 \\ &= \frac{1}{4}\left(n\cdot\frac{1}{n}+\sum_i\sum_{j\neq i} P(I_i=1,I_j=1)\right)-\frac{1}{4} \\ &= \frac{1}{4}\sum_i\sum_{j\neq i} P(I_i=1,I_j=1). \end{aligned}$$

Hence,

$$\mathrm{Var}(P)=\begin{cases} 0 & \text{if } n=1 \\ \frac{1}{4} & \text{if } 1<n<4 \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

(c) Let C denote the length of the cycle containing an arbitrarily chosen value, say the first. The predicate there are no pairs is equivalent to the predicate there are no cycles of length 2. Then, defining $S_n=\{x\in\mathbb{Z}: x\neq 2, 1\leq x\leq n\}$, for $n>0$

$$\begin{aligned} Q_n &= \sum_{i\in S_n} P(C=i)\cdot Q_{n-i} \\ &= \frac{1}{n}\sum_{i\in S_n} Q_{n-i}. \end{aligned}$$

Using base case $Q_0=0$,

i	1	2	3	4	5	6	7	8
Q_i	1	$1/2$	$1/2$	$5/8$	$5/8$	$29/48$	$29/48$	$233/384$

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9. Let C_1 denote the size of the cycle that contains 1.

(a) With base case $M_0=0$,

$$\begin{aligned} M_n &= E[N] \\ &= \sum_{i=1}^{n} E[N|C_1=i]\cdot P(C_1=i) \\ &= \sum_{i=1}^{n}(1+M_{n-i})\cdot\frac{1}{n} \\ &= 1+\frac{1}{n}\sum_{j=0}^{n-1} M_j. \end{aligned}$$

(b) There are i values in a cycle of size i, and $i \cdot \frac{1}{i} = 1$, so the term in the summation corresponding to a given value contributes that value's part in its cycle.

$$\begin{aligned}
E[N] &= \sum_{j=1}^{n} E\left[\frac{1}{C_j}\right] \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} P(C_j = i) \cdot \frac{1}{i} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{i} \\
&= \sum_{i=1}^{n} \frac{1}{i}
\end{aligned}$$

This result can be confirmed by induction to be the solution to the recurrence relation (9a).
When $n = 0$,

$$M_0 = \sum_{j=0}^{-1} \frac{1}{j} = 0$$

as required.
Suppose $M_i = \sum_{j=1}^{i} \frac{1}{j}$ for $0 \le i \le k$ and arbitrary integer k. The below shows that $M_{k+1} = \sum_{i=1}^{k+1} \frac{1}{i}$.

$$\begin{aligned}
M_{k+1} &= 1 + \frac{1}{k+1} \sum_{i=0}^{k} M_i \\
&= 1 + \frac{1}{k+1} \sum_{i=1}^{k}\sum_{j=1}^{i} \frac{1}{j} \\
&= 1 + \frac{1}{k+1} \sum_{j=1}^{k}\sum_{i=j}^{k} \frac{1}{j} \\
&= 1 + \frac{1}{k+1} \sum_{j=1}^{k} \frac{k-j+1}{j} \\
&= 1 + \frac{1}{k+1} \sum_{j=1}^{k} \left(-1 + \frac{k+1}{j}\right) \\
&= 1 - \frac{k}{k+1} + \sum_{j=1}^{k} \frac{1}{j} \\
&= \sum_{j=1}^{k+1} \frac{1}{j}
\end{aligned}$$

(c) Let D denote the event that $1, 2, ..., k$ are all in the same cycle, supposing $1 \le k \le n$.

If D occurs, the size of the cycle containing 1 must be at least k. Hence, conditioning on C_1 and using the relationship

$$\binom{i}{j-1} + \binom{i}{j} = \binom{i+1}{j},$$

$$\begin{aligned}
P(D) &= \sum_{x=k}^{n} P(C_1 = x) P(D|C_1 = x) \\
&= \sum_{x=k}^{n} \frac{1}{n} \left(\frac{x-1}{n-1} \cdot \ldots \cdot \frac{x-(k-1)}{n-(k-1)} \right) \\
&= \frac{(n-k)!}{n!} \sum_{x=k}^{n} \frac{(x-1)!}{(x-k)!} \\
&= \frac{(n-k)!}{n!} \sum_{x=k}^{n} \binom{x-1}{x-k} (k-1)! \\
&= \frac{k!}{k} \frac{(n-k)!}{n!} \sum_{y=0}^{n-k} \binom{y+k-1}{y} \\
&= \frac{1}{k} \binom{n}{k}^{-1} \left[\sum_{y=0}^{n-k} \binom{y+k-1}{y} \right] \\
&= \frac{1}{k} \binom{n}{k}^{-1} \left[\binom{0+k-1}{0} + \sum_{y=1}^{n-k} \binom{y+k-1}{y} \right] \\
&= \frac{1}{k} \binom{n}{k}^{-1} \left[1 + \sum_{y=1}^{n-k} \left(\binom{y+k}{y} - \binom{y-1+k}{y-1} \right) \right] \\
&= \frac{1}{k} \binom{n}{k}^{-1} \left[1 + \sum_{y=1}^{n-k} \binom{y+k}{y} - \sum_{y=0}^{n-k-1} \binom{y+k}{y} \right] \\
&= \frac{1}{k} \binom{n}{k}^{-1} \left[1 + \binom{n-k+k}{n-k} - \binom{0+k}{0} \right] \\
&= \frac{1}{k} \binom{n}{k}^{-1} \left[1 + \binom{n}{k} - 1 \right] \\
&= \frac{1}{k}.
\end{aligned}$$

(d) Let E denote the event that $1, 2, ..., k$ is a cycle, supposing $1 \le k \le n$.
The probability of this event is the probability 1 is in a cycle of size k, $P(C_1 = k) = \frac{1}{n}$; and the other values in that cycle are $2, ..., k$, which denotes one of the $\binom{n-1}{k-1}$ possible cycles of size k.

Putting this together,

$$\begin{aligned}
P(E) &= \frac{1}{n}\binom{n-1}{k-1}^{-1} \\
&= \frac{(k-1)!(n-k)!}{n(n-1)!} \\
&= \frac{k!}{k}\frac{(n-k)!}{n!} \\
&= \frac{1}{k}\binom{n}{k}^{-1}.
\end{aligned}$$

10. Let M denote the number of maximal increasing subsequences in a random permutation of n values. Let I_j indicate that j is the beginning of a maximal subsequence for $1 \leq j \leq n$.

    The probability that j begins a maximal increasing subsequence is independent of whether k, for $k > j$, also begins a maximal increasing subsequence.

    Hence, using the independence of I_j and I_k for $j \neq k$,

$$\begin{aligned}
\mathrm{Var}(M) &= \mathrm{Var}\left(\sum_j I_j\right) \\
&= \sum_j \mathrm{Var}(I_j) \\
&= \sum_j \left(E(I_j^2) - E(I_j)^2\right) \\
&= \sum_j \left(\frac{1}{j} - \left(\frac{1}{j}\right)^2\right) \\
&= \sum_j \frac{j-1}{j^2}.
\end{aligned}$$

11. Let I_i, for $1 \leq i < n$ indicate that $X(i) < X(i+1)$ for a random permutation X. Then, the number of rises in a random permutation of $n > 0$ values is

$$N = \sum_{i=1}^{n-1} I_i.$$

    (a)

$$\begin{aligned}
E[N] &= \sum_{i=1}^{n-1} E[I_i] \\
&= \sum_{i=1}^{n-1} P(X(i) < X(i+1)) \\
&= \sum_{i=1}^{n-1} \frac{1}{2} \\
&= \frac{n-1}{2}
\end{aligned}$$

(b) It is clear that a random permutation of $n < 2$ elements has no variance in the number of rises since there are none. Consider instead the number of the rises in a random permutation of $n \geq 2$ elements.

A random permutation with consecutive rises at a given position has three numbers in increasing order, of which there are $\binom{n}{3}$ choices, with the remaining $n-3$ elements occurring in any order. Hence, counting the number of permutations with consecutive rises at a given position, for $i < n-1$,

$$P(I_i = 1, I_{i+1} = 1) = \frac{\binom{n}{3}(n-3)!}{n!} = \frac{1}{6}.$$

A random permutation with rises at two given non-adjacent positions has a pair of increasing numbers at the first position, of which there are $\binom{n}{2}$ choices, with another pair at the second position, of which there are $\binom{n-2}{2}$ choices, and the remaining $n-4$ numbers occurring in any order. Hence, counting the number of permutations with rises at two given non-adjacent positions, for $j > i+1$,

$$P(I_i = 1, I_j = 1) = \frac{\binom{n}{2}\binom{n-2}{2}(n-4)!}{n!} = \frac{1}{4}.$$

Hence, for $n \geq 2$,

$$\begin{aligned}
E[N^2] &= E\left[\left(\sum_{i=1}^{n-1} I_i\right)^2\right] \\
&= E\left[\sum_{i=1}^{n-1} I_i^2 \; + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n-1} I_i I_j\right] \\
&= \sum_{i=1}^{n-1} P(I_i = 1) + 2\sum_{i=1}^{n-1}\sum_{j>i}^{n-1} P(I_i = 1, I_j = 1) \\
&= \frac{n-1}{2} + 2\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} P(I_i = 1, I_j = 1) \\
&= \frac{n-1}{2} + 2\sum_{i=1}^{n-2} P(I_i = 1, I_{i+1} = 1) + 2\sum_{i=1}^{n-2}\sum_{j=i+2}^{n-1} P(I_i = 1, I_j = 1) \\
&= \frac{n-1}{2} + 2(n-2)\cdot\frac{1}{6} + (n-2)(n-3)\cdot\frac{1}{4}
\end{aligned}$$

so

$$\begin{aligned}
\mathrm{Var}(N) &= E[N^2] - E[N]^2 \\
&= \frac{n-1}{2} + \frac{n-2}{3} + \frac{(n-2)(n-3)}{4} - \left(\frac{n-1}{2}\right)^2 \\
&= \frac{n+1}{12}.
\end{aligned}$$

12. Letting

$$I_{i,j} = \begin{cases} 1 & \text{if } p_i < p_j \\ 0 & \text{otherwise} \end{cases},$$

define the position of i in the weighted random permutation $p_i = 1 + \sum_{j \neq i} I_{j,i}$.

(a)

$$\begin{aligned} E[p_i] &= 1 + \sum_{j \neq i} E[I_{j,i}] \\ &= 1 + \sum_{j \neq i} P(I_{j,i} = 1) \\ &= 1 + \sum_{j \neq i} \frac{\lambda_j}{\lambda_i + \lambda_j}. \end{aligned}$$

(b) Given, for $i \neq j, k$ and $j \neq k$,

$$\begin{aligned} P(I_{j,i} = 1, I_{k,i} = 1) &= P(I_{j,i} = 1, I_{k,i} = 1, I_{j,k} = 1) + P(I_{j,i} = 1, I_{k,i} = 1, I_{j,k} = 0) \\ &= \frac{\lambda_j}{\lambda_i + \lambda_j + \lambda_k} \cdot \frac{\lambda_k}{\lambda_i + \lambda_k} + \frac{\lambda_k}{\lambda_i + \lambda_j + \lambda_k} \cdot \frac{\lambda_j}{\lambda_i + \lambda_j} \\ &= \frac{\lambda_j \lambda_k}{\lambda_i + \lambda_j + \lambda_k} \left(\frac{1}{\lambda_i + \lambda_k} + \frac{1}{\lambda_i + \lambda_j} \right), \end{aligned}$$

we have

$$\begin{aligned} & \sum_{\substack{j,k \neq i \\ k \neq j}} \left(E\left[I_{j,i} I_{k,i}\right] - E\left[I_{j,i}\right] E\left[I_{k,i}\right] \right) \\ = \; & \sum_{\substack{j,k \neq i \\ k \neq j}} \left(\frac{\lambda_j \lambda_k}{\lambda_i + \lambda_j + \lambda_k} \left(\frac{1}{\lambda_i + \lambda_k} + \frac{1}{\lambda_i + \lambda_j} \right) - \frac{\lambda_j}{\lambda_i + \lambda_j} \cdot \frac{\lambda_k}{\lambda_i + \lambda_k} \right) \\ = \; & \sum_{\substack{j,k \neq i \\ k \neq j}} \left(\frac{\lambda_j \lambda_k}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)} \left(\frac{2\lambda_i + \lambda_j + \lambda_k}{\lambda_i + \lambda_j + \lambda_k} - 1 \right) \right) \\ = \; & \lambda_i \sum_{\substack{j,k \neq i \\ k \neq j}} \frac{\lambda_j \lambda_k}{(\lambda_i + \lambda_j)(\lambda_i + \lambda_k)(\lambda_i + \lambda_j + \lambda_k)} \end{aligned}$$

and, together with

$$\begin{aligned} \sum_{j \neq i} \left(E\left[I_{j,i}^2\right] - E\left[I_{j,i}\right]^2 \right) &= \sum_{j \neq i} \left(E\left[I_{j,i}\right] - E\left[I_{j,i}\right]^2 \right) \\ &= \sum_{j \neq i} \left(E\left[I_{j,i}\right] \left(1 - E\left[I_{j,i}\right]\right) \right) \\ &= \sum_{j \neq i} \frac{\lambda_j}{\lambda_i + \lambda_j} \cdot \frac{\lambda_i}{\lambda_i + \lambda_j} \\ &= \lambda_i \sum_{j \neq i} \frac{\lambda_j}{\left(\lambda_i + \lambda_j\right)^2}, \end{aligned}$$

we have

$$\begin{aligned}
\mathrm{Var}(p_i) &= \mathrm{Var}(p_i - 1) \\
&= \mathrm{Var}\left(\sum_{j\neq i} I_{j,i}\right) \\
&= E\left[\left(\sum_{j\neq i} I_{j,i}\right)^2\right] - E\left[\sum_{j\neq i} I_{j,i}\right]^2 \\
&= \sum_{\substack{j,k\neq i\\ k\neq j}} \left(E\left[I_{j,i}I_{k,i}\right] - E\left[I_{j,i}\right]E\left[I_{k,i}\right]\right) + \sum_{j\neq i}\left(E\left[I_{j,i}^2\right] - E\left[I_{j,i}\right]^2\right) \\
&= \lambda_i \sum_{j\neq i} \frac{\lambda_j}{\lambda_i+\lambda_j}\left(\frac{1}{\lambda_i+\lambda_j} + \sum_{k\neq i,j}\frac{\lambda_k}{(\lambda_i+\lambda_k)(\lambda_i+\lambda_j+\lambda_k)}\right) [1].
\end{aligned}$$

References

[1] joriki (https://math.stackexchange.com/users/6622/joriki). Variance of weighted random permutations. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/4846976 (version: 2024-01-18).