# Modelling neutron star mass and pressure using coupled ODEs


Awwab Butt, ID: 11091269

School of Physics and Astronomy
The University of Manchester

Second Year Theory of Computing Project Phys20872
26 April 2024


**Abstract**

This reports models the mass and pressure of neutron stars as a pair of coupled ODEs, and solves them numerically for a various equations of state (EOS) corresponding to different models of the star to find the mass and radius for given boundary conditions. The effects of the effects Tolman–Oppenheimer–Volkoff relativistic corrections on the Newtonian structure ODEs was also investigated and found to be significant for the mass and density ranges of neutron stars. The maximum mass for a pure neutron star modeled as an ideal Fermi gas of degenerate neutrons was found to be: $0.822\,M_{\odot}$ with $R = 8.2 \cdot 10^5\,cm$, which is approximately consistent with the literature [3] [6, p. 16] with values of: $0.71\,M_{\odot}$ with $R = 9.5 \cdot 10^5\,cm$ and $0.712\,M_{\odot}$ with $R = 9.14 \cdot 10^5\,cm$ respectively. For a pure neutron star modeled using the empirical nuclear interaction model the maximum mass was a found to be: $1.932\,M_{\odot}$ with $R = 1.065 \cdot 10^6\,cm$, which is consistent with mass bounds from observation and theory found in the literature [2] [5] with bounds of: $2.2\,M_{\odot} < M < 2.7\,M_{\odot}$ and $2.01^{+0.04}_{-0.04} < M/M_{\odot} < 2.16^{+0.17}_{-.015}$ respectively.

# 1 Introduction

Neutron stars are the massive stellar remnants of enormous stars. They are formed when stars with masses of $M > 8\,M_\odot$ at the end of their life span undergo a supernova, a result of iron fusion not providing a positive energy release because of the relative nuclear binding energies [4, p.1-7]. This results in it becoming unable to support hydro-static equilibrium resulting in a gravity causing the star to implode. This super heats and pressurises the core leading electron-capture and a rapid release of neutrinos and energy resulting in a shock wave causing the star to violently expel its outer shell leaving only the core, either a neutron star or black hole [1, p. 2]. Neutron stars are made up of mostly neutrons and are prevented from further collapsing by neutron degeneracy pressure keeping it in hydro-static equilibrium, a consequence o the Pauli exclusion principle. The maximum mass these stars can have is called the Tolman-Oppenheimer-Volkov limit, and is estimated to be $2.01^{+0.04}_{-0.04} < M/M_\odot < 2.16^{+0.17}_{-.015}$ in a recent paper [5].

The aim of this report is to numerically solve the coupled ODEs describing the pressure and mass of neutron stars as functions of distance $r$ from the centre given some boundary conditions. The solution depends on the model used to the model the neutron star but, all of the models for the neutron star used in this report use the assumption that the star is stationary, non-rotating and in a stable hydro-static equilibrium.

In this report we also discuss general relativistic correction factors to the coupled structure ODEs called the Tolman–Oppenheimer–Volkoff equation and how this compares with the uncorrected versions.

# 2 Theory

## 2.1 Solving coupled ODEs numerically

Often ODEs are difficult, or impossible to solve analytically therefore it is important to be able to solve them numerically. The method used in this report is the Runge–Kutta method to the $4^{th}$ order (RK4), as this provides a good balance between the accuracy and computational complexity when compared to other methods like Euler's method. RK4 has an error of $\mathcal{O}(h^4)$ compared to Euler's method with $\mathcal{O}(h)$ where $h$ is the step size [9, p. 102, 106] . Note that we dealt with the singularities by approximating it as the RK4 step at a small positive positive increment $dx$ from 0, the value used here was $1 \cdot 10^{-12}$.

## 2.2 Root finding for inverse functions

It can difficult to find the inverse of a function analytically therefore, we use a root finding algorithm to find them numerically. This report uses the Newton-Raphson method.

## 2.3 Constants

Constants were sourced from the Particle Data Group constants database [10]. The nucleon mass $m_N$ was defined as average of the proton and neutron rest masses.

## 2.4 Newtonian Structure equations

From Newton's law of gravity and the hydro-static equilibrium condition, we can derive the coupled Newtonian structure equations for pressure $p$ and mass $m$ as functions of the distance from the centre $r$, for a stable non-rotating star.

$$\frac{dp}{dr} = \frac{-G\epsilon(r)m(r)}{r^2c^2}. \tag{1}$$

$$\frac{dm}{dr} = \frac{4\pi r^2\epsilon(r)}{c^2}. \tag{2}$$

Where $c$ is the speed of light, $G$ is the gravitational constant and $\epsilon(r)$ is the energy density, from the energy mass relation $\rho(r) = \frac{\epsilon(r)}{c^2}$.

These coupled ODEs can then be solved numerically for the boundary conditions $p(0) = p_0$, $m(0) = 0$ where $p_0$ is referred to as the central pressure. The distance $r$ for which the $p$ is less than or equal to 0 corresponds to the radius $R$ and Mass $(m(r))$ for a given central pressure $p_0$.

### 2.4.1 Tolman-Oppenheimer-Volkoff correction

In the high mass 'short' distance distance regime of a neutron star's internal structures, general relativistic effects have a very significant effect especially as the density increases. Therefore general relativistic corrections need to be applied to the Newtonian equations for pressure (1) to model them accurately. These were originally derived by Volkoff and Oppenheimer in the 1930s [8].

$$\frac{dp}{dr} = \frac{-G\epsilon(r)m(r)}{r^2c^2}\left[1 + \frac{\epsilon(r)}{p(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}. \tag{3}$$

The coupled mass equation (2) however remains unchanged.

### 2.4.2 Modelling Stars as idealised Fermi gasses

A stationary non-rotating neutron dwarf's matter can be treated as an Idealised Fermi gas of degenerate neutrons [6, p.14]. The total energy density $\epsilon$ is:

$$\epsilon = \epsilon_n(k_F) + n \cdot m_N c^2. \tag{4}$$

Where $n$ is the neutron number density, $m_N$ is the nucleon mass which is and $k_F$ is the Fermi momentum. The energy density contribution from the degenerate neutrons $\epsilon_{elec}$ and pressure $p$ of this Fermi gas can be written as functions of the normalised Fermi momentum $x$, $x = \frac{k_F}{m_e c}$.

$$\epsilon_n(x) = \frac{\epsilon_0}{8}\left[(2x^3 + x)(1 + x^2)^{\frac{1}{2}} - arcsinh(x)\right], \tag{5}$$

$$p(x) = \frac{\epsilon_0}{24}\left[(2x^3 - 3x)(1 + x^2)^{\frac{1}{2}} + 3 \cdot arcsinh(x)\right], \tag{6}$$

$$\epsilon_0 = \frac{m_n^5 c^4}{\pi^2 \hbar^3}. \tag{7}$$

### 2.5 Polytropic approximation

In the non-relativistic limit, $x << 1$, (6) can be approximated as a polytropic equation of state (EOS) of the form:

$$p(\epsilon) = K_{nonrel}\epsilon^{\frac{5}{3}}, \tag{8}$$

$$K_{nonrel} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_N c^2}\right). \tag{9}$$

Equations (6) and (5) can also be approximated as:

$$\epsilon_n = \frac{\epsilon_0}{3}x^3 \qquad p(x) = \frac{\epsilon_0}{15}x^5 \tag{10}$$

In the relativistic limit , $x >> 1$ (6) can be written as the following polytropic EOS:

$$p(\epsilon) = K_{rel}\epsilon^{\frac{4}{3}}, \tag{11}$$

$$K_{rel} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2}{m_N c^2}\right). \tag{12}$$

Using a general polytrope of the form $p = K\epsilon^\gamma$ as the EOS, the $\epsilon(r)$ dependence in the Newtonian equations (1), (2) can be replaced and a polytropic approximation form of the structure equations can be found:

$$\frac{dp(r)}{dr} = -\frac{R_0 p(r)^{\frac{1}{\gamma}}\bar{m}(r)}{r^2 K^{\frac{1}{\gamma}}}, \tag{13}$$

$$\frac{d\bar{m}(r)}{dr} = \frac{4\pi r^2}{c^2 M_\odot}\left(\frac{p(r)}{K}\right)^{\frac{1}{\gamma}}. \tag{14}$$

Here $R_0$ is half the Schwarzschild radius $\left(\frac{GM_\odot}{c^2}\right)$, $M_\odot$ is a solar-mass and and $\bar{m}$ is the dimensionless mass, solar mass ratio: $\bar{m} = \frac{m}{M_\odot}$.

## 2.6 Nuclear interactions

To model the nuclear interactions inside a neutron star the symmetric empirical interaction model described in [6, p.20-22]. For this model $c$ was set to 1 and units of fm and MeV are used. Symmetric meaning $n_p = n_n$ the number densities are the same. The energy density for this model is:

$$\frac{\epsilon(n)}{n} = m_N + \langle E_0 \rangle u^{2/3} + \frac{A}{2}u + \frac{B}{\sigma + 1}u^{\sigma}, \quad u = \frac{n}{n_0}. \tag{15}$$

The average ground state kinetic energy per nucleon is:

$$\langle K_0 \rangle = \frac{3}{10 m_N} \left( \frac{3\pi^2 \hbar^3 n_0}{2} \right)^{\frac{2}{3}} = 22.1\, MeV \tag{16}$$

The constants found are:

$$A = -118.2\, MeV,\ B = 65.38\, MeV,\ \sigma = 2.112,\ \langle E_0 \rangle = 22.1\, MeV\ S_0 = 20\, MeV, \tag{17}$$

$$S_0 = 20\, MeV, \quad n_0 = 0.16\, nucloens/fm^3. \tag{18}$$

$$p(n) = n_0 \left[ \frac{2}{3} \langle E_0 \rangle u^{5/3} + \frac{A}{2}u^2 + \frac{B\sigma}{\sigma + 1}u^{\sigma+1} \right] \tag{19}$$

$$\epsilon(n, 1) = n \left[ m_N + 2^{2/3} \langle K_0 \rangle u^{2/3} + \frac{A}{2}u + \frac{B}{1 + \sigma}u^{\sigma} + \left( S_0 - (2^{2/3} - 1)\langle E_0 \rangle \right) u \right] \tag{20}$$

$$p(n, \alpha) = p(n, 0) + n_0 \alpha \left[ \left( 2^{2/3} - 1 \right) \langle K_0 \rangle \left( \frac{2u^{\frac{5}{3}}}{2} - u^2 \right) + S_0 u^2 \right] \tag{21}$$

$$\alpha = \frac{N - Z}{A} \tag{22}$$

By setting $\alpha = 1$ the equations now correspond to a pure neutron star so using (20) and (21) and EoS for this pure neutron star with nuclear interactions can be calculated numerically. Then it can be used to numerically calculate the EOS and solve the coupled TOV ODEs to find the mass and radius.

## 2.7 Numerically finding EOS

The relativistic limits makes it easy to find an equation of state where $p$ or $\epsilon$ are functions of each other, however it is not possible to do for (5) and (6) or any other two complicated functions. However one can solve for the EOS numerically by performing a root finding algorithm on an equation of the form:

$$y = p(x) - p(r). \tag{23}$$

Here $x$ is not necessarily just the normalised Fermi momentum just a variable both $\epsilon$ and $p$ are functions of. Running a root finding algorithm on (23) for a given $p(r)$ yields the normalised Fermi momentum $x$ corresponding to the position $r$. This $x$ can be used to find the $\epsilon(x)$ for the position, r which corresponds to $\epsilon(r)$ which has effectively found the EoS numerically. This energy density $\epsilon(r)$ can then be used to solve the coupled ODEs.

# 3 Results and Analysis

## 3.1 Neutron Star

Bellow is a comparison of the TOV and Newtonian structure ODEs solutions in the polytropic non-relativistic limit for a range of central pressures $2 \cdot 10^{21}\, dyne/cm^2$ to $4 \cdot 10^{24}\, dyne/cm^2$. Evidently the Newtonian structure equations agree closely with the TOV solutions for small $p_0$ and begin to diverge as it gets higher and the density/mass increases. This is as expected as its know Newton's law of gravity is a good approximation of general relativity in the small mass regime, and this also highlights the importance of including the TOV corrections to model such a massive object accurately.

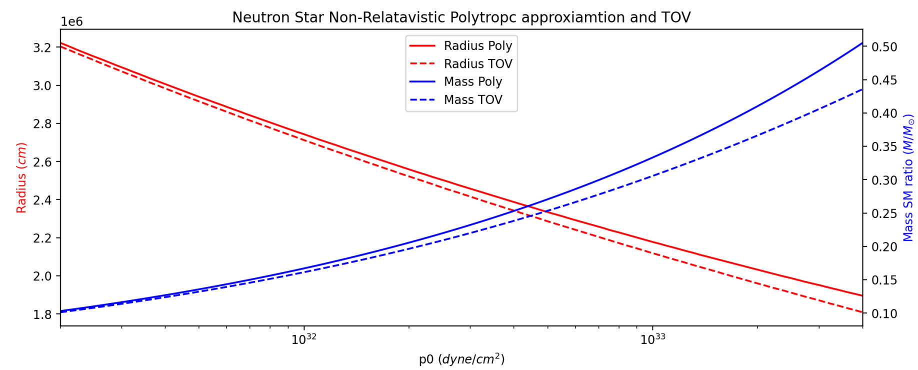


Figure 1: Comparison of the TOV and the Newtonian coupled equations over a range of $p_0$ of $2 \cdot 10^{21}\, dyne/cm^2$ to $4 \cdot 10^{24} dyne/cm^2$ for a pure neutron star.

Now the complete Fermi gas pressure and energy density equations (6), (5) for a neutron star and TOV ODEs were used to produce Fig. 2 and Fig. 3 for central pressures $p_0$ of between $1 \cdot 10^{20}\ dyne/cm^2$ to $1 \cdot 10^{40}\ dyne/cm^2$. The complete Fermi gas EOS equations are valid in the nor and relativistic limits as they don't rely on the polytropic approximation and the EOS is solved numerically. The shape of the graphs are in line with expectations from the literature [6] [7]. The maximum mass that was calculated was $0.822\, M_{\odot}$, with a radius of $8.2 \cdot 10^5\, cm$, with a central pressure of $5.462 \cdot 10^{35}\, dyne/cm^2$. These are approximately in agreement with values in the literature [6] which also references the original Oppenheimer Volkoff paper [3] with values of the maximum masses and radii of $0.712\, M_{\odot}$, $9.140 \cdot 10^5\, cm$,$0.71\, M_{\odot}$, $9.5 \cdot 10^5\, cm$. Although these values are roughly consistent there is a significant discrepancy.

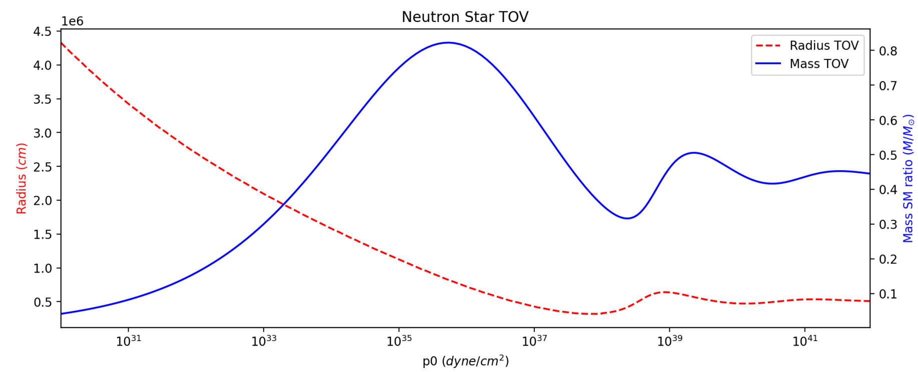


Figure 2: Pure neutron neutron star mass and radius against central pressure for a range of $1 \cdot 10^{20}\, dyne/cm^2$ to $1 \cdot 10^{40}\, dyne/cm^2$, calculated using TOV and the complete Fermi gas equations.

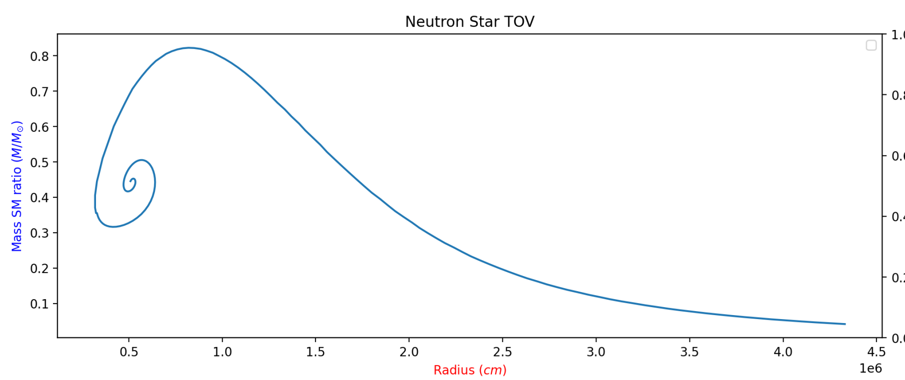


Figure 3: Mass against radius of pure neutron stars with EOS calculated numerically using the complete Fermi gas equations and with the TOV corrections.

Lastly, the figure below represents the nuclear interaction model, using the equations (20) and (21) with $\alpha = 1$ corresponding to a pure neutron star for the EOS and uses the TOV ODEs. THe maximum mass found was: $1.932\, M_\odot$ with $R = 1.065 \cdot 10^6\, cm$. This is consistent with the literature [2] [5] with bounds of: $2.2\, M_\odot < M < 2.7\, M_\odot$ and $2.01^{+0.04}_{-0.04} < M/M_\odot < 2.16^{+0.17}_{-.015}$.

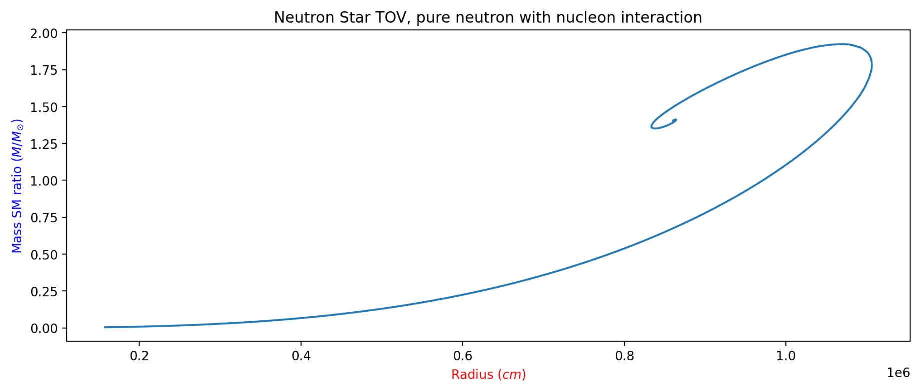


Figure 4: Mass against radius of pure neutron stars with EOS calculated numerically from the nuclear interaction equations with the TOV corrections.

# 4 Conclusion

In conclusion the findings in our report are approximately consistent with the literature for the maximum pure neutron Fermi gas star mass of $0.822\,M_{\odot}$, [6] has $0.712\,M_{\odot}$,[7] has $M \approx 0.8\,M_{\odot}$, [3] has $0.71\,M_{\odot}$. However, although the approximate shapes of the graphs are accurate, there are non-negligible discrepancies particularly from the [6]version and suggests that there is a systematic error in the implementation that is distorting it slightly.

There are many improvements that could be made to our report, for the purposes of this report the neutron star was approximated as a stationary pure idealised Fermi gas of neutrons, and used a simplified nuclear interaction model. Although simple, it did achieve a relatively good approximation for the maximum neutron star mass of $1.932\,M_{\odot}$ with $R = 1.065 \cdot 10^{6}\,cm$, which was consistent with modern estimates of mass bounds $2.01^{+0.04}_{-0.04} < M/M_{\odot} < 2.16^{+0.17}_{-.015}$ [5]. The nuclear interaction model was very simplified and could be updated to a more accurate one, like for example incorporating the contributions of protons and electrons present in a real neutron star. In reality it is highly unlikely that a neutron star forms with 0 angular momentum and often neutrons stars are spinning extremely fast and effects of rotation are significant, therefore its a good next step to model this.

# References


[1] P. Cerda-Duran, N. Elias-Rosa, L. Rezzolla, P. Pizzochero, D. I. Jones, N. Rea, and I. Vidaña. *Neutron Stars Formation and Core Collapse Supernovae*, pages 1–56. Springer International Publishing, Cham, 2018.

[2] V. Kalogera and G. Baym. The maximum mass of a neutron star. *Astrophys. J. Lett.*, 470:L61–L64, 1996.

[3] J. R. Oppenheimer and G. M. Volkoff. On massive neutron cores. *Phys. Rev.*, 55:374–381, Feb 1939.

[4] A. Y. Potekhin. The physics of neutron stars. *Physics-Uspekhi*, 53(12):1235–1256, Dec. 2010.

[5] L. Rezzolla, E. R. Most, and L. R. Weih. Using gravitational-wave observations and quasi-universal relations to constrain the maximum mass of neutron stars. *The Astrophysical Journal Letters*, 852(2):L25, Jan. 2018.

[6] I. Sagert, M. Hempel, C. Greiner, and J. Schaffner-Bielich. Compact stars for undergraduates. *European Journal of Physics*, 27(3):577–610, Apr. 2006.

[7] R. R. Silbar and S. Reddy. Neutron stars for undergraduates. *American Journal of Physics*, 72(7):892–905, June 2004.

[8] S. A. T. Stuart L. Shapiro. *General Relativity*, chapter 5, pages 106–126. John Wiley & Sons, Ltd, 1983.

[9] D. Walker. *Computational Physics.* Mercury Learning and Information, Berlin, Boston, 2022.

[10] R. L. Workman and Others. Review of Particle Physics. *PTEP*, 2022:083C01, 2022.