1. Problem Statement

Consider a general electrostatic problem with body force and traction applied on the boundary, whose strong form is given by:

Strong Form

$$\sigma_{ij,j} + b_i = \rho \ddot{u}_i \quad in\ \Omega$$

$$u_i = g_i \quad on\ \Gamma_g$$

$$\sigma_{ij} n_j = h_i \quad on\ \Gamma_h$$

With initial condition

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \mathbf{0}$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \mathbf{0}$$

$$\boldsymbol{a}(\boldsymbol{x}, 0) = \mathbf{0}$$

2. Finite Element Formulation

Test the strong form by test function, and integrate over the domain yields the weak form.

Weak Form

Find $u_i \in S \subset H^1$, $u_i = g_i$ on Γ_g, such that $\forall w_i \in V \subset H^1$, $w_i = 0$ on Γ_g

$$\int_\Omega \rho \ddot{u}_i w_i d\Omega + \int_\Omega w(i,j)\sigma_{ij} d\Omega = \int_\Omega w_i b_i d\Omega + \int_{\Gamma_h} w_i h_i d\Gamma$$

Where $u(k,l) = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)$, $\sigma_{ij} = C_{ijkl} u(k,l)$

Galerkin Form

Let u_i^h and w_i^h be the approximation of u_i and w_i respectively.

Find $u_i^h \in S^h \subset H^1$, $u_i^h = g_i$ on Γ_g, such that $\forall w_i^h \in V^h \subset H^1$, $w_i^h = 0$ on Γ_g

$$\int_\Omega \rho \ddot{u}_i^h w_i^h d\Omega + \int_\Omega w^h(i,j)\sigma_{ij}^h d\Omega = \int_\Omega w_i^h b_i d\Omega + \int_{\Gamma_h} w_i^h h_i d\Gamma$$

Where $u^h(k,l) = \frac{1}{2}\left(u_{k,l}^h + u_{l,k}^h\right)$, $\sigma_{ij}^h = C_{ijkl} u^h(k,l)$

For a sphere subjected to internal pressure, complexity of the problem is significantly reduced by spherical symmetric. By the symmetric of geometry and loading condition, material points of sphere could only deform in radial direction, and experiences no shear. Then,

$$u_1 = u_1(r)$$

$$u_2 = u_3 = 0$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial r}$$

$$\varepsilon_{22} = \varepsilon_{33} = \frac{u_1}{r}$$

$$\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

It can be seen that the original 3D problem is reduced to 1D, together with traction/natural boundary condition, galerkin formulation reduced to:

Find $u^h \in S^h \subset H^1$, such that $\forall w^h \in V^h \subset H^1$,

$$\int_\Omega \rho \ddot{u}_i^h w_i^h d\Omega + \int_\Omega w^h(i,j)\sigma_{ij}^h d\Omega = \int_{\Gamma_h} w_i^h h_i d\Gamma$$

Matrix Form

Discretize the domain into non-overlapping elements with equal length, i.e. Let $\Omega = \bigcup_e \Omega^e$. Hence the continuous integrals are discretized as:

$$\sum_e \int_{\Omega^e} \rho \ddot{u}_i^h w_i^h d\Omega + \sum_e \int_{\Omega^e} w^h(i,j)\sigma_{ij}^h d\Omega = \sum_e \int_{\Omega^e \cap \Gamma_h} w_i^h h_i d\Omega$$

In each element Ω^e, u^h and w^h are approximated by linear combination of two linear functions which exactly interpolate exact solution at nodal point:

$$u^h = \left[\frac{1}{2}(1-\xi), \frac{1}{2}(1+\xi)\right]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \boldsymbol{N}\boldsymbol{d}$$

Similarly,

$$w^h = \left[\frac{1}{2}(1-\xi), \frac{1}{2}(1+\xi)\right]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \boldsymbol{N}\boldsymbol{c}$$

$$u_{,x}^h = \boldsymbol{N}_{,x}\boldsymbol{d} = \boldsymbol{N}_{,\xi}\frac{\partial \xi}{\partial x}\boldsymbol{d} = \left[-\frac{1}{l^e}, \frac{1}{l^e}\right]\boldsymbol{d}$$

$$w_{,x}^h = \boldsymbol{N}_{,x}\boldsymbol{c} = \boldsymbol{N}_{,\xi}\frac{\partial \xi}{\partial x}\boldsymbol{c} = \left[-\frac{1}{l^e}, \frac{1}{l^e}\right]\boldsymbol{c}$$

$$\ddot{u}^h = \boldsymbol{N}\ddot{\boldsymbol{d}}$$

Then transfer the domain and boundary integrations to spherical coordinate, and plug matrix forms above yields:

$$\sum_e \int_{\Omega^e} \rho \boldsymbol{c}^{eT} \boldsymbol{N}^T \boldsymbol{N} \ddot{\boldsymbol{d}}^e d\Omega + \sum_e \int_{\Omega^e} w^h(i,j) \sigma_{ij}^h d\Omega = \sum_e \int_{\Omega^e \cap \Gamma_h} \boldsymbol{c}^{eT} \mathbf{N}^T \mathbf{h} \mathrm{d}\Omega$$

For isotropic linear elasticity, constitutive relation gives:

$$\sigma_{ij} = \boldsymbol{D}\boldsymbol{\varepsilon} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

Where indices i, j, k represents r, θ, ϕ respectively.

In this case, constitute relation is reduced to:

$$\sigma_{ij} = \boldsymbol{D}\boldsymbol{\varepsilon} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda \\ \lambda & \lambda+2\mu & \lambda \\ \lambda & \lambda & \lambda+2\mu \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial r} \\ \frac{u_1}{r} \\ \frac{u_1}{r} \end{bmatrix} = \boldsymbol{D}\boldsymbol{B}\boldsymbol{d}$$

Where $B = \begin{bmatrix} \boldsymbol{N}_{,r} \\ \frac{\boldsymbol{N}}{r} \\ \frac{\boldsymbol{N}}{r} \end{bmatrix} = \begin{bmatrix} -\frac{1}{l^e} & \frac{1}{l^e} \\ \frac{1}{2r}(1-\xi) & \frac{1}{2r}(1+\xi) \\ \frac{1}{2r}(1-\xi) & \frac{1}{2r}(1+\xi) \end{bmatrix}$

Plug this in, do the integration in spherical coordinate, and cancel out the common c^{eT} term gives:

$$\sum_e 4\pi \int_{\Omega^e} \rho \boldsymbol{N}^T \boldsymbol{N} r^2 dr\, \ddot{\boldsymbol{d}}^e + \sum_e 4\pi \int_{\Omega^e} B^T D B r^2 dr\, \boldsymbol{d}^e = \sum_e 4\pi \int_{\Omega^e \cap \Gamma_h} \mathbf{N}^T \boldsymbol{h} \mathrm{r}^2 \mathrm{dr}$$

By $dr = \frac{dr}{d\xi} d\xi = \frac{l^e}{2} d\xi$

$$\sum_e 2\pi\rho l^e \int_{\Omega^e} \boldsymbol{N}^T \boldsymbol{N} r^2 d\xi\, \ddot{\boldsymbol{d}}^e + \sum_e 2\pi l^e \int_{\Omega^e} B^T D B r^2 d\xi\, \boldsymbol{d}^e = \sum_e \boldsymbol{f}_e$$

Element Mass Matrix

$$\boldsymbol{M}^e = 2\pi\rho l^e \int_{\Omega^e} \boldsymbol{N}^T \boldsymbol{N} r^2 d\xi$$

Element Stiffness Matrix

$$\boldsymbol{K}^e = 2\pi l^e \int_{\Omega^e} B^T D B r^2 d\xi$$

Force Vector (Traction only applied on the inner surface)

$$\boldsymbol{f}_e = \begin{cases} 4\pi R_i^2 P \begin{bmatrix} 1 \\ 0 \end{bmatrix} & for\ m = 1 \\ 0 & otherwise \end{cases}$$

In the integral, r could be expressed in terms of ξ by isoparametric mapping

$$r = \boldsymbol{N}\boldsymbol{r}$$

Integrands of integrals are not polynomial, gauss quadrature cannot compute the integral exactly, but still applies. In this case, 3 points gauss quadrature is applied for numerical integration of element stiffness matrix and mass matrix:

$$\xi_1 = -\sqrt{\frac{3}{5}}, w_1 = \frac{5}{9}$$

$$\xi_2 = 0, w_2 = \frac{8}{9}$$

$$\xi_3 = \sqrt{\frac{3}{5}}, w_3 = \frac{5}{9}$$

After doing global assembly, we yield semi-discretized equation as follow:

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{K}\boldsymbol{d} = \boldsymbol{f}$$

3. Newmark Method

Newmark Method is applied to advance time forward, which gives as follow:

$$\boldsymbol{d}_{n+1} = \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}[(1-2\beta)\boldsymbol{a}_n + 2\beta \boldsymbol{a}_{n+1}]$$

$$\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \Delta t[(1-\gamma)\boldsymbol{a}_n + \gamma \boldsymbol{a}_{n+1}]$$

Plug these two equations into semi-discretize equation:

$$\boldsymbol{M}\boldsymbol{a}_{n+1} + \boldsymbol{K}\left\{\boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}[(1-2\beta)\boldsymbol{a}_n + 2\beta \boldsymbol{a}_{n+1}]\right\} = \boldsymbol{f}_{n+1}$$

Rearrange

$$(\boldsymbol{M} + \beta \Delta t^2 \boldsymbol{K})\boldsymbol{a}_{n+1} = \boldsymbol{f}_{n+1} - \boldsymbol{K}\left\{\boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}[(1-2\beta)\boldsymbol{a}_n]\right\}$$

$$\boldsymbol{M}^* \boldsymbol{a}_{n+1} = \boldsymbol{f}^*$$

Where

$$\boldsymbol{M}^* = \boldsymbol{M} + \beta \Delta t^2 \boldsymbol{K}$$

$$\boldsymbol{f}^* = \boldsymbol{f}_{n+1} - \boldsymbol{K}\left\{\boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}[(1-2\beta)\boldsymbol{a}_n]\right\}$$

Notice that, at the beginning of each time step, forcing term f_{n+1} is known from boundary condition, d_n, v_n and a_n are computed from previous time step. In other words, $\boldsymbol{M}^*$ and $\boldsymbol{M}^*$are all known. hence $\boldsymbol{a}_{n+1}$ could be directly obtained by solving system of linear equations directly by:

$$\boldsymbol{a}_{n+1} = \boldsymbol{M}^{*-1}\boldsymbol{f}^*$$

Then displacement d_{n+1} and velocity v_{n+1} at new time could be obtained directly by Newmark Method:

$$\boldsymbol{d}_{n+1} = \boldsymbol{d}_n + \Delta t \boldsymbol{v}_n + \frac{\Delta t^2}{2}[(1-2\beta)\boldsymbol{a}_n + 2\beta \boldsymbol{a}_{n+1}]$$

$$\boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \Delta t[(1-\gamma)\boldsymbol{a}_n + \gamma \boldsymbol{a}_{n+1}]$$

Initially, sphere is at rest, displacement and velocity at $t = 0$ are all 0, assume initial acceleration is also 0, then $d_n = v_n = a_n = 0$. With these initial conditions, we could marching time forward to solve the problem numerically at different time.

4. Stability and Error

Accuracy and stability of Newmark method depends on the choice of γ and β.

Table1. Parameters of Newmark family

Method	β	γ
Average Acceleration	1/4	1/2
Central Difference	0	1/2

As discussed in class, for undamped case, stability condition of Newmark is

Unconditionally Stable

$$\frac{1}{2} \leq \gamma \leq 2\beta$$

Conditionally Stable

$$\gamma \geq \frac{1}{2}$$

$$\beta < \frac{\gamma}{2}$$

$$\Omega_{crit} = \left(\frac{\gamma}{2} - \beta\right)^{-\frac{1}{2}}$$

For accuracy, Newmark method is of second order of accuracy if $\gamma = \frac{1}{2}$, and of first order of accuracy otherwise.

Stability condition and accuracy of two methods are summarized in the table below.

Table2. Stability and Accuracy of Newmark family

Method	Stability	Order of Accuracy
Average Acceleration	Unconditional	2
Central Difference	$\Omega_{crit} = 2$	2

Consider the case of central difference, critical time step is

$$\Delta t_{crit} = \frac{2}{\omega} = \frac{2}{\sqrt{\lambda_{max}}} \geq \frac{2}{\sqrt{\lambda^e_{max}}}$$

Where λ^e_{max} is the largest eigenvalue at element level, to save the computation power of computing eigenvalues of large system.

5. Numerical Result and Discussion

With all the obtained nodal displacements, displacements and stresses at all locations in the domain at all times could be found by interpolation, which in parametric domain is:

$$d = \boldsymbol{N}\boldsymbol{d}|_{\xi=\xi_{interp}}$$

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{B}\boldsymbol{d}|_{\xi=\xi_{interp}}$$

Relation between a point within ith element and that in parametric domain is given by:

$$\frac{x - x_i}{x_{i+1} - x_i} = \frac{\xi - (-1)}{1 - (-1)}$$

$$\frac{x - x_i}{h} = \frac{\xi + 1}{2}$$

Where h is element size.

In this report, numerical results at $r = 13,15,17$ are presented.

5.1. Stability of Central Difference and Average Acceleration Method

Theory shows that central difference method is conditional stable, average acceleration method is unconditionally stable. Parameters of numerical study on this is summarized in the table below.

Table3. Parameter of numerical study on stability.

Method	Number of Elements	Time Step	Test Point (r)
Central Difference	80	$1.02\Delta t_{crit}$, $0.98\Delta t_{crit}$, $0.5\Delta t_{crit}$	13, 15, 17
Average Acceleration	80	0.1, 0.01, 0.001	13, 15, 17

For central difference, stability of central difference with time step a little bit larger (1.02), a little bit smaller (0.98), or much smaller (0.5) than the critical time step is tested.

For Average Acceleration, compare with critical time step of central difference, $\Delta t = 0.001$ is the time step works for both method; but $\Delta t = 0.1$ *and* 0.01 is much larger than the time step of central difference.

Following figures shows the results of stability analysis

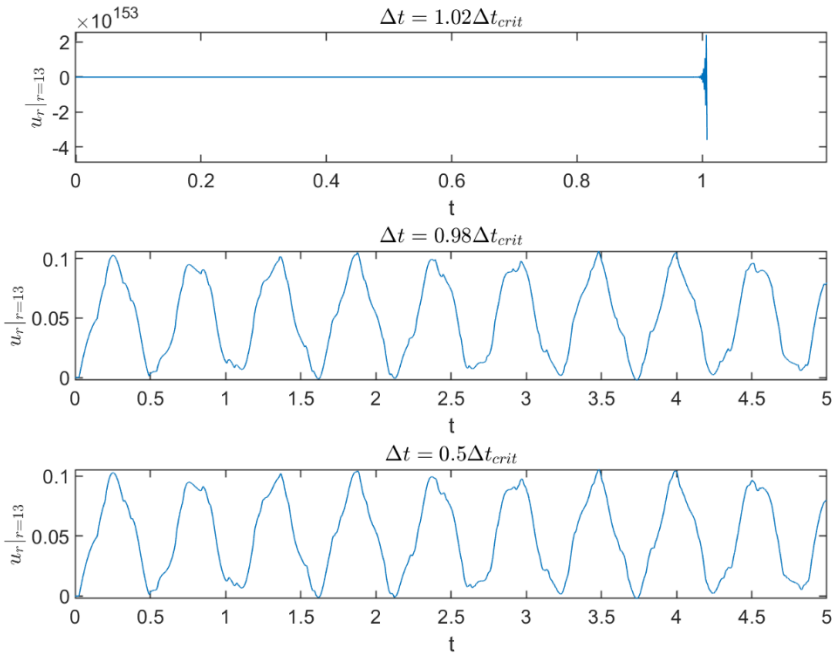


Fig.1. Displacement at r =13 solved by central difference.

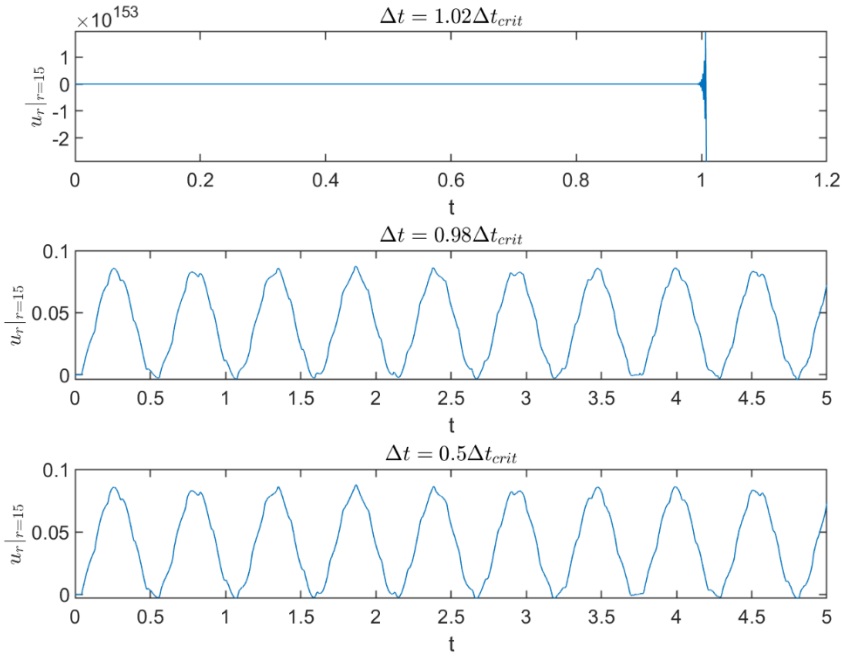


Fig.2. Displacement at r =15 solved by central difference.

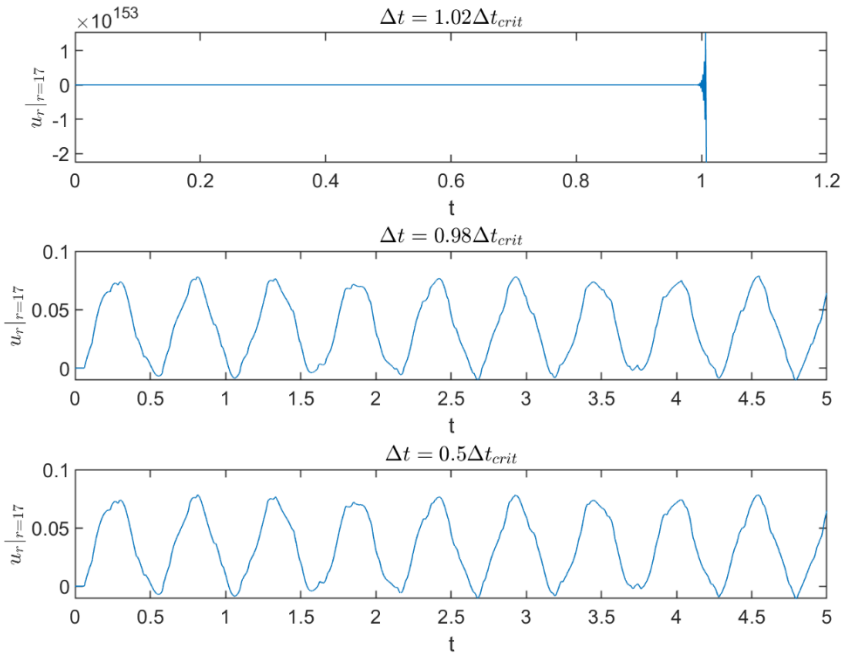


Fig.3. Displacement at r =17 solved by central difference.

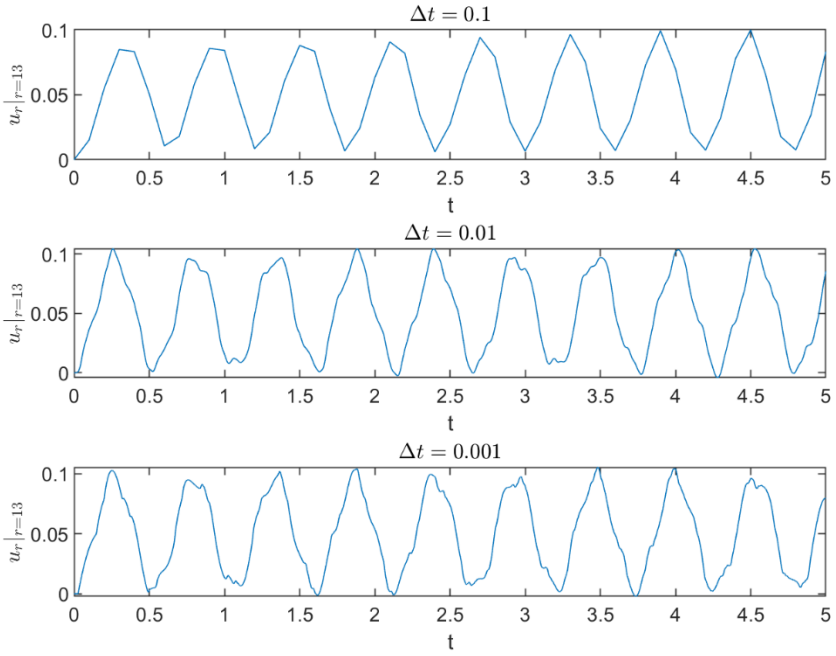


Fig.4. Displacement at r =13 solved by average acceleration.

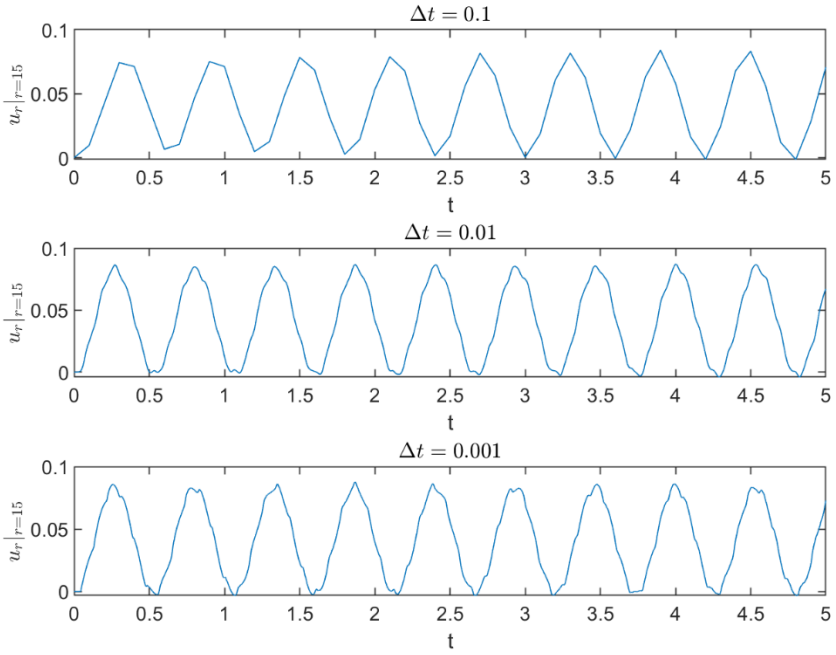


Fig.5. Displacement at r =15 solved by average acceleration.

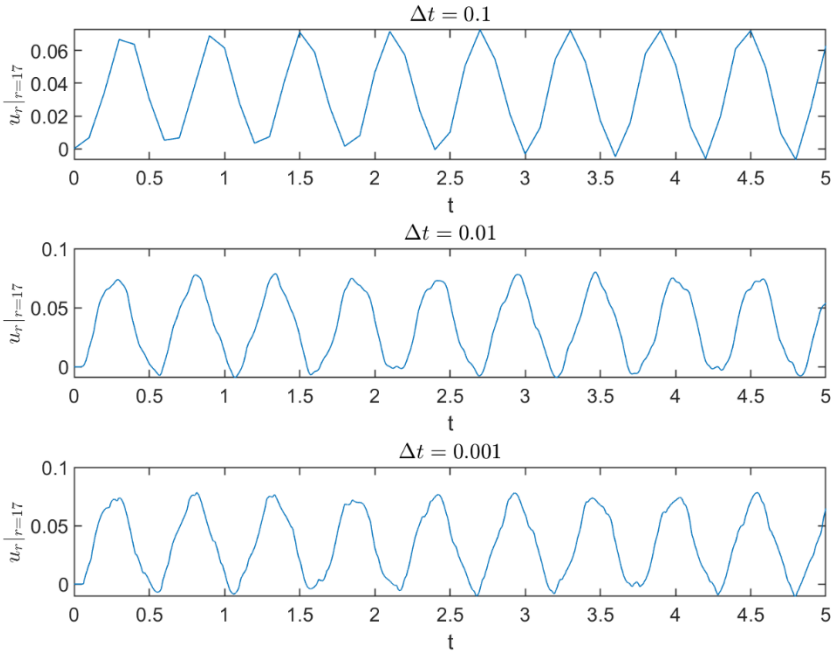


Fig.6. Displacement at r =17 solved by average acceleration.

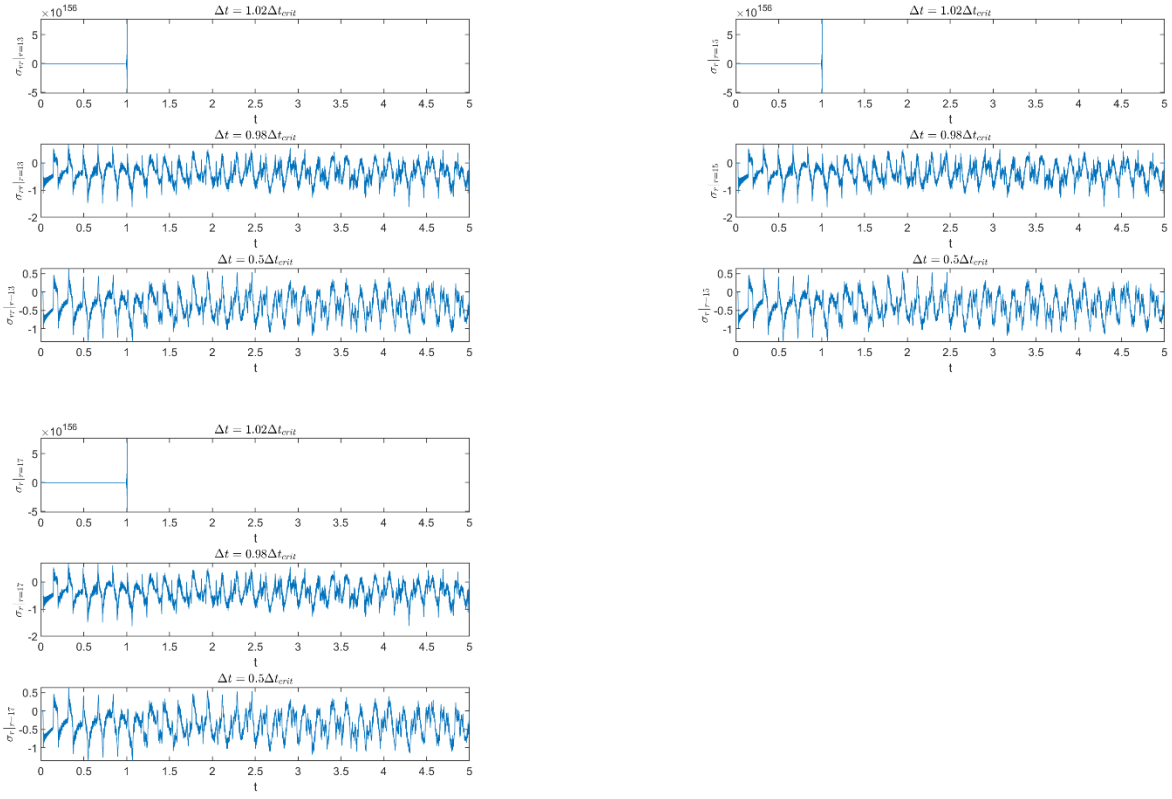


Fig7. Radial stresses at r = 13, 15, 17 solved by central difference.

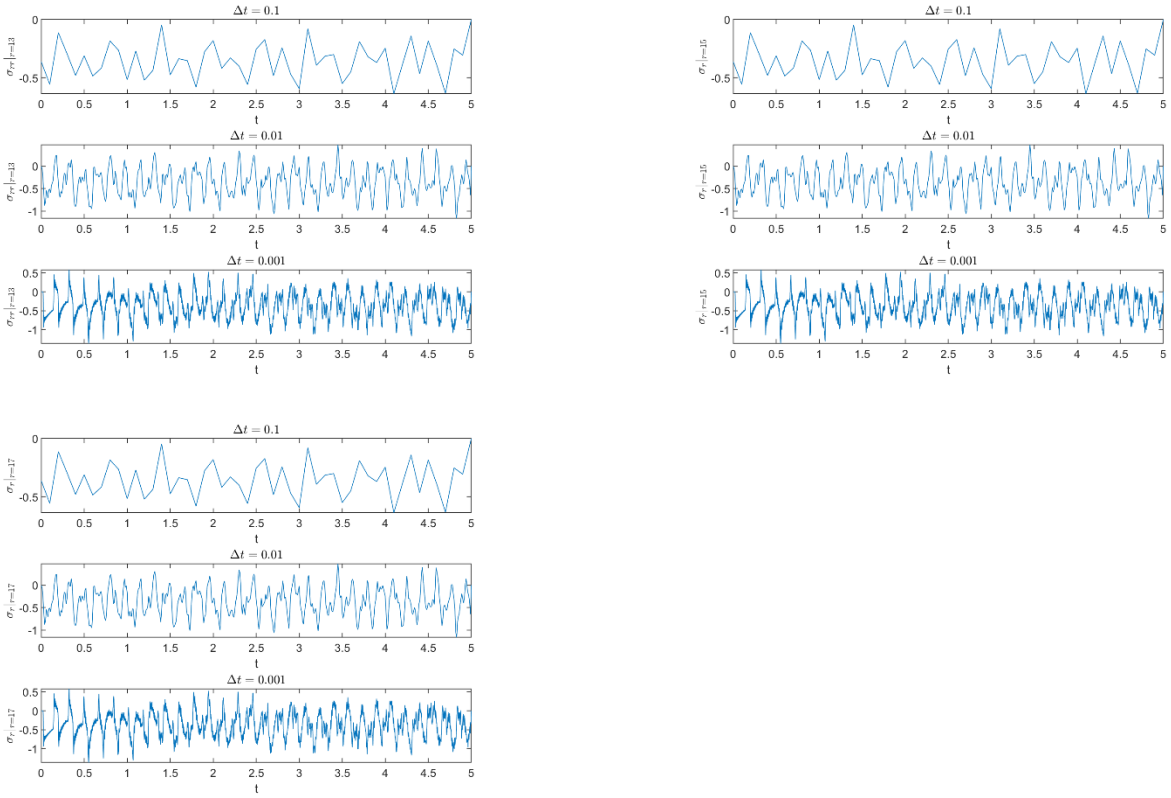


Fig8. Radial stresses at r = 13, 15, 17 solved by average acceleration.

From the plots, it can be seen that, for central difference method, when the time step is larger than the critical time step, solution will below up even if it is only a little bit larger than the critical value. And solution is stable when time step is smaller than the critical time step. Thus central difference is *conditionally stable*, whose stability depends on the size of time step.

For average acceleration, dt of 0.1, 0.01 and 0.001 are all gives stable solution. The time step which gives below up solution in central difference still gives stable solution in average acceleration. Thus the stability of average acceleration does not depend on time step, which implies *unconditionally stable*.

5.2. Effect of Time Step

From previous section, time step will affect the stability of central difference method, while average acceleration method has no stability issue with time step.

In this section, displacement and stresses at t = 5 which given by two methods with different time step are compared.

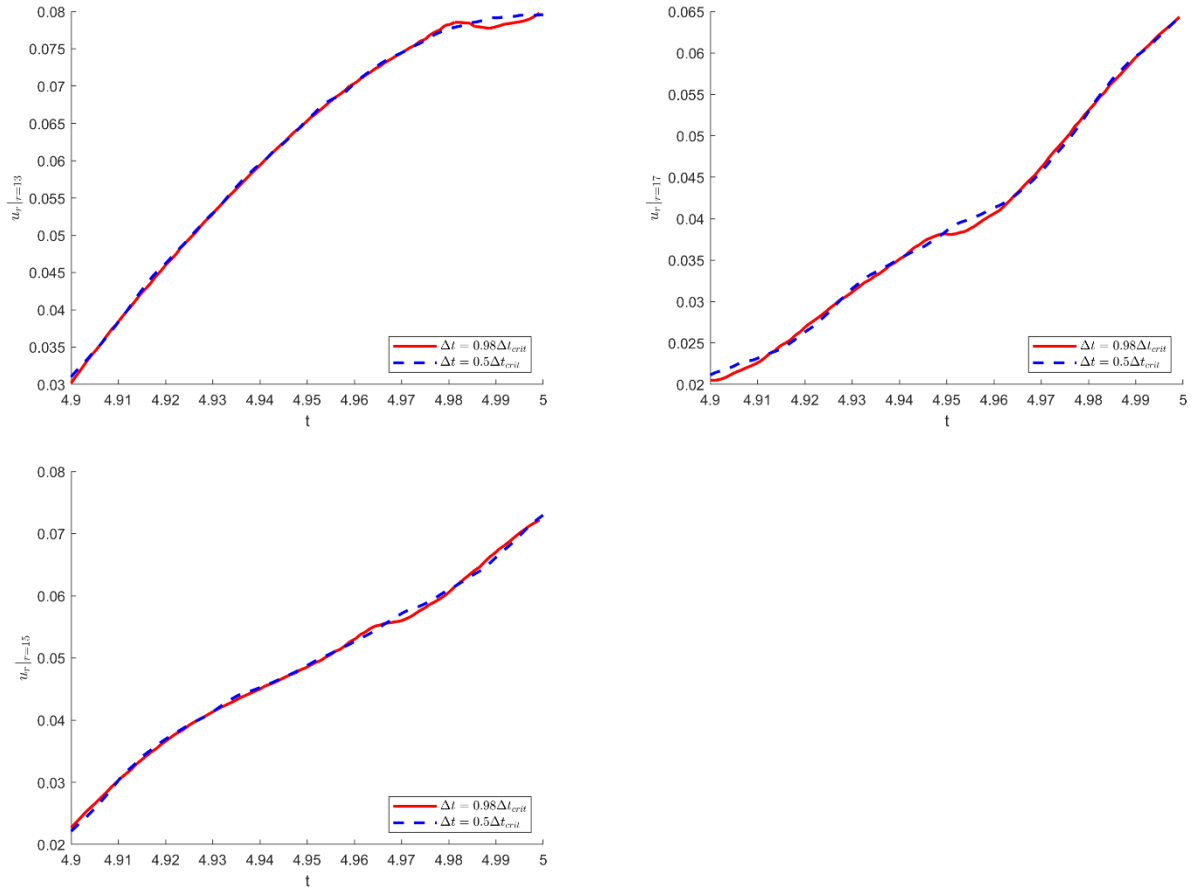
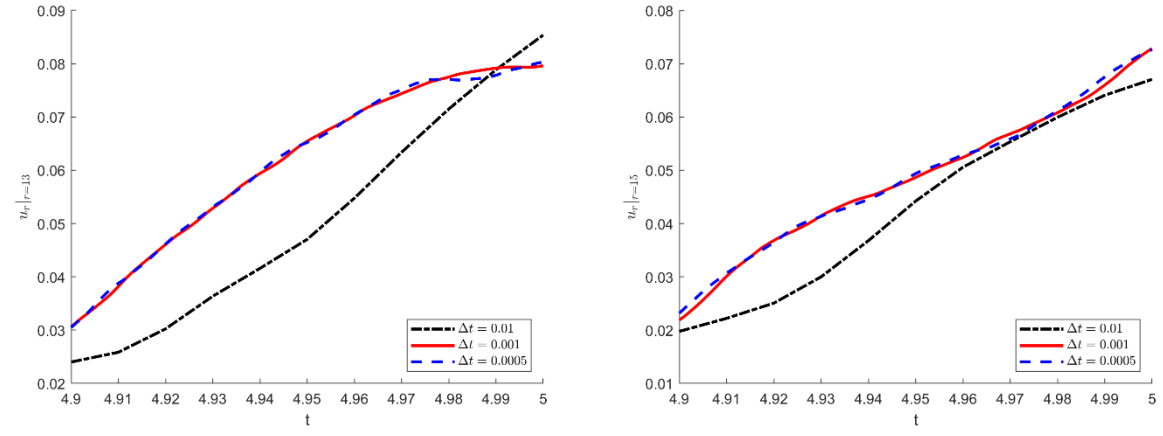


Fig9. Displacement solved by central difference with difference time step at various location.



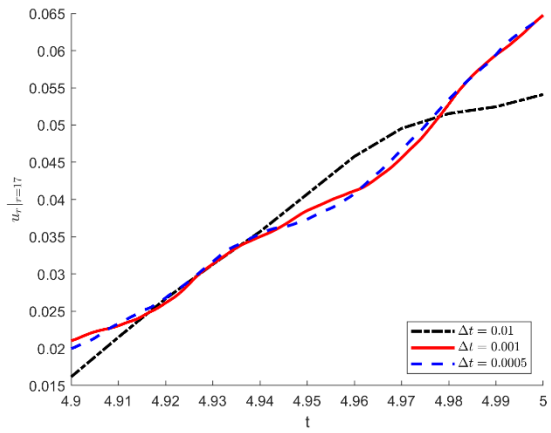


Fig10. Displacement solved by average acceleration with difference time step at various location.

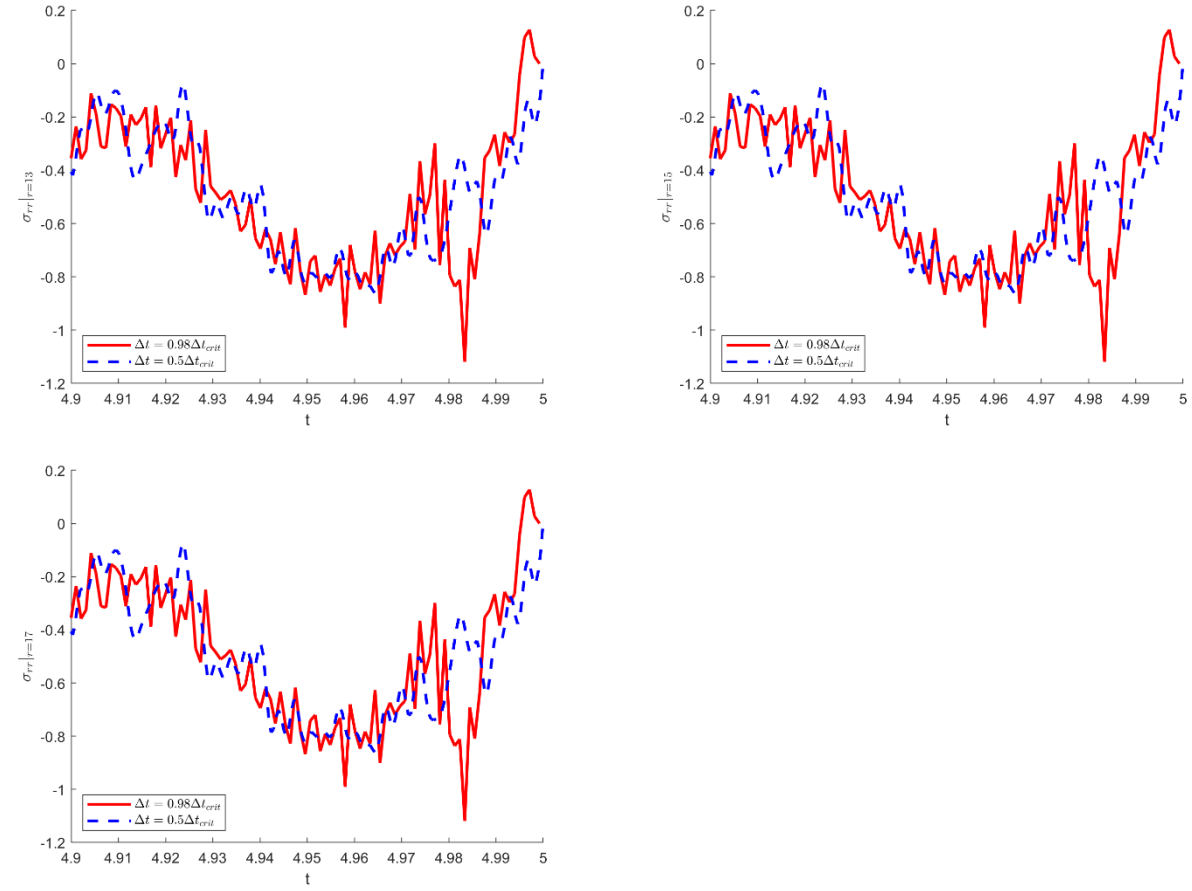


Fig11. Radial stress solved by central difference with difference time step at various location.

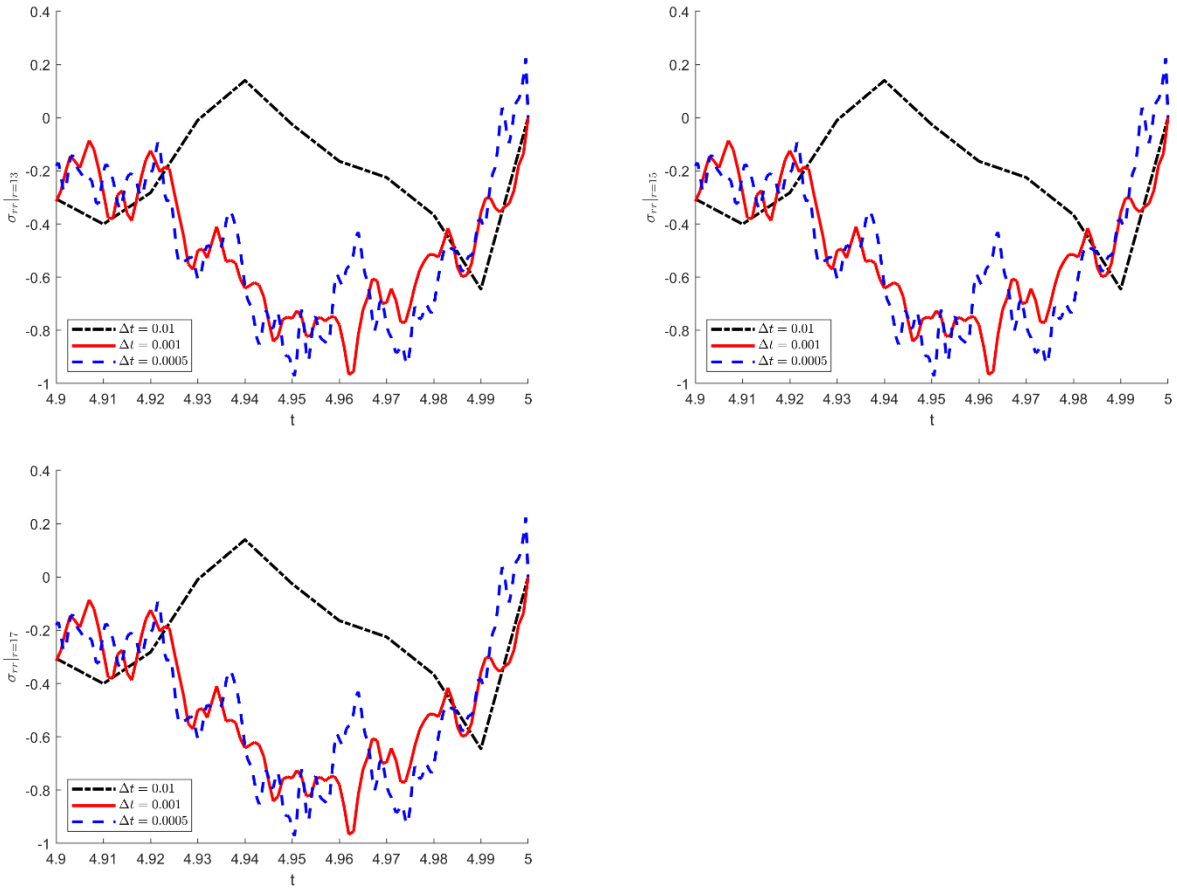


Fig12. Radial Stress solved by average acceleration with difference time step at various location.

Comparing plots for average acceleration method, it can be seen that, even if it is unconditionally stable, Δt should also keep small for solution to be accurate.

For central difference method, $0.98\Delta t_{crit}$ and $0.5\Delta t_{crit}$ both give similar results; and this is the same case for average acceleration method ($\Delta t = 0.01, 0.005$).

From the theory, accuracy is second order in time, it is expected that smaller Δt have better solution. However, if only the trend of transient solution is interested, selecting Δt close to critical time step may a good choice to balance the accuracy of solution and computation time.

5.3. Effect of Mesh Size

To study the effect of mesh refinement, keep the time step Δt unchanged as 0.0001 for all cases, whose resolution is considered to be fine enough to get rid of the influence of time step. Parameters used in the study of effect of mesh refinement is summarized in the table below.

Table4. Parameters of numerical study on mesh refinement.

Method	Number of Elements	Time Step
Central Difference	10, 20, 40, 80	0.0001
Average Acceleration	10, 20, 40, 80	0.0001

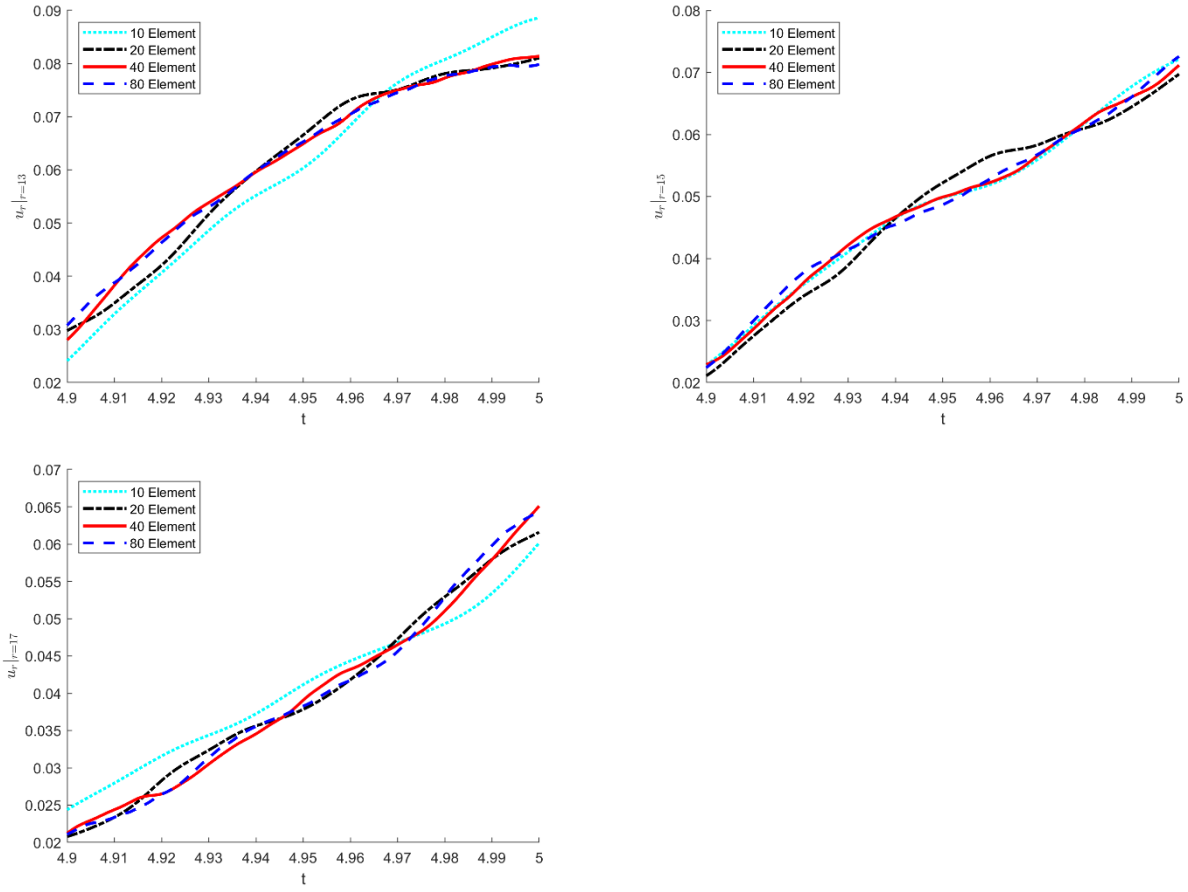


Fig13. Displace solved by central difference with time step of 0.0001, various number of spatial elements at various location.

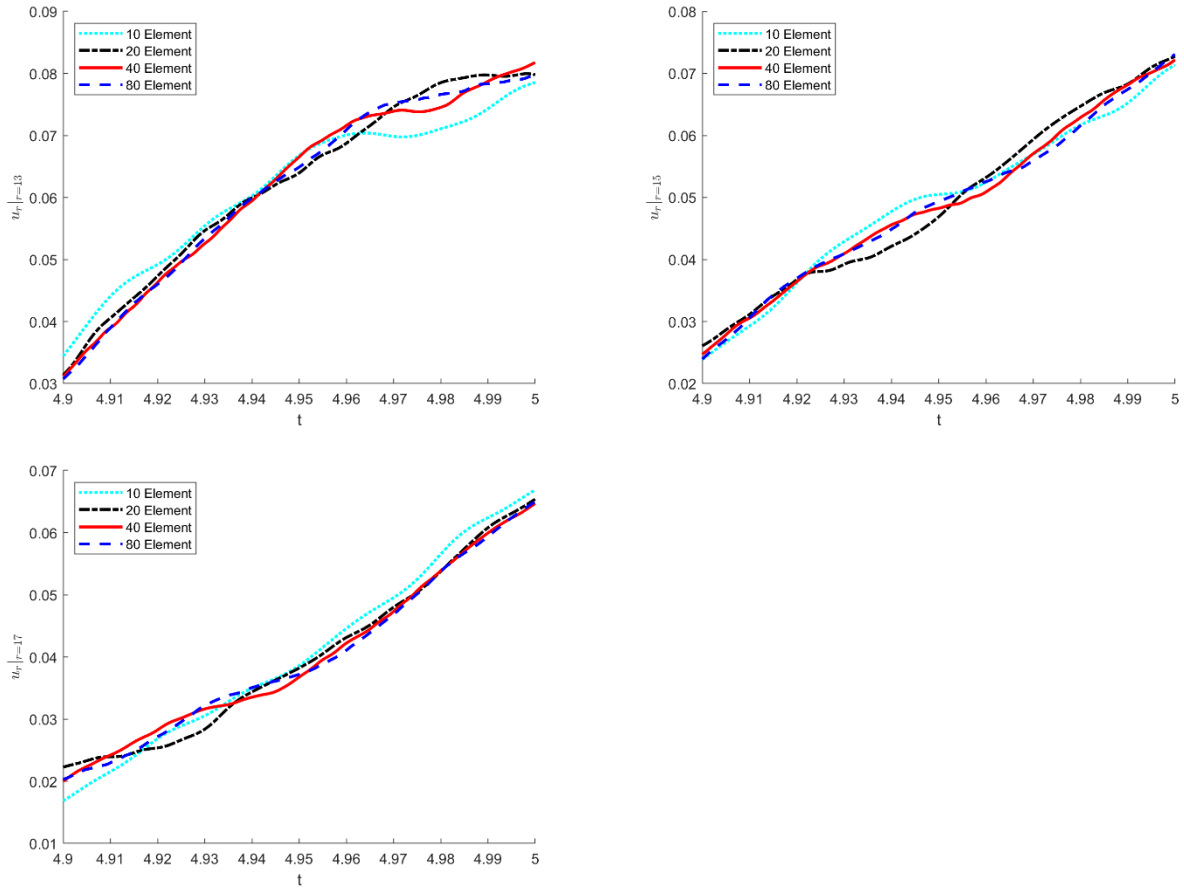
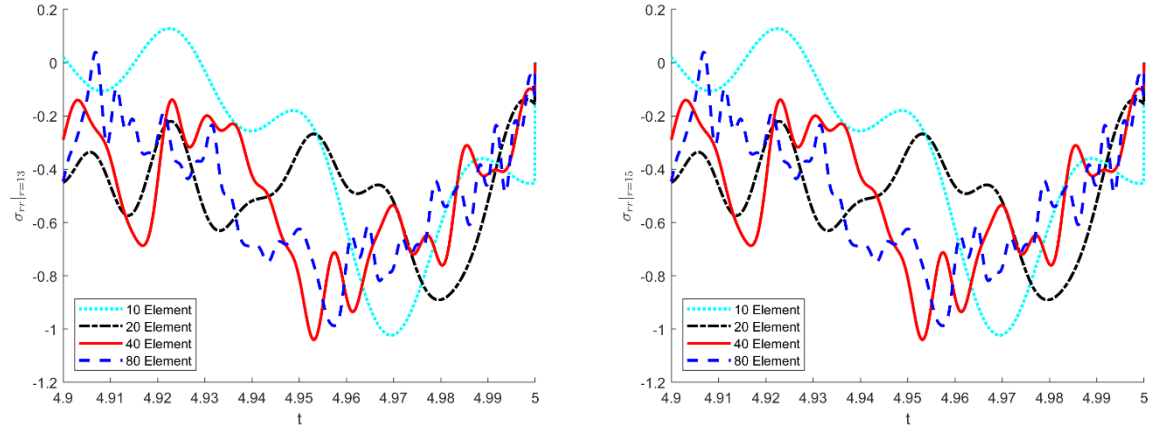


Fig13. Displace solved by average acceleration with time step of 0.0001, various number of spatial elements at various location.



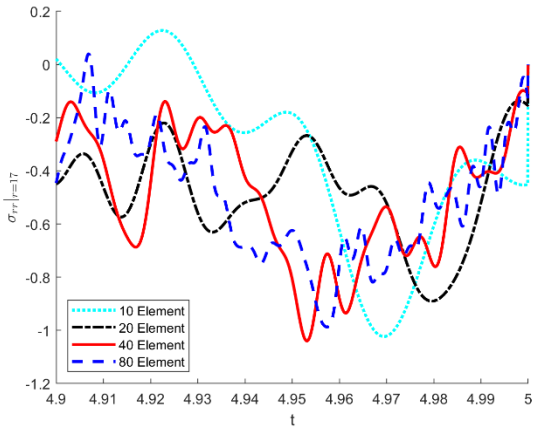


Fig14. Radial Stress solved by central difference with time step of 0.0001, various number of spatial elements at various location.

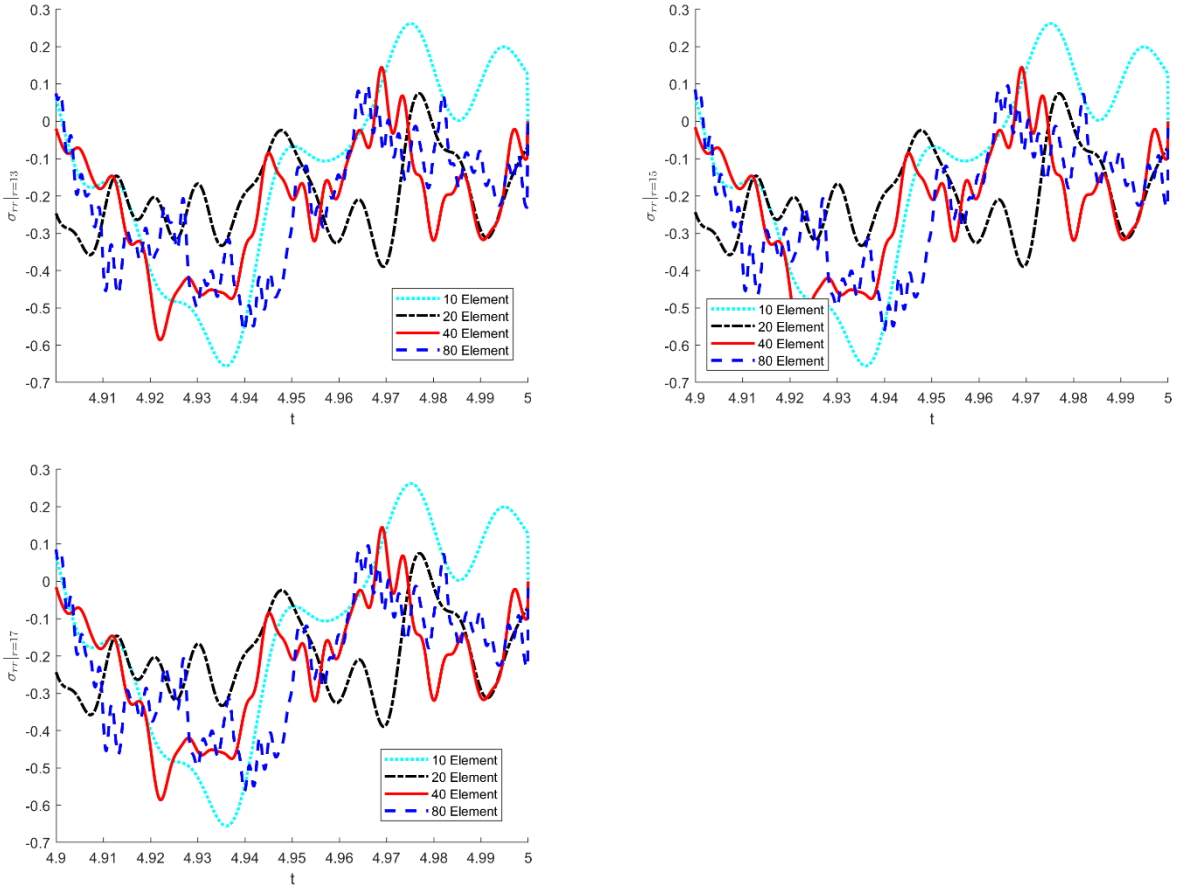


Fig14. Radial Stress solved by average acceleration with time step of 0.0001, various number of spatial elements at various location.

From theory, it is expected that numerical solution becomes better with mesh refinement. From the displacement plot, with the mesh refinement, displacement converges, as line of 40 elements and 80 elements almost coincide with each other. This could also be observed from plot of stress that stress solved from10 element differs a lot from that solved by more elements.

6. Summary

In terms of stability, central difference method is conditional stable, thus when marching the time forward, time step should be smaller than the critical time step in order to keep the numerical wave speed not travel faster than the physical wave speed. Although average acceleration method does not have restriction on time step, it is still necessary to use the time step low to capture the transient physics and keep solution to be accurate. Hence, to obtained both stable and accurate solution, general guideline is to refine the mesh and time step.