MAE 290B. Homework 3

Winter 2019

Due Thursday, Feb. 21, in class

Note: *You are expected to provide legible and clear solutions for the homework. Otherwise points will be deducted. Computer programs must be* **written on your own***, included in the manuscript and also submitted through TritonEd. Plots and sketches must be labeled and the axis well defined. Staple the document and number the pages. Torn-out pages are not allowed.*

1. The linear diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}\,, \tag{1}$$

with $\alpha > 0$ is to be solved. The spatial derivative is discretized using a fourth-order, central finite difference and forward Euler is used for time integration.

a) Derive the modified differential equation and obtain the order of accuracy of the method in time and space.

b) Show that the numerical method is consistent.

c) Eq.(1) is to be solved in the domain $0 < x < 1.$ with the numerical method specified in a). The diffusion coefficient $\alpha = 10^{-3}$, the domain is periodic in x and the initial condition is $T(x, 0) = \sin 4\pi x$. Plot the temperature distribution at $t = 1, 10$ and 100 with $N = 41$ spatial points.

d) Let us study the importance of the spatial resolution by looking at the effect of changing Δx. Fix Δt, vary N and save the value of $T(x = 0.55, t = 1)$. Plot the absolute error $E(\Delta x) = |T(x = 0.55, t = 1)_{\Delta x} - T(x = 0.55, t = 1)_{\Delta x = 0.001}|$ as a function of Δx and estimate the spatial accuracy based on these results. Compare your estimation with the spatial order of accuracy obtained analytically in a).

e) Let us study now the importance of the temporal resolution by looking at the effect of changing Δt. Fix Δx, vary Δt and save the value of $T(x = 0.55, t = 1)$. Plot the absolute error $E(\Delta t) = |T(x = 0.55, t = 1)_{\Delta t} - T(x = 0.55, t = 1)_{\Delta t = 0.0001}|$ as a function of Δt and estimate the temporal accuracy of the method based on these results. Compare your estimation with the temporal order of accuracy obtained analytically in a).

2. The linear advection-diffusion equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}\,, \tag{2}$$

with u and α as positive constants, is to be numerically solved. A RKW3 (*Numerical Renaissance. Section 10.4.1.3*) scheme is to be used for the time integration with a second-order, central approximation for the spatial derivatives.

a) Perform an analysis of the spatial discretization to obtain the modified wavenumber/s k' as a function of the wavenumber k.

b) Obtain the amplification factor of the numerical method using Fourier modes with modified wave number/s k'.

c) Perform a stability analysis of the method. What is the restriction on the CFL number for the numerical method to be stable when $\alpha = 0$? What is the restriction on the diffusion number for the numerical method to be stable when $u = 0$? Can you come up with a simple combined restriction for arbitrary values of u and α?

d) Eq.(2) is going to be numerically integrated in a domain $0 \leq x \leq 1$. The initial condition is

$$T(x,0) = \begin{cases} 0 & x \leq 0 \\ \sin(\pi x)\left(\cos(5\pi x) + \sin(20\pi x)\right) & 0 \leq x \leq 1 \\ 0 & 1 \leq x\,. \end{cases} \tag{3}$$

i. Advection: The parameters are $u = 0.5$, $\alpha = 0$. The boundary condition is $T(x = 0, t) = 0$. Solve for $T(x,t)$ over the time interval $0 \leq t \leq 2$. Explain how would you choose Δx and Δt based on accuracy and stability. Compare the numerical solution with the exact one at $t = 0, 0.5, 1, 2$ s and discuss the evolution of $T(x,t)$. The artificial right boundary might need some special treatment. Use first homogeneous Dirichlet and then use first-order backward finite difference. Plot both cases and describe the differences.

ii. Diffusion: The parameters are $u = 0$, $\alpha = 0.01$. The boundary conditions are $T(x = 0, t) = 0$ and $T(x = 1, t) = 1$. Solve for $T(x,t)$ over the time interval $0 \leq t \leq 10$. Explain how would you choose Δx and Δt based on accuracy and stability. Plot and discuss the evolution of $T(x,t)$ at $t = 0, 0.025, 0.1, 0.5, 1, 10$ s. Is the characteristic diffusion time equal for all the wavenumbers?