MAE 290B. Homework 4

Winter 2019

Due **Tuesday, March 5**, in class

Note: *You are expected to provide legible and clear solutions for the homework. Otherwise points will be deducted. Computer programs must be* **written on your own***, included in the manuscript and also submitted through TritonEd. Plots and sketches must be labeled and the axis well defined. Staple the document and number the pages. Torn-out pages are not allowed.*

1. Consider the paraxial Helmholtz equation

$$\frac{\partial \phi}{\partial y} = \frac{-i}{2k}\frac{\partial^2 \phi}{\partial x^2}\,, \tag{1}$$

which looks like the unsteady heat equation except that y **is the time-like variable** and the coefficient of the second derivative is imaginary. In this equation, ϕ is a complex variable representing the phase and amplitude of the wave and k is the wave number equal to $2\pi/\lambda$, where λ is the wavelength. Having a single-frequency wave source at $y = 0$ (a laser beam aperture, for example), this equation describes spatial evolution of the wave as it propagates in the y-direction. Since y is the time-like variable, an initial condition at $y = 0$ is required to close the equation. Consider the following initial condition for the problem:

$$\phi(x, 0) = \exp\left[-\frac{(x-5)^2}{4}\right] + \exp\left[-\frac{(x-15)^2}{4} + 10ix\right] \tag{2}$$

Assume $k = 10$ and note that $i = \sqrt{-1}$. This condition corresponds to two beam sources at $x = 5$ and $x = 15$ with the later beam making an angle of $10/k$ radians with the x-axis. Furthermore, assume a finite domain in the x-direction defined by $0 \le x \le 20$ with the following boundary conditions: $\phi(0, y) = \phi(20, y) = 0$.

a) Consider second-order central difference for discretization in the x-direction. What value of Δx would you choose? (*Hint*: Plot the initial condition.)

b) What method would you choose to advance the equation in the y-direction? Using Δx from part (a), what will be the maximum stable Δy?.

c) Using second-order central difference in the x-direction and an appropriate method of your choice for y, obtain the solution of the paraxial wave equation for $0 \le y \le 35$.

d) Plot $|\phi|^2$ as a function of x and y using a contour plot routine, such as *pcolor* in Matlab. Include a colorbar. What you should observe is reflection of one source and its interference with the other source as it propagates through the domain. (*Tip*: to visualize properly the solution remove the grid lines from the plot using the *pcolor* options.)

2. The inviscid Burger's equation is a canonical non-linear partial differential equation in fluid dynamics which can develop discontinuities (shock-waves). Written in conservative form, which is especially suitable for numerical integration, the equation reads

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0\,.$$

Consider the following initial condition

$$u(x,0) = \begin{cases} e^{-\frac{(x-1)^2}{0.18}} & x \le 2 \\ 0 & 2 \le x\,. \end{cases} \tag{3}$$

in the domain $0 \le x \le 5$ and $0 \le t \le 10$ with the boundary condition $u(x = 0, t) = 0$.

a) Use RKW3 to obtain the solution at $t = 0, 0.5, 1, 1.5, 5, 10$. Employ a second-order, central approximation for the spatial derivatives. *Tip*: You can use the structure of the code in homework 3.1c.

b) Repeat part 2.a) with upwind first-order FD approximation for the spatial derivatives.

c) Discuss numerical inaccuracies in your solutions from part a) and b). What alternative methods would you suggest to solve this nonlinear equation? Justify your answer.