MAE 290B. Homework 1

Winter 2018

Due Jan. 24, in class

Note: *You are expected to provide legible and clear solutions for the homework. Otherwise points will be deducted. Computer programs must be* **written on your own** *and submitted through TritonEd. Plots and sketches must be labeled and the axis well defined. Staple the document and number the pages. Torn-out pages are not allowed.*

1. A function $f(x)$ is known at points x_i, $i = 1, 2, \ldots, n$.

a) Obtain a second-order accurate central approximation to $f''(x)$ at x_i using the Taylor-series method. Give the truncation error (including coefficient).

b) Due to the finite-precision of the computers the evaluation of f at the grid node k has a roundoff error, i.e. $f_k = y_k + e_k$ where e_k is the roundoff error and y_k is the exact value. It is known that $|e_k| < \epsilon$ and $|f^{(4)}(x)| < M$. Denote the error in computing $f''(x)$ with the second-order accurate central formula by $E(\Delta x, f)$, and recall that $E(\Delta x, f)$ is the summation of the roundoff and truncation error. Find the optimal step size (Δx) based on ϵ and M, to minimize the total error $E(\Delta x, f)$.

c) Use points x_i, x_{i+1}, and x_{i+2} to obtain the best approximation to $f'(x)$ at x_i. Give the truncation error (including coefficient).

2. The following IVP is to be numerically integrated to obtain y at $t = 500$ s:

$$y' = \left(-\frac{1}{\tau} + i\omega\right) y \quad ; \quad y(0) = 1\,. \tag{1}$$

Use the θ method for solving $y' = f(y)$ as follows:

$$y_{n+1} = y_n + h[\theta f_{n+1} + (1-\theta) f_n]\,, \tag{2}$$

where $\theta \in [0, 1]$ is a fixed parameter.

a) Using the model problem $y' = \lambda y$ perform a stability analysis for $\theta = \{0, \frac{1}{2}, 1\}$. How does the choice of θ affect the stability? Sketch the stability regions for the three cases. Recall that we are assuming that $\lambda_R \leq 0$ i.e the exact solution is bounded.

b) Write a computer program that implements the θ method. Use it to obtain the solution for the initial times $0 \leq t \leq 50$ and the later times $450 \leq t \leq 500$ with a user-prescribed θ, ω and h. Let $\omega = 0.25$ s^{-1}, $\tau = 400$ s and consider two cases: $\theta = 1/4$ and $\theta = 3/4$. Vary the time step $h = 0.01, 0.1, 1$ s and discuss the influence on the solution. Compare the two methods to the exact solution in terms of accuracy, stability and convergence.

c) Assume the characteristic decay time $\tau \to \infty$ so that Eq. (1) simplifies to

$$y' = i\omega y \quad ; \quad y(0) = 1\,.$$

Analytically obtain the amplitude and phase error of the method with $\theta = 3/4$ and time step h. Simplify the error expressions assuming that $\omega h << 1$.

3. A third-order Runge-Kutta scheme (RK3) is used to integrate the model linear problem:

$$y' = \lambda y \quad ; \quad y(0) = 1 \ . \tag{3}$$

a) Obtain the stability restriction on the time step h for $\lambda \in \mathbb{C}$. Show the solution as a stability diagram. What is the restriction on h when $\lambda \in \mathbb{R}$?

b) Write a program that integrates Eq.(3) using a RK3 scheme (T. Bewley *Numerical Renaissance* Eq.(10.37) or Eq.(10.39)). Vary the time step $h = 0.01, 0.1, 1\ \mathrm{s}$ and compare the results with the solutions of problem 2b.