1. $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}$

For fourth-ordered center FD

$$\frac{\partial u}{\partial t} = \frac{-u(x+2\Delta x) + 8u(x+\Delta x) - 8u(x-\Delta x) + u(x-2\Delta x)}{12\Delta x}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{-u(x+2\Delta x) + 16u(x+\Delta x) - 30u(x) + 16u(x-\Delta x) - u(x-2\Delta x)}{12\Delta x^2}$$

Then Semi-discretized Equation is:

$$\frac{\partial u}{\partial t} = \frac{-c}{12\Delta x}\left[-u(x+2\Delta x) + 8u(x+\Delta x) - 8u(x-\Delta x) + u(x-2\Delta x)\right]$$
$$+ \frac{\nu}{12\Delta x^2}\left[-u(x+2\Delta x) + 16u(x+\Delta x) - 30u(x) + 16u(x-\Delta x) - u(x-2\Delta x)\right]$$

Write $u = \hat{u}e^{ikx}$

Then $u(x+\Delta x) = \hat{u}e^{ik(x+\Delta x)} = \hat{u}e^{ikx}\cdot e^{ik\Delta x}$

Similarly: $u(x+2\Delta x) = \hat{u}e^{ikx}\cdot e^{i2k\Delta x}$

$u(x-\Delta x) = \hat{u}e^{ikx}\cdot e^{-ik\Delta x}$

$u(x-2\Delta x) = \hat{u}e^{ikx}\cdot e^{-i2k\Delta x}$

plug them into Equation Above.

$$\frac{\partial \hat{u}}{\partial t}\cancel{e^{ikx}} = \frac{-c}{12\Delta x}\left[-\hat{u}\cancel{e^{ikx}}\cdot e^{i2k\Delta x} + 8\hat{u}\cancel{e^{ikx}}\cdot e^{ik\Delta x} - 8\hat{u}\cancel{e^{ikx}}e^{-ik\Delta x} + \hat{u}\cancel{e^{ikx}}\cdot e^{-i2k\Delta x}\right]$$
$$+ \frac{\nu}{12\Delta x^2}\left[-\hat{u}\cancel{e^{ikx}}\cdot e^{i2k\Delta x} + 16\hat{u}\cancel{e^{ikx}}e^{ik\Delta x} - 30\hat{u}\cancel{e^{ikx}} + 16\hat{u}\cancel{e^{ikx}}\cdot e^{-ik\Delta x} - \hat{u}\cancel{e^{ikx}}\cdot e^{-i2k\Delta x}\right]$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t} = \hat{u}\left(\frac{-c}{12\Delta x}\right)\left[\underbrace{-e^{i2k\Delta x} + e^{-i2k\Delta x}}_{-2i\sin 2k\Delta x} + \underbrace{8e^{ik\Delta x} - 8e^{-ik\Delta x}}_{8(2i\sin k\Delta x)}\right]$$
$$+ \hat{u}\left(\frac{\nu}{12\Delta x^2}\right)\left[\underbrace{-e^{i2k\Delta x} - e^{-i2k\Delta x}}_{-2\cos 2k\Delta x} + \underbrace{16(e^{ik\Delta x} + e^{-ik\Delta x})}_{32\cos k\Delta x} - 30\right]$$

$$\Rightarrow \frac{\partial \hat{u}}{\partial t} = \hat{u}\left[i \frac{c}{6\Delta x} \sin 2k\Delta x - i \frac{4}{3} \frac{c}{\Delta x} \sin k\Delta x \right]$$

$$+ \hat{u}\left[-\frac{1}{6} \frac{\nu}{\Delta x^2} \cos 2k\Delta x + \frac{8}{3} \frac{\nu}{\Delta x^2} \cos k\Delta x - \frac{5}{2} \frac{\nu}{\Delta x^2} \right]$$

$$= \lambda \hat{u}$$

thus $\lambda = i \frac{c}{\Delta x}\left[\frac{1}{6} \sin 2k\Delta x - \frac{4}{3} \sin k\Delta x \right] + \frac{\nu}{\Delta x^2}\left[-\frac{1}{6} \cos 2k\Delta x + \frac{8}{3} \cos k\Delta x - \frac{5}{2} \right]$

$$\Rightarrow \lambda\Delta t = i \frac{c\Delta t}{\Delta x}\left[\frac{1}{6} \sin 2k\Delta x - \frac{4}{3} \sin k\Delta x \right] + \frac{\nu\Delta t}{\Delta x^2}\left[-\frac{1}{6} \cos 2k\Delta x + \frac{8}{3} \cos k\Delta x - \frac{5}{2} \right]$$

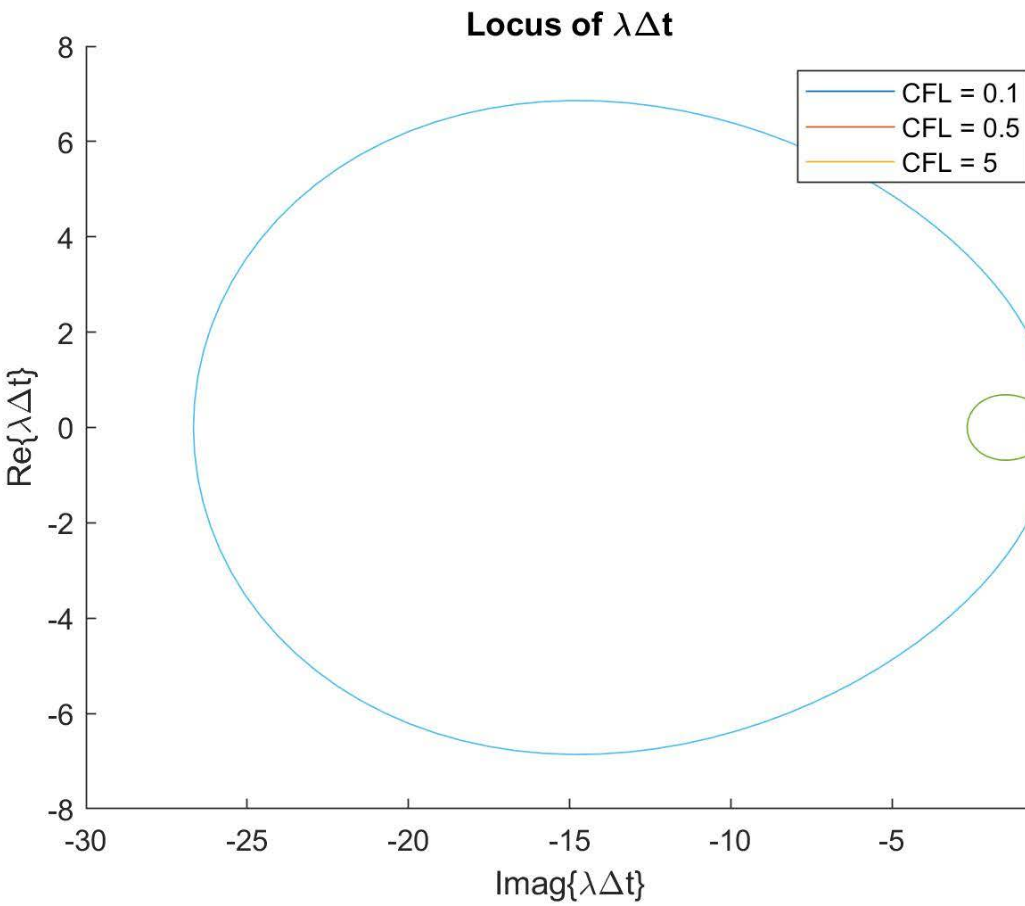
since $\frac{\nu\Delta t}{\Delta x^2} = \frac{c\Delta t}{\Delta x} \frac{\nu}{\Delta x c} = \frac{CFL}{Re}$

$$\Rightarrow \lambda\Delta t = \underbrace{i\, CFL \left[\frac{1}{6} \sin 2k\Delta x - \frac{4}{3} \sin k\Delta x \right]}_{\lambda_I \Delta t} + \overbrace{\frac{CFL}{Re}\left[-\frac{1}{6} \cos 2k\Delta x + \frac{8}{3} \cos k\Delta x - \frac{5}{2} \right]}^{\lambda_R \Delta t}$$

where $k\Delta x = \frac{2\pi n}{L} \frac{L}{N} = \frac{2\pi n}{N}$, $n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$

$$\Rightarrow k\Delta x = -\pi, \ldots, \pi - \frac{2\pi}{N}$$

Problem1. Plot



$$\frac{du}{dt} = H(u) + Lu$$

Problem 2

2 (a) Exact Taylor Expansion of u^{n+1} with respect to u^n is

$$u^{n+1} = u^n + \Delta t \left.\frac{du}{dt}\right|_{u=u^n} + \frac{\Delta t^2}{2}\left.\frac{d^2u}{dt^2}\right|_{u=u^n} + \dots$$

By $\left.\frac{du}{dt}\right|_{u=u^n} = H(u^n) + Lu^n$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(Lu^n + H(u^n)\right)$$

$$= L\frac{du^n}{dt} + \frac{dH(u)}{du}\left.\frac{du}{dt}\right|_{u=u^n}$$

$$= L\frac{du^n}{dt} + \frac{dH(u^n)}{du}\left(Lu^n + H(u^n)\right)$$

$$= L\left(Lu^n + H(u^n)\right) + \frac{dH(u^n)}{du}\left(Lu^n + H(u^n)\right)$$

$$= \left[L + \frac{dH(u)}{du}\right]\left[Lu^n + H(u^n)\right]$$

$$\Rightarrow \boxed{u^{n+1} = u^n + \Delta t\left[H(u^n) + Lu^n\right] + \frac{\Delta t^2}{2}\left[L^2u^n + LH(u^n) + \frac{dH(u^n)}{du}Lu^n + \frac{dH(u^n)}{du}H(u^n)\right]}$$

For two step RK-IMEX we have:

$$u^* = u^n + \Delta t\,\alpha_1 Lu^n + \Delta t\,\beta_1 Lu^* + \Delta t\,\gamma_1 H(u^n)$$

$$\Rightarrow (1 - \Delta t\,\beta_1 L)\,u^* = u^n + \Delta t\,\alpha_1 Lu^n + \Delta t\,\gamma_1 H(u^n)$$

$$\Rightarrow u^* = (1 - \Delta t\,\beta_1 L)^{-1}\left(u^n + \Delta t\,\alpha_1 Lu^n + \Delta t\,\gamma_1 H(u^n)\right)$$

Expand $(1-\Delta t\,\beta_1 L)^{-1}$ By Taylor series And plug into Eqn:

$$u^* = \left(1 + \Delta t\,\beta_1 L + \Delta t^2\beta_1^2L^2 + \cdots\right)\left(u^n + \Delta t\,\alpha_1 Lu^n + \Delta t\,\gamma_1 H(u^n)\right)$$

$$\Rightarrow u^* = u^n + \Delta t\,\alpha_1 L u^n + \Delta t\,\gamma_1 H(u^n)$$
$$+ \Delta t\,\beta_1 L u^n + \Delta t^2 \beta_1 \alpha_1 L^2 u^n + \Delta t^2 \beta_1 \gamma_1 L H(u^n)$$
$$+ \Delta t^2 \beta_1^2 L^2 u^n + \Delta t^3 \beta_1^2 L^3 u^n + \Delta t^3 \beta_1^2 \gamma_1 L^2 H(u^n) + \dots$$

Step 2: $u^{n+1} = u^* + \Delta t\,\alpha_2 L u^* + \Delta t\,\beta_2 L u^{n+1} + \underline{\Delta t\,\gamma_2 H(u^*)} + \Delta t\,\zeta_1 H(u^n)$

since $H(u^*) = H(u^n) + \frac{dH(u)}{du}(u^* - u^n) + \dots$

$$\rightarrow u^{n+1} = u^* + \Delta t\,\alpha_2 L u^* + \Delta t\,\beta_2 L u^{n+1} + \left[\Delta t\,\gamma_2 H(u^n) + \Delta t\,\gamma_2 \frac{dH}{du}(u^* - u^n)\right]$$
$$+ \Delta t\,\zeta_1 H(u^n)$$

$$\rightarrow u^{n+1} = (1 - \Delta t\,\beta_2 L)^{-1} u^* + \overbrace{(1 - \Delta t\,\beta_2 L)^{-1} \Delta t\,\alpha_2 L u^*}^{(\Delta t\,\alpha_2 L + \Delta t^2 \beta_2 \alpha_2 L^2 + \Delta t^3 \beta_2^2 \alpha_2 L^3) u^*}$$
$$+ (1 - \Delta t\,\beta_2 L)^{-1}\left[\Delta t\,\gamma_2 H(u^n) + \Delta t\,\gamma_2 \frac{dH}{du}(u^* - u^n)\right]$$
$$+ (1 - \Delta t\,\beta_2 L)^{-1} \Delta t\,\zeta_1 H(u^n)$$

Again Expand $(1 - \Delta t\,\beta_2 L)^{-1}$ By taylor series And plug in u^*:

$$u^{n+1} = (1 + \Delta t\,\beta_2 L + \Delta t^2 \beta_2^2 L^2 + \Delta t\,\alpha_2 L + \Delta t^2 \beta_2 \alpha_2 L^2 + \Delta t^3 \beta_2^2 \alpha_2 L^3)\Big[u^n + \Delta t\,\alpha_1 L u^n$$
$$+ \Delta t\,\gamma_1 H(u^n) + \Delta t\,\beta_1 L u^n + \Delta t^2 \beta_1 \alpha_1 L^2 u^n + \Delta t^2 \beta_1 \gamma_1 L H(u^n) + \Delta t^2 \beta_1^2 L^2 u^n$$
$$+ \Delta t^3 \beta_1^2 L^3 u^n + \Delta t^3 \beta_1^2 \gamma_1 L^2 H(u^n)\Big]$$
$$+ (1 + \Delta t\,\beta_2 L + \Delta t^2 \beta_2^2 L^2)\Big[\Delta t\,\gamma_2 H(u^n) + \Delta t\,\gamma_2 \frac{dH}{du}\big(\Delta t\,\alpha_1 L u^n + \Delta t\,\gamma_1 H(u^n)$$
$$+ \Delta t\,\beta_1 L u^n + \Delta t^2 \beta_1 \alpha_1 L^2 u^n + \Delta t^2 \beta_1 \gamma_1 L H(u^n) + \Delta t^2 \beta_1^2 L^2 u^n + \Delta t^3 \beta_1^2 L^3 u^n$$
$$+ \Delta t^3 \beta_1^2 \gamma_1 L^2 H(u^n)\big)\Big]$$
$$+ (1 + \Delta t\,\beta_2 L + \Delta t^2 \beta_2^2 L^2)(\Delta t\,\zeta_1 H(u^n))$$

Expand All of these terms, And Ignore terms not in exact Taylor Expansion:

u^n, $H(u^n)$, Lu^n, L^2u^n, $LH(u^n)$, $\frac{\partial H}{\partial u}Lu^n$, $\frac{\partial H}{\partial u}H(u^n)$

$$
\begin{aligned}
u^{n+1} = \Big\{ & \left[u^n + \Delta t\,\alpha_1 L u^n + \Delta t\,\gamma_1 H(u^n) + \Delta t\,\beta_1 L u^n + \Delta t^2 \beta_1 \alpha_1 L^2 u^n + \Delta t^2 \beta_1 \gamma_1 L H(u^n) + \Delta t^2 \beta_1^2 L^2 u^n\right] \\
& + \left[\Delta t\,\beta_2 L u^n + \Delta t^2 \beta_2 \alpha_1 L^2 u^n + \Delta t^2 \beta_2 \gamma_1 L H(u^n) + \Delta t^2 \beta_2 \beta_1 L^2 u^n\right] \\
& + \left[\Delta t\,\alpha_2 L u^n + \Delta t^2 \alpha_2 \alpha_1 L^2 u^n + \Delta t^2 \alpha_2 \gamma_1 L H(u^n) + \Delta t^2 \alpha_2 \beta_1 L^2 u^n\right] \\
& + \Delta t^2 \beta_2^2 L^2 u^n \\
& + \left[\Delta t^2 \beta_2 \alpha_2 L^2 u^n\right] \\
& + \left[\Delta t\,\gamma_2 H(u^n) + \Delta t^2 \gamma_2 \alpha_1 \frac{\partial H}{\partial u} L u^n + \Delta t^2 \gamma_2 \gamma_1 \frac{\partial H}{\partial u} H(u^n) + \Delta t^2 \gamma_2 \beta_1 \frac{\partial H}{\partial u} L u^n\right] \\
& + \left[\Delta t^2 \beta_2 \gamma_2 L H(u^n)\right] \\
& + \left[\Delta t\,\zeta_1 H(u^n) + \Delta t^2 \beta_2 \zeta_1 L H(u^n)\right]
\end{aligned}
$$

Compare with Exact Taylor Expansion

① u^n term: $1 = 1$

② $H(u^n)$ term: $\Delta t\,\gamma_1 + \Delta t\,\zeta_1 + \Delta t\,\gamma_2 = \Delta t$

③ Lu^n term: $\Delta t\,\alpha_1 + \Delta t\,\beta_1 + \Delta t\,\beta_2 + \Delta t\,\alpha_2 = \Delta t$

④ L^2u^n term: $\Delta t^2 \beta_1 \alpha_1 + \Delta t^2 \beta_1^2 + \Delta t^2 \beta_2 \alpha_1 + \Delta t\,\beta_2 \beta_1 + \Delta t^2 \alpha_2 \alpha_1$

$+ \Delta t^2 \alpha_2 \beta_1 + \Delta t^2 \beta_2 \alpha_2 + \Delta t^2 \beta_2^2 = \frac{\Delta t^2}{2}$

⑤ $LH(u^n)$ term: $\Delta t^2 \beta_1 \gamma_1 + \underline{\Delta t^2 \beta_2 \gamma_1} + \Delta t^2 \alpha_2 \gamma_1 + \underline{\Delta t^2 \beta_2 \gamma_2} + \Delta t^2 \beta_2 \zeta_1 = \frac{\Delta t^2}{2}$

⑥ $\frac{\partial H}{\partial u} Lu^n$ term: $\Delta t^2 \gamma_2 \alpha_1 + \Delta t^2 \gamma_2 \beta_1 = \frac{\Delta t^2}{2}$

⑦: $\frac{\partial H}{\partial u} H(u^n)$ term $\Delta t^2 \gamma_2 \gamma_1 = \frac{\Delta t^2}{2}$

Thus in total we have 6 equation with 7 unknowns

$H(u_n)$ ① $\gamma_1 + \gamma_2 + z_1 = 1$

$L(u)$ ② $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 1$

$L^2(u)$ ③ $\beta_1\alpha_1 + \beta_1^2 + \alpha_1\beta_2 + \beta_1\beta_2 + \alpha_1\alpha_2 + \alpha_2\beta_1 + \beta_2\alpha_2 + \beta_2^2 = \frac{1}{2}$

$LH(u_n)$ ④ $\beta_1\gamma_1 + \beta_2\gamma_1 + \alpha_2\gamma_1 + \beta_2\gamma_2 + \beta_2 z_1 = \frac{1}{2}$

$\frac{\partial H}{\partial u}L(u)$ ⑤ $\alpha_1\gamma_2 + \beta_1\gamma_2 = \frac{1}{2}$

$\frac{\partial H}{\partial u}H(u_n)$ ⑥ $\gamma_1\gamma_2 = \frac{1}{2}$

Unknowns: $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, z_1$.

Problem 2

(b)(ii) since we have 7 undetermined variables but only have 6 equations, This problem is undetermined.

two step RK-θ Gives:

$$u^{n+1} = u^n + \Delta t\left[(\alpha_1+\alpha_2+\beta_1+\beta_2)Lu^n + (\gamma_1+\gamma_2+\zeta_1)H(u^n)\right]$$
$$+\frac{\Delta t^2}{2}\Big[(\alpha_1\alpha_2+\alpha_1\beta_1+\alpha_1\beta_2+\alpha_2\beta_1+\alpha_2\beta_2+\beta_1^2+\beta_2^2+\beta_1\beta_2)L^2u^n$$
$$+(\beta_1\gamma_1+\beta_2\gamma_1+\beta_2\gamma_2+\alpha_2\gamma_1+\beta_2\zeta_1)LH(u^n)$$
$$+(\alpha_1\gamma_2+\beta_1\gamma_2)\frac{\partial H}{\partial u}Lu^n+\gamma_1\gamma_2\frac{\partial H}{\partial u}H(u^n)\Big]$$

To see what will happen when for $Re|\lambda \Delta t| \to \infty$ we can only look at viscous term, since it solely contributed to the real part of λ, which is related with viscous term.

Problem 2

(b) Then Apply this two-step RK-θ Scheme to Model Equation $\frac{du}{dt} = \lambda u$

Thus $U^* = U^n + \Delta t\, \alpha_1 \lambda U^n + \Delta t\, \beta_1 \lambda U^*$

$\Rightarrow U^* = (1-\Delta t\, \beta_1 \lambda)^{-1}(1+\alpha_1 \lambda \Delta t)U^n$

plug this into $U^{n+1} = U^* + \Delta t\, \alpha_2 \lambda U^* + \Delta t\, \beta_2 \lambda U^{n+1}$

$\Rightarrow U^{n+1} = (1-\beta_2 \lambda \Delta t)^{-1}(1+\alpha_2 \lambda \Delta t)(1-\Delta t\, \beta_1 \lambda)^{-1}(1+\alpha_1 \lambda \Delta t)U^n$

Then Amplification factor gives:

$$\sigma = \frac{U^{n+1}}{U^n} = \frac{(1+\alpha_2 \lambda \Delta t)(1+\alpha_1 \lambda \Delta t)}{(1-\beta_2 \lambda \Delta t)(1-\beta_1 \lambda \Delta t)}$$

For $Re(\lambda \Delta t) = \lambda \Delta t \to \infty$. If $\sigma \to 0$, it need denominator of σ should be much larger than numerator, or numerator goes to 0; Since the former one is not valid, to let the order of magnitude of Denominator much larger than numerator, it requires either $\alpha_2 \lambda \Delta t$ or $\alpha_1 \lambda \Delta t$, or both of them vanish, This imply $\alpha_2 = 0$ or $\alpha_1 = 0$ or Both $\alpha_1 = \alpha_2 = 0$

unknowns $\alpha_1, \beta_1, \gamma_1, \gamma_2, \xi_1$,

Problem 2. (b) (ii)

Set $\alpha_2 = 0$, and $\beta_2 = \beta_1$. 6 Equation becomes:

① $\gamma_1 + \gamma_2 + \xi_1 = 1$

② $\alpha_1 + 2\beta_1 = 1$

③ $\beta_1\alpha_1 + \beta_1^2 + \alpha_1\beta_1 + \beta_1^2 + \beta_1^2 = \frac{1}{2}$

$\Rightarrow \beta_1(3\beta_1 + 2\alpha_1) = \frac{1}{2}$

④ $\beta_1\gamma_1 + \beta_1\gamma_1 + \beta_1\gamma_2 + \beta_1\xi_1 = \frac{1}{2}$

$\Rightarrow \beta_1(2\gamma_1 + \gamma_2 + \xi_1) = \frac{1}{2}$

⑤ $\alpha_1\gamma_2 + \beta_1\gamma_2 = \frac{1}{2}$

⑥ $\gamma_1\gamma_2 = \frac{1}{2}$

With the Aid of Matlab, we find

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \gamma_2 \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}-1 \\ 1-\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 1-\sqrt{2} \end{bmatrix}$$

And $\alpha_2 = 0$, $\beta_2 = \beta_1 = 1 - \frac{\sqrt{2}}{2}$

Problem 3

3.(a) Apply model Eqn $\frac{\partial u}{\partial t} = \overset{\equiv H(u)}{\lambda u}$ to Analysis the stability of two step RK-θ with any H term:

$$U^* = U^n + \gamma_1 H(U^n) \Delta t$$
$$= U^n + \gamma_1 \lambda U^n \Delta t$$

$$U^{n+1} = U^* + [\gamma_2 H(U^*) + \zeta_1 H(U^n)] \Delta t$$
$$= U^* + [\gamma_2 \lambda U^* + \zeta_1 \lambda U^n] \Delta t$$

$$= U^n + \gamma_1 \lambda \Delta t U^n + \gamma_2 \lambda \Delta t (U^n + \gamma_1 \lambda \Delta t U^n)$$
$$+ \zeta_1 \lambda \Delta t U^n$$

$$\rightarrow \sigma = \frac{U^{n+1}}{U^n} = 1 + \gamma_1 \lambda \Delta t + \gamma_2 \lambda \Delta t + \gamma_1 \gamma_2 (\lambda \Delta t)^2 + \zeta_1 \lambda \Delta t$$

$$= 1 + (\gamma_1 + \gamma_2 + \zeta_1) \lambda \Delta t + \gamma_1 \gamma_2 (\lambda \Delta t)^2$$

From 2(b) (ii). we have

$$\gamma_1 + \gamma_2 + \zeta_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 - \sqrt{2} = 1$$

$$\gamma_1 \gamma_2 = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$$

$$\rightarrow \sigma = 1 + \lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^2$$

for stability A Require $|\sigma| \leq 1$

Since H only contribute to imaginary part, we can write

$\lambda \Delta t = i \lambda_I \Delta t$, $\sigma = 1 + i \lambda_I \Delta t + \frac{1}{2} (i \lambda_I \Delta t)^2$

$$\rightarrow |\sigma| = \left| \left[1 - \frac{1}{2} (\lambda_I \Delta t)^2 \right] + i \lambda_I \Delta t \right| \leq 1$$

Problem 3

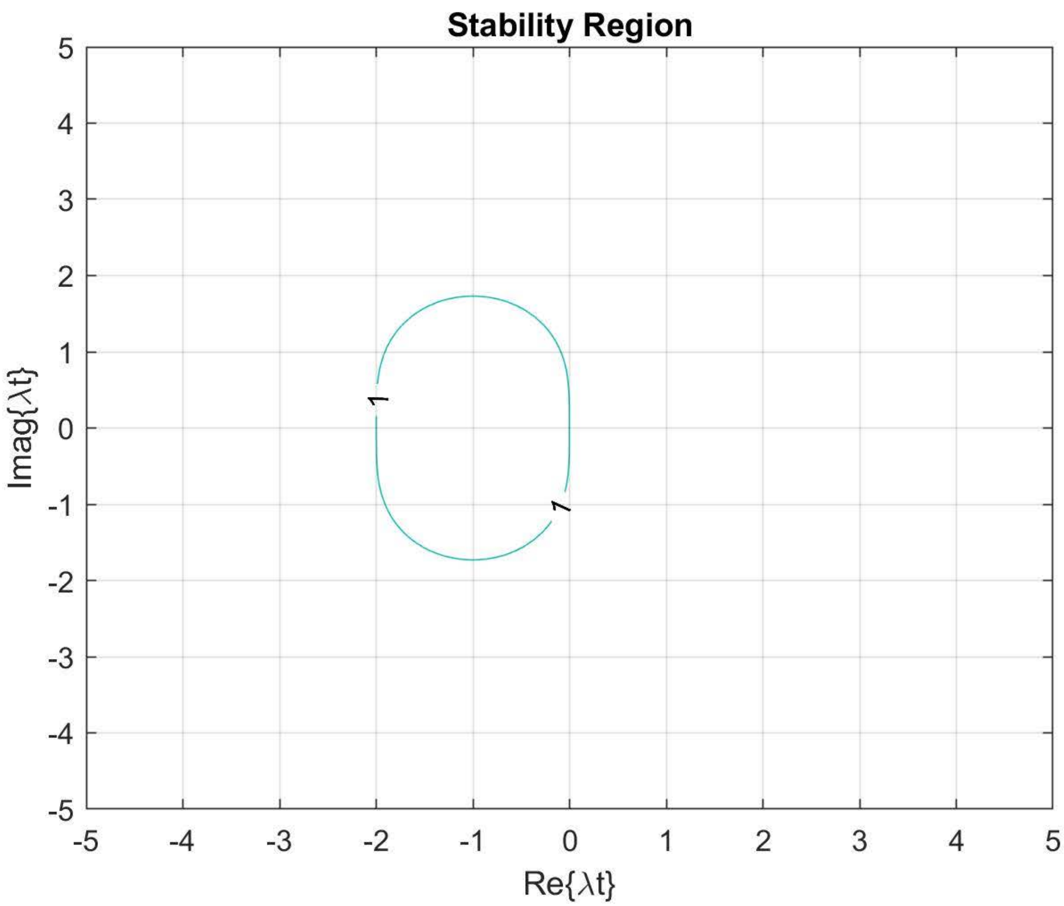
3. (a) (continued) $|\sigma| = \sqrt{\left[1-\frac{1}{2}(\lambda_I \Delta t|^2)\right]^2 + (\lambda_I \Delta t|^2} \leq 1$

$$1 + \frac{1}{4}(\lambda_I \Delta t|^4 - (\lambda_I \Delta t|^2 + (\lambda_I \Delta t|^2 \leq 1$$

$$\longrightarrow \frac{1}{4}(\lambda_I \Delta t|^4 \leq 0$$

Then the step RK-0 is unconditional unstable for Any terms.

$\Rightarrow$ No stable Region



Stability Region
5
4
3
2
1
0
-1
-2
-3
-4
-5
Imag{λt}
-5
-4
-3
-2
-1
0
1
2
3
4
5
Re{λt}
1
1

Problem 3

3. (b) Apply stability Analysis to Model eq. As what has been done in 2(b), we find

$$\sigma = \frac{u^{n+1}}{u^n} = \frac{(1+\alpha_2\lambda\Delta t)(1+\alpha_1\lambda\Delta t)}{(1-\beta_2\lambda\Delta t)(1-\beta_1\lambda\Delta t)}$$

$$= \frac{1+(\sqrt{2}-1)\lambda\Delta t}{\left[1-(1-\frac{\sqrt{2}}{2})\lambda\Delta t\right]^2}$$

Since L term only contribute to Real part of $\lambda\Delta t$

we can write $\lambda\Delta t = \lambda_R\Delta t$

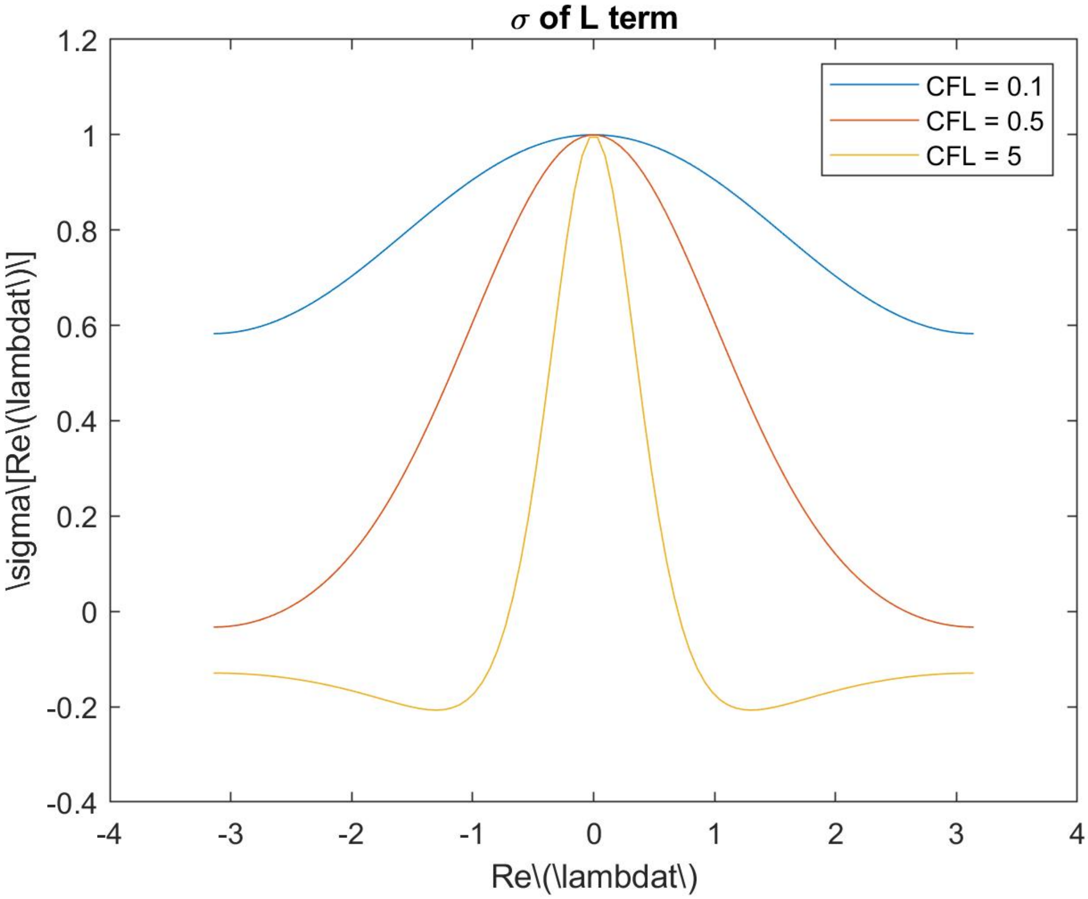
Then $\sigma = \dfrac{1+(\sqrt{2}-1)\lambda_R\Delta t}{\left[1-(1-\frac{\sqrt{2}}{2})\lambda_R\Delta t\right]^2}$

From problem 1, we have

$$\lambda_R\Delta t = \frac{CFL}{Re}\left[-\frac{1}{6}\cos 2k\Delta x + \frac{8}{3}\cos k\Delta x - \frac{5}{2}\right]$$

Then σ can be found for this discretization

Problem3 (b) Plot



Problem 3

3.(c) For stability analysis we can apply RK-θ method to model eqn $\frac{du}{dt} = \lambda u = i\lambda_I u + \lambda_R u = H(u) + Lu$

Then $u^* = u^n + [\alpha_1 L u^n + \beta_1 L u^*]\Delta t + \gamma_1 H(u^n)\Delta t$

$= u^n + \alpha_1 \lambda_R \Delta t\, u^n + \beta_1 \lambda_R \Delta t\, u^* + i\gamma_1 \lambda_I \Delta t\, u^n$

$\rightarrow (1-\beta_1 \lambda_R \Delta t)\, u^* = (1 + \alpha_1 \lambda_R \Delta t + i\gamma_1 \lambda_I \Delta t)\, u^n$

$\rightarrow u^* = (1-\beta_1 \lambda_R \Delta t)^{-1}(1 + \alpha_1 \lambda_R \Delta t + i\gamma_1 \lambda_I \Delta t)\, u^n$

plug this into next step.

$u^{n+1} = u^* + (\alpha_2 L u^* + \beta_2 L u^{n+1})\Delta t + [\gamma_2 H(u^*) + \zeta_1 H(u^n)]\Delta t$

$= u^* + \alpha_2 \lambda_R \Delta t\, u^* + \beta_2 \lambda_R \Delta t\, u^{n+1}$

$+ i\gamma_2 \lambda_I \Delta t\, u^* + i\zeta_1 \lambda_I \Delta t\, u^n$

$\Rightarrow (1-\beta_2 \lambda_R \Delta t)\, u^{n+1} = (1 + \alpha_2 \lambda_R \Delta t + i\gamma_2 \lambda_I \Delta t)\, u^* + i\zeta_1 \lambda_I \Delta t\, u^n$

$\Rightarrow u^{n+1} = \frac{1 + \alpha_2 \lambda_R \Delta t + i\gamma_2 \lambda_I \Delta t}{1-\beta_2 \lambda_R \Delta t} \frac{1 + \alpha_1 \lambda_R \Delta t + i\gamma_1 \lambda_I \Delta t}{1-\beta_1 \lambda_R \Delta t}\, u^n$

$+ \frac{i\zeta_1 \lambda_I \Delta t}{1-\beta_2 \lambda_R \Delta t}\, u^n$

$\Rightarrow \sigma = \frac{u^{n+1}}{u^n} = \frac{(1 + \alpha_2 \lambda_R \Delta t + i\gamma_2 \lambda_I \Delta t)(1 + \alpha_1 \lambda_R \Delta t + i\gamma_1 \lambda_I \Delta t)}{(1-\beta_2 \lambda_R \Delta t)(1-\beta_1 \lambda_R \Delta t)}$

$+ \frac{i\zeta_1 \lambda_I \Delta t}{1-\beta_2 \lambda_R \Delta t}$

Problem 3

3.(c) (continued). Apply the condition $\alpha_2 = 0$, $\beta_1 = \beta_2$

$$\sigma = \frac{(1+i\gamma_2\lambda_I\Delta t)(1+\alpha_1\lambda_R\Delta t + i\gamma_1\lambda_I\Delta t) + i\zeta_1\lambda_I\Delta t(1-\beta_1\lambda_R\Delta t)}{(1-\beta_1\lambda_R\Delta t)^2}$$

$$= \frac{1}{(1-\beta_1\lambda_R\Delta t)^2}\Big[1+\alpha_1\lambda_R\Delta t + i\gamma_1\lambda_I\Delta t + i\gamma_2\alpha_1\lambda_I\Delta t\,\lambda_R\Delta t$$
$$+ (-\gamma_2\gamma_1\lambda_I\Delta t\,\lambda_I\Delta t) + i\zeta_1\lambda_I\Delta t - i\zeta_1\beta_1\lambda_I\Delta t\,\lambda_R\Delta t\Big]$$

$$= \frac{1+\alpha_1\lambda_R\Delta t - \gamma_1\gamma_2(\lambda_I\Delta t)^2}{(1-\beta_1\lambda_R\Delta t)^2} + i\,\frac{\lambda_I\Delta t(\gamma_1+\gamma_1\gamma_2\lambda_R\Delta t + \zeta_1 - \zeta_1\beta_1\lambda_R\Delta t)}{(1-\beta_1\lambda_R\Delta t)^2}$$

When $\gamma_1 = \gamma_2 = \frac{\sqrt{2}}{2}$, $\beta_1 = 1-\frac{\sqrt{2}}{2}$, $\zeta_1 = 1-\sqrt{2}$, $\alpha_1 = \sqrt{2}-1$

$$\lambda_I = CFL\left(\frac{1}{6}\sin 2k\Delta x - \frac{4}{3}\sin k\Delta x\right)$$

$$\lambda_R = \frac{CFL}{Re}\left(-\frac{1}{6}\cos 2k\Delta x + \frac{8}{3}\cos k\Delta x - \frac{5}{2}\right)$$

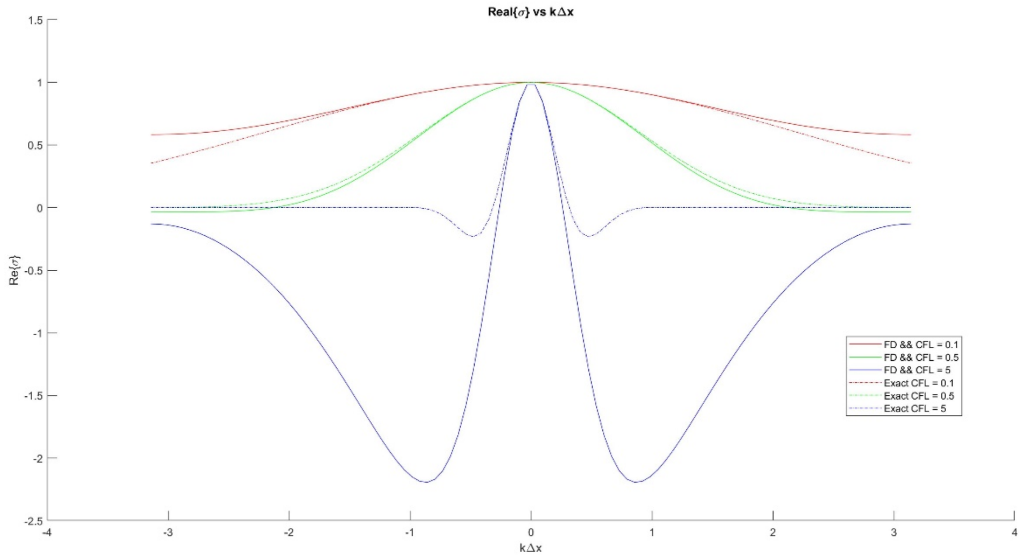
$$k\Delta x = -\pi, \dots, \pi - \frac{2\pi}{N}$$

Thus, for $v^n = 1$, $v^{n+1} = \sigma = \frac{1+\alpha_1\lambda_R\Delta t - \gamma_1\gamma_2(\lambda_I\Delta t)^2}{(1-\beta_1\lambda_R\Delta t)^2}$

$$+ i\,\frac{\lambda_I\Delta t(\gamma_1+\gamma_1\gamma_2\lambda_R\Delta t + \zeta_1 - \zeta_1\beta_1\lambda_R\Delta t)}{(1-\beta_1\lambda_R\Delta t)^2}$$

Then we can use Matlab to plot σ against $k\Delta x$

Problem3 (c) Plot



Problem 3

(d) Compare the σ given by FD and Exact fourier.

- for CFL = 0.1. FD scheme works well for Low wavenumber. but undershoot for high wave number
- for CFL = 0.5. FD scheme Also works well for Low wavenumber. but overshoot for high wavenumber
- for CFL = 5. FD scheme only converge for limited region of wavenumber. ※

The reason for this. when CFL is large. the range of $\lambda \Delta t$ will getting larger and larger, and go further and further beyond the stable Region of the scheme

```
close all;clear;clc;
%Problem1
CFL = [0.1 0.5 5];
Re = 1;
kx = linspace(-pi,pi,100);
figure;hold on;
for i = 1:3
Real_lambdat(:,i) = (CFL(i)/Re).*((-
1/6).*cos(2*kx)+(8/3).*cos(kx)-5/2);
Imag_lambdat(:,i) = CFL(i).*((1/6).*sin(2*kx)-
(4/3).*sin(kx));
plot(Real_lambdat,Imag_lambdat);
end
legend('CFL = 0.1','CFL = 0.5','CFL = 5'); hold off;
title('Locus of
\lambda\Deltat');xlabel('Imag\{\lambda\Deltat\}');ylabel('R
e\{\lambda\Deltat\}');

%Problem3 (a)
x = linspace(-5,5,100); y = x; [X,Y]=meshgrid(x,y);
Z = abs(1+(X+i*Y)+(1/2)*(X+i*Y).^2);
figure; contour(X,Y,Z,[1,1],'ShowText','on');
title('Stability Region');
xlabel('Re\{\lambdat\}'); ylabel('Imag\{\lambdat\}'); grid
on;

%Problem3 (b)
sigma_b = (1+(sqrt(2)-1).*Real_lambdat)./(1-(1-
sqrt(2)/2).*Real_lambdat).^2;
figure; plot(kx,sigma_b); title('\sigma of L
term');xlabel('Re\{\lambdat\}');
ylabel('\sigma\{Re\(\lambdat\)\}'); legend('CFL = 0.1','CFL
= 0.5','CFL = 5');
ylim([-0.4 1.2])

%Problem3 (c)
alpha = sqrt(2)-1; beta1 = 1-sqrt(2)/2; gamma1 = sqrt(2)/2;
gamma2 = gamma1;
zeta = sqrt(2)-1;
%sigma = Amplification factor
sigma = (1+alpha.*Real_lambdat-
gamma1.*gamma2.*(Imag_lambdat).^2)./(1-
beta1.*Real_lambdat).^2 ...
+1i*Imag_lambdat.*(gamma1+gamma1.*gamma2.*Real_lambdat+zeta
-zeta.*beta1.*Real_lambdat)./(1-beta1.*Real_lambdat).^2;
for i = 1 : 3
sigma_exact(:,i) = exp((1i.*kx-kx.^2)*CFL(i));
end
```

```
figure; hold on; p1 = plot(kx,sigma); p2 =
plot(kx,sigma_exact,'-.');
set(p1,{'color'},{[1 0 0];[0 1 0];[0 0 1]});
set(p2,{'color'},{[1 0 0];[0 1 0];[0 0 1]});
xlabel('k\Deltax'); ylabel('Re\{\sigma\}');
title('Real\{\sigma\} vs k\Deltax');
legend('FD && CFL = 0.1','FD && CFL = 0.5','FD && CFL =
5','Exact CFL = 0.1','Exact CFL = 0.5','Exact CFL = 5');
ylim([-2.5 1.5])
```