MAE 290C, Spring 2019
HOMEWORK 1
Due Fri 04/19/2019 11:59PM (dropbox, email, google drive)
Provide source codes used to solve all questions

Problem 1. Let us consider the 1D linear dissipative convection problem

$$\partial_t u = -c\partial_x u + \nu \partial_{xx} u = H(u) + Lu = \lambda u \tag{1}$$

as a model to study time integration schemes for the Navier-Stokes equation, where we will treat $H(u) = c\partial_x u$ explicitly and $Lu = \nu\partial_{xx}u$ implicitly, and λ is an eigenvalue that depends on the spatial discretization method.

1. Find λ when the spatial derivatives are discretized with fourth-order centered finite difference schemes. Plot the locus of $\lambda\Delta t$ in the complex plane for $c\Delta t/\Delta x = 0.1,\ 0.5,\ 5$.

2. Let us derive a two-step RK-θ scheme to integrate equation (1) in time. The scheme should have the form

$$u^* = u^n + (\alpha_1 Lu^n + \beta_1 Lu^*)\Delta t + \gamma_1 H(u^n)\Delta t \tag{2}$$

$$u^{n+1} = u^* + (\alpha_2 Lu^* + \beta_2 Lu^{n+1})\Delta t + \left[\gamma_2 H(u^*) + \zeta_1 H(u^n)\right]\Delta t \tag{3}$$

   (a) Find the conditions required for the scheme (2) – (3) to be second order accurate. Write these conditions in the form of algebraic equations for the parameters $\alpha_i,\ \beta_i,\ \gamma_i, \zeta_1$ with $i = 1, 2$.

   (b) Show that these conditions lead to an undertermined problem. There are several solution branches, but an interesting one is obtained by setting β_1 and β_2 as free parameters. This branch admits a solution where $\sigma = \|u^{n+1}/u^n\| \to 0$ for $Re(\lambda\Delta t) \to \infty$, which is interesting for stability reasons and is also easy to find.

      i. Show that $\sigma = \|u^{n+1}/u^n\| \to 0$ for $Re(\lambda\Delta t) \to \infty$ implies that either $\alpha_1 = 0$ or $\alpha_2 = 0$.

      ii. Using this information, let us find a balanced scheme stemming from this branch with $\beta_1 = \beta_2$. Find the values of $\alpha_i,\ \beta_i,\ \gamma_i, \zeta_1$ with $i = 1, 2$ for that balanced scheme.

3. Now let us study the stability of this second-order scheme.

   (a) Plot the stability region of the explicit part of the scheme, together with the three loci of $\lambda\Delta t$ from question 1.

   (b) Plot $\sigma[Re(\lambda\Delta t)]$ for the implicit part of the scheme.

   (c) Finally, let us compare σ for the whole scheme with the exact value,

$$\sigma_{exact} = \exp\left\{\left[-ik\Delta x - \frac{(k\Delta x)^2}{Re_\Delta x}\right]\frac{c\Delta t}{\Delta x}\right\},$$

   where $Re_{\Delta x} = c\Delta x/\nu$. Plug in $\lambda(c, \nu, \Delta x, k)$ from the fourth-order finite difference scheme into σ for the whole RK2 scheme and particularize for $Re_{\Delta x} = 1$. Plot the resulting $\sigma(k\Delta x, c\Delta t/\Delta x)$ as a function of $k\Delta x\,\epsilon\,[-\pi, \pi)$ for $c\Delta t/\Delta x = 0.1,\ 0.5,\ 5$.
   Hint: it is alright to do this step numerically (e.g. set $u^n = 1$ and perform one iteration of the RK2-θ scheme for different values of $k\Delta x$, then plot). However, be careful to use the whole scheme and not its Taylor expansion, which is only valid for small $\lambda\Delta t$.

   (d) Discuss the results of the previous question, relating them to the plots you obtained in question 3.a.