# MAE 290C, Spring 2019
## HOMEWORK 3
**Due Friday May 10 11:59PM (dropbox, email, google drive)**
**Provide source codes used to solve all questions**

**Problem 1.** Let us consider non-dimensional Navier-Stokes equations for incompressible flow in non-dimensional form

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + Re^{-1}\nabla^2 \mathbf{u}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

in a 2D periodic domain $0 \le x < 2\pi$ and $0 \le y < 2\pi$. In order to illustrate the fractional step method, let us integrate the non-linear terms with an explicit Euler method and the linear terms including the pressure with an implicit Euler method, to arrive at

$$M \begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} A & G\Delta t \\ D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^n - \mathbf{H}^n \Delta t \\ 0 \end{bmatrix}, \tag{3}$$

where $A = I - Re^{-1}\Delta t L$, $G$ is the discretized gradient operator, $D$ is the discretized divergence operator, $L = D \cdot G$ is the discretized Laplacian operator and $H$ are the discretized non-linear terms. To keep the illustration simple, let us consider a Fourier spatial discretization in both the $x$ and $y$ directions, with $N_x = N_y = 32$, $Re = 1000$ and $\Delta t = 0.01$.

1. Write a code to determine block matrices $A$, $G$, and $D$, in wavenumber space ($\alpha$, $\beta$) and assemble matrix $M$. Plot each of these 4 matrices vs ($\alpha$, $\beta$).

   <u>*Hints*</u>*:* Plot matrices in 2D using the MATLAB command `imagesc` or a method that yields a similar output. Note that matrix $A$ has a size $2n \times 2n$ where $n = N_x N_y$, $D$ has a size $n \times 2n$ and $G = D^T$. You can define these matrices by means of `for` or `do` loops but you may also find the `sparse`, `diag` and `repmat` commands useful if you are working with MATLAB. For instance, the identity matrix term in the $A$ block of $M$ is generated by

   ```
   %
   II_2n = diag(ones(2*n,1));
   %
   eye_in_A = sparse(1:2*n,1:2*n,II_2n,3*n,3*n);
   %
   ```

   and the $\partial_x$ term in $G$ can be generated by

   ```
   %
   alp = [0:Nx/2-1 -Nx/2:-1];
   %
   ialp = repmat(i*alp,[1,Ny]);
   %
   Galp = sparse(1:n    ,2*n+1:3*n,ialp,3*n,3*n);
   %
   ```

2. The resulting matrix $M$ is singular. In this Fourier representation, the singularity is easy to grasp since $M$ has a row of zeros. Which one? (i.e. which value of ($\alpha$, $\beta$)?) What does this mean physically in terms of the pressure?

3. The artificial compressibility method removes the singularity of $M$ by solving the approximate problem

$$\begin{bmatrix} A & G\Delta t \\ D & \lambda I \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{n} - \mathbf{H}^n \Delta t \\ \lambda p^n \end{bmatrix}. \tag{4}$$

Write a code to assemble the new $M$ matrix of the artificial compressibility method for $\lambda = 0.01$. Plot the matrix and calculate the eigenvalue closest to zero.

4. The fractional step method is equivalent to an approximate $LU$ factorization of the original problem

$$\begin{bmatrix} A & G\Delta t \\ D & 0 \end{bmatrix} \approx \begin{bmatrix} A & ABG\Delta t \\ D & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ D & -DBG\Delta t \end{bmatrix} \begin{bmatrix} I & BG\Delta t \\ 0 & I, \end{bmatrix} \tag{5}$$

with $B \approx inv(A)$, leading to the scheme:

$$A\mathbf{u}^* = \mathbf{u}^n - \mathbf{H}^n \Delta t \tag{6}$$

$$DBGp^{n+1} = \frac{D\mathbf{u}^*}{\Delta t}, \tag{7}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^* + BGp^{n+1}\Delta t. \tag{8}$$

It turns out that working with a Fourier spatial discretization makes computing $inv(A)$ trivial since $A$ is a diagonal matrix. Thus, we will use an *exact* fractional step replacing $B$ with $inv(A)$ in the scheme above, and then use the solution as reference to compute the errors of the artificial compressibility scheme.

(a) Assuming that $u^n = \sin x \sin y$ and $v^n = \cos x \cos y$, which satisfies continuity, show that the non-linear terms are

$$u\partial_x u + v\partial_y u = \sin(2x)/2, \tag{9}$$

$$u\partial_x v + v\partial_y v = -\sin(2y)/2. \tag{10}$$

so that $\mathbf{u}^n - \mathbf{H}^n\Delta t$ does not satisfy continuity and therefore $\mathbf{u}^*$ does not satisfy continuity either. Fourier transform $\mathbf{u}^n$ and the non-linear terms and perform an exact iteration of the fractional step method. Assume $p^n = 0$ everywhere. There is no aliasing in this case given that we chose $N_x$ and $N_y$ to be large enough. Plot the intermediate solution in the physical domain $u^*(x, y)$, $v^*(x, y)$ and $p^*(x, y)$. Plot the solution in the physical domain $u^{n+1}(x, y)$, $v^{n+1}(x, y)$ and $p^{n+1}(x, y)$. How does it differ from the intermediate solution? (*Hint:* Matrix $DBG$ in 7 has one row of zeros for the same reason that $M$ has one. But this indetermination is not important because the product $BGp^{n+1}$ in 8 cancels it. So, it is ok to choose a bogus value of $\widehat{p}_{\alpha=0,\beta=0}$.)

(b) Perform one iteration with the artificial compressibility scheme. Plot the solution in the physical domain $u^{n+1}(x, y)$, $v^{n+1}(x, y)$ and $p^{n+1}(x, y)$.

(c) Plot the L2-norm of the errors in $u$, $v$ and $p$ as a function of $\Delta t$ and $\lambda$. Vary $\Delta t$ in the range $[10^{-6}, 10^{-1}]$ and $\lambda$ in the range $[10^{-4}, 0.1]$.

(d) **Extra credit (kind of difficult):** Incidentally, it also turns out that $A$ being diagonal in Fourier discretizations also makes any matrix $B$ produce an exact fractional step as long as $B$ is diagonal, non-singular and $B(n + j, n + j) = B(j, j)$ for $j = 1, n$. Prove this property for extra credit and try it numerically with $B(j, j)$, $j = 1, n$ defined as random matrix. The lesson to learn here is that depending on your spatial discretization scheme, there may be many simple ways to make the projection operation

$$\mathbf{u}^{n+1} = \left[I - BG\left(DBG\right)^{-1} D\right] \mathbf{u}^{n+1}$$

accurate or even exact, and that choosing $B$ to be the Taylor expansion of $inv(A)$ might not always be the most efficient method.