$$u_{N-j} = e^{-i\omega t}\left[e^{ik\Delta x j} + r e^{i(\pi - k\Delta x)j}\right]$$

$j=0$

1. $u_N = 0$

$$0 = e^{-i\omega t}\left[e^0 + r e^0\right] = e^{-i\omega t}(1+r)$$

$\Rightarrow \quad 0 = 1 + r$

$\Rightarrow \quad \boxed{r = -1}$

2. $\underset{j=0}{U_N} = 2\underset{j=1}{U_{N-1}} - \underset{j=2}{U_{N-2}}$

$$\cancel{e^{-i\omega t}}[e^0 + re^0] = 2\cancel{e^{-i\omega t}}[e^{ik\Delta x} + re^{i(\pi - k\Delta x)}] - \cancel{e^{-i\omega t}}[e^{i2k\Delta x} + re^{i2(\pi - k\Delta x)}]$$

$$\Rightarrow \quad 1+r = 2e^{ik\Delta x} - 2re^{-ik\Delta x} - e^{i2k\Delta x} - re^{-i2k\Delta x}$$

$$\Rightarrow \quad r(1 + 2\cos k\Delta x - 2i\sin k\Delta x + \cos(2k\Delta x) - i\sin(2k\Delta x)) = -1 + 2\cos k\Delta x + 2i\sin k\Delta x - \cos(2k\Delta x) - i\sin(2k\Delta x)$$

$$\Rightarrow \quad \boxed{r = \frac{[-1 + 2\cos k\Delta x - \cos(2k\Delta x)] - i[-2\sin k\Delta x + \sin(2k\Delta x)]}{[1 + 2\cos k\Delta x + \cos(2k\Delta x)] - i[2\sin k\Delta x + \sin(2k\Delta x)]}}$$

$$\begin{array}{cccc} j=0 & j=1 & j=2 & j=3 \end{array}$$

3. $U_N = 3U_{N-1} - 3U_{N-2} + U_{N-3}$

$$\cancel{e^{-i\omega t}}[e^0 + re^0] = 3\cancel{e^{-i\omega t}}[e^{ik\Delta x} + re^{i(\pi - k\Delta x)}]$$
$$-3\cancel{e^{-i\omega t}}[e^{i2k\Delta x} + re^{i2(\pi - k\Delta x)}]$$
$$+\cancel{e^{-i\omega t}}[e^{i3k\Delta x} + re^{i3(\pi - k\Delta x)}]$$

$$\Rightarrow \quad 1+r = 3e^{ik\Delta x} - 3e^{i2k\Delta x} + e^{i3k\Delta x} - 3re^{-ik\Delta x} - 3re^{-i2k\Delta x} - re^{-i3k\Delta x}$$

$$\Rightarrow \quad r(1 + 3\cos k\Delta x - 3i\sin k\Delta x + 3\cos(2k\Delta x) - 3i\sin(2k\Delta x) + \cos(3k\Delta x) - i\sin(3k\Delta x))$$
$$= 3\cos k\Delta x + 3i\sin k\Delta x - 3\cos(2k\Delta x) - 3i\sin(2k\Delta x) + \cos(3k\Delta x) + i\sin(3k\Delta x) - 1$$

$$\Rightarrow \quad \boxed{r = \frac{[-1 + 3\cos k\Delta x - 3\cos(2k\Delta x) + \cos(3k\Delta x)] - i[-3\sin k\Delta x + 3\sin(2k\Delta x) - \sin(3k\Delta x)]}{[1 + 3\cos k\Delta x + 3\cos(2k\Delta x) + \cos(3k\Delta x)] - i[3\sin k\Delta x + 3\sin(2k\Delta x) + \sin(3k\Delta x)]}}$$

$$\overset{j=0}{U_N} = \overset{j=1}{U_{N-1}}$$

4.

$$\cancel{e^{-i\omega t}}[e^0 + re^0] = \cancel{e^{-i\omega t}}[e^{ik\Delta x} + re^{i(\pi - k\Delta x)}]$$

$$\Rightarrow \quad 1 + r = \cos k\Delta x + i\sin k\Delta x - r\cos k\Delta x + ri\sin k\Delta x$$

$$\Rightarrow \quad r(1 + \cos k\Delta x - i\sin k\Delta x) = -1 + \cos k\Delta x + i\sin k\Delta x$$

$$\Rightarrow \quad \boxed{r = \frac{-1 + \cos k\Delta x + i\sin k\Delta x}{1 + \cos k\Delta x - i\sin k\Delta x}}$$

5. $U_N = -U_{N-1}$

$\cancel{e^{-i\omega t}}[e^0 + re^0] = -\cancel{e^{-i\omega t}}[e^{ik\Delta x} + re^{i(\pi - k\Delta x)}]$

$\Rightarrow \quad 1 + r = -\cos k\Delta x - i\sin k\Delta x + r\cos k\Delta x - ri\sin k\Delta x$

$\Rightarrow \quad r(1 - \cos k\Delta x + i\sin k\Delta x) = -1 - \cos k\Delta x - i\sin k\Delta x$

$\Rightarrow \quad \boxed{r = \dfrac{-(1 + \cos k\Delta x + i\sin k\Delta x)}{1 - \cos k\Delta x + i\sin k\Delta x}}$

6. $\frac{d \overset{j=0}{U_N}}{dt} = -c\, \frac{\overset{j=0}{U_N} - \overset{j=1}{U_{N-1}}}{\Delta x}$

$$\frac{d}{dt}\left(e^{-i\omega t}(e^0 + re^0)\right) = -\frac{c}{\Delta x}\Big[e^{-i\omega t}(e^0 + re^0) - e^{-i\omega t}\left(e^{ik\Delta x} + re^{i(\pi - k\Delta x)}\right)\Big]$$

$$\Rightarrow \quad -i\omega\, \cancel{e^{-i\omega t}}(1+r) = -\frac{c}{\Delta x}\cancel{e^{-i\omega t}}\left[1+r-\left(e^{ik\Delta x} - re^{-ik\Delta x}\right)\right]$$

By $\quad \omega = -\frac{c}{\Delta x}\sin k\Delta x$

$$\Rightarrow \quad -i\left(\cancel{-\frac{c}{\Delta x}}\sin k\Delta x\right)(1+r) = \left(\cancel{-\frac{c}{\Delta x}}\right)\left[1+r-\left(e^{ik\Delta x} - re^{-ik\Delta x}\right)\right]$$

$$\Rightarrow \quad -i\sin k\Delta x - i(\sin k\Delta x)r = 1+r - \cos k\Delta x - i\sin k\Delta x + r\cos k\Delta x - ir\sin k\Delta x$$

$$r\left(1+\cos k\Delta x + \cancel{i\sin k\Delta x} - \cancel{i\sin k\Delta x}\right) = -\cancel{i\sin k\Delta x} - 1 + \cos k\Delta x + \cancel{i\sin k\Delta x}$$

$$\Rightarrow \quad \boxed{r = \frac{\cos k\Delta x - 1}{1+\cos k\Delta x}}$$

7. $$\frac{\partial u_N}{\partial t} = -c\,\frac{3u_N - 4u_{N-1} + u_{N-2}}{2\Delta x}$$

$$\frac{d}{dt}\left(e^{-i\omega t}(e^0 + re^0)\right) = -\frac{c}{2\Delta x}\Big[3e^{-i\omega t}(e^0 + re^0) - 4e^{-i\omega t}\left(e^{ik\Delta x} + re^{i(\pi - k\Delta x)}\right) + e^{-i\omega t}\left(e^{i2k\Delta x} + re^{i2(\pi - k\Delta x)}\right)$$

$$\Rightarrow -i\omega \cancel{e^{-i\omega t}}(1+r) = \frac{-c}{2\Delta x}\Big[3(1+r)\cancel{e^{-i\omega t}} - 4\cancel{e^{-i\omega t}}\left(e^{ik\Delta x} - re^{-ik\Delta x}\right) + \cancel{e^{-i\omega t}}\left(e^{i2k\Delta x} + re^{-i2k\Delta x}\right)\Big]$$

by $\omega = -\frac{c}{\Delta x}\sin k\Delta x$

$$\Rightarrow -i\left(\cancel{\frac{-c}{\Delta x}}\right)(\sin k\Delta x)(1+r) = \left(\cancel{-\frac{c}{\Delta x}}\right)\frac{1}{2}\Big[3 + 3r - 4\cos k\Delta x - 4i\sin k\Delta x + 4r\cos k\Delta x - 4ir\sin k\Delta x + \cos 2k\Delta x + i\sin 2k\Delta x + r\cos(2k\Delta x) - ri\sin(2k\Delta x)\Big]$$

$$\Rightarrow r\left[3 + 4\cos k\Delta x + \cos(2k\Delta x) - 4i\sin k\Delta x - i\sin(2k\Delta x) + 2i\sin k\Delta x\right]$$
$$= -3 + 4\cos k\Delta x - \cos(2k\Delta x) + 4i\sin k\Delta x - i\sin(2k\Delta x) - 2i\sin k\Delta x$$

$$\Rightarrow \boxed{r = \frac{[-3 + 4\cos k\Delta x - \cos(2k\Delta x)] + i[2\sin k\Delta x - \sin(2k\Delta x)]}{[3 + 4\cos k\Delta x + \cos(2k\Delta x)] - i[2\sin k\Delta x + \sin(2k\Delta x)]}}$$

To minimize Reflection, we need to minimize r.

From Numerical Result, it can be seen that Linear Extrapolating, Quadratic Extrapolating, First order upwind and second order upwind boundary conditions have no reflection,

Homogenous Neumann boundary condition observed small oscillation / reflection ; However, Homogenous boundary condition and Anti-symmetric boundary condition have serious Reflection,

Compare Numerical Result with Analytical study of Reflection it can be seen that they are consistent with each other,

In this case, for RK3 to be stable, we need

$$c < \sqrt{3}\frac{\Delta x}{\Delta t}$$

From initial condition, $w = A = 0.1$ or 3

$$\Rightarrow |k\Delta x| = \left|\frac{w}{c}\right|\Delta x = \frac{A}{\frac{\sqrt{3}\Delta x}{\Delta t}}\cdot\Delta x = \frac{A\Delta t}{\sqrt{3}} = 0.017 \text{ or } 5.8\times10^{-4}$$

$k\Delta x$ for both of this two case are very close to origin.

Thus BC 2, 3, 6, 7 have $|H|$ of nearly $0 \Rightarrow$ no Reflection

1 have $|H| = 1 \Rightarrow$ total Reflection

4 have $|H|$ slightly larger, $\Rightarrow$ minor Reflection

5 have $|r| \gg 1 \Rightarrow$ Serious Reflection

Also. This is Also the reason for Reflection of $A=\infty$ is less than $A=3$ for Homogenous Neumann boundary condition