# 1 Definitions

In linear programming problem we wish to optimize(minimize/maximize) a linear function given a set of linear inequalities

## 1.1 Linear function

The linear function to optimize is written in terms of $n$ numbers $a_1, a_2, ..., a_n$ and variables $x_1, x_2, ..., x_n$ and is written as

$$f(x_1, x_2, ..., x_n) = a_1x_1 + a_2x_2 + ... + a_nx_n = \sum_{j=1}^{n} a_jx_j$$

## 1.2 Linear equality and inequality

If $b$ is a real number, and $f$ is a linear function as defined above, then

$$f(x_1, x_2, ..., x_n) = b$$

is a **linear equality.** The following are linear inequalities

$$f(x_1, x_2, ..., x_n) \leq b$$
$$f(x_1, x_2, ..., x_n) \geq b$$

## 1.3 Linear constraints

These are either linear equalities or linear inequalities. We do not however allow strict inequalities.

## 1.4 Linear-programming problem

Is the problem of optimizing a linear function subset to a set of linear constraints. The optimization can be minimization or maximization and the problem to be solved is respectively called a minimization/maximization linear program.

## 1.5 Standard form

Informally it is maximization of a linear function subject to a set of linear *inequalities*. Formally, we are given $n$ real numbers $c_1, c_2, ..., c_n$ and $m$ real numbers $b_1, b_2, ..., b_m$. This gives a total of $m \cdot n$ real numbers $a_{ij}$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. The the linear program is to find $n$ real number $x_1, x_2, ..., x_n$ that satisfies

$$\begin{array}{ccccl} \text{maximize} & \sum_{j=1}^{n} c_jx_j & & & \\ \text{s.t} & \sum_{j=1}^{n} a_{ij}x_j & \leq & b_i & \text{for } i = 1, 2, ..., m \\ & x_j & \geq & 0 & \text{for } j = 1, 2, ..., n \end{array}$$

The function to maximize is the **objective function** and the $n + m$ inequalites are the **constraints**. The last constraint is called **nonnegativity constraint**. Standard form requires nonnegativity constraint. It can also be written in terms of matrices.

The setting of variables that satisfies the *constrains* is called *feasible* and a setting that of variables that fails to satisfy at least 1 is called ***infeasible.***A feasible solution with maximal value is called an

### 1.6 Optimal solution

Feasible solution that is the maximum over all feasible solutions.

### 1.7 Slack form

Informally it is maximization of a linear function subject to a set of linear *equalities*.

Basically standard form but where the only inequality constraints are the nonnegative constraints, and the other inequalities are converted into equality constraint.

### 1.8 Feasible solution

A linear program consists of $n$ variables and $m$ constraints. Any setting of the $n$ variables that satisfies the $m$ constraints is a *feasible solution*.

### 1.9 Convex region

The points of a line segment between any two points in the region will also be in the region.

### 1.10 Objective function

The function we wish to maximize. Remember for both slack and standard form, we wish to maximize.

### 1.11 Feasible region

The setting of feasible solutions form an $n$-dimensional convex region.

### 1.12 Objective value

The value of the objective function in the feasible region. We can search for the best objective value in the feasible region. This is however problematic when the feasible region consists of infinite number of points.

*It is the value of a feasible solution.*

### 1.13 Basic variables

The variables on the left hand side of a LP in slack form

### 1.14 Nonbasic variables

The variables on the right hand side of a LP in slack form.

### 1.15 Basic solution

This is computed for each basic variable in slack form. It is achieved by setting each nonbasic variable to 0 and computing the equality constraint.

### 1.16 Tight

Setting of nonbasic variables that causes the basic variable to become 0

### 1.17 Violates

Setting of nonbasic variables that causes the basic varaible to become negative.

### 1.18 Basic solution

Set all nonbasic variables to 0 and compute basic variable on the LHS

### 1.19 Basic feasible solution

If the basic solution is also feasible.

### 1.20 Pivot

Choose a nonbasic variable, $x_e$, called the entering variable and a basic variable $x_l$, the leaving variable, and exchange their role after increasing $x_e$ as much as possible.

## 2 Simple linear program in 2D

Consider the linear program on page 846 of CLRS. The set of feasible solutions to the linear program for which we have$x_1 + x_2 = z$ that the sum is a particular value $z$ are those that lie on a line with slope $-1$. Intuitively, if we draw the $x_1, x_2$ coordinate system, then if either increases, the other must decrease by the same amount. For instance, $x_1 + x_2 = 0$ is a line that only touches the feasible region in $(x_1, x_2) = (0, 0)$. Generally, for any $z$, the line $x_1 + x_2 = z$ in the $x_1 - x_2$ coordinate system, and the intersection with the feasible region is a set of feasible solutions. Since the feasible region is bounded, there must be a $z$, for which $x_1 + x_2$ is the highest. That is we want a line $x_1 + x_2 = z$

with maximal $z$ that is within the feasible region. Any point in this line must be optimal, because this line has maximal value $z$.

The line $x_1 + x_2 = z$ with maximum value $z$ will be on the border of the feasible region, since this will be the absolute highest values of $x_1, x_2$. The line can then intersect the feasible region in a single vertex, or a line segment. In case of a line segment, then either endpoint of the line segments are optimal solutions.

## 3 Informal simplex

1. Write the linear program from standard form to slack form
   (a) The linear equalities form *basic variables*
   (b) Basic variables are expresses in terms of *nonbasic variables*
2. We move from one vertex to another in the feasible region by making nonbasic variables basic and basic variables nonbasic. This is called a *pivot*

## 4 Converting linear programs to standard form

It is always possible to convert a minimization/maximization linear program with inequalities into standard form linear program. There are the following reasons for the linear program not being in standard form

1. It is a minimaztion problem
2. No nonnegativity constraint
3. Equality constraint
4. Greater than equal inequalities

We convert a linear program $L$ into an *equivalent* linear program $L'$ in standard form. By equivalent, the following is meant

1. For a maximization linear program $L$
   (a) If there exists a feasible solution $\bar{x}$ in $L$ with objective value $z$, then there exists a feasible solution $\bar{x'}$in $L'$ with objective value $z$.
   (b) For each feasible solution $\bar{x'}$ in $L'$ with objective value $z$ there is an feasible solution $\bar{x}$in $L$ with objective value $z$.
2. For a minimization linear program $L$
   (a) If there exists a feasible solution $\bar{x}$ in $L$ with objective value $z$, then there exists a feasible solution $\bar{x'}$in $L'$ with objective value $z$.

(b) For each feasible solution $\bar{x'}$ in $L'$ with objective value $-z$ there is an feasible solution $\bar{x}$in $L$ with objective value $z$.

The equivalence of the maximization linear program is easy to see why holds - they share optimal solutions. For the minimization, the smallest value $z$ will be the largest value amongst $-z$. And the largest value of negative number $-z$ will be the one with smallest $z$.

## 4.1 How to convert

Overcoming each of the 4 violations to standard form has to show the objective value can be converted, and that the feasible solution can be converted.

1. To convert a minimization program - negate coefficients of the objective function
   Why does this work?

   (a) We find an objective value $-z$, so we can covert it to $z$
   We want to minimize an objective function of the form $\sum_{j=1}^{n} c_j x_j = c_1x_1 + c_2x_2 + ... + c_nx_n$. If we negate each coefficient $c_j$ we get $-c_1x_1 - c_2x_2 - ... - c_nx_n$. Maximizing this will give us the smallest negative number $-z$. This is just be a minus in front of minimizing the objective function and getting $z$.

   (b) We have that $\bar{x} = \bar{x'}$
   The variables for the feasible solution in $L'$ must be the same as $L$, since we get $-z$, and we negated the coefficients of the objective function. Instead of minimizing $x_1 + x_2$, we maximize $-x_1 - x_2$ and find $-z$. We only get $-z$ since we negated coefficients.

2. Missing nonnegativity constraint - Replace every occurrence of $x_j$ with $x'_j - x''_j$. In the objective function we transform $c_jx_j \mapsto c_jx'_j - c_jx''_j$ , and replace constraints $a_{ij}x_j \mapsto a_{ij}x'_j - a_{ij}x''_j$ and add the nonnegative constraint $x'_j, x''_j \geq 0$.

   (a) How are the feasible solutions equivalent?
   The feasible solution $\hat{x}$ corresponds to a feasible solution $\bar{x}$ by simply setting $\bar{x_j} = \hat{x'_j} - \hat{x}''_j$. Now the other way, any feasible solution $\bar{x}$ corresponds to the feasible solution $\hat{x}$by setting $\hat{x}'_j = \bar{x}_j$and $\hat{x}''_j = 0$ if $\bar{x_j} \geq 0$. Otherwise when $\bar{x_j} < 0$ we can set $\hat{x}''_j = -\bar{x}_j$and $\hat{x}'_j = 0$.

   (b) How are the objective values equivalent?
   For the case where $\bar{x_j} \geq 0$ then $\bar{x_j} = \hat{x'_j} - \hat{x}''_j$ is the exact same expression. And when $\bar{x_j} < 0$ then $\bar{x}_j = \hat{x'_j} - \hat{x}'_j = -\hat{x}'_j = \bar{x}_j$ so we maximize the same objective function and an objective value in $L'$ must be an objective value in $L$.

3. Dealing with equality constraint - Replace all functions $g(x_1, ..., x_n) = b$ with two inequalities $g(x_1, x_2, ..., x_n) \geq b$ and $g(x_1, x_2, ..., x_n) \leq b$
   (a) The feasible solutions must be equivalent since we do not change the variables and the double inequality ensures we only get a solution when $g(x_1, ..., x_n) = b$
   (b) The objective value is also the same since the double inequality ensures we only have a solution when $g(x_1, ..., x_n) = b$.
4. Dealing with greater than equal - multiply these constraints through with $-1$. This flips the inequality and negates coefficients on both sides (this fixes all greater than equal from 3).
   (a) This is an equivalent inequality and we do not change the variables so we have an equivalent feasible solution
   (b) Again, the constraint is equivalent so we must get the same objective value in both solutions

# 5 Converting LP to slack form

We replace all inequalities that are not nonnegative constraints with a **slack variable** $s$ and a new nonnegative constraint

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j \tag{1}$$

$$s \geq 0 \tag{2}$$

The variable is called slack since it measures the difference between $b_i$and the sum. Remember the constraint was normally of the form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \tag{3}$$

so the slack captures how much the sum is off from the constraint. Why are these equivalent? Well if (1) and (2) holds, then $b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$ which implies (3). Likewise we can write 3 as $b_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0$ and the split it into (1) and (2). This means we get an equivalent $L'$ by the rewrite. We will usally write the slack variable as $x_{n+1} = b_i - \sum_{j=1}^{n} a_{ij} x_j$.

# 6 Common problems as LP

## 6.1 Single pair shortest path

The problem gives us a graph $G = (V, E)$ and a weight function $w$ mapping edges to real values. We wish to find the shortest path from source vertex $s$ to destination vertex $t$. The value of the shortest path is called $d_t$.

Bellman-Ford will at termination find the value $d_v$for each each vertex such that $d_v \leq d_u + w(u, v)$ (remember, the shortest path either skips $(u, v)$ so strict inequality holds or it consists of a path $s \rightsquigarrow u \rightarrow v \rightsquigarrow t$. This becomes the linear program

$$\begin{array}{rlcll} \text{maximize} & d_t & & & \\ s.t & d_v & \leq & d_u + w(u,v) & \text{for each edge } (u,v) \in E \\ & d_s & = & 0 & \end{array}$$

Note that this is a maximization problem instead of minimization. The reason is that minimizing would give a solution where each $d_u = 0$, since this satisfies each constraint. However, due to the inequality constraint, the algorithm will find the maximum $d_v$ for which the inequality constraint holds for each vertex. So this does find the shortest path. There are $|V|$ variables and $|E|+1$ constraints.

### 6.2 Maximum flow

A flow is a real valued function $f : V \times V \mapsto \mathbb{R}$ that satisfies capacity constraint and flow conservation. A max flow is a flow that maximizes the value of a flow. We can use this to write a linear program. We assume the $c(u, v) = 0$ if $(u, v) \notin E$ and we also assume there are no antiparallel edges

$$\begin{array}{rccll} \text{maximize} & \sum_{v \in V} f_{sv} & - & \sum_{v \in V} f_{vs} & \\ s.t & f_{uv} & \leq & c(u,v) & \text{for each } u,v \in V \\ & \sum_{v \in V} f_{vu} & = & \sum_{v \in V} f_{vu} & \text{for each } u \in V \setminus \{s,t\} \\ & f_{uv} & \geq & + & \text{for each } u,v \in V \end{array}$$

We see this simply states the rules of the flow. The flow has for each pair of vertices a variable $uv$ so we have $V^2$variables. From constraint line 1 and 3 we have $2|V|^2$ constrains. The second line only loops over $u$ so it has $|V| - 2$ constraints making the total constraints $2|V|^2 + |V| - 2$ constraints.

## 7 Simplex algorithm

Each iteration converts one slack form into an equivalent slack form. How it does that is described later. Thereafter the objective value of the associated feasible solution is no less than the previous iteration. We achieve this as follows

1. Chose a nonbasic variable such that if we increase it from 0 then the objective value will increase too. How much we can increase the nonbasic variable can be limited by other constraints.

2. Raise the chosen nonbasic variable until a basic variable becomes 0

3. Rewrite by exchanging roles of the basic variable that became 0 and the nonbasic variable

when the solution becomes obvious it stops.

# 8 Duality

Instead of solving the original linear program, called the **primal**, it turns out we can formulate a **dual** problem whose objective we minimize and get an identical solution

$$\begin{array}{rlcll} \text{minimize} & \sum_{i=1}^{m} b_i y_i & & & \\ s.t & \sum_{i=1}^{m} a_{ij} y_i & \geq & c_j & \text{for } j = 1, 2, ..., n \\ & y_i & \geq & 0 & \text{for } i = 1, 2, ..., m \end{array}$$

Compare this to the *primal*

$$\begin{array}{rlcll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t} & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \text{for } i = 1, 2, ..., m \\ & x_j & \geq & 0 & \text{for } j = 1, 2, ..., n \end{array}$$

then we have done the following

1. Change maximization to minimization
2. Exchange roles of coefficients on the RHS - instead of having $c_j$ in the objective function, it is a constraint, and each constraint $b_i$ is in the objective function.
3. $\geq$ are changed to $\leq$.
4. Each $m$ constraints in the primal has an associated $y_i$ in the dual
5. Each of the $n$ constraints in the dual has an associated $x_j$in the primal

## 8.1 Lemma: Weak Duality

Let $\bar{x}$ be a feasible solution in the primal and $\bar{y}$a feasible solution in the dual. Then it holds that

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i$$

**Proof**

We have that $\sum_{i=1}^{m} a_{ij}y_i \geq c_j$for $j = 1, 2, ..., n$ and we can insert this into the above

$$\begin{aligned}\sum_{j=1}^{n} c_j\bar{x}_j &\leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\bar{y}_i\right)\bar{x}_j \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\bar{y}_i \cdot \bar{x}_j\right) \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\bar{y}_i \cdot x_j\right) \\ &= \sum_{i=1}^{m}\bar{y}_i\left(\sum_{j=1}^{n} a_{ij} \cdot x_j\right) \\ &\leq \sum_{i=1}^{m}\bar{y}_i \cdot b_j\end{aligned}$$

## 8.2 Corollary

Let $\bar{x}$ be a feasible solution to the primal and $\bar{y}$to the corresponding dual. Then if

$$\sum_{j=1}^{n} c_j\bar{x}_j = \sum_{i=1}^{m} b_i\bar{x}_i$$

then $\bar{x}$ and $\bar{y}$ are optimal solutions to the primal and dual respectively.

## Proof

The previous lemma showed us that $\sum_{j=1}^{n} c_j\bar{x}_j \leq \sum_{i=1}^{m} b_i\bar{y}_i$ and thereby the objective value of the primal cannot exceed the objective value of the dual. Likewise the dual can never be less than the primal. Since the dual is a minimization and the primal is a maximization problem, then if $\bar{x}$ and $\bar{y}$ have the same objective value then this says neither can be improved - the dual has reached it minimum, so the primal cannot be better. Or the dual has reached is max so the dual cannot be less. So when they are equal for the feasible solutions $\bar{x}$ and $\bar{y}$ , they have both reached a minimum or maximum.

## 8.3 Theorem

Suppose simplex returns the values $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ for the primal linear program with nonbasic and basic variables returned to be N, B respectively. Let $c'$ denote the coefficients in the final slack form and $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ defined by

$$\bar{y}_i = \begin{cases} -c'_{n+1} & \text{if } (n+1) \in N \\ 0 & \text{otherwise} \end{cases}$$

then $\bar{x}$ is an optimal solution to the primal and $\bar{y}$to the dual and we have

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{x}_i$$