Gaussians & Assignment 1 Handout

Probabilistic Decision Making VU – Practicals

(Reinforcement Learning KU)

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A **probability space** is a triple

$$(\Omega, \mathcal{F}, \mathbb{P})$$

σ-Algebra

Let $\Omega \neq \emptyset$ be a set and let $\mathcal{F} \subseteq \mathcal{P}(\Omega)$. Then $\mathcal{F}$ is a **σ-algebra** over Ω if

- $\Omega \in \mathcal{F}$
- $A \in \mathcal{F} \implies A^c := \Omega \setminus A \in \mathcal{F}$ **(closed under complement)**
- $A_1, A_2, \dots \in \mathcal{F} \implies \bigcup_{i \geq 1} A_i \in \mathcal{F}$ **(closed under countable union)**

Probability Measure

Let $(\Omega, \mathcal{F})$ be a measurable space. A map

$$\mathbb{P} : \mathcal{F} \to [0, 1]$$

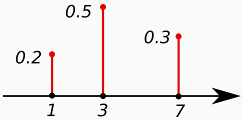
is a **probability measure** if

- $\mathbb{P}(\Omega) = 1$
- for pairwise disjoint $A_1, A_2, \dots \in \mathcal{F}$: $\mathbb{P}\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} \mathbb{P}(A_i)$

Probability Mass Function (PMF)

Let $X : \Omega \to \mathcal{X}$ be a **discrete** RV with countable state space $\mathcal{X}$. The **probability mass function** (PMF) $p_X : \mathcal{X} \to [0, 1]$ is defined as

$$p_X(x) := \mathbb{P}_X(\{x\})$$

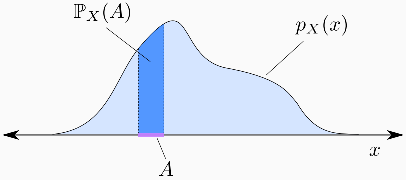


https://commons.wikimedia.org/wiki/File:Discrete_probability_distrib.svg

Probability Density Function (PDF)

Let $X : \Omega \to \mathcal{X}$ be a RV with state space $\mathcal{X} = \mathbb{R}$. **If it exists**, a function $p_X : \mathbb{R} \to [0, \infty]$ is a **density** of X if for any event A,

$$\int_A p_X(x)\, \mathrm{d}x = \mathbb{P}_X(A).$$

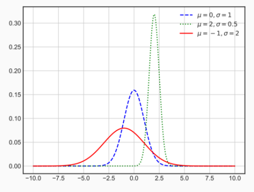


Gaussian Distribution

Univariate Gaussian

Gaussian PDF with **mean** $\mu \in \mathbb{R}$ and **variance** $\sigma^2 > 0$:

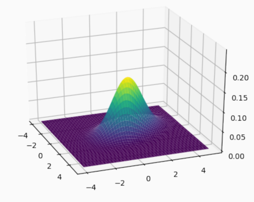
$$p_X(x) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$



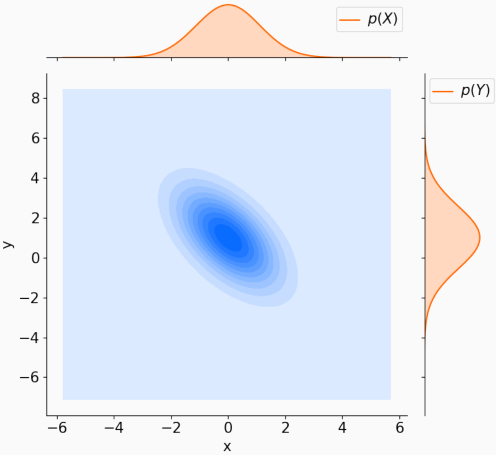
Multivariate Gaussian

Mean $\boldsymbol{\mu} \in \mathbb{R}^D$, symm. pos. def. **covariance** $\Sigma \in \mathbb{R}^{D \times D}$:

$$\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \Sigma) = \frac{1}{\sqrt{(2\pi)^D |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}-\boldsymbol{\mu})\right)$$

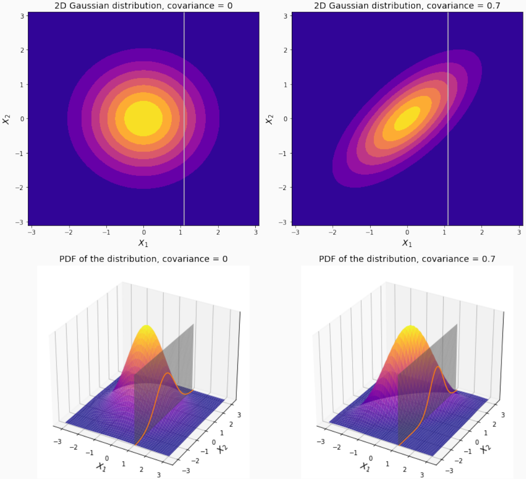


- **Marginals of Gaussians** are again Gaussians !



Properties of Gaussians – cont.

- **Conditionals of Gaussians** are again Gaussians ❗

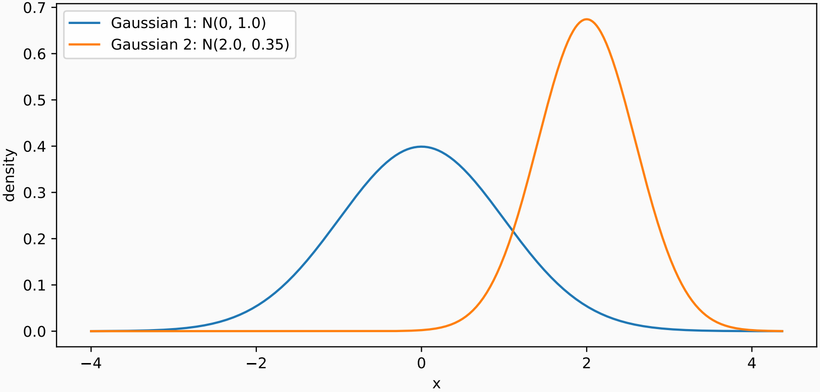


Source: `https://vzahorui.net/regression/gaussian-process/`

- **Product of two Gaussian PDFs** is again a (scaled) Gaussian PDF ❗

$$\mathcal{N}(\mathbf{x}; \mathbf{a}, A)\,\mathcal{N}(\mathbf{x}; \mathbf{b}, B) = Z \cdot \mathcal{N}(\mathbf{x}; \mathbf{c}, C)$$

where $C := (A^{-1} + B^{-1})^{-1}$, $\mathbf{c} := C(A^{-1}\mathbf{a} + B^{-1}\mathbf{b})$, and $Z := \mathcal{N}(\mathbf{a}; \mathbf{b}, A + B)$.



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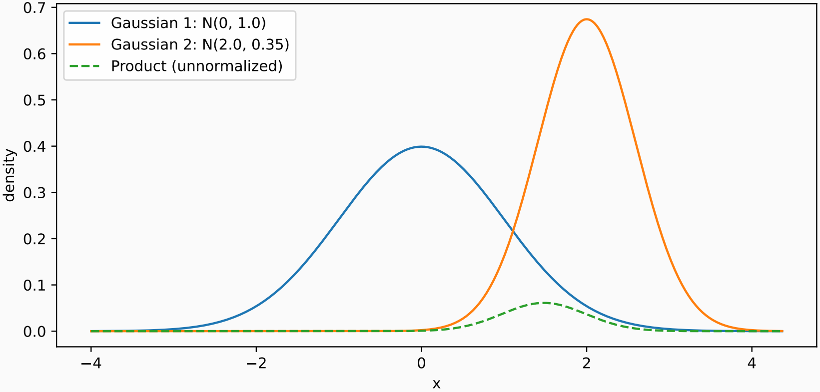

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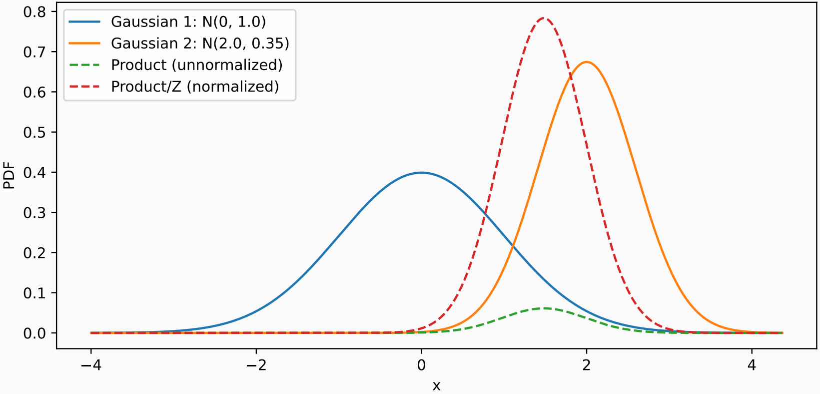

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- **Note:** Product of Gaussian **PDFs** $\neq$ Product of Gaussian **RVs** ❗

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