Network Science (VU) (706.703)

Introduction to Dynamical Systems

Denis Helic

HCC, TU Graz

December 16, 2025

Outline

Dynamical Systems

Systems evolving in time

Definition of dynamical systems

- We now first focus now on dynamical systems in a non-network context
- We also concentrate on the deterministic systems of continous real-valued variables evolving in continous time t
- A simple example is a system described by a single variable $x(t)$
- The variable evolves according to a first-order differential equation:

$$\frac{dx}{dt} = \dot{x} = f(x)$$

- Henceforth, we will denote the time derivative of x with $\dot{x}$

Definition of dynamical systems

- $f(x)$ is some specified function that describes the behavior of x
- Typically we also have initial conditions (for an initial value problem)
- The value $x(t_0)$ at some initial time t_0

Definition of dynamical systems

- We can have dynamical systems with two variables:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

- We can extend this approach to even more variables

General framework

- A dynamical system with n variables:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \dots, x_n) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \dots, x_n)
\end{aligned}
$$

General framework

- We might have also the right side dependence on t, e.g:

$$\dot{x}_1 = f_1(x_1, t)$$

- However, we can easily rewrite this equation in one without dependence on t, but with one extra variable

General framework

$$x_2 = t \implies \dot{x}_2 = 1$$

- And we also have: $x_2(0) = 0$

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) = 1 \end{aligned}$$

General framework

- Another extension would be to consider systems governed by higher derivatives
- It turns out that these can always be reduced to simpler cases
- However, we need to introduce extra variables

General framework

- The examples were all examples of **linear** systems because all of the x_i on the right hand side are to the first power only
- Otherwise the systems are **nonlinear**
- Nonlinear terms are products, powers, e.g. $x_1 x_2$, x_1^2, and so on
- Further nonlinear terms are (nonlinear) functions of x_i, e.g. $sin x_i$, or $log x_i$, and so on
- With nonlinearity the study of even such simple dynamical systems covers a broad range of interesting scientific situations

Exponential growth/decay equation

- Linear systems with a single variable exhibit exponential growth/decay behavior
- For example exponential growth equation

$$\dot{x} = kx$$

- Where $k > 0$ is the growth rate
- We might have the following initial condition: $x(0) = x_0$

Exponential growth/decay equation

- Such simple systems can be solved analytically by separating variables and integrating

$$
\begin{aligned}
\frac{dx}{dt} &= kx \\
\frac{dx}{x} &= kdt \\
\int \frac{dx}{x} &= \int kdt
\end{aligned}
$$

Exponential growth/decay equation

- Solving integrals:

$$
\begin{aligned}
ln x &= kt + c \\
x &= e^{kt} e^{c} = C e^{kt}
\end{aligned}
$$

Exponential growth/decay equation

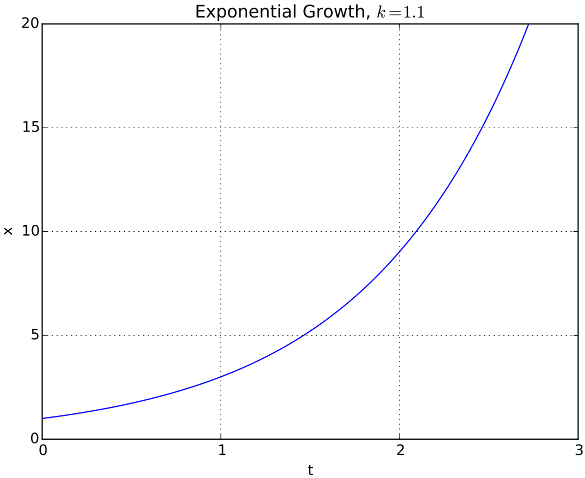
- The constant C is calculated from the initial conditions
- For $t = 0$ we have $x(0) = x_0$

$$\begin{aligned} x_0 &= Ce^{k \cdot 0} = C \cdot 1 \\ C &= x_0 \end{aligned}$$

- The final solution

$$x = x_0 e^{kt}$$

Exponential growth/decay equation



Exponential growth/decay equation

- Similarly exponential decay equation

$$\dot{x} = -\lambda x$$

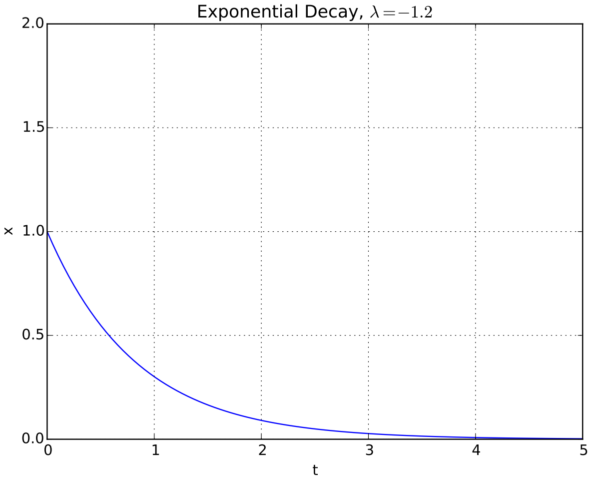
- Where $\lambda > 0$ is the decay rate
- We might have the following initial condition: $x(0) = x_0$

Exponential growth/decay equation

- Again, by separating variables, integrating and calculating integration constants from the initial conditions
- The final solution:

$$x = x_0 e^{-\lambda t}$$

Exponential growth/decay equation



Problems with analytical solutions

- In principle, we can always solve the equation from above by separating the variables and integrating:

$$
\begin{aligned}
\frac{dx}{dt} &= f(x) \\
\int_{x_0}^{x} \frac{dx'}{f(x')} &= t - t_0
\end{aligned}
$$

Problems with analytical solutions

- In practice, the integral may not exist in the closed form
- For cases with two or more variables it is not even in principle possible to find solution in a general case
- We will see later that for the network cases we typically have n variables: one variable per node
- Thus, except in some special cases a full analytical solution is typically not possible
- We can of course always integrate equations numerically or simulate
- But, combining these methods with some geometric and analytical techniques provides us with more qualitative insight

Fixed Points

Equilibrium

Fixed points

- A fixed point is a steady state of the system
- Any value of the variable(s) for which the system is stationary
- The system does not change over time
- Equilibrium

Fixed points

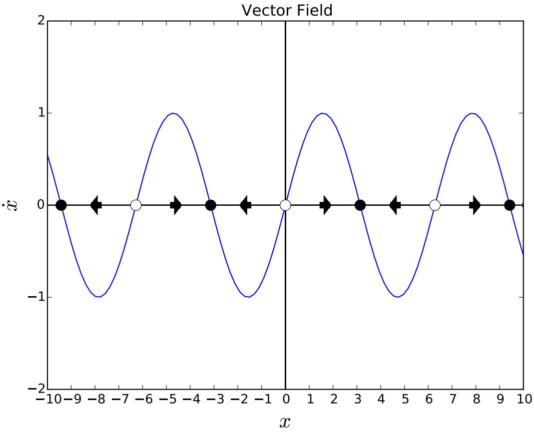
- For example in a system with one variable x a fixed point x^* is any point for which the function $f(x)$ does not change:

$$f(x^*) = 0$$

- This makes $\frac{dx}{dt} = 0$, and x does not move
- Thus, if in the evolution of the system we reach a fixed point the system stays there forever

Vector field

- We plot $\dot{x}$ vs x, e.g. $\dot{x} = sinx$



Logistic Growth

Limited growth

Logistic growth equation

- The simplest population growth model is the exponential growth model: $\dot{N} = rN$, with $r > 0$ being the growth rate
- This model predicts the exponential growth: $N = N_0 e^{rt}$, where N_0 is the population at time $t = 0$
- Of course, such exponential growth can not go forever
- For population larger then some (positive) carrying capacity K the growth rate becomes actually negative
- The death rate is higher than the birth rate

Logistic growth equation

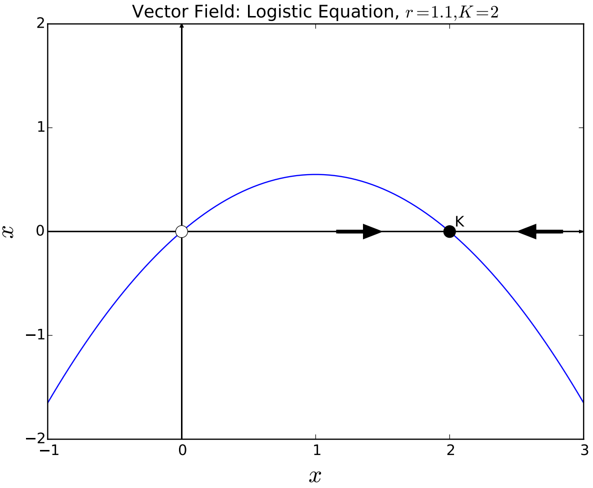
- To model the effects of overcrowding and limited resources we will assume that per capita growth rate $\frac{\dot{N}}{N}$ decreases when N is sufficiently large
- A mathematically convenient solution is to assume that per capita growth rate $\frac{\dot{N}}{N}$ decreases linearly with N

$$\frac{\dot{N}}{N} = r(1 - \frac{N}{K})$$

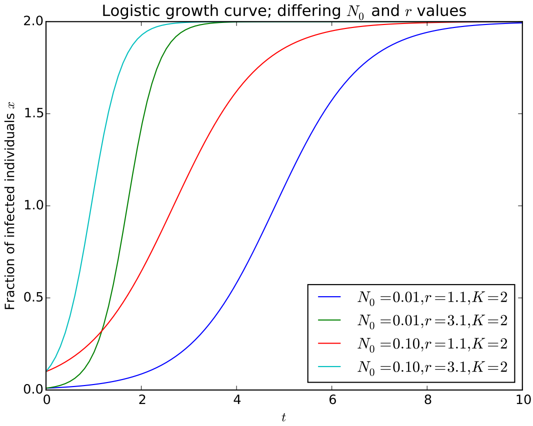
$$\dot{N} = rN(1 - \frac{N}{K})$$

- This is the *logistic growth equation*

Vector field: logistic equation



Logistic growth curve



Multivariate Linear Systems

More then one variable

Two-dimensional linear system

- A two-dimensional linear system is of the form:

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2\end{aligned}$$

- a, b, c, d are parameters

Two-dimensional linear system

- In matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Two-dimensional linear system

- The system is linear also in another sense
- If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions so is any linear combination: $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$
- $\dot{\mathbf{x}} = 0$ when $\mathbf{x} = 0$
- $\mathbf{x}^* = 0$ is always a fixed point for any choice of $\mathbf{A}$

Solutions for two-dimensional linear systems

- Generalizing from the one-dimensional linear system, the solutions for a two-dimensional linear systems will be of the form:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

- This corresponds to an exponential growth/decay alongside the line spanned by the vector $\mathbf{v}$

Solutions for two-dimensional linear systems

- Let us find the solutions
- We substitute $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$\lambda e^{\lambda t}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v}$$

- Canceling $e^{\lambda t}$ we get:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Solutions for two-dimensional linear systems

- The straight line solutions are eigenvectors of $\mathbf{A}$
- The growth rate/decay is given by the eigenvalues of $\mathbf{A}$
- If the corresponding eigenvalue is smaller than zero we have an exponential decay alongside that eigenvector
- If the corresponding eigenvalue is greater than zero we have an exponential growth alongside that eigenvector
- Larger eigenvalue is a fast eigendirection, smaller eigenvalue is a slow eigendirection
- These are **eigensolutions**

Solutions for two-dimensional linear systems

- If $\lambda_1 \neq \lambda_2$ the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent
- Then any initial condition $\mathbf{x}_0$ can be written as linear combination of eigenvectors:

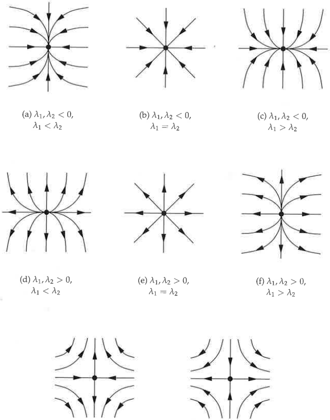
$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

- The general solution for $\mathbf{x}(t)$:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

- It is a linear combination of solutions to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, i.e. it is itself a solution
- It satisfies the initial conditions: it is the only solution

Flows in two-dimensional linear systems



(a) $\lambda_1, \lambda_2 < 0$, $\lambda_1 < \lambda_2$

(b) $\lambda_1, \lambda_2 < 0$, $\lambda_1 = \lambda_2$

(c) $\lambda_1, \lambda_2 < 0$, $\lambda_1 > \lambda_2$

(d) $\lambda_1, \lambda_2 > 0$, $\lambda_1 < \lambda_2$

(e) $\lambda_1, \lambda_2 > 0$, $\lambda_1 = \lambda_2$

(f) $\lambda_1, \lambda_2 > 0$, $\lambda_1 > \lambda_2$

(g) $\lambda_1 < 0 < \lambda_2$

(h) $\lambda_2 < 0 < \lambda_1$

Flows in two-dimensional linear systems

- If $\mathbf{A}$ is not symmetric eigenvectors are not orthogonal
- This transforms the axes, but the behavior is similar
- A new interesting behavior might emerge if the eigenvalues are complex
- This gives an oscillation around a fixed point, which either grows or decays
- It spirals inwards or outwards around the fixed point
- In certain cases there is a stable oscillatory behavior: limit cycle