Introduction

Probabilistic Decision Making — Lecture 1

1st October 2025

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Absolve Probabilistic Decision Making
Continue doing Orthodox Statistics

A Reasoning Task

$\forall X: \text{gummy\_bear}(X) \land \text{red}(X) \rightarrow \text{eaten}(X)$

$\forall X\, \forall Y: \text{gummy\_bear}(X) \land \text{stick\_together}(X, Y) \rightarrow \text{gummy\_bear}(Y)$

$\text{gummy\_bear}(\text{Berta})$

$\text{stick\_together}(\text{Berta}, \text{Hugo})$

$\text{red}(\text{Hugo})$

Question: Can you predict whether Hugo will be eaten?

A Reasoning Task cont'd

Facts:

$\text{gummy\_bear}(\text{Berta})$

$\text{stick\_together}(\text{Berta}, \text{Hugo})$

$\text{red}(\text{Hugo})$

Hence:

$\text{gummy\_bear}(\text{Berta}) \wedge \text{stick}(\text{Berta}, \text{Hugo}) \rightarrow \text{gummy\_bear}(\text{Hugo})$

$\text{gummy\_bear}(\text{Hugo}) \wedge \text{red}(\text{Hugo}) \rightarrow \text{eaten}(\text{Hugo})$

Yes, Hugo will be eaten. Excellent, we know how to use Logic as a reasoning framework!

Another Reasoning Task

$$
\begin{aligned}
a^2 &= b^2 + c^2 \\
a &= e + 3f^2
\end{aligned}
$$

Question: Given that $b = 4$, $e = 2$ and $f = 1$, can you predict c?

Another Reasoning Task cont'd

$$
\begin{aligned}
a &= e + 3f^2 \\
&= 2 + 3 = 5
\end{aligned}
$$

further

$$
\begin{aligned}
c^2 &= a^2 - b^2 \\
c^2 &= 25 - 16 = 9
\end{aligned}
$$

hence

$$
c = +3 \ \text{ or } \ -3
$$

Excellent, we know how to use Systems of Equations as a reasoning framework! We are also not disturbed by the fact that we got a non-unique answer ($c = \pm 3$).

Yet Another Reasoning Task

a	b	c	$p(a, b, c)$
0	0	0	0.1
0	0	1	0.05
0	1	0	0.2
0	1	1	0.05
1	0	0	0.01
1	0	1	0.09
1	1	0	0.2
1	1	1	0.3

Question: Given that $c = 1$, can you predict a?

...and this is why we do this course

You are familiar with two rigorous reasoning systems (logic, systems of equations), but you are probably not really aware that

Probability is just Rigorous Reasoning under Uncertainty!

Yet Another Reasoning Task cont'd

Question: Given that $c = 1$, can you predict a?

First note that b is not mentioned, hence we need to get rid of it by **marginalization** (summing it out):

a	b	c	$p(a,b,c)$
0	0	0	0.1
0	0	1	0.05
0	1	0	0.2
0	1	1	0.05
1	0	0	0.01
1	0	1	0.09
1	1	0	0.2
1	1	1	0.3

$\Rightarrow$

a	c	$p(a,c)$
0	0	0.3
0	1	0.1
1	0	0.21
1	1	0.39

Yet Another Reasoning Task cont'd

Question: Given that $c = 1$, can you predict a?

Now that we have a reduced model over only a and c, we want to inject the information $c = 1$ by **conditioning**:

$$p(a \mid c) = \frac{p(a, c)}{p(c)}$$

a	c	$p(a, c)$
0	0	0.3
0	1	0.1
1	0	0.21
1	1	0.39

$\Longrightarrow$

c	$p(c)$
0	0.51
1	0.49

a	c	$p(a \mid c)$
0	0	0.59
1	0	0.41
0	1	**0.2**
1	1	**0.8**

(Results are rounded)

Yet Another Reasoning Task cont'd

Question: Given that $c = 1$, can you predict a?

Answer: When $c = 1$ then $a = \begin{cases} 0 & \text{with probability } 0.2 \\ 1 & \text{with probability } 0.8 \end{cases}$

Again, we got a non-unique solution. Are you <u>now</u> concerned by that fact? If yes, why?

The solutions are equipped with weights (probabilities) expressing our (un)certainty.

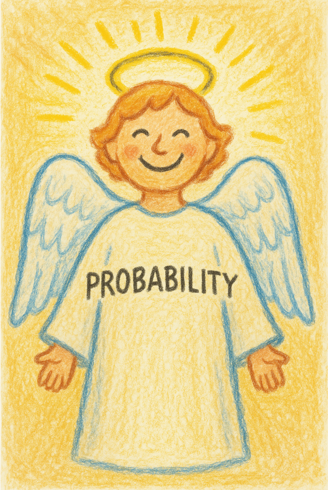
What this course is about

- probability is not (only) about p-values and dubious statistical tests!
- *"Probability [is] a faithful guardian of common sense."*

  Pearl (1988)
- *"The true logic of this world is the calculus of probabilities."*

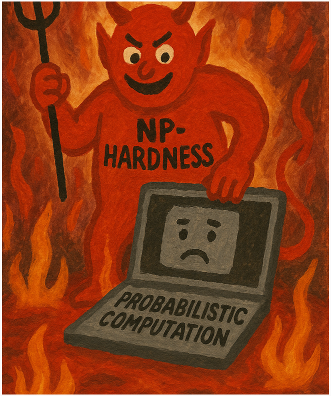
  Maxwell (1850)
- *"Probability is but common sense reduced to computation."*

  Laplace (1814)
- **in this course we recognize and deal with probability as rigorous reasoning under uncertainty!**



What this course is also about

- while probability is—in principle—rigorous and optimal reasoning under uncertainty, it is also a **computational nightmare**
- this makes it a really interesting research topic!
- **we will learn methods and techniques to tackle these challenges**



Organization

VU vs. VO + KU

- the new course **Probabilistic Decision Making** is actually intended as a VU (integrated lecture and practicals)
- however, in some curricula, still the old courses are listed:
  - **Reinforcement Learning**, Lecture (VO)
  - **Reinforcement Learning**, Practicals (KU)
- transition solution for this year:
  - same content for the two course formats
  - same semester hours (3) and ECTS (5)
  - **VU: one grade**
  - **VO+KU: two grades**
- **you can do either PDM VU or RL KU, but not both**

Assignment Sheets

- 4 assignment sheets over the Semester
- same for PDM (VU) and RL (KU)
- selective submission interviews will be conducted
  - after every submission (scope: single assignment)
  - after exam (scope: all assignments and exam)
- groups of 2 students (groups of 1 allowed on voluntary basis)
- **review paragraph:** an obligatory review paragraph needs to be included, about the distribution of work within the group (see assignment sheets for details)
- **no LLMs allowed!** Evident LLM usage yields **invalid grade due to deception.**

Assignment Sheets cont'd

	Handout	Handin	Exercise-Points
Assignment Sheet 1	24.10.2025	14.11.2025	20
Assignment Sheet 2	14.11.2025	05.12.2025	25
Assignment Sheet 3	05.12.2025	09.01.2026	35
Assignment Sheet 4	09.01.2026	26.01.2026	20

Assignment Sheets cont'd

- all submissions will be uploaded to TeachCenter
- first page of the submission must be the **cover sheet**

Assignment X

Probabilistic Decision Making, WS 2025/26

Team Members		
Last name	First name	Matriculation number
Schmidhuber	Jürgen	12345678
LeCun	Yann	90123456

Exams

PDM VU

- final exam on the 28^{th} of January, 9:45, i11
- duration 1 hour
- alternatively: home exam with oral checkup (depends on final student numbers)

RL VO

- 6 exam dates (TBD)
- duration 2 hours

Grading Scheme

Any grade (VU, VO, KU) will be determined by the **percentage** of achieved points, relative to the max. points:

85	-	100	very good (1)
70	-	84.9	good (2)
60	-	69.9	satisfactory (3)
50	-	59.9	sufficient (4)
0	-	49.9	insufficient (5)

For the VU:

$$\text{total points} = 0.65 \times \text{exercise points} + 0.35 \times \text{exam points}$$

Lectures and Practical Sessions

- **Lectures:** Wednesdays, 9:45-11:30, i11
  2 blocks of 45 minutes, 10 minutes break
- **Practicals:** Fridays, 12:30-14:00, i11
- **Lecturers:** Robert Peharz, Thomas Wedenig
- **Teaching Assistant:** Leon Tiefenböck
- attendance in the lectures and practicals is **not** compulsory
- the sessions (lectures and practicals) will be recorded and be posted on TeachCenter
- attending the sessions in person is strongly recommended :D

Content (roughly, subject to change)

- **October: Basic Probability Theory**
  - probability spaces, measure theory
  - random variables, distribution functions
  - probabilistic inference
  - learning, basic models
- **November: Basics of Probabilistic Machine Learning**
  - graphical models
  - mixture models
  - decision theory
  - missing data
- **December and January: Advanced Topics**
  - generative models
  - approximate inference (variational, Monte Carlo)
  - Gaussian processes and Bayesian optimization
  - reinforcement learning

Contact and Communication

- **in person:** use the lectures and practical sessions
- **Q&A-sessions:** extra sessions to ask questions and discuss the course content (will be recorded as well)
  - Friday 31. October, 11:00-12:30, i11
  - Friday 5. December, 11:00-12:30, i11
  - Friday 23. January, 11:00-12:30, i11
- discussion forum on Teach Center
- if really needs to be email: (robert.peharz@tugraz.at, thomas.wedenig@tugraz.at)

Today: Why Probabilistic Decision Making?

Why should decision making under uncertainty be based on probability?

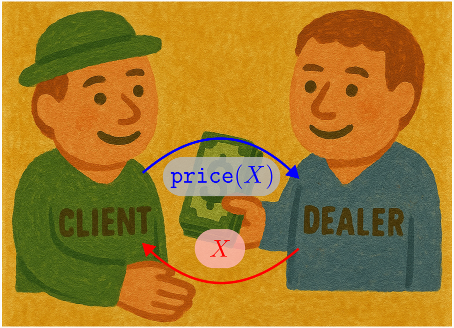
Today, we will discuss a classical argument:

The Dutch Book Argument

Fair Prices

Situations with Uncertain Payments

- assume a situation with an **uncertain payment** X
- to receive X one needs to pay first a **fixed price** $\texttt{price}(X)$
- basically, a **betting game**
- let us think of two agents, the **client** and the **dealer**:
  - the client pays a **deterministic** $\texttt{price}(X)$ to the dealer
  - the dealer pays **uncertain** X to the client



A Coin Game (I)

Example

- the client pays $\texttt{price}(X) = 1$ Euro
- a fair coin is flipped
  - if the coin shows 'heads', the dealer pays $X = 4$ Euro
  - if the coin shows 'tails', the dealer pays $X = 0$ Euro

What do you prefer? Being the client or the dealer?

A Coin Game (II)

Let's change the payoff a bit:

- the client pays $\texttt{price}(X) = 1$ Euro
- a standard coin is flipped
  - if the coin shows 'heads', the dealer pays $X = \cancel{4}\ 1$ Euro
  - if the coin shows 'tails', the dealer pays $X = 0$ Euro

What do you prefer now? Being the client or the dealer?

A Coin Game (III)

Example

Let's change the payoff one more time:

- the client pays $\texttt{price}(X) = 1$ Euro
- a standard coin is flipped
  - if the coin shows 'heads', the dealer pays $X = \cancel{4}\ \cancel{1}\ 2$ Euro
  - if the coin shows 'tails', the dealer pays $X = 0$ Euro

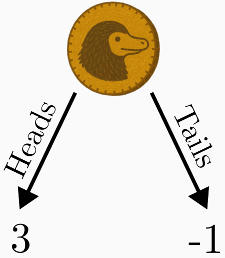
What do you prefer now? Being the client or the dealer?

Coin Games summarized

Example

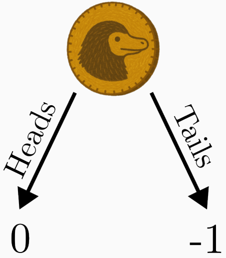
- the net wins for the client are summarized below
- multiply with -1 to get the net wins for the dealer ($=$ net losses for client)

<u>Coin Game I</u>



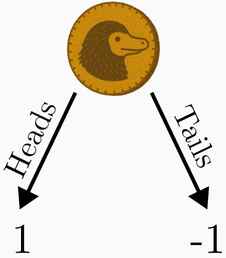
unfair for dealer

<u>Coin Game II</u>



unfair for client

<u>Coin Game III</u>



fair

Fair Price

The **fair price** of an uncertain payment X is the amount $\texttt{price}(X)$ at which an **agent**—according to its **information status**—is **indifferent** of the two situations

- pay $\texttt{price}(X)$ and receive X, or **(client)**
- receive $\texttt{price}(X)$ and pay X. **(dealer)**

Note that the fair price indeed depends on the information status, or assumptions, of the agent. E.g., if they assume that the coin is loaded, they might come up with different fair prices.

Also note that the existence of fair prices are a (weak?) assumption we make here.

Fair Prices (I)

Example

- client pays $\texttt{price}(X)$ Euro
- a fair die is rolled.
- if the die shows $1 - 4$, the client gets nothing
- if the die shows 5, the client gets 4 Euro
- if the die shows 6, the client gets 8 Euro

What is—according to your assessment—the fair $\texttt{price}(X)$?

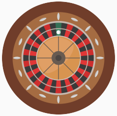
Fair Prices (I) **Example**

- client pays $\texttt{price}(X)$ Euro
- a fair die is rolled.
- if the die shows $1 - 4$, the client gets nothing
- if the die shows 5, the client gets 4 Euro
- if the die shows 6, the client gets 8 Euro

What is—according to your assessment—the fair $\texttt{price}(X)$?

(Likely one should assume $\texttt{price}(X) = 2$ Euro)

Fair Prices (II) **Example**

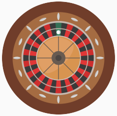


- client pays $\texttt{price}(X) = 1$ Euro
- game of roulette is played (numbers 0-36).
- if the ball ends up on a Red or Black number, the client gets nothing
- if the ball ends up on the Green Zero, the client gets X Euro

What is—according to your assessment—the value for X in order that $\texttt{price}(X) = 1$ is the fair price?

Fair Prices (II)

Example



- client pays $\texttt{price}(X) = 1$ Euro
- game of roulette is played (numbers 0-36).
- if the ball ends up on a Red or Black number, the client gets nothing
- if the ball ends up on the Green Zero, the client gets X Euro

What is—according to your assessment—the value for X in order that $\texttt{price}(X) = 1$ is the fair price?

(Likely one should assume $X = 37$ Euro)

Linearity of Fair Prices

You can probably accept the following proposal:

constant $\times$ **fair = fair**

- If $\texttt{price}(X)$ is the fair price for X, then
- $\alpha\,\texttt{price}(X)$ is the fair price for αX, for any $\alpha \in \mathbb{R}$.

Because:

- you wouldn't deem a bet unfair just because we switch from Euro to Yen, which is just some conversion factor $\alpha > 0$
- accepting $\alpha = -1$ is the definition of fair, client and dealer are swapped
- together with the argument above, we would accept any $\alpha < 0$
- $\alpha = 0$ is of course also fine

You can probably also accept:

fair + fair = fair

- If $\texttt{price}(X)$ is the fair price for X and
- $\texttt{price}(Y)$ is the fair price for Y,
- then $\texttt{price}(X) + \texttt{price}(Y)$ is the fair price for $X + Y$.

Because you would accept both bets one after the other.

Consequently:

Fair prices are linear.

- If $\texttt{price}(X_i)$ is the fair price of X_i, then
- $\sum_{i=1}^{K} \alpha_i \, \texttt{price}(X_i)$ is the fair price for $\sum_{i=1}^{K} \alpha_i X_i$, for any $\alpha_1, \ldots, \alpha_K \in \mathbb{R}$.

So far, we were assuming that

- fair prices exist for uncertain payments and that
- fair prices behave linearly

These assumptions will lead to some interesting insights.

Dutch Books

Odds

Assume an uncertain process (e.g. dice game, coin flip, horse race, etc.) and let A be a possible **outcome** or **event**. The **odds** o_A is a real number assigned to A.

It is the ratio between the payoff for a bet on the event A and the client's wager. That is, the odds describes the bet

$$\texttt{price}(X) \quad \text{for} \quad X = \begin{cases} o_A \times \texttt{price}(X) & \text{if } A \text{ happens} \\ 0 & \text{otherwise} \end{cases}$$

Fair Odds

Odds are **fair** if they describe fair bets.

An agent's fair odds describe the agent's assessment about an uncertain situation. They can be seen as the agent's policy when they find it acceptable to take either side of the bet.

- a dealer offers the odds

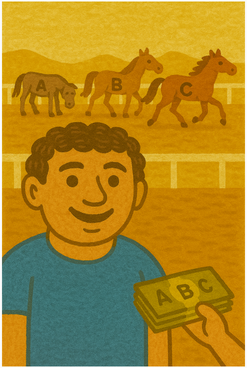
$$o_A = 2 \quad \text{for} \quad A = \text{'coin shows heads'}$$

- the client buys a bet on A for 5 Euro
- if the coin shows 'tails', the dealer pays nothing, if it shows 'heads' the dealer pays

$$5 \times o_A = 10 \text{ Euro}$$

- most people would agree that $o_A = 2$ is fair, i.e. they find it acceptable to be on either side of the bet
- for $o_A < 2$ one will prefer to be the dealer, for $o_A > 2$ one prefers to be the client

- Tim Van Belgie is an expert in horse races
- he offers you the following **odds**
  - Horse A wins: $o_A = 50$
  - Horse B wins: $o_B = 5$
  - Horse C wins: $o_C = 1.25$
- Tim just loves betting, so these are his fair odds
- in particular, he will accept any bet in any direction



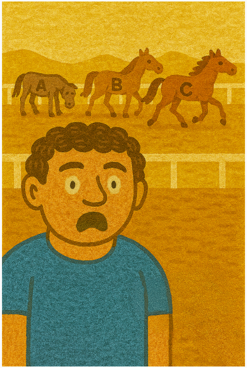
Do you note anything peculiar?

A Horse Race cont'd

Example

We offer Tim the following combined bet:

-10 Euro	on Horse A	($o_A = 50$)
-100 Euro	on Horse B	($o_B = 5$)
-400 Euro	on Horse C	($o_C = 1.25$)



outcome	**we win**	
Horse A:	$-10 \times 50 + 510$	$= 10$
Horse B:	$-100 \times 5 + 510$	$= 10$
Horse C:	$-400 \times 1.25 + 510$	$= 10$

We found a so-called Dutch Book—a combination of bets that wins with certainty! In other words: it yields a guaranteed loss for Tim.

What is wrong with Tim's odds?

What is wrong with Tim's odds?

In short, they don't obey the laws of probability!

Note: Our treatment of probabilities is a bit handwavy today, but we will define them formally in the upcoming lecture.

Events

Let $A_1, \ldots, A_K$ outcomes of a random process, that are

- **mutual exclusive:** at most one A_i can happen,
- **exhaustive:** at least one A_i must happen,
- i.e. exactly one A_i happens.

Further, consider **composite events**:

- $A_i \cup A_j$ meaning A_i or A_j happens
- $A_i \cup A_j \cup A_k$ meaning A_i or A_j or A_k happens
- $\vdots$ $\vdots$

Probability Distribution

A probability distribution $\mathbb{P}$ assigns real numbers to each event as follows:

- $\sum_{i=1}^{K} \mathbb{P}(A_i) = 1$ for exclusive and exhaustive A_i
- $\mathbb{P}(A) \geq 0$ for all A
- if A and B are mutually exclusive: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Note that we don't assume any data here and we are not estimating anything. Probabilities are just any numbers obeying above rules!

Probabilities

Example

Let $\{A_i\}_{i=1}^{6}$ be the possible outcomes of a die game, i.e. A_i encodes that the die shows the number i. Let a probability distribution be given as

$$
\begin{array}{lll}
\mathbb{P}(A_1) = 0.05 & \mathbb{P}(A_2) = 0.05 & \mathbb{P}(A_3) = 0.1 \\
\mathbb{P}(A_4) = 0.2 & \mathbb{P}(A_5) = 0.2 & \mathbb{P}(A_6) = 0.4
\end{array}
$$

Here, the die is of course manipulated, preferring higher numbers.

- the probability of “die shows an even number” is

$$\mathbb{P}(A_2 \cup A_4 \cup A_6) = 0.05 + 0.2 + 0.4 = 0.65$$

- the probability of “die shows a Fibonacci number”‘ is

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_5) = 0.05 + 0.05 + 0.1 + 0.2 = 0.4$$

Let $\mathbb{P}$ be **any** probability distribution for some set of events. Odds are **coherent with $\mathbb{P}$** if

$$o_A \coloneqq \frac{1}{\mathbb{P}(A)} \qquad \mathbb{P}(A) = \frac{1}{o_A},$$

i.e. if odds and probabilities are reciprocal to each other.

Generally, odds are **coherent** if they are coherent with **some** $\mathbb{P}$, otherwise they are called **incoherent**.

Dutch Book Argument
A Dutch Book for a set of odds exists **if and only if** the odds are incoherent.

Note that the argument does **not at all** talk about "how to construct good odds" or "what good odds are". It tells us that odds are admissible if and only of they correspond some **arbitrary** probability distribution.

In other words, **the Dutch Book Argument forces us to accept probability as the framework of our reasoning!**

The Dutch Book Argument: Proof

Let's prove the direction: incoherent odds $\Rightarrow$ Dutch Book exists.
The other direction is left to the reader ;)

Incoherent odd violate at least one of the following constraints:

1. $$\mathbb{P}(A) \geq 0 \quad \text{meaning} \quad o_A = \frac{1}{\mathbb{P}(A)} \geq 0$$
2. For mutually exclusive events A, B:
$$\mathbb{P}(A \cup B) = \frac{1}{o_{A\cup B}} = \frac{1}{o_A} + \frac{1}{o_B} = \mathbb{P}(A) + \mathbb{P}(B)$$
3. For mutually exclusive and exhaustive $\{A_i\}_{i=1}^K$:
$$\sum_{i=1}^K \mathbb{P}(A_i) = \sum_{i=1}^K \frac{1}{o_{A_i}} = 1$$

1. $o_A \geq 0$

- let $o_A < 0$ for some event A
- buy a bet on A for -1 Euro (i.e. you receive 1 Euro)
- if event A happens, you need to pay o_A Euro
- but o_A is negative, hence you receive money again
- net win at least 1 Euro

2. $\frac{1}{o_{A \cup B}} = \frac{1}{o_A} + \frac{1}{o_B}$ for mutually exclusive events A, B

$\mathbb{P}(A) = \frac{1}{o_A}$ is the fair price for winning 1 Euro in the case of A.

- first assume $\frac{1}{o_{A \cup B}} > \frac{1}{o_A} + \frac{1}{o_B}$, i.e. $\mathbb{P}(A \cup B) > \mathbb{P}(A) + \mathbb{P}(B)$
- sell a bet on $A \cup B$ for $\mathbb{P}(A \cup B)$
- buy a bet on A for $\mathbb{P}(A)$ and a bet on B for $\mathbb{P}(B)$
- thus, we got $\mathbb{P}(A \cup B) - \mathbb{P}(A) - \mathbb{P}(B) > 0$ Euro
- then, if either A or B happens we both receive and pay one Euro
- in total we win $\mathbb{P}(A \cup B) - \mathbb{P}(A) - \mathbb{P}(B) > 0$ Euro

For the case $\frac{1}{o_{A \cup B}} < \frac{1}{o_A} + \frac{1}{o_B}$ just swap buying and selling of bets.

3. $\sum_{i=1}^{K} \frac{1}{o_{A_i}} = 1$ for mutually exclusive and exhaustive $\{A_i\}_{i=1}^{K}$

Again recall that $\mathbb{P}(A) = \frac{1}{o_A}$ is the fair price for winning 1 Euro in the case of A.

- first assume that $\sum_{i=1}^{K} \frac{1}{o_{A_i}} < 1$
- buy bets on each A_i for $\mathbb{P}(A_i)$, spending $\sum_{i=1}^{K} \frac{1}{o_{A_i}} < 1$ Euro
- since one A_i must happen, we win 1 Euro

For the case $\sum_{i=1}^{K} \frac{1}{o_{A_i}} > 1$ just swap buying and selling of bets.

- fair odds describe decision making under uncertainty
- underlying assumptions:
  - fair prices always exist
  - fair prices are linear
- sanity check: are odds coherent, i.e. derived from **some** probability distribution?
- **incoherent odds $\Leftrightarrow$ Dutch Book exists (guaranteeing loss)**
- the argument is **normative**: it doesn't say anything about what "good" odds are—only that you can rule out all incoherent odds
- **only odds derived from probability are admissible**

Image Credits

- screen shot from Matrix, The Movie
- generated by ChatGPT:
  - tumbleweed
  - probability angel
  - computational nightmare
  - client and dealer
  - niffler coin
  - horse race
- https://www.svgrepo.com/svg/50195/dice
- htps://www.svgrepo.com/svg/98957/roulette