Assignment 1

Probabilistic Decision Making VU, WS 2025/26

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Task 1–Probability Spaces [5 Points]

Task 1.1 [1.5 Points]

For each of the following candidates $(\Omega, \mathcal{F})$, decide whether $\mathcal{F}$ is a σ-algebra on the set Ω. Justify your answer in each case. $P(A)$ denotes the powerset of A, i.e., the set of all subsets of A.

(a) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}\}$.

(b) $\Omega = \mathbb{R}$, $\mathcal{F} = P(\Omega)$.

(c) $\Omega = \mathbb{N}$, $\mathcal{F} = \{A \subseteq \Omega : A \text{ finite or } \Omega \setminus A \text{ finite}\}$.

(d) $\Omega = \mathbb{N}$, $\mathcal{F} = \{A \subseteq \Omega : A \text{ countable or } \Omega \setminus A \text{ countable}\}$.

1.1

For this task, we are using the definition (from the slides) of a σ-algebra, where $\mathcal{F}$ is called a σ-algebra over Ω if:

1. $\Omega \in \mathcal{F}$
2. $\mathcal{F}$ is closed under complement: if $A \in \mathcal{F}$, then also $A^c = \Omega \setminus A \in \mathcal{F}$
3. $\mathcal{F}$ is closed under countable unions: if $A_1, A_2, \ldots \in \mathcal{F}$, then also $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

It also follows that:

- $\emptyset \in \mathcal{F}$
- $\mathcal{F}$ is closed under countable intersections: if $A_1, A_2, \ldots \in \mathcal{F}$, then also $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

(a)

$\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}\}$

Yes, this is a σ-algebra.

The complement of each set is also listed in $\mathcal{F}$:

$$\begin{aligned} \emptyset &\leftrightarrow \Omega \\ \{1, 2\} &\leftrightarrow \{3, 4, 5, 6\} \\ \{3, 4\} &\leftrightarrow \{1, 2, 5, 6\} \\ \{5, 6\} &\leftrightarrow \{1, 2, 3, 4\} \end{aligned}$$

Also, $\mathcal{F}$ is closed under countable unions:

$$
\begin{aligned}
\{1,2\} \cup \{3,4\} &= \{1,2,3,4\} \\
\{1,2\} \cup \{5,6\} &= \{1,2,5,6\} \\
\{3,4\} \cup \{5,6\} &= \{3,4,5,6\} \\
\{1,2\} \cup \{3,4\} \cup \{5,6\} &= \Omega \\
\{1,2,3,4\} \cup \{5,6\} &= \Omega \\
\{1,2,5,6\} \cup \{3,4\} &= \Omega \\
\{3,4,5,6\} \cup \{1,2\} &= \Omega
\end{aligned}
$$

Any set $\cup\, \Omega = \Omega$.
Any set $\cup\, \emptyset =$ the set itself.
Thus, $\mathcal{F}$ satisfies all properties of a σ-algebra.

(b) $\Omega = \mathbb{R}$, $\mathcal{F} = P(\Omega)$

Yes, this is a σ-algebra.

The power set of any set is always a σ-algebra. This is because:

1. By definition of the powerset, $\Omega = \mathbb{R}$ is a subset of itself, and so $\mathbb{R} \in \mathcal{P}(\mathbb{R})$.
2. Let $A \in \mathcal{F} = \mathcal{P}(\mathbb{R})$. Then $A \subseteq \mathbb{R}$, and that means that the complement $\mathbb{R} \setminus A$ is also a subset of $\mathbb{R}$. Therefore $\mathbb{R} \setminus A \in \mathcal{P}(\mathbb{R}) = \mathcal{F}$.
3. Let A_i be a countable family with $A_i \in \mathcal{F}$ for all i. Then each $A_i \subseteq \mathbb{R}$, and so the union $\bigcup_{i=1}^{\infty} A_i$ is again a subset of $\mathbb{R}$. Hence,

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{P}(\mathbb{R}) = \mathcal{F}.$$

(c) $\Omega = \mathbb{N}$, $\mathcal{F} = \{A \subseteq \Omega : A$ finite or $\Omega \setminus A$ finite$\}$

No, this is not a σ-algebra.

For the complements, we have two options:

1. If A is finite, the complement is cofinite, still $\in \mathcal{F}$.
2. If A is cofinite, the complement is finite, still $\in \mathcal{F}$.

But $\mathcal{F}$ is not closed under countable unions.
Let A_k be finite sets of $\mathbb{N}$ with one element each, being odd numbers:

$$A_1 = \{1\}, A_2 = \{3\}, A_3 = \{5\}, \ldots$$

If we take their countable union,

$$\bigcup_{k=1}^{\infty} A_k = \{1, 3, 5, \ldots\},$$

this set becomes infinite, and also the complement of A_k the even numbers is infinite too. This breaks the definition of a σ-algebra even though it is closed under complements it is not closed under countable unions.

(d) $\Omega = \mathbb{N}$, $\mathcal{F} = \{A \subseteq \Omega : A$ countable or $\Omega \setminus A$ countable$\}$

Yes, this is a σ-algebra.

Every subset of $\Omega = \mathbb{N}$ is countable since $\mathbb{N}$ itself is countable by definition. By taking the same argument as in we defined in (b) that $\mathcal{F} = \mathcal{P}(\Omega)$, which is always a σ-algebra, we conclude $\mathcal{F}$ to be a σ-algebra.

1. It contains Ω, so $\Omega \in \mathcal{P}(\Omega)$.
2. If $A \in \mathcal{P}(\Omega)$, then its complement $\Omega \setminus A$ is also a subset of Ω, so $\Omega \setminus A \in \mathcal{P}(\Omega)$.
3. Any countable union of subsets of Ω is again a subset of Ω, thus $\bigcup_{i=1}^{\infty} A_i \in \mathcal{P}(\Omega)$.

The countable union of countable sets is again countable and therefore $\mathcal{F} = \mathcal{P}(\Omega)$ is a σ-algebra.

Task 1.2 [1.5 Points]

For each of the following candidates $(\Omega, \mathcal{F})$, decide whether $\mathcal{F}$ is a σ-algebra on the set Ω. If you think $\mathcal{F}$ is not a σ-algebra on Ω, provide the smallest σ-algebra on Ω that contains all sets $\in \mathcal{F}$.

(a) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4, 5, 6\}, \{1, 3\}, \{2, 4, 5, 6\}\}$.

(b) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\emptyset, \Omega, \{2, 4, 6\}, \{1, 3, 5\}, \{1, 2, 3\}, \{4, 5, 6\}\}$.

(c) $\Omega = \{1, 2, 3, 4, 5\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, \{5\}, \{1, 2, 5\}\}$.

1.2

We again use the definition above from the slide, closure under complements, closure under countable unions.

(a) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4, 5, 6\}, \{1, 3\}, \{2, 4, 5, 6\}\}$

No, this is not a σ-algebra.

Basically the unions of the sets are missing which again introduce new complement sets. Smallest σ-algebra:

$$\Big\{\emptyset,\ \Omega,\ \{1\},\ \{2\},\ \{3\},\ \{4,5,6\},\ \{1,2\},\ \{1,3\},\ \{1,4,5,6\}, \{2,3\},$$
$$\{2,4,5,6\},\ \{3,4,5,6\},\ \{1,2,3\},\ \{1,2,4,5,6\},\ \{1,3,4,5,6\},\ \{2,3,4,5,6\}\Big\}.$$

For each set we have the complement and F is now closed under countable unions.

(b) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\emptyset, \Omega, \{2, 4, 6\}, \{1, 3, 5\}, \{1, 2, 3\}, \{4, 5, 6\}\}$

No, this is not a σ-algebra.

Again the unions of the sets are missing which introduce new complement sets. Smallest σ-algebra:

$$\Big\{\emptyset,\ \Omega, \{2,4,6\}, \{1,3,5\}, \{1,2,3\}, \{4,5,6\}, \{1,2,3,4,6\}, \{5\},$$
$$\{2,4,5,6\}, \{1,3\}, \{1,2,3,5\}, \{4,6\}, \{1,3,4,6\}, \{2,5\}, \{2\}, \{1,3,4,5,6\}\Big\}.$$

For each set we have the complement and F is now closed under countable unions.

(c) $\Omega = \{1,2,3,4,5\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1,2\}, \{3,4,5\}, \{1,2,3,4\}, \{5\}, \{1,2,5\}\}$

No, this is not a σ-algebra.

Here only one complement $\{3,4\}$ of $\{1,2,5\}$ is missing.
So the smalles σ-algebra:

$$\Big\{\emptyset, \Omega, \{1,2\}, \{3,4,5\}, \{1,2,3,4\}, \{5\}, \{1,2,5\}, \{3,4\}\Big\}.$$

For each set we now have the complement and F is closed under countable unions.

Task 1.3 [1 Point]

For each of the following $(\Omega, \mathcal{F}, \mathbb{P})$, decide whether it is a valid probability triple. Justify your answer in each case.

(a) $\Omega = \{1,2,3,4,5\}, \quad \mathcal{F} = P(\Omega), \quad \mathbb{P}(A) = \sum_{n\in A} p_n$, with $(p_1, \ldots, p_5) = (0.2, 0.3, 0.4, 0.1, 0)$.

(b) $\Omega = \mathbb{N} \setminus \{0\}, \quad \mathcal{F} = P(\Omega), \quad \mathbb{P}(A) = \sum_{n\in A} \frac{1}{2^n}$.

(c) $\Omega = \mathbb{N} \setminus \{0\}, \quad \mathcal{F} = P(\Omega), \quad \mathbb{P}(A) = \sum_{n\in A} \frac{1}{n^2}$.

(d) $\Omega = [0,1], \quad \mathcal{F} = P(\Omega), \quad \mathbb{P}(A) = 0.6\,\delta_0(A) + 0.4\,\delta_1(A) \quad$ with $\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$

1.3

For each candidate $(\Omega, \mathcal{F}, \mathbb{P})$ we will decide if it is a probability triple which is again defined in the slides as follows:

- Let Ω be a non-empty set (sample space),
- $\mathcal{F}$ be a sigma-algebra over Ω
- $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ be a function (probability measure) with
  - $\mathbb{P}(\Omega) = 1$
  - For any disjoint $A_1, A_2, A_3, \ldots$ from $\mathcal{F}$,
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i) \quad \text{(sigma-additivity)}$$

  (Note that this includes finite unions as well, by setting all but finitely many $A_i = \emptyset$.)

Any such triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

(a) $\Omega = \{1, 2, 3, 4, 5\}, \mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}(A) = \sum_{n \in A} p_n,\ (p_1, \ldots, p_5) = (0.2, 0.3, 0.4, 0.1, 0)$

Yes, this is a probability space.

It holds that $0 \leq p_n \leq 1$ and
$$\sum_{n=1}^{5} p_n = 0.2 + 0.3 + 0.4 + 0.1 + 0 = 1$$
and so $\mathbb{P}(\Omega) = 1$.

(b) $\Omega = \mathbb{N} \setminus \{0\}, \quad \mathcal{F} = \mathcal{P}(\Omega), \quad \mathbb{P}(A) = \sum_{n \in A} 2^{-n}$

Yes, this is a probability space.

This is a geometric series with
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = 1,$$
so $\mathbb{P}(\Omega) = 1$ and also $0 \leq \mathbb{P}(A) \leq 1$ for all A.

(c) $\Omega = \mathbb{N} \setminus \{0\}, \mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}(A) = \sum_{n \in A} n^{-2}$

No, this is not a probability space.

We have again a known series called the Basel Problem
$$\mathbb{P}(\Omega) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6} > 1,$$
so $\mathbb{P}(\Omega) \neq 1$.

(d) $\Omega = [0,1], \quad \mathcal{F} = \mathcal{P}(\Omega), \quad \mathbb{P}(A) = 0.6\,\delta_0(A) + 0.4\,\delta_1(A)$

Yes, this is a probability space.

Clearly $0 \le \mathbb{P}(A) \le 1$ and $\mathbb{P}(\Omega) = 0.6 + 0.4 = 1$. If A_i are disjoint, at most one A_i contains 0 and at most one contains 1, where

$$\begin{aligned}\mathbb{P}\left(\bigcup_i^2 A_i\right) &= 0.6\,\delta_0\left(\bigcup_i^2 A_i\right) + 0.4\,\delta_1\left(\bigcup_i^2 A_i\right) \\ &= \sum_i \big(0.6\,\delta_0(A_i) + 0.4\,\delta_1(A_i)\big) \\ &= 0.6 \times 1 + 0.4 \times 1 \\ &= 1\end{aligned}$$

so $\mathbb{P}(\Omega) = 1$.

Task 1.4 [1 Points]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \mathbb{Q})$ be two probability triples, i.e., Ω is a non-empty sample space, $\mathcal{F}$ is a σ-algebra on Ω, and $\mathbb{P}, \mathbb{Q} : \mathcal{F} \to [0,1]$ are probability measures.

(a) Assume that

$$\mathbb{P}(A) = \mathbb{Q}(A) \quad \forall A \in \mathcal{F} \text{ where } \mathbb{P}(A) \le \frac{1}{2}.$$

Prove the following statement:

$$\mathbb{P}(A) = \mathbb{Q}(A) \quad \forall A \in \mathcal{F}.$$

(b) Assume now instead that

$$\mathbb{P}(A) = \mathbb{Q}(A) \quad \forall A \in \mathcal{F} \text{ where } \mathbb{P}(A) < \frac{1}{2}.$$

Do we still have $\mathbb{P}(A) = \mathbb{Q}(A)$ for all $A \in \mathcal{F}$? If you think so, prove this. If you believe the opposite, provide a counterexample, i.e., a concrete example of $\Omega, \mathcal{F}, \mathbb{P}$ and $\mathbb{Q}$ that do not satisfy this.

1.4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \mathbb{Q})$ be probability spaces.

(a)

We know that the probability of the set A $\mathbb{P}(A)$ + its complement $\mathbb{P}(A^c)$ together must equal to $\mathbb{P}(\Omega) = 1$ by the definition of probability spaces.

$$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1.$$

So it has to hold that either $\mathbb{P}(A) \le \frac{1}{2}$ or $\mathbb{P}(A^c) \le \frac{1}{2}$.
If $\mathbb{P}(A) \le \frac{1}{2}$, the assumption already gives us $\mathbb{Q}(A) = \mathbb{P}(A)$.
But if the complement $\mathbb{P}(A^c) \le \frac{1}{2}$, we also have from the assumption that $\mathbb{P}(A^c) = \mathbb{Q}(A^c)$ holds. That means that we can define $\mathbb{Q}(A)$ as $1 - \mathbb{Q}(A^c)$ So yes the statement is correct.

(b)

The statement can fail with the strict inequality now simply because we exclude $\mathbb{P}(A) = \frac{1}{2}$. We can easily construct a counterexample and show that it not holds.

Let $\Omega = \{1, 2\}$, $\mathcal{F} = \mathcal{P}(\Omega)$.
We define $\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \frac{1}{2}$ where obviously $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$ holds automatically to be a proper probability measure.
On the counterpart we now define $\mathbb{Q}(\{1\}) = \frac{2}{3}$, $\mathbb{Q}(\{2\}) = \frac{1}{3}$, and again $\mathbb{Q}(\emptyset) = 0$, $\mathbb{Q}(\Omega) = 1$. Then both $\mathbb{P}$ and $\mathbb{Q}$ are probability measures.

Now the only set with $\mathbb{P}(A) < \frac{1}{2}$ is $A = \emptyset$. And since we defined $\mathbb{P}(\emptyset) = 0$ but also $\mathbb{Q}(\emptyset) = 0$ it shows that

$$\mathbb{P}(A) = \mathbb{Q}(A) \quad \text{for all } A \text{ with } \mathbb{P}(A) < \tfrac{1}{2}$$

is satisfied. But actually we have defined that $\mathbb{P}(\{1\}) = \frac{1}{2}$ and also $\mathbb{Q}(\{1\}) = \frac{2}{3}$ where clearly $\frac{1}{2} \neq \frac{2}{3}$ so $\mathbb{P} \neq \mathbb{Q}$ on $\mathcal{F}$. It does not hold when the assumption uses $<$ instead of $\leq$.

Task 2 : Sigma-Algebras

Task 2.1

A set of sets E is a sigma-algebra over a finite set Ω iff E meets these conditions,

1. $\Phi, \Omega \in E$
2. If a set $A \in E$ then $A^c \in E$
3. if sets $A_1, A_2, ... \in E$ then also $\bigcup_{i=1}^{\infty} A_i \in E$

Algorithm 1 To check of the given set of sets E is a $\sigma-$algebra over a finite set Ω

Require: Sample space Ω, collection of sets $\mathcal{E}$

Let $\varnothing \leftarrow \Omega - \Omega$
if $\Omega \notin \mathcal{E}$ **then**
    return false
end if
if $\varnothing \notin \mathcal{E}$ **then**
    return false
end if
for all $A \in \mathcal{E}$ **do**
    if $\Omega - A \notin \mathcal{E}$ **then**
        return false
    end if
end for
$U \leftarrow \varnothing$
for all $A \in \mathcal{E}$ **do**
    $U \leftarrow U \cup A$
end for
if $U \notin \mathcal{E}$ **then**
    return false
end if
return true

The time the implemented algorithm takes is $\mathbf{O}(|\mathbf{E}|^{\mathbf{2}})$

Test Cases:

Ω	E	**Is Sigma Algebra?**
$\{1,2,3,4,5,6\}$	$\varnothing$, Ω	True
$\{1,2,3,4,5,6\}$	$\varnothing$, $\{1,3,5\}$, $\{2,4,6\}$, Ω	True
$\{1,2,3,4,5,6\}$	$\varnothing$, $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{3,4,5,6\}$, $\{1,2,5,6\}$, $\{1,2,3,4\}$, Ω	True
$\{1,2,3,4,5,6\}$	$\varnothing$, $\{1\}$, $\{2,3,4,6\}$, Ω	False
$\{1,2,3,4,5,6\}$	$\mathcal{P}(\Omega)$	True

Task 2.2

Algorithm 2 Construction of the Smallest Sigma-Algebra Containing a Given Collection of Sets

Require: Sample space Ω, collection of sets $\mathcal{E}$
Ensure: Smallest sigma-algebra on Ω containing all sets in $\mathcal{E}$ that are subsets of Ω

Initialize smallest_sigma_algebra $\leftarrow$ [ ]
Initialize valid_sets $\leftarrow$ [ ]
Let $\varnothing \leftarrow \Omega - \Omega$
for each A in $\mathcal{E}$ **do**
    if $A \subseteq \Omega$ **then**
        Append A to valid_sets
    end if
end for
Append all elements of valid_sets to smallest_sigma_algebra
if $\Omega \notin$ valid_sets **then**
    Append Ω to smallest_sigma_algebra
end if
if $\varnothing \notin$ valid_sets **then**
    Append $\varnothing$ to smallest_sigma_algebra
end if
for each A in valid_sets **do**
    Let $A^c \leftarrow \Omega - A$
    if $A^c \notin$ smallest_sigma_algebra **then**
        Append A^c to smallest_sigma_algebra
    end if
end for
return smallest_sigma_algebra

Test Cases:

Ω	E	**Smallest Sigma Algebra**
$\{1,2,3,4,5,6\}$	$\{1,2\}$, $\{8,4\}$	$\varnothing$, $\{1,2\}$, $\{3,4,5,6\}$, Ω
$\{1,2,3,4,5,6\}$	$\{1\}$	$\varnothing$, $\{1\}$, $\{2,3,4,5,6\}$, Ω
$\{1,2,3,4,5,6\}$	$\{3,4,5\}$	$\varnothing$, $\{3,4,5\}$, $\{1,2,6\}$, Ω
$\{1,2,3,4,5,6\}$	$\varnothing$	$\varnothing$, Ω
$\{1,2,3,4,5,6\}$	Ω	$\varnothing$, Ω

Task 2.3

If F_1 and F_2 are two sigma algebras over Ω
To Check if $F_1 \cap F_2$ is a sigma algebra over Ω

1. Since, $\Omega \in F_1$ and $\Omega \in F_2$, therefore, $\Omega \in F_1 \cap F_2$ holds True.
2. If $A \in F_1$ and $A \in F_2$, then $A \in F_1 \cap F_2$. Since F_1 and F_2 are sigma-algebras, $A^c \in F_1$ and $A^c \in F_2$ and hence $A^c \in F_1 \cap F_2$
3. If $A_1, A_2, ... \in F_1 \cap F_2$ then $A_1, A_2, ... \in F_1$ and $A_1, A_2, ... \in F_2$.

   Since *F_1* and *F_2* are sigma-algebras, $\bigcup_{i=1}^{\infty} A_i \in F_1$ and $\bigcup_{i=1}^{\infty} A_i \in F_2$.

   Hence $\bigcup_{i=1}^{\infty} A_i \in F_1 \cap F_2$ holds True.

Since all the conditions meet, $F_1 \cap F_2$ is a sigma algebra over Ω

Task 2.4

If F_1 and F_2 are two sigma algebras over Ω_1 and Ω_2
To Check if $F_1 \cup F_2$ is a sigma algebra over $\Omega_1 \cup \Omega_2$

Since $\Omega_1 \in F_1$ and $\Omega_2 \in F_2$, we assume $\Omega_1 \cup \Omega_2 \in F_1 \cup F_2$, but this is not True.

Let us assume $\Omega_1 = \{1\}$ and $\Omega_2 = \{2\}$

Then $F_1 = \{\Phi, \{1\}\}$ and $F_2 = \{\Phi, \{2\}\}$

$F_1 \cup F_2 = \{\Phi, \{1\}, \{2\}\}$ and does not contain $\Omega_1 \cup \Omega_2 = \{1, 2\}$

Therefore, $\Omega_1 \cup \Omega_2 \notin F_1 \cup F_2$

Hence we can say, $F_1 \cup F_2$ is not a sigma algebra over $\Omega_1 \cup \Omega_2$

Task 3 Distribution Functions [5 Points]

Task 3.1 [3 points]

Let

$$\mathcal{N}(x;\mu,\sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

denote the univariate *Gaussian* probability density function (pdf) with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

Show that the product of two Gaussian pdfs is again an (unnormalized) Gaussian, i.e.,

$$\mathcal{N}(x;a,\sigma_a^2)\,\mathcal{N}(x;b,\sigma_b^2) = Z\cdot\mathcal{N}(x;\mu_{ab},\sigma_{ab}^2) \quad \text{with} \quad \mu_{ab} = \frac{\sigma_b^2 a + \sigma_a^2 b}{\sigma_a^2+\sigma_b^2} \quad \text{and} \quad \sigma_{ab}^2 = \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2+\sigma_b^2},$$

where $Z = \int \mathcal{N}(x;a,\sigma_a^2)\,\mathcal{N}(x;b,\sigma_b^2)\,dx$ is the *partition function*, which can also be written as evaluating a different Gaussian at the point a: $Z = \mathcal{N}(a;b,\sigma_a^2+\sigma_b^2)$. Clearly show all steps in your derivation. Your final result should exactly recover the equation $\mathcal{N}(x;a,\sigma_a^2)\,\mathcal{N}(x;b,\sigma_b^2) = \mathcal{N}(a;b,\sigma_a^2+\sigma_b^2)\,\mathcal{N}(x;\mu_{ab},\sigma_{ab}^2)$ with μ_{ab},σ_{ab}^2 as above.

Hints:

- Write down the product using the definition above and make use of the rules of the exp function to transform the product into a single exponential form.
- After some algebraic manipulation, transform the exponent such that the term x^2 has no leading coefficient in the numerator, which should allow you to read off μ_{ab} and σ_{ab}^2.
- Complete the square in the exponent. The quantity that is added to the resulting quadratic will be related to $N(a;b,\sigma_a^2+\sigma_b^2)$.
- **Remark:** Here, we are multiplying two Gaussian density functions (pointwise). This is not the same as multiplying the corresponding Gaussian random variables.

3.1

$$\begin{aligned}
\mathcal{N}(x;a,\sigma_a^2)\,\mathcal{N}(x;b,\sigma_b^2) &= \left(\frac{1}{\sqrt{2\pi\sigma_a^2}}\exp\left(-\frac{(x-a)^2}{2\sigma_a^2}\right)\right)\left(\frac{1}{\sqrt{2\pi\sigma_b^2}}\exp\left(-\frac{(x-b)^2}{2\sigma_b^2}\right)\right)\\
&= \frac{1}{\sqrt{2\pi\sigma_a^2}}\frac{1}{\sqrt{2\pi\sigma_b^2}}\exp\left(-\frac{(x-a)^2}{2\sigma_a^2}\right)\exp\left(-\frac{(x-b)^2}{2\sigma_b^2}\right)\\
&= \frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}}\exp\left(-\frac{(x-a)^2}{2\sigma_a^2}\right)\exp\left(-\frac{(x-b)^2}{2\sigma_b^2}\right)
\end{aligned}$$

(Now using the first hint we know that $e^{-a}e^{-b} = e^{-a-b}$), so we get

$$= \frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}}\exp\left(-\frac{(x-a)^2}{2\sigma_a^2} - \frac{(x-b)^2}{2\sigma_b^2}\right) \tag{1}$$

Now we can expand the squares in the exponent to get the first and second binomial formulas

$$\begin{aligned}(x-a)^2 &= x^2 - 2ax + a^2,\\ (x-b)^2 &= x^2 - 2bx + b^2.\end{aligned} \tag{2}$$

and substitute these into the exponent term of what we got in (1)

$$\begin{aligned}
&-\frac{(x-a)^2}{2\sigma_a^2} - \frac{(x-b)^2}{2\sigma_b^2}\\
&= -\frac{1}{2\sigma_a^2}(x^2 - 2ax + a^2) - \frac{1}{2\sigma_b^2}(x^2 - 2bx + b^2)\\
&= -\frac{1}{2\sigma_a^2}x^2 + \frac{\cancel{2}a}{\cancel{2}\sigma_a^2}x - \frac{a^2}{2\sigma_a^2} - \frac{1}{2\sigma_b^2}x^2 + \frac{\cancel{2}b}{\cancel{2}\sigma_b^2}x - \frac{b^2}{2\sigma_b^2}\\
&= -\left(\frac{1}{2\sigma_a^2} + \frac{1}{2\sigma_b^2}\right)x^2 + \left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x - \left(\frac{a^2}{2\sigma_a^2} + \frac{b^2}{2\sigma_b^2}\right).
\end{aligned} \tag{3}$$

With the second hint we factor out the $-\frac{1}{2}$ to make the structure clearer to have x^2 with no leading coefficient in the numerator and also negate the +/- signs

$$= -\frac{1}{2}\left[\left(\frac{1}{\sigma_a^2} + \frac{1}{\sigma_b^2}\right)x^2 - 2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x + \left(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\right)\right]. \tag{4}$$

We can rewrite the current exponent by factoring out $\frac{1}{\sigma_{ab}^2}$ from the quadratic terms and introduce the combined variance σ_{ab}^2 like

$$\frac{1}{\sigma_{ab}^2} := \frac{1}{\sigma_a^2} + \frac{1}{\sigma_b^2}. \tag{5}$$

If we invert both sides we obtain

$$\sigma_{ab}^2 = \frac{1}{\frac{1}{\sigma_a^2} + \frac{1}{\sigma_b^2}} = \frac{1}{\frac{\sigma_a^2 + \sigma_b^2}{\sigma_a^2\sigma_b^2}} = \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2 + \sigma_b^2}. \tag{6}$$

Substituting (5) in (4) we get

$$\begin{aligned}
&= -\frac{1}{2}\left[\frac{1}{\sigma_{ab}^2}x^2 - 2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x + \left(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\right)\right]\\
&= -\frac{1}{2\sigma_{ab}^2}\left[x^2 - 2\sigma_{ab}^2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x + \sigma_{ab}^2\left(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\right)\right].
\end{aligned} \tag{7}$$

So the whole product is now

$$\mathcal{N}(x; a, \sigma_a^2)\,\mathcal{N}(x; b, \sigma_b^2) = \frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}}\,\exp\left\{-\frac{1}{2\sigma_{ab}^2}\left[x^2 - 2\sigma_{ab}^2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x + \sigma_{ab}^2\left(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\right)\right]\right\}. \tag{8}$$

Now we compare the exponent in (7) with the standard Gaussian exponent form

$$-\frac{1}{2\sigma_{ab}^2}\left[x^2 - 2\mu_{ab}x\right].$$

The coefficients of x^2 and x must match. From (7):
The coefficient of x^2 is 1, which already matches.
The coefficient of $-2x$ is $\sigma_{ab}^2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)$. So we can now alos read off

$$\mu_{ab} = \sigma_{ab}^2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right). \tag{9}$$

Substitute (6) into (9)

$$\begin{aligned}
\mu_{ab} &= \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2 + \sigma_b^2}\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right) \\
&= \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2 + \sigma_b^2}\left(\frac{a\sigma_b^2 + b\sigma_a^2}{\sigma_a^2\sigma_b^2}\right) \\
&= \frac{\sigma_b^2 a + \sigma_a^2 b}{\sigma_a^2 + \sigma_b^2}.
\end{aligned} \tag{10}$$

Then (7) becomes

$$-\frac{1}{2\sigma_{ab}^2}\left[x^2 - 2\sigma_{ab}^2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x + \sigma_{ab}^2\left(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\right)\right]. \tag{11}$$

and the quadratic term inside the brackets is actually just

$$x^2 - 2\sigma_{ab}^2\left(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\right)x = x^2 - 2\mu_{ab}x. \tag{12}$$

With the third hint we can now complete the square for the expression $x^2 - 2\mu_{ab}x$

$$\begin{aligned}
x^2 - 2\mu_{ab}x &= x^2 - 2\mu_{ab}x + (\mu_{ab}^2 - \mu_{ab}^2) \\
&= (x^2 - 2\mu_{ab}x + \mu_{ab}^2) - \mu_{ab}^2 \\
&= (x - \mu_{ab})^2 - \mu_{ab}^2.
\end{aligned} \tag{13}$$

We substitute this back into (11) and get

$$x^2 - 2\sigma_{ab}^2\Big(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\Big)\, x + \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big) = (x - \mu_{ab})^2 - \mu_{ab}^2 + \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big). \tag{14}$$

and the whole exponent becomes

$$\begin{aligned} -\frac{1}{2\sigma_{ab}^2}\left[x^2 - 2\sigma_{ab}^2\Big(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\Big)\, x + \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\right] &= -\frac{1}{2\sigma_{ab}^2}\left[(x - \mu_{ab})^2 - \mu_{ab}^2 + \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\right] \\ &= -\frac{(x - \mu_{ab})^2}{2\sigma_{ab}^2} + \frac{1}{2\sigma_{ab}^2}\left(\mu_{ab}^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\right). \end{aligned} \tag{15}$$

We can see that this separates into an x-dependent and a constant part just as in the standard Gaussian exponent form we mentioned above. Finally insert (15) into (8):

$$\begin{aligned} \mathcal{N}(x; a, \sigma_a^2)\, \mathcal{N}(x; b, \sigma_b^2) &= \frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}} \exp\Big(-\frac{(x - \mu_{ab})^2}{2\sigma_{ab}^2} + \frac{1}{2\sigma_{ab}^2}\big(\mu_{ab}^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\big)\Big) \\ &= \frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}} \exp\Big(\frac{1}{2\sigma_{ab}^2}\big(\mu_{ab}^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\big)\Big) \exp\Big(-\frac{(x - \mu_{ab})^2}{2\sigma_{ab}^2}\Big). \end{aligned} \tag{16}$$

Now we multiply and divide by $\sqrt{2\pi\sigma_{ab}^2}$ to normalize it

$$\begin{aligned} \mathcal{N}(x; a, \sigma_a^2)\, \mathcal{N}(x; b, \sigma_b^2) &= \underbrace{\frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}} \sqrt{2\pi\sigma_{ab}^2} \exp\Big(\frac{1}{2\sigma_{ab}^2}\big(\mu_{ab}^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\big)\Big)}_{=:Z} \\ &\quad \times \underbrace{\frac{1}{\sqrt{2\pi\sigma_{ab}^2}} \exp\Big(-\frac{(x - \mu_{ab})^2}{2\sigma_{ab}^2}\Big)}_{=\mathcal{N}(x;\mu_{ab},\sigma_{ab}^2)} \\ &= \underline{Z\, \mathcal{N}(x; \mu_{ab}, \sigma_{ab}^2).} \end{aligned} \tag{17}$$

Now to get the final term $\mathcal{N}(x; a, \sigma_a^2)\, \mathcal{N}(x; b, \sigma_b^2) = \mathcal{N}(a; b, \sigma_a^2 + \sigma_b^2)\, \mathcal{N}(x; \mu_{ab}, \sigma_{ab}^2)$ with μ_{ab}, σ_{ab}^2 we first simplify the constant exponent term in Z. We already know σ_{ab}^2 from (6). Now simplify the exponent $\frac{1}{2\sigma_{ab}^2}(\mu_{ab}^2 - \sigma_{ab}^2\big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\big))$ using (9) and getting rid of the μ_{ab} term

$$\begin{aligned} \mu_{ab}^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big) &= \sigma_{ab}^4\Big(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\Big)^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big) \\ \Rightarrow \quad \frac{1}{2\sigma_{ab}^2}\big(\mu_{ab}^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\big) &= \frac{1}{2\sigma_{ab}^2}\big(\sigma_{ab}^4\Big(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\Big)^2 - \sigma_{ab}^2\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)\big) \\ &= \frac{\sigma_{ab}^2\Big(\frac{a}{\sigma_a^2} + \frac{b}{\sigma_b^2}\Big)^2}{2} - \frac{\Big(\frac{a^2}{\sigma_a^2} + \frac{b^2}{\sigma_b^2}\Big)}{2}. \end{aligned} \tag{18}$$

Now we plug in (6) for σ_{ab}^2:

$$\begin{aligned}\sigma_{ab}^2\Big(\frac{a}{\sigma_a^2}+\frac{b}{\sigma_b^2}\Big)^2 &= \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2+\sigma_b^2}\left(\frac{a}{\sigma_a^2}+\frac{b}{\sigma_b^2}\right)^2 \\ &= \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2+\sigma_b^2}\left(\frac{a\sigma_b^2+b\sigma_a^2}{\sigma_a^2\sigma_b^2}\right)^2 \\ &= \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2+\sigma_b^2}\cdot\frac{(a\sigma_b^2+b\sigma_a^2)^2}{\sigma_a^4\sigma_b^4} \\ &= \frac{(a\sigma_b^2+b\sigma_a^2)^2}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}. \end{aligned} \tag{19}$$

If we insert it back into (18) we have

$$\frac{\frac{(a\sigma_b^2+b\sigma_a^2)^2}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}}{2}-\frac{\left(\frac{a^2}{\sigma_a^2}+\frac{b^2}{\sigma_b^2}\right)}{2}=\frac{1}{2}\left[\frac{(a\sigma_b^2+b\sigma_a^2)^2}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}-\left(\frac{a^2}{\sigma_a^2}+\frac{b^2}{\sigma_b^2}\right)\right] \tag{20}$$

We put the second term over the common denominator $\sigma_a^2\sigma_b^2$:

$$\frac{a^2}{\sigma_a^2}+\frac{b^2}{\sigma_b^2}=\frac{a^2\sigma_b^2+b^2\sigma_a^2}{\sigma_a^2\sigma_b^2}. \tag{21}$$

and recieve

$$\begin{aligned}&\frac{1}{2}\left[\frac{(a\sigma_b^2+b\sigma_a^2)^2}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}-\frac{a^2\sigma_b^2+b^2\sigma_a^2}{\sigma_a^2\sigma_b^2}\right] \\ &=\frac{1}{2}\left[\frac{(a\sigma_b^2+b\sigma_a^2)^2}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}-\frac{(a^2\sigma_b^2+b^2\sigma_a^2)(\sigma_a^2+\sigma_b^2)}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}\right] \\ &=\frac{1}{2}\,\frac{(a\sigma_b^2+b\sigma_a^2)^2-(a^2\sigma_b^2+b^2\sigma_a^2)(\sigma_a^2+\sigma_b^2)}{(\sigma_a^2+\sigma_b^2)\sigma_a^2\sigma_b^2}. \end{aligned} \tag{22}$$

Now we have to simplify the numerator as this is just a binomial formula and expand

$$(a\sigma_b^2+b\sigma_a^2)^2=a^2\sigma_b^4+2ab\sigma_a^2\sigma_b^2+b^2\sigma_a^4 \tag{23}$$

and

$$(a^2\sigma_b^2+b^2\sigma_a^2)(\sigma_a^2+\sigma_b^2)=a^2\sigma_b^2\sigma_a^2+a^2\sigma_b^4+b^2\sigma_a^4+b^2\sigma_a^2\sigma_b^2. \tag{24}$$

So the the numerator becomes

$$
\begin{aligned}
&(a\sigma_b^2 + b\sigma_a^2)^2 - (a^2\sigma_b^2 + b^2\sigma_a^2)(\sigma_a^2 + \sigma_b^2) \\
&= \left(a^2\sigma_b^4 + 2ab\sigma_a^2\sigma_b^2 + b^2\sigma_a^4\right) - \left(a^2\sigma_b^2\sigma_a^2 + a^2\sigma_b^4 + b^2\sigma_a^4 + b^2\sigma_a^2\sigma_b^2\right) \\
&= a^2\sigma_b^4 + 2ab\sigma_a^2\sigma_b^2 + b^2\sigma_a^4 - a^2\sigma_b^2\sigma_a^2 - a^2\sigma_b^4 - b^2\sigma_a^4 - b^2\sigma_a^2\sigma_b^2 \\
&= 2ab\sigma_a^2\sigma_b^2 - a^2\sigma_a^2\sigma_b^2 - b^2\sigma_a^2\sigma_b^2 \\
&= \sigma_a^2\sigma_b^2\left(2ab - a^2 - b^2\right) \\
&= -\sigma_a^2\sigma_b^2\left(a^2 - 2ab + b^2\right) \\
&= -\sigma_a^2\sigma_b^2\left(a - b\right)^2.
\end{aligned} \tag{25}
$$

Now we can plug (25) back into (22), and finally obtain the standard gaussian exponent form of Z

$$
\begin{aligned}
&\frac{1}{2}\,\frac{-\cancel{\sigma_a^2\sigma_b^2}\,(a-b)^2}{(\sigma_a^2 + \sigma_b^2)\cancel{\sigma_a^2\sigma_b^2}} \\
&= -\frac{(a-b)^2}{2(\sigma_a^2 + \sigma_b^2)}.
\end{aligned} \tag{26}
$$

Now since we have the exponent already only the normalization factor in Z needs to brought to standard form, which we can simplify again with replacing σ_{ab}^2, as follows

$$
\begin{aligned}
\frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}}\sqrt{2\pi\sigma_{ab}^2} &= \frac{1}{2\pi\sqrt{\sigma_a^2\sigma_b^2}}\sqrt{2\pi\frac{\sigma_a^2\sigma_b^2}{\sigma_a^2+\sigma_b^2}} \\
&= \frac{1}{2\pi\sqrt{\cancel{\sigma_a^2\sigma_b^2}}}\frac{\sqrt{2\pi}\sqrt{\cancel{\sigma_a^2\sigma_b^2}}}{\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{1}{2\pi}\frac{\sqrt{2\pi}\sqrt{1}}{\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{1}{2\pi}\frac{\sqrt{2\pi}\,1}{\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{1}{2\pi}\frac{\sqrt{2\pi}}{\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{\sqrt{2\pi}}{2\pi\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{\cancel{\sqrt{2\pi}}}{(\sqrt{2\pi})^{\cancel{2}}\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_a^2+\sigma_b^2}} \\
&= \frac{1}{\sqrt{2\pi(\sigma_a^2+\sigma_b^2)}}. \qquad (27)
\end{aligned}
$$

Combining (26) and (27) into Z, we obtain

$$
Z = \frac{1}{\sqrt{2\pi(\sigma_a^2+\sigma_b^2)}}\exp\left(-\frac{(a-b)^2}{2(\sigma_a^2+\sigma_b^2)}\right). \qquad (28)
$$

which is exactly the Gaussian density

$$
Z = \mathcal{N}\big(a; b, \sigma_a^2+\sigma_b^2\big). \qquad (29)
$$

and so we obtained the final result where we exactly recovered the equation

$$
\mathcal{N}(x; a, \sigma_a^2)\,\mathcal{N}(x; b, \sigma_b^2) = \mathcal{N}\big(a; b, \sigma_a^2+\sigma_b^2\big)\,\mathcal{N}\big(x; \mu_{ab}, \sigma_{ab}^2\big)
$$

Task 3.2 [2 points]

Let $p : \mathbb{R} \to \mathbb{R}$ denote the probability density function of a continuous random variable X, given as

$$p(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1. The corresponding *cumulative distribution function* (CDF) is defined as

   $$F(x) = \int_{-\infty}^{x} p(z)\, dz.$$

   Write down $F(x)$ as a simple function of x that does not involve an integral (i.e., solve the definite integral).

2. Compute $\mathbb{P}_X([-0.5, 0.5] \cup [1.5, 2])$ using only the CDF F and the properties of a probability measure.

3. Compute $\mathbb{P}_X(\{1\})$. Is it the same as $p(1)$?

4. Prove or refute: $\mathbb{P}_X([0, 1]) = \mathbb{P}_X([0, 1))$.

5. Analytically compute $\mathbb{E}_X[X]$.

6. We can sample from $p(x)$ using the *inverse transform sampling* trick: First, sample $u \sim \text{Unif}([0, 1])$ and then compute $x = F^{-1}(u)$ where F^{-1} denotes the inverse of F. The result x is a proper sample from p. Write down $F^{-1}(u)$ and implement this sampling procedure in `monte_carlo.py` (function `F_inv`).

7. Use this sampling procedure to estimate $\mathbb{E}_X[X]$ via Monte Carlo: For all $N \in \{100, 200, 300, \ldots, 10000\}$, compute the sample mean $\hat{\mathbb{E}}_N[X] := \frac{1}{N}\sum_{i=1}^{N} x_i$ where x_i are i.i.d. samples from p. Plot the sample mean as a function of N (i.e., N is shown on the x-axis, and the corresponding sample mean on the y-axis). Draw a horizontal line at the true expectation. Include this plot in your report.

3.2

The density is

$$p(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1.

CDF $F(x)$. Since we know that x lives in [0,2] we can directly conclude that $x \leq 0 = 0$.

$$F(x) = \int_{-\infty}^{x} p(z)\,dz = \int_{0}^{x} \frac{z}{2}\,dz = \frac{x^2}{4}$$

$$F(x) = \begin{cases} 0 & x \leq 0, \\ \frac{x^2}{4} & 0 < x < 2, \\ 1 & x \geq 2. \end{cases}$$

2.

Compute $\mathbb{P}_X([-0.5, 0.5] \cup [1.5, 2])$ using F. Since $[-0.5, 0.5]$ and $[1.5, 2]$ are disjoint we have

$$\mathbb{P}_X([-0.5, 0.5] \cup [1.5, 2]) = \big(F(0.5) - F(-0.5)\big) + \big(F(2) - F(1.5)\big).$$

With using our defined F from above

$$F(-0.5) = 0$$

$$F(0.5) = \frac{0.5^2}{4} = \frac{1}{16}$$
$$F(1.5) = \frac{1.5^2}{4} = \frac{9}{16}$$
$$F(2) = 1$$

So we have

$$\begin{aligned} \mathbb{P}_X([-0.5, 0.5] \cup [1.5, 2]) &= \frac{1}{16} + \left(1 - \frac{9}{16}\right) \\ &= \frac{1}{16} + \frac{7}{16} \\ &= \frac{8}{16} \\ &= \frac{1}{2} \end{aligned}$$

3.

Compute $\mathbb{P}_X(\{1\})$ and compare to $p(1)$.
For a pdf with continuous random variables we know that just a single point is always 0 because it has no width and probability density $\neq$ density!

$$\mathbb{P}_X(\{1\}) = 0.$$

But $p(1) = \frac{1}{2}$ and that's why $\mathbb{P}_X(\{1\}) \neq p(1)$.

4.

Prove or refute: $\mathbb{P}_X([0,1]) = \mathbb{P}_X([0,1))$.
We have $[0,1] = [0,1) \cup \{1\}$, so

$$\mathbb{P}_X([0,1]) = \mathbb{P}_X([0,1)) + \mathbb{P}_X(\{1\}).$$

From 3. where we have said that $\mathbb{P}_X(\{1\}) = 0$, so it follows that $\mathbb{P}_X([0,1]) = \mathbb{P}_X([0,1))$ is true.

5.

Compute $\mathbb{E}_X[X]$.

$$\begin{aligned}
\mathbb{E}[X] = \int_x p(x)\,x\,dx &= \int_0^2 \frac{x}{2} \cdot x\,dx \\
&= \frac{1}{2}\int_0^2 x^2\,dx \\
&= \frac{1}{2} \cdot \left(\frac{2^3}{3} - 0\right) \\
&= \frac{1}{2} \cdot \frac{2^3}{3} \\
&= \frac{1}{2} \cdot \frac{8}{3} \\
&= \frac{8}{6} \\
&= \frac{4}{3}
\end{aligned}$$

So the expectation $\mathbb{E}_X[X]$ is $\frac{4}{3}$.

6.

We are given the cumulative distribution function (CDF) of X:

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \dfrac{x^2}{4}, & 0 < x < 2, \\ 1, & x \geq 2. \end{cases}$$

$F(x)$ increases continuously only on the interval $0 < x < 2$ and on this interval we have

$$u = \frac{x^2}{4}.$$

and u ranges from 0 to 1 in that interval because

$$F(0) = 0, \quad F(2) = 1 \quad \Rightarrow \quad 0 < u < 1.$$

So to get the inverse we need to get x on one side

$$\begin{aligned} u &= \frac{x^2}{4} \\ 4u &= x^2 \\ 4u &= x^2. \\ x &= 2\sqrt{u} \end{aligned}$$

Including the intervals for each case for completeness, we now have the inverse function defined as

$$F^{-1}(u) = x = \begin{cases} 0, & u = 0, \\ 2\sqrt{u}, & 0 < u < 1, \\ 2, & u = 1. \end{cases}$$

7.