Advanced Mathematics

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Sequences and series of functions

Example of a sequence and a limit

Decimal representation of a rational number:

$$\frac{7}{6} = \frac{6+1}{6} = 1 + \frac{1}{6} = 1 + \frac{1}{10}\frac{10}{6} = 1 + \frac{1}{10}\left(1 + \frac{4}{6}\right) =$$

$$1+0.1+\frac{1}{100}\frac{40}{6} = 1+0.1+\frac{1}{10^2}\left(6 + \frac{4}{6}\right) = 1+0.1+0.06+\frac{1}{10^3}\frac{4}{6} = \cdots$$

Sequence of real numbers

$$a_0 = 1; \quad a_1 = 1.1; \quad a_2 = 1.16; \quad a_3 = 1.166; \ldots$$

Distance ($n > 1$):

$$\left|a_n - \frac{7}{6}\right| = \frac{4}{6 \times 10^n} \to 0 \quad \text{if} \quad n \to \infty \;\leftrightarrow\; \lim_{n\to\infty} a_n = \frac{7}{6}$$

Def.: The sequence $\{a_n\}_{n=1}^{\infty}$ has a limit a if and only if for $\forall \epsilon > 0$, there exists a natural number N_ϵ (depends on ϵ) such that

$$|a_n - a| < \epsilon, \quad \forall n > N_\epsilon$$

For the previous example: Given ϵ we have

$$\left| a_n - \frac{7}{6} \right| = \frac{4}{6 \times 10^n} < \epsilon \quad \Rightarrow \quad n > \lg\left(\frac{4}{6\epsilon}\right)$$

Thus, we can select

$$N_\epsilon = \left\lfloor \lg\left(\frac{4}{6\epsilon}\right) \right\rfloor + 1$$

Calculation of limits

Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

$$\lim_{n\to\infty}(a_n + b_n) = a + b$$

$$\lim_{n\to\infty}(a_n b_n) = ab$$

$$\lim_{n\to\infty} a_n^k = (\lim_{n\to\infty} a_n)^k = a^k$$

If $f : \mathbb{R} \to \mathbb{R}$, $f \in C$ (a continuous function), then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(a)$$

Sequences of functions. Pointwise convergence

Given a set $D \subset \mathbb{R}$ and a family of indexed functions $f_n : D \to R$, we can define a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$.

For each fixed $x \in D$, $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of real numbers.

Def.: (pointwise convergence) The sequence of functions $\{f_n\}_{n=0}^{\infty}$ converges to $f(x)$ pointwise on D iff $\forall x \in D$ the sequence of real numbers $\{f_n(x)\}_{n=0}^{\infty}$ converges to the number $f(x)$

$$\lim_{n\to\infty} f_n(x) = f(x)$$

In other words: $\{f_n\}_{n=0}^{\infty}$ converges to $f(x)$ pointwise iff $\forall \epsilon > 0$ $\exists N_{x,\epsilon} \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for $n > N_{x,\epsilon}$.

Uniform convergence

This is a stronger type of convergence.

Def.: $\{f_n\}_{n=0}^{\infty}$ converges to $f(x)$ uniformly iff $\forall \epsilon > 0$ and $\forall x \in D$ $\exists N_\epsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for $n > N_\epsilon$.

Note: N_ϵ is independent on x.

Criterion of non-convergence: If $\forall N \in \mathbb{N}$ $\exists \epsilon > 0$, $n > N$, and $x_0 \in D$ such that

$$|f_n(x_0) - f(x_0)| > \epsilon$$

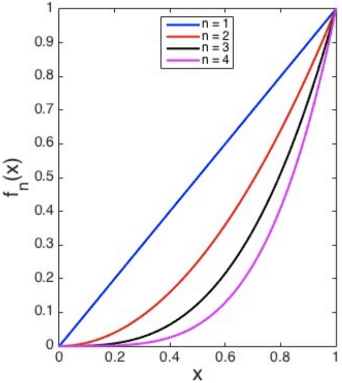
Supreme criterion for uniform convergence

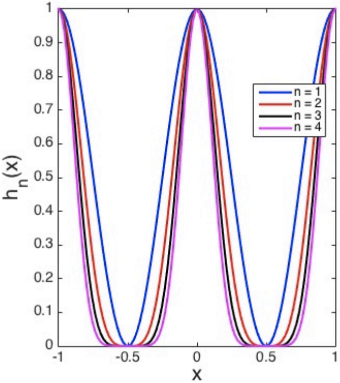
$$f_n(x) \xrightarrow{uniform} f(x) \quad \text{iff} \quad \lim_{n\to\infty} \left(\sup_{x \in D} |f_n(x) - f(x)| \right) = 0$$

Problem 1.1

Consider $f_n(x) = x^n$, $x \in [0,1]$ and $h_n(x) = (cos(\pi x))^{2n}$

1. Represent graphically for $n = 1, 2, 3$





2. Pointwise convergence

$$f(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & \text{if } \ x \in [0,1) \\ 1, & \text{if } \ x = 1 \end{cases}$$

$$f(x) = \lim_{n\to\infty} (cos(\pi x))^{2n} = \begin{cases} 1, & \text{if } \ x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

3. Uniform convergence.

$$\lim_{n\to\infty} \left(\sup_{x\in[0,1]} |x^n - f(x)| \right) = \lim_{n\to\infty} \left(\sup_{x\in[0,1)} \{x^n\} \right) = 1$$

Thus, it does not converge uniformly on $[0,1]$. However, if $x \in [0,a]$ $(0 < a < 1)$, then

$$\lim_{n\to\infty} \left(\sup_{x\in[0,a]} |x^n - f(x)| \right) = \lim_{n\to\infty} \left(\sup_{x\in[0,a]} \{x^n\} \right) = \lim_{n\to\infty} a^n = 0$$

Thus, the sequence converges uniformly on $[0, a]$!

General property: A sequence of continuous functions cannot converge to a noncontinuous function (the previous example).

Problem 1.4b. $f_n = \frac{\sin(nx)}{1+nx}$. Show that it converges uniformly on $[a, \infty)$, $a > 0$.

Pointwise: $(|\sin(nx)| \leq 1)$

$$f(x) = \lim_{n\to\infty} \frac{\sin(nx)}{1 + nx} = 0$$

Uniform:

$$L = \lim_{n\to\infty} \left(\sup_{x\in[a,\infty]} \left| \frac{\sin(nx)}{1 + nx} - 0 \right| \right) = \lim_{n\to\infty} \left(\sup_{x\in[a,\infty]} \frac{|\sin(nx)|}{1 + nx} \right)$$

Let's find the extreme points (max/min) of $\frac{\sin(nx)}{1+nx}$.

$$f_n' = \frac{n\cos(nx)(1+nx) - n\sin(nx)}{(1+nx)^2} = 0$$

Thus, the extreme points satisfy to

$$\cos(nx^*)(1+nx^*) = \sin(nx^*) \quad \text{or} \quad \cos^2(nx^*)(1+nx^*)^2 = \sin^2(nx^*)$$

which is equivalent to

$$\sin^2(nx^*) = \frac{(1+nx^*)^2}{1+(1+nx^*)^2}$$

Thus,

$$\sup_{x\in[a,\infty]} \frac{|\sin(nx)|}{1+nx} = \max\left\{ \frac{|\sin(na)|}{1+na}, \frac{\sqrt{\sin^2(nx^*)}}{1+nx^*} \right\} =$$

$$= \max\left\{\frac{|\sin(na)|}{1+na}, \frac{1}{\sqrt{1+(1+nx^*)^2}}\right\} =$$

$$= \max\left\{\frac{|\sin(na)|}{1+na}, \frac{1}{\sqrt{1+(1+nx_1^*)^2}}\right\}$$

where x_1^* is the first extreme value ($x_1^* \geq a$). Now

$$\lim_{n\to\infty}\left(\max\left\{\frac{|\sin(na)|}{1+na}, \frac{1}{\sqrt{1+(1+nx_1^*)^2}}\right\}\right) = 0$$

and thus the sequence converges uniformly.

Convergence of integrals

Theorem: If a sequence of functions $\{f_n(x)\}$ converges uniformly to a function f on the interval $[a, b]$ and the integrals $I_n = \int_a^b f_n(x)\, dx$ and $I = \int_a^b f(x)\, dx$ exist, then

$$\lim_{n\to\infty} I_n = I.$$

Problem 1.7: Find $\lim_{n\to\infty} \int_0^1 \frac{ne^x}{n+x} dx$

1. Pointwise $(f_n(x) = \frac{ne^x}{n+x})$

$$\lim_{n\to\infty} \frac{ne^x}{n+x} = e^x \lim_{n\to\infty} \frac{n}{n+x} = e^x \equiv f(x)$$

2. Uniform:

$$\lim_{n\to\infty}\left(\sup_{x\in[0,1]}\left|\frac{ne^x}{n+x}-e^x\right|\right)=\lim_{n\to\infty}\left(\sup_{x\in[0,1]}\frac{xe^x}{n+x}\right)$$

It's easy to check that $\left(\frac{xe^x}{n+x}\right)'>0$. Thus, the function reaches its maximum at the right end: $x=1$. Therefore,

$$\lim_{n\to\infty}\left(\sup_{x\in[0,1]}\frac{xe^x}{n+x}\right)=\lim_{n\to\infty}\left(\frac{e}{n+1}\right)=0$$

3. Limit: (we now can swap the limit and integral)

$$\lim_{n\to\infty}\int_0^1\frac{ne^x}{n+x}dx=\int_0^1\left(\lim_{n\to\infty}\frac{ne^x}{n+x}\right)dx=\int_0^1 e^x dx=e-1$$

Series of real numbers

In the first example of this chapter:

$$s_0 = 1, \quad s_1 = 1 + 0.1, \quad s_2 = 1 + 0.1 + 0.06, \ldots$$

then $\lim_{n\to\infty} s_n = \frac{7}{6}$.

In general: $\sum_{n=0}^{\infty} a_n$ is a series of real numbers; $s_n = \sum_{k=0}^{n} a_n$ is a partial sum.

Geometric series: $\sum_{n=0}^{\infty} a^n$. Then $s_n = 1 + a + a^2 + \cdots + a^n$

$$1 + as_n = s_{n+1} = s_n + a^{n+1} \quad \Rightarrow \quad s_n = \frac{1 - a^{n+1}}{1 - a}$$

If $0 \leq a < 1$

$$\sum_{n=0}^{\infty} a^n = \lim_{n\to\infty} s_n = \frac{1}{1 - a}$$

Generic series $s_n = \sum_{k=0}^{n} a_k$.

Convergence. Assume the sequence of partial sums converges: $s_n \to s$. Then $s_n - s_{n-1} = a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0$$

Thus, the necessary condition for convergence is: $a_n \to 0$.

Integral criterium: If $f(x)$ is a positively defined and monotonously decaying function and

$$\int_a^\infty f(x)\,dx < \infty \quad \text{then} \quad \sum_{n=a}^{\infty} f(n) < \infty$$

i.e., converges.

Series of functions

Let $f_n : D \to \mathbb{R}$ be a family of functions. Then we can define the series:

$$\sum_{k=1}^{\infty} f_k(x)$$

We can also define partial sums:

$$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

Then (if exists) $s(x) = \lim_{n\to\infty} s_n(x) = \sum_{k=1}^{\infty} f_k(x)$.

Convergence: A series converges pointwise (uniformly) if the corresponding sequence $s_n(x)$ converges pointwise (uniformly).

Weierstrass criterium: If $f_n(x) \leq a_n \; \forall n > N$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Problem 1.8a: Study the pointwise and uniform convergence of $\sum_{n=0}^{\infty} x^n$, $x \in [0,1]$.

$$s_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

1. Pointwise:

$$s(x) = \sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} s_n = \begin{cases} \frac{1}{1-x}, & \text{if } 0 \le x < 1 \\ \infty, & \text{if } x = 1 \end{cases}$$

The series converges for $x \in [0,1)$.

2. Uniform: Consider $x \in [0,b]$, $0 < b < 1$. Let's apply the Weierstrass criterium:

$$f_n(x) = x^n \le b^n, \quad \sum_{n=1}^{\infty} b^n = \frac{1}{1-b} < 0 \quad \Rightarrow \quad \text{uniform convergence.}$$

Note that for $x \in [0,1)$ the series does not converge uniformly. Let's show it.

$$\lim_{n\to\infty}\left(\sup_{x\in[0,1)}|s_n(x)-s(x)|\right)=\lim_{n\to\infty}\left(\sup_{x\in[0,1)}\left|\frac{1-x^{n+1}}{1-x}-\frac{1}{1-x}\right|\right)=$$

$$=\lim_{n\to\infty}\left(\sup_{x\in[0,1)}\frac{x^{n+1}}{1-x}\right)=\infty$$

Note $\sup=\infty$ is reached when $x\to 1$.