Advanced Mathematics

Second order differential equations. Laplace Transform.

1. **(*)** The solutions of the characteristic equation associated to a linear, second order ODE with constant coefficients are $\lambda_1 = -1$ and $\lambda_2 = 0$. Find the differential equation and its general solution. Do the same for the case $\lambda_1 = -1$ y $\lambda_2 = 2$.

2. Find the general solutions of the following differential equations:
   1) $y'' - y = 0$ 2) $y'' - 8y' + 15y = 0$ 3) $y'' - 3y' - 10y = 0$. 4) $y'' - 6y' + 25y = 0$
   5) $2y'' + 2y' + 3y = 0$ 6)$y'' + 2y' + y = 0$ 7) $y'' + 4y = 0$

3. Find the general solutions of the following non-homogeneous differential equations:
   1) $y'' + 3y' - 10y = 6e^{4x}$ 2) $y'' - 4y' + 4y = 2e^{2t} + \frac{t}{2}$ 3) $y'' - 3y' + 2y = 14\sin(2x) - 18\cos(2x)$
   4) $y'' + 4y = 3\sin(x)$ 5) $y'' - 2y' + 2y = e^x \sin(x)$ 6) $2y'' - 4y' - 8y = -50\cos(3x) - 40\sin(3x)$.

4. Solve the following *boundary value problem*:
$$x'' + x' - 6x = 0,\ x(0) = 1 \text{ and } x(\infty) = 0\,.$$

5. Solve the following initial and boundary value problems:
   1) $y'' - 5y' + 6y = 0, \quad y(1) = e^2,\ y'(1) = 3e^2$ 2) $y'' - 6y' + 9y = 0, \quad y(0) = 3,\ y'(0) = 11$
   3) $2y'' - y' + 2y = e^{4x}, \quad y(0) = 3,\ y'(0) = 2$ 4) $y'' - y' - 5y = 1, \quad \lim_{x\to\infty} y(x) = -\frac{1}{5}$
   5) $y'' - 5y' + 6y = 2e^{-2t}(9\sin(2t) + 4\cos(2t)), \quad \lim_{t\to\infty} y(t) = 0$.

6. **(*)** Assume that the roots of the polinomial $\lambda^2 + a\lambda + b = 0$ have negative real part. Prove that every solution to the ODE $x'' + ax' + bx = 0$ satisfies $\lim_{t\to\infty} x(t) = 0$.

7. **(*)** Let a, b and c be three positive constants. Prove that the difference between any two of the solutions of the equation $ax'' + bx' + cx = g(t)$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, converges to zero when $t \to \infty$.

8. Consider the ODE $x'' + a_1x' + a_0x = q(t)$, with $a_0 \neq 0$, where q is a polinomial of degree 2. Prove that there is a particular solution of the equation which is also a polinomial of degree 2.

9. Consider the ODE $x'' + a_1x' + a_0x = q(t)$.
   a) If $q(t) = e^{\alpha t}$, prove that the equation admits a particular solution of one of this forms: $Ae^{\alpha t}$ when α is not a root of the characteristic equation, $Ate^{\alpha t}$, when α is a *simple* root of the characteristic equation, and $At^2e^{\alpha t}$ in the case that α is a *double* root of the characteristic equation.
   b) If $q(t) = \cos\alpha t$, prove that the equation admits a particular solution of the forms $A\cos\alpha t + B\sin\alpha t$ or $t(A\cos\alpha t + B\sin\alpha t)$, if $\pm\alpha i$ are roots of the characteristic equation.

10. **Damped Oscilator** A disk, fixed to a mass m, is inmersed in a fluid that performs a damping force (ie. friction) $-b\frac{dx}{dt}$. The elastic restoring force of a spring is $-kx$. Write and solve the equation for the damped spring (See Figure **??**).

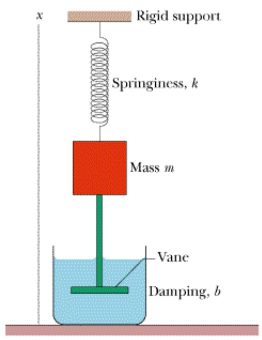

Figure 1: Damped oscilator

11. **(*)** In an RLC circuit (Resistor, inductor, capacitor, see Figure **??**) the potential drop on each element are given by:
Resistance: $V_R = -RI$
Inductor: $V_L = -L\frac{dI}{dt}$
Capacitor: $\frac{\mathrm{d}V_C}{\mathrm{d}t} = -\frac{1}{C}I(t)$
(where R, L, and C are constants and I is the current, dependent on time). If $E(t)$ denotes the voltage of a battery, find the differential equation of the system under Kirchoff laws. Take the derivative ot the final expression and compare with the previous exercise.

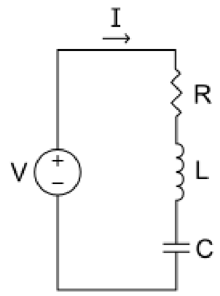

Figure 2: RLC circuit

12. a) **(*)** Given $f : \mathbb{R} \to [0, \infty)$. Assume that $Lf(s_0) := \int_0^\infty f(t)e^{-s_0 t}dt$ exists for some $s_0 > 0$ and also $\lim_{t\to\infty} e^{-s_0 t} f(t) = 0$. Prove that, for all $s > s_0$ the following integral has a finite value:
$$Lf(s) = \int_0^\infty f(t)e^{-st}dt.$$
The function Lf is the *Laplace Transform* of the function f. Verify that the domain of Lf is at least $[s_0, \infty)$.
b) Verify the following table of transforms:
1) $f(x) = 1$ has $Lf(s) = \frac{1}{s}$ 2) $f(x) = x$ has $Lf(s) = \frac{1}{s^2}$
3) $f(x) = e^{-\alpha x}$ has $Lf(s) = \frac{1}{s+\alpha}$ 4) $f(x) = \sin \beta x$ has $Lf(s) = \frac{\beta}{s^2+\beta^2}$.
5) $f(x) = \cos \beta x$ has $Lf(s) = \frac{s}{s^2+\beta^2}$. 6) Find $Lf(s)$, where $f(x) = e^{3x} \sin x$.

13. Prove the following properties of the Laplace transform:
a) For functions f and g and $\alpha, \beta \in \mathbb{R}$ we have that $L(\alpha f + \beta g) = \alpha Lf + \beta Lg$.
b) If f is twice differentiable, then $Lf'(s) = sLf(s) - f(0)$ and $Lf''(s) = s^2 Lf(s) - sf(0) - f'(0)$.

14. Find the Laplace transform of the following functions:
$$f(t) = \begin{cases} x \sin x, & x \geq 3 \\ 0, & x < 3 \end{cases}, \quad \text{and} \quad g(t) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x \geq 1. \end{cases}$$

15. Consider the functions:
1) $Lf(s) = \frac{1}{s+3}$ 2) $Lf(s) = \frac{4}{s^2+3s+1}$ 3) $Lf(s) = \frac{s}{s^2-4s+3}$
4) $Lf(s) = \frac{s-7}{25+(s-7)^2}$ 5) $Lf(s) = \frac{s+1}{s^2+1}$.
in each case find f, ie. the inverse Laplace transform.

16. **(*)** Let $H(s) = \frac{1}{s+3}$, $H(s) = \frac{4}{s^2+3s+1}$ and $H(s) = \frac{s}{s^2-4s+3}$ be three *transference functions* of electric circuits. Find the differential equations for those circuits.

17. Solve the following initial value problems by means of the Laplace transform:
a) $y'' - 6y' + 5y = 0,\ y(0) = 3, y'(0) = 11$ b) $y'' + 4y' + 5y = 0,\ y(0) = 1, y'(0) = 0$
c) $\begin{cases} \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x} \\ y(0) = 1, y'(0) = 0 \end{cases}$ d) $\begin{cases} \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x} \\ y(0) = 1, y'(0) = 0 \end{cases}$
e) $\begin{cases} \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} + 3 \\ y(0) = 0, y'(0) = 0 \end{cases}$

LAPLACE TRANSFORM TABLE

The following functions of the independent variable t have Laplace transforms given by the functions of s on the right:

1. If $f(t) = 1, \forall t$ then $Lf(s) = \frac{1}{s}$
2. If $f(t) = \sin(\alpha t)$, then $Lf(s) = \frac{\alpha}{s^2+\alpha^2}$
3. If $f(t) = \cos(\alpha t)$, then $Lf(s) = \frac{s}{s^2+\alpha^2}$
4. If $f(t) = e^{-\alpha t}$, then $Lf(s) = \frac{1}{s+\alpha}$
5. If $f(t) = \sinh(\alpha t)$, then $Lf(s) = \frac{\alpha}{s^2-\alpha^2}$
6. If $f(t) = \cosh(\alpha t)$, then $Lf(s) = \frac{s}{s^2-\alpha^2}$
7. If $f(t) = e^{-\alpha t}\sin(\beta t)$, then $Lf(s) = \frac{\beta}{(s+\alpha)^2+\beta^2}$
8. If $f(t) = e^{-\alpha t}\cos(\beta t)$, then $Lf(s) = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$
9. In general, given $f(t)$, then $L[e^{-\alpha t}f(t)](s) = Lf(s+\alpha)$
10. If $f(t) = t^n$, then $Lf(s) = \frac{\Gamma(n+1)}{s^{n+1}}$, ($\Gamma$ is the Euler's Gamma function).
11. If $f(t) = te^{-\alpha t}$, then $Lf(s) = \frac{1}{(s+\alpha)^2}$
12. If $f(t) = t\sin(\alpha t)$, then $Lf(s) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$
13. If $f(t) = t\cos(\alpha t)$, then $Lf(s) = \frac{s^2-\alpha^2}{(s^2+\alpha^2)^2}$
14. In general, given $f(t)$, then $L[t^n f(t)](s) = (-1)^n \frac{\partial^n Lf(s)}{\partial s^n}$