ADVANCED MATHEMATICS

Rings, Fields and Polinomials I

1. Show if the following sets have a ring structure, and verify if they are conmutative, with unity, integral domains or fields.

   (a) The positive integers.

   (b) The integers that are a multiple of 7.

   (c) $\{0, 1, -1, i, -i\}$.

   (d) $\mathcal{M}_{2\times 3}(\mathbb{R})$.

   (e) $\mathcal{M}_{2\times 2}(\mathbb{Z}_3)$.

   (f) $\mathbb{Z} \times \mathbb{Z}_3 \times 2\mathbb{Z}$.

   (g) $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$.

   (h) The set of polinomials $\{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ of $\mathbb{R}[x]$.

2. Prove that a non-empty set B of a ring A is a *subring* if for every pair $b, b' \in B$ we have that $b - b' \in B$ and $bb' \in B$. Additionally, show that B is an *ideal* of A if for every pair $b, b' \in B$, $a \in A$, it holds that $b - b'$, ab and ba are in B.

3. Prove that the set $A = \{0, 2, 4, 6, 8\}$ is a subring of $\mathbb{Z}_{10}$. Is A an ideal of $\mathbb{Z}_{10}$? Compute the table of A for the multiplication and see if A has a neutral element for the product. Is A a Field?

4. Show that the set $B := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$ is a subring of $\mathcal{M}_2(\mathbb{R})$. Prove that the set I of matrices of the form $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is an ideal of B. Is I an ideal of $\mathcal{M}_2(\mathbb{R})$?

5. An element b of a ring B is **a divisor of zero** if $b \neq 0$ and there exists $a \neq 0$ with $a \in B$ such that $ab = 0$. We say that $a \in B$ is **nilpotent** if $a \neq 0$ and there exists an integer $n > 1$ such that $a^n = 0$. Prove that, if a es nilpontent, then it is a divisor of zero.

6. (a) Consider the ring B of exercise 4. Prove that every non-zero element of the ideal I of exercise 4 is nilpotent.
   (b) Find the nilpotent elements of the ring $\mathbb{Z}_{12}$.

7. Show that the set B of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, with $a, b \in \mathbb{R}$ is a subring with unity of $\mathcal{M}_2(\mathbb{R})$. Let $f : B \to \mathbb{C}$ be the mapping defined by $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + bi$. Prove that f is a ring isomorphism. Show that then B is a Field.

8. Compute the quotient and remainder of the division:

   (a) $x^4 + 3x^3 + 2x^2 + x + 4$ by $3x^2 + 2x$ in $\mathbb{Z}_5[x]$.

   (b) x^{10} by $x^2 + 1$ in $\mathbb{Z}_2[x]$.

   (c) $x^4 + 3x^3 + 2x^2 + x + 4$ by $x^2 + 2x$ in $\mathbb{Z}[x]$.

   (d) $x^4 + 3x^3 + 2x^2 + x + 4$ by $3x^2 + 2x$ in $\mathbb{Q}[x]$.

9. Given $m, n, p \in \mathbb{N}$, with $p > 1$, show the equivalence of:

   (a) $m|n$.

   (b) $p^m - 1 \,|\, p^n - 1$.

   (c) $x^{p^m-1} - 1 \,|\, x^{p^n-1} - 1$.

10. Compute the greatest common divisor of each of the following pairs of polinomials and write them as $a(x)f(x) + b(x)g(x)$:

    (a) $f(x) = x^3 - 1$, $g(x) = x^4 - x^3 + x^2 + x - 2$, in $\mathbb{Q}[x]$;

    (b) $f(x) = x^2 + 1$, $g(x) = x^3 + 2x - i$, in $\mathbb{C}[x]$;

    (c) $f(x) = x^3 + x + 1$, $g(x) = x + 1$ in $\mathbb{Z}_3[x]$;

    (d) $f(x) = x^3 + x + 1$, $g(x) = x + 1$ in $\mathbb{Z}_5[x]$;

    (e) $f(x) = x^4 + x^3 - x^2 + x - 2$, $g(x) = x^3 + 6x^2 + x + 1$ in $\mathbb{Q}[x]$;

    (f) $f(x) = x^4 + x^3 + x^2 + x$, $g(x) = x^2 + x - 1$ in $\mathbb{Z}_3[x]$;

    (g) $f(x) = x^5 + 5x^4 + 3x^3 + 2x + 1$, $g(x) = x^4 + 3$ in $\mathbb{Z}_7[x]$;

11. Find all the zeros in $\mathbb{Z}_5$ of the polinomials $f(x) = x^5 + 3x^3 + x^2 + 2x \in \mathbb{Z}_5[x]$ and $g(x) = x^5 - x \in \mathbb{Z}_5[x]$.

12. Which of the following polinomials have multiple roots?

    (a) $g(x) = x^4 - x^3 + x^2 + x - 2$, in $\mathbb{Q}[x]$;

    (b) $g(x) = x^3 + 2x - i$, in $\mathbb{C}[x]$;

    (c) $f(x) = x^3 + x + 1$ in $\mathbb{Z}_3[x]$;

    (d) $f(x) = x^3 + x + 1$ in $\mathbb{Z}_5[x]$;

    (e) $f(x) = 3x^4 + 6x^3 + 5x^2 + 4x + 2$ in $\mathbb{Q}[x]$;

    (f) $f(x) = x^4 + x^3 + x^2 + x$ in $\mathbb{Z}_3[x]$;

    (g) $f(x) = x^5 + 5x^4 + 3x^3 + 2x + 1$ in $\mathbb{Z}_7[x]$;