Advanced Mathematics

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Chapter 2

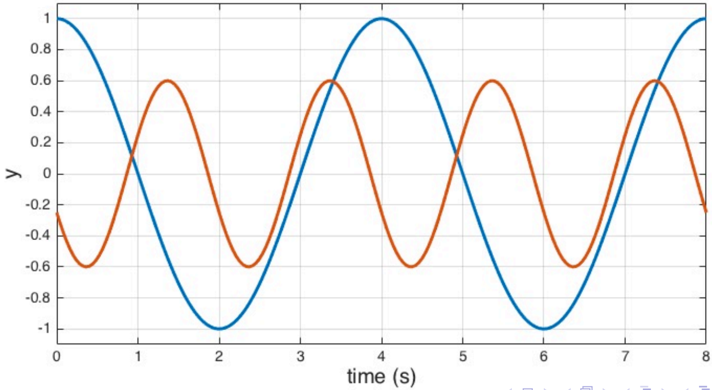
Fourier series and Fourier transform

Periodic functions

Let's consider a function (or as we shall call it a signal)

$$s(t) = A\cos(\omega t + \phi)$$

blue : $A = 1,\ \omega = 2\pi\frac{1}{4},\ \phi = 0$ red : $A = 0.6,\ \omega = 2\pi\frac{1}{2},\ \phi = 2$



Def.: A signal $s(t)$ is periodic iff $\exists T > 0$ s.t. $s(t+T) = s(t)$.

Note: If $s(t+T) = s(t)$ then $s(t+2T) = s(t)$ etc.

Def.: $T_0 = \min\{T\}$ is called the period of $s(t)$

In the example above: for the blue curve $T = 4$ s; for red $T = 2$ s. Thus, the blue function (signal) makes one cycle per each 4 s.

The function $s(t) = A\cos\left(2\pi\frac{t}{T} + \phi\right)$ has period T.

Def.: $f = \frac{1}{T}$ (1/s) is called the frequency of $s(t)$. It is measured in Hertz

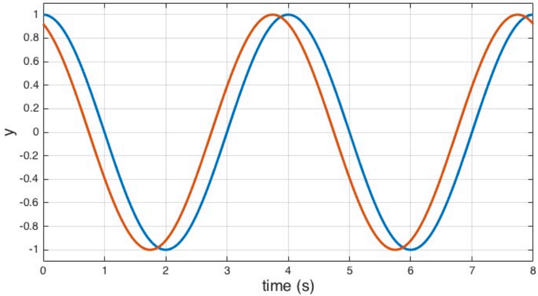
$$1 \text{ Hz} = \frac{1}{1 \text{ s}}$$

For example 4 Hz means 4 cycles per second.

Def.: The magnitude $\varphi(t) = 2\pi f t + \phi$ measured in radians is called the phase or angle of the signal.

Def.: The value $\omega = 2\pi f$ measured in [rad/s] is called the angular velocity of the signal.

Given two signals $s_1(t) = \cos(\omega t)$ and $s_2(t) = \cos(\omega t + \phi)$, we can define the phase shift ϕ between them.



The time delay τ can be calculated by:

$$s_1(t) = s_2(t-\tau) = \cos(\omega t) = \cos(\omega(t-\tau)+\phi) \Rightarrow \omega\tau = \phi \Rightarrow \tau = \frac{\phi}{\omega}$$

Def.: Given a signal $s(t) = A\cos(\omega t + \phi)$ the magnitude A is called the amplitude of the signal.

Problem 2.1a Find the period of $\sin(ax)$

By definition:

$$\sin(a(x+T)) = \sin(ax+aT) = [\text{ if } aT = 2\pi] = \sin(ax+2\pi) = \sin(ax)$$

Thus, $T = \frac{2\pi}{a}$. From the other side we know $\omega = a = 2\pi f = \frac{2\pi}{T}$, which gives the same result.

Problem 2.1c Find the period of e^{iax}

We have

$$e^{iax} = e^{iax+2\pi i} = e^{ia(x+2\pi/a)} \Rightarrow T = \frac{2\pi}{a}$$

Orthogonal functions

Def.: Two functions $f(x)$ and $g(x)$ are called orthogonal if their scalar product is equal to zero:

$$\langle f, g \rangle = \int_a^b f^*(x) g(x)\, dx = 0$$

Problem 2.2a Show that $\{\cos nx, \sin nx\}$ $(n \geq 0)$ are orthogonal in $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \cos nx \sin nx\, dx = -\frac{1}{n} \int_{-\pi}^{\pi} \cos nx\, d \cos nx = -\frac{\cos^2 nx}{2n}\Big|_{-\pi}^{\pi} = 0$$

Problem 2.3 Prove

$$\int_{-T/2}^{T/2} f(t)\,dt = \int_0^T f(t)\,dt = \int_\alpha^{\alpha+T} f(t)\,dt$$

$$\int_{-T/2}^{T/2} f(t)\,dt = \int_{-T/2}^{0} f(t)\,dt + \int_0^{T/2} f(t)\,dt = [u = t - T] =$$

$$= \int_{T/2}^{T} f(u+T)\,du + \int_0^{T/2} f(t)\,dt = \int_0^T f(t)\,dt$$

$$\int_\alpha^{T+\alpha} f(t)\,dt = \int_\alpha^0 f(t)\,dt + \int_0^T f(t)\,dt + \int_T^{T+\alpha} f(t)\,dt$$

The last integral ($u = t - T$):

$$\int_T^{T+\alpha} f(t)\,dt = \int_0^\alpha f(u+T)\,du = -\int_\alpha^0 f(t)\,dt$$

Fourier series

Let's consider a signal $s(t)$ such that $s(t+2\pi) = s(t)$ (2π periodic). In 1807 Jean-Baptiste Joseph Fourier proposed that

$$s(t) = \sum_{n=0}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

where a_n, b_n are some real numbers. Later it was called the Fourier series.

This series allows representing $s(t)$ as a composition of simple sin and cosine functions.

The series must converge. Thus, for $n \gg 1$, a_n and b_n should be small. Therefore, $s(t) \approx \sum_{n=0}^{N} (a_n \cos(nt) + b_n \sin(nt))$, if N is big enough.

Calculation of a_n and b_n

Let's start from the definition and integrate it (assuming that the series converges uniformly):

$$\int_{-\pi}^{\pi} s(t)\,dt = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \left(a_n \cos(nt) + b_n \sin(nt)\right)\,dt$$

Using Problem 2.2 we get the first coefficient:

$$\int_{-\pi}^{\pi} s(t)\,dt = 2\pi a_0 \quad \Rightarrow \quad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} s(t)\,dt$$

Now we can multiply by $\cos(kt)$ and integrate

$$\int_{-\pi}^{\pi} s(t)\cos(kt)\,dt = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \cos(kt)\left(a_n \cos(nt) + b_n \sin(nt)\right)dt$$

Due to the function orthogonality, only for $k = n$ we have nonzero term:

$$\int_{-\pi}^{\pi} s(t)\cos(nt)\,dt = a_n \int_{-\pi}^{\pi} \cos^2(nt)\,dt = a_n \int_{-\pi}^{\pi} \frac{1+\cos(2nt)}{2}\,dt = \pi$$

Thus,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\cos(nt)\,dt \quad n > 0$$

We can do the same with sin:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} s(t)\sin(nt)\,dt \quad n \geq 0$$

Note, $b_0 = 0$. Then we have: Given a signal $s : [-\pi, \pi] \to \mathbb{R}$ (could be not periodic) its Fourier series is

$$s(t) = a_0 + \sum_{n=1}^{\infty} (a_n\cos(nt) + b_n\sin(nt))$$

where a_n and b_n are calculated by the equations given above.

Problem 2.5a Find Fourier series for $s(t) = |t|$ on $[-\pi, \pi]$.

We note that this is an even function. Thus, $b_n = 0$. Let's find a_n:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|\, dt = \frac{1}{\pi}\int_0^{\pi} t\, dt = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \cos(nt)\, dt = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\, dt = [\text{by parts}] =$$

$$= \frac{2}{\pi n}\left[t\sin(nt)\Big|_0^{\pi} - \int_0^{\pi} \sin(nt)\, dt\right] = \frac{2}{\pi n^2}\cos(nt)\Big|_0^{\pi} = \frac{2[(-1)^n - 1]}{\pi n^2} =$$

$$= -\frac{4}{\pi n^2}\begin{cases} 0, & \text{if} \quad n = 2k \\ 1, & \text{if} \quad n = 2k+1 \end{cases}$$

Matlab script for Problem 2.5a

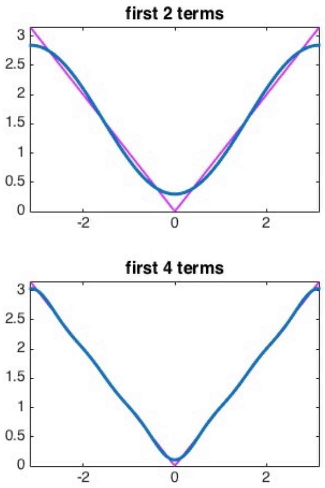
```
%% Fourier Series for s(t) = |t| on [-pi,pi]
t = linspace(-pi,pi,200); % time
s = abs(t);               % signal

a0 = pi/2; f0 = ones(1,200); % constant term

k = (1:5)';
a = -4./(pi*(2*k-1).^2);  % values for a_n
e = cos((2*k-1)*t);       % cos(nt) functions

% Make drawing
figure('color','w','position',[100 100 800 500])
sp = a0*f0;
for k = 1:4
    subplot(2,2,k)
    plot(t,s,'m','LineWidth',2)
    hold on
    sp = sp + a(k)*f(k,:);
```

Graphic representation for Problem 2.5a



Fourier series: General case

Let $s(t)$ be defined on the interval $[0, T]$. The Fourier series:

$$s(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right)\right)$$

$$a_0 = \frac{1}{T}\int_0^T s(t)\, dt, \quad a_n = \frac{2}{T}\int_0^T s(t) \cos\left(\frac{2\pi n}{T} t\right)\, dt$$

$$b_n = \frac{2}{T}\int_0^T s(t) \sin\left(\frac{2\pi n}{T} t\right)\, dt$$

Theorem (*Convergence*). If a Fourier series of a function $s(t)$ with period T converges uniformly, then the series converges to $s(t)$.

Convergence of Fourier series

Theorem: Let $s : \mathbb{R} \to \mathbb{R}$, $s(t+T) = s(t)$ be a piecewise continuous (finite number of step-type discontinuities are allowed). If for $t \in [0, T]$ there exist lateral derivatives $s'(t^-)$ and $s'(t^+)$, then the Fourier series converges pointwise to

$$\frac{s(t^+) + s(t^-)}{2}$$

Note: If $s(t)$ is derivable, then its Fourier series converges pointwise to $s(t)$. Indeed, in this case we have

$$(s(t^+) + s(t^-))/2 = s(t).$$

Parseval's equality

If $s \in L_2$ is 2π periodic and continuous, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} s^2(t)\,dt = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(formally it can be obtained by squaring the formula and integrating using the orthogonality).

Problem 2.8a: Use the Fourier series of x^2 on $[-\pi, \pi]$ to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

1. Fourier series. The function is even, therefore $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{x^3}{6\pi}\bigg|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) \, dx = [\text{two times by parts}] = \frac{4(-1)^n}{n^2}$$

Thus

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}$$

2. If $x = 0$ we have

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

3. If $x = \pi$ we have $\cos(\pi n) = (-1)^n$ and then

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

4. Let's apply the Parceval's equality:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\, dx = \frac{2\pi^4}{5} = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty} \frac{1}{n^4}$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k+1)^2} - \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{(2k)^2} \Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$