AMPLIACIÓN DE MATEMÁTICAS

Fourier Series and Transforms.

1. Find the period of the functions $\sin(ax)$, $\cos(ax)$ and e^{iax}.

2. Verify orthogonality of the following families of functions on the respective intervals:
   a) $\{\cos(nx), \sin(nx)\}_{n\geq 0}$ on $[-\pi, \pi]$ b) $\{e^{inx}\}_{n\in\mathbb{Z}}$ on $[-\pi, \pi]$
   c) $\{\cos(nx)\}_{n\geq 0}$ and $\{\sin(nx)\}_{n\geq 1}$ on $[0, \pi]$.

3. * Let f be a periodic function of period T and continuous with the possible exception of a finite set of points in $[0, T]$. Show that:
$$\int_{-T/2}^{T/2} f(t)dt = \int_0^T f(t)dt = \int_\alpha^{\alpha+T} f(t)dt$$
   for every $\alpha \in \mathbb{R}$.

4. Let $f \in C[-\pi, \pi]$ be a 2π-periodic and differentiable function. Show that if f is even (that is $f(-x) = f(x)$), then we can write:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$
   and $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, if f is odd (that is $f(-x) = -f(x)$).

5. Find the Fourier series on the interval $[-\pi, \pi]$, of the following functions:
   a) $f(x) = |x|$ b) $f(x) = \cos^3(x)$ c) $f(x) = e^x$ d) $f(x) = |\sin(x)|$ e) $f(x) = \sin^5(x)$.

6. * Let $\{x_n\}$ be a numeric sequence convergent to a number x. Let
$$\tau_k = \frac{x_1 + x_2 + \ldots.. + x_k}{k},\ k \in \mathbb{N},$$
   be the sequence of *Césaro means* of $\{x_n\}$. Show that the sequence $\{\tau_k\}$ converges to x.
   (**Fejer's Theorem** asserts that the sequence of Césaro means of a Fourier series of a continuous, 2π-periodic function uniformly converges to the function).

7. For each of the following functions and intervals: a) Draw the graph, b) Justify the existence of the Fourier series, c) Compute the Fourier coefficients.
   1) $f(x) = x^2,\ x \in [-\pi, \pi]$. 2) $f(x) = |x|,\ x \in (-\pi, \pi)$. 3) $f(x) = |\sin(x)|,\ x \in (-\pi, \pi)$.
   4) $f(x) = x,\ x \in (-\pi, \pi)$. 5) $f(x) = x,\ x \in (0, 2\pi)$. 6) $f(x) = x^2,\ x \in (0, 2\pi)$.
   7) $f(x) = x(\pi - x),\ x \in [0, \pi]$.

8. Using the results of the previous exercise find:
   from 1) the value of
$$\sum_{n=1}^{\infty} \frac{1}{n^2},\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2},\quad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},\quad \sum_{n=1}^{\infty} \frac{1}{n^4},\quad \sum_{n=1}^{\infty} \frac{1}{(2n+1)^4};$$
   from 2), the value of
$$\sum_{n=1}^{\infty} \frac{1}{n^2},\qquad \sum_{n=1}^{\infty} \frac{1}{n^4};$$
   from (7), the value of
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}.$$

9. Let $f(x) = (x-2)^2, x \in [0,4]$ and $g(x) = |x|^3, x \in [-3,3]$. Find the series of these functions in terms of sines and cosines.

10. Find expressions in terms of sines and cosines (Fourier series) of the following functions:
a) $f(t) = \begin{cases} -1 & \text{if} \quad -T/2 < t < 0 \\ 1 & \text{if} \quad 0 < t < T/2 \end{cases}$, $\quad f \quad T$-periodic.

b) $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x \leq 1/2 \\ 0 & \text{if} \quad 1/2 < x \leq 1 \end{cases}$, $\quad f \quad$ 1-periodic.

c) $f(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 1 \\ 2-x & \text{if} \quad 1 < x \leq 2 \end{cases}$, $\quad f \quad$ 2 -periodic.

11. * Let f be a $2\pi-$periodic function such that its Fourier series converges pointwise to $f(x)$ for every $x \in [-\pi, \pi]$. If the sequences of Fourier coefficients $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ verify that $\sum_{n=0}^{\infty} |a_n| < \infty$ and $\sum_{n=1}^{\infty} |b_n| < \infty$, show that the Fourier series uniformly converges to f on $[-\pi, \pi]$.

12. * Let $f : \mathbb{R} \to \mathbb{R}$ be an even function such that $\int_{-\infty}^{\infty} |f(t)| dt < \infty$. Show that the Fourier transform of the function f is *real* (that is, $F[f](\lambda) \in \mathbb{R}$ for all $\lambda \in \mathbb{R}$). If f is odd, verify that $F[f](\lambda)$ is pure imaginary (that is $\text{Re}\{F[f](\lambda)\} = 0$).

13. Compute the Fourier transform of the following functions:
a) $\chi_{[-\delta,\delta]}(x) = \begin{cases} 1 & \text{if} \quad x \in [-\delta, \delta] \\ 0 & \text{in other cases} \end{cases}$, $\qquad$ b) $f(x) = \cos(\alpha x)\chi_{[-\pi,\pi]}(x)$
c) $f(t) = \begin{cases} k & \text{if} \quad -T \leq t < 0 \\ -k & \text{if} \quad 0 \leq t < T \\ 0 & \text{if} \quad t \notin [-T, T] \end{cases}$ $\qquad$ d) $f(x) = \begin{cases} x+\pi & \text{if} \quad -\pi \leq x \leq 0 \\ \pi - x & \text{if} \quad 0 \leq x \leq \pi \\ 0 & \text{in other case} \end{cases}$

14. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, with $f_n(x) = \cos(2\pi\alpha x)\chi_{[-\frac{n}{\alpha},\frac{n}{\alpha}]}(x)$. Draw the graph of $F[f_n]$ and then compute the pointwise limit of the sequence of functions.

15. Let $h(t) = \begin{cases} Ae^{-\alpha t} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$ where A and α are positive parameters.
Show that $F[h](\lambda) = \frac{A}{\alpha + i\lambda}$ (Butterworth filter). **(*)** Design an **RC** circuit such that its *transference function* is precisely $\frac{3}{4+i\lambda}$ (The second part of the exercise can be done after having studied the next topic, namely *Ordinary Differential Equations*).

16. For each case, show that the functions f and g are the same, even though they are given with different expressions:
a) $f(x) = \sin(x)\chi_{[-\pi,\pi]}(x) \quad$ and $\quad f(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{\sin(s\pi)}{1-s^2} \sin(sx)\, ds$
b) $g(x) = \sin(x)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(x) \quad$ and $\quad g(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{s\cos(\frac{s\pi}{2})}{1-s^2} \sin(sx)\, ds.$

17. * Let $f : \mathbb{R} \to \mathbb{R}$ be a *signal* that is filtered by a *low pass filter* $\chi_{[-\delta,\delta]}$. Compute the frequency component f_δ of f bounded to the strip $[-\delta, \delta]$. (**Hint:** $f_\delta(t) = (f * g)(t)$ where g is the filter over the time domain).

18. * Let $f(t) = e^{-t}\chi_{[0,\infty)}(t)$.
a) Show that $f(at) * f(bt) = \frac{f(at)-f(bt)}{b-a}$, for $a, b \in (0, \infty)$.
b) Conclude that $f(at) * f(at) = tf(at)$.