Advanced Mathematics

Convergence of sequences and series of functions

1. Consider the following sequences of functions on the corresponding intervals:

$$f_n(x) = x^n \quad \text{for} \quad x \in [0,1]; \qquad h_n(x) = (\cos \pi x)^{2n} \quad \text{for } x \in \mathbb{R}$$

   a) Draw the functions for $n = 1, 2, 3$.

   b) Study the pointwise and uniform convergence of each sequence of functions.

2. Study the pointwise and uniform convergence in $x \in [0,1]$ of the sequences of functions:

$$f_n(x) = \frac{x}{1+nx} \quad \text{and} \quad g_n(x) = \frac{1}{1+nx}.$$

3. Study the pointwise and uniform convergence of the following sequences of functions:

   a) $f_n(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{1}{n} \\ \frac{-x}{n-1} + \frac{1}{n-1} & \text{for } \frac{1}{n} \le x \le 1 \end{cases}$

   b) $f_n(x) = \dfrac{1-x^n}{1+x^n}$ for $1 \le x < \infty$

   c) $f_n(x) = x - x^n$ for $x \in [0,1]$

   d) $f_n(x) = (1-x)^n$ for $0 \le x \le 1$.

4. a) Let $f_n(x) = xe^{-nx}$, $x \ge 0$. Show that this sequence converges uniformly on $[0,\infty)$.
   b) Let $f_n(x) = \dfrac{\sin(nx)}{1+nx}$, $x \ge 0$. Show that for every $a > 0$ the sequence converges uniformly on $[a,\infty)$, but it does not on $[0,\infty)$.
   c) Let $f_n(x) = \dfrac{nx}{1+nx}$, $x \ge 0$. Show that for all $a > 0$ the converges uniformly on $[a,\infty)$, but it does not on $[0,a]$.

5. Show that the sequence $\dfrac{x^n}{1+x^n}$ does not converge uniformly on $[0,2]$.

6. Study the pointwise and uniform convergence of the sequence $f_n(x) = n^2 x e^{-nx^2}$ on the interval $[0,1]$.

7. Find $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{ne^x}{n+x} dx$.

8. Study the pointwise and uniform convergence of the following series of functions:

   a) $\displaystyle\sum_{n=0}^{\infty} x^n, \quad x \in [0,1]$ $\qquad$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2}, \quad x \in \mathbb{R}$ $\qquad$ c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x^2}{x^2+1}\right)^n, \quad x \in \mathbb{R}$

9. Rewrite the following integrals in terms of series of functions:

$$\int_1^a \frac{\sin t}{t} \mathrm{d}t \quad \text{and} \quad \int_1^a \frac{e^{-x^2}}{x} \mathrm{d}x$$