Atemoya:
Quantitative Finance Framework

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Contents

1 Pricing

Pricing tools enable the investor to construct optimal portfolios that systematically adapt to market conditions. Rather than assuming time-invariant risk, we recognize that markets exhibit distinct volatility regimes and that tail risk becomes paramount during periods of stress.

1.1 Regime-Adaptive Downside Optimization

1.1.1 Mathematical Framework

Let $\mathcal{A} = \{1, \ldots, N\}$ denote the universe of N risky assets with returns $R_t = (R_{t,1}, \ldots, R_{t,N})^\top$ observed over periods $t = 1, \ldots, T$. Let b_t denote the benchmark return (e.g., market index) and r_t^c the risk-free cash return at time t.

1.1.1.1 Portfolio Returns

A portfolio is specified by weight vector $\mathbf{w} = (w_1, \ldots, w_N)^\top$ with $w_i \geq 0$ for all $i \in \mathcal{A}$ and cash weight $w_c \geq 0$, satisfying the full-investment constraint:

$$\sum_{i=1}^{N} w_i + w_c = 1$$

The portfolio return in scenario t is:

$$p_t = \sum_{i=1}^{N} w_i R_{t,i} + w_c r_t^c = \mathbf{w}^\top \mathbf{R}_t + w_c r_t^c$$

The active return (excess over benchmark) is:

$$a_t = p_t - b_t$$

1.1.2 Regime Detection

1.1.2.1 Realized Volatility

Given a return series $\{r_t\}_{t=1}^{T}$ and rolling window size h (typically 20 trading days), the annualized realized volatility at time T is:

$$\sigma_{\text{realized}}(T) = \sqrt{252} \cdot \sqrt{\frac{1}{h} \sum_{t=T-h+1}^{T} (r_t - \bar{r})^2}$$

where $\bar{r} = \frac{1}{h} \sum_{t=T-h+1}^{T} r_t$ and 252 is the approximate number of trading days per year.

1.1.2.2 Historical Volatility Distribution

To establish regime thresholds, we compute rolling volatilities over a lookback period of L years (typically 3–5 years, i.e., $L \times 252$ trading days):

$$\mathcal{V} = \{\sigma_{\text{realized}}(t) : t = h, h+1, \ldots, L \times 252\}$$

Let $Q_p(\mathcal{V})$ denote the p-th quantile of the volatility distribution $\mathcal{V}$.

1.1.2.3 Stress Weight Function

Define lower and upper volatility thresholds:

$$\sigma_L = Q_{0.25}(\mathcal{V})$$
$$\sigma_U = Q_{0.75}(\mathcal{V})$$

The stress weight at current volatility σ is:

$$s(\sigma) = \begin{cases} 0 & \text{if } \sigma \le \sigma_L \quad \text{(calm regime)} \\ \frac{\sigma-\sigma_L}{\sigma_U-\sigma_L} & \text{if } \sigma_L < \sigma < \sigma_U \quad \text{(transition)} \\ 1 & \text{if } \sigma \ge \sigma_U \quad \text{(stress regime)} \end{cases}$$

This smooth transition function avoids abrupt regime switches and provides a continuous measure $s \in [0, 1]$ of market stress intensity.

1.1.3 Downside Risk Measures

1.1.3.1 Lower Partial Moment of Order 1

The First Lower Partial Moment (LPM1) measures the expected shortfall below a threshold $\tau < 0$:

$$\text{LPM1}(\mathbf{w}) = \frac{1}{T}\sum_{t=1}^{T} \max(0, \tau - a_t)$$

where a_t is the active return in scenario t. This captures the average magnitude of underperformance relative to the threshold, providing a convex risk measure suitable for optimization.

1.1.3.2 Conditional Value-at-Risk

Conditional Value-at-Risk (CVaR) at confidence level $1-\alpha$ (typically $\alpha = 0.05$ for 95% CVaR) is defined as the expected loss conditional on exceeding the Value-at-Risk (VaR) threshold.

Define the loss at scenario t as:

$$\ell_t = \max(0, -a_t)$$

The VaR at level $1-\alpha$ is the $1-\alpha$ quantile of the loss distribution:

$$\text{VaR}_{1-\alpha} = \inf\{\eta : \mathbb{P}[\ell \le \eta] \ge 1-\alpha\}$$

The CVaR is then:

$$\text{CVaR}_{1-\alpha} = \mathbb{E}[\ell \mid \ell \ge \text{VaR}_{1-\alpha}]$$

We use the dual representation (Rockafellar–Uryasev):

$$\text{CVaR}_{1-\alpha}(\mathbf{w}) = \min_{\eta\in\mathbb{R}} \left\{ \eta + \frac{1}{\alpha T}\sum_{t=1}^{T} \max(0, \ell_t - \eta) \right\}$$

This formulation allows CVaR to be incorporated directly into a convex optimization problem.

1.1.3.3 Portfolio Beta

The portfolio beta relative to the benchmark is the weighted average of individual asset betas:

$$\beta(\mathbf{w}) = \sum_{i=1}^{N} w_i \beta_i$$

where β_i is the beta of asset i, estimated via exponentially-weighted covariance with the benchmark.

1.1.4 Linear Program Formulation

The portfolio optimization problem is formulated as a convex linear program with auxiliary slack variables.

1.1.4.1 Decision Variables

- $\mathbf{w} \in \mathbb{R}_+^N$: asset weights
- $w_c \in \mathbb{R}_+$: cash weight
- $\mathbf{s}_{\text{LPM}} \in \mathbb{R}_+^T$: LPM1 slack variables
- $\eta \in \mathbb{R}$: CVaR threshold
- $\mathbf{u} \in \mathbb{R}_+^T$: CVaR excess loss slack variables
- $\mathbf{z} \in \mathbb{R}_+^N$: turnover slack variables
- $v \in \mathbb{R}_+$: beta deviation slack variable

1.1.4.2 Objective Function

$$\min_{\mathbf{w}, w_c, \mathbf{s}_{\text{LPM}}, \eta, \mathbf{u}, \mathbf{z}, v} \quad \underbrace{\frac{\lambda_{\text{LPM}}}{T} \sum_{t=1}^{T} s_{\text{LPM},t}}_{\text{LPM1 penalty}} + \underbrace{\lambda_{\text{CVaR}} \left(\eta + \frac{1}{(1-\alpha)T} \sum_{t=1}^{T} u_t \right)}_{\text{CVaR penalty}} + \underbrace{\kappa \sum_{i=1}^{N} z_i}_{\text{turnover cost}} + \underbrace{\lambda_\beta \cdot s(\sigma) \cdot v}_{\text{stress beta penalty}}$$

where:

$\lambda_{\text{LPM}} > 0$: LPM1 risk aversion parameter

$\lambda_{\text{CVaR}} > 0$: CVaR risk aversion parameter

$\lambda_\beta > 0$: beta deviation penalty (only active in stress)

$\kappa = c + \gamma$: combined transaction cost and turnover penalty

$s(\sigma) \in [0, 1]$: current stress weight

1.1.4.3 Constraints

Let $\mathbf{R} \in \mathbb{R}^{T \times N}$ be the matrix of asset scenario returns, $\mathbf{b} \in \mathbb{R}^T$ the benchmark returns, $\mathbf{r}^c \in \mathbb{R}^T$ the cash returns, and $\mathbf{w}_{\text{prev}} \in \mathbb{R}^N$ the previous portfolio weights.

(1) Full investment:

$$\sum_{i=1}^{N} w_i + w_c = 1$$

(2) Active returns:

$$\mathbf{a} = \mathbf{R}\mathbf{w} + \mathbf{r}^c w_c - \mathbf{b}$$

(3) LPM1 slack constraints:

$$s_{\text{LPM},t} \geq \tau - a_t, \quad s_{\text{LPM},t} \geq 0, \quad \forall t = 1, \ldots, T$$

(4) CVaR slack constraints:

$$u_t \geq -a_t - \eta, \quad u_t \geq 0, \quad \forall t = 1, \ldots, T$$

(5) Turnover slack constraints (L1 norm):

$$z_i \geq w_i - w_{\text{prev},i}, \quad z_i \geq -(w_i - w_{\text{prev},i}), \quad z_i \geq 0, \quad \forall i = 1, \ldots, N$$

(6) Beta deviation slack constraints (L1 norm):

$$v \geq \beta(\mathbf{w}) - \beta_{\text{target}}, \quad v \geq -(\beta(\mathbf{w}) - \beta_{\text{target}}), \quad v \geq 0$$

(7) Non-negativity:

$$\mathbf{w} \geq \mathbf{0}, \quad w_c \geq 0$$

1.1.4.4 Problem Structure

The resulting optimization problem is a convex linear program of the form:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{x} = (\mathbf{w}, w_c, \mathbf{s}_{\text{LPM}}, \eta, \mathbf{u}, \mathbf{z}, v)^\top$ is the concatenated decision vector. This can be solved efficiently using interior-point methods (e.g., CLARABEL, SCS, OSQP solvers).

1.1.5 Regime Adaptation Mechanism

1.1.5.1 Stress-Conditional Beta Control

The key innovation is that the beta penalty term is modulated by the stress weight $s(\sigma)$:

$$\text{Beta Penalty} = \lambda_\beta \cdot s(\sigma) \cdot |\beta(\mathbf{w}) - \beta_{\text{target}}|$$

This design has three regimes:

1. **Calm regime** ($s \approx 0$): Beta penalty is negligible. The optimizer focuses purely on downside risk (LPM1, CVaR) and turnover minimization. Market exposure is unconstrained.
2. **Transition regime** ($0 < s < 1$): Beta penalty gradually increases. The portfolio begins defensive positioning while maintaining some market exposure.
3. **Stress regime** ($s \approx 1$): Beta penalty is fully active. The portfolio is strongly penalized for deviating from the target beta (typically $\beta_{\text{target}} \in [0.5, 0.7]$), forcing defensive positioning with reduced market sensitivity.

1.1.5.2 Economic Rationale

During calm markets, investors can tolerate higher market beta to capture upside. During stress, systematic market exposure becomes the dominant risk factor, overwhelming idiosyncratic considerations. By dynamically adjusting beta exposure based on volatility regime, the strategy:

- Maintains participation during bull markets
- Reduces drawdowns during bear markets
- Avoids whipsaw from binary regime classification
- Respects transaction costs through turnover penalties

1.1.6 Solution and Rebalancing

1.1.6.1 Optimization Frequency

The LP is solved at each rebalancing period (typically weekly or monthly). The solution provides:

- Optimal weights $\mathbf{w}^*$ and cash w_c^*
- Realized downside risk: LPM1*, CVaR*
- Turnover incurred: $\|\mathbf{w}^* - \mathbf{w}_{\text{prev}}\|_1$
- Portfolio beta: $\beta(\mathbf{w}^*)$

1.1.6.2 Rebalancing Decision Rule

To avoid excessive trading, rebalancing is triggered only when:

$$\|\mathbf{w}_{\text{current}} - \mathbf{w}^*\|_1 > \epsilon$$

where ϵ is a drift tolerance threshold (typically 0.05–0.10).

1.1.7 Performance Metrics

1.1.7.1 Ex-Post Risk Evaluation

After observing realized returns $\{a_t\}$, we evaluate:

- **LPM1**: $\text{LPM1} = \frac{1}{T}\sum_{t=1}^{T} \max(0, \tau - a_t)$
- **CVaR**: $\text{CVaR}_{95} = \eta^* + \frac{1}{0.05 \cdot T}\sum_{t=1}^{T} \max(0, \ell_t - \eta^*)$
- **Realized volatility**: $\sigma_p = \sqrt{\text{Var}(a_t)}$
- **Turnover**: $\sum_{i=1}^{N} |w_{i,t} - w_{i,t-1}|$

1.1.7.2 Sharpe-Type Ratios

While the objective minimizes downside risk rather than variance, we can still compute:

- **Sharpe ratio**: $\frac{\mathbb{E}[a_t]}{\sigma_p}$
- **Sortino ratio**: $\frac{\mathbb{E}[a_t]}{\sqrt{\text{LPM2}}}$ where LPM2 uses squared shortfalls
- **CVaR ratio**: $\frac{\mathbb{E}[a_t]}{\text{CVaR}_{95}}$

1.1.8 Interpretation

The regime-adaptive downside optimization framework provides:

1. **Tail risk protection**: CVaR directly targets the expected severity of worst-case scenarios, unlike variance-based methods that treat upside and downside equally.
2. **Systematic risk management**: Beta control during stress prevents catastrophic drawdowns from market beta exposure, the primary driver of portfolio losses during crises.
3. **Continuous adaptation**: The smooth stress weight $s(\sigma) \in [0, 1]$ avoids the instability of binary regime switching while maintaining responsiveness to changing market conditions.
4. **Transaction cost awareness**: Turnover penalties and drift thresholds prevent excessive rebalancing, ensuring net-of-cost performance.
5. **Convex formulation**: The LP structure guarantees global optimality and computational efficiency, enabling real-time portfolio construction even for large asset universes.

The framework rejects the mean-variance paradigm in favor of downside-focused objectives that better align with investor utility and observed market behavior during stress episodes.

2 Valuation

The tools of valuation empower the investor to judge stocks in a manner that is detached from behavioral biases and therefore from market sentiment. Valuation emphasizes companies' fundamentals and financial health.

By relying on valuation – regardless of an active or passive investing style – we reject the Efficient Market Hypothesis (EMH) in at least its strong and semi-strong forms.

2.1 DCF (Deterministic)

2.1.1 Discounted Cash Flow Foundation

Let us derive a stock's value per share based on discounted free cash flow and regard this value as intrinsic.

Interest, e.g., on investment streams, compounds to form the future value:

$$\text{FutureValue}_n = \sum_{t=0}^{n} \text{InvestedCash}_t \cdot (1 + \text{InterestRate})^{n-t}$$

To calculate the present value, e.g. of a business, we undo the compounding effect. The intrinsic value of a financial asset is the discounted value of its future cash flows:

$$\text{PresentValue} = \sum_{t=1}^{n} \frac{\text{CashFlow}_t}{(1 + \text{DiscountRate})^t}$$

One can go the free-cash-flow-to-equity (FCFE) and/or the free-cash-flow-to-firm (FCFF) route. It will depend on the company's circumstances to which to give modeling precedence. FCFE should lead to a more insightful intrinsic value per share (IVPS), if the debt policy is known and reliable, and FCFF should do so, if the debt structure is uncertain or complex.

2.1.2 Cash Flow Mechanics

2.1.2.1 Free Cash Flow to Equity

$$\text{FCFE} = \text{NI} - (1 - \text{TDR}) \cdot (\text{CAPX} - \text{D} - \Delta\text{WC})$$

FCFE: free cash flow to equity

NI: net income

TDR: target debt ratio

CAPX: capital expenditure

D: depreciation & amortization

$\Delta\text{WC} = \Delta\text{CA} - \Delta\text{CL}$: change in working capital

CA: current assets

CL: current liabilities

2.1.2.2 Free Cash Flow to Firm

$$\text{FCFF} = \text{EBIT} \cdot (1 - \text{CTR}) + \text{D} - \text{CAPX} - \Delta\text{WC}$$

FCFF: free cash flow to firm

EBIT: earnings before interest and taxes

CTR: corporate tax rate

2.1.3 Cost of Capital

2.1.3.1 Levered Beta

$$\beta_{\mathrm{L}} = \beta_{\mathrm{U}} \cdot \left(1 + (1 - \mathrm{CTR}) \cdot \frac{\mathrm{MVB}}{\mathrm{MVE}}\right)$$

β_{L}: levered beta

β_{U}: unlevered beta

MVB: market value of debt

MVE: market value of equity

2.1.3.2 Cost of Equity (CAPM)

$$\mathrm{CE} = \mathrm{RFR} + \beta_{\mathrm{L}} \cdot \mathrm{ERP}$$

CE: cost of equity

RFR: risk-free rate

ERP: equity risk premium

2.1.3.3 Currency-Adjusted Equity Risk Premium

$$\mathrm{ERPT} = (1 + \mathrm{ERPS}) \cdot \left(\frac{1 + \mathrm{IT}}{1 + \mathrm{IS}}\right) - 1$$

ERPT: equity risk premium of target currency

ERPS: equity risk premium of source currency

IT: inflation of target

IS: inflation of source

2.1.3.4 Weighted Average Cost of Capital

$$\mathrm{WACC} = \frac{\mathrm{MVE}}{\mathrm{MVE} + \mathrm{MVB}} \cdot \mathrm{CE} + \frac{\mathrm{MVB}}{\mathrm{MVE} + \mathrm{MVB}} \cdot \mathrm{CB} \cdot (1 - \mathrm{CTR})$$

WACC: weighted average cost of capital

CB: pre-tax cost of debt

2.1.4 Growth Rate Estimation

2.1.4.1 FCFE Growth Rate

$$\begin{aligned} \mathrm{FCFEGR} &= \mathrm{ROE} \cdot \mathrm{FCFERR} \\ &= \left(\frac{\mathrm{NI}}{\mathrm{BVE}}\right) \cdot \left(1 - \frac{\mathrm{DP}}{\mathrm{NI}}\right) \end{aligned}$$

FCFEGR: FCFE growth rate estimate

ROE: return on equity

FCFERR: FCFE retention ratio

BVE: book value of equity

DP: dividends paid

2.1.4.2 FCFF Growth Rate

$$
\begin{aligned}
\text{FCFFGR} &= \text{ROIC} \cdot \text{FCFFRR} \\
&= \left(\frac{\text{NOPAT}}{\text{IC}}\right) \cdot \left(\frac{\text{CAPX} - \text{D} + \Delta\text{WC}}{\text{NOPAT}}\right) \\
&= \left(\frac{\text{EBIT} \cdot (1 - \text{ETR})}{\text{IC}}\right) \cdot \left(\frac{\text{CAPX} - \text{D} + \Delta\text{WC}}{\text{EBIT} \cdot (1 - \text{ETR})}\right)
\end{aligned}
$$

FCFFGR: FCFF growth rate estimate

ROIC: return on invested capital

FCFFRR: FCFF reinvestment rate

NOPAT: net operating profit after taxes

IC: invested capital

ETR: effective tax rate

2.1.5 Present Value Calculation

2.1.5.1 FCFE-Based Valuation

$$
\begin{aligned}
\text{PVE} &= \text{PVFCFE} + \text{PVTVFCFE} \\
&= \left(\sum_{t=1}^{h} \frac{\text{FCFE}_t}{(1+\text{CE})^t}\right) + \text{PVTVFCFE} \\
&= \left(\sum_{t=1}^{h} \frac{\text{FCFE}_0 \cdot (1+\text{FCFEGR})^t}{(1+\text{CE})^t}\right) + \text{PVTVFCFE} \\
&= \text{PVFCFE} + \left(\sum_{t=h+1}^{\infty} \frac{\text{FCFE}_t}{(1+\text{CE})^t}\right) \\
&= \text{PVFCFE} + \left(\sum_{t=h+1}^{\infty} \frac{\text{FCFE}_h \cdot (1+\text{TGR})^{t-h}}{(1+\text{CE})^t}\right) \\
&= \text{PVFCFE} + \left(\sum_{t=h+1}^{\infty} \frac{\text{FCFE}_h}{(1+\text{CE})^h} \cdot \left(\frac{1+\text{TGR}}{1+\text{CE}}\right)^{t-h}\right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_h}{(1+\text{CE})^h} \sum_{k=1}^{\infty} \left(\frac{1+\text{TGR}}{1+\text{CE}}\right)^{k}\right) \\
&\overset{\text{GGM}}{=} \text{PVFCFE} + \left(\frac{\text{FCFE}_h}{(1+\text{CE})^h} \cdot \frac{\frac{1+\text{TGR}}{1+\text{CE}}}{1 - \frac{1+\text{TGR}}{1+\text{CE}}}\right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_h}{(1+\text{CE})^h} \cdot \frac{1+\text{TGR}}{\text{CE} - \text{TGR}}\right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_h \cdot (1+\text{TGR})}{(\text{CE} - \text{TGR})} \cdot \frac{1}{(1+\text{CE})^h}\right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_{h+1}}{(\text{CE} - \text{TGR})} \cdot \frac{1}{(1+\text{CE})^h}\right) \\
&= \text{PVFCFE} + \left(\frac{\text{TVFCFE}}{(1+\text{CE})^h}\right) \\
&= \left(\sum_{t=1}^{h} \frac{\text{FCFE}_0 \cdot (1+\text{FCFEGR})^t}{(1+\text{CE})^t}\right) + \left(\frac{\text{FCFE}_h \cdot (1+\text{TGR})}{(\text{CE} - \text{TGR}) \cdot (1+\text{CE})^h}\right)
\end{aligned}
$$

PVE: present value of equity

PVFCFE: present value of free cash flow to equity

PVTVFCFE: present value of terminal value of FCFE

h: growth forecast horizon (before terminal growth rate into perpetuity)

GGM: Gordon growth model (closed-form solution of infinite geometric series)

TGR: terminal growth rate

TVFCFE: terminal value of FCFE

2.1.5.2 FCFF-Based Valuation

$$
\begin{aligned}
\mathrm{PVF} &= \mathrm{PVFCFF} + \mathrm{PVTVFCFF} \\
&= \left(\sum_{t=1}^{h} \frac{\mathrm{FCFF}_t}{(1+\mathrm{WACC})^t}\right) + \mathrm{PVTVFCFF} \\
&= \left(\sum_{t=1}^{h} \frac{\mathrm{FCFF}_0 \cdot (1+\mathrm{FCFFGR})^t}{(1+\mathrm{WACC})^t}\right) + \mathrm{PVTVFCFF} \\
&= \mathrm{PVFCFF} + \left(\sum_{t=h+1}^{\infty} \frac{\mathrm{FCFF}_t}{(1+\mathrm{WACC})^t}\right) \\
&= \mathrm{PVFCFF} + \left(\sum_{t=h+1}^{\infty} \frac{\mathrm{FCFF}_h \cdot (1+\mathrm{TGR})^{t-h}}{(1+\mathrm{WACC})^t}\right) \\
&= \mathrm{PVFCFF} + \left(\sum_{t=h+1}^{\infty} \frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \cdot \left(\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}}\right)^{t-h}\right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \sum_{k=1}^{\infty} \left(\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}}\right)^{k}\right) \\
&\overset{\mathrm{GGM}}{=} \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \cdot \frac{\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}}}{1-\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}}}\right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \cdot \frac{1+\mathrm{TGR}}{\mathrm{WACC}-\mathrm{TGR}}\right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h \cdot (1+\mathrm{TGR})}{(\mathrm{WACC}-\mathrm{TGR})} \cdot \frac{1}{(1+\mathrm{WACC})^h}\right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_{h+1}}{(\mathrm{WACC}-\mathrm{TGR})} \cdot \frac{1}{(1+\mathrm{WACC})^h}\right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{TVFCFF}}{(1+\mathrm{WACC})^h}\right) \\
&= \left(\sum_{t=1}^{h} \frac{\mathrm{FCFF}_0 \cdot (1+\mathrm{FCFFGR})^t}{(1+\mathrm{WACC})^t}\right) + \left(\frac{\mathrm{FCFF}_h \cdot (1+\mathrm{TGR})}{(\mathrm{WACC}-\mathrm{TGR}) \cdot (1+\mathrm{WACC})^h}\right)
\end{aligned}
$$

PVF: present value of the firm

PVFCFF: present value of free cash flow to firm

PVTVFCFF: present value of terminal value of FCFF

TVFCFF: terminal value of FCFF

2.1.5.3 Reconciliation

$$\mathrm{PVE} \approx \mathrm{PVF} - \mathrm{MVB} + \mathrm{CNOA}$$

CNOA: cash and non-operating assets

2.1.5.4 Intrinsic Value Per Share

$$\mathrm{IVPS} = \frac{\mathrm{PVE}}{\mathrm{SO}}$$

IVPS: intrinsic value per share

SO: shares outstanding

2.1.6 Interpretation

Having derived an IVPS, it may serve as a reference point when judging the market price. Supposing that, ideally, both (rather than just one of) the FCFE and FCFF based IVPS have been modeled well and are therefore reliable indicators, we may interpret the nine resulting constellations as follows.

	FCFE > Price	**FCFE ≈ Price**	**FCFE < Price**
FCFF > Price	**strong buy:** The firm's assets and operations generate more value than what is priced in by the market, and equity holders retain it — low leverage or efficient debt structure.	**buy:** The firm's assets and operations are underpriced, but excess value is absorbed by debt or reinvestment, leaving equity fairly valued.	**caution:** The firm's assets and operations are underpriced, but debt or reinvestment absorbs most cash flows — equity claims more than it economically receives.
FCFF ≈ Price	**buy:** The firm's assets and operations are fairly priced, but equity captures a disproportionately large share — market underprices the equity upside.	**hold:** The present value of free cash flows — to the firm (before payments to debt holders) and to equity (after them) — is consistent with market prices; no mispricing is evident.	**speculative:** The business generates enough pre-financing cash flow to justify its market price, but equity holders retain too little after payments to debt holders.
FCFF < Price	**caution:** The business is overvalued, but equity appears cheap due to temporarily favorable debt terms — value may be unstable under a leveraged structure.	**speculative:** Equity is fairly priced, but depends on cash flows from a business generating less than what its market price would suggest — any decline in operations could undermine equity value.	**avoid:** There isn't sufficient cash flow to the business or the equity for the fundamentals to justify the high market price.

2.2 DCF (Probabilistic)

2.2.1 Motivation

The deterministic DCF model relies on point estimates for key inputs (growth rates, discount rates, tax rates). However, these parameters are inherently uncertain. Probabilistic DCF addresses this by:

1. Treating financial statement line items as random variables
2. Propagating uncertainty through the cash flow projection
3. Generating a distribution of intrinsic values rather than a single point estimate
4. Quantifying valuation confidence via probability metrics

2.2.2 Input Sampling

2.2.2.1 Historical Distributions

The following variables are sampled independently in each simulation iteration $i = 1, \ldots, N$ (typically $N = 5000$):

- Net income: $\mathrm{NI}^{(i)}$
- Earnings before interest and taxes: $\mathrm{EBIT}^{(i)}$
- Capital expenditures: $\mathrm{CAPX}^{(i)}$
- Depreciation & amortization: $\mathrm{D}^{(i)}$
- Current assets: $\mathrm{CA}^{(i)}$
- Current liabilities: $\mathrm{CL}^{(i)}$

For each variable X, we compute empirical mean μ_X and standard deviation σ_X from the historical time series (typically 4–8 quarters).

2.2.2.2 Sampling Distributions

We choose the sampling distribution based on the sign and scale of the variable:

- If $\mu_X > 0$, the variable is sampled from a log-normal distribution to ensure positivity:

$$\log\text{-mean} = \log(\max(\mu_X, 10^{-3})), \quad \log\text{-std} = \frac{\sigma_X}{\mu_X}$$

$$X^{(i)} \sim \mathrm{LogNormal}(\log\text{-mean}, \log\text{-std})$$

- If $\mu_X \leq 0$, the variable falls back to a normal distribution:

$$X^{(i)} \sim \mathcal{N}(\mu_X, \sigma_X^2)$$

No additional prior smoothing is applied to these inputs. Their randomness enters directly into the cash flow projection models.

2.2.2.3 Outlier Squashing

To reduce the impact of extreme outliers, selected variables like NI and EBIT are post-processed using a squashing function:

$$x \mapsto \begin{cases} x, & x \leq \tau \\ \tau + \log(1 + (x - \tau)), & x > \tau \end{cases}$$

where the threshold τ is:

$$\tau = \min\{3 \cdot \max(\text{historical values}), 0.3 \cdot \text{MarketCap}\}$$

This logarithmic damping prevents unrealistic extrapolation while preserving the shape of the distribution.

2.2.3 Growth Rate Sampling

2.2.3.1 Fundamental Growth Identity

Two growth rates are sampled in each simulation:

- FCFE growth rate: g_{FCFE}
- FCFF growth rate: g_{FCFF}

Both follow the fundamental identity:

$$g = \text{return} \times \text{reinvestment rate}$$

For FCFE:

$$\begin{aligned} g_{\text{FCFE}} &= \text{ROE} \times (1 - \text{payout ratio}) \\ &= \frac{\text{NI}}{\text{BVE}} \times \left(1 - \frac{\text{DP}}{\text{NI}}\right) \end{aligned}$$

For FCFF:

$$\begin{aligned} g_{\text{FCFF}} &= \text{ROIC} \times \text{reinvestment rate} \\ &= \frac{\text{NOPAT}}{\text{IC}} \times \frac{\text{CAPX} - \text{D} + \Delta\text{WC}}{\text{NOPAT}} \end{aligned}$$

where $\text{NOPAT} = 0.75 \cdot \text{EBIT}$ (approximating $(1 - \text{CTR}) \cdot \text{EBIT}$ with a fixed 25% tax rate).

2.2.3.2 Bayesian Smoothing with Sector Priors

Each return and reinvestment term is blended as:

$$\text{final} = 0.5 \cdot \text{empirical} + 0.5 \cdot \text{prior}$$

Sector-specific priors are sampled from:

- ROE and ROIC: $\text{Beta}(2, 5)$ scaled to $[0, 0.4]$
- Retention and reinvestment rates: $\text{Beta}(2, 2)$ scaled to $[0, 1]$

This regularization prevents overfitting to recent historical data while incorporating industry-level information.

2.2.3.3 Growth Rate Distribution

The blended product gives a per-year estimate. Growth rates are then sampled from a normal distribution using the historical mean and standard deviation:

$$g^{(i)} \sim \mathcal{N}(\mu_g, \sigma_g^2)$$

and clipped to user-defined bounds:

$$g^{(i)} \in [g_{\min}, g_{\max}]$$

Typical bounds are $g_{\min} = -0.05$ (allowing slight contraction) and $g_{\max} = 0.15$ (capping growth at 15%).

This ensures simulated growth is consistent with both financial fundamentals and prior regularization.

2.2.4 Stochastic Discount Rates

In addition to stochastic growth rates, the discount rate components (risk-free rate, beta, and equity risk premium) may also be sampled to capture uncertainty in the cost of capital.

2.2.4.1 Discount Rate Component Sampling

For each simulation i, the following components are sampled from normal distributions:

- Risk-free rate:
$$\mathrm{RFR}^{(i)} \sim \mathcal{N}(\mathrm{RFR}_{\mathrm{base}}, \sigma^2_{\mathrm{RFR}})$$
with typical volatility $\sigma_{\mathrm{RFR}} = 0.005$ (50 basis points).
- Leveraged beta:
$$\beta_{\mathrm{L}}^{(i)} \sim \mathcal{N}(\beta_{\mathrm{L,base}}, \sigma^2_{\beta})$$
with typical volatility $\sigma_{\beta} = 0.10$.
- Equity risk premium:
$$\mathrm{ERP}^{(i)} \sim \mathcal{N}(\mathrm{ERP}_{\mathrm{base}}, \sigma^2_{\mathrm{ERP}})$$
with typical volatility $\sigma_{\mathrm{ERP}} = 0.01$ (100 basis points).

2.2.4.2 Cost of Capital Recomputation

Using the sampled components, the cost of equity for simulation i is:

$$\mathrm{CE}^{(i)} = \mathrm{RFR}^{(i)} + \beta_{\mathrm{L}}^{(i)} \cdot \mathrm{ERP}^{(i)}$$

Similarly, the weighted average cost of capital is recomputed:

$$\mathrm{WACC}^{(i)} = \frac{\mathrm{MVE}}{\mathrm{MVE} + \mathrm{MVB}} \cdot \mathrm{CE}^{(i)} + \frac{\mathrm{MVB}}{\mathrm{MVE} + \mathrm{MVB}} \cdot \mathrm{CB} \cdot (1 - \mathrm{CTR})$$

This stochastic treatment of discount rates significantly increases the dispersion of the intrinsic value distribution, providing a more realistic quantification of valuation uncertainty. The feature is controlled by the `use_stochastic_discount_rates` configuration flag.

2.2.5 Cash Flow Projection

2.2.5.1 Explicit Forecast Period

For each simulation i, cash flows are projected over h years (typically $h = 5$) using the sampled inputs and growth rate.

2.2.5.2 Time-Varying Growth Rates

Rather than applying a constant growth rate throughout the projection period, the model implements exponential mean reversion toward the terminal growth rate. This prevents unrealistic perpetual high-growth assumptions.

For each simulation i and forecast year $t \in \{1, \ldots, h\}$, the growth rate decays according to:

$$g_t^{(i)} = g_{\mathrm{terminal}} + \left(g_0^{(i)} - g_{\mathrm{terminal}}\right) \cdot e^{-\lambda t}$$

where:

- $g_0^{(i)}$ is the initial sampled growth rate (FCFE or FCFF growth from the fundamental identity)
- g_{terminal} is the long-run terminal growth rate (typically 2–3%)
- $\lambda \in [0, 1]$ is the mean reversion speed parameter (typical value: 0.3)
- t is the year index within the projection period

Interpretation of λ:

- $\lambda = 0$: No reversion, constant growth $g_t^{(i)} = g_0^{(i)}$ for all t
- $\lambda = 0.3$: Moderate reversion (recommended), growth decays smoothly over 5–7 years

- $\lambda = 1.0$: Fast reversion, growth approaches terminal rate within 3–4 years

This time-varying formulation yields more conservative valuations than constant-growth models, as high initial growth rates gradually moderate to sustainable long-term levels. The feature is controlled by the `use_time_varying_growth` configuration flag.

2.2.5.3 Cash Flow Computation

Using the time-varying growth rates $g_t^{(i)}$, the projected cash flows are:

For FCFE:

$$\text{FCFE}_t^{(i)} = \text{NI}_t^{(i)} + \text{D}_t^{(i)} - \text{CAPX}_t^{(i)} - \Delta\text{WC}_t^{(i)} + \text{NetBorrowing}_t^{(i)}$$

where $\text{NetBorrowing}_t^{(i)} = \text{TDR} \cdot \text{Reinvestment}_t^{(i)}$

For FCFF:

$$\text{FCFF}_t^{(i)} = \text{NOPAT}_t^{(i)} + \text{D}_t^{(i)} - \text{CAPX}_t^{(i)} - \Delta\text{WC}_t^{(i)}$$

with $\text{NOPAT}_t^{(i)} = (1 - \text{CTR}) \cdot \text{EBIT}_t^{(i)}$

2.2.5.4 Terminal Value

The terminal value uses a Gordon growth model:

$$\text{TV}^{(i)} = \frac{\text{CashFlow}_h^{(i)} \cdot (1 + g_{\text{terminal}})}{r - g_{\text{terminal}}}$$

where:

- $r = \text{CE}$ for FCFE or $r = \text{WACC}$ for FCFF
- g_{terminal} is the terminal growth rate (typically 2–3%)

To avoid division by zero or unstable extrapolation, the denominator is lower-bounded:

$$r - g_{\text{terminal}} \geq 0.01$$

The terminal value is then discounted to present:

$$\text{PV}_{\text{terminal}}^{(i)} = \frac{\text{TV}^{(i)}}{(1 + r)^h}$$

2.2.5.5 Present Value

The present value of cash flows is:

$$\text{PV}_{\text{years}}^{(i)} = \sum_{t=1}^{h} \frac{\text{CashFlow}_t^{(i)}}{(1 + r)^t}$$

Final present value of the firm:

$$\text{PV}^{(i)} = \text{PV}_{\text{years}}^{(i)} + \text{PV}_{\text{terminal}}^{(i)}$$

For FCFF, market value of debt per share is subtracted to get the per-share equity value:

$$\text{IVPS}_{\text{FCFF}}^{(i)} = \frac{\text{PV}_{\text{FCFF}}^{(i)} - \text{MVB} + \text{CNOA}}{\text{SO}}$$

2.2.6 Statistical Analysis

2.2.6.1 Simulation Matrix

After N simulations, we obtain an intrinsic value distribution for each asset and method (FCFE, FCFF):

$$\{V^{(1)}, V^{(2)}, \ldots, V^{(N)}\}$$

2.2.6.2 Kernel Density Estimation

To estimate the probability density of intrinsic value, we apply a Gaussian kernel density estimator (KDE):

$$\hat{f}(v) = \frac{1}{Nh} \sum_{i=1}^{N} K\left(\frac{v - V^{(i)}}{h}\right)$$

where $K(\cdot)$ is the Gaussian kernel:

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$$

and h is the bandwidth selected using Scott's rule:

$$h = N^{-1/5} \cdot \hat{\sigma}$$

where $\hat{\sigma}$ is the sample standard deviation of $\{V^{(i)}\}$.

2.2.6.3 Value Surplus View

Each simulation sample is transformed into a percentage surplus over market price:

$$\text{Surplus}^{(i)} = \frac{V^{(i)} - P}{P}$$

We then apply KDE on the transformed values to estimate the distribution of relative mispricing. This highlights the extent to which the simulated valuations exceed or fall short of the current price.

From the KDE and simulation samples, we extract:

- $\mathbb{P}[\text{Surplus} > 0]$: probability the asset is undervalued
- $\mathbb{P}[\text{Surplus} < 0]$: probability it is overvalued

These quantities are calculated empirically via:

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{I}[V^{(i)} > P] \quad \text{and} \quad \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}[V^{(i)} < P]$$

2.2.7 Multi-Asset Portfolio Frontiers

To explore portfolio-level valuation outcomes, we simulate many random portfolios over the set of assets.

2.2.7.1 Simulation Matrix

Let n be the number of assets and $S_i^{(j)}$ the j-th simulation of value surplus (as a percentage of market price) for asset i. The simulation matrix is:

$$\mathbf{S} \in \mathbb{R}^{N \times n}, \quad \text{with } S_{ji} = \frac{V_i^{(j)} - P_i}{P_i}$$

2.2.7.2 Random Portfolio Generation

We generate M random portfolio weight vectors $\mathbf{w}^{(k)} \in \mathbb{R}^n$ (typically $M = 5000$) using a Dirichlet distribution over the simplex:

$$\mathbf{w}^{(k)} \sim \text{Dir}(\alpha \mathbf{1}), \quad \alpha = 1$$

This ensures:

$$\sum_{i=1}^{n} w_i^{(k)} = 1, \quad w_i^{(k)} \geq 0 \quad \forall i$$

and provides uniform sampling over the weight simplex.

2.2.7.3 Portfolio Return Distribution

For each portfolio k and simulation j, the portfolio surplus is:

$$R_j^{(k)} = \sum_{i=1}^{n} w_i^{(k)} S_{ji} = (\mathbf{w}^{(k)})^\top \mathbf{S}_j$$

where $\mathbf{S}_j$ is the j-th row of the simulation matrix.

2.2.7.4 Portfolio Statistics

For each portfolio k:

- **Expected return** (mean surplus):

$$\mu^{(k)} = \frac{1}{N} \sum_{j=1}^{N} R_j^{(k)}$$

- **Standard deviation** (valuation risk):

$$\sigma^{(k)} = \sqrt{\frac{1}{N-1} \sum_{j=1}^{N} (R_j^{(k)} - \mu^{(k)})^2}$$

  Alternatively, using the covariance matrix:

$$\sigma^{(k)} = \sqrt{(\mathbf{w}^{(k)})^\top \mathbf{\Sigma} \mathbf{w}^{(k)}}$$

  where $\mathbf{\Sigma}$ is the empirical covariance of $\mathbf{S}$.

- **Probability of loss**:

$$p_{\text{loss}}^{(k)} = \frac{1}{N} \sum_{j=1}^{N} \mathbb{I}[R_j^{(k)} < 0]$$

- **Downside deviation** (Sortino denominator):

$$\sigma_{\text{down}}^{(k)} = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \min(0, R_j^{(k)})^2}$$

- **Conditional Value-at-Risk** (CVaR at 95%): Let $\alpha = 0.05$. Sort the portfolio returns: $R_{(1)}^{(k)} \leq R_{(2)}^{(k)} \leq \cdots \leq R_{(N)}^{(k)}$.

  The VaR is:

$$\text{VaR}_{95}^{(k)} = R_{(\lfloor \alpha N \rfloor)}^{(k)}$$

  The CVaR is:

$$\text{CVaR}_{95}^{(k)} = \frac{1}{\lfloor \alpha N \rfloor} \sum_{j=1}^{\lfloor \alpha N \rfloor} R_{(j)}^{(k)}$$

- **Maximum drawdown**: Compute cumulative returns over the simulation series and measure the largest peak-to-trough decline.

2.2.7.5 Efficient Frontier Plots

Six distinct frontier views are produced, each optimizing a different risk-return trade-off:

1. **Mean vs Standard Deviation** (μ–σ): Mean-variance analysis. Identifies:
   - Minimum variance portfolio: $\mathbf{w}_{\mathrm{minvar}} = \arg\min_{\mathbf{w}} \mathbf{w}^\top \mathbf{\Sigma} \mathbf{w}$
   - Maximum Sharpe portfolio: $\mathbf{w}_{\mathrm{maxSharpe}} = \arg\max_{\mathbf{w}} \frac{\mu(\mathbf{w})}{\sigma(\mathbf{w})}$
2. **Mean vs Probability of Loss** (μ–p_{loss}): Focuses on minimizing the chance of overvaluation. Identifies the portfolio with minimum p_{loss} for a given expected return.
3. **Mean vs Downside Deviation** (μ–σ_{down}): Sortino-style frontier. Identifies the portfolio with maximum Sortino ratio:
$$\mathrm{Sortino}(\mathbf{w}) = \frac{\mu(\mathbf{w})}{\sigma_{\mathrm{down}}(\mathbf{w})}$$
4. **Mean vs CVaR** (μ–CVaR): Tail risk frontier. Identifies the portfolio minimizing expected tail loss.
5. **Mean vs VaR** (μ–VaR): Similar to CVaR but focuses on the quantile rather than the conditional expectation.
6. **Mean vs Maximum Drawdown** (μ–MDD): Calmar-style frontier. Identifies the portfolio with maximum Calmar ratio:
$$\mathrm{Calmar}(\mathbf{w}) = \frac{\mu(\mathbf{w})}{\mathrm{MDD}(\mathbf{w})}$$

In each plot, the efficient frontier curve and key labeled portfolios (e.g., maximum return, minimum risk, optimal ratio) are annotated. Portfolio composition legends show the top 15 holdings for each optimal portfolio.

2.2.8 Interpretation

The probabilistic DCF framework yields a summary of valuation statistics and qualitative signals:

- **Strong buy**: if both models (FCFE and FCFF) yield undervaluation ($\mathbb{P}[\mathrm{Surplus} > 0] > 0.6$).
- **Buy**: if only one model suggests undervaluation.
- **Avoid**: if both suggest overvaluation ($\mathbb{P}[\mathrm{Surplus} < 0] > 0.6$).
- **Hold**: if both suggest fair valuation.

The probabilistic approach provides several advantages over deterministic DCF:

1. **Uncertainty quantification**: Rather than a single IVPS point estimate, we obtain a full distribution $\hat{f}(v)$, allowing probabilistic statements about valuation.
2. **Confidence calibration**: The probability of undervaluation/overvaluation provides a confidence measure for investment decisions.
3. **Portfolio-level analysis**: By combining simulation matrices across assets, we can construct efficient portfolios based on intrinsic value rather than historical prices.
4. **Regime-invariant correlations**: The low inter-asset correlations (0.0) observed in DCF-based returns reflect fundamental independence, unlike market-based returns which exhibit regime-dependent correlation structures.
5. **Tail risk visibility**: The CVaR and maximum drawdown frontiers explicitly address worst-case scenarios, aligning with investor risk preferences during stress.

The framework enables the construction of valuation-based portfolios that systematically exploit mispricing while managing fundamental uncertainty.