Atemoya

github.com/cb-g/atemoya

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1 Valuation

The tools of valuation empower the investor to judge stocks in a manner that is detached from behavioral biases and therefore from market sentiment. Valuation emphasizes companies' fundamentals and financial health.

By relying on valuation - regardless of an active or passive investing style - we reject the Efficient Market Hypothesis (EMH) in at least its strong and semi-strong forms.

1.1 DCF (deterministic)

Let us derive a stock's value per share based on discounted free cash flow and regard this value as intrinsic:

Interest, e.g., on investment streams, compounds to form the future value.

$$\mathrm{FutureValue}_n = \sum_{t=0}^{n} \mathrm{InvestedCash}_t \cdot (1 + \mathrm{InterestRate})^{n-t}$$

To calculate the present value, e.g. of a business, we undo the compounding effect. The intrinsic value of a financial asset is the discounted value of its future cash flows.

$$\mathrm{PresentValue} = \sum_{t=1}^{n} \frac{\mathrm{CashFlow}_t}{(1 + \mathrm{DiscountRate})^t}$$

One can go the free-cash-flow-to-equity (FCFE) and/or the free-cash-flow-to-firm (FCFF) route. It will depend on the company's circumstances to which to give modeling precedence. FCFE should lead to a more insightful intrinsic value per share (IVPS), if the debt policy is known and reliable, and FCFF should do so, if the debt structure is uncertain or complex.

1.1.1 Sensitivity

In general, the terminal growth rate and the discount rate (not solely but) significantly affect the model outputs.

1.1.2 Mechanics

$$\mathrm{FCFE} = \mathrm{NI} - (1 - \mathrm{TDR}) \cdot (\mathrm{CAPX} - \mathrm{D} - (\mathrm{CA} - \mathrm{CL}))$$

FCFE: free cash flow to equity

NI: net income

TDR: target debt ratio

CAPX: capital expenditure

D: depreciation

CA: current assets

CL: current liabilities

$$\mathrm{FCFF} = \mathrm{EBIT} \cdot (1 - \mathrm{CTR}) + \mathrm{B} - \mathrm{CAPX} - (\mathrm{CA} - \mathrm{CL})$$

FCFF: free cash flow to firm

EBIT: earnings before interest and taxes

CTR: corporate tax rate

B: debt

$$\beta_{\mathrm{L}} = \beta_{\mathrm{U}} \cdot \left(1 + (1 - \mathrm{CTR}) \cdot \frac{\mathrm{MVB}}{\mathrm{MVE}}\right)$$

β_{L}: levered beta

β_{U}: unlevered beta

MVB: market value of debt

MVE: market value of equity

$$\mathrm{CE} = RFR + \beta_{\mathrm{L}} \cdot \mathrm{ERP}$$

CE: cost of equity

RFR: risk-free rate

ERP: equity risk premium

$$\mathrm{ERPT} = (1 + \mathrm{ERPS}) \cdot \left(\frac{1 + \mathrm{IT}}{1 + \mathrm{IS}} \right) - 1$$

ERPT: equity risk premium of target

ERPS: equity risk premium of source

IT: inflation of target

IS: inflation of source

$$\mathrm{WACC} = \frac{\mathrm{MVE}}{\mathrm{MVE} + \mathrm{MVB}} \cdot \mathrm{CE} + \frac{\mathrm{MVB}}{\mathrm{MVE} + \mathrm{MVB}} \cdot \mathrm{CB} \cdot (1 - \mathrm{CTR})$$

WACC: weighted average cost of capital

CB: pre-tax cost of debt

$$\begin{aligned} \mathrm{FCFEGR} &= \mathrm{ROE} \cdot \mathrm{FCFERR} \\ &= \left(\frac{\mathrm{NI}}{\mathrm{BVE}} \right) \cdot \left(1 - \frac{\mathrm{DP}}{\mathrm{NI}} \right) \end{aligned}$$

FCFEGR: FCFE growth rate estimate

ROE: return on equity

FCFERR: FCFE retention ratio

BVE: book value of equity

DP: dividends paid

$$\begin{aligned} \mathrm{FCFFGR} &= \mathrm{ROIC} \cdot \mathrm{FCFFRR} \\ &= \left(\frac{\mathrm{NOPAT}}{\mathrm{IC}} \right) \cdot \left(\frac{\mathrm{CAPX} - \mathrm{D} + (\mathrm{CA} - \mathrm{CL})}{\mathrm{NOPAT}} \right) \\ &= \left(\frac{\mathrm{EBIT} \cdot (1 - \mathrm{ETR})}{\mathrm{IC}} \right) \cdot \left(\frac{\mathrm{CAPX} - \mathrm{D} + (\mathrm{CA} - \mathrm{CL})}{\mathrm{EBIT} \cdot (1 - \mathrm{ETR})} \right) \\ &= \left(\frac{\mathrm{EBIT} \cdot \left(1 - \frac{\mathrm{ITE}}{\mathrm{EBIT}}\right)}{\mathrm{IC}} \right) \cdot \left(\frac{\mathrm{CAPX} - \mathrm{D} + (\mathrm{CA} - \mathrm{CL})}{\mathrm{EBIT} \cdot \left(1 - \frac{\mathrm{ITE}}{\mathrm{EBIT}}\right)} \right) \end{aligned}$$

FCFFGR: FCFF growth rate estimate

ROIC: return on invested capital

FCFFRR: FCFF reinvestment rate

NOPAT: net operating profit after taxes

IC: invested capital

ETR: effective tax rate

ITE: income tax expense

$$
\begin{aligned}
\text{PVE} &= \text{PVFCFE} + \text{PVTVFCFE} \\
&= \left(\sum_{t=1}^{h} \frac{\text{FCFE}_t}{(1+\text{CE})^t} \right) + \text{PVTVFCFE} \\
&= \left(\sum_{t=1}^{h} \frac{\text{FCFE}_0 \cdot (1+\text{FCFEGR})^t}{(1+\text{CE})^t} \right) + \text{PVTVFCFE} \\
&= \text{PVFCFE} + \left(\sum_{t=h+1}^{\infty} \frac{\text{FCFE}_t}{(1+\text{CE})^t} \right) \\
&= \text{PVFCFE} + \left(\sum_{t=h+1}^{\infty} \frac{\text{FCFE}_h \cdot (1+\text{TGR})^{t-h}}{(1+\text{CE})^t} \right) \\
&= \text{PVFCFE} + \left(\sum_{t=h+1}^{\infty} \frac{\text{FCFE}_h}{(1+\text{CE})^h} \cdot \left(\frac{1+\text{TGR}}{1+\text{CE}} \right)^{t-h} \right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_h}{(1+\text{CE})^h} \sum_{k=1}^{\infty} \left(\frac{1+\text{TGR}}{1+\text{CE}} \right)^{k} \right) \\
&\overset{\text{GGM}}{=} \text{PVFCFE} + \left(\frac{\text{FCFE}_h}{(1+\text{CE})^h} \cdot \frac{\frac{1+\text{TGR}}{1+\text{CE}}}{1-\frac{1+\text{TGR}}{1+\text{CE}}} \right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_h}{(1+\text{CE})^h} \cdot \frac{1+\text{TGR}}{\text{CE}-\text{TGR}} \right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_h \cdot (1+\text{TGR})}{(\text{CE}-\text{TGR})} \cdot \frac{1}{(1+\text{CE})^h} \right) \\
&= \text{PVFCFE} + \left(\frac{\text{FCFE}_{h+1}}{(\text{CE}-\text{TGR})} \cdot \frac{1}{(1+\text{CE})^h} \right) \\
&= \text{PVFCFE} + \left(\frac{\text{TVFCFE}}{(1+\text{CE})^h} \right) \\
&= \left(\sum_{t=1}^{h} \frac{\text{FCFE}_0 \cdot (1+\text{FCFEGR})^t}{(1+\text{CE})^t} \right) + \left(\frac{\text{FCFE}_h \cdot (1+\text{TGR})}{(\text{CE}-\text{TGR}) \cdot (1+\text{CE})^h} \right)
\end{aligned}
$$

PVE: present value of equity

PVFCFE: present value of free cash flow to equity

PVTVFCFE: present value of terminal value of free cash flow to firm

h: growth forecast horizon (before terminal growth rate into perpetuity)

GGM: Gordon growth model (closed-form solution of infinite geometric series)

TGR: terminal growth rate

TVFCFE: terminal value of free cash flow to firm

$$
\begin{aligned}
\mathrm{PVF} &= \mathrm{PVFCFF} + \mathrm{PVTVFCFF} \\
&= \left(\sum_{t=1}^{h} \frac{\mathrm{FCFF}_t}{(1+\mathrm{WACC})^t} \right) + \mathrm{PVTVFCFF} \\
&= \left(\sum_{t=1}^{h} \frac{\mathrm{FCFF}_0 \cdot (1+\mathrm{FCFFGR})^t}{(1+\mathrm{WACC})^t} \right) + \mathrm{PVTVFCFF} \\
&= \mathrm{PVFCFF} + \left(\sum_{t=h+1}^{\infty} \frac{\mathrm{FCFF}_t}{(1+\mathrm{WACC})^t} \right) \\
&= \mathrm{PVFCFF} + \left(\sum_{t=h+1}^{\infty} \frac{\mathrm{FCFF}_h \cdot (1+\mathrm{TGR})^{t-h}}{(1+\mathrm{WACC})^t} \right) \\
&= \mathrm{PVFCFF} + \left(\sum_{t=h+1}^{\infty} \frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \cdot \left(\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}} \right)^{t-h} \right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \sum_{k=1}^{\infty} \left(\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}} \right)^{k} \right) \\
&\overset{\mathrm{GGM}}{=} \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \cdot \frac{\frac{1+\mathrm{TGR}}{1+\mathrm{WACC}}}{1 - \frac{1+\mathrm{TGR}}{1+\mathrm{WACC}}} \right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h}{(1+\mathrm{WACC})^h} \cdot \frac{1+\mathrm{TGR}}{\mathrm{WACC}-\mathrm{TGR}} \right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_h \cdot (1+\mathrm{TGR})}{(\mathrm{WACC}-\mathrm{TGR})} \cdot \frac{1}{(1+\mathrm{WACC})^h} \right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{FCFF}_{h+1}}{(\mathrm{WACC}-\mathrm{TGR})} \cdot \frac{1}{(1+\mathrm{WACC})^h} \right) \\
&= \mathrm{PVFCFF} + \left(\frac{\mathrm{TVFCFF}}{(1+\mathrm{WACC})^h} \right) \\
&= \left(\sum_{t=1}^{h} \frac{\mathrm{FCFF}_0 \cdot (1+\mathrm{FCFFGR})^t}{(1+\mathrm{WACC})^t} \right) + \left(\frac{\mathrm{FCFF}_h \cdot (1+\mathrm{TGR})}{(\mathrm{WACC}-\mathrm{TGR}) \cdot (1+\mathrm{WACC})^h} \right)
\end{aligned}
$$

PVF: present value of the firm

PVFCFF: present value of free cash flow to firm

PVTVFCFF: present value of terminal value of free cash flow to firm

TVFCFF: terminal value of free cash flow to firm

$$\mathrm{PVE} \approx \mathrm{PVF} - \mathrm{B} + \mathrm{CNOA}$$

CNOA: cash and non-operating assets

$$\mathrm{IVE} = \mathrm{PVE}$$

IVE: intrinsic value of equity

$$\mathrm{IVE} = \mathrm{PVF} - \mathrm{B} + \mathrm{CNOA}$$

$$\mathrm{IVPS} = \frac{\mathrm{IVE}}{\mathrm{SO}}$$

IVPS: intrinsic value per share

SO: shares outstanding

1.1.3 Interpretation

Having derived an IVPS, it may serve as a reference point when judging the market price. Supposing that, ideally, both (rather than just one of) the FCFE and FCFF based IVPS have been modeled well and are therefore reliable indicators, we may interpret the nine resulting constellations as follows.

	FCFE > Price	**FCFE ≈ Price**	**FCFE < Price**
FCFF > Price	**strong buy:** The firm's assets and operations generate more value than what is priced in by the market, and equity holders retain it — low leverage or efficient debt structure.	**buy:** The firm's assets and operations are underpriced, but excess value is absorbed by debt or reinvestment, leaving equity fairly valued.	**caution:** The firm's assets and operations are underpriced, but debt or reinvestment absorbs most cash flows — equity claims more than it economically receives.
FCFF ≈ Price	**buy:** The firm's assets and operations are fairly priced, but equity captures a disproportionately large share — market underprices the equity upside.	**hold:** The present value of free cash flows — to the firm (before payments to debt holders) and to equity (after them) — is consistent with market prices; no mispricing is evident.	**speculative:** The business generates enough pre-financing cash flow to justify its market price, but equity holders retain too little after payments to debt holders.
FCFF < Price	**caution:** The business is overvalued, but equity appears cheap due to temporarily favorable debt terms — value may be unstable under a leveraged structure.	**speculative:** Equity is fairly priced, but depends on cash flows from a business generating less than what its market price would suggest — any decline in operations could undermine equity value.	**avoid:** There isn't sufficient cash flow to the business or the equity for the fundamentals to justify the high market price.

1.2 DCF (probabilistic)

1.2.1 Sensitivity and mechanics

1.2.1.1 Inputs

The following variables are sampled independently in each simulation: net income (NI), EBIT, capital expenditures (CAPX), depreciation (D), current assets (CA), current liabilities (CL).

We get an empirical mean μ and standard deviation σ from their historical time series.

We choose the sampling distribution depending on the sign and scale of the variable:

- If $\mu > 0$, the variable is sampled from a log-normal distribution:

$$\log\text{-mean} = \log(\max(\mu, 10^{-3})), \quad \log\text{-std} = \sigma/\mu$$

$$X^{(i)} \sim \text{LogNormal}(\log\text{-mean}, \log\text{-std})$$

- If $\mu \leq 0$, the variable falls back to a normal distribution:

$$X^{(i)} \sim \mathcal{N}(\mu, \sigma^2)$$

No additional prior smoothing is applied to these inputs. Their randomness enters directly into the cash flow projection models.

To reduce the impact of outliers, selected variables like NI and EBIT are post-processed using a squashing function:

$$x \mapsto \begin{cases} x, & x \leq \tau \\ \tau + \log(1 + (x - \tau)), & x > \tau \end{cases}$$

where τ is the minimum of:

- three times the historical maximum of the variable, and
- thirty percent of market capitalization.

1.2.1.2 Growth rate

Two growth rates are sampled in each simulation:

- FCFE growth rate (g_{FCFE})
- FCFF growth rate (g_{FCFF})

Both follow the identity:

$$g = \text{return} \times \text{reinvestment rate}$$

For FCFE:

- Return on equity: $\text{ROE} = \text{NI}/\text{BVE}$
- Retention ratio: $1 - \text{DP}/\text{NI}$

For FCFF:

- Return on invested capital: $\text{ROIC} = \text{NOPAT}/\text{IC}$, where $\text{NOPAT} = 0.75 \cdot \text{EBIT}$
- Reinvestment rate: $(\text{CAPX} - \text{D} + \Delta\text{WC})/\text{NOPAT}$

Each return and reinvestment term is blended as:

$$\text{final} = 0.5 \cdot \text{empirical} + 0.5 \cdot \text{prior}$$

Priors are sampled from:

- $\text{Beta}(2, 5)$ scaled to $[0, 0.4]$ for ROE and ROIC
- $\text{Beta}(2, 2)$ scaled to $[0, 1]$ for retention and reinvestment rates

The blended product gives a per-year estimate. Growth rates are then sampled from a normal distribution using the historical mean and standard deviation, and clipped to user-defined bounds.

$$g \in [g_{\min},\ g_{\max}]$$

This ensures simulated growth is consistent with both financial fundamentals and prior regularization.

1.2.1.3 Discounting and terminal value

Each simulation projects cash flows over h years using the sampled inputs and growth rate.

For FCFE:

$$\mathrm{FCFE}_t = \mathrm{NI}_t + \mathrm{D}_t - \mathrm{CAPX}_t - \Delta\mathrm{WC}_t + \mathrm{NetBorrowing}_t$$

where $\mathrm{NetBorrowing}_t = \mathrm{TDR} \cdot \mathrm{Reinvestment}_t$

For FCFF:

$$\mathrm{FCFF}_t = \mathrm{NOPAT}_t + \mathrm{D}_t - \mathrm{CAPX}_t - \Delta\mathrm{WC}_t$$

with $\mathrm{NOPAT}_t = (1 - \mathrm{CTR}) \cdot \mathrm{EBIT}_t$

Each cash flow is discounted using the cost of equity:

$$\mathrm{COE} = \mathrm{RFR} + \beta_l \cdot \mathrm{ERP}$$

where β_l is the levered beta:

$$\beta_l = \beta_u \cdot \left(1 + (1 - \mathrm{CTR}) \cdot \frac{\mathrm{MVB}}{\mathrm{MVE}}\right)$$

The present value of cash flows is:

$$\mathrm{PV}_{\text{years}} = \sum_{t=1}^{h} \frac{\mathrm{CashFlow}_t}{(1 + \mathrm{COE})^t}$$

The terminal value is calculated using a Gordon growth model:

$$\mathrm{TV} = \frac{\mathrm{CashFlow}_h \cdot (1 + g_{\text{terminal}})}{\mathrm{COE} - g_{\text{terminal}}}$$

To avoid division by zero or unstable extrapolation, the denominator is lower-bounded:

$$\mathrm{COE} - g_{\text{terminal}} \geq 0.01$$

The terminal value is then discounted to present:

$$\mathrm{PV}_{\text{terminal}} = \frac{\mathrm{TV}}{(1 + \mathrm{COE})^h}$$

Final present value of the firm:

$$\mathrm{PV}_{\text{firm}} = \mathrm{PV}_{\text{years}} + \mathrm{PV}_{\text{terminal}}$$

For FCFF, market value of debt per share is subtracted to get the per-share equity value.

1.2.1.4 Single-asset value-surplus

For each asset and valuation model (FCFE, FCFF), we obtain n simulated intrinsic values $V^{(1)}, \ldots, V^{(n)}$.

To estimate the distribution of intrinsic value, we apply a Gaussian kernel density estimator (KDE):

$$\hat{f}(v) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{v - V^{(i)}}{h}\right)$$

where $K(\cdot)$ is the Gaussian kernel, and h is the bandwidth selected using Scott's rule.

The KDE is used in two distinct views:

- Intrinsic Value View: We estimate the probability density $\hat{f}(v)$ of the intrinsic value V directly from the simulation output. This view is used to visualize how likely different valuations are, independent of the current market price.
- Surplus Percentage View: Each simulation sample is transformed into a percentage surplus over market price:

  $$\text{Surplus}^{(i)} = \frac{V^{(i)} - P}{P}$$

  We then apply KDE on the transformed values to estimate the distribution of relative mispricing. This highlights the extent to which the simulated valuations exceed or fall short of the current price.

From the KDE, we extract:

- $\mathbb{P}[\text{Surplus} > 0]$: probability the asset is undervalued.
- $\mathbb{P}[\text{Surplus} < 0]$: probability it is overvalued.

These quantities are calculated empirically via:

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{I}[V^{(i)} > P] \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}[V^{(i)} < P]$$

1.2.1.5 Multi-asset value-surplus frontier

To explore portfolio-level valuation outcomes, we simulate many random portfolios over the set of assets.

Let n be the number of assets and $S_i^{(j)}$ the j-th simulation of value surplus (as a percentage of market price) for asset i. The simulation matrix is:

$$\mathbf{S} \in \mathbb{R}^{m \times n}, \quad \text{with } S_{ji} = \frac{V_i^{(j)} - P_i}{P_i}$$

We generate N random portfolio weight vectors $\mathbf{w}^{(k)} \in \mathbb{R}^n$ using a Dirichlet distribution over the simplex, i.e., $\sum_i w_i^{(k)} = 1$.

For each portfolio k:

- The expected return (mean surplus) is computed as:

  $$\mu^{(k)} = \mathbb{E}_j[\mathbf{w}^{(k)} \cdot \mathbf{S}^{(j)}]$$

- The standard deviation (valuation risk) is:

  $$\sigma^{(k)} = \sqrt{\mathbf{w}^{(k)T}\, \Sigma\, \mathbf{w}^{(k)}}$$

  where Σ is the empirical covariance of $\mathbf{S}$ across simulations.
- The probability of loss is:

  $$\mathbb{P}^{(k)}[\mathbf{w}^{(k)} \cdot \mathbf{S}^{(j)} < 0]$$

  computed empirically from the simulation matrix.

Two portfolio views are produced:

- Mean vs Standard Deviation (μ–σ): Visualizes the valuation return–risk tradeoff. An analytic minimum-risk portfolio is computed from Σ via:

$$\mathbf{w}_{\text{min-risk}} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

- Mean vs Loss Probability (μ–p_{loss}): Evaluates portfolio robustness under valuation uncertainty. This view emphasizes portfolios that minimize the chance of overvaluation (i.e., surplus < 0).

In both plots, efficient frontiers and key labeled portfolios (e.g., maximum return, minimum risk, minimum loss probability) are annotated and saved as PDF and SVG files.

1.3 Interpretation

The module yields a summary of valuation statistics and qualitative signals:

- strong buy: if both models (FCFE and FCFF) yield undervaluation.
- buy: if only one model suggests undervaluation.
- avoid: if both suggest overvaluation.
- hold: if both suggest fair valuation.

2 Pricing

Seeing merit in pricing tools is supposing that markets are at least occasionally inefficient.

2.1 MPT

2.1.1 Mechanics and Interpretation

$$\mathbb{E}[R_p] = \sum_{i=1}^{n} w_i \cdot \mathbb{E}[R_i] = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \mathbb{E}[R_1] \\ \mathbb{E}[R_2] \\ \vdots \\ \mathbb{E}[R_n] \end{bmatrix} = \mathbf{w}^\top \mathbf{r}$$

$\mathbb{E}[R_p]$: expected return of the portfolio

w_i: weight of asset i in the portfolio

$\mathbb{E}[R_i]$: expected return of asset i

$\mathbf{w}$: vector of portfolio weights

$\mathbf{r}$: vector of expected asset returns

$$\begin{aligned}
\mathrm{Var}(R_p) &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \cdot \mathrm{Cov}(R_i, R_j) \\
\sigma_p^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \cdot \sigma_{ij} \\
&= \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \mathrm{Cov}(R_1, R_1) & \mathrm{Cov}(R_1, R_2) & \cdots & \mathrm{Cov}(R_1, R_n) \\ \mathrm{Cov}(R_2, R_1) & \mathrm{Cov}(R_2, R_2) & \cdots & \mathrm{Cov}(R_2, R_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(R_n, R_1) & \mathrm{Cov}(R_n, R_2) & \cdots & \mathrm{Cov}(R_n, R_n) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
&= \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \mathbf{w}^\top \mathbf{\Sigma} \mathbf{w}
\end{aligned}$$

$\mathrm{Var}(R_p) = \sigma_p^2$: variance of the portfolio return

$\mathrm{Cov}(R_i, R_j) = \sigma_{ij}$: covariance between returns of asset i and asset j

$\mathbf{\Sigma}$: covariance matrix of asset returns

We have a quadratic program, i.e., an optimization problem with a quadratic objective function and linear constraints. Specifically, since Σ is symmetric and positive semidefinite we have a convex quadratic program.

$$\begin{aligned}
\text{minimize} \quad & \mathbf{w}^\top \mathbf{\Sigma} \mathbf{w} \\
\text{subject to} \quad & \mathbf{w}^\top \mathbf{r} = \mu \\
& \mathbf{w}^\top \mathbf{1} = 1
\end{aligned}$$

$\mathbf{r}$: vector of expected asset returns

μ: target expected return of the portfolio

$\mathbf{1}$: vector of ones

Alternatively and equivalently, one could formulate the problem as maximizing expected return subject to a fixed level of portfolio variance:

$$\begin{aligned} \text{maximize} \quad & \mathbf{w}^\top \mathbf{r} \\ \text{subject to} \quad & \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \leq \sigma^2 \\ & \mathbf{w}^\top \mathbf{1} = 1 \end{aligned}$$

σ^2: upper bound on acceptable portfolio variance

At the optimum, the gradient of the objective function must lie in the span, i.e., equals a linear combination of the gradients of the constraints:

$$\nabla_{\mathbf{w}} \left(\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \right) \in \operatorname{span} \left\{ \nabla_{\mathbf{w}} \left(\mathbf{w}^\top \mathbf{r} \right), \nabla_{\mathbf{w}} \left(\mathbf{w}^\top \mathbf{1} \right) \right\}$$

$$\nabla_{\mathbf{w}} \left(\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \right) = \lambda \nabla_{\mathbf{w}} \left(\mathbf{w}^\top \mathbf{r} \right) + \gamma \nabla_{\mathbf{w}} \left(\mathbf{w}^\top \mathbf{1} \right)$$

λ: Lagrange multiplier associated with the target expected return constraint $\mathbf{w}^\top \mathbf{r} = \mu$

γ: Lagrange multiplier associated with the full investment constraint (i.e. size of cash position larger than zero if constraint is relaxed) $\mathbf{w}^\top \mathbf{1} = 1$

To enforce that the solution lies on these constraint surfaces (i.e., satisfies the specified values μ and 1), we introduce a Lagrangian function:

$$\mathcal{L}(\mathbf{w}, \lambda, \gamma) = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} - \lambda(\mathbf{w}^\top \mathbf{r} - \mu) - \gamma(\mathbf{w}^\top \mathbf{1} - 1)$$

Then take gradients with respect to each variable:

$$\begin{aligned} \nabla_{\mathbf{w}} \mathcal{L} &= 2\boldsymbol{\Sigma} \mathbf{w} - \lambda \mathbf{r} - \gamma \mathbf{1} = 0 \\ \nabla_{\lambda} \mathcal{L} &= \mathbf{w}^\top \mathbf{r} - \mu = 0 \\ \nabla_{\gamma} \mathcal{L} &= \mathbf{w}^\top \mathbf{1} - 1 = 0 \end{aligned}$$

$\mathcal{L}$: scalar function encoding both the objective and the constraints

μ: target expected return specified by the investor

The Lagrangian formulation packages both the directional optimality condition and the feasibility conditions.

The multiplier λ can be interpreted economically as the marginal increase in portfolio variance required to achieve one additional unit of expected return. In regions of the efficient frontier where λ is small, the investor can obtain significantly higher returns for only a modest increase in risk — making those portfolios especially attractive. Conversely, where λ is large, the return gains come at the cost of disproportionately higher risk, indicating diminishing trade-off efficiency.

To construct the efficient frontier, we solve the mean-variance optimization problem for a range of target expected returns μ. Each solution yields an optimal portfolio $\mathbf{w}^*(\mu)$, along with its corresponding portfolio variance $\sigma_p^2(\mu)$. Plotting the pairs $(\sigma_p(\mu), \mu)$ traces out the efficient frontier.

In practice, portfolio optimization under MPT is typically performed using log returns (also called continuously compounded returns), derived from adjusted closing prices. Log returns offer several practical advantages:

Additivity from multiplicative prices: Since asset prices evolve multiplicatively, taking logarithms converts returns into an additive structure. This allows multi-period log returns to be computed by simple summation, simplifying time aggregation and modeling.

Approximate normality: Log returns tend to be more symmetrically distributed and closer to normal than simple returns, especially for short time intervals. This aligns better with MPT's reliance on variance as a risk measure. MPT does not assume returns are normally distributed. But normality is often assumed in applications to justify using only mean and variance — because for normal distributions, these fully describe risk and return.

Covariance stability: When estimating the covariance matrix $\mathbf{\Sigma}$, log returns generally lead to more stable and well-behaved estimates compared to arithmetic returns.

For a time series of adjusted prices P_t, the log return at time t is defined as:

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$$

P_t: adjusted closing price of the asset at time t

So far, we have permitted asset weights to become negative in order to reach our optimization goal, i.e., short-selling is allowed in the portfolio. When we require the weights to be non-negative, we are effectively adding an inequality constraint to our quadratic program and so we must apply the Karush-Kuhn-Tucker conditions (stationarity, primal feasibility, dual feasibility, complementary slackness), thereby extending the Lagrangian.

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