Post-suppositions and semantic theory*

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Abstract I explore two alternatives to post-suppositions for generating cumulative readings of sentences with multiple modified numerals. The first uses higher-order dynamic generalized quantifiers (GQs), functions whose 'trace' is itself typed as a dynamic GQ. Such functions, though a hop-and-a-skip up the type hierarchy from dynamic GQs, are already available to any standard-issue Montagovian dynamic semanticist, and dealing with them compositionally requires nothing beyond whatever machinery already underwrites quantifiers in object position, scope ambiguity, and so on (though the analysis is presented using a handy continuations-oriented tower notation). I build on this theory by exploring a type-theoretic elaboration of it (technically, in terms of subtype polymorphism) that rules out arguably unattested 'pseudo-cumulative' readings.

Second, I show that these steps (higher-order dynamic GQs subtyping) are unnecessary if the usual 'point-wise' dynamic semantics of Dynamic Predicate Logic or Compositional DRT, where propositions are typed as relations on assignments, is replaced with an *update semantics*, where propositions are typed as functions from sets of assignments into sets of assignments. Conservative update-semantic meanings for modified numerals — direct analogs of their point-wise counterparts — automatically yield cumulative readings and fail to derive pseudo-cumulative readings. This gives an argument for an update semantics in the anaphoric domain (so far as I know the first of its kind).

I compare these two kinds of analyses with each other and consider their relation to post-suppositions in semantic theory more generally, concluding: (i) the theories canvassed here have modest empirical advantages over post-suppositions; (ii) the update-theoretic account offers the most direct route to an empirically adequate analysis of modified numerals; (iii) the reasons for this turn out to be specific to modified numerals; theories of other 'post-suppositional' phenomena are better formulated with higher-order GQs.

1 Cumulative readings of modified numerals

1.1 Basic data

Example (1) has a *cumulative* reading (Krifka 1999, cf. also Scha 1981), which asks us to consider all the boys who saw movies, and all the movies seen by boys. Tallied up, there should be precisely three such boys, and five such movies. Thus, (1) functions as a true description of Scenario A (Figure 1, left).

(1) Exactly three boys saw exactly five movies.

Of course, (1) also has a (surface-scope) distributive reading, which tells us that there's exactly three boys who each saw exactly five movies (so, perhaps, 15 total movies seen). I discuss distributive readings in Section 4.3 and Section 6.3.

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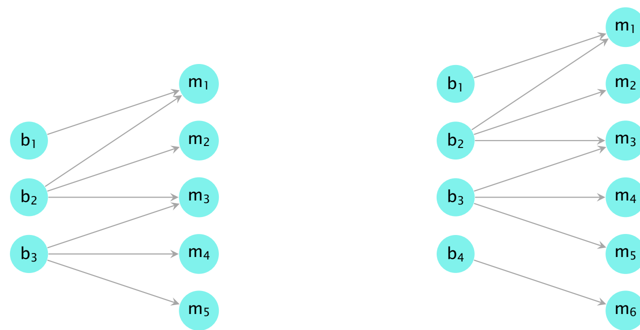

Scenario A Scenario B

Figure 1: Some boys seeing some movies. After Brasoveanu (2013: 155).

As Brasoveanu (2013) points out (cf. also Krifka 1999, Landman 2000, Ferreira 2007), the cumulative reading of (1) is *stronger than* "the largest number of boys who between them saw exactly five movies is three". These weaker truth-conditions, which I'll call *pseudo-cumulative*, serve as an accurate description of Scenario B (Figure 1, right), where there's a way to choose three boys who between them saw exactly five movies, but no way to choose four or more. (In fact, in Scenario B there are two pluralities of exactly three boys who between them saw exactly five movies: $\mathsf{b}_1 \sqcup \mathsf{b}_2 \sqcup \mathsf{b}_3$ and $\mathsf{b}_1 \sqcup \mathsf{b}_3 \sqcup \mathsf{b}_4$.[1])

The pseudo-cumulative truth-conditions happen to be satisfied in Scenario A as well as Scenario B. Might a pseudo-cumulative reading of (1) be the reason we hear it as a true description of Scenario A, rather than the existence of a cumulative reading per se? Brasoveanu (2013: 157–8) argues against this, pointing out that it's straightforward to hear (1)'s negation (2) as a true description of Scenario B and as a false description of Scenario A (we're interested only in non-distributive readings here; if you don't trust your ability to zoom in on the non-distributive reading, adding *between them* will force it). If (1) only had a pseudo-cumulative reading, this would be unexpected; instead, we'd expect (2) to be necessarily false in both scenarios.

(2) It isn't true that exactly three boys saw exactly five movies. (Brasoveanu 2013: 157, ex. 5)

In fact, it isn't clear the pseudo-cumulative "reading" is a possible construal of (1) at all (and Brasoveanu gives some reason to think it is not). I return to this point in Section 4 and again in Section 6.2.

1.2 The puzzle

An adequate semantics for modified numerals should at least generate cumulative readings (and should arguably fail to generate pseudo-cumulative readings as well). The puzzle posed by this data is that the weaker pseudo-cumulative reading looks to be the

1 $\ulcorner a \sqcup b \urcorner$ is a plurality consisting of a and b. See Section 2.1 for more on plurals and plurality.

only one derived given what look to be fairly anodyne assumptions about the (dynamic) semantics of modified numerals.

To illustrate, an intuitive dynamic meaning assignment for *exactly*^{v} *three boys* is (3). A "random" plurality of boys is stored in v by the dynamic existential quantifier $\mathbf{E}^{v}\,\mathbf{boys}$, piped to a nuclear scope argument k (⌜;⌝ names the operation of dynamic conjunction), maximized by the maximization operator $\mathbf{M}_{v}$, and finally measured by the cardinality test $\mathbf{3}_{v}$ (formal meat is put on these bones in Section 2.2).[2] In other words, (3) identifies the largest plurality of boys satisfying some property k and checks that it has three members — which seems like a basically reasonable thing for *exactly*^{v} *three boys* to be doing! A corresponding dynamic meaning for *saw exactly*^{u} *five movies* is given in (4). (There exist a variety of ways to compositionally integrate quantifiers in these and other cases. See Section 3.4 for discussion of this point.) Combining (3) and (4) with functional application delivers (5).

(3) $[\![\text{exactly}^{v}\ \text{three boys}]\!] = \lambda k.\, \mathbf{M}_{v}\,(\mathbf{E}^{v}\,\mathbf{boys}\,;\,k\,v)\,;\,\mathbf{3}_{v}$

(4) $[\![\text{saw exactly}^{u}\ \text{five movies}]\!] = \lambda v.\, \mathbf{M}_{u}\,(\mathbf{E}^{u}\,\mathbf{movs}\,;\,\mathbf{saw}\,u\,v)\,;\,\mathbf{5}_{u}$

(5) $\mathbf{M}_{v}\,(\mathbf{E}^{v}\,\mathbf{boys}\,;\,\mathbf{M}_{u}\,(\mathbf{E}^{u}\,\mathbf{movs}\,;\,\mathbf{saw}\,u\,v)\,;\,\mathbf{5}_{u})\,;\,\mathbf{3}_{v}$

As Brasoveanu observes, the issue with logical forms like (5) — and thus with the otherwise sensible-looking meaning assignments in (3) and (4) — is that the outer existential operator $\mathbf{E}^{v}\,\mathbf{boys}$ ends up introducing and storing in v pluralities of boys *who between them saw exactly five movies*. Maximization by $\mathbf{M}_{v}$ subsequently delivers the biggest plurality or pluralities of boys who between them saw exactly five movies. In Scenario A there is one such plurality, $\mathsf{b}_1 \sqcup \mathsf{b}_2 \sqcup \mathsf{b}_3$. In Scenario B there are two, $\mathsf{b}_1 \sqcup \mathsf{b}_2 \sqcup \mathsf{b}_3$ and $\mathsf{b}_1 \sqcup \mathsf{b}_3 \sqcup \mathsf{b}_4$. The cardinality test $\mathbf{3}_{v}$ requires the maximal entity or entities stored in v to have exactly three members. This test is passed in both Scenario A and Scenario B — in both cases, the biggest pluralities of boys who saw exactly five movies are three-boys-large — even though the desired cumulative reading is only true at Scenario A. Thus, (5) represents only the weaker pseudo-cumulative reading of (1).

1.3 Post-suppositions as a solution

In contrast with the pseudo-cumulative (5), a truly cumulative reading is generated when $\mathbf{5}_{u}$ occurs outside the scope of $\mathbf{M}_{v}$, as in (6). Without any interference from $\mathbf{5}_{u}$, $\mathbf{M}_{v}$ in (6) builds the maximal plurality of boys who saw movies, and $\mathbf{M}_{u}$ builds the maximal plurality of movies seen by those boys. In Scenario A, there are three such boys and five such movies, and the cardinality tests are satisfied. In Scenario B, there are *four* such boys, and *six* such movies, and the cardinality tests are unhappy. Thus, (6) represents the cumulative reading of (1) — true in Scenario A, false in Scenario B.

(6) $\mathbf{M}_{v}\,(\mathbf{E}^{v}\,\mathbf{boys}\,;\,\mathbf{M}_{u}\,(\mathbf{E}^{u}\,\mathbf{movs}\,;\,\mathbf{saw}\,u\,v))\,;\,\mathbf{5}_{u}\,;\,\mathbf{3}_{v}$

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2 Application associates to the left, and parentheses are omitted when possible. For example, I write ⌜$\mathbf{saw}\,u\,v$⌝ in lieu of ⌜$((\mathbf{saw}\,(u))\,(v))$⌝ and ⌜$K\,;\,L\,;\,M$⌝ in lieu of ⌜$(K\,;\,L)\,;\,M$⌝ or ⌜$K\,;\,(L\,;\,M)$⌝ (since $(;)$ is associative).

The question, of course, is how to derive logical forms like (6) from a reasonable semantics for modified numerals. As (3), (4), and (5) demonstrate, this isn't a trivial task: one of the modified numerals has to take scope over the other, and so, it seems, one of the modified numerals' cardinality tests has to get caught within the scope of the other's maximization operator (this seems like a good place to mention that inverse-scoping *exactly*^{u} *five movies* over *exactly*^{v} *three boys* derives a distinct pseudo-cumulative reading, on which the biggest number of movies seen by exactly three boys is five). In fact, considering related data, Krifka (1999) concludes that a satisfactory analysis is *impossible* so long as we assume that any sentence gives rise to a total ordering of its scope-bearing elements (though Krifka works within a static semantics and is accordingly not directly concerned with pseudo-cumulative readings):

> The problem cases discussed here clearly require a representation in which NPs are not scoped with respect to each other. Rather, they ask for an interpretation strategy in which all the NPs in a sentence are somehow interpreted on a par.
> (Krifka 1999: 262)

In line with Krifka's assessment, Brasoveanu (2013) proposes to derive meanings like (6) by admitting a degree of parallel processing into the semantics, arguing that the cardinality tests of modified numerals comprise a distinguished class of *post-suppositional* meanings, whose semantic integration is 'delayed' until an appropriate evaluation context is established. For (1), a post-suppositional logical form might be (7), where the post-suppositional cardinality tests are underlined.

(7) $\quad \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)\,;\underline{\mathbf{5}_u})\,;\underline{\mathbf{3}_v}$

Just as in (5), $\underline{\mathbf{5}_u}$ in (7) occurs within the "syntactic scope" of $\mathbf{M}_v$ — and so we can, perhaps, imagine how this kind of logical form might be compositionally derived. However, $\underline{\mathbf{5}_u}$'s post-suppositional status keeps it from being interpreted there. Instead, (7) can be considered a kind of short-hand for the *bi-dimensional* logical form in (8), where dynamic existential quantification and maximization happen in the first dimension, and the cardinality tests are relegated to the second dimension:

(8) $\quad \big(\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)),\mathbf{3}_v\,;\mathbf{5}_u\big)$

A compositional architecture for deriving meanings like (8) is presented in Section 5.1 (my version of a post-suppositional theory differs in certain respects from Brasoveanu, who gives a semantics for a first-order metalanguage stated in terms of certain representational properties of the metalanguage, whereas I will directly map phrase markers to typed model-theoretic values — but those differences are orthogonal to the points I make in this paper). For now I simply note that, as a semantic value is compositionally assembled, post-supposed meanings are accumulated as dynamic conjuncts in a second dimension, independently of the utterance's main, 'suppositional' content.[3]

3 Similar formal mechanisms and, to an extent, empirical motivations, underlie Karttunen & Peters's (1979) bi-dimensional theory of presupposition, Rooth's (1985) bi-dimensional treatment of association with focus, Abusch's (1994) bi-dimensional treatment of exceptionally scoping indefinites, and Potts's (2005) bi-dimensional treatment of certain not-at-issue content. See Section 5.2 for more on this point.

These two dimensions — immediate supposition and delayed post-supposition — can be reconciled in a variety of ways, either at the 'meta-level' via appropriately construed notions of truth and entailment, or by directly defining operators that shuttle content between the two dimensions (Brasoveanu explores both options). It's a simple matter, for example, to define a mapping from (8) to (6), thereby deriving a cumulative meaning.[4]

1.4 This paper

This paper introduces two distinct alternatives to post-suppositions for deriving cumulative readings of constructions like (1), and then compares them with the post-suppositional theory, and each other. I first show how such readings can be generated using higher-order dynamic generalized quantifiers (GQs), functions whose 'trace' — the thing they 'leave behind' when they take scope — is itself typed as a dynamic GQ. So if $\mathtt{e}$ and $\mathtt{t}$ are, respectively, the types of dynamic individuals and dynamic propositions, and $\mathtt{Q} ::= (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$ the type of a garden-variety dynamic GQ, the type of a higher-order dynamic GQ can be given as $(\mathtt{Q} \to \mathtt{t}) \to \mathtt{t}$.[5] Such functions are already available in principle to any Montagovian dynamic semanticist, given the customary recursively defined type hierarchy (i.e., for any two types A and B, $A \to B$ is also a type).

Higher-order dynamic GQs allow $\mathbf{5}_u$ (the cardinality test associated with *exactly*^{u} *five movies*) to out-scope both $\mathbf{E}^v$ and $\mathbf{M}_v$ (the existential and maximization operators associated with *exactly*^{v} *three boys*) — even as *exactly*^{v} *three movies* out-scopes *exactly*^{u} *five movies*! This derives the cumulative reading of (1): if $\mathbf{5}_u$ isn't in the scope of $\mathbf{E}^v$ and $\mathbf{M}_v$, it doesn't interfere with the process of constructing a maximal movie-watching plurality of boys. Compositionally, this turns out to require nothing beyond whatever already allows scope-taking throughout the grammar; a higher-order GQ takes scope normally, and then so does its 'trace'. Figure 2 sketches the basic idea. (I'll suggest in Section 3.1 that this strategy has some things in common with semantic approaches to scope reconstruction.) Though I'll present the theory using a continuations-oriented tower notation for expository reasons (cf. Barker & Shan 2008, 2014) — in particular, towers make higher-order GQs far easier to work with and reason about than do linearized λ-terms — it's ultimately consistent with any theory of scope-taking, including Quantifier Raising, Flexible Types (Hendriks 1993), type-logical approaches, etc.

Though higher-order dynamic GQs allow us to generate cumulative readings, this comes at a price: arguably unattested pseudo-cumulative readings are freely generated as well. This happens when (helping ourselves to a representational metaphor), the entirety of *exactly*^{u} *five movies* takes scope under the 'trace' of *exactly*^{v} *three boys*, as in Figure 3. In order to rule out pseudo-cumulative readings, I explore a type-theoretic elaboration of the higher-order GQ theory, in terms of *subtype polymorpshim* (e.g., Pierce 2002). In effect, this elaboration renders cumulative readings well-typed and pseudo-

4 There's a minor fudge here. The cardinality tests occur in different orders in (8) and (6). This isn't an accident. The order in (6) is useful for comparing with (5), and the order in (8) is what's actually generated by the post-suppositional semantics, as we'll see. No harm done: $\mathbf{5}_u\,;\mathbf{3}_v$ and $\mathbf{3}_v\,;\mathbf{5}_u$ turn out to be equivalent updates.

5 $\ulcorner A \to B \urcorner$ names the type of functions from type A onto type B. Types associate to the right: thus, $A \to B \to C$ is equivalent to $A \to (B \to C)$. When I write $\ulcorner A ::= B \urcorner$, I mean that type A is being defined as B.

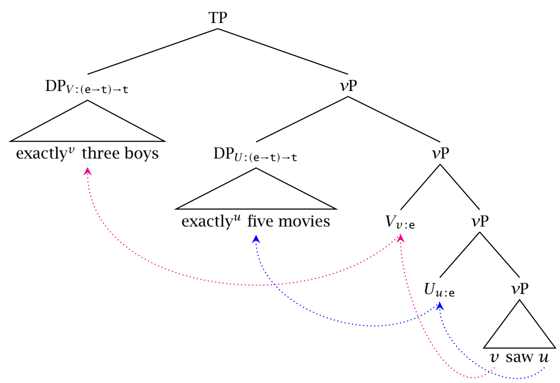

Figure 2: An LF dramatizing how the higher-order dynamic GQ analysis works to derive cumulative readings with multiple modified numerals. I use ⌜$\mathrm{Op}_{v:\tau}\ X$⌝ as short-hand for ⌜Op $[\lambda_{v:\tau}\ X]$⌝ — i.e., to indicate that Op binds a variable v of type τ in X. Both quantifiers (higher-order dynamic GQs) take scope normally, and then so do their traces (regular dynamic GQs). (Movement of *exactly^v three boys* to Spec-TP can be seen as an overt movement, while the remaining movements are all covert.) The end result is that the cardinality tests are deployed in the scope positions of the modified numerals, and the maximization and dref-introduction operators in the scope positions of the traces. This results in the cumulative reading.

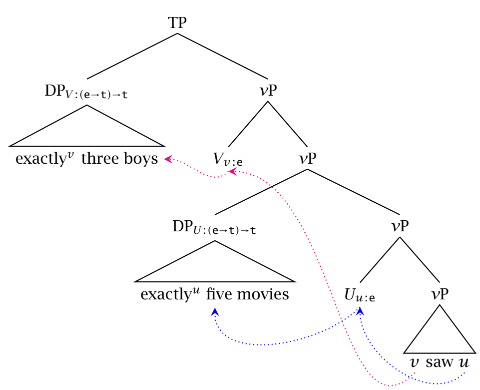

Figure 3: Deriving the pseudo-cumulative reading with higher-order dynamic GQs. *Exactly^u five movies* takes scope under the higher-order trace of *exactly^v three boys.* This derivation can be ruled out by being rendered not-well-typed. This is undertaken in Section 4.

cumulative readings not-well-typed. Though taking this tack requires clear stipulations above and beyond what is needed to generate cumulative readings, I argue that parallel stipulations are present in the post-suppositional theory.

Second, I show that both of these steps — higher-order dynamic GQs for generating cumulative readings and subtype polymorphism for avoiding pseudo-cumulative readings — can be dispensed with, if we move from thinking of dynamic propositions as 'point-wise' relations on assignments — i.e., with $\mathtt{t} ::= a \to a \to t$ (where a is the type of assignment functions) — to thinking of them as 'global' update functions on sets of assignments — i.e., with $\mathtt{u} ::= (a \to t) \to a \to t$ the type of update-theoretic propositions. It turns out that direct update-theoretic analogs of garden-variety dynamic GQs, type $(\mathtt{e} \to \mathtt{u}) \to \mathtt{u}$, automatically give rise to cumulative readings, and automatically fail to generate pseudo-cumulative readings. In fact, in an update semantics there's no need for *any* stipulations beyond the expected meanings for modified numerals.

The reason for these successes turns out to be somewhat surprising, and (I think) independently interesting. The more global perspective afforded by an update semantics allows maximization operators to survey the entire context as it is progressively assembled in the course of a complex dynamic update. In particular, in an update-theoretic analysis of (1), the maximization operator associated with *exactly*^{u} *five movies* ends up constructing the maximal sum of movies seen by any boys whatsoever. This contrasts with the point-wise/relational semantics, in which the maximization operator associated with *exactly*^{u} *five movies* constructs, *for each plurality of boys*, the maximal sum of movies seen by *those boys*. This is the key difference between the point-wise and update-theoretic dynamic theories, on which their different predictions *vis à vis* modified numerals turns. Overall, these points amount to an argument for an update-style dynamic semantics in the anaphoric domain (so far as I know the first of its kind; see, e.g., Veltman 1996 for arguments in favor of update semantics in the modal domain).

On either of the theories presented in this paper, there is no need to jettison the standard assumption that any reading of a sentence with more than one quantificational expression can be attributed to a total ordering of its scope-bearing elements. Moreover, both the higher-order GQ and update-theoretic accounts seem to improve on post-suppositional analyses, which do not account for the influence of islands on the availability of cumulative readings without additional stipulations about how post-suppositional content "projects" out of islands.

Finally, because the update-theoretic account of cumulative readings needs no stipulations to rule out pseudo-cumulative readings, it appears preferable to the higher-order dynamic GQ theory. One may wonder, then, why we go to the trouble of investigating both kinds of theories here. The reason, as I discuss at the end of the paper, is that other ostensibly 'post-suppositional' phenomena turn out to be best understood in terms of higher-order dynamic GQs. I consider the case of *dependent indefinites*, concluding that a theory using higher-order dynamic meanings achieves better empirical coverage than a post-suppositional theory, and that its virtues cannot be replicated merely by going update-theoretic. I conclude that higher-order and update-theoretic perspectives on 'post-suppositional' content are complementary, and each deserves a place at the table.

2 Formal preliminaries

2.1 Pluralities

Like Brasoveanu (2013) (and many others following Link 1983), I'll assume the domain of individuals (type e) includes both atomic and non-atomic entities. For concreteness, we'll suppose (with Schwarzschild 1996) that pluralities are sets, with atomic entities (type e_{at}) construed as singleton sets, sum formation construed as set union (i.e., $x \sqcup y \coloneqq x \cup y$), and the domain of individuals defined inductively as the sum-closure of e_{at}:

(9) $e ::= e_{\mathsf{at}} \mid e \sqcup e$

The part-of relation is characterized in terms of subset-hood: $x \leqslant y \coloneqq x \subseteq y$, and the proper part-of relation is defined in the usual way: $x < y \coloneqq x \leqslant y \wedge y \nleqslant x$. Formulas like $\ulcorner x \leqslant_{\mathsf{at}} y \urcorner$ ('x is an atomic part of y') are shorthand for $\ulcorner x \leqslant y \wedge \forall x' \leqslant x : x' = x \urcorner$. We can measure plural individuals by counting their atomic parts: $\#x \coloneqq |\{x' \mid x' \leqslant_{\mathsf{at}} x\}|$ (with set-based pluralities, $\#x$ is guaranteed to be equivalent to $|x|$).

Finally, we'll assume (with Brasoveanu) that the static properties and relations on which dynamic predicates are built are cumulatively closed. For example, if $\mathsf{boys}\,x$ and $\mathsf{boys}\,y$, then $\mathsf{boys}\,(x \sqcup y)$. If $\mathsf{saw}\,x\,y$ and $\mathsf{saw}\,x'\,y'$, then $\mathsf{saw}\,(x \sqcup x')\,(y \sqcup y')$.

2.2 Dynamics

Those are all the assumptions we'll make about the underlying static machinery. Here, now, is a brief introduction to dynamic interpretation in a Montagovian setting. The basic setup is essentially that of Muskens (1996)'s CDRT (for related perspectives see Heim 1982, Barwise 1987, Rooth 1987, Groenendijk & Stokhof 1991a, Muskens 1995, Brasoveanu 2007), though I will make somewhat more direct use of variables in what follows. Readers comfortable with CDRT can safely skip or skim this subsection.

Our basic types are individuals (type e, as defined in (9)), truth values ($t ::= \{0, 1\}$), and variables ($\mathcal{V} ::= \{u, v, ...\}$). Derivatively, we also have (partial) assignment functions (type $a ::= \mathcal{V} \to e$).[6] For convenience, I define the following type synonyms: $\mathsf{e} ::= \mathcal{V}$ and $\mathsf{t} ::= a \to a \to t$; dynamic individuals are just variables, and dynamic propositions are relations on assignments (more on this in the next paragraph). The set of compositionally relevant types τ is then defined inductively as follows:

(10) $\tau ::= \mathsf{e} \mid \mathsf{t} \mid \tau \to \tau$

Dynamic propositions are relations on assignments. The basic idea is that sentences are devices for *changing the context*, possibly in nondeterministic ways: fed an input assignment i, a dynamic proposition K returns a set of *updated* assignments $\{j \mid i K j\}$ as possible outputs (where it helps, I'll use a more iconic infix notation for dynamic propositions — i.e., $\ulcorner i K j \urcorner$ in lieu of $\ulcorner K\, i\, j \urcorner$). Thus, verbs and verbal projections denote functions from sequences of arguments into dynamic propositions. For example, the

6 Partiality of assignments will actually turn out to be useful down the road. See fns. 13 and 23.

meaning of a transitive verb like *saw* is given in (11). **Saw** has type $\mathtt{e} \to \mathtt{e} \to \mathtt{t}$; it takes two type-$\mathtt{e}$ arguments and returns a dynamic proposition, i.e., a relation on assignments.[7]

(11) $i\,(\textbf{saw}\,u\,v)\,j \coloneqq i = j \wedge \mathsf{saw}\,i_u\,i_v$ $\qquad \textbf{saw} : \mathtt{e} \to \mathtt{e} \to \mathtt{t}$

In more detail, **saw** simply checks that the incoming assignment i maps v and u to a seer-seen pair (to save parentheses, I write $\ulcorner i_v \urcorner$ for $\ulcorner i\,v \urcorner$). If this condition is satisfied, i is passed along unchanged; otherwise, no updated assignment is returned — in other words, there's no j such that $i\,(\textbf{saw}\,u\,v)\,j$ — and the update is said to have *failed*.

The ability to output assignments means dynamic propositions can store information about the values of variables. For example, *a*^{v} *linguist saw a*^{u} *philosopher* gets associated with an update K characterized in (12) (though I do not indicate how this meaning is compositionally derived here; see Section 2.3). Discourse referent (aka *dref*) introduction is *assignment modification*: at any j output by (12), v stores some (atomic) linguist x, and u some (atomic) philosopher y, such that x saw y ($\ulcorner i^{v \to x} \urcorner$ is the assignment mapping v to x and otherwise just like i).

(12) $i\,K\,j = \exists x \in \mathsf{ling} : \exists y \in \mathsf{phil} : \mathsf{saw}\,y\,x \wedge j = (i^{v \to x})^{u \to y}$

When dynamic propositions are relations on assignments, dynamic conjunction amounts to relation composition, defined in (13): $i\,(L\,;R)\,j$ iff there's a path through L and R that begins at i and ends at j. (Thinking, equivalently, of a relation on assignments as a function from an assignment into a set of assignments, $L;R$ amounts to $\lambda i.\bigcup_{h \in L i} R\,h$: the assignments h output by L at i are fed *point-wise* to R, and the resulting sets of R-updated assignments unioned.) This guarantees that R is evaluated in the anaphoric context established by L — in other words, the drefs introduced by L are visible to R (which allows binding even in the absence of scope/LF c-command).

(13) $i\,(L\,;R)\,j \coloneqq \exists h : i\,L\,h\,R\,j$ $\qquad (;) : \mathtt{t} \to \mathtt{t} \to \mathtt{t}$

To extract a truth-condition from a dynamic proposition, we quantify over output assignments, as in (14). Relative to an input assignment i, K counts as **true** iff K yields any output whatsoever given i as an input. Otherwise, K is said to *fail* at i. (This 'text-level' existential quantification corresponds to Heim's (1982) global existential closure operation.) Dynamic counterparts of tautologies and contradictions can be defined as in (15). (For any K, $K\,;\mathbf{T} \equiv \mathbf{T}\,;K \equiv K$, and $K\,;\mathbf{F} \equiv \mathbf{F}\,;K \equiv \mathbf{F}$. These facts will be useful later.)

(14) K is **true** at i iff $\exists j.\,i\,K\,j$

(15) $i\,\mathbf{T}\,j \coloneqq i = j \qquad i\,\mathbf{F}\,j \coloneqq i \neq i$ $\qquad \mathbf{T}, \mathbf{F} : \mathtt{t}$

Correctly, (12) is **true** at any i iff a linguist saw a philosopher (else, there aren't any assignments mapping v and u to a linguist-philosopher pair standing in the seeing relation, as no such pair exists!). $\mathbf{T}$ is **true** at any i whatsoever, and $\mathbf{F}$ isn't **true** at any i.

7 Static constants in the meta-language are written with sans serif, and dynamic constants with **bold sans**.

2.3 Dynamic GQs and the pseudo-cumulative reading

Just as static frameworks have static GQs with type $(e \to t) \to t$, so dynamic frameworks have a class of *dynamic* GQs, with the corresponding dynamic type $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$ (see, e.g., Chierchia 1992, 1995, Kanazawa 1994). For example, *every*^{v} *linguist* might be assigned a meaning like (16). This function takes as an argument a dynamic property k (type $\mathtt{e} \to \mathtt{t}$), and returns a dynamic proposition (type $\mathtt{t}$).

(16) $i\,(\mathbf{ev.ling}^{v}\,k)\,j \coloneqq i = j \wedge \forall x \in \mathsf{ling} : \exists h : i^{v \mapsto x}\,(k\,v)\,h$ $\qquad$ $\mathbf{ev.ling}^{v} : (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$

In more detail, the job of $\mathbf{ev.ling}^{v}$ is to ensure that every (atomic) linguist satisfies the nuclear scope k (notice that this happens somewhat indirectly: for each atomic linguist x, $k\,v$ is evaluated at an incoming assignment function $i^{v \mapsto x}$ that maps v to x).

The dynamic GQ mooted for *exactly*^{v} *three boys* in (3) — which gives rise only to the pseudo-cumulative reading — is repeated below:

(3) $\lambda k.\,\mathbf{M}_{v}\,(\mathbf{E}^{v}\,\mathbf{boys}\,;\,k\,v)\,;\,\mathbf{3}_{v}$ $\qquad$ type: $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$

This entry has three key parts: the dynamic existential quantifier $\mathbf{E}^{v}\,\mathbf{boys}$, the maximization operator $\mathbf{M}_{v}$, and the cardinality test $\mathbf{3}_{v}$. $\mathbf{E}^{v}\,\mathbf{boys}$, defined in (17), first stores various pluralities of boys in v.[8] In (3), these boy-pluralities are subsequently piped to the scope argument k. Next, the maximality operator $\mathbf{M}_{v}$, defined in (18), ensures that only those assignments that harbor mereologically maximal values for v are retained.[9] Finally, the cardinality test $\mathbf{3}_{v}$, defined in (19), requires these maximal values for v to be composed of exactly three individuals. The superscripted variable in $\mathbf{E}^{v}\,\mathbf{boys}$ is intended to flag that it introduces a dref, while $\mathbf{M}_{v}$ and $\mathbf{3}_{v}$ are anaphoric, in the sense that they place additional constraints on an already-introduced dref.

(17) $i\,(\mathbf{E}^{v}\,\mathbf{boys})\,j \coloneqq \exists x \in \mathsf{boys} : j = i^{v \mapsto x}$ $\qquad$ $\mathbf{E}^{v}\,\mathbf{boys} : \mathtt{t}$

(18) $i\,(\mathbf{M}_{v}\,K)\,j \coloneqq i\,K\,j \wedge \neg\exists h : i\,K\,h \wedge j_{v} < h_{v}$ $\qquad$ $\mathbf{M}_{v} : \mathtt{t} \to \mathtt{t}$

(19) $i\,(\mathbf{3}_{v})\,j \coloneqq i = j \wedge \#i_{v} = 3$ $\qquad$ $\mathbf{3}_{v} : \mathtt{t}$

We're now in position to verify that (5) only derives pseudo-cumulative truth conditions. In (20), I repeat the crucial subformula of (5) (i.e., with the outer maximization operator stripped off). First, we store various boy-pluralities in v. Next, we associate those boys with the maximal pluralities of movies they saw, which we store in u (e.g., the maximal plurality of movies seen by $\mathsf{b}_3 \sqcup \mathsf{b}_4$ in Scenario B is $\mathsf{m}_3 \sqcup \mathsf{m}_4 \sqcup \mathsf{m}_5 \sqcup \mathsf{m}_6$). Finally, we retain only those assignments whose maximal movie-pluralities are five movies large.

(20) $\mathbf{E}^{v}\,\mathbf{boys}\,;\,\mathbf{M}_{u}\,(\mathbf{E}^{u}\,\mathbf{movs}\,;\,\mathbf{saw}\,u\,v)\,;\,\mathbf{5}_{u}$

Once (20) is processed, v contains pluralities of boys who saw exactly five movies. In the pseudo-cumulative Scenario B, there are three such boys: $\mathsf{b}_1 \sqcup \mathsf{b}_2 \sqcup \mathsf{b}_3$, $\mathsf{b}_2 \sqcup \mathsf{b}_3$, and

8 In full generality, $i\,(\mathbf{E}^{v}\,P)\,j \coloneqq \exists x : i^{v \mapsto x}\,(P\,v)\,j$, with $\mathbf{E}^{v} : (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$.

9 Cases like Krifka's (1999: 262) *In Guatemala, at most three percent of the population owns at least seventy percent of the land* may suggest, as Krifka (1999: 267) argues, that maximization is informational, rather than mereological (cf. von Fintel, Fox & Iatridou 2014, Schlenker 2012). I pass over this complication here.

$\mathsf{b}_1 \sqcup \mathsf{b}_3 \sqcup \mathsf{b}_4$. Subsequent maximization by $\mathbf{M}_v$ tosses out $\mathsf{b}_2 \sqcup \mathsf{b}_3$, leaving $\mathsf{b}_1 \sqcup \mathsf{b}_2 \sqcup \mathsf{b}_3$ and $\mathsf{b}_1 \sqcup \mathsf{b}_3 \sqcup \mathsf{b}_4$, after which $\mathbf{3}_v$ tosses out any maximal boy-pluralities containing more or fewer than three boys (which in this case has no effect). Thus, when all is said and done, (5) succeeds in Scenario B, and so we have failed to derive the cumulative reading.

3 The cumulative reading, via higher-order dynamic GQs

3.1 Interlude: scope reconstruction and higher-order GQs

I'll shortly suggest a theory of modified numerals in which a higher-order dynamic GQ takes scope, and then so does its trace. Before jumping into that, it will be useful to consider how something parallel happens in semantic treatments of scope reconstruction (e.g., Cresti 1995; cf. also Sternefeld 2001). Consider in this regard example (21). It readily allows a reading with its quantifiers scoping under negation and in their surface order (i.e., a reading denying the surface-scope construal of *every linguist owns a villa*).[10]

(21) Every linguist doesn't own a villa. ✓ not ≫ every ≫ a

Deriving this reading requires reconstructing *every linguist* to a position under *don't* (given the standard assumption that *don't* doesn't take scope). On a semantic approach to scope reconstruction, this means that the overt movement of *every linguist* (from a *v*P-internal position to Spec-TP) leaves a higher-order trace — i.e., $(e \to t) \to t$ in a static semantics. In addition (again given standard assumptions), the quantified object *a villa* is uninterpretable in situ and must undergo quantifier raising, ultimately adjoining to a node of type t. Putting these two movements together gives the LF on the left-hand side of Figure 4 (reminder: I use $\ulcorner \mathrm{Op}_{v:\tau}\ X \urcorner$ as short-hand for $\ulcorner \mathrm{Op}\ [\lambda_{v:\tau}\ X] \urcorner$ — i.e., to mean that Op binds a variable v of type τ in X).

This LF assigns the quantifiers *inverse scope* (since *every linguist* reconstructs into its base position, under the QR'd object). The reading of interest — with surface-scoped quantifiers — requires one additional movement. This is shown in the right-hand LF of Figure 4. The additional movement (this time leaving a lower-order trace x in the *v*P-internal base position) means that *every linguist* will still undergo scope reconstruction, but now only as far as its intermediate trace. This derives the *not* ≫ *every* ≫ *a* reading. We thus conclude that the higher-order trace of *every linguist* is able to take scope independently of the quantifier that birthed it.

A common way to think about scope reconstruction is that quantifiers can leave a trace with the same type as the quantifier. In such situations, the usual relationship between a quantifier and its scope is inverted — the quantifier serves as an argument to its scope rather than a functor over its scope. In the static case, the quantifier will have type $Q ::= (e \to t) \to t$, and its scope will have type $Q \to t$.

Perfectly equivalently, however, we're free to think of a quantifier that binds a higher-order trace as having the higher-order type $(Q \to t) \to t$! For example, in LFs like those

10 It is, I suppose, possible to deny the existence of this reading by admitting only the weaker *not* ≫ *a* ≫ *every* construal, which is true whenever the putative *not* ≫ *every* ≫ *a* construal is (cf. Reinhart 1976: 193). Defusing this objection just requires replacing *a villa* with a non-monotonic quantifier like *exactly two villas*.

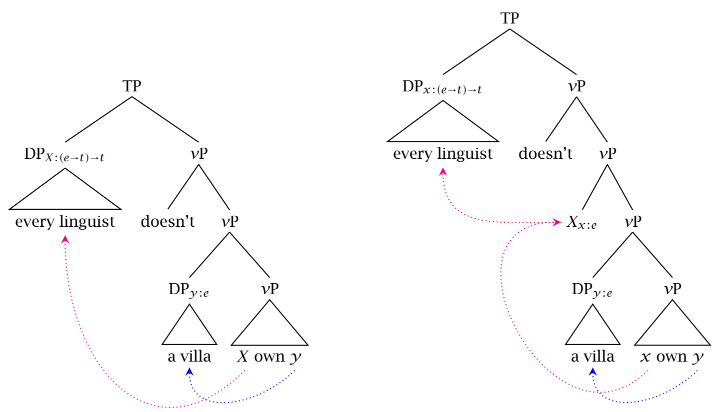

Figure 4: Semantic scope reconstruction for the *not* ≫ *every* ≫ *a* reading of *every linguist doesn't own a villa.* On the left, an overt movement of *every linguist* (leaving a higher-order trace) and an obligatory covert scoping of *a villa* yields *not* ≫ *a* ≫ *every*. On the right, an additional scoping of the higher-order trace to a position above *a villa* yields *not* ≫ *every* ≫ *a*.

depicted in Figure 4, we could view *every linguist* as having the (static) higher-order GQ meaning in (22), where the underbraced position is the familiar static GQ meaning that we usually associate with *every linguist*.

(22) $\lambda c.\, c\, \underbrace{[\![\text{every linguist}]\!]}_{Q ::= (e\to t)\to t}$ type: $(Q \to t) \to t$

The *only* effect of this move in the present case is to restore the usual function-argument relationship to the quantifier and its scope — the meanings derived for Figure 4 with a meaning like (22) are no different than the meanings derived with a lower-typed *every linguist*. Notice in this respect that (22) can be derived from a standard GQ meaning via an application of Lift (e.g., Partee 1986). In general, higher-order meanings generated by Lift don't let us derive more meanings compositionally — they just build the same meanings in different ways.

3.2 Two kinds of higher-order dynamic GQs

These are the two pieces that our first account of modified numerals will exploit. In the first place, the modified numeral will inhabit a higher type than standard dynamic GQs: $(\mathtt{Q} \to \mathtt{t}) \to \mathtt{t}$ in lieu of $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$. Secondly, the modified numeral's trace, itself of type $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$, will take scope.

An initial attempt at a higher-order dynamic GQ meaning for *exactly*^{v} *three boys* is given in (23) below. Analogously to (22), this meaning is simply the result of applying Lift to the garden-variety dynamic GQ meaning defined earlier in (3).

(23) $\lambda c.\, c\, \underbrace{(\lambda k.\, \mathbf{M}_v\, (\mathbf{E}^v\, \mathbf{boys}\, ;\, k\, v)\, ;\, \mathbf{3}_v)}_{\mathtt{Q} ::= (\mathtt{e}\to\mathtt{t})\to\mathtt{t}}$ type: $(\mathtt{Q} \to \mathtt{t}) \to \mathtt{t}$

Of course, because (23) is simply the Lift of the dynamic GQ (3), it won't allow us to generate any readings that we couldn't already generate with (3).

However, the new scope argument c in a higher-order dynamic GQ creates a new evaluation context to which $\mathbf{3}_v$ can be attached — that is, though $\mathbf{3}_v$ is attached under c in (23), it could just as well be attached outside c. This is the key that unlocks cumulative readings. We posit the higher-order dynamic GQ meaning in (24) for *exactly*^{v} *three boys* and, analogously, the meaning in (25) for *exactly*^{u} *five movies*.[11] These meanings differ from (23) in a small but significant way: the cardinality test which scopes under the higher-order scope argument c in (23) is evaluated after $c\,(\lambda k.\dots)$ in (24) and (25).

(24) $[\![\text{exactly}^v \text{ three boys}]\!] \coloneqq \lambda c.\, c\,(\underbrace{\lambda k.\, \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\, k\,v)}_{\mathtt{Q} ::= (\mathtt{e}\to\mathtt{t})\to\mathtt{t}})\,;\, \mathbf{3}_v$ $\qquad$ type: $(\mathtt{Q}\to\mathtt{t})\to\mathtt{t}$

(25) $[\![\text{exactly}^u \text{ five movies}]\!] \coloneqq \lambda c.\, c\,(\underbrace{\lambda k.\, \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\, k\,u)}_{\mathtt{Q} ::= (\mathtt{e}\to\mathtt{t})\to\mathtt{t}})\,;\, \mathbf{5}_u$ $\qquad$ type: $(\mathtt{Q}\to\mathtt{t})\to\mathtt{t}$

Higher-order dynamic GQs like (24) and (25) are more powerful than both regular dynamic GQs like (3), and Lift's of dynamic GQs like (23) — that is, they can be used to compositionally derive *strictly more* meanings (specifically, as we'll explore in the next few sections, cumulative readings). One way to appreciate this extra richness is to notice that we can define a mapping from (24) into (3), but not the reverse. Consider (26).

(26) $^{\cup}M \coloneqq \lambda k.\, M\,(\lambda m.\, m\,k)$ $\qquad$ $(\cup) : ((\mathtt{Q}\to\mathtt{t})\to\mathtt{t})\to\mathtt{Q}$

This mapping from higher-order dynamic GQs into regular dynamic GQs blurs distinctions — e.g., $^{\cup}$(23) and $^{\cup}$(24) are both equivalent to (3). Therefore, the mapping is not injective, and not invertible. Correspondingly, there is no general way to map regular dynamic GQs like (3) to higher-order meanings like (24). In other words, (24) and (25) are *irreducibly* higher-order (unlike the Lift-ed higher-order quantifiers (22) and (23), which have functionally equivalent dynamic GQ counterparts).

3.3 Cumulative readings, the hard way

In one sense, the higher-order dynamic GQ theory of cumulative readings begins and ends here — lexical entries like (24) and (25) can be used to generate cumulative meanings for (1) straightaway. For example, given an LF like Figure 2 (on page 6), we derive the meaning in (27), where $\ulcorner\mathbb{B}\urcorner$ and $\ulcorner\mathbb{M}\urcorner$ are abbreviations for (24) and (25). The first line of (27) wears its derivation on its sleeve: both higher-order GQs take scope (in their surface order), and then so do their traces (again, in their surface order). After a rather tedious series of elementary β-reductions, we arrive, at last, at the cumulative reading.

(27) $\mathbb{B}\,(\lambda V.\, \mathbb{M}\,(\lambda U.\, V\,(\lambda v.\, U\,(\lambda u.\, \mathbf{saw}\,u\,v))))$
$\rightsquigarrow \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\, \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\, \mathbf{saw}\,u\,v))\,;\, \mathbf{5}_u\,;\, \mathbf{3}_v$

11 I don't give an entry for $[\![\text{exactly}^v]\!]$ here, but it is natural to model it as a function from a degree and a dynamic property to a higher-order dynamic GQ like the one defined in (24). In other words, $[\![\text{exactly}^v]\!]\, 3\, \mathbf{boys} \equiv$ (24), and so $[\![\text{exactly}]\!]$ has type $d \to (\mathtt{e}\to\mathtt{t}) \to (\mathtt{Q}\to\mathtt{t})\to\mathtt{t}$. "Bare" uses of numerals like *three boys* could be accounted for (as is fairly standard) via a silent existential-like determiner, e.g., one that takes a degree and a dynamic property as arguments and returns a garden-variety dynamic GQ — crucially, one without any $\mathbf{M}$ operators.

Given that higher-order dynamic GQs like (24) and (25) and their traces both take scope, there are predicted to be exactly six different scopings derivable when a sentence contains two modified numeral DPs (given that a higher-order dynamic GQ has to scope over its trace).[12] In fact, no fewer than four of these scopings turn out to derive equivalent cumulative meanings for (1)! (The remaining two scopings derive two distinct pseudo-cumulative readings. See Section 4 for discussion.) The first of these is of course (27), and the remaining three are listed in (28)–(30).[13]

(28) $\mathbb{B}\,(\lambda V.\,\mathbb{M}\,(\lambda U.\,U\,(\lambda u.\,V\,(\lambda v.\,\mathbf{saw}\,u\,v))))$
$\rightsquigarrow \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{saw}\,u\,v))\,;\mathbf{5}_u\,;\mathbf{3}_v$

(29) $\mathbb{M}\,(\lambda U.\,\mathbb{B}\,(\lambda V.\,V\,(\lambda v.\,U\,(\lambda u.\,\mathbf{saw}\,u\,v))))$
$\rightsquigarrow \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v))\,;\mathbf{3}_v\,;\mathbf{5}_u$

(30) $\mathbb{M}\,(\lambda U.\,\mathbb{B}\,(\lambda V.\,U\,(\lambda u.\,V\,(\lambda v.\,\mathbf{saw}\,u\,v))))$
$\rightsquigarrow \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{saw}\,u\,v))\,;\mathbf{3}_v\,;\mathbf{5}_u$

For example, reversing the relative scopes of the 'traces' V and U in (27) delivers (28), which amounts to doing dref introduction and maximization for movie-seeing boys and boy-seen movies in the opposite order of (27) — differences from (27) are highlighted in blue. We can also switch the relative scopes of the higher-order dynamic GQs themselves, as in (29); the upshot here is to reverse the orders of the cardinality tests $\mathbf{3}_v$ and $\mathbf{5}_u$. And, of course, we can combine these two manipulations simultaneously, as in (30).

In each of (27)–(30), we end up introducing some boys and some movies (though not always in that order), retaining the ones who stand in the seeing relation, maximizing, and (finally) counting. In none of these cases do we count before we maximize, and so we derive the cumulative reading across-the-board.[14]

This immediate success is encouraging, but there's some work to be done. First: we have yet to consider (and rule out) the remaining two scopings, which derive (distinct) pseudo-cumulative readings. Second: while carrying out multiple involved β-reduction chains might convince you that higher-order dynamic GQs are working as I say they are — at least for the cases considered in (27)–(30) — this is probably not particularly illuminating for readers who wish to understand *why* higher-order dynamic GQs behave in this way, or who wish to poke and prod to see whether things might go awry in other cases not considered here. Higher-order types are genuinely tricky to reason about, and confusion will only multiply as we consider various predictions and elaborations of the

12 In general, a sentence with n higher-order modified numerals allows $\frac{(2n)!}{2^n}$ different scopings. Precisely $(n!)^2$ of these will amount to equivalent cumulative readings, and the remainder will turn out to be different varieties of pseudo-cumulative readings. We'll see a general way to rule out pseudo-cumulative readings in Section 4.

13 A subtlety about indices should be noted. If the subject and object DPs bear the same index, an impossible (and quite bizarre) reading results, with maximal pluralities of self-watching movies, which are simultaneously three- and five- movies large. This possibility can be ruled out by requiring $\mathbf{E}^v$ to presuppose the novelty of v, i.e., with $i\,(\mathbf{E}^v\,P)\,j$ defined only if $v \notin \mathrm{Dom}\,i$ (cf. Heim 1982). (See also footnote 23.)

14 Two points. First, one could speculate that this (limited) degree of scopal *commutativity* has something to do with Krifka's (1999) intuition that cumulative readings result from scopal *independence*. (The update-theoretic meanings for modified numerals will end up *totally* commutative! See Section 6 for more on this point.) Second, I am not committed to all of these derivations being possible. Additional constraints such as Fox's (2000) Scope Economy might well rule out (28)–(30) on the grounds that they fail to generate new interpretations.

basic account. To help make this undertaking as accessible as possible, to clarify the basic underpinnings of the higher-order dynamic GQ theory, and to dispense with those pesky β-reductions, Section 3.4 introduces *towers* as a bit of notational sugar.

3.4 Towers for scope-takers and scope-taking

This section introduces a tower notation, pioneered in continuations-based approaches to scope-taking (Barker 2002, Shan & Barker 2006, Barker & Shan 2008, 2014; see also Elbourne 2011: 92–4 for an informal introduction, and Charlow 2014 for a generalization of the approach). Let me emphasize at the outset that towers are in this case nothing more than an *expository* device. As pointed out by Szabolcsi (2010: 32) (cf. also Szabolcsi 2011), though continuations- and LF- based approaches may initially appear unrelated, they are in actuality two (largely) equivalent ways of building scope arguments for things that take scope.[15] The reason to use towers here is two-fold: (i) towers make it easy to see how the higher-order entries for modified numerals allow cardinality tests to take scope separately from dref introduction and maximization; (ii) unlike β-reduction, tower combination converges virtually instantaneously on a logical form. Formal details are sketched only in broad strokes here; see Appendix A for a more detailed treatment.

Our entries for modified numerals have multiple 'levels'. In the inner, underbraced levels of (24) and (25) (repeated below) lives a dynamic GQ, which introduces and then maximizes a plurality of boys or movies satisfying some property k. Cardinality tests, meanwhile, live on a higher plane, separate from the higher-order meaning's type-$\mathtt{Q}$ core. This separation allows the cardinality tests to take scope *separately from* dref introduction and maximization — allowing cumulative readings to be derived — even as all these pieces coexist within unitary meanings for modified numeral DPs.[16]

(24) $[\![\text{exactly}^v \text{ three boys}]\!] := \lambda c.\, c\,(\underbrace{\lambda k.\, \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\, k\,v)}_{\mathtt{Q} ::= (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}})\,;\, \mathbf{3}_v$ type: $(\mathtt{Q} \to \mathtt{t}) \to \mathtt{t}$

(25) $[\![\text{exactly}^u \text{ five movies}]\!] := \lambda c.\, c\,(\underbrace{\lambda k.\, \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\, k\,u)}_{\mathtt{Q} ::= (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}})\,;\, \mathbf{5}_u$ type: $(\mathtt{Q} \to \mathtt{t}) \to \mathtt{t}$

This separation becomes much easier to see when we adopt an alternative notation for expressions that take scope. Following Barker & Shan (2008, 2014) we write linear expressions like $\lambda k. \ldots k\,x \ldots$ in terms of *towers*, as $\frac{\ldots[\;]\ldots}{x}$. Figure 5 gives some examples of how the tower convention applies to the different sorts of modified numeral meanings

15 A difference in the present case may bear mentioning. LF-oriented (that is, syntactic) treatments of scope must establish some kind of formal link between a scope-taker and its trace. Generally, this happens via assignment function modification (e.g., the classic treatment of Heim & Kratzer 1998). In the presence of type-$\mathtt{Q}$ traces, then, assignment functions must be able to harbor values of type $\mathtt{Q}$. However, in a dynamic setting this has the potential to lead to paradoxical conclusions about the cardinalities of types a and $\mathtt{Q}$ (as pointed out by Muskens 1995: 179–80 for a slighty different case). This issue doesn't arise in non-LF approaches to scope.

16 Other researchers have proposed higher-order functions to allow a single expression to make different parts of its meaning felt in different places. See, e.g., de Swart's (2000) account of split-scope effects for negative quantifiers, Barker & Shan's (2008) account of donkey anaphora out of DP (which relies on higher-order static meanings for quantified DPs), and especially Bumford's (2016) account of certain "split-scope" effects for definite descriptions (which builds in part on earlier versions of the present work).

$$\underbrace{\begin{array}{c} \mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;[\;])\,;\mathbf{3}_v \\ \hline v \end{array}}_{\lambda k.\,\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;k\,v)\,;\mathbf{3}_v} \qquad \underbrace{\begin{array}{c} [\;] \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;[\;])\,;\mathbf{3}_v \\ \hline v \end{array}}_{\lambda c.\,c\,(\lambda k.\,\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;k\,v)\,;\mathbf{3}_v)} \qquad \underbrace{\begin{array}{c} [\;]\,;\mathbf{3}_v \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;[\;]) \\ \hline v \end{array}}_{\lambda c.\,c\,(\lambda k.\,\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;k\,v))\,;\mathbf{3}_v}$$

Figure 5: Three instantiations of the tower notation for various regular and higher-order dynamic GQs. From left to right, the towers represent the basic meaning for dynamic GQs (only deriving pseudo-cumulative readings), the inessentially higher-order meaning (the Lift of the left-most tower, which again only derives pseudo-cumulative meanings), and — our actual proposal — the irreducibly higher-order meaning (which allows us to derive cumulative readings).

we've considered. The basic dynamic GQ meaning for *exactly*^{v} *three boys* becomes the left-most tower. The Lift-ed higher-order dynamic GQ becomes the middle tower (here, we apply the tower convention twice; first to $\lambda c. \ldots$ and second to $\lambda k. \ldots$). Finally, the irreducibly higher-order meaning — our proposal — becomes the right-most tower (again, we apply the tower convention twice).

The tower notation makes evident the separation inherent in the irreducibly higher-order dynamic GQs. In the right-most tower of Figure 5, the cardinality check occupies its own tier, separate from dref introduction and maximization. In the left two towers, by contrast, these three components all occupy the same level. Thus, the irreducibly higher-order dynamic GQ scopally distinguishes parts of its meaning that the other towers conflate. This highlights how the irreducibly higher-order GQ alone allows the top-level cardinality test to take scope separately from the mid-level dynamic effects.

Analogously to the tower convention for values, we can adopt a tower notation for types. Anything of type $(A \to B) \to C$ is a potential scope-taker, something that functions locally as an A, takes scope over a B, and ultimately yields a C. Any type of this form can be written as $\frac{C \mid B}{A}$. In most of the cases considered here, B and C will be identical types, in which case I'll simply write $\frac{B}{A}$. Thus, for example, the type of a standard dynamic GQ can be written $\mathtt{Q} ::= \frac{\mathtt{t}}{\mathtt{e}}$, and the type of a higher-order dynamic GQ can be written $\frac{\mathtt{t}}{\mathtt{Q}}$.

Besides making it clear how the irreducibly higher-order dynamic GQ differs from its counterparts in Figure 5, tower notation helps clarify how higher-order scope-takers interact with each other to derive meanings for complex expressions. Consider Figure 6, which presents a towers-based derivation of sentence (1), alongside Figure 7, which summarizes the key conventions used for manipulating tower representations.[17]

In Figure 6, the higher-order subject and object DPs are recast as towers (applying the tower conventions for values and types). The transitive verb, in turn, enters the derivation twice-Lift-ed, which enables it to tower-compose with its higher-order object and subject. (Because the Lift operation is polymorphic, a Lift-ed expression presupposes nothing about what sorts of things it can take scope over or in. I use ⌜?⌝, which can be resolved to any type, as a way to index this polymorphism.) The derivation proceeds by

17 In all of the towers-based derivations, to keep things as simple as possible, I'll abstract away from the overt movement of the subject DP from a *v*P-internal position to Spec-TP (cf. Figures 2 and 3).

$$
\begin{array}{c} \texttt{t} \\ \hline \texttt{t} \\ \hline \texttt{e} \\ \text{exactly}^{v}\ \text{three boys} \\ [\ ]\,;\mathbf{3}_v \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\ ]) \\ \hline v \end{array}
\left(\begin{array}{c} ? \\ \hline ? \\ \hline \texttt{e}\to\texttt{e}\to\texttt{t} \\ \text{saw} \\ [\ ] \\ \hline [\ ] \\ \hline \mathbf{saw} \end{array}
\quad
\begin{array}{c} \texttt{t} \\ \hline \texttt{t} \\ \hline \texttt{e} \\ \text{exactly}^{u}\ \text{five movies} \\ [\ ]\,;\mathbf{5}_u \\ \hline \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\ ]) \\ \hline u \end{array} \right)
$$

$$
\xrightarrow{\text{Combine}} \begin{array}{c} [\ ]\,;\mathbf{5}_u\,;\mathbf{3}_v \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\ ])) \\ \hline \mathbf{saw}\,u\,v \end{array}
$$

$$
\xrightarrow{\text{Lower}} \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v))\,;\mathbf{5}_u\,;\mathbf{3}_v
$$

Figure 6: Deriving the cumulative reading of *exactly^v three boys saw exactly^u five movies* using higher-order dynamic GQs. *Saw* enters the derivation twice-Lift-ed. Two applications of Combine glue the sentence together (with both occurences of the polymorphic type ? resolved to t), and the result is finally Lower-ed into the cumulative logical form.

$$
\begin{array}{c} A \\ \text{exp} \\ x \end{array} \xrightarrow{\text{Lift}} \begin{array}{c} ? \\ \hline A \\ \text{exp} \\ [\ ] \\ \hline x \end{array}
\qquad\Bigg|\qquad
\begin{array}{c} D \\ \hline C \\ \hline A\to B \\ \text{left} \\ g\,[\ ] \\ \hline h\,[\ ] \\ \hline f \end{array}
\quad
\begin{array}{c} D \\ \hline C \\ \hline A \\ \text{right} \\ i\,[\ ] \\ \hline j\,[\ ] \\ \hline x \end{array}
\xrightarrow{\text{Combine}}
\begin{array}{c} D \\ \hline C \\ \hline B \\ \text{left right} \\ g\,[i\,[\ ]] \\ \hline h\,[j\,[\ ]] \\ \hline f\,x \end{array}
\qquad\Bigg|\qquad
\begin{array}{c} A \\ \hline A \\ \hline A \\ \text{exp} \\ f\,[\ ] \\ \hline g\,[\ ] \\ \hline x \end{array}
\xrightarrow{\text{Lower}}
\begin{array}{c} A \\ \text{exp} \\ f\,[g\,[x]] \end{array}
$$

Figure 7: The rules of the road: Lift, Combine, and Lower. In order: Lift creates a polymorphic tower (with the polymorphic 'result' type represented as ⌜?⌝); Combine applies a function to an argument on the bottom level and plugs upper-right levels into upper-left levels (a symmetric version of Combine, with the function-argument relationship reversed, is omitted); Lower ends a derivation by collapsing a tower into a linear term. Here, Combine and Lower are characterized for three-level towers. The fully general forms of these rules work for towers of arbitrary height.

iteratively Combine-ing the towers — at each turn, doing functional application on the bottom levels, and plugging the upper-right levels into the upper-left levels (as Figure 7 suggests, tower combination is defined only when the types on upper levels match). The result is a composite three-story tower (a higher-order dynamic generalized *propositional* quantifier, for those keeping score), which is finally Lower-ed into a cumulative logical form corresponding to (27).

The remaining cumulative derivations, summarized in (28)–(30), can be replicated in terms of towers via various kinds of Lift-style manipulations (again, see Appendix A for more formal detail). For example, (28), which reverses the relative scopes of the subject and object 'traces', can be derived by Lift-ing the subject and object in different ways, as in (31). Notice in particular that, post-Lift, the object's dref introduction and maximization operators occupy a higher tier than those of the subject. The effect here is to give the object's dref introduction and maximization operators scope over those of the subject, just as if the object's type-Q trace was taking inverse scoping over the subject's (in general, things on higher levels out-scope things on lower levels).

(31)
$$\begin{array}{c} [\,]\,;\mathbf{3}_v \\ \hline [\,] \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\,]) \\ \hline v \end{array} \left(\begin{array}{c} [\,] \\ \hline [\,] \\ \hline [\,] \\ \hline \mathbf{saw} \end{array} \quad \begin{array}{c} [\,]\,;\mathbf{5}_u \\ \hline \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\,]) \\ \hline [\,] \\ \hline u \end{array} \right)$$

$$\xrightarrow{\text{Combine, Lower}} \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{saw}\,u\,v))\,;\mathbf{5}_u\,;\mathbf{3}_v$$

As in Figure 6, Combine does functional application on the bottom levels, while plugging upper-right levels into upper-left levels. The result after Lower-ing is identical to the logical form given in (28). Similar tower derivations, using Lift to generate new scopings, can be given for the remaining cumulative logical forms in (29) and (30).

4 Pseudo-cumulative readings

4.1 Deriving pseudo-cumulative readings

As we have seen, higher-order dynamic meanings for modified numeral DPs generate six different scopings (and six logical forms) for sentences with two modified numerals. Four of these amount to equivalent cumulative readings, which seems like progress. But this comes at a price: the remaining two derivations represent distinct pseudo-cumulative readings. First, let's see what these amount to in terms of the linear notation:

(32) $\mathbb{B}\,(\lambda V.\,V\,(\lambda v.\,\mathbb{M}\,(\lambda U.\,U\,(\lambda u.\,\mathbf{saw}\,u\,v))))$
$\rightsquigarrow \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)\,;\mathbf{5}_u)\,;\mathbf{3}_v$

(33) $\mathbb{M}\,(\lambda U.\,U\,(\lambda u.\,\mathbb{B}\,(\lambda V.\,V\,(\lambda v.\,\mathbf{saw}\,u\,v))))$
$\rightsquigarrow \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{saw}\,u\,v)\,;\mathbf{3}_v)\,;\mathbf{5}_u$

Pseodu-cumulative derivations happen when one higher-order modified numeral scopes under the other's type-Q 'trace'. Here, the key scope-bearing elements and their reflexes in the resulting logical forms are highlighted in blue. In (32), which corresponds to the pseudo-cumulative LF from Figure 3 (page 6), $\mathbb{M}$ scopes under V. The resulting reading merely requires the biggest pluralities of boys who between them saw exactly five movies to be three-boys-large. In (33), $\mathbb{B}$ scopes under U. The resulting reading merely requires the biggest pluralities of movies seen by exactly three boys to be five-movies-large.

As before, towers help us appreciate what's going on. The result in (32), for example, can be replicated by Lift-ing the subject's $\mathbf{M}_v$ and $\mathbf{E}^v\,\mathbf{boys}$ operators to a position where they scope over $\mathbf{5}_u$, as in (34). A couple applications of Combine, followed by a Lower, is all that's needed to derive a pseudo-cumulative logical form.

(34)
$$\begin{array}{c} [\ ]\,;\mathbf{3}_v \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\ ]) \\ \hline [\ ] \\ \hline v \end{array} \left(\begin{array}{c} [\ ] \\ \hline [\ ] \\ \hline [\ ] \\ \hline \mathbf{saw} \end{array} \quad \begin{array}{c} [\ ] \\ \hline [\ ]\,;\mathbf{5}_u \\ \hline \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\ ]) \\ \hline u \end{array} \right)$$

$$\xrightarrow{\text{Combine, Lower}} \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{saw}\,u\,v)\,;\mathbf{5}_u)\,;\mathbf{3}_v$$

A corresponding towers-based derivation of (33) is possible, too. This time, we Lift the object's $\mathbf{M}_u$ and $\mathbf{E}^u\,\mathbf{movs}$ operators to a higher level than $\mathbf{3}_v$, Combine, and Lower.

Higher-order meanings for modified numerals correctly generate cumulative readings. But does the ability to generate pseudo-cumulative readings represent an countervailing over-generation concern? Brasoveanu (2013) argues that it does, pointing to the infelicity of texts like (35): since the 'surface-scope' pseudo-cumulative reading derived in (32) and (34) is true in situations like Scenario B, where four boys saw six movies, it should be reasonable to highlight the possibility of additional boy-movie-seeings in a follow-up sentence. But (35) doesn't seem reasonable — it seems contradictory. Similarly, if pseudo-cumulative readings existed, (36) should count as a felicitous elaboration of *exactly three boys saw exactly five movies* (the sentence in parentheses rules out interference from the surface-scope distributive construal). This looks to be incorrect.

(35) Exactly three boys saw exactly five movies. #Perhaps there was another boy that saw a different movie, but I didn't notice him. (Brasoveanu 2013: 157, ex. 4)

(36) #What's more, at least four boys saw movies. (But none saw more than three.)

To these points, one might add that *exactly three boys saw exactly five movies* simply seems impossible to hear as true in Scenario B, which itself militates quite strongly against the existence of a pseudo-cumulative reading.

The issue of pseudo-cumulative readings becomes even more acute when we consider sentences with more than two modified numerals. For example, on the higher-order GQ approach, (37) turns out to admit no fewer than 90 different scopings (see fn. 12)!

(37) Exactly three boys watched exactly four movies with exactly five girls.

Exactly 36 of these scopings amount to equivalent cumulative readings (fn. 12), but the remainder is comprised of various pseudo-cumulative readings (including readings mixing cumulativity and pseudo-cumulativity!). Such ambiguity is clearly not in evidence for such constructions, which only seem to allow purely cumulative readings, purely distributive readings, and readings with various mixtures of cumulativity and distributivity (distributive readings are discussed in Sections 4.3 and 6.3).

4.2 Avoiding the pseudo-cumulative reading with subtyping

Pseudo-cumulative readings result when one higher-order modified numeral takes scope under another's type-$\mathtt{Q}$ 'trace'. In this section, I'll sketch a general way to rule out these kinds of scopings, in terms of something called *subtype polymorphism* (aka subtyping).

The intuition behind subtyping is simple: types can exhibit mereological structure. For example, while it's reasonable (both linguistically and intuitively) to think of *eventualities* as entities — i.e., as members of type e (e.g., Bach 1986, Parsons 1990), it does not necessarily make much sense to think of all the members of type e (e.g., humans) as eventualities. This distinction can be reflected in terms of a subtyping relationship, as in (38): v, the type of eventualities, is a subtype of e, the type of entities, but the converse does not hold. In prose, if x has type v, we know that x has type e, but if x has type e, we cannot on that basis alone know whether x has type v.[18]

(38) $v \sqsubset e \qquad e \not\sqsubset v$

(See, e.g., Johnson & Bayer 1995, Bernardi & Szabolcsi 2008 for linguistically oriented investigations of subtyping, in terms of the related notion of partially ordered categories.)

Subtyping can be used to prevent modified numerals from taking scope under other modified numerals' traces.[19] We begin by distinguishing two types, $\mathtt{t}$ and $\mathtt{T}$, and extending the subtyping relation as in (39). Intuitively, we can think of $\mathtt{t}$ as the type of a 'not necessarily complete' sentence, and of $\mathtt{T}$ as the type of a 'definitely complete' sentence. Since $\mathtt{t} \sqsubset \mathtt{T}$, we can always decide that a not-necessarily-complete sentence (type $\mathtt{t}$) is definitely complete (type $\mathtt{T}$), but we can't, in general, conclude the reverse.

(39) $\mathtt{t} \sqsubset \mathtt{T} \qquad \mathtt{T} \not\sqsubset \mathtt{t}$

Subtyping relations entail further subtyping relationships between functions, with a general rule given in (40). Notice that we require $A' \sqsubset A$ (i.e., the subtyping relation is *contravariant* for function arguments). Here's why. If a function $f : A \to B$ meets an argument $x : A'$, the fact that $A' \sqsubset A$ lets us conclude that $x : A$, in which case $f\,x$ is well-typed (and has type B). Thus we conclude that f accepts any $x : A'$ (because in such cases it is always possible to conclude that $x : A$). Since tower types $\frac{B}{A}$ are just abbreviations for $(A \to B) \to B$, we also have the subtyping relation between type towers in (41) (for similicity, we assume that $\sqsubset$ is a partial order, i.e., antisymmetric).

(40) $A \to B \sqsubset A' \to B' \iff A' \sqsubset A \wedge B \sqsubset B'$

(41) $\frac{B}{A} \sqsubset \frac{B'}{A'} \iff A \sqsubset A' \wedge B = B'$ (i.e., $A \sqsubset A' \wedge B \sqsubset B' \wedge B' \sqsubset B$)

18 See Pierce (2002: 392) for a typing calculus with polymorphism and subtyping.

19 Kubota & Uegaki (2009) exploit a related strategy to force benefactive inferences in Japanese to take widest scope (see Barker, Bernardi & Shan 2010 for discussion and elaboration of Kubota & Uegaki's approach).

$$\underset{\text{e3b's}}{\begin{array}{c}\mathtt{T}\\\hline\mathtt{t}\\\hline\mathtt{e}\end{array}}\left(\underset{\text{saw}}{\begin{array}{c}?\\\hline ?\\\hline \mathtt{e}\to\mathtt{e}\to\mathtt{t}\end{array}}\quad\underset{\text{e5m's}}{\begin{array}{c}\mathtt{T}\\\hline\mathtt{t}\\\hline\mathtt{e}\end{array}}\right)\xrightarrow{\textsf{Combine}}\begin{array}{c}\mathtt{T}\\\hline\mathtt{t}\\\hline\mathtt{t}\end{array}\xrightarrow{\textsf{Lower}}\begin{array}{c}\mathtt{T}\\\hline\mathtt{t}\end{array}\xrightarrow{\sqsubset}\begin{array}{c}\mathtt{T}\\\hline\mathtt{T}\end{array}\xrightarrow{\textsf{Lower}}\mathtt{T}$$

Figure 8: An intermediate stage in the derivation of the cumulative reading of *exactly*$^{\nu}$ *three boys saw exactly*^{u} *five movies*, using subtyping. The towers compose via Combine. We collapse the bottom two stories with Lower to yield a two-story tower. Because $\mathtt{t} \sqsubset \mathtt{T}$, we can conclude that this expression is type $\frac{\mathtt{T}}{\mathtt{T}}$, and can thus be Lowered once more into something with type T, concluding the derivation and yielding the same result as Figure 6.

$$\underset{\text{e3b's}}{\begin{array}{c}\mathtt{T}\\\hline\mathtt{t}\\\hline ?\\\hline\mathtt{e}\end{array}}\left(\underset{\text{saw}}{\begin{array}{c}?\\\hline ?\\\hline ?\\\hline \mathtt{e}\to\mathtt{e}\to\mathtt{t}\end{array}}\quad\underset{\text{e5m's}}{\begin{array}{c}?\\\hline\mathtt{T}\\\hline\mathtt{t}\\\hline\mathtt{e}\end{array}}\right)\xrightarrow{\textsf{Combine}}\text{type error!}$$

Figure 9: An attempted pseudo-cumulative derivation of *exactly*$^{\nu}$ *three boys saw exactly*^{u} *five movies.* As in (34), we Lift the subject's second-level effects to the third level, in an attempt to give them scope over the object's cardinality test. However, the third-level types for the subject (t) and the object (T) are *incompatible.* This results in a type error, and the derivation fails.

These preliminaries out of the way, we propose that the type of **saw** (and other transitive verbs) is $\mathtt{e} \to \mathtt{e} \to \mathtt{t}$, and that higher-order modified numerals have the (tower) types depicted in Figure 8. The central idea is that the mid-level dynamic effects of modified numerals (dref introduction and maximization) take scope over and within a not-necessarily-complete sentence, while their top-level effects (cardinality tests) take scope over and within a definitely-complete sentence. As Figure 8 bears out, this doesn't disrupt the derivation of cumulative readings. As in Figure 6, the towers compose via Combine to give a three-story composite tower. This is then Lower-ed into a two-story tower by collapsing the bottom two levels to yield a tower with type $\frac{\mathtt{T}}{\mathtt{t}}$ (as in Figure 7, Lower requires the types of the collapsed levels to match). The fact that $\mathtt{t} \sqsubset \mathtt{T}$ lets us conclude that $\frac{\mathtt{T}}{\mathtt{t}} \sqsubset \frac{\mathtt{T}}{\mathtt{T}}$ (see (41)). With matching types on the top and bottom levels, we Lower once more to yield something of type T, thereby deriving the cumulative reading.

By contrast, the pseudo-cumulation derivation no longer goes through. See Figure 9. As in (34), we begin by lifting the subject's mid-level effects (dref introduction and maximization) onto the third story. However, a problem immediately arises: the third-level types of the subject and the object are *incompatible* (cf. Figure 7, where Combine requires upper-level types to match). In other words, the third-level effects of the subject and the object have irreconcilable differences: the former are expecting to take scope over a possibly-incomplete sentence (i.e., something of type t), but the latter insist on returning a definitely-complete sentence (i.e., something of type T)! As such, the derivation triggers a type error and fails.[20]

This fundamental incompatibility between the type expectations of mid-level effects (dref introduction and maximization) and top-level effects (cardinality tests) works to rule out *all possible* pseudo-cumulative readings. This includes 'inverse-scope' pseudo-cumulative readings like (33), along with the vast menagerie of pseudo-cumulative readings previously derived for sentences like (37) with three or more modified numeral DPs. Because mid-level effects cannot scope over top-level effects, cardinality tests must always take widest scope, and pseudo-cumulative readings are correctly ruled out.

As before, though the tower notation helps us appreciate this result, it's not playing a crucial role in the explanation. In (42), we see a linear representation of the cumulative reading. Here, given the type assignments to higher-order modified numerals in Figure 8, the 'trace' V of *exactly*^{v} *three boys* will be of type $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$, and therefore expects its (underbraced) argument to have type $\mathtt{e} \to \mathtt{t}$. Because $\mathbb{M}$ doesn't occur within V's scope in (42), this expectation is met.[21] However, in the pseudo-cumulative (43), V's expectations are unsatisfiable. Because $\mathbb{M}$ has a type of the form $(A \to \mathsf{T}) \to \mathsf{T}$, V's (underbraced) argument will be of type $\mathtt{e} \to \mathsf{T}$, and not of type $\mathtt{e} \to \mathtt{t}$. This results in a type error.

(42) $\mathbb{B}\,(\lambda V.\,\mathbb{M}\,(\lambda U.\,V\,(\underbrace{\lambda v.\,U\,(\lambda u.\,\textbf{saw}\,u\,v)}_{\mathtt{e}\to\mathtt{t}})))$ type: T

(43) $\mathbb{B}\,(\lambda V.\,V\,(\underbrace{\lambda v.\,\mathbb{M}\,(\lambda U.\,U\,(\lambda u.\,\textbf{saw}\,u\,v))}_{\mathtt{e}\to\mathsf{T}}))$ type error!

It is also possible to appreciate how subtyping rules out pseudo-cumulative readings on a more granular level. Consider the type assignments in (44). Crucially, $\mathbf{M}_v$ insists that its argument has type $\mathtt{t}$, and $\mathbf{3}_v$ insists that it is of type T. Dynamic conjunction is polymorphic: it can either have type $\mathtt{t} \to \mathtt{t} \to \mathtt{t}$, or $\mathsf{T} \to \mathsf{T} \to \mathsf{T}$.

(44) $\textbf{saw} : \mathtt{e} \to \mathtt{e} \to \mathtt{t}$ $\quad \mathbf{E}^{v}\,\textbf{boys} : \mathtt{t}$ $\quad \mathbf{3}_v : \mathsf{T}$

$\mathbf{M}_v : \mathtt{t} \to \mathtt{t}$ $\quad (\,;\,) : A \to A \to A \mid A \in \{\mathtt{t}, \mathsf{T}\}$

The types in (44) are consistent with (in fact, entail) the types posited for higher-order modified numerals in Figure 8. And they have a rather remarkable further consequence: they render pseudo-cumulative *logical forms* not-well-typed! Consider (45) and (46).

(45) $\mathbf{M}_v\,(\underbrace{\mathbf{E}^{v}\,\textbf{boys}\,;\,\mathbf{M}_u\,(\mathbf{E}^{u}\,\textbf{movs}\,;\,\textbf{saw}\,u\,v)}_{\mathtt{t}})\,;\,\mathbf{5}_u\,;\,\mathbf{3}_v$ type: T

(46) $\mathbf{M}_v\,(\underbrace{\mathbf{E}^{v}\,\textbf{boys}\,;\,\mathbf{M}_u\,(\mathbf{E}^{u}\,\textbf{movs}\,;\,\textbf{saw}\,u\,v)\,;\,\mathbf{5}_u}_{\mathsf{T}})\,;\,\mathbf{3}_v$ type error!

20 The actual proof that Figure 8 yields a type error is a bit more involved than these informal remarks indicate. We confine our attention to the bottom three levels of the subject and object towers, each of which has a type of the form $\frac{B}{A} ::= \frac{B|B}{A}$, for $B \in \{\mathtt{t}, \mathsf{T}\}$. When B is $\mathtt{t}$ (as with the subject DP), we have $\frac{\mathtt{t}|\mathtt{t}}{A} \sqsubset \frac{\mathsf{T}|\mathtt{t}}{A}$ (by the subtyping rule for functions). When B is T (as with the object DP), we have $\frac{\mathsf{T}|\mathsf{T}}{A} \sqsubset \frac{\mathsf{T}|\mathtt{t}}{A}$. Though these inferred types are identical, they are not compatible: the subject's tower (type $\frac{\mathsf{T}|\mathtt{t}}{A}$) expects to scope over something of type $\mathtt{t}$, but the object's tower (type $\frac{\mathsf{T}|\mathtt{t}}{A}$) insists on returning something of type T. These demands cannot both be met, and so the derivation necessarily fails.

21 $\mathbb{M}$ also expects *its* argument to be of type $((\mathtt{e} \to \mathtt{t}) \to \mathtt{t}) \to \mathsf{T}$. No problem: $((\mathtt{e} \to \mathtt{t}) \to \mathtt{t}) \to \mathtt{t} \sqsubset ((\mathtt{e} \to \mathtt{t}) \to \mathtt{t}) \to \mathsf{T}$.

Per (44), $\mathbf{M}_v$ wants its argument to have type $\mathtt{t}$. Though this is possible in the cumulative logical form (45) (the underbraced portion can be assigned type $\mathtt{t}$), the situation is different in the pseudo-cumulative (46). There, the occurrence of $\mathbf{5}_u$ (type T) acts like a ratchet, forcing the immediately preceding dynamic conjunction to be instantiated with type $\mathsf{T} \to \mathsf{T} \to \mathsf{T}$, and ultimately forcing the entire underbraced portion to have type T. This is, of course, contrary to the wishes of $\mathbf{M}_v$, and a type error results.

4.3 Distributive readings

With subtyping, *exactly*^{v} *three boys saw exactly*^{u} *five movies* is predicted to have just one reading — a cumulative one. As Brasoveanu (2013: 176–80) emphasizes, however, there's at least one more reading for the sentence — a (surface-scope) *distributive* construal — and it turns out to require the object's cardinality test to end up in the scope of the subject's trace (precisely the situation which the absence of pseudo-cumulative readings led us to deem impossible)! This section shows how distributive readings are derived with a simple addition: a distributivity operator that 'resets' its scope to type $\mathtt{t}$.

The (surface-scope) distributive interpretation of *exactly*^{v} *three boys saw exactly*^{u} *five movies* is about a maximal plurality of boys, *each of whom* saw exactly five movies. As the gloss suggests, in this reading, the subject introduces various pluralities of boys, which the distributivity operator catches and divides into atom-sized chunks, each of which is required to have seen exactly five movies. (The pluralities of boys who each saw exactly five movies are then maximized and measured.) This can be captured with a logical form like (47), which relies on a distributivity operator $\mathbf{D}_v$, defined in (48).

(47) $\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;\mathbf{D}_v\,(\lambda v'.\,\mathbf{M}_u\,(\mathbf{E}^u\,\textbf{movs}\,;\textbf{saw}\,u\,v')\,;\mathbf{5}_u))\,;\mathbf{3}_v$

(48) $i\,(\mathbf{D}_v\,k)\,j \coloneqq i = j \wedge \forall x \leqslant_{\mathsf{at}} i_v : \exists h : i^{v \to x}\,(k\,v)\,h$ $\qquad \mathbf{D}_v : (\mathsf{e} \to \mathsf{T}) \to \mathtt{t}$

Definition (48) has two key parts: semantic, and type-theoretic. I discuss these in order. Semantically, $\mathbf{D}_v$ is rather mundane: it universally quantifies over the atoms of a plurality stored in v, and requires each of them to satisfy some property k — compare the entry for $\textbf{every.ling}^v$ given in (16). This definition is based on Charlow (2014: 100), where null distributivity operators apply to non-atomic (dynamic) individuals and turn them into scope-taking universal quantifiers (the lexical semantics of the null distributivity operator can then be given as $\lambda v.\,\mathbf{D}_v$).

The definition of $\mathbf{D}_v$ yields a reasonable distributive meaning for (47). But is this logical form actually *derivable* when $\mathtt{t} \sqsubset \mathsf{T}$, and higher-order modified numerals are assigned the types in Figure 8? (Alternatively, given the type assignments in (44), we might ask whether (47) is itself well-typed.) It is. (And it is.) The key is the type assigned to $\mathbf{D}_v$, $(\mathsf{e} \to \mathsf{T}) \to \mathtt{t}$. Unlike anything else we've seen so far, $\mathbf{D}_v$ has the ability to dial back the type, from a definitely-complete T to a not-necessarily-complete $\mathtt{t}$. This allows higher-order modified numerals to occur within $\mathbf{D}_v$'s scope, even as $\mathbf{D}_v$ occurs within the scope of a modified numeral's type-$(\mathsf{e} \to \mathtt{t}) \to \mathtt{t}$ 'trace'.[22]

22 Notice in addition that the subtyping relation entails that $\mathbf{D}_v$ also has type $(\mathsf{e} \to \mathtt{t}) \to \mathtt{t}$. This guarantees, correctly, that $\mathbf{D}_v$ *may* take scope over a modified numeral's cardinality test, but is not *obliged* to do so.

A derivation of the distributive reading is given in (49). The left-most tower is gotten by applying the null distributivity operator to (the bottom story of) the tower for *exactly*^{v} *three boys*, then invoking the λ-theoretic equivalence of $\mathbf{D}_v$ and $\lambda k.\,\mathbf{D}_v\,(\lambda v'.\,k\,v')$ — i.e., $\frac{\mathbf{D}_v\,(\lambda v'.[\ ])}{v'}$ in the tower notation — and finally lifting $\mathbf{D}_v$ to the third level where it scopes over $\mathbf{5}_u$. After Combine and Lower, we're left with (47), the distributive logical form.

(49)
$$\begin{array}{c} [\ ]\,;\mathbf{3}_v \\ \hline \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\ ]) \\ \hline \mathbf{D}_v\,(\lambda v'.[\ ]) \\ \hline [\ ] \\ \hline v' \end{array} \left(\begin{array}{c} [\ ] \\ \hline [\ ] \\ \hline [\ ] \\ \hline [\ ] \\ \hline \mathbf{saw} \end{array} \quad \begin{array}{c} [\ ] \\ \hline [\ ] \\ \hline [\ ]\,;\mathbf{5}_u \\ \hline \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\ ]) \\ \hline u \end{array} \right)$$

Though the object's top-level effects end up in the scope of the subject's mid-level effects, this doesn't result in a type error. $\mathbf{D}_v$ returns something of type $\mathtt{t}$, which the 'mid-level' effects of *exactly*^{v} *three boys* are happy to take scope over.[23]

As above, the tower presentation corresponds to what's observed with linearized λ-terms like (50), as well as logical forms like (51) (when we pair (48) with the more granular type assignments in (44)). As in (49), though the object DP's cardinality test ratchets the type to T, $\mathbf{D}_v$ dials it back to type $\mathtt{t}$, allowing the end results to be well-typed.

(50) $\mathbb{B}\,(\lambda V.\,V\,(\underbrace{\lambda v.\,\mathbf{D}_v\,(\lambda v'.\,\mathbb{M}\,(\lambda U.\,U\,(\lambda u.\,\mathbf{saw}\,u\,v'))))}_{\mathtt{e}\to\mathtt{t}}))$ type: T

(51) $\mathbf{M}_v\,(\underbrace{\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{D}_v\,(\lambda v'.\,\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v')\,;\mathbf{5}_u)}_{\mathtt{t}})\,;\mathbf{3}_v$ type: T

I'll also note that (48) allows for mixed distributive-cumulative readings, which seems to be a good prediction. Cases like (37), repeated below, have a reading characterizing situations in which there are exactly three atomic boys x, such that x's movie-watchings with girls implicated exactly four movies and exactly five girls.

(37) Exactly three boys watched exactly four movies with exactly five girls.

Mixed readings have derivations more complicated, but not different in kind, from ones we've already seen. The post-verbal modified numerals compose as in cumulative derivations like Figure 8, after which the distributive subject swoops in, as in (49).

4.4 Wrapping up and looking ahead

All told, higher-order modified numerals offer a reasonably attractive theory of cumulative and distributive readings. When the cardinality test is cleaved off from mid-level

23 Two points. (i) If desired, the *inverse-scope* distributive reading can be derived by applying the null distibutivity operator to the bottom level of *exactly*^{u} *five movies*, then Lift-ing $\mathbf{D}_u$ high enough to take (inverse) scope over $\mathbf{3}_v$. This flexibility may be undesirable: inverse-scope distributive readings are known to be relatively restricted (e.g., Reinhart 2006, Szabolcsi 2010). (ii) If, on the other hand, $\mathbf{D}_u$ takes (inverse) scope over $\mathbf{M}_v$ but *under* $\mathbf{3}_v$, the result will be undefined: since $\mathbf{D}_u$ tosses out any drefs generated in its scope, and the modified numeral subject presupposes the novelty of index v (cf. fn. 13), $\mathbf{3}_v$ will end up searching in vain for a referent.

dynamic effects, cumulative readings are immediately generated using nothing more exotic than garden-variety scope-taking. Pseudo-cumulative readings can be ruled out with subtyping, and the account can be refined with a suitable meaning and type assignment for scopal distributivity operators.

That said, it is clear that the subtyping regime suggested here is rather stipulative. Why, after all, should top-level effects be associated with type $\mathtt{T}$, and mid-level effects with type $\mathtt{t}$? Why, after all, should distributivity operators have type $(\mathtt{e} \to \mathtt{T}) \to \mathtt{t}$?

In the following sections, I'll compare higher-order modified numerals with a post-suppositional theory, and then with a leaner theory couched in terms of *update semantics*. In Section 5, I'll argue that the stipulations about (sub)typing which prove crucial to the higher-order dynamic GQ theory are mirrored in the post-suppositional account, and that the post-suppositional account needs to make *additional* stipulations to account for the behavior of modified numerals on islands. In Section 6, I'll show that all of these stipulations can be put to bed with update-theoretic dynamic meanings for modified numerals, which gives a strong argument in favor of going the update-theoretic route, at least in this case. And in Section 7, I'll ask whether, given these results, there still might be some use for higher-order dynamic GQs (and answer in the affirmative).

5 Post-suppositions

5.1 Composition with post-suppositions

I begin by sketching a Montagovian type logic with post-suppositions (a compositionality-friendly re-formulation of Brasoveanu's post-suppositional dynamic logic). The basic idea is to add a second dimension of meaning (following, among others, Karttunen & Peters 1979, Rooth 1985, Abusch 1994, Potts 2005), and to put cardinality tests there. We begin by enriching the set of compositionally relevant types τ with ordered pairs of values and dynamic propositions, along with functions fro and to enriched types:[24]

(52) $\tau^+ ::= \tau \mid \tau \times \mathtt{t} \mid \tau^+ \to \tau^+$

Post-suppositional meanings for modified numerals can then be defined as in (53). This denotation is an ordered pair whose first member is a tower, and whose second member is a dynamic proposition. Crucially, the cardinality test is relegated to the second, post-suppositional dimension, separate from dref introduction and maximization.

(53) $[\![\text{exactly}^v \text{ three boys}]\!]^+ := \left(\dfrac{\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\;])}{v}, \mathbf{3}_v \right)$ $\quad$ type: $\dfrac{\mathtt{t}}{\mathtt{e}} \times \mathtt{t}$

Expressions lacking post-suppositions may become (trivially) post-suppositional via the general mapping defined in (54), where $\mathbf{T}$ is the dynamic tautology defined in (15). For example, applying η to $\mathbf{saw}$ yields $(\mathbf{saw}, \mathbf{T})$.

(54) $\eta\, x := (x, \mathbf{T})$ $\quad$ $\eta : A \to (A \times \mathtt{t})$

24 $S \times T$ is the type of ordered pairs whose first member has type S and whose second member has type T.

$$\frac{C}{A \to B} \times \mathtt{t} \qquad \frac{C}{A} \times \mathtt{t} \qquad\qquad \frac{C}{B} \times \mathtt{t}$$

$$\text{left} \qquad \text{right} \quad \xrightarrow{\mathsf{Combine}^{+}} \quad \text{left right}$$

$$\left(\frac{g\,[\;]}{f}, p\right) \quad \left(\frac{h\,[\;]}{x}, q\right) \qquad\qquad \left(\frac{g\,[h\,[\;]]}{f\,x}, p\,;q\right)$$

Figure 10: Tower combination with post-suppositions. In the left dimension, the towers compose as usual — i.e., via $\mathsf{Combine}$. In the right dimension, post-supposed context is accumulated via dynamic conjunction.

Given these meaning assignments, we can derive a meaning for *exactly^{v} three boys saw exactly^{u} five movies* as in (55). This derivation relies on $\mathsf{Combine}^{+}$, characterized in Figure 10, to glue its pieces together. $\mathsf{Combine}^{+}$ is a post-suppositional extension of tower combination — it does $\mathsf{Combine}$ in the first dimension, and *dynamic conjunction* of post-supposed content in the second dimension. The other post-suppositional tower operations, Lift^{+} and Lower^{+} (omitted here), work similarly: they perform Lift and Lower in the first dimension, leaving post-supposed content untouched. For example, in (55) the meaning for *saw* is derived by applying η and then Lift^{+} to $\textbf{saw}$.

(55)
$$\left(\frac{\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,;[\;])}{v}, \mathbf{3}_v\right)\left(\left(\frac{[\;]}{\textbf{saw}}, \mathbf{T}\right)\left(\frac{\mathbf{M}_u\,(\mathbf{E}^u\,\textbf{movs}\,;[\;])}{u}, \mathbf{5}_u\right)\right)$$
$$\xrightarrow{\mathsf{Combine}^{+},\ \mathsf{Lower}^{+}} (\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,; \mathbf{M}_u\,(\mathbf{E}^u\,\textbf{movs}\,; \textbf{saw}\,u\,v)), \mathbf{3}_v\,; \mathbf{5}_u)$$

After $\mathsf{Combine}^{+}$ and Lower^{+}, we get a bi-dimensional logical form. In the first dimension, we introduce the maximal pluralities of boys who saw movies, stored in v, and the maximal pluralities of movies seen by boys, stored in u. In the second, post-suppositional dimension, we dynamically conjoin the two cardinality tests, $\mathbf{3}_v$ and $\mathbf{5}_u$. (Notice that $\mathbf{T}$ has dropped out: $\mathbf{3}_v\,; \mathbf{T}\,; \mathbf{5}_u$ is equivalent to $\mathbf{3}_v\,; \mathbf{5}_u$.)

We can imagine a variety of ways that post-suppositions might be integrated with the main semantic content of an expression. One possibility is to upgrade the notion of truth as in (56): to check whether a pair of a dynamic proposition K and a post-supposed dynamic proposition L is $\textbf{true}^{+}$ at an assignment i, we simply check whether the dynamic conjunction $K\,;L$ is **true** at i. Thus, checking the bi-dimensional logical form derived in (55) is $\textbf{true}^{+}$ amounts to checking whether (57) is **true**. Because the cardinality tests both occur last in (57), the post-suppositional meaning derived in (55) has the truth conditions of the cumulative reading, which is the desired result.[25]

(56) (K, L) is $\textbf{true}^{+}$ at $i \iff K\,;L$ is **true** at i

(57) $\mathbf{M}_v\,(\mathbf{E}^v\,\textbf{boys}\,; \mathbf{M}_u\,(\mathbf{E}^u\,\textbf{movs}\,; \textbf{saw}\,u\,v))\,; \mathbf{3}_v\,; \mathbf{5}_u$

25 This architecture has something akin to cross-dimensional anaphora: drefs introduced in the first dimension are piped into the second dimension when the two are reconciled by $\textbf{true}^{+}$ (cf. Karttunen & Peters 1979). See Sudo (2014) for an account of presupposition projection in quantified sentences based on a similar idea.

In addition to being cashed out at the 'meta-level' — e.g., in the definition of truth — we might, with Brasoveanu (2013), imagine that post-suppositions can be discharged by lexical items. It's straightforward, for example, to define an operator that 'resets' an expression's post-suppositions by dynamically conjoining its first and second dimensions, as in (58). A further application of η will deliver $(K \,;L, \mathbf{T})$.

(58) $\quad (K, L)^{\bullet} \coloneqq K \,; L \qquad\qquad (\bullet) : (\mathtt{t} \times \mathtt{t}) \to \mathtt{t}$

Operators like (58) that reify post-suppositions are necessary. Because cardinality tests occupy a separate, post-suppositional dimension, they will by default percolate upwards indefinitely, not interacting with any 'suppositional' content. (It may be helpful to recall that such behavior is the central motivation for bi-dimensionality in alternative semantics; cf., e.g., Rooth 1985, 1996, Kratzer & Shimoyama 2002.) But post-supposed cardinality tests *can* be out-scoped — for example, by distributivity operators, as we have seen in Section 4.3. Thus, as Brasoveanu (2013: 176–80) points out, distributivity operators in a post-suppositional world must be able to unwind post-suppositions. (In alternative semantics, expressions like *only* play an analogous role, shuttling content from the second dimension into the first.)

A distributivity operator $\mathbf{D}_v^+$ with this behavior is defined in (59). $\mathbf{D}_v^+$ takes scope over a post-suppositional sentence and cashes out the post-suppositional content $(k\,v')_2$ by dynamically conjoining it with the non-post-supposed content $(k\,v')_1$. (The subscripted numerals are projection functions on ordered pairs: $(p, q)_1 \coloneqq p$, and $(p, q)_2 \coloneqq q$. $\mathbf{D}_v$ is the distributivity operator defined in (48), though since we have no use for subtyping in the post-suppositional framework, its type here is simply $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$.) If $\mathbf{D}_v^+$ takes scope over *exactly*^{u} *five movies*, the latter's post-supposition, $\mathbf{5}_u$, ends up in the scope of $\mathbf{D}_v$, as required for the distributive reading and depicted in (60).

(59) $\quad \mathbf{D}_v^+\, k \coloneqq \mathbf{D}_v\, (\lambda v'.\, (k\,v')_1 \,; (k\,v')_2) \qquad\qquad \mathbf{D}_v^+ : (\mathtt{e} \to (\mathtt{t} \times \mathtt{t})) \to \mathtt{t}$

(60) $\quad \big(\mathbf{M}_v\, (\mathbf{E}^v\, \textbf{boys} \,; \mathbf{D}_v\, (\underbrace{\lambda v'.\, \mathbf{M}_u\, (\mathbf{E}^u\, \textbf{movs} \,; \textbf{saw}\, u\, v') \,; \mathbf{5}_u})), \mathbf{3}_v\big)$

I pass over the details of how $\mathbf{D}_v^+$ compositionally works to derive distributive logical forms like (60) (interested readers can consult Appendix B for the formal details). The key point here is that $\mathbf{D}_v^+$ reifies post-suppositions, allowing them to be discharged with something other than maximal scope.

5.2 Comparing post-suppositions and higher-order dynamic GQs

The post-suppositional account has some striking similarities with the theory that relies on higher-order dynamic GQs and subtyping. The top-most scopal tier exploited by the latter plays a theoretical role analogous to the second, post-suppositional dimension of the former. Both moves afford a degree of separation between a modified numeral's dref introduction and maximization operators and its cardinality test, which allows cumulative readings to be generated.

Moreover, in both theories, the flow of information out of the separate realm where cardinality tests abide must sometimes be explicitly managed (or, one might argue, stipulated). In the higher-order theory, distributivity operators reset the type in a way that allows any cardinality tests in their scope to be discharged non-maximally. In the post-suppositional theory, distributivity operators shuttle post-suppositional content into the first dimension, to similar effect.

There is also an important difference between the theories. The post-suppositional account treats post-supposed content as *independent of* regular meaning. This predicts that post-supposed content should 'project' out of islands — as noted above, the default in bi-dimensional theories is for second-dimensional content to project upwards indefinitely. Indeed, island-insensitivity is a hallmark of other empirical domains where bi-dimensionality has been invoked (association with focus, non-restrictive relative clauses, presupposition, and exceptionally scoping indefinites):

(61) John only gripes [when *MARY* leaves the lights on]. (after Rooth 1996)

(62) John gripes [when Mary, *who's a talented linguist*, leaves the lights on].

(63) John gripes [when *the King of France* leaves the lights on].

(64) John gripes [when *a famous expert on binding* leaves the lights on].

In each of these cases, the *italicized* expression makes some semantic effect felt outside of the [bracketed] island (sometimes obligatorily, and sometimes optionally). In (61), *MARY* associates with *only*, despite an intervening island boundary. In (64), the indefinite can receive a widest-scope interpretation. Similar remarks apply to (62) and (63).

In contrast, the higher-order dynamic GQ theory predicts that, since cardinality tests actually *take scope*, they should be unable to do so out of islands. This latter prediction appears to be correct. Cases like (65) and (66), with one modified numeral embedded inside a [bracketed] relative clause island, don't allow cumulative readings.[26]

(65) Exactly^{v} three boys waved to a^{z} girl [who owns exactly^{u} five donkeys].

(66) Exactly^{v} three boys waved to every^{z} girl [who owns exactly^{u} five donkeys].

Post-suppositions generate unattested cumulative representations for such cases: e.g., (67) for (65), which is **true**$^{+}$ iff the boys who waved to a girl who owns some donkeys are three, and the donkeys such that some boys waved to a girl who own them are five.

(67) $(\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{E}^z\,\mathbf{girl}\,;\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{donks}\,;\mathbf{owns}\,u\,z)\,;\mathbf{waved\text{-}to}\,z\,v),\mathbf{3}_v\,;\mathbf{5}_u)$

Could the absence of cumulative readings in such cases be due to a silent distributivity operator that guards the edge of the island, and which prevents any post-supposed content from projecting further? There's reason to be skeptical. Collectively interpreted relative clauses also disrupt cumulative readings, as evidenced by (68).

(68) Exactly three boys waved to some girls [who had surrounded exactly five donkeys].

Of course, instead of a distributivity operator, the post-suppositional account may stipulate that a 'reifying' operator like $(\bullet)$ guards the edges of islands, thereby halting

26 Brasoveanu (2013: 176–8) suggests that quantificational operators like *every* discharge any post-suppositional content accrued in their scope, but not their restrictor. So cases like (66) should still allow cumulative readings.

the upward projection of post-suppositional content. Yet it is still plausible that this would fail to have the desired effect: while there is a good deal of evidence that second-dimensional content *can* be discharged by operators of this sort, there is significantly less evidence that it *must* be. Consider in this respect (69) and (70).

(69) [JOHN only gripes [when MARY leaves the lights on]]$_C$, and [MARY only gripes [when JOHN leaves the lights on]]$_{\sim C}$

(70) Every^i boy would love [if a^j famous expert on binding read some^k paper of his$_i$].
$\checkmark \text{a}^j \gg \text{every}^i \gg \text{some}^k \gg \text{if}$

In both cases, an island-external operator discharges *something* inside an island, but not *everything*. In the second sentence of (69), *only* associates with *JOHN*. This does not, however, prevent the higher 'focus interpretation operator' $\sim C$ (Rooth 1992, 1996) from *also* associating with *JOHN* (as required for the two conjuncts to form a good discourse pair). In (70), there are two indefinites on an island. The first be interpreted with widest scope, even as the second is captured by *every^i boy* (so the reading in question is about a specific expert, and boys who favor a specific paper of theirs; cf. Abusch 1994). These cases suggest that association with 'reification' operators like $(\bullet)$, *only*, and *every^i boy* is (at least partially) optional, even when the associated expression is inside an island.[27]

A final point. The higher-order dynamic GQ theory fails to generate island-escaping cumulative readings, which looks to be correct. Here, I'll briefly sketch how it generates the attested interpretations of constructions like (65), (66), and (68) — i.e., with the island-embedded modified numeral's cardinality tests trapped on the island. The type assignments in (44) entail that, e.g., *some* has the polymorphic type $(\mathtt{e} \to A) \to (\mathtt{e} \to A) \to A$, for $A \in \{\mathtt{t}, \mathtt{T}\}$. So if the relative clause under *some* contains a modified numeral, the resulting DP will be of type $(\mathtt{e} \to \mathtt{T}) \to \mathtt{T}$. This DP can either take inverse scope over the modified numeral subject (or even just over its type-$(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$ 'trace', given that $\mathtt{t} \sqsubset \mathtt{T}$), or we may posit a type-resetting distributivity operator — see (48) — on the subject.

I conclude that the higher-order theory of modified numerals modestly improves on the post-suppositional theory *vis à vis* islands. The latter theory must stipulate an operator like $(\bullet)$ to corral any post-suppositions generated on an island. Even so, data like (69) and (70) suggest that such operators may fail to have the intended effect.

6 Update semantics

6.1 Formal preliminaries

Higher-order meanings for modified numerals derive cumulative readings, using nothing more exotic than scope. With a little bit of prodding — incorporating a new type $\mathtt{T}$, along

27 Such behavior is not in fact predicted by standard formulations of, e.g., alternative semantics (Rooth 1996, Wold 1996). In Charlow (2014, 2016), I argue that such deficits are not deep features of these approaches to island-insensitivty (though see, e.g., Krifka 2006, Beck 2006, Kotek & Erlewine 2016 for alternative views). Appendix B sketches how the compositional theory of post-suppositions assumed here actually predicts that association with a reifying operator outside an island is optional.

with the subtyping relation $\mathtt{t} \sqsubset \mathtt{T}$, and suitable stipulations about the types of higher-order modified numerals and distributivity operators — we rule out pseudo-cumulative readings and rule in distributive readings. And because the theory is oriented around scope, we don't over-generate readings for modified numerals on islands.

It turns out, however, that for the specific case of cumulative readings with modified numerals, the account can be considerably simplified — and made considerably more explanatory — if we move to an *update semantics* (e.g., Heim 1982, Dekker 1993, 1994, 2012, Veltman 1996, Groenendijk, Stokhof & Veltman 1996, Bittner 2001). In an update semantics, dynamic propositions are functions from sets of assignments into sets of assignments, type $\mathtt{u} ::= (a \to t) \to a \to t$, rather than 'point-wise' relations on assignments, type $\mathtt{t} ::= a \to a \to t$. Intuitive $(\mathtt{e} \to \mathtt{u}) \to \mathtt{u}$ dynamic GQ meanings for modified numerals — fairly direct analogs of their point-wise counterparts — generate cumulative readings and fail to generate pseudo-cumulative readings. No higher-order meanings, or special stipulations about subtyping, distributivity operators, or islands required.

We begin by definining $\mathtt{u}$ as the type of update-theoretic dynamic propositions (type $\mathtt{e}$ is unchanged) in (71), and re-defining the set of compositionally relevant types τ as in (72). I refer to things of type $a \to t$ as 'contexts' in what follows.

(71) $\mathtt{e} ::= \mathcal{V} \qquad \mathtt{u} ::= (a \to t) \to a \to t$

(72) $\tau ::= \mathtt{e} \mid \mathtt{u} \mid \tau \to \tau$

Update-theoretic characterizations of basic meanings like **saw**, $\mathbf{E}^v$ **boys**, and $\mathbf{3}_v$ are given in (73), (74), and (75). (Similar to the point-wise semantics, I use a more iconic post-fix notation: $s\,[\mathcal{K}] := \mathcal{K}\,s$.) These meanings work as follows: $\mathbf{saw}\,u\,v$ updates a context s by retaining only those assignments in s mapping v and u to a seer-seen pair; $\mathbf{E}^v$ **boys** updates a context by individually updating each of its assignments with a dref pointing to some plurality of boys; and $\mathbf{3}_v$ updates a context by retaining those assignments that map v to a plurality with three atomic parts.

(73) $s\,[\mathbf{saw}\,u\,v] := \{i \in s \mid \mathsf{saw}\,i_u\,i_v\}$ $\qquad$ $\mathbf{saw} : \mathtt{e} \to \mathtt{e} \to \mathtt{u}$

(74) $s\,[\mathbf{E}^v\,\mathbf{boys}] := \{i^{v \mapsto x} \mid i \in s, x \in \mathsf{boys}\}$ $\qquad$ $\mathbf{E}^v\,\mathbf{boys} : \mathtt{u}$

(75) $s\,[\mathbf{3}_v] := \{i \in s \mid \#i_v = 3\}$ $\qquad$ $\mathbf{3}_v : \mathtt{u}$

These functions are in an important sense the 'same' as their point-wise counterparts, in the specific sense that they are simple injections of the point-wise meanings into the richer type-space of update functions. In particular, we may define a 'lifting' function $\uparrow$ that maps a point-wise dynamic proposition into an update-theoretic one, as in (76) below (here, to simplify the definitions, I treat relations on assignments as functions from assignments into sets of assignments). The lifting function $\uparrow$ is injective, and hence invertible. The inverse of $\uparrow$, $\downarrow$, is defined in (77). Thus, ${}^{\downarrow\uparrow}K \equiv K$ holds for all $K : \mathtt{t}$, but it is not in general the case that ${}^{\uparrow\downarrow}\mathcal{K} \equiv \mathcal{K}$ for all $\mathcal{K} : \mathtt{u}$. We'll return to this point below.

(76) ${}^{\uparrow}K := \lambda s.\bigcup_{i \in s} K\,i$ $\qquad$ $\uparrow : \mathtt{t} \to \mathtt{u}$

(77) ${}^{\downarrow}\mathcal{K} := \lambda i.\{i\}\,[\mathcal{K}]$ $\qquad$ $\downarrow : \mathtt{u} \to \mathtt{t}$

Each of the update-theoretic meanings in (73), (74), and (75) is the result of applying $\uparrow$ to the corresponding point-wise meaning. For example, the point-wise meaning for $\textsf{\textbf{saw}}\,u\,v$ was defined as $\lambda i.\{j \mid i = j \wedge \textsf{saw}\,i_u\,i_v\}$ (again treating relations as functions into sets). Applying $\uparrow$ to this gives $\lambda s.\bigcup_{i\in s}\{j \mid i = j \wedge \textsf{saw}\,i_u\,i_v\}$, which is equivalent to the definition in (73). Similar remarks apply to (74) and (75).[28]

In addition to these meanings, we define the update-theoretic maximization operator as in (78). Maximization of v relative to some update $\mathcal{K}$ and context s works by updating s with $\mathcal{K}$, and retaining only those assignments in the resulting context that harbor mereologically maximal values for v.

(78) $s\,[\mathbf{M}_v\,\mathcal{K}] \coloneqq \{i \in s\,[\mathcal{K}] \mid \neg\exists j \in s\,[\mathcal{K}] : i_v < j_v\}$ $\qquad\mathbf{M}_v : \mathsf{u} \to \mathsf{u}$

Strikingly, the update-theoretic $\mathbf{M}_v$ and its point-wise counterpart are in fact *the same function* (though with different types). This becomes apparent when (18), the point-wise definition for $\mathbf{M}_v\,K$, is re-written in set-theoretic terms, as in (79). This formula has the same form as (78) — given that $s\,[\mathcal{K}]$ is post-fix notation for $\mathcal{K}\,s$, the update-theoretic $\mathbf{M}_v$ and the point-wise (79) are perfect alphabetic variants of each other.

(79) $\lambda K.\lambda i.\{j \in K\,i \mid \neg\exists h \in K\,i : j_v < h_v\}$ $\qquad$ type: $\mathtt{t} \to \mathtt{t}$

The only truly sui generis piece in our update semantics is the (standard) definition of dynamic conjunction. Because updates are functions from contexts into contexts (compare: relations between assignment functions), update-theoretic dynamic conjunction of L and R means updating the input context with L and R in succession.

(80) $s\,[L\,;R] \coloneqq s\,[L]\,[R]$ $\qquad (;) : \mathsf{u} \to \mathsf{u} \to \mathsf{u}$

Dynamic conjunction amounts to function composition (compare: the point-wise notion of relation composition) — that is, $s\,[L\,;R]$ is equivalent to $R\,(L\,s)$.

6.2 Modified numerals and the cumulative reading

With these basic pieces in place, we may define an update-theoretic entry for modified numeral DPs as in (81). I write $[\cdot]$ for the update-theoretic interpretation function (as opposed to $[\![\cdot]\!]$, the point-wise dynamic interpretation function).

(81) $[\text{exactly}^v \text{ three boys}] \coloneqq \lambda k.\mathbf{M}_v\,(\mathbf{E}^v\,\textsf{\textbf{boys}}\,;k\,v)\,;\mathbf{3}_v$ $\qquad$ type: $(\mathsf{e} \to \mathsf{u}) \to \mathsf{u}$

This definition has the same form as the point-wise definition given back in (3) — which, recall, only derived pseudo-cumulative readings. The differences are in the types and

28 Of course, (74) doesn't give a general characterization of $s\,[\mathbf{E}^v\,P]$, for arbitrary $P : \mathsf{e} \to \mathsf{u}$. Generalized recipes for lifing to and lowering from update-theoretic meanings can be given by a pair of mutually recursive functions $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$, defined below. As with $\uparrow$ and $\downarrow$, $\lceil\lfloor x\rfloor\rceil = x$ (for any x), but $\lfloor\lceil x\rceil\rfloor$ is not in general equivalent to x. Applying $\lfloor\cdot\rfloor$ to the point-wise definition of $\mathbf{E}^v$ (fn. 8) gives $s\,[\mathbf{E}^v\,P] \coloneqq \bigcup_{i\in s}\{\{i^{v\to x}\}\,[P\,v] \mid x : e\}$.

$$\lfloor x\rfloor \coloneqq \begin{cases} x & \textbf{if } x : \mathsf{e} \\ {\uparrow}x & \textbf{if } x : \mathtt{t} \\ \lambda y.\lfloor x\lceil y\rceil\rfloor & \textbf{if } x : A \to B \end{cases} \qquad \lceil x\rceil \coloneqq \begin{cases} x & \textbf{if } x : \mathsf{e} \\ {\downarrow}x & \textbf{if } x : \mathsf{u} \\ \lambda y.\lceil x\lfloor y\rfloor\rceil & \textbf{if } x : A \to B \end{cases}$$

under the hood. This function has type $(\mathtt{e} \to \mathtt{u}) \to \mathtt{u}$ instead of $(\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$, and $\mathbf{E}^{v}\,\textbf{boys}$ and $\mathbf{M}_{v}$ have been replaced with their update-theoretic counterparts from (74) and (78).

A towers-based derivation for *exactly*^{v} *three boys saw exactly*^{u} *five movies* is given in (82). We convert the modified numerals into towers, Lift the transitive verb, Combine, and Lower. Using $\mathbb{B}$ and $\mathbb{M}$ to abbreviate the meanings of the subject and object DPs, as before, the result is equivalent to the linearized term $\mathbb{B}\,(\lambda v.\,\mathbb{M}\,(\lambda u.\,\textbf{saw}\,u\,v))$ — i.e., a completely run-of-the-mill case of quantifiers taking scope in their surface order. The resulting meaning has the same form as the point-wise logical form representing the pseudo-cumulative reading — unsurprisingly, given that the meanings for modified numerals have the same form as their point-wise counterparts.

(82)
$$\frac{\mathbf{M}_{v}\,(\mathbf{E}^{v}\,\textbf{boys}\,;[\ ]\,;\mathbf{3}_{v})}{v}\left(\frac{[\ ]}{\textbf{saw}}\;\frac{\mathbf{M}_{u}\,(\mathbf{E}^{u}\,\textbf{movs}\,;[\ ]\,;\mathbf{5}_{u})}{u}\right)$$
$$\xrightarrow{\text{Combine, Lower}}\ \mathbf{M}_{v}\,(\mathbf{E}^{v}\,\textbf{boys}\,;\mathbf{M}_{u}\,(\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v\,;\mathbf{5}_{u})\,;\mathbf{3}_{v})$$

Unlike its pseudo-cumulative counterpart, however, the meaning derived in (82) correctly represents the *cumulative reading*. The crucial difference between the point-wise pseudo-cumulative logical form and the update-theoretic cumulative logical form can be appreciated by considering the meanings each of them assigns to the sub-formula $\mathbf{M}_{u}\,(\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v)$, given in (83) and (84) below. In stating (83) I use a set-theoretic notation for the relational semantics, and in stating (84) I use a regular, non-postfix notation for the underbraced sub-formula; these notational moves help highlight the parallels between the two meanings. In fact, the two formulas are essentially identical! Their principal difference is in the types of their input arguments.[29] In the point-wise system, the input i is a single assignment function. In the update-theoretic system, the input s is a *set* of assignment functions — i.e., an entire context.

(83) **Point-wise:**
$$\lambda i.\,\{j \in (\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v)\,i \mid \neg\exists h \in \underbrace{(\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v)\,i} : j_u < h_u\}$$

(84) **Update-theoretic:**
$$\lambda s.\,\{j \in (\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v)\,s \mid \neg\exists h \in \underbrace{(\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v)\,s} : j_u < h_u\}$$

This is the crucial difference between the two kinds of dynamic theories, on which their different predictions turn. In a point-wise system, maximization over u happens many times over — once per input assignment. Thus, a plurality of movies m that looks maximal relative to some input assignment i may turn out looking not-so-maximal relative to some other input i'. Nevertheless, m will be stored as a possible value for u.

In the update-theoretic setting, by contrast, maximization over u happens *only once*. Since the incoming context s harbors all possible pluralities of boys, $(\mathbf{E}^{u}\,\textbf{movs};\textbf{saw}\,u\,v)\,s$ contains all the movies seen by any plurality of boys, whatsoever. It is with respect to this global boys-seeing-movies context that the movies are maximized. So we end up,

29 The point-wise and update-theoretic meanings for $\mathbf{E}^{u}\,\textbf{movs}\,;\textbf{saw}\,u\,v$ are distinct, but isomorphic: the latter is equivalent to ↑-ing the former, and the former is equivalent to ↓-ing the latter.

correctly, storing only the maximal plurality of movies seen by any plurality of boys (and, higher up, the maximal plurality of boys who saw this maximal plurality of movies). Thus, the update-theoretic meaning derived in (82) represents the cumulative reading: we find the biggest pluralities of boys who saw movies, and movies seen by boys, then check that those boys and movies have, respectively, three and five atomic parts.

Things could have been otherwise with some different lexical entries. If our update-theoretic definition of maximization (or, for that matter, dynamic conjunction) had been replaced with a 'lifting' of its point-wise counterpart (see fn. 28 for a generalization of $\uparrow$ that applies to functions), we'd be back to deriving only pseudo-cumulative readings! The update semantics derives cumulative readings precisely because its maximization operators survey entire contexts. This global perspective isn't available to point-wise meanings, which process one assignment at a time, nor to $\uparrow$-derived update-theoretic meanings, which are functionally equivalent to their point-wise counterparts.[30] In fact, to turn the logical form in (82) into a pseudo-cumulative one, it'd be enough to replace $\mathbf{M}_v\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)$ with ${}^{\uparrow\downarrow}\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)$.

Slightly more formally, the essential feature of the update-theoretic account — the reason it derives cumulative readings that the point-wise account can't derive without higher-order dynamic GQs (with subtyping to rule out pseudo-cumulative readings) — is that it exploits *non-distributive* updates. An update $\mathcal{K}$ is distributive iff for any context s, $s\,[\mathcal{K}] = \bigcup_{i \in s} \{i\}\,[\mathcal{K}]$ — i.e., iff 'globally' updating s with $\mathcal{K}$ is equivalent to updating $\mathcal{K}$ one-by-one with each of the assignments in s, and unioning the results (e.g., van Benthem 1989, Groenendijk & Stokhof 1991b, Rothschild & Yalcin 2015). Our update-theoretic $\mathbf{M}_v\,\mathcal{K}$ is not, in general, distributive — its ability to generate cumulative readings turns precisely on the fact that it swallows contexts whole. ($\mathcal{K}\,;\mathcal{K}'$ in turn is distributive only if neither conjunct contains maximization operators).

Standard dynamic accounts of anaphora — including update-theoretic varieties (e.g., Groenendijk & Stokhof 1991b, Rawlins 2006) — are either point-wise or distributive. And while there exist well-known arguments that non-distributivity is an essential feature of update-theoretic accounts of *modality* (e.g., Veltman 1996, Groenendijk, Stokhof & Veltman 1996, Büring 1998), I'm at present unaware of any arguments for updates that are non-distributive with respect to *assignment functions*. The ease with which the update-theoretic account of modified numerals generates cumulative readings, and fails to generate pseudo-cumulative readings, is thus a novel argument for such an approach.

6.3 Distributive readings

So far, the cumulative reading is the only one the update semantics derives. (Inverse-scoping the object over the subject, for example, derives the cumulative reading all over again, via a logical form that reverses the relative scopes of its dref introduction and maximization operators, and does the cardinality checks in opposite orders. Thus, the

30 Interestingly, the same point applies to dynamic accounts that follow Groenendijk & Stokhof (1990) in treating sentence meanings as functions from assignment functions and 'right-contexts' to truth values, i.e., with type $a \to (a \to t) \to t$ (see also de Groote 2006). Even though this propositional type is ismorphic to u, these theories process incoming assignment functions one-by-one, as the type $a \to (a \to t) \to t$ indicates.

entries for modified numerals given here are scopally commutative — cf. fn. 14.) Adding distributive readings is straightforward. We begin by defining a distributivity operator in (85), the update-theoretic analog of the pointwise entry defined in (48).[31]

(85) $s\,[\mathbf{D}_v\,k] := \{i \in s \mid \forall x \leqslant_{\mathsf{at}} i_v : \exists j \in \{i^{v \to x}\}\,[k\,v]\}$ $\qquad$ $\mathbf{D}_v : (\mathsf{e} \to \mathsf{u}) \to \mathsf{u}$

The entry works as expected, retaining only those assignments in the input context relative to which every atomic part of v has the property k. (As before, the meaning of the null distributivity operator can then be given as $\lambda v.\,\mathbf{D}_v$.)

A simple derivation using this operator is given in (86). The tower for the subject is gotten by applying the null distributivity operator to the bottom level of the tower for *exactly*^{v} *three boys*, expanding $\mathbf{D}_v$ to $\lambda k.\,\mathbf{D}_v\,(\lambda v'.\,k\,v')$, and then applying the tower convention. Combine and Lower glue the pieces together, leaving us with a logical form that adequately represents the (surface-scope) distributive reading of the sentence.

(86)
$$\begin{array}{c}\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\;]\,;\mathbf{3}_v)\\ \hline \mathbf{D}_v\,(\lambda v'.\,[\;])\\ \hline v'\end{array}\left(\begin{array}{c}[\;]\\ \hline [\;]\\ \hline \mathbf{saw}\end{array}\;\;\begin{array}{c}[\;]\\ \hline \mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\;]\,;\mathbf{5}_u)\\ \hline u\end{array}\right)$$

$$\xrightarrow{\text{Combine, Lower}} \mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;\mathbf{D}_v\,(\lambda v'.\,\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v'\,;\mathbf{5}_u))\,;\mathbf{3}_v)$$

The main advantage this distributivity operator has over the post-suppositional and higher-order point-wise versions is that there is no need to make any bespoke decisions about how distributivity interacts with subtyping or post-suppositional content. We simply upgrade the point-wise entry into an update-theoretic one, and we are done.

6.4 Discussion

The update-semantic account of modified numerals looks preferable to both of the enrichments to the point-wise theory that we have explored. Unlike the theory oriented around higher-order dynamic GQs, there is no need to appeal to higher-order meanings for modified numerals — although the type $(\mathsf{e} \to \mathsf{u}) \to \mathsf{u}$ is larger than $(\mathsf{e} \to \mathsf{t}) \to \mathsf{t}$, it is far smaller than than $(\mathsf{Q} \to \mathsf{t}) \to \mathsf{t}$ — nor to stipulate anything about subtyping, or about how things like distributivity operators interact with subtyping. Unlike the post-suppositional theory, there is no need to cleave off a separate dimension of post-suppositional content, with attendant stipulations about how things like distributivity operators interact with post-suppositions. In addition, because the update-theoretic account is, like the higher-order dynamic GQ theory, fundamentally oriented around scope, we correctly have it that cumulative readings should be sensitive to islands (more on this point shortly).

In closing this section, I'd like to highlight some comparatively subtle points about the how the update semantics is formulated, which I take to be revealing about the general enterprise. Let's begin with two apparent puzzles. First, consider the example in

31 Like all of our entries besides maximization and dynamic conjunction, the $\mathbf{D}_v$ is equivalent (modulo subtyping) to the result of applying $\lfloor\cdot\rfloor$ (see fn. 28) to the point-wise distributivity operator defined in (48).

(87), with what we might suppose is its update-theoretic meaning (on the next line). The underbraced part of this formula has the meaning in (84). Consequently, maximization over movies seen happens with respect to a context containing *all the boys*. Thus we seem to derive, as a default, a reading saying that the maximal number of movies seen by any boys whatsoever is five — which is to say, a meaning we usually think of as requiring inverse scope! Meanwhile, the regular 'surface-scope-cumulative' reading (there exists a plurality of boys B such that B saw exactly five movies) is nowhere in sight.

(87) Some^v boys saw exactly^u five movies.
$\mathbf{E}^v\,\mathbf{boys}\,;\underbrace{\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)}\,;\mathbf{5}_u$

A related apparent issue is seen in (88), with a modified numeral inside a coordinate structure island, alongside what we might take to be its update-theoretic logical form. Again, the logical form gives something akin to an 'inverse-scope' reading (exactly five movies were seen by some boys, and those boys cheered), instead of the intuitive 'surface-scope' reading (there's a plurality of boys who saw exactly five movies, and who cheered). In this case, the situation is rather direr than (87) — not only do we fail to generate an intuitively available construal, but we also generate an impossible 'inverse-scope-like' interpretation, despite the island-boundedness of the modified numeral.

(88) Some^v boys [saw exactly^u five movies] and cheered.
$\mathbf{E}^v\,\mathbf{boys}\,;\underbrace{\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;\mathbf{saw}\,u\,v)}\,;\mathbf{5}_u\,;\mathbf{cheered}\,v$

These problems are only apparent, in that they rely on intuitive, but in the end mistaken ideas about the lexical semantics of *some*^{v}. In particular, the logical forms in (87) and (88) presume that $[\text{some}^v] := \lambda P.\lambda k.\mathbf{E}^v\,P\,;k\,v$ (see fn. 28 for a general definition of $\mathbf{E}^v\,P$). In actuality, our meaning for *some*^{v} should not be defined in this way, but rather as the *lift* of the point-wise *some*^{v} into the space of update-theoretic meanings (the reader is again referred to fn. 28 for a generalized definition of lifting).

(89) $[\text{some}^v] := \lambda P.\lambda k.{}^{\uparrow}[\![\text{some}^v]\!]\,(\lambda v.{}^{\downarrow}P\,v)\,(\lambda v.{}^{\downarrow}k\,v)$ type: $(\mathsf{e} \to \mathsf{u}) \to (\mathsf{e} \to \mathsf{u}) \to \mathsf{u}$

The occurence of $\downarrow$ over the restrictor P and scope k guarantee that both P and k will be processed *distributively* — that is, both will see the incoming context one assignment at a time. This correctly prevents any maximization operators in P or k from illegitimately acquiring 'scope' over the dynamic existential quantifier introduced by *some*.

In sum, though our semantics for *modified numerals* exploits non-distributive meanings — i.e., meanings that are irreducibly update-theoretic, and not just update-typed counterparts of point-wise meanings — nothing else does or should (pending evidence to the contrary). Though this is stipulative in a sense, I actually take it to be a conservative way of proceeding. When we move from a less expressive setting like the point-wise semantics to a richer setting like update semantics, as many insights as possible should be retained from the less expressive theory — namely, by defining a way to inject less-expressive values into functional equivalents in the richer type-space. Only those things that irreducibly live in the richer type space should be directly defined in it.

7 Discussion

7.1 Interactions with modals

Modified numerals have certain peculiar interactions with modals. There is, for example, an asymmetry in how (90) and (91) are naturally interpreted: (90) places an upper bound on the number people Jasper's allowed to invite, while (91) places a lower bound on the number of books Jasper's allowed to read (e.g., Nouwen 2010, Kennedy 2015).

(90) Jasper is allowed to invite at most ten people. (Brasoveanu 2013: 188, ex. 123)

(91) Jasper should read at least ten books. (Brasoveanu 2013: 185, ex. 118)

Brasoveanu (2013) argues that data of this sort — in particular, (90) — gives independent motivation for the post-suppositional theory of modified numerals. In this section, I'll briefly sketch how the update-theoretic account generates such readings, as well (as does the higher-order dynamic GQ theory, though I do not detail this here). This suggests that *whatever* features of a theory allow it to explain cumulative readings with modified numerals can also be leveraged to explain their interactions with modal operators. Our strategy follows Brasoveanu closely, especially in treating possibility modals like modified numerals, and necessity modals like distributive quantifiers.

Our first step will be to 'intensionalize' the update semantics. We begin by admitting drefs that point to pluralities of worlds (i.e., modal bases), and re-conceiving of individual drefs as pointing to *individual concepts* — i.e., functions from worlds to (plural) individuals. With these pieces in place, an entry for a transitive verb will look as follows:

(92) $s\,[\,\mathbf{invite}_w\,u\,v\,] := \{i \in s \mid \forall w' \leqslant_{\mathsf{at}} i_w : \mathsf{invite}_{w'}\,i_{u,w'}\,i_{v,w'}\}$ $\qquad \mathbf{invite}_w : \mathsf{e} \to \mathsf{e} \to \mathsf{u}$

Here, **invite** is indexed to a plurality of worlds w. The context is updated by checking that, at each atomic part w' of w, the value of v at w' saw the value of u at w' (to save parens, $\ulcorner i_{v,w'} \urcorner$ is used to mean the value of the invidual concept i_v at w'). Assignments in s with appropriate values for w, u, and v are kept. The rest are discarded.

The entries for dref introduction, maximization, and cardinality checks are upgraded in a similar way. $\mathbf{E}^v\,\mathbf{ppl}$ stores in v a people-valued individual concept (to simplify the definitions, I will assume there are people in any relevant world). $\mathbf{M}_{v,w}$ requires that the pluralities returned by v at the atomic parts of w be maximal in $s\,[\mathcal{K}]$. Cardinality tests like $\leqslant\!\mathbf{10}_{v,w}$ measure the size of a dref across the atomic parts of w.

(93) $s\,[\,\mathbf{E}^v\,\mathbf{ppl}\,] := \{i^{v \to x} \mid \forall w : \mathsf{ppl}_w\,x_w\}$ $\qquad \mathbf{E}^v_w\,\mathbf{ppl} : \mathsf{u}$

(94) $s\,[\,\mathbf{M}_{v,w}\,\mathcal{K}\,] := \{i \in s\,[\mathcal{K}] \mid \neg\exists j \in s\,[\mathcal{K}] : \exists w' \leqslant_{\mathsf{at}} j_w : j_{v,w'} > i_{v,w'}\}$ $\qquad \mathbf{M}_{v,w} : \mathsf{u} \to \mathsf{u}$

(95) $s\,[\,\leqslant\!\mathbf{10}_{v,w}\,] := \{i \in s \mid \forall w' \leqslant_{\mathsf{at}} i_w : i_v\,w' \leqslant 10\}$ $\qquad \leqslant\!\mathbf{10}_{v,w} : \mathsf{u}$

We use these pieces to give intensionalized meanings for modified numerals, as in (96). Apart from the subscripted w's, (96) has the same form as the update-theoretic meaning defined earlier in (81). Specifically, the meaning for $\mathbf{M}_{v,w}$ guarantees that intensionalized modified numerals are still *non-distributive*, crucially updating *whole contexts.*

(96) $[\text{at most}^v_w \text{ ten people}] := \lambda k.\,\mathbf{M}_{v,w}\,(\mathbf{E}^v\,\mathbf{ppl}\,;k\,v)\,;\leqslant\!\mathbf{10}_{v,w}$ $\qquad$ type: $(\mathsf{e} \to \mathsf{u}) \to \mathsf{u}$

Finally, we give an entry for the possibility modal *allowed* in (97). (Here, we rely on our old update-theoretic definitions of maxmization and dref introduction; there's no need to further intensionalize the already intensional.) The modal is indexed to $\mathsf{Acc}_{@}$, a dynamic encoding of a Kripkean accessibility relation/Kratzerian modal base (relative to some index of evaluation @). Fixing a prejacent K, the modal introduces the maximal plurality of accessible worlds in which K holds, then checks that, post-maximization, at least one world remains.

(97) $[\text{allowed}^{w}_{\mathsf{Acc}_{@}}] \coloneqq \lambda K.\, \mathbf{M}_w\,(\mathbf{E}^w\,\mathsf{Acc}_{@}\,;K)\,;\geqslant\!\mathbf{1}_w$ type: $\mathsf{u} \to \mathsf{u}$

The entry in (97) has same basic form as a modified numeral. Therefore, we may expect that it gives rise to 'cumulative' readings when a modified numeral DP ends up in its scope. And this is in fact what happens. Consider (98), our translation of (90) (for simplicity, we're treating *Jasper invites* as an unanalyzed unit).

(98) $\mathbf{M}_w\,(\mathbf{E}^w\,\mathsf{Acc}_{@}\,;\underbrace{\mathbf{M}_{v,w}\,(\mathbf{E}^v\,\textbf{ppl}\,;\textbf{jasper-invites}_w\,v)}\,;\leqslant\!\mathbf{10}_{v,w})\,;\geqslant\!\mathbf{1}_w$

As with prior update-theoretic cumulative readings, the non-distributive character of the underbraced formula is crucial: this formula swallows the incoming context whole, which here contains every accessible plurality of worlds. This means that $\mathbf{M}_{v,w}$ returns the maximal individual concepts ranging over people Jasper invited at any accessible world. Per (94), an individual concept stored at v is maximal relative to w, $\mathcal{K}$, and s iff no other individual concept stored at v in any $j \in s\,[\mathcal{K}]$ picks out a bigger plurality at any atomic part of j_w. Thus, post-maximization, v selects, for any accessible world, the maximal sum of people Jasper invited there. The maximal v in hand, $\leqslant\!\mathbf{10}_{v,w}$ requires that v has at most ten individuals throughout w. In sum, there are no accessible worlds where Jasper invites more than ten people. This is (90)'s upper-bounded reading.

Meanwhile, lower-bounded readings with necessity modals like *should* are also generated if (with Brasoveanu) we assume that *should*'s semantics contains a $\mathbf{D}_w$ operator, as in (99) (the vacuous abstraction $\lambda w'$ is due to the fact that this $\mathbf{D}_w$ operator doesn't take scope like nominal $\mathbf{D}$-operators; even so, $\mathbf{D}_w$ guarantees that any w's in K evaluate to atomic parts of the world-pluralities introduced by $\mathbf{E}^w\,\mathsf{Acc}_{@}$). This entry yields (100) as our translation of (91).

(99) $[\text{should}^{w}_{\mathsf{Acc}_{@}}] \coloneqq \lambda K.\, \mathbf{M}_w\,(\mathbf{E}^w\,\mathsf{Acc}_{@})\,;\mathbf{D}_w\,(\lambda w'.\,K)$

(100) $\mathbf{M}_w\,(\mathbf{E}^w\,\mathsf{Acc}_{@})\,;\mathbf{D}_w\,(\lambda w'.\,\mathbf{M}_{v,w}\,(\mathbf{E}^v\,\textbf{books}\,;\textbf{jasper-reads}_w\,v)\,;\geqslant\!\mathbf{10}_{v,w})$

Here, we introduce a maximal plurality of accessible worlds, then distributively check that each accessible world is one in which Jasper reads at least ten books. This correctly represents the lower-bounded, minimal-requirements reading of (91).[32]

32 The account may modestly improve on Brasoveanu's (2013). Because Brasoveanu's distributivity operators can and do apply to any propositionally-typed thing, he predicts the existence of a reading for (90) akin (91) — i.e., with a distributivity operator scoping over the prejacent, which results in a meaning paraphrasable as "inviting at most ten people is something Jasper's allowed to do". It is dubious whether such readings exist (e.g., Nouwen 2010), and for this reason Brasoveanu proposes (with Nouwen) that they are blocked by the existence of a simpler construction — namely, one with a bare numeral in lieu of a modified numeral. Because our null distribitivity operators (semantics: $\lambda v.\,\mathbf{D}_v$) apply to *variables*, a parallel problem does not arise here.

7.2 Dependent indefinites as higher-order

The last substantive thing I'd like to do is show that, despite its real positive points, the update-theoretic account is not the be-all and end-all of how to theorize about empirical phenomena that seem to be in some sense 'post-suppositional'. Indeed, I'll suggest that when we broaden our view beyond cumulative readings of modified numerals, there are reasons to think that something like an account phrased in terms of higher-order dynamic GQs may be useful. Of course, one needn't *choose between* an update-theoretic account of cumulative readings and higher-order accounts of other phenomena: higher-order dynamic GQs and update-theoretic meanings can comfortably coexist within a single grammar (so long, anyway, as a consistent kind of dynamic semantics is assumed throughout; or, alternatively, if a general lifting method is provided for turning point-wise values into update-theoretic ones — as, for example, in fn. 28).

Let's work our way up to a puzzle. *Dependent indefinites* are licensed only when they occur within the scope of a distributive quantifier (see Henderson 2014 for a refinement of this somewhat crude characterization). In Hungarian, for example, partially reduplicated indefinites are dependent. Thus, the sentences in (101) are unambiguous: their indefinites must be interpreted with narrow scope (unlike their English translations).

(101) Minden gyerek olvasott egy-egy / hét-hét könyvet. (Farkas 1997, ex. 34)
every child read a-RED / seven-RED book-ACC
'Every child read a / seven book(s).'

Henderson (2014) proposes a post-suppositional analysis of dependent indefinites using dynamic plural logic, where propositions are relations on *sets* of assignments. (Dynamic plural logic can be used to keep track of how quantificational dependences are introduced and elaborated in discourse. See, e.g., van den Berg 1996, Brasoveanu 2007). Henderson's semantics for dependent indefinites is comprised of a standard dynamic existential quantifier, coupled with a post-suppositional test that checks whether the indefinite has varied with respect to some other operator (i.e., if multiple witnesses for the indefinite were activated within a distributive quantifier's scope; such tests are possible to formulate within a dynamic plural logic, though I pass over the details here). In terms of the post-suppositional theory from Section 5, this might amount to a semantics like (102) for the dependent indefinite, which grows into a post-suppositional sentence meaning like (103). (I write ⌜$\mathtt{t_{pl}}$⌝ for the type of dynamic plural logic propositions.)

(102) $[\![\text{a-a}^u_v \text{ book}]\!] := (\lambda k.\, \mathbf{E}^u\, \mathbf{book} \,;\, k\,u,\, \mathbf{dep}_{v,u})$ type: $((\mathtt{e} \to \mathtt{t_{pl}}) \to \mathtt{t_{pl}}) \times \mathtt{t_{pl}}$

(103) $(\mathbf{ev\text{-}child}^v\, (\lambda v'.\, \mathbf{E}^u\, \mathbf{book} \,;\, \mathbf{read}\, u\, v'),\, \mathbf{dep}_{v,u})$ type: $\mathtt{t_{pl}} \times \mathtt{t_{pl}}$

In (103), the dependent indefinite introduces an index u, and is anaphoric on an index v. The post-suppositional $\mathbf{dep}_{v,u}$ requires u to vary with v as the distributive quantifier $\mathbf{ev\text{-}child}^v$ activates different children and the book(s) they read. This happens only if the books read vary with (i.e., depend on) the children who did the book-reading.

Kuhn (2016), however, takes issue with the post-suppositional account — along lines that may be familiar at this point. In particular, Kuhn notes that, at least in Hungarian, dependent indefinites on an island can't be licensed by a distributive quantifier outside the island. For instance, (104) is simply judged ungrammatical.[33]

(104) *Minden nyelvész szeretné, ha [két-két elmélet megdőlne].
every linguist would.like if two-two theories would.fall
'Every linguist would like if two-two theories were refuted.'

Kuhn (2016) argues on this basis that dependent indefinites must actually *take scope over* their licensors. Kuhn assigns to dependent indefinites meanings like (105), which turn into logical forms like (106) (here I gloss over the details of the implementation and take certain liberties with my presentation of the formalism.) In (106), various pluralities of theories are introduced, then distributively quantified over by Q. After this happens, the pluralities of theories that witness the truth of $Q\,(\lambda v' \ldots)$ are restricted to those that are two-theories-large. The dependency test $\mathbf{dep}_{v,u}$ finally checks that those two-movie pluralities depend on (i.e., vary with) v.

(105) $[\![\text{two-two}^{u}_{v} \text{ theories}]\!] \coloneqq \lambda k.\, \mathbf{E}^{v}\,\textbf{theories}\,;\, k\,v\,;\, \mathbf{2}_u\,;\, \mathbf{dep}_{v,u}$ type: $(\mathtt{e} \to \mathtt{t_{pl}}) \to \mathtt{t_{pl}}$

(106) $\mathbf{E}^{v}\,\textbf{theories}\,;\, Q\,(\lambda v' \ldots)\,;\, \mathbf{2}_u\,;\, \mathbf{dep}_{v,u}$ type: $\mathtt{t_{pl}}$

Crucially, in (106) the indefinite has scoped over the distributive quantifier Q, which guarantees that $\mathbf{E}^{v}\,\textbf{theories}$ is processed before Q. Kuhn 2016 argues that, if a dependent indefinite scopes under its licensor, the requirements of $\mathbf{dep}_{u,v}$ cannot be satisfied. Thus, a dependent indefinite must take scope over a licensor, and the island-boundedness of dependent indefinite licensing is derived.

An important issue with this account, as Kuhn himself points out (citing examples due to an anonymous referee), is that it straightforwardly predicts that it should be impossible for a dependent indefinite to be bound into by its licensor: for the dependent indefinite to be licensed, it needs to take scope over its licensor, which should make it impossible for the licensor to bind any pronouns inside the dependent indefinite.[34] This prediction is incorrect, as cases like (107) and (108) bear out.

(107) Minden rendező benevezte két-két filmjét.
Every director entered two-two film-POSS.3SG-ACC
'Every^{i} director entered two films of his$_{i}$ (in the competition).'
(Kuhn 2016, ex. 104)

(108) Mindenki meglátogatott egy-egy rokont és aztán írt nekik.
Everybody visited a-a relative and then wrote them
'Everybody^{i} visited a relative$_{i}$ and then wrote them.'
(Farkas 1997, ex. 57)

33 I thank Anna Szabolcsi for discussion and help constructing Hungarian examples.

34 It is crucial that a (dependent) indefinite's NP restrictor be evaluated before its scope — since, for example, indefinites in the restrictor of *two-two* can bind pronouns in its scope.

(i) Minden tanár megengedte két-két diáknak aki rossz választ adott hogy kijavítsa.
Every teacher allowed two-two student-DAT who wrong answer-ACC gave that correct.DEFOBJ
'Every teacher allowed two students who gave a^{i} wrong answer to fix it$_{i}$.'

In both (107) and (108), a dependent indefinite is licensed, despite a pronoun inside its restrictor being bound by the very quantifier that licenses it (for (108) I assume that the relational noun *rokont* 'relative' has a covert pronoun bound by the distributive subject). A semantics like (105) predicts that this is impossible.

A scope-based theory of dependent plurals that allows their licensors to bind into them is straightforward to construct with higher-order dynamic GQs. I briefly develop an analysis of (107) within a point-wise dynamic semantics (in fact, basic features of dependent indefinites can be captured without dynamic plural logic, though the central features of the theory of dependent indefinites given here are certainly compatible with dynamic plural logic). To begin with, I define a dependent-indefinite-friendly meaning for distributive quantifiers in (109), which introduces a dref ranging over atomic directors, and which is anaphorically linked via u to the dependent indefinite in its scope.

(109) $[\![\text{every}^{v}_{u}\ \text{director}]\!] := \dfrac{\mathbf{\Pi}_{v,u}\,\textbf{drctr}\,(\mathbf{E}^{v}\,\textbf{drctr}\,;[\ ])}{v}$ type: $\dfrac{\mathtt{t}}{\mathtt{e}}$

(110) $i\,(\mathbf{\Pi}_{v,u}\,\textbf{drctr}\,K)\,j := (\forall x \in \mathsf{drctr} : \exists h : i K h \wedge h_v = x) \wedge j = i^{u \to \sqcup\{h_u \mid i K h\}}$

$\mathbf{\Pi}^{u}_{v}\,\textbf{drctr} : \mathtt{t} \to \mathtt{t}$

In brief, (109) works by constructing a dynamic proposition K that contains all the directors, and all the witnesses for any indefinites in its scope (eventually, the films directors entered in the competition). The outer layer of quantification contributed by $\mathbf{\Pi}_{v,u}$ subsequently checks that every (atomic) director is associated with some successful output in K (which gives us the expected universal force), while at the same time re-associating u with the *mereological fusion* of all the values taken on by u within K. Thus, after processing $\mathbf{\Pi}_{v,u}$, u will come to store a plural individual comprised of, for example, the films of all the directors that were entered (by them) in the competition.[35]

With this meaning in hand, dependent indefinites can be treated as higher-order dynamic GQs. A representative entry for an indefinite with a bound-into restrictor is given in (111). The lower two levels of this tower are just the meaning one might give to *two*^{u} *films of his*$_{v}$. On the top level, though, lives a second cardinality check, which requires there to be *more than two* individuals stored in u (which, as we'll see shortly, is a relatively coarse way to check that u has varied in the scope of $\mathbf{\Pi}_{v,u}$).

(111) $[\![\text{two-two}^{u}\ \text{films of his}_{v}]\!] := \dfrac{[\ ]\,;\,{>}\mathbf{2}_{u}}{\dfrac{\mathbf{E}^{u}\,(\textbf{films-of}\,v)\,;\,\mathbf{2}_{u}\,;\,[\ ]}{u}}$ type: $\dfrac{\mathtt{t}}{\dfrac{\mathtt{t}}{\mathtt{e}}}$

Putting these two pieces together gives the derivation in Figure 11. Because of the separation in the higher-order tower for the dependent indefinite, it scopes *both over*

35 A general meaning for $\mathbf{\Pi}_{v,u}$ is given below. This entry is inspired by the weak conservativity-based dynamic GQs of, e.g., Chierchia (1992, 1995), Kanazawa (1994). Ultimately, (i) should be generalized by replacing u with a set of variables, in order to allow a single universal to license multiple dependent indefinites in its scope.

(i) $i\,(\mathbf{\Pi}_{v,u}\,P\,K)\,j := (\forall x : \forall g : i^{v \to x}\,(P\,v)\,g \Rightarrow \exists h : i K h \wedge h_v = x) \wedge j = i^{u \to \sqcup\{h_u \mid i K u\}}$

$$\begin{array}{c} [\,] \\ \hline \mathbf{\Pi}_{v,u}\,\textbf{drctr}\,(\textbf{E}^{v}\,\textbf{drctr}\,;[\,]) \\ \hline v \end{array} \left(\begin{array}{c} [\,] \\ \hline [\,] \\ \hline \textbf{entered} \end{array} \quad \begin{array}{c} [\,]\,;>\mathbf{2}_u \\ \hline \textbf{E}^{u}\,(\textbf{films-of}\,v)\,;\mathbf{2}_u\,;[\,] \\ \hline u \end{array} \right)$$

$$\xrightarrow{\text{Combine, Lower}} \underbrace{\mathbf{\Pi}_{v,u}\,\textbf{drctr}\,(\textbf{E}^{v}\,\textbf{drctr}\,;\textbf{E}^{u}\,(\textbf{films-of}\,v)\,;\mathbf{2}_u\,;\textbf{entered}\,u\,v)}_{\text{every director entered two films of his, } u \,\to\, \text{the films thereby entered}}\,;>\mathbf{2}_u$$

Figure 11: A derivation using a higher-order dynamic GQ meaning for the dependent indefinite. The underbraced update checks that every director entered two films of his, and adds a dref u pointing to a plural individual comprising the films entered by directors. Subsequently, $>\mathbf{2}_u$ checks that at least three films are stored in u. This gives a rough-and-ready approximation of the variation requirement of dependent indefinites, within a 'first-order' dynamic system (i.e., one where propositions are type $\texttt{t} ::= a \to a \to t$), and while allowing *every director* to bind into the dependent indefinite's restrictor, even as the latter's dependency test is scopally "postponed".

and under the universally quantified subject DP. More specifically, the simple indefinite part of its semantics takes scope under the subject (allowing the pronoun in $\textbf{films-of}\,v$ to be bound), while the dependency-enforcing cardinality check $>\mathbf{2}_u$ takes scope over the subject. If the top-level cardinality check didn't scope over the subject, $\mathbf{2}_u$ and $>\mathbf{2}_u$ would make conflicting demands about the pluralities stored in u. However, when $\mathbf{\Pi}_{v,u}$ intervenes between $>\mathbf{2}_u$ and $\mathbf{2}_u$, it resets u to the mereological fusion of the two-film pluralities entered by directors. This allows $>\mathbf{2}_u$ to be satisfied, so long as more than one two-film plurality was entered by a director. This gives a reasonable approximation of the variation/dependency requirement of dependent indefinites.

In sum, higher-order dynamic GQs give a natural account of the island-sensitivity of dependent indefinite licensing (since the dependency test needs to acquire scope over the licensor, and it cannot do so from within an island), while at the same time allowing dependent indefinites to be bound into, as required by examples like (107) and (108). Moreover, is difficult to see how an explanation based instead on (irreducibly) update-theoretic values would go, since $\mathbf{2}_u$ and $>\mathbf{2}_u$ are fundamentally incompatible without an intervening distributive quantifier. Such intervention is straightforward to achieve with higher-order meanings for dependent indefinites (since we can separate $>\mathbf{2}_u$ and $\mathbf{2}_u$, by placing them on different 'levels'), but looks impossible if dependent indefinites are defined as 'two-level' update-theoretic dynamic GQs — i.e., with type $\frac{\texttt{u}}{\texttt{e}}$.[36]

7.3 Conclusion

This paper has offered two account of modified numerals, both of which can be used to derive cumulative readings in sentences with multiple modified numeral DPs. In the

36 There may even be a use for subtyping here, as well. As, e.g., Farkas (1997) points out, dependent indefinites in Hungarian and Romanian cannot be licensed by *modal* quantifiers. One way to implement this within the present theory would be to elaborate a subtyping regime that allows us to control the kinds of propositions over which the dependency checks of dependent indefinites can take scope.

theory based on higher-order dynamic GQs, cardinality tests are scopally distinguished from maximization and dref introduction. Using higher-order meanings for modified numerals, cumulative readings can be derived as in (27) (repeated below), where $\ulcorner\mathbb{B}\urcorner$ abbreviates $[\![$exactly^v three boys$]\!]$, and $\ulcorner\mathbb{M}\urcorner$ abbreviates $[\![$exactly^u five movies$]\!]$.

(27) $\mathbb{B}\,(\lambda V.\,\mathbb{M}\,(\lambda U.\,V\,(\lambda v.\,U\,(\lambda u.\,\textbf{saw}\,u\,v))))$ type: $\texttt{t}$
$\rightsquigarrow \textbf{M}_v\,(\textbf{E}^v\,\textbf{boys}\,;\textbf{M}_u\,(\textbf{E}^u\,\textbf{movs}\,;\textbf{saw}\,u\,v))\,;\textbf{5}_u\,;\textbf{3}_v$

With (irreducibly) update-theoretic meanings for modified numerals, matters are rather simpler. The inherently global, non-distributive character of update-theoretic maximality operators (and dynamic conjunction) means that mundane derivations yield cumulative readings, despite cardinality tests ending up within the scope of maximization operators:

(112) $\mathbb{B}\,(\lambda v.\,\mathbb{M}\,(\lambda u.\,\textbf{saw}\,u\,v))$ type: $\texttt{u}$
$\rightsquigarrow \textbf{M}_v\,(\textbf{E}^v\,\textbf{boys}\,;\textbf{M}_u\,(\textbf{E}^u\,\textbf{movs}\,;\textbf{saw}\,u\,v)\,;\textbf{5}_u)\,;\textbf{3}_v$

The higher-order account of cumulative readings improves on a post-suppositional account in a couple respects. Because the theory is oriented around scope and scope-taking, the island-sensitivity of cumulative readings is predicted. Moreover, the underlying compositional machinery is quite conservative: a higher-order modified numeral simply takes scope, followed by its ‘trace’.

The update-theoretic treatment of modified numerals does even better. Whereas the higher-order theory must be enriched with subtyping and specific assumptions about the types assigned to modified numerals and distributivity operators — stipulations that I’ve argued are paralleled in a post-suppositional theory — the update-theoretic account needs no stipulations beyond the intuitive meanings it gives to modified numerals.

As I argued in the last section, I don’t think the moral of the story is that update semantics ‘wins’. Rather, theories based on update semantics and theories based on higher-order (dynamic) GQs are *complementary*. While update semantics underwrites a satisfying theory of modified numerals, higher-order dynamic GQs help us make sense of dependent indefinites — specifically, how they can scope over a licensor that binds into their restrictor NP. If we discover some empirical phenomena that look post-suppositional, and which aren’t sensitive to islands, then the best analysis may very well turn out to be verily post-suppositional. Like a lot of things, it depends.

A Tower combination

Tower combination is underwritten by three freely applying type-shifters: Lift, Combine, and Lower. Lift is the familiar mapping from values into scope-takers defined in (113).

(113) $\mathsf{Lift}\,x \coloneqq \lambda k.\,k\,x$

A recursive definition of Combine is given in (114) (this formulation is due to Bumford 2016). If either L or R is in the domain of the other, $\mathsf{Combine}\,L\,R$ amounts to type-driven functional application. Otherwise, L and R are scoped, and their 'traces' are $\mathsf{Combine}$'d. Because $\mathsf{Combine}$ is recursive, it can glue together towers of arbitrary height.

(114) $$\mathsf{Combine}\,L\,R := \begin{cases} L\,R \textbf{ if } R \in \mathrm{Dom}\,L \\ R\,L \textbf{ if } L \in \mathrm{Dom}\,R \\ \lambda k.L\,(\lambda l.R\,(\lambda r.k\,(\mathsf{Combine}\,l\,r))) \textbf{ otherwise} \end{cases}$$

The last piece, Lower, is also defined recursively. If doing so is defined, Lower applies a tower X to a trivial scope argument, namely the identity function. Otherwise we recurse, applying X to a scope argument constructed by Lower-ing X's 'trace'. As with $\mathsf{Combine}$, the recursive formulation of Lower allows it to apply to towers of arbitrary height.

(115) $$\mathsf{Lower}\,X := \begin{cases} X\,(\lambda x.x) \textbf{ if defined} \\ X\,(\lambda x.\mathsf{Lower}\,x) \textbf{ otherwise} \end{cases}$$

It's possible, albeit tedious, to check that Lift, $\mathsf{Combine}$, and Lower yield the higher-order meaning in (116) for the VP of *exactly^v three boys saw exactly^u five movies*, the higher-order meaning in (117) for the entire sentence, and the dynamic proposition in (118) for the sentence's final, post-Lower meaning. (As in the main text, I use ⌜$\mathbb{B}$⌝ to abbreviate ⟦exactly^v three boys⟧, and ⌜$\mathbb{M}$⌝ to abbreviate ⟦exactly^u five movies⟧).

(116) $\mathsf{Combine}\,(\mathsf{Lift}\,(\mathsf{Lift}\,\textbf{saw}))\,\mathbb{M}$
$= \lambda c.\mathbb{M}\,(\lambda U.c\,(\lambda k.U\,(\lambda u.k\,(\textbf{saw}\,u))))$

(117) $\mathsf{Combine}\,\mathbb{B}\,(\mathsf{Combine}\,(\mathsf{Lift}\,(\mathsf{Lift}\,\textbf{saw}))\,\mathbb{M})$
$= \lambda c.\mathbb{B}\,(\lambda V.\mathbb{M}\,(\lambda U.c\,(\lambda k.V\,(\lambda v.U\,(\lambda u.k\,(\textbf{saw}\,u\,v))))))$

(118) $\mathsf{Lower}\,(\mathsf{Combine}\,\mathbb{B}\,(\mathsf{Combine}\,(\mathsf{Lift}\,(\mathsf{Lift}\,\textbf{saw}))\,\mathbb{M}))$
$= \mathbb{B}\,(\lambda V.\mathbb{M}\,(\lambda U.V\,(\lambda v.U\,(\lambda u.\textbf{saw}\,u\,v))))$

By β-reducing and applying the tower convention, one can verify that (116), (117), and (118) correspond to the various stages of Figure 6 (page 17).

Manipulations like Lift and Lower can be performed 'inside' towers. For example, to 'internally' Lift a tower X — as was required in, e.g., (34) — we can proceed as follows:

(119) $\mathsf{Combine}\,(\mathsf{Lift}\,\mathsf{Lift})\,X = \lambda c.X\,(\lambda x.c\,(\lambda k.k\,x))$

Internally Lower-ing towers — as, for example, in Figure 8 (page 21) — works similarly.

B Post-suppositions, compositionally

I implement the compositional theory of post-suppositions with something called a Writer monad (e.g., Wadler 1995, Giorgolo & Asudeh 2012). The Writer monad is a pair of functions η and $\gg\!=$, which we also take to be freely-applying type-shifters. For notational ease, we introduce a type constructor P, such that $\mathsf{P}A ::= (A \times \mathtt{t})$ ('a post-suppositional A is a pair of an A and a dynamic proposition'), defining η and $\gg\!=$ as follows:

(120) $\eta\,a := (a, \mathbf{T})$ $\qquad\qquad \eta : A \to \mathsf{P}A$

(121) $(a, p) \gg\!= k := ((k\,a)_1, p\,;(k\,a)_2)$ $\qquad\qquad (\gg\!=) : \mathsf{P}A \to (A \to \mathsf{P}B) \to \mathsf{P}B$

The $\gg=$ function turns a post-suppositional pair into something that *takes scope*. Specifically, applying $\gg=$ unarily to some (a, p) gives $\lambda k.\,(a, p) \gg= k$, which is equivalent to the tower $\frac{(a,p) \gg= \lambda a'.[\,]}{a'}$. Based on this, we observe that the post-supposition-friendly $\mathsf{Combine}^+$ operation (Figure 10, page 26), is derivable from η, $\gg=$, $\mathsf{Combine}$, and Lower:

(122) $$\frac{[\,]}{\eta}\left(\frac{(m,p) \gg= \lambda m'.[\,]}{m'}\;\frac{(n,q) \gg= \lambda n'.[\,]}{n'}\right) \xrightarrow{\text{Combine, Lower}} (\mathsf{Combine}\, m\, n, p\, ;q)$$

A post-suppositional derivation of the (surface-scope) distributive reading is complex, but included for completeness. To arrive at the subject's tower, we apply $\lambda v.\,\mathbf{D}_v^+$ to the post-suppositional subject via $\mathsf{Combine}^+$, perform some routine λ-expansions and applications of the tower convention, and internally Lift the subject such that m (the eventual location of $\mathbf{D}_v^+$) scopes over the object's post-suppositions. We $\mathsf{Combine}$ and then Lower, applying η after certain Lower-ings so that everything types out (specifically, when M and n hit the ground floor). We're left with a post-suppositional logical form where $\mathbf{5}_u$ occurs, as expected, within the scope of $\mathbf{D}_v$.

(123) $$\begin{array}{c} \left(\frac{\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,;[\,])}{\mathbf{D}_v^+}, \mathbf{3}_v\right) \gg= \lambda M.[\,] \\ \hline M\,(\lambda m.[\,]) \\ \hline m\,(\lambda v'.[\,]) \\ \hline [\,] \\ \hline v' \end{array} \left(\begin{array}{c} [\,] \\ \hline [\,] \\ \hline [\,] \\ \hline [\,] \\ \hline \mathbf{saw} \end{array}\;\; \begin{array}{c} [\,] \\ \hline [\,] \\ \hline \left(\frac{\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,;[\,])}{u}, \mathbf{5}_u\right) \gg= \lambda n.[\,] \\ \hline n\,(\lambda u'.[\,]) \\ \hline u' \end{array}\right)$$

$$\xrightarrow{\text{Combine, Lower, }\eta} (\mathbf{M}_v\,(\mathbf{E}^v\,\mathbf{boys}\,; \mathbf{D}_v\,(\lambda v'.\mathbf{M}_u\,(\mathbf{E}^u\,\mathbf{movs}\,; \mathbf{saw}\,u\,v')\,; \mathbf{5}_u)), \mathbf{3}_v)$$

Last, I demonstrate that association with a $(\bullet)$ operator isn't obligatory. Let's assume a structure like [$\bullet$ [island]] — i.e., with an operator that discharges post-suppositions guarding the island's edge. If the island is associated with a post-suppositional meaning m, the following derivation is available:

(124) $$\frac{[\,]}{\eta}\left(\frac{[\,]}{(\bullet)}\left(\frac{[\,]}{\eta}\;\frac{m \gg= \lambda p'.[\,]}{p'}\right)\right) \xrightarrow{\text{Combine, Lower}} m$$

Though nothing scopes out of the island, the island itself takes scope via $\gg=$. In the end, $(\bullet)$ has no effect, and the post-suppositions of m live on.

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