Chapter IV

Situation-Theoretic Account of Valid Reasoning with Venn Diagrams

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Venn diagrams are widely used to solve problems in set theory and to test the validity of syllogisms in logic. Since elementary school we have been taught how to draw Venn diagrams for a problem, how to manipulate them, how to interpret the resulting diagrams, and so on. However, it is a fact that Venn diagrams are not considered valid proofs, but heuristic tools for finding valid formal proofs. This is just a reflection of a general prejudice against visualization which resides in the mathematical tradition. With this bias for linguistic representation systems, little attempt has been made to analyze any nonlinguistic representation system despite the fact that many forms of visualization are used to help our reasoning.

The purpose of this chapter is to give a semantic analysis for *a* visual representation system—the Venn diagram representation system.[1] We were mainly motivated to undertake this project by the discussion of multiple forms of representation presented in Chapter I More specifically, we will clarify the following passage in that chapter, by presenting Venn diagrams as a formal system of representations equipped with its own syntax and semantics:

> As the preceding demonstration illustrated, Venn diagrams provide us with a formalism that consists of a standardized system of representations, together with rules of manipulating them. ... We think it should be possible to give an information-theoretic analysis of this system, ...

In the following, the formal system of Venn diagrams is named VENN. The analysis of VENN will lead to interesting issues which have their ana-

[1]This chapter is limited to the use of Venn diagrams to test the validity of syllogisms from traditional logic.

logues in other deductive systems. An interesting point is that VENN, whose primitive objects are diagrammatic, not linguistic, casts these issues in a different light from linguistic representation systems. Accordingly, this VENN system helps us to realize what we take for granted in other more familiar deductive systems. Through comparison with symbolic logic, we hope the presentation of VENN contributes some support to the idea that valid reasoning should be thought of in terms of manipulation of information, not just in terms of manipulation of linguistic symbols.

To support our claim that this use of Venn diagrams is a standard representation system, we aim to develop the syntax and the semantics of this formal system in the following way:

In §1, the primitive objects are clarified for this system and well-formed diagrams are defined. Several interesing issues arise from the fact that the primitive objects of VENN are diagrammatic. For example, unlike with linguistic representation systems, we need an extra relation among tokens of the same type. Also, we need to specify a relation among diagrams which look very similar to each other.

In §2, the semantics of VENN is developed with the help of situation-theoretic tools. First, a homomorphic relationship is formalized between Venn diagrams drawn, say, on a piece of a paper, and information conveyed in syllogisms. It is this relation which allows us to represent certain facts (about which we aim to reason) in terms of certain diagrams and to tell what a diagram conveys. What it is for one diagram to follow from other diagram(s) is definable by the relation of the contents of the diagrams.

In §3, we define what it is to obtain one diagram from other diagrams in this system, and introduce five ways of manipulating diagrams. This establishes the syntax of this system.

The soundness of this system is proved in §4. That is, whenever diagram D is obtainable (as defined in §3) from a set of diagrams Δ, diagram D follows (as defined in §2) from Δ.

In the last section, §5, this system is proved complete. That is, this system, along with its own transformation rules, allows us to obtain any diagram D from a set of diagrams Δ, if D follows from Δ.

1 Syntax

Preliminary Remarks

Let us assume that any representation system aims to represent information about the situation about which we want to reason. VENN, which we are about to examine, adopts diagrams as its medium to effect this representation. We aim to examine the formalism with which Venn diagrams provide us in the following respects:

1. What are the formation rules of meaningful units in this system?
2. What are the meaningful units of this system about?

These two questions help us to answer whether this Venn diagram system is a standard formal representation system or not. If this system is deductive (which we want to claim), then one more question should be answered:

3. What are the rules for manipulating the objects of this system?

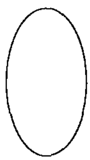
These questions will be discussed in §1, §2 and §3, respectively. In order to address the first point, i.e. what are the formation rules, we need to specify the set of primitive objects of which a meaningful unit in this system consists. Before this syntactic discussion begins, let us consider some of the features we want to incorporate into this representation system.

Let us think of the information which this system aims to convey. The following are examples:

All unicorns are red.
No unicorns are red.
Some unicorns are red.
Some unicorns are not red.

These four pieces of information have something in common. That is, all of these are about some relation between the following sets–the set of unicorns and the set of red things. Each piece of information shows a different relation between these two sets. Therefore, the Venn diagram system needs to represent the following: sets and relations between sets.

A set is represented by a differentiable closed curve which does not self-intersect, as follows:

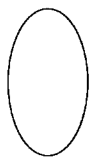


However, a main question is whether we want to have an infinite number of different closed curve-types *or* only one closed curve-type. An analogy with sentential logic might be helpful in this matter. In sentential logic we are given an infinite sequence of sentence symbols, A_1, A_2, ... An atomic sentence of English is translated into a sentence symbol. When we translate different atomic sentences of English into the language of sentential logic, we choose different sentence symbols. It does not matter which sentence symbols we use, as long as we use different symbol-types for different English sentence-types. Another important point is that after choosing a sentence symbol (type), say A_{17}, for a certain English sentence (type), we have to keep using this sentence symbol, the 17th sentence symbol, for the translation of this English sentence-type.

Therefore, if the first alternative—to have different closed curve-types—is chosen for this system, then we can say, just as in the language of sentential logic, that tokens of the same closed curve-type represent the same set. However, in this system we would have to accept the following counterintuitive aspect: very similar looking (or even identical looking) closed curve-tokens might belong to different closed curve-types. Even though we think it is theoretically possible to have such a system, we decide to choose the other alternative, that is, to have only one closed curve-type.

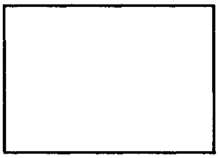
Now one question is how to represent different sets by closed curves. Of course, we want to say that in VENN different sets are represented by different closed curves. However, these different closed curves are tokens of the same type–the closed curve-type, since we have only one closed curve-type. In the case of sentential logic, we have an infinite number of sentence symbol-types to represent an infinite number of English sentence-types. Therefore, different English sentence-types are represented by different sentence symbol-types. In VENN we have only one closed curve-type to represent an infinite number of sets. A main problem is how to tell whether given closed curves represent different sets or not. Of course, we cannot rely on how these tokens look, since every token of a closed curve belongs to one and the same type. It seems obvious that we need an extra mechanism to keep straight the relation among tokens of a closed curve, unlike in the case of sentential logic. This point will be dicussed after the primitive objects are introduced.

Suppose that the following closed curve represents the set of unicorns:



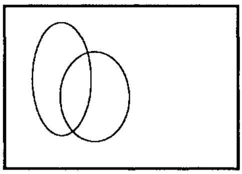
Accordingly, this closed curve makes a distinction between the set of unicorns and anything else. Strictly speaking, the area enclosed by the closed curve, not the closed curve itself, represents the set of unicorns. It will be good if we can treat anything else as a set as well. However, there is no such set as the set of non-unicorns, unless there is a background set. Therefore, we want to introduce a way to represent the background set in each case. Whatever sets we want to represent by closed curves, we can always come up with a background set which is large enough to include all the members of the sets represented by the drawn closed curves.

A background set is represented by a rectangle, as follows:



As in the case of closed curves, we also need some mechanism in order to make sure whether tokens of a rectangle represent the same background set or not.

For the given example, we will draw two closed curves within a rectangle to represent three sets: the set of unicorns, the set of red things and the background set. However, in order to represent the four pieces of information mentioned above, we should be able to represent the following sets: the set of non-red unicorns and the set of red unicorns.[2] We might need to represent the set of red non-unicorns and the set of non-red non-unicorns as well, depending upon the information we want to convey. The moral is to draw closed curves in such a way that we should be able to represent all of these sets in one diagram:



In addition to the background set, the set of unicorns and the set of red things, the overlapping closed curves make a distinction among the set of red unicorns, the set of non-red unicorns, the set of red non-unicorns and the set of non-red non-unicorns. This feature should be incorporated not only into the syntax of this system, that is, into the formation rules, but into the semantics of this system in the following way: Two sets represented by two disjoint areas do not share any element. And a background set, which a rectangle represents, is divided exhaustively by the sets represented by the enclosed areas which are included in the rectangle.[3]

So far, we discussed how this system represents sets. Now what we need is a way to represent relations between sets. For example, the information

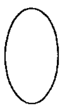
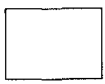
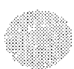
[2] "All unicorns are red" conveys the information that the set of non-red unicorns is empty, while "Some unicorns are not red" conveys the opposite information, that is, the set of non-red unicorns is not empty. "No unicorn is red" says that the set of red unicorns is empty, while "Some unicorns are red" says the opposite.

[3] After establishing the semantics, we can prove that these two desired features are expressed in this semantics. For more detail, refer to Shin [1990].

that all unicorns are red conveys information about a certain relation between the set of unicorns and the set of red things. That is, every member of the former set is also a member of the latter set. However, this relation can be expressed in terms of the set of non-red unicorns. That is, the set of non-red unicorns is empty. In the previous paragraph, we suggested that this system should represent the set of non-red unicorns. Therefore the problem of representing relations between sets reduces to the problem of representing the emptiness or non-emptiness of sets. For the emptiness of a set, we shade the whole area which represents the set. In order to represent that a set is not empty, we put down $\otimes$ in the area representing the set. If the set is represented by more than one area, we draw $\otimes$ in each area and connect the $\otimes$'s by lines. For this, we adopt the expression $\otimes^n$ $(n \geq 1)$ and call it an X-sequence. Each X-sequence consists of a finite number of X's and (possibly) lines. The formation rules deal with each object in detail.

Primitive Objects

We assume we are given the following sequence of distinct diagrammatic objects to which we give names as follows:

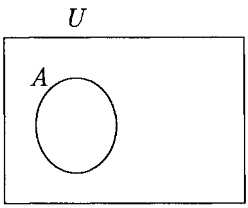
Diagrammatic Objects	Name
	closed curve
	rectangle
	shading
∞	X

How can we talk about diagrams or parts of diagrams of this system? In the case of linguistic representation systems, we adopt the convention to use quotation marks. For this visual representation system, we suggest writing down a letter for the name of each diagram and each closed curve and each rectangle. Introducing letters as names (not as part of the language but as a convention for our convenience) solves our problem of how to mention diagrams, rectangles or closed curves. But, how do we mention the rest of our system, i.e., shadings and X's?

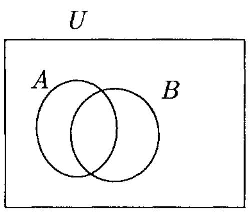
First, let me introduce three terms for our further discussion—region, basic region and minimal region. By *region*, we mean any enclosed area in a diagram. By *basic region*, we mean a region enclosed by a rectangle or by a closed curve. By *minimal region*, we mean a region within which no other region is included. The set of regions of a diagram D (let us name it $\mathrm{RG}(D)$) is the smallest set satisfying the following clauses:

1. Any basic region of diagram D is in $\mathrm{RG}(D)$.
2. If R_1 and R_2 are in set $\mathrm{RG}(D)$, then so are the intersection of R_1 and R_2, the union of R_1 and R_2, and the difference between R_1 and R_2.

For reasons that we will see soon, we need to refer to the regions of a diagram. We can name the regions which are made up of rectangles or closed curves by using the names of the rectangles and closed curves. For example, in the following diagram,



there are three regions—a part enclosed by a rectangle, U, a part enclosed by a closed curve, A, and a part enclosed by U and and not by A. As said above, let us name a region after the name of the closed curve or the rectangle which encloses it. So, the first region is region U and the second is region A. Since the third region is the difference between region U and region A, we name it region $U - A$. Let us think of the case in which some closed curves overlap with each other, as in the following example:



We refer to the region intersected by both region A and region B as region A-*and*-B, and refer to the region which is the union of region A and region B as region $A + B$.

To implement these ideas, we get the following convention of naming the regions:

1. A basic region enclosed by a closed curve, say, A, or enclosed by a rectangle, say, U, is named region A or region U.

   Let R_1 and R_2 be regions. Then,

2. A region which is the intersection of R_1 and R_2 is named R_1-*and*-R_2.

3. A region which is the union of R_1 and R_2 is named $R_1 + R_2$.

4. A region which is the difference between R_1 and R_2 is named $R_1 - R_2$.

Recalling the definition of set RG(D) (the set of regions of diagram D) given above, we know that this convention of naming the regions exhausts the cases.

Let us go back to the question of how to talk about shadings and X's. As we will see soon, any shading or any X of any diagram (at least any interesting diagram) is in some region. Now, in order to mention these constituents of our language we can refer to them in terms of the names of the smallest regions. For example, we can refer to a shading or an X which is in region A (where A is the smallest region with these constituents) as the shading in region A or the X in region A.

Before moving to the formation rules of this system, we need to discuss one more point mentioned in the preliminary remarks: a relation among closed curve-tokens and a relation among rectangle-tokens. Suppose that the following are given to us:

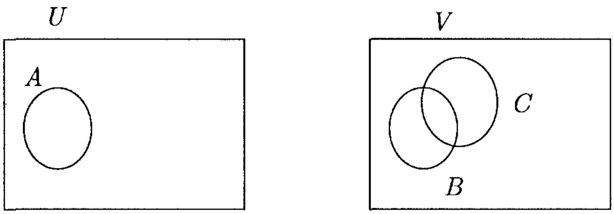
How can we tell whether these two closed curves represent different sets or not? It depends on whether a user of the Venn diagrams intends to represent the same set or different sets by these two tokens of the closed curve. Accordingly, the relation in which we are interested is the relation that holds among closed curve-tokens or among rectangle-tokens which represent the same set. Let us name this relation a counterpart relation. Then, we can think of the following features for this special relation: First of all, this relation should be an equivalence relation among basic regions of given diagrams. Second, a counterpart relation holds only among tokens of the same type—among closed curves-tokens and among rectangle-tokens. Third, since a user would not draw two closed curves within one diagram to represent the same set, we want to say that within one diagram a counterpart relation does not hold among distinct basic regions.

Given diagrams $D_1, \ldots, D_n$, let a counterpart relation (let us call it set *cp*) be an equivalence relation on the set of basic regions of $D_1, \ldots, D_n$ satisfying the following:

1. If $\langle A, B\rangle \in cp$, then both A and B are either closed curves or rectangles.

2. If $\langle A, B\rangle \in cp$, then <u>either</u> A is identical to B <u>or</u> A and B are in different diagrams.

Within one diagram, every basic region enclosed by a closed curve or a rectangle has only one counterpart, that is, itself. Therefore, we have only one *cp* set. However, when more than one diagram is given, there would not be a unique set *cp*. For example, in the following diagrams,



all of the following sets satisfy the conditions of set *cp*:

1. $\{\langle U,U\rangle, \langle V,V\rangle, \langle A,A\rangle, \langle B,B\rangle, \langle C,C\rangle\}$
2. $\{\langle U,U\rangle, \langle V,V\rangle, \langle A,A\rangle, \langle B,B\rangle, \langle C,C\rangle, \langle U,V\rangle, \langle V,U\rangle\}$
3. $\{\langle U,U\rangle, \langle V,V\rangle, \langle A,A\rangle, \langle B,B\rangle, \langle C,C\rangle, \langle A,B\rangle, \langle B,A\rangle\}$
4. $\{\langle U,U\rangle, \langle V,V\rangle, \langle A,A\rangle, \langle B,B\rangle, \langle C,C\rangle, \langle A,C\rangle, \langle C,A\rangle\}$
5. $\{\langle U,U\rangle, \langle V,V\rangle, \langle A,A\rangle, \langle B,B\rangle, \langle C,C\rangle, \langle U,V\rangle, \langle V,U\rangle, \langle A,B\rangle, \langle B,A\rangle\}$
6. $\{\langle U,U\rangle, \langle V,V\rangle, \langle A,A\rangle, \langle B,B\rangle, \langle C,C\rangle, \langle U,V\rangle, \langle V,U\rangle, \langle A,C\rangle, \langle C,A\rangle\}$

Among these equivalence relations, a user chooses set *cp* for each occasion. For example, if a user intends to represent the same set with A and C and the same set with U and V, then this user chooses the sixth equivalence relation as set *cp*. Some user might intend to represent the same set by A and B and the same set by U and V. In this case, the fifth relation above will be the set *cp* the user chooses. Or, some user might intend to represent different sets by each closed curve and by each rectangle. In this case, the user chooses set *cp* such that all of its elements consist of a basic region and itself—the first relation above.

Well-Formed Diagrams

We assumed that any finite combination of diagrammatic objects is a diagram. However, not all of the diagrams are well-formed diagrams, just as not all of the expressions are well-formed formulas in sentential logic or first-order logic. The set of well-formed diagrams, say $\mathcal{D}$, is the smallest set satisfying the following rules:

1. Any rectangle drawn in the plane is in set $\mathcal{D}$.

2. If D is in the set $\mathcal{D}$, then if D' results by adding a closed curve interior to the rectangle of D by the partial-overlapping rule (described below), then D' is in set $\mathcal{D}$.

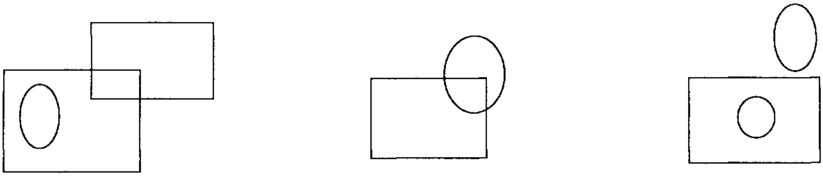
   Partial-overlapping rule: A new closed curve should overlap *every* existent minimal region, but *only* once and *only* part of each minimal region.

3. If D is in the set $\mathcal{D}$, and if D' results by shading some entire region of D, then D' is in set $\mathcal{D}$.

4. If D is in the set $\mathcal{D}$, and if D' results by adding an X to a minimal region of D, then D' is in set $\mathcal{D}$.

5. If D is in the set $\mathcal{D}$, and if D' results by connecting existing X's by lines (where each X is in different regions), then D' is in set $\mathcal{D}$.

According to this recursive definition, every well-formed diagram should have one and only one rectangle. It also tells us that if there is any closed curve on a diagram, it should be in the rectangle. Therefore, this definition rules out all the following diagrams as ill-formed:



Let me illustrate through examples how the partial-overlapping rule in clause 2 works. By the first clause, for any well-formed diagram there should be a rectangle in it. Let us name it U as follows:

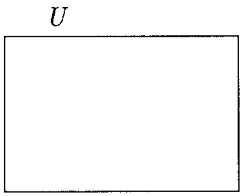


Diagram a

First, let us try to draw a new closed curve, A. How does this partial-overlapping rule work? In diagram a, there is only one minimal region—the region U. The new closed curve A should be drawn to overlap a part of this existent region.
Therefore,

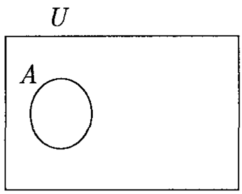


Diagram b

Next, we are going to draw another new closed curve, B, on this sheet of paper. In diagram b , there are two minimal regions—region A and region $U - A$. According to this partial-overlapping rule, the new closed curve B should overlap each of these two regions, but only partially and only once. That is,

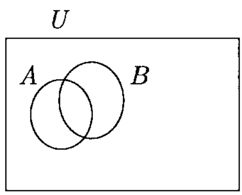


Diagram c

Therefore, the following diagrams are ruled out:

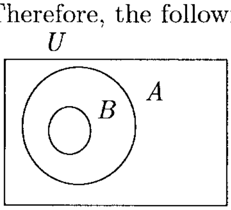


Diagram 1

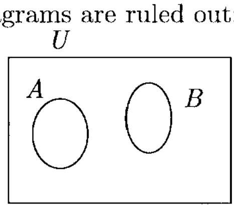


Diagram 2

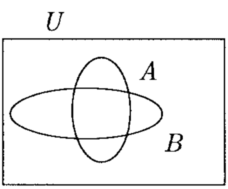


Diagram 3

In Diagram 1, one minimal region—region $U - A$—is not overlapped by the new closed curve B. In Diagram 2, one minimal region—region A—is not overlapped by B at all. In Diagram 3, minimal region $U - A$ is overlapped by the new closed curve B twice.

Next, we want to draw one more closed curve, C. All the following diagrams are eliminated:

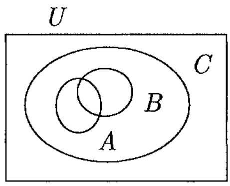


Diagram 4

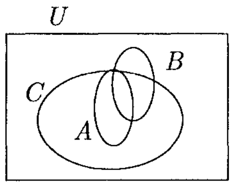


Diagram 5

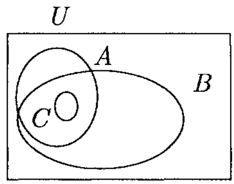


Diagram 6

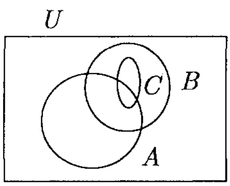


Diagram 7

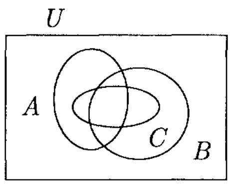


Diagram 8

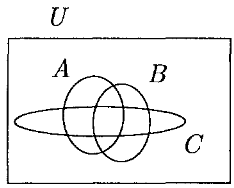


Diagram 9

Diagram c on the previous page has four minimal regions: region $U - (A + B)$, region $A - B$, region A-*and*-B and region $B - A$. We have to make sure that the third closed curve, C, should overlap every part of these four minimal regions. That is,

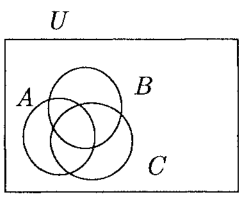
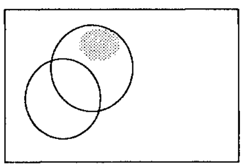
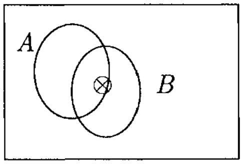


Diagram d

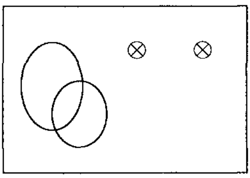
Clause 3 says that, a shading, if there is any, should fill up region(s). Therefore, the following is not well-formed:

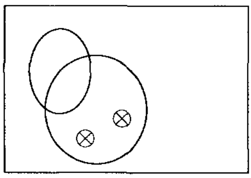


Clause 4 tells us that the following diagram cannot be well-formed, since an X is not in a minimal region. It is in region B.
However, region B is not a minimal region:

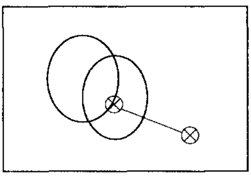


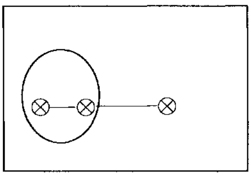
Clause 4 and Clause 5 together tell us that if there is any X-sequence, each X of the sequence should be in a minimal region. Clause 5 also tells us that each X of an X-sequence should be in a different minimal region. The following are well-formed:

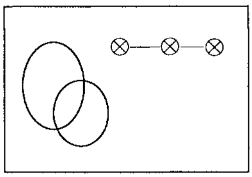




On the other hand, the following cannot be well-formed:







From now on, let us abbreviate "well-formed diagram" by "*wfd.*"

2 Semantics

As discussed in the preliminary remarks, this representation system aims to represent sets and certain relations among those sets. We think that two kinds of homomorphisms are needed for this representation system, one for the representation of sets and the other for the representation of relations among sets. Based on these two homomorphisms (which are defined in below), we will formalize what it is for one diagram to follow other diagrams.

From regions to sets

Each basic region, which is made by a closed curve or a rectangle, represents a set. Since we are concerned with *wfds*, let us define the set BRG to be the set of all basic regions of *wfds*. Let $\mathcal{D}$ be a set of *wfds*. That is,

$$\text{BRG} = \{\text{a basic region of } D \mid D \in \mathcal{D}\}.$$

Let U be a non-empty domain. Then, for any function f such that

$$f\colon \text{BRG} \to \mathcal{P}(U), \text{ where if } \langle A, B\rangle \in cp, \text{ then } f(A) = f(B),$$

we can extend this function to get the mapping from set RG—the set of all regions of diagrams in set $\mathcal{D}$—to $\mathcal{P}(U)$. This extended relation is a homomorphism between regions and sets. That is, $\overline{f} : \text{RG} \to \mathcal{P}(U)$, where

$$\overline{f} = \begin{cases} f(A) & \text{if } A \in \text{BRG} \\ \overline{f}(A_1) - \overline{f}(A_2) & \text{if } A = A_1 - A_2 \\ \overline{f}(A_1) \cap \overline{f}(A_2) & \text{if } A = A_1\text{-}and\text{-}A_2 \\ \overline{f}(A_1) \cup \overline{f}(A_2) & \text{if } A = A_1 + A_2 \end{cases}$$

From facts to facts

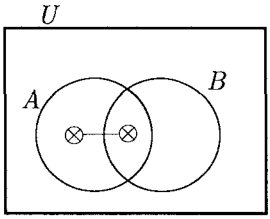
Among the primitive objects of this system listed in §1, rectangle-tokens and closed curve-tokens make up regions, and these regions represent sets as seen above. The other objects — a shading and an X — represent certain facts about the sets represented by the regions in which these constituents are drawn. The following are important representational relations in this system:

- A shaded region represents the empty set.
- A region with an X-sequence represents a non-empty set.

How do we define this representational relation? We want to define a function between facts about regions of a diagram and facts about sets of a situation. There are many facts about regions of a diagram. However, as said before, not all of them are representing facts. The function we have in mind is concerned only with the representing facts in diagrams. What are the representing facts in this system? The two kinds of representational relations listed above show us what the representing facts are:

1. A region, say A, is shaded.
2. An X-sequence is in a region, say A.

We need some remark about the second statement. This statement can be ambiguous in the following reason: If an X-sequence is in region A, then we can say that the X-sequence is also in region B if region A is a part of region B. For example, it is true that the X-sequence, $\otimes^2$, is in region A in the following diagram:



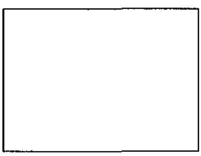
However, it is also true that the X-sequence is in region $A+B$, or in region U. In this case, we are concerned with the smallest region with the X-sequence. Since each $\otimes$ of the X-sequence is in a minimal region in a *wfd*, the union of the minimal regions with $\otimes$ is the smallest region with the X-sequence. Now, we can express the representing facts (listed above) by means of situation-theoretic terminology—infons—as follows:

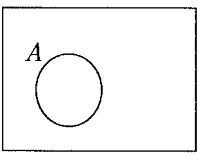
(1) $\langle\langle$shaded, A; 1$\rangle\rangle$.
(2) $\langle\langle$In, $\otimes^n, A$; 1$\rangle\rangle$, where
$A = A_1 + \cdots + A_n$ and
for every $1 \leq i \leq n$, A_i is a minimal region with $\otimes$.

What are the facts that these representing facts represent? Based upon the homomorphism $\overline{f}$ defined above, let us express these represented facts by means of situation-theoretic terminology:

(1′) $\langle\langle$Empty, $\overline{f}(A)$; 1$\rangle\rangle$
(2′) $\langle\langle$Empty, $\overline{f}(A)$; 0$\rangle\rangle$

If these were all representing facts, then we would have to say that the following two diagrams contain the same representing facts:





However, we want to say that these two diagrams represent different facts. Intuitively, the diagram on the right represents the set region A represents along with the background set, while the left one does not represent any set, except the background set. Therefore, we want to add one more representing fact:

(3) $\langle\langle$Region, A; 1$\rangle\rangle$

The following fact is represented by this representing fact:

(3′) $\langle\langle \text{Set}, \overline{f}(A); 1\rangle\rangle$

Let $\mathcal{D}$ be a set of *wfds*. Let $\mathcal{D}$-*Facts* be the set of the facts about diagrams in $\mathcal{D}$. That is,

$$\mathcal{D}\text{-}Facts = \{\alpha \mid D \models \alpha \text{ and } D \in \mathcal{D}\}, \text{ where}$$
$$D \models \alpha \text{ iff } \alpha \text{ is true of the diagram } D.$$

Let $\mathcal{S}$-*Facts* be as follows: (Suppose that U is a non-empty domain.)

$$S\text{-}Facts = \{\alpha \mid \alpha = \langle\langle \text{Empty}, a; i\rangle\rangle \text{ or } \langle\langle \text{Set}, a; 1\rangle\rangle\}, \text{ where}$$
$$i \in \{0, 1\} \text{ and } a \in \mathcal{P}(U).$$

Since not all of the facts about diagrams are representing facts, the homomorphism between $\mathcal{D}$-*Facts* and $\mathcal{S}$-*Facts* must be partial. Define a homomorphism h from facts about diagrams to facts about sets as follows:

$$h(\langle\langle \text{Shaded}, A; 1\rangle\rangle) = \langle\langle \text{Empty}, \overline{f}(A); 1\rangle\rangle,$$
$$h(\langle\langle \text{In}, \otimes^n, A; 1\rangle\rangle) = \langle\langle \text{Empty}, \overline{f}(A); 0\rangle\rangle.$$
$$h(\langle\langle \text{Region}, A; 1\rangle\rangle) = \langle\langle \text{Set}, \overline{f}(A); 1\rangle\rangle.$$

Content of a diagram

In a deductive system, the semantics of the system allows one to define what it is for a sentence (of the language) to follow from a set of other sentences (of the language). In the case of sentential logic, *wff* α follows from a set Γ of *wffs* if and only if every truth assignment which satisfies every member of Γ also satisfies α. In the case of first-order logic, sentence α follows from a set Γ of sentences if and only if every model of Γ is also a model of α. In either case, a definition for a logical consequence seems to fit our intuition that a conclusion follows from the premises if the truth of the premises guarantees the truth of the conclusion.

What we want is for the semantics of VENN to define a similar kind of inference between a *wfd* and a set of *wfds*. What is it for one *wfd* to follow from a set of other *wfds*, using the analogy of deductive systems? We think that in this representation system, the content of a *wfd* is a counterpart of truth assignment (in sentential logic) or of structure (in first-order logic). As a logical consequence relation between *wffs* is defined in terms of the truth values of *wffs*, we can expect a similar consequence relation between *wfds* to be defined in terms of the contents of *wfds*, as follows:

(1) *Wfd* D follows from a set Δ of *wfds* ($\Delta \models D$) iff the content of the diagrams in Δ involves the content of the diagram D.

Therefore, we need to formalize the content of a diagram and the involvement relation between the contents of diagrams.

What is the content of a diagram? By the two homomorphisms defined above, we can draw a diagram to represent certain facts of the situations about which we aim to reason. Also, we can talk about what a diagram represents—the content of a diagram. Therefore, the content of a *wfd* D, $Cont(D)$, is defined as the set of the represented facts:

$$Cont(D) = \{h(\alpha) \mid D \models \alpha \text{ and } D \in \mathcal{D}\}, \text{ where}$$
$$h \text{ is the homomorphism defined above.}$$

Suppose Δ is a set of *wfds*. Then, the content of the diagrams in this set (say, $Cont(\Delta)$) is the union of the contents of every diagram in Δ. So,

$$Cont(\Delta) = \bigcup_{D \in \Delta} Cont(D)$$

What does it mean that the content of *wfds* in Δ involves the content of *wfd* D? Let us express this as $Cont(\Delta) \Longrightarrow Cont(D)$. Here, we use Barwise and Etchemendy's infon algebra scheme to define the relation between the contents of diagrams.

Let U be a set such that it is a universe of objects. Let Sit be a subset of $\mathcal{P}(\mathcal{P}(U))$ such that it is closed under $\cup$ and $-$. We define a situation s to be $s \in Sit$—a set of subsets of U closed under $\cup$ and $-$. Let σ be a basic infon such that $\sigma = \langle\!\langle R, a; i \rangle\!\rangle$, where $R \in \{\text{Empty}, \text{Set}\}$, a is a set and $i \in \{0, 1\}$. We define what it means for a basic infon σ to be supported by one of these situations s, as follows:

$$\begin{array}{lll} s \models \langle\!\langle \text{Empty}, x; 1 \rangle\!\rangle & \text{iff} & x \in s \text{ and } x = \emptyset. \\ s \models \langle\!\langle \text{Empty}, x; 0 \rangle\!\rangle & \text{iff} & x \in s \text{ and } x \neq \emptyset. \\ s \models \langle\!\langle \text{Set}, x; 1 \rangle\!\rangle & \text{iff} & x \in s. \end{array}$$

Let Σ_1 and Σ_2 be sets of infons. We define the involvement relation as follows:

$$(2)\ \Sigma_1 \Longrightarrow \Sigma_2 \qquad \text{iff} \qquad \forall_{s \in Sit}(\forall_{\alpha \in \Sigma_1} s \models \alpha \rightarrow \forall_{\beta \in \Sigma_2} s \models \beta)$$

The formal scheme expressed in (2) reshapes our intuitive idea on the inference of a *wfd* from a set of *wfds*, expressed in (1), as follows:

Definition: *Wfd D follows* from set of *wfds* Δ ($\Delta \models D$) iff every situation which supports every member of $Cont(\Delta)$ also supports every member of $Cont(D)$ ($Cont(\Delta) \Longrightarrow Cont(D)$).

Recall that our definition of the content of a *wfd* tells us the information that the diagram conveys. The content of the diagrams in Δ involves the content of the diagram D if and only if the information of diagram D is extractable from the information of the diagrams in Δ. Therefore, the above definition for $\Delta \models D$ reflects our intuition that a valid inference is a process of extracting certain information from given information.

3 Rules of Transformation

In this section, we define what it means to obtain a diagram from other diagrams.

Definition: Let Δ be a set of *wfds* and D be a *wfd*. *Wfd* D is *obtainable* from a set Δ of *wfds* ($\Delta \vdash D$) iff there is a sequence of *wfds* $< D_1, \ldots, D_n >$ such that $D_n \equiv D$[4] and for each $1 \leq k \leq n$ *either*
(a) there is some D' such that $D' \in \Delta$ and $D' \equiv D_k$, *or*
(b) there is some D' such that for some $i, j < k$, a rule of transformation allows us to get D' from either D_i or D_j (or both) and $D' \equiv D_k$.

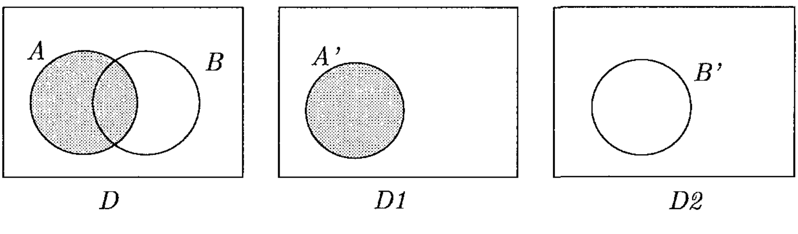
We are concerned with transformations only from *wfds* to *wfds*. Therefore, let us assume that given diagrams are always well-formed. Also, it is assumed that we should not get an ill-formed diagram and should apply each rule within this limit so as to get only *wfds*.[5]

R1: *The rule of erasure of a diagrammatic object*

We may copy[6] a *wfd* omitting a diagrammatic object, that is, a closed curve or a shading or a whole X-sequence. Let us go through examples for erasing each object.

(i) When we erase a closed curve, certain regions disappear. Shadings drawn in these regions are erased as well so that the resulting diagram is a *wfd*. In the following cases, the transformation from the left figure into the right one is done by the application of this rule.

(Case 1)

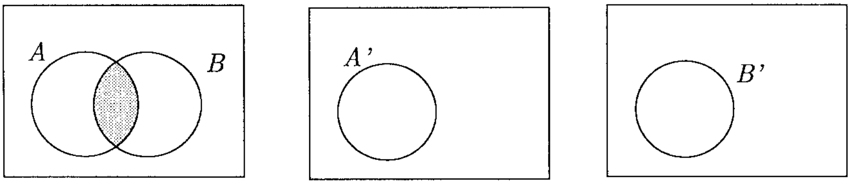


[4] D_1 is equivalent to D_2 if and only if (i) every basic region of D_1 has a counterpart region in D_2 and vice versa, (ii) every shading of D_1 has a counterpart shading in D_2 and vice versa, (iii) every X-sequence of D_1 has a counterpart X-sequence in D_2 and vice versa. For a more rigorous definition, refer to §2.3.2 of Shin [1990].

[5] Another alternative: We could formulate each rule in such a way as to prevent us from getting any ill-formed diagrams.

[6] If diagram D_1 is a *copy* of diagram D_2, then two diagrams are equivalent to each other. For more detail, refer to Shin [1990].

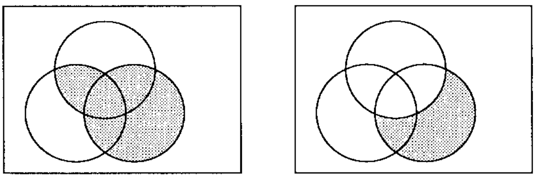
(Case 2)



D

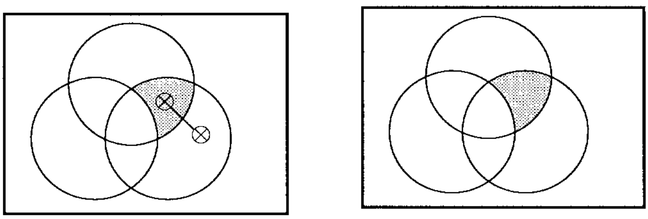
(ii) When we omit the shading in some region, we should erase the entire shading in a minimal region. Otherwise, we get an ill-formed diagram. In the following case, this rule allows us to transform the diagram on the left to the diagram on the right.

(Case 3)



(iii) The erasure of a whole X-sequence allows the transformation from the left figure to the right one.

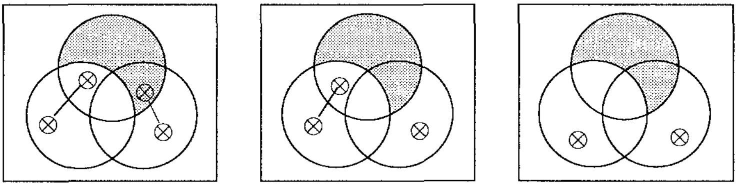
(Case 4)



R2: *The rule of erasure of the part of an X-sequence*

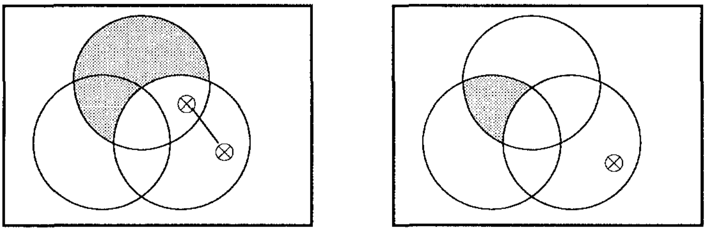
We may copy a *wfd* omitting any part of an X-sequence only if that part is in a shaded region. That is, we may erase ⊗- or -⊗, only if the ⊗ in ⊗- or -⊗ is in a shaded region. Let us compare the following two cases:

(Case 5)



D *D1* *D2*

(Case 6)

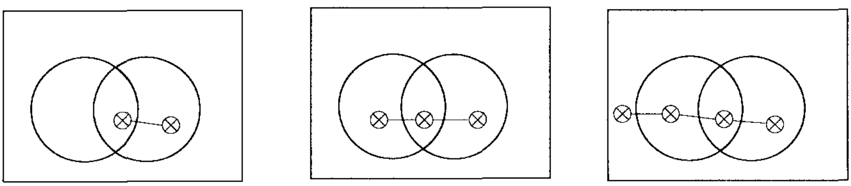


The transformation in Case 5 is legitimate by this rule. However, we do not have any rule to allow the transformation in Case 6. The part of the X-sequence which is erased in the diagram on the right, that is, -⊗, is not in the shaded part of the diagram on the left. Therefore, Rule 2 does not allow this partial erasure. Rule 1 is concerned only with the erasure of a whole X-sequence, not with a proper sub-part of an X-sequence.

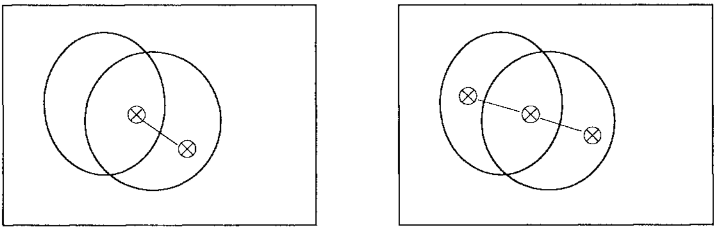
R3: *The rule of spreading X's*

If *wfd D* has an X-sequence, then we may copy *D* with ⊗ drawn in some other region and connected to the existing X-sequence. For example,

(Case 7)



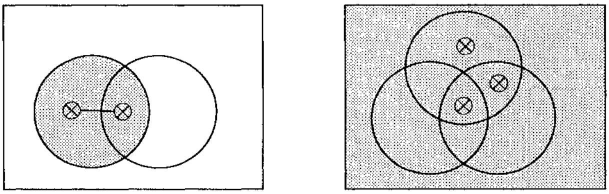
(Case 8)



R4: *The rule of conflicting information*

If a diagram has a region with both a shading and an X-sequence, then we may transform this diagram to any diagram. This rule allows us to transform the diagram on the left to the diagram on the right.

(Case 9)



R5: *The rule of unification of diagrams*

We may unify two diagrams, D_1 and D_2, into one diagram, call it D, if the given *cp* relation contains the ordered pair of the rectangle of D_1 and the rectangle of D_2.

D is the *unification* of D_1 and D_2 if the following conditions are satisfied:

1. The rectangle and the closed curves of D_1 are copied[7] in D.
2. The closed curves of D_2 which do not stand in the given *cp* relation to any of the closed curves of D_1 are copied in D. (Note: Since D is a *wfd*, the partial overlapping rule should be observed.)
3. For any region A shaded in D_1 or D_2, the $\overline{cp}$-related region[8] to A of D should be shaded.

[7]Let A and B be closed curves. If A is a *copy* of B, then $\langle regionA, region\rangle \in cp$.

[8]Given set cp, set $\overline{cp}$ is a binary relation on RG such that $\overline{cp}$ is the smallest set satisfying the following:

1. If $\langle A, B\rangle \in cp$, then $\langle A, B\rangle \in \overline{cp}$.

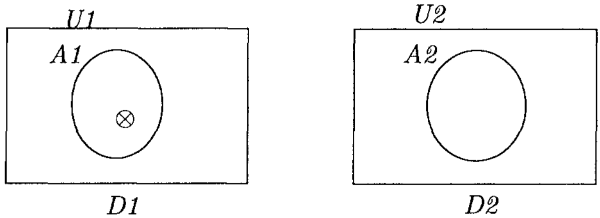
Suppose that $\langle A, B\rangle \in \overline{cp}$ and $\langle C, D\rangle \in \overline{cp}$.

2. If $A + C \in$ RG and $B + D \in$ RG, then $\langle A + C, B + D\rangle \in \overline{cp}$, $\langle A\text{-}and\text{-}C, B\text{-}and\text{-}D\rangle \in \overline{cp}$, $\langle A - C, B - D\rangle \in \overline{cp}$ and $\langle C - A, D - B\rangle \in \overline{cp}$.

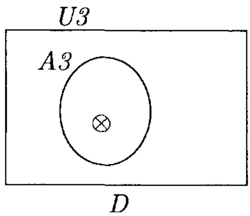
4. For any region A with an X-sequence in D_1 or D_2, an X-sequence should be drawn in the $\overline{cp}$-related regions to A of D.

Let me illustrate this rule through several examples.

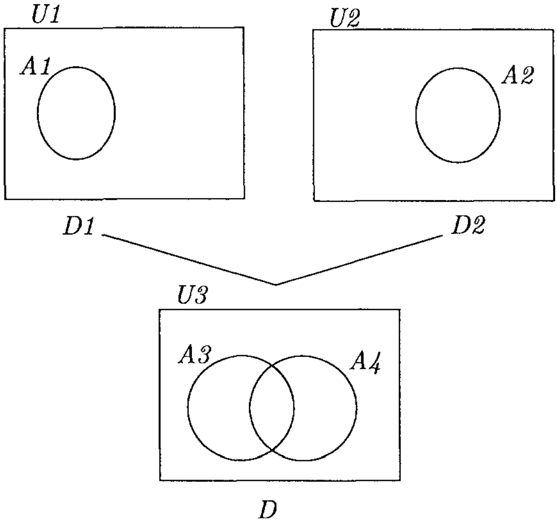
(Case 10) Two diagrams, D_1 and D_2, are given, where $\langle U_1, U_2\rangle$, $\langle A_1, A_2\rangle$ $\in cp$:



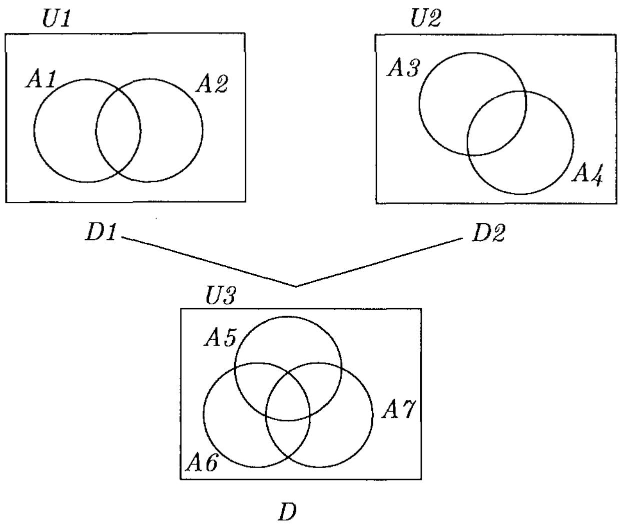
Since the given cp relation holds between the rectangle of D_1 and the rectangle of D_2, we can unify two diagrams. First, we copy the rectangle and the closed curve of D_1 and name them U_3 and A_3 respectively. Accordingly, $\langle U_1, U_3\rangle$, $\langle A_1, A_3\rangle \in cp$. Since the closed curve of D_2, i.e. A_2, is cp-related to the closed curve of D_1, i.e. A_1, we do not add any closed curve. An X in region A_1 of D_1 should be drawn in region A_3 of D, since A_1 and A_3 are $\overline{cp}$-related and A_3 is a minimal region. Therefore, we obtain diagram D:



(Case 11) Let $\langle U_1, U_2\rangle \in cp$, and $\langle A_1, A_2\rangle \notin cp$. Since A_2 is not cp-related to any closed curve in D_1, we draw the cp-related closed curve, i.e. A_4, in D, to get diagram D as follows (notice that the partial overlapping rule is observed):

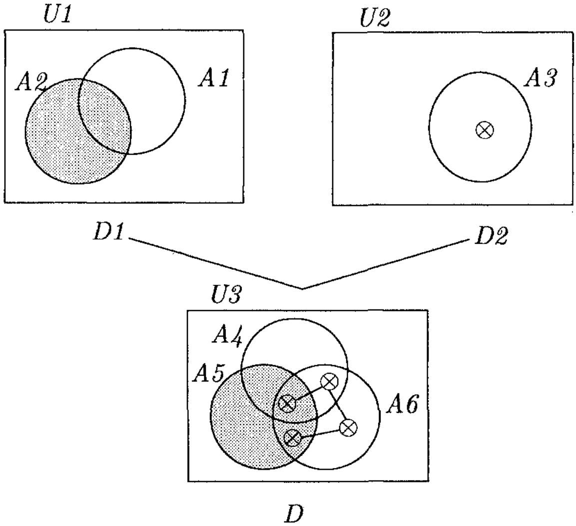


(Case 12) Let $\langle U_1, U_2\rangle \in cp$, $\langle A_1, A_4\rangle \in cp$, $\langle A_2, A_3\rangle \notin cp$. By definition of set cp, $\langle A_1, A_2\rangle \notin cp$, $\langle A_3, A_4\rangle \notin cp$. Accordingly, $\langle A_1, A_3\rangle \notin cp$ (since cp is an equivalence relation). After copying the rectangle and the closed curves of D_1, we need to copy closed curve A_3. Therefore, the following holds: $\langle U_1, U_3\rangle$, $\langle U_2, U_3\rangle$, $\langle A_1, A_5\rangle$, $\langle A_4, A_5\rangle$, $\langle A_2, A_6\rangle$, $\langle A_3, A_7\rangle \in cp$.



Now, we will see how this rule transforms diagrams with shadings or X-sequences into one diagram:

(Case 13) Let $\langle U_1, U_2 \rangle \in cp$ and $\langle A_1, A_3 \rangle$, $\langle A_2, A_3 \rangle \notin cp$.



Suppose that closed curve A_4 is the copy of A_1, A_5 is of A_2 and A_6 is of A_3. Accordingly, $\langle A_1, A_4 \rangle$, $\langle A_2, A_5 \rangle$, $\langle A_3, A_6 \rangle \in cp$. Clause 3 of the unification rule tells us that the shading in region A_2 of D_1 should be copied in the $\overline{cp}$-related region, i.e. region A_5, in diagram D. Our syntactic rule prevents us from copying the X in region A_3 into the $\overline{cp}$-related region, i.e. region A_6, in the unified diagram, since region A_6 is not a minimal region any more. Clause 4 of the unification rule guides us to draw $\otimes^4$ in the $\overline{cp}$-related region, i.e. A_6, to get a well-formed unified diagram D.

4 Soundness

We defined what it is for one diagram to follow from other diagrams ($\Delta \models D$). We also defined what it is for one diagram to be obtained from other diagrams ($\Delta \vdash D$). Now, we raise the question about the soundness of this representation system. Whenever one *wfd* D is obtainable from a set Δ of *wfds* (i.e., $\Delta \vdash D$), is it the case that the content of diagrams in Δ involves the content of D (i.e., $\Delta \models D$)? That is, we want to prove that if $\Delta \vdash D$, then $\Delta \models D$.

Proof: Suppose that $\Delta \vdash D$. By definition, there is a sequence of *wfds* $< D_1, \ldots, D_n >$ such that $D_n \equiv D$ and for each $1 \leq k \leq n$ *either* (a) there is some D' such that $D' \in \Delta$ and $D' \equiv D_k$, *or* (b) there is some D' such that for some $i, j < k$, a rule of transformation allows us to get D' from either D_1 or D_2 (or both) and $D' \equiv D_k$. We show by induction on the length of a sequence of *wfds* that for any diagram D obtainable from Δ, the content of the diagrams in Δ involves the content of D.

(Basis Case) This is when the length of the sequence is 1. That is, $D_1 \equiv D$. Since there is no previous diagram in this sequence, it should be the case that there is some D' such that $D' \in \Delta$ and $D' \equiv D_1$. Since $D_1 \equiv D$ and $D' \equiv D_1$,

1. $D' \equiv D$ (since $\equiv$ is symmetric and transitive)
2. $Cont(D') = Cont(D)$ (by 1 and corollary 8.2 of Appendix B)
3. $Cont(D') \subseteq Cont(\Delta)$ (since $D' \in \Delta$, by the definition of $Cont(\Delta)$)
4. $Cont(D) \subseteq Cont(\Delta)$ (by 2 and 3)
5. $Cont(\Delta) \Longrightarrow Cont(D)$ (by 4 and the definition of $\Longrightarrow$)

Therefore, $\Delta \models D$.

(Inductive Step) Suppose that for any *wfd* D if D has a length of a sequence less than n, then $\Delta \models D$. We want to show that if *wfd* D has a length of a sequence n then $\Delta \models D$. That is, $D_n \equiv D$. If there is some D' such that $D' \in \Delta$ and $D' \equiv D_n$, then as we proved in the basis case, $\Delta \models D$. Otherwise, it must be the case that there is some D' such that for some $i, j < n$, a rule of transformation allows us to get D' from either D_1 or D_2 (or both) and $D' \equiv D_n$. By our inductive hypothesis, $\Delta \models D_i$ and $\Delta \models D_j$. That is, $Cont(\Delta) \Longrightarrow Cont(D_i)$ and $Cont(\Delta) \Longrightarrow Cont(D_j)$. Therefore, $Cont(\Delta) \Longrightarrow (Cont(D_i) \cup Cont(D_j))$. Since each rule of transformation is valid,[9] if D' is obtained by either D_i or D_j, then $(Cont(D_i) \cup Cont(D_j)) \Longrightarrow Cont(D')$. By the transitivity of the involvement relation, it is the case that $Cont(\Delta) \Longrightarrow Cont(D')$. Since $D' \equiv D_n$ and $D_n \equiv D$, $D' \equiv D$. Hence, $Cont(D') = Cont(D)$. Accordingly, $Cont(\Delta) \Longrightarrow Cont(D)$. Therefore, $\Delta \models D$. □

5 Completeness

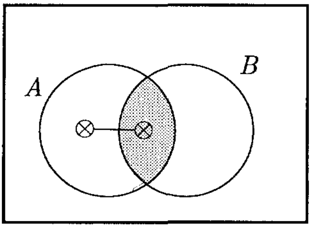
In this section, we raise the question about the completeness of this representation system. Whenever the content of diagrams in Δ involves the content of D (i.e., $\Delta \models D$), is it the case that one *wfd* D is obtainable from a set Δ of *wfds* (i.e., $\Delta \vdash D$) in this system?

Closure Content

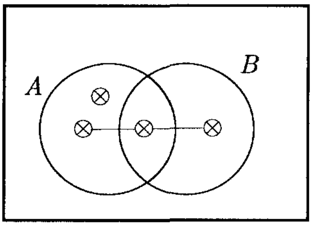
According to the definition of the content of a diagram in §2, given a homomorphism, the representing facts of a diagram determine the content of the diagram. However, according to the definition of the involvement in

[9] For the proof, refer to Shin [1990].

§2, the content of a diagram might involve more than its content itself. For example, the following diagram supports only two infons—$\langle\langle$In, $\otimes^2, A; 1\rangle\rangle$ and $\langle\langle$Shaded, $A\text{-}and\text{-}B; 1\rangle\rangle$:



Hence, $Cont(D) = \{\langle\langle$Empty, $\overline{f}(A); 0\rangle\rangle$, $\langle\langle$Empty, $\overline{f}(A\text{-}and\text{-}B);\ 1\rangle\rangle\}$. However, the content of this diagram involves other sets of infons as well. For example, $\{\langle\langle$Empty, $\overline{f}(A - B); 0\rangle\rangle$, $\langle\langle$Empty, $\overline{f}(A + B); 0\rangle\rangle\}$ is involved by the content of this diagram. That is, the content of the diagram above involves the content of the following diagram:



If we unify these two diagrams by the unification rule, then the content of the unified diagram is involved by the content of these two diagrams. Here, we are interested in the maximal content involved by the content of a given diagram. Let *Closure-Cont*(D) be a set such that it is the maximal content involved by the content of diagram D. That is,

$$Closure\text{-}Cont(D) = \{\alpha \mid Cont(D) \Longrightarrow \{\alpha\}\ \}.$$

By the definition of *Closure-Cont*(D), we have $Cont(D) \Longrightarrow$ *Closure-Cont*(D). We also know that $Cont(D)$ is a subset of *Closure-Cont*(D). Therefore, by the definition of involvement relation, *Closure-Cont*$(D) \Longrightarrow Cont(D)$. We can prove[10] that for any diagram D which does not convey conflicting information[11] VENN allows us to obtain a diagram whose content is the maximal content of D.

[10]Refer to Theorem 9 of Appendix C of Shin [1990].

[11]That is, D does not have a region both with a shading and an X-sequence. If D has a region both a shading and an X-sequence, then by rule of confliction information we can get any diagram. It is also uninteresting.

Maximal Representation

Let us extend this idea to a set of diagrams. In §2.3 we defined the content of diagrams in Δ (i.e., $Cont(\Delta)$) as the union of the contents of diagrams in Δ. Here we define the closure content of diagrams in Δ as the set of the maximal content involved by $Cont(\Delta)$. That is,

$$Closure\text{-}Cont(\Delta) = \{\alpha \mid Cont(\Delta) \Longrightarrow \{\alpha\}\ \}$$

We know that $Cont(\Delta) \Longrightarrow Closure\text{-}Cont(\Delta)$ and $Closure\text{-}Cont(\Delta) \Longrightarrow Cont(\Delta)$. By the induction of the cardinality of set $Cont(\Delta)$, we can prove the following interesting theorem[12]:

Maximal Representation Theorem
Given set Δ of diagrams, there is a diagram D such that $\Delta \vdash D$, where $Cont(D) = Closure\text{-}Cont(\Delta)$.

Completeness

Now, we want to prove that if $\Delta \models D$ then $\Delta \vdash D$.

Proof: Suppose that $\Delta \models D$. That is, $Cont(D)$ is involved by $Cont(\Delta)$. Hence, $Cont(D) \subseteq (Closure\text{-}Cont(\Delta))$. The Maximal Representation Theorem tells us that diagram D' whose content is $Closure\text{-}Cont(\Delta)$ is obtainable from Δ. Therefore, $Cont(D) \subseteq Cont(D')$. It suffices to show that $\{D'\} \vdash D$.
If $Cont(D) = Cont(D')$, then D' is what we want.
If $Cont(D) \subset Cont(D')$, check what $Cont(D') - Cont(D)$ is. What could be a difference in content between two diagrams? In other words, what kind of diffferent representing facts can two diagrams support? We can think of three kinds of representing facts which can make a difference in content between two diagrams: Whether one region is shaded or not, whether one region has an X-sequence or not, and whether one closed curve exists or not.[13] In each case, we apply the following procedure:

(i) If $\langle\langle \text{Empty}, a; 1\rangle\rangle \in (Cont(D') - Cont(D))$, then apply the rule of erasure of a primitive object to D' to erase a shading.

(ii) If $\langle\langle \text{Empty}, a; 0\rangle\rangle \in (Cont(D') - Cont(D))$, then apply the rule of erasure of a primitive object to D' and erase an X-sequence.

(iii) If $\langle\langle \text{Set}, a; 1\rangle\rangle \in (Cont(D') - Cont(D))$(where there is a basic region A such that $f(A) = a$), then apply the rule of erasure of a primitive object to erase closed curve A from D'.

[12]For the proof, refer to Appendix C of Shin [1990].

[13]Since both diagrams are well-formed, it is impossible that two diagrams have the equivalent basic regions but not the equivalent regions.

We removed (from diagram D') the representing facts which make a difference in these two sets. Hence, we get diagram D such that $\{D'\} \vdash D$ where $Cont(D') - Cont(D)$, that is, $Cont(D) \subseteq (Closure\text{-}Cont(\Delta))$. (Completeness) $\square$