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A Cross-Sectional Score for the Relative Performance of an Allocation

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Abstract- **The aim of this paper is to propose an innovative score measuring the relative performance – in terms of return – of an asset allocation with respect to the alternative allocations offered to the manager. Intuitively, this score is defined as the percentage of alternative allocations outperformed by the manager's allocation. In particular, considering the case of a manager investing according to the zero-dollar long/short equally weighted strategy, we study in details the computation and the properties of this score and we deal with the related combinatorial issues when the number of assets is large.**

Keywords: **Performance measure; Order statistics; Generalized hyperbolic distribution**
JEL Codes: **C13, C16, C46, C62**

1. Introduction

During the last two decades, the explosive growth of the asset management industry came with an increasing interest in the analysis of investment performance. The research in this area is axed on Sharpe-like ratios proposed in the 60's [Sharpe (1966), Treynor (1965), Jensen (1968)] and it is expanding by developing the notion of performance as a reward counter-balanced by some risk. The main innovations focused on the definition and modeling of risk [Shadwick and Keating (2002), Darolles et al. (2009)]. Practically, the performance of a portfolio manager, over a given period, is usually computed as the ratio of his excess return over a risk measure [Grinblatt et al. (1994)]. The managers are then ranked according to these ratios, and the manager providing the highest and steadiest returns receives the best score. These measures are convenient because they require no assumption on the strategy of the portfolio managers. However, they suffer major drawbacks. First, these measures are relative to a peer's performance and irrelevant if no peer is found. We generally assume that the best score corresponds to a "good" portfolio allocation, with no guarantee on the goodness of this allocation. Secondly, as they are a ratio of two random variables, they suffer significant estimation errors [Lo (2002) and Christie (2007) among others] which prevent any performance comparison to be significant.

In this paper, we differentiate three elements affecting the performance of a manager: the set of investable portfolios offered to this manager, the period of investment and the allocation method used. In general, when we measure the performance of a manager we aim to measure the performance of his allocation method, i.e. his ability to correctly choose his allocation among his set of investable portfolios and for any period. We can achieve this goal by fixing the two first elements. First, the set of investable portfolios is defined by the set of constraints on the portfolio positions. Those are usually stated by the investment policy of the manager. For instance, they can be affected by short-sale or diversification restrictions. Remark that, using Sharpe-like ratios, this issue is dealt with by comparing managers having the same constraints and investing with similar strategies, in other words these managers share the same set of investable portfolios. Secondly, the period of investment affects the return provided by an allocation method. Indeed, over two different periods, the use of the returns in order to compare the performance of two managers is known to be delicate. A manager using a certain allocation method would obtain different returns over the different periods while his allocation method may not be in question. This issue can be tackled by normalizing the returns with a proxy of the risk for the different periods. However, in the following, we propose an alternative way to deal with the normalization of the returns. Basically, we normalize an invested portfolio's return by providing the percentage of investable portfolios that it outperforms. This normalized return, called score, has the advantage to require no peer to be interpreted and it is cross-sectional in the asset returns as it is cross-sectional in the set of the investable portfolio. This score is the subject of this paper. Finally, the performance of an allocation method over several periods can be obtained by computing its average score over these periods. This performance measure would then be a cross-sectional performance measure and an alternative to the classical performance measures based on Markowitz framework [Markowitz (1952)]. Such a performance measure will be the object of a companion paper.

First, we define the new score presented in the previous paragraph. For a given investment strategy, this score provides the percentage of investable portfolios which are outperformed by the portfolio in consideration. By construction it is independent of the period considered. In addition, for a given set of investable portfolio defined by the investment strategy, it reflects the goodness in the choice of an allocation. Thus, it quantifies the ability of the portfolio manager to choose his portfolio. Secondly, we study in details the computation and the properties of this score for the Zero-Dollar Long/Short Equally Weighted (LSEW) strategy especially in the case of a very large number of assets. Then, in order to deal with the combinatorial issue inherent to the computation of the score, we introduce an exchangeability assumption on the asset returns. The validity of this assumption is verified empirically in Section 4. In the general framework of returns characterized by generalized hyperbolic distributions [Eberlein et al. (1995) and Prause (1999)] we show that the score is independent of the mean, the variance and the covariance of the returns. In other words, it depends only on the shape parameters of the distribution which characterizes the skewness and the kurtosis of the returns, and justifies working with this class of distributions.

The paper is organized as follows. In Section 2, we introduce the score which quantifies the quality of an allocation over a given period. In Section 3, we study and compute this score under fair assumptions. Considering the LSEW strategy and assuming that the returns are characterized by generalized hyperbolic distributions, we detail the influence of the distribution parameters on the score. Section 4 investigates the relevance of the assumptions. Section 5 concludes.

2. A Cross-Sectional Score of Portfolio Performance

In this section, we introduce a new score quantifying the quality of an allocation. Consider a portfolio manager whose investment policy defines a finite set of portfolios. To provide an objective measure of his allocation performance, we compare the return of his portfolio with the returns of all other investable portfolios. If his portfolio outperforms $S\%$ of all portfolios, we say that it scores $S, S \in [0,1]$. This score S will be the measure of the manager performance that we investigate in details. Because each investable portfolio has a score, the score is cross-sectional over this set. Moreover, this score is interesting because it is independent of the period considered, and thus of the market conditions. In addition, because the set of portfolios is already defined and the period identical to all portfolios, it focuses only on the goodness of the allocation choice and does not need to be compared to a peer's portfolio performance.

We introduce now some formalism. Denote Γ the set of the investable portfolios induced by the manager's strategy, and $\gamma = (\gamma(1), \ldots, \gamma(n))'$ a vector of weights, in a market of n assets, where $\gamma(i)$ is the weight associated with asset $i, i = 1, \ldots n$. In practice, the assets are indivisible and any endowment is finite, so the number of investable portfolios is finite. Moreover, in this paper, we focus on the case where Γ is a closed set under taking additive inverses. Such a proceeding leads to a set of portfolios which enables a manager to bet on a market direction as well as on its inverse. This feature is typical of absolute return funds which invest in long and short positions to adapt any market condition. This property of Γ provides a simple interpretation of the score as a score of 50% corresponds to a portfolio return of 0% and any portfolio with a score S_1 can be related to a symmetric portfolio – having opposite weights – with a score $1 - S_1$. We propose now a way to compute the performance measure S. Given an invested portfolio $\gamma \in \Gamma$, if $N(\gamma)$ is the number of portfolios outperformed by γ, then the performance S associated with γ is

$$S(\gamma) = \frac{N(\gamma)}{|\Gamma|} \tag{1}$$

The computation of $S(\gamma)$ requires the identification of all investable portfolios outperformed by γ. As soon as $|\Gamma|$ is large, this computation is not direct. To deal with this issue, we introduce the relevant theoretical framework.

We consider a market of n assets whose returns $X = (X_1, \ldots, X_n)'$ have the joint density f and where Y' is the transpose of Y. The marginal density of $X_i, i \in \{1, \ldots, n\}$ is denoted f_i, and the vector of order statistics induced by X is $X_{(n)} = (X_{(1)}, X_{(2)}, \ldots, X_{(n)})'$. Let be a portfolio $\gamma \in \Gamma$, it returns $\gamma' X$, then for any realization $x = (x_1, x_2, \ldots, x_n)'$, $x_{(n)}$ being a permutation of the elements of x, it exists a portfolio $\tilde{\gamma} \in \Gamma$ such that

$$\gamma' x = \tilde{\gamma}' x_{(n)}$$

In the following, we denote g the density of $\gamma' X$ and $u_{\tilde{\gamma}}$ the density of $\tilde{\gamma}' X_{(n)}$. As Γ is finite, there exists an optimal portfolio γ_o which provides the highest return for a given realization x. Its order statistic is denoted $\tilde{\gamma}_o$ and its return is equal to

$$\gamma_o' x = \tilde{\gamma}_o' x_{(n)} \tag{2}$$

It is helpful to remark that the return of any portfolio $\gamma \in \Gamma$ can be expressed relatively to the return of the optimal portfolio $\tilde{\gamma}_o$. This means that there exists a parameter $k \in [-1,1]$ such that:

$$\gamma' x = k \tilde{\gamma}_o' x_{(n)} \tag{3}$$

By definition, the optimal portfolio γ_o scores $S=1$ and its opposite - which is the worst portfolio - scores $S=0$.

Thus, to obtain an approximation of $S(\gamma_i)$ for a given portfolio γ_i, we approximate the number of portfolios $N(\gamma_i)$ by the expected number of portfolios returning less than k_i times the return of the optimal portfolio. We denote this expected number $\bar{N}(k_i)$:

$$\begin{aligned}\bar{N}(k_i) &= E\left(\left|\left\{\gamma\in\Gamma \,\middle|\, \gamma' X \le k_i\,\tilde{\gamma}_o' X_{(n)}\right\}\right|\right)\\ &= \sum_{\gamma\in\Gamma} P\left(\gamma' X \le k_i\,\tilde{\gamma}_o' X_{(n)}\right)\\ &= \sum_{\gamma\in\Gamma}\left(\sum_{\tilde{\gamma}\in\Gamma} P\left(\tilde{\gamma}' X_{(n)} \le k_i\,\tilde{\gamma}_o' X_{(n)}\right) P\left(\gamma' X = \tilde{\gamma}' X_{(n)}\right)\right)\\ &= \left(\sum_{\tilde{\gamma}\in\Gamma} P\left(\tilde{\gamma}' X_{(n)} \le k_i\,\tilde{\gamma}_o' X_{(n)}\right)\right)\left(\sum_{\gamma\in\Gamma} P\left(\gamma' X = \tilde{\gamma}' X_{(n)}\right)\right)\end{aligned} \tag{4}$$

Observing that $\sum_{\gamma\in\Gamma} P\left(\gamma' X = \tilde{\gamma}' X_{(n)}\right) = 1$, we obtain

$$\begin{aligned}\bar{N}(k_i) &= \sum_{\tilde{\gamma}\in\Gamma} P\left(\tilde{\gamma}' X_{(n)} \le k_i\,\tilde{\gamma}_o' X_{(n)}\right)\\ &= \sum_{\tilde{\gamma}\in\Gamma} P\left(\left(\tilde{\gamma}' - k_i\,\tilde{\gamma}_o'\right) X_{(n)} \le 0\right)\\ &= \sum_{\gamma\in\Gamma} P\left(\left(\gamma' - k_i\,\tilde{\gamma}_o'\right) X_{(n)} \le 0\right)\end{aligned} \tag{5}$$

If we denote f_{γ,k_i} the density of $\left(\gamma' - k_i\,\tilde{\gamma}_o'\right) X_{(n)}$, the relationship (5) becomes:

$$\bar{N}(k_i) = \sum_{\gamma\in\Gamma} \int_{-\infty}^{0} f_{\gamma,k_i}(y)\,dy \tag{6}$$

Plugging relationship (6) in equation (1) provides an approximation of the score for any portfolio γ_i returning k_i times the return of the optimal portfolio γ_o:

$$\bar{S}(k_i) = \frac{\bar{N}(k_i)}{|\Gamma|} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \int_{-\infty}^{0} f_{\gamma,k_i}(y)\,dy \tag{7}$$

As soon as the number of assets is large, the enumeration of the portfolios of Γ is laborious and the computation of (7) remains difficult. To achieve this computation, we introduce the technical assumption that the asset returns are exchangeable and provide the resulting expression of $\bar{S}$ in the next proposition.

We recall that a sequence of random variables is exchangeable if, for any permutation of these random variables, the joint probability distribution of the rearranged sequence is the same as the joint probability of the original sequence [Arellano-Valle and Genton (2007)]. In particular, a sequence of independent and identically distributed (i.i.d.) random variables is exchangeable.

Proposition 1: Let X be an exchangeable random vector (Assumption A_0). Denote $X_{(n)}$ its corresponding vector of order statistics, γ_o the optimal portfolio and γ a portfolio returning k times the return of γ_o, then the approximated score $\bar{S}$ for a portfolio γ is equal to

$$\bar{S}(k) = \int_{-\infty}^{0} \left(g * h_k\right)(y)\,dy \tag{8}$$

where * stands for the convolution product; g is the density of $\gamma' X$ and h_k the density of $-k\tilde{\gamma}_o' X_{(n)}$, where $\tilde{\gamma}_o$ is the ordered representation of the optimal portfolio γ_o.

Proof: The proof of this proposition is postponed in Appendix 5.2.

The validity of assumption (A_0) is verified empirically in Section 4.

3. Estimation of the Score

In this section, we thoroughly study the score S in the case of the zero-dollar long/short equally weighted strategy (LSEW). Assuming the generic case where the assets' returns follow a multivariate generalized hyperbolic distribution, we investigate the effects of the distributions parameters on the score. Next, we illustrate the cases of the 130/30 and Hamming strategies.

3.1 The Case of the LSEW Strategy

The LSEW strategy consists in investing in portfolios which are long/short (i.e. include both long and short positions), zero-dollar (the value of the long positions is equal to the value of the short positions) and equally weighted (each position has the same value in absolute value). In addition, the leverage of these portfolios is fixed to 2:1. The notation 2:1 means that the amount of capital backing the portfolio represents 50% of the portfolio value. It is the minimum amount required under the U.S. Regulation (namely Regulation T). As a consequence, in our case, the sum of the absolute values of the weights of the portfolio equals 2. The LSEW strategy is particularly interesting because it is the one used to track the momentum effect in most of the literature [Jeegadesh and Titman (1993), Rouwenhorst (1998), Chan et al. (2000), Okunev and White (2003), Kazemi et al. (2009) and Billio et al. (2011) among others]. This LSEW strategy is also the base of most of the relative value strategies (arbitrage) which take advantage of the mispricing between two assets [see Gatev et al. (1999) for the case of pair trading]. Note that the consideration of equal weights is not as limiting as it seems in the choice of the portfolio. On the contrary, as shown in Demiguel et al. (2009), the errors in estimating the means and covariances of the assets' returns penalize the mean-variance optimization enough to erase the diversification gain and provide portfolios with a lower out-of-sample Sharpe ratio than the naive equally weighted portfolio.

For instance, in a market of 4 assets (A,B,C,D), there are 6 LSEW portfolios. We represent them in Table 1. Note that there are $|\Gamma| = \frac{n!}{\left(\frac{n}{2}!\right)^2}$ LSEW portfolios in a market of *n* assets. So, the number of portfolios increases exponentially with *n*. As an illustration of the combinatorial issue, a market of 30 assets leads to 1.55 10^8 portfolios which would require 4.33 Go of memory to be stocked and prevent any enumeration.

In addition, the optimal portfolio is then long the $\frac{n}{2}$ assets which perform the best and is short the $\frac{n}{2}$ assets which perform the worst:

$$\tilde{\gamma}_o(i) = \begin{cases} -\frac{2}{n} & ,if \quad i \le n/2 \\ \frac{2}{n} & ,if \quad i > n/2 \end{cases}$$

In practice, the computation of $\bar{S}$ using the expression (8) requires to determine the density g corresponding to a linear combination of n random variables, the density h_k corresponding to the linear combination of n order statistics and the convolution product between g and h_k. For the computation of h_k, we use the methodology developed by Arellano-Valle and Genton (2007). Nevertheless, their result is difficult to apply as soon as n is large. In that case Monte Carlo simulations are appropriate. Through an example, we carry out the computation of $\bar{S}$.

Let consider a simulated market of 10 assets - inducing $|\Gamma| = 252$ LSEW portfolios - whose returns are i.i.d., and follow a Gaussian distribution with mean 0 and variance 0.01. Then, the density g is the sum of 10 independent Gaussian densities, and we calculate the density h_k using Monte Carlo simulations, computing $\gamma' x - k\tilde{\gamma}_o' x_{(n)}$ for each realization x, with $\gamma \in \Gamma$ and $\tilde{\gamma}_o$ the optimal portfolio obtained by ranking the 10 returns. In Figure 1, we represent $\bar{S}$ as a function of *k*. We remark that the score $\bar{S}$ of a portfolio γ_i providing $k_i = 60\%$ of the return of the optimal portfolio is $\bar{S} = 92\%$. This means that only 8% of the LSEW portfolios provide an higher return than γ_i, on average.

Financial asset returns are well known to have distributions which are asymmetric and leptokurtic. Thus, it is important to be able to compute $\bar{S}$ when the asset returns are modeled by distributions more complex than the Gaussian one. As shown in Eberlein et al. (1995), Prause (1999) and Fajardo et al. (2009), among others, a multivariate generalized hyperbolic distribution can be considered due to its flexibility and its good fitting for financial asset returns. We exhibit such an example in Section 4 showing the superiority of the fit obtained with the generalized hyperbolic distribution over the Gaussian distribution. Thus, in the following, we assume that the observations $X = (X_1, \ldots, X_n)'$ are characterized by a multivariate generalized hyperbolic distribution and we identify the distribution's parameters which affect $\bar{S}$.

A multivariate generalized hyperbolic distributions $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$ can be represented as a normal mean-variance mixture [Barndorff-Nielsen et al. (1982)], and is characterized by six parameters: the mean $\mu \in R^n$, the variance-covariance matrix $\Sigma \in R^{n\times n}$, the skewness parameter, $\kappa \in R^n$, and the shape parameters λ, χ and ψ. In the following, we use this very flexible class of distributions to characterize the assets on a market since it contains a lot of well known distributions (Laplace, Student-t, normal inverse Gaussian, inverse Gaussian, etc.).

Under the GHD assumption and (A_0), the vector X is an exchangeable random vector characterized by a multivariate generalized hyperbolic distribution, and $\Sigma = \sigma^2\left[(1-\rho) I_n + \rho 1_n 1_n'\right]$ where σ is the variance of X and ρ is the correlation between X_i and X_j, $i, j \in \{1, \ldots, n\}$, [Arellano-Valle and Genton (2007)].

Proposition 2: Let X be an exchangeable random vector (Assumption A_0) distributed according to a multivariate generalized hyperbolic distribution $GH_n(\lambda, \chi, \psi, \mu, \Sigma, \kappa)$ (Assumption A_1), $X_{(n)}$ being the random vector of its order statistics and γ_o the optimal portfolio. Then , we have

$$\bar{S}(k) = \int_{-\infty}^{0} v_k(y)\,dy \qquad (9)$$

with v_k the density function of $Z - k\tilde{\gamma}_o' U_{(n)}$ where Z is an elliptically contoured random variable[1] $EC_1\left(0, \frac{4}{n}, \varphi^{(1)}\right)$, and

[1] As stated in Schmidt (2003), "let X be an n-dimensional random vector and $\Sigma \in R^{n\times n}$ be a symmetric positive semi-definite matrix. If $X-\mu$, for some $\mu \in R^n$, possesses a characteristic function of the form $\varphi_{X-\mu}(t) = \varphi(t'\Sigma t)$ for some function $\varphi: R^+ \to R$, then X is said to be elliptically distributed with parameters μ (location), Σ (dispersion), and ϕ. The density function, if existent, of an elliptically contoured distribution has

$U_{(n)}$ is the vector of order statistics induced by $U_n \sim EC_n\left(0, I_n, \varphi^{(n)}\right)$ with the density generator $\varphi^{(m)}$ given by

$$\varphi^{(m)}(u) = C_m \frac{K_{\lambda-\frac{m}{2}}\left(\sqrt{\psi(\chi+u)}\right)}{\left(\sqrt{\chi+u}\right)^{\frac{m}{2}-\lambda}} \tag{10}$$

with C_m a normalizing constant, and K_ν the modified Bessel function of the third kind.

Proof: The proof of this proposition is postponed in Appendix 5.3.

We remark that the score $\bar{S}$ depends only on the shape parameters λ, χ and ψ which confirms the limitations implied by working in the Gaussian framework. Therefore, in the case of Gaussian i.i.d. returns as presented in Figure 1, different means and different variances and correlations would lead to the same function $\bar{S}(k)$. In Section 4, Figure 4, we compare the score $\bar{S}_N$ obtained with a Gaussian distribution of the returns and the score $\bar{S}_{NIG}$ obtained with a Normal Inverse Gaussian (NIG) distribution[2] – a sub-class of generalized hyperbolic distribution – fitted on market data. We observe that – by investing randomly in a LSEW portfolio – extreme scores are more likely in a market where the returns follow a NIG distribution than in a market where the returns follow a Gaussian distribution. As a consequence a risk-lover manager chooses to invest in the NIG market where his allocation has a higher probability to provide a return similar to the optimal portfolio's return. Here, this opportunity is counterbalanced by the higher risk to provide a return similar to the worst portfolio's return.

3.2 The Case of the 130/30 and 'Hamming' Strategies

The score presented in the previous section is used with the LSEW strategy and it can be applied to any other strategy. In this section, we present the score obtained with two other equally weighted strategies: the 130/30 strategy and the Hamming strategy. The 130/30 strategy is long (or short) 130% of the portfolio and short (or long) 30%. All the portfolios generated have a leverage of 1.6:1. The Hamming strategy consists in investing in equally weighted position which can be long or short without constraint on the number of long or short positions. Here, the weights of the portfolios are adjusted in a way such that they all have a leverage of 2:1.

In Figure 1, we represent $\bar{S}$ as a function of k for these two strategies along with the LSEW strategy.

We observe that:

- for $k < 0$, $\bar{S}_{130/30} > \bar{S}_{LSEW} > \bar{S}_{Hamming}$: so, by choosing randomly a portfolio, the probability to obtain a low score is higher for the 130/30 strategy, next followed by the LSEW strategy and finally by the Hamming strategy.
- for $k > 0$, $\bar{S}_{Hamming} > \bar{S}_{LSEW} > \bar{S}_{130/30}$: so, by choosing randomly a portfolio, the probability to obtain a high score is higher for the 130/30 strategy, next followed by the LSEW strategy and finally by the Hamming strategy.

The symmetry of the strategies implies that the probability to get a high score $(k > 0)$ by choosing randomly a portfolio is off-set by the probability to get a low score $(k < 0)$. An investor who is concerned by the score of her invested portfolio would prefer the Hamming strategy if she is risk averse and the 130/30 if she is risk lover.

4. Empirical Relevance of the Assumptions (A_0) and (A_1)

Let consider a market whose returns follow an arbitrary random vector X. In order to verify that the assumptions (A_0) and (A_1) are not too strong to be relevant, we compare the score $\bar{S}$ computed assuming (A_0) and (A_1) and the score $\breve{S}(k)$ computed as the average percentage of portfolios returning less than k times the return of the optimal portfolio, using the relationship (1). Practically, to obtain $\breve{S}(k)$, we need to enumerate all the LSEW portfolios. In our example, we restrict ourselves to a market of 10 assets, corresponding to 252 LSEW portfolios. The market is composed by the 10 Datastream™ sectorial world indices, with their monthly returns, from January 1975 to May 2008. The Datastream™ codes of the indices are reported in Appendix 5.1. To compute $\bar{S}$, we assume that the asset returns are stationary, exchangeable and characterized by a generalized hyperbolic (GH) distribution. Here, we fit the assets' returns with a Normal Inverse Gaussian (NIG) distribution $(\lambda = -0.5)$ for the reasons explained earlier. Because the assets returns are assumed to be exchangeable, they all have the same distribution which is estimated through a fit over the concatenation of all the assets returns. Even if this proceeding does not allow to estimate the correlation between the assets, its estimation is not required for the computation of the score as shown in Proposition 2. The estimation has been performed using the Matlab package developed by Saket Sathe which is available on-line in the Matlab© Central web site: http://www.mathworks.com.

In order to illustrate the accuracy of our choice, we propose in Figure 3 the Q-Q plots corresponding to the adjustments of a Gaussian distribution and a NIG one over the

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the following form: $f(x) = |\Sigma|^{-1/2} g\left((x-\mu)'\Sigma^{-1}(x-\mu)\right) \quad x \in R$ for some density generator function $g: R^+ \to R^+$. Observe that the name "elliptically contoured" distribution is related to the elliptical contours of the density f."

[2] To simplify the estimations, we consider the NIG and Variance-Gamma distributions which are both sub-classes of generalized hyperbolic distribution. In the following, we choose the NIG distribution because it provides the best fit over the considered data.

distribution of the pooled assets returns. The Q-Q plots clearly show the superiority of the fit obtained using the NIG distribution. In Table 2, we exhibit the p-values of the Kolmogorov-Smirnov test considering the empirical sample of 4010 returns (10 (assets)×401 (months)). Under the null hypothesis, we first assume that the empirical sample is drawn from the Gaussian distribution, and next from the NIG distribution.

The test validates the choice of the NIG distribution for the returns (p-value higher that 0.05). In order to illustrate the impact of the distribution's choice for the returns, we provide the scores $\bar{S}$ issued from the Gaussian hypothesis denoted $\bar{S}_N$, and from the NIG hypothesis denoted $\bar{S}_{NIG}$. Both scores are computed using Monte Carlo simulations using the 4010 returns. In Figure 4, we represent $\bar{S}_{NIG}$ with the blue line, $\breve{S}$ with the red line, and $\bar{S}_N$ with the black dot line.

We observe that $\bar{S}_{NIG}$ and $\breve{S}$ coincide. The blue line covers the red one almost everywhere. Thus, it seems that the assumptions (A_0) and (A_1) used to compute $\bar{S}_{NIG}(k)$ do not create any relevant bias in the computation of the score. When we assume that the data set comes from a Gaussian random vector - which is invalidated in Table 2 - we observe a difference between $\bar{S}_N$ (black dashed line) and $\breve{S}$ (red line). The score $\bar{S}_N$ underestimates $\breve{S}$ for negative k and overestimates it for positive k. Thus, $\bar{S}_{NIG}$ can be considered as a better approximation of $\breve{S}$ than $\bar{S}_N$.

5. Conclusion

This paper proposes an innovative way to quantify the goodness of an allocation through a cross-sectional score. While most of the previous works on performance measure require a peer system to appreciate a manager's performance, this approach permits to be free of this constraint. This score is a cornerstone in the construction of a cross-sectional performance measure which would take into account phenomena such as the momentum effect. Such a performance measure would be an alternative to the classical performance measures based on Markowitz' framework and is the subject of a next paper.

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Appendix

5.1 Datastream Codes of the Data used in this Paper

The Datastream™codes of the World sectorial Datastream™indices are: OILGSWD, BMATRWD, INDUSWD, CNSMGWD, HLTHCWD, CNSMSWD, TELCMWD, UTILSWD, FINANWD, TECNOWD.

5.2 Proof of Proposition 1

Let X be an absolutely continuous exchangeable random vector, $X_{(n)}$ be the corresponding random vector of its order statistics. Let be $\gamma_i \in \Gamma$ any portfolio, γ_o the optimal portfolio and g the density of $\gamma_i' X$, then we have

$$P\left(\gamma_i' X = y\right) = \sum_{\gamma\in\Gamma} P\left(\gamma' X_{(n)} = y\right) P\left(\gamma_i' X = \gamma' X_{(n)}\right) \quad \text{(A-1)}$$

As X is an exchangeable random vector, then γ has the same probability to be the representation of γ_i in terms of order statistics, thus

$$P\left(\gamma_i' X = \gamma' X_{(n)}\right) = \frac{1}{|\Gamma|} \quad \text{(A-2)}$$

Plugging relationship (A-1) in expression (A-2) leads to

$$P\left(\gamma_i' X = y\right) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} P\left(\gamma' X_{(n)} = y\right) \quad \text{(A-3)}$$

Denoting u_γ the density function of $\gamma' X_{(n)}$, we remark that

$$g = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} u_\gamma \quad \text{(A-4)}$$

From (7), we know that if the portfolio γ_i returns k_i times the return of the optimal portfolio γ_o, we have

$$\bar{S}(k_i) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\int_{-\infty}^{0} f_{\gamma,k_i}(y)\,dy \quad \text{(A-5)}$$

where f_{γ,k_i} is the density function of $\left(\gamma - k_i \tilde{\gamma}_o\right)' X_{(n)}$. Denoting h_{k_i} the density of $-k_i \tilde{\gamma}_o' X_{(n)}$, we obtain

$$\bar{S}(k_i) = \frac{1}{|\Gamma|}\int_{-\infty}^{0}\sum_{\gamma\in\Gamma}\left(u_\gamma * h_{k_i}\right)(y)\,dy \quad \text{(A-6)}$$

Using the property of distributivity of the convolution product, the relationship (A-6) can be rewritten as follows:

$$\bar{S}(k_i) = \frac{1}{|\Gamma|}\int_{-\infty}^{0}\left(\left(\sum_{\gamma\in\Gamma} u_\gamma\right) * h_{k_i}\right)(y)\,dy \quad \text{(A-7)}$$

Now, from (A-4), we have $\sum_{\gamma\in\Gamma} u_\gamma = |\Gamma| g$, and the relationship (A-7) becomes:

$$\bar{S}(k_i) = \frac{1}{|\Gamma|}\int_{-\infty}^{0}\left(|\Gamma| g * h_{k_i}\right)(y)\,dy = \int_{-\infty}^{0}\left(g * h_{k_i}\right)(y)\,dy \quad \text{(A-8)}$$

The proof of Proposition 1 is complete.

5.3 Proof of Proposition 2

Let $X = \left(X_1, \ldots, X_n\right)'$ be an absolutely continuous exchangeable random vector distributed according to the multivariate generalized hyperbolic distribution

$GH_n\left(\lambda,\chi,\psi,\mu,\Sigma,\kappa\right)$, $X_{(n)}$ be the random vector of its order statistics, $\gamma\in\Gamma$ be a portfolio and γ_o the optimal portfolio. In Proposition 1, we established that $\bar{S}$ depends on $\left(g*h_k\right)$. Here, we need to study separately g, the distribution of $\gamma' X$, and h_k, the distribution of $\tilde{\gamma}_o' X_{(n)}$. We begin with the study of g in Corollary 1.

Corollary 1: Let $X=\left(X_1,\ldots,X_n\right)'$ be an absolutely continuous exchangeable random vector distributed according to the multivariate generalized hyperbolic distribution $GH_n\left(\lambda,\chi,\psi,\mu,\Sigma,\kappa\right)$ and $\gamma\in\Gamma$ be any LSEW portfolio, then $\gamma' X$ is distributed according to an elliptically contoured distribution such that

$$\gamma' X \sim EC_1\left(0,\sigma^2\left(1-\rho\right)\frac{4}{n},\varphi^{(1)}\right) \qquad \text{(B-1)}$$

where the density generator $\varphi^{(1)}$ is given by

$$\varphi^{(1)}(u)=C_1\frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\psi(\chi+u)}\right)}{\left(\sqrt{\chi+u}\right)^{\frac{1}{2}-\lambda}} \qquad \text{(B-2)}$$

with C_1 a normalizing constant and K_ν the modified Bessel function of the third kind.

In other words, $\gamma' X$ follows a symmetric generalized hyperbolic distribution with location 0 and scale $\sigma^2\left(1-\rho\right)\frac{4}{n}$.

Proof of Corollary 1:

From McNeil et al. (2005), we know that the generalized hyperbolic distributions are closed under linear transformation. So, if $X\sim GH_n\left(\lambda,\chi,\psi,\mu,\Sigma,\kappa\right)$ and $Y=\gamma' X$ where $\gamma\in R^n$, then

$$Y\sim GH_1\left(\lambda,\chi,\psi,\gamma'\mu,\gamma'\Sigma\gamma,\gamma'\kappa\right) \qquad \text{(B-3)}$$

In our case, we have

- γ is a LSEW portfolio, so $\gamma' 1_n=0$, thus $\gamma'\mu=0$ and $\gamma'\kappa=0$
- the random variables are exchangeable, so $\Sigma=\sigma^2\left[\left(1-\rho\right)I_n+\rho 1_n 1_n'\right]$, where σ is the scale and ρ is the correlation

Consequently, $\gamma' X$ is distributed as follows

$$\gamma' X\sim GH_1\left(\lambda,\chi,\psi,0,\sigma^2\left(1-\rho\right)\frac{4}{n},0\right) \qquad \text{(B-4)}$$

i.e. $\gamma' X$ follows a symmetric generalized hyperbolic distribution.

From Schmidt (2003) (p.54, definition 3.2.12), we know that the symmetric generalized hyperbolic distribution $GH_n\left(\lambda,\chi,\psi,\mu,\Sigma,0\right)$ is the elliptically contoured distribution $EC_n\left(\mu,\Sigma,\varphi^{(n)}\right)$ where the density generator $\varphi^{(n)}$ is given by

$$\varphi^{(n)}(u)=C_n\frac{K_{\lambda-\frac{n}{2}}\left(\sqrt{\psi(\chi+u)}\right)}{\left(\sqrt{\chi+u}\right)^{\frac{n}{2}-\lambda}} \qquad \text{(B-5)}$$

with C_n a normalizing constant defined in Schmidt (2003) (formula 5.3) and K_ν the modified Bessel function of the third kind. So, in our case, we have

$$\gamma' X \sim EC_1\left(0,\sigma^2\left(1-\rho\right)\frac{4}{n},\varphi^{(1)}\right) \qquad \text{(B-6)}$$

where

$$\varphi^{(1)}(u)=C_1\frac{K_{\lambda-\frac{1}{2}}\left(\sqrt{\psi(\chi+u)}\right)}{\left(\sqrt{\chi+u}\right)^{\frac{1}{2}-\lambda}} \qquad \text{(B-7)}$$

The proof of Corollary 1 is complete.

Now, we investigate the distribution of $\tilde{\gamma}_o' X_{(n)}$:

Corollary 2: Let $X=\left(X_1,\ldots,X_n\right)'$ be an absolutely continuous exchangeable random vector distributed according to the multivariate generalized hyperbolic distribution $GH_n\left(\lambda,\chi,\psi,\mu,\Sigma,\kappa\right)$, $X_{(n)}$ be the random vector of its order statistics and $\tilde{\gamma}_o\in\Gamma$ be the order statistics representation of the optimal portfolio, then $\tilde{\gamma}_o' X_{(n)}$ is distributed according to an elliptically contoured distribution such that

$$\tilde{\gamma}_o' X_{(n)} \overset{d}{=} \sigma\sqrt{1-\rho}\,\tilde{\gamma}_o' U_{(n)} \qquad \text{(B-8)}$$

where $\rho\in\left[0,1\right)$ and $U_{(n)}$ is the vector of order statistics induced by the spherically contoured random vector $U\sim EC_n\left(0,I_n,\varphi^{(n)}\right)$ with $\varphi^{(n)}$ given by

$$\varphi^{(n)}(u)=C_n\frac{K_{\lambda-\frac{n}{2}}\left(\sqrt{\psi(\chi+u)}\right)}{\left(\sqrt{\chi+u}\right)^{\frac{n}{2}-\lambda}} \qquad \text{(B-9)}$$

with C_n a normalizing constant and K_ν the modified Bessel function of the third kind.

Proof of Corollary 2:

From Arellano-Valle and Genton (2007) (Corollary 1), we have

$$\tilde{\gamma}_o' X_{(n)} \overset{d}{=} \left(\tilde{\gamma}_o' X \,\middle|\, \Delta X\geq 0\right) \qquad \text{(B-10)}$$

where Δ is such that $\Delta X = \left(X_2 - X_1 , X_3 - X_2 , \ldots, X_n - X_{n-1} \right)'$. We note that $\Delta\Delta' = \left(\delta_{i,j} \right)$, δ being the Kronecker product, with $\delta_{i,i} = 2$, $\delta_{i-1,i} = \delta_{i+1,i} = -1$ and $\delta_{i,j} = 0$ otherwise. The generalized hyperbolic distributions are closed under linear transformation and X is an exchangeable random vector, so we have

$$\Delta X \sim GH_{n-1}\left(\lambda, \chi, \psi, 0, \sigma^2 \left(1-\rho \right) \Delta\Delta', 0 \right) \qquad \text{(B-11)}$$

Thus, from Schmidt (2003) as seen in Corollary 1, ΔX follows an elliptically contoured distribution

$$\Delta X \sim EC_{n-1}\left(0, \sigma^2 \left(1-\rho \right) \Delta\Delta', \varphi^{(n-1)} \right) \qquad \text{(B-12)}$$

where

$$\varphi^{(n-1)}(u) = C_{n-1} \frac{K_{\lambda - \frac{n-1}{2}}\left(\sqrt{\psi(\chi + u)} \right)}{\left(\sqrt{\chi + u} \right)^{\frac{n-1}{2} - \lambda}} \qquad \text{(B-13)}$$

Since $\tilde{\gamma}_o$ is a LSEW portfolio, relationship (B-6) holds. So, from expression (B-6) and expression (B-12), we have

$$\begin{cases} \tilde{\gamma}_o' X \sim EC_1\left(0, \sigma^2 \left(1-\rho \right) \frac{4}{n}, \varphi^{(1)} \right) \\ \Delta X \sim EC_{n-1}\left(0, \sigma^2 \left(1-\rho \right) \Delta\Delta', \varphi^{(n-1)} \right) \end{cases} \qquad \text{(B-14)}$$

which are the intermediary results obtained in the proof of Corollary 3 in Arellano-Valle and Genton (2007). Thus, Corollary 3 can be used here, and we extend it to generalized hyperbolic distributions. It follows

$$\tilde{\gamma}_o' X_{(n)} \overset{d}{=} \sigma\sqrt{1-\rho}\, \tilde{\gamma}_o' U_{(n)} \qquad \text{(B-15)}$$

where $U_{(n)}$ is the vector of order statistics induced by the spherically contoured random vector $U \sim EC_n\left(0, I_n, \varphi^{(n)} \right)$ and $\rho \in \left[0,1 \right)$.

The proof of Corollary 2 is complete.

Now, we prove Proposition 2. From Corollary 1 and denoting $Z \sim EC_1\left(0, \frac{4}{n}, \varphi^{(1)} \right)$, we have

$$\gamma' X \overset{d}{=} \sigma\sqrt{1-\rho}\, Z \qquad \text{(B-16)}$$

Then, from Corollary 2 and relationship (B-16), we have

$$\gamma' X - k\, \tilde{\gamma}_o' X_{(n)} \overset{d}{=} \sigma\sqrt{1-\rho}\left(Z - k\, \tilde{\gamma}_o' U_{(n)} \right) \qquad \text{(B-17)}$$

Let denote v_k the density function of $Z - k\, \tilde{\gamma}_o' U_{(n)}$. From (B-17), we have the following expression of $\bar{S}(k)$:

$$\bar{S}(k) = \int_{-\infty}^{0} \left(g * h_k \right)(y) dy = \int_{-\infty}^{0} v_k(y)\, dy \qquad \text{(B-18)}$$

So, $\bar{S}(k)$ is independent of μ, σ, ρ and κ.

The proof of Proposition 2 is complete.

References

1. Arellano-Valle, R. and Genton, M. (2007) ‘On the exact distribution of linear combinations of order statistics from dependent random variables’, Journal of Multivariate Analysis, Vol. 97, pp. 1876-1894.
2. Barndorff-Nielsen, O.E., Kent, J. and Sørensen, M. (1982), ‘Normal variance-mean mixture and z-distributions’, International Statistical Review, Vol. 50, pp. 145-159.
3. Billio, M., Calès, L. and Guégan, D. (2011) ‘Portfolio symmetry and momentum’, European Journal of Operational Research, Vol. 214, pp. 759-767.
4. Chan, K., Alaudeen, H. and Wilson, T. (2000), ‘Profitability of momentum strategies in the international equity markets’, Journal of Financial and Quantitative Analysis, Vol. 35, pp. 153-172.
5. Christie, S. (2007) Beware the Sharpe Ratio, Macquarie Applied Finance Centre, Macquarie University, Sydney, Australia.
6. Darolles, S., Gourieroux, C. and Jasiak, J. (2009) ‘L-performance with an application to hedge funds’, Journal of Empirical Finance, Vol. 16, pp. 671-685
7. Demiguel, V., Garlappi, L. and Uppal, R. (2009), ‘Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy ?’, The Review of Financial Studies, Vol. 22, pp. 1915-1953.
8. Eberlein, E. and Keller, U. (1995), ‘Hyperbolic distributions in finance’, Bernoulli, Vol. 1, pp. 281-299.
9. Fajardo, J. and Farias, A. (2009), ‘Multivariate affine generalized hyperbolic distributions: An empirical investigation’, International Review of Financial Analysis, Vol. 18, pp. 174-184.
10. Gatev, E., Goetzmann, W. and Rouweunhorst, G. (1999) Pairs Trading: Performance of a Relative Value Arbitrage Rule, Working Paper 7032, National Bureau of Economic Research, United States.
11. Grinblatt, M. and Titman, S. (1994), ‘A study of monthly mutual funds returns and performance evaluation techniques’, Journal of Financial and Quantitative Analysis, Vol. 29, pp. 419-444.
12. Jeegadesh, N. and Titman, S. (1993), ‘Returns to buying winners and selling losers: Implications for stock market efficiency’, Journal of Finance, Vol. 48, pp. 65-91.
13. Jensen, M. (1968), ‘The performance of mutual funds in the period 1945-1964’, Journal of Business, Vol. 42, pp. 167-247.
14. Kazemi, H., Schneeweis, T. and Spurgin, R. (2009), ‘Momentum in asset returns: are commodity returns a special case ? ’, Journal of Alternative Investments, Vol. 10, pp. 23-36.
15. Lo, A. (2002), ‘The Statistics of Sharpe ratios’, Financial Analysts Journal, July/August, pp. 36-52.
16. Markowitz, H. (1952), ‘Portfolio Sselection’, Journal of Finance, Vol. 7, pp. 77-91.
17. McNeil, A., Frey, R. and Embrechts, P. (2005) Quantitative risk management: Concepts, techniques and tools, Princeton University Press.

18. Okunev, K. and White, D. (2003), 'Do momentum-based strategies still work in foreign currency markets ?', Journal of Financial and Quantitative Analysis, Vol. 38, pp. 425-447.
19. Prause, K. (1999) The Generalized Hyperbolic Model: Estimation, Financial Derivatives and Risk Measures, Ph.D. Dissertation, University of Freiburg, Germany.
20. Rouwenhorst, K. (1998), 'International momentum strategies', Journal of Finance, Vol. 53, pp. 267-284.
21. Schmidt, R. (2003) Dependencies of Extreme Events in Finance, Ph.D. Dissertation, University of Ulm, Germany http://stats.lse.ac.uk/schmidt.
22. Shadwick, W. and Keating, C. (2002), 'A universal performance measure', Journal of Performance Measure, Spring, pp. 59-84.
23. Sharpe, W. (1966) 'Mutual fund performance', Journal of Business, Vol. 39, pp. 119-138.
24. Treynor, J. (1965) 'How to rate management of investment funds', Harvard Business Review, Vol. 43, pp. 63-75.

Table 1: The set $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ is the set of the LSEW portfolios that can be built in a market of 4 assets, here $\{A, B, C, D\}$.

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
Asset A	1/2	−1/2	1/2	−1/2	1/2	−1/2
Asset B	1/2	1/2	−1/2	1/2	−1/2	−1/2
Asset C	−1/2	−1/2	−1/2	1/2	−1/2	1/2
Asset D	−1/2	1/2	1/2	−1/2	1/2	1/2

Table 2: P-values obtained with the Kolmogorov-Smirnov tests between the empirical distribution of the returns of the 10 Datastream world sectorial indices pooled all together and a fitted Gaussian distribution, and between the same empirical distribution and a fitted NIG distribution.

	Gaussian dist.	NIG dist.
p-value	$2.0175\ 10^{-5}$	0.7846

Figure 1: Representation of $\bar{S}$ with respect to k.

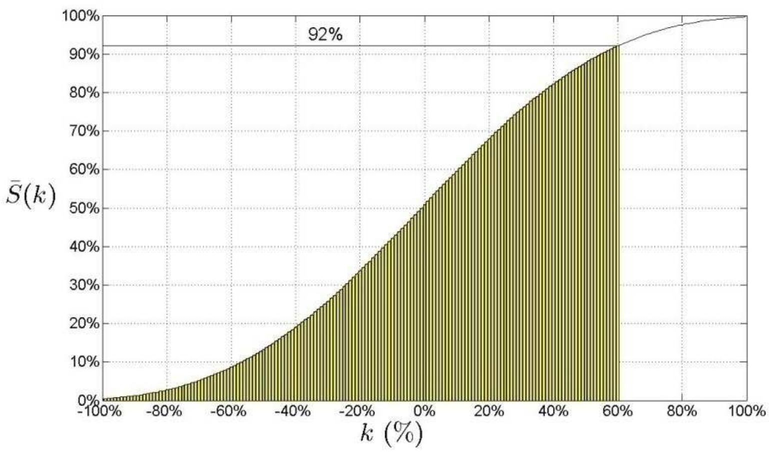

Figure 2: Representation of $\bar{S}$ with respect to k, as defined in (8), for three different investment strategies: the LSEW strategy, the 130/30 strategy and the Hamming strategy.

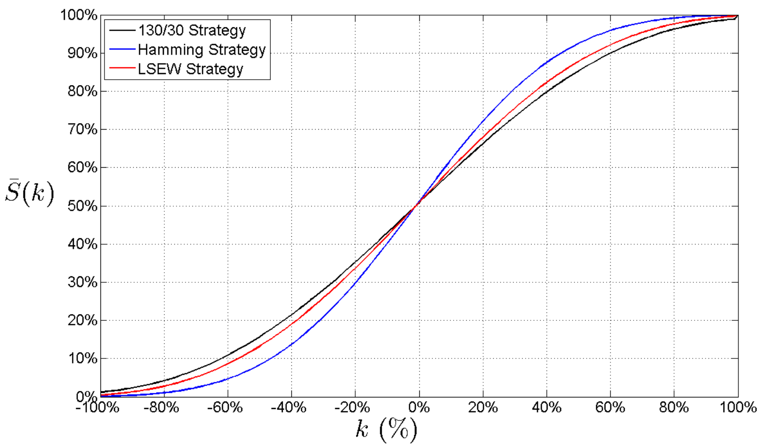

Figure 3: Comparison of the *Q-Q* plots obtained by plotting the empirical distribution of the returns of the 10 Datastream world sectorial indices pooled all together against a fitted Gaussian distribution (in the left plot) and against a fitted NIG distribution (in the right plot). It shows the superiority of the fit obtained using the NIG distribution.

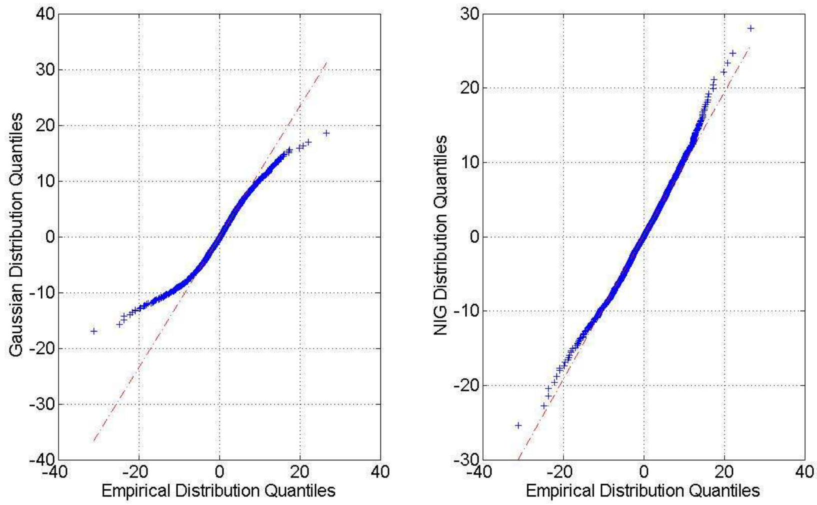

Figure 4: Representations of $\breve{S}(k)$, $\bar{S}_{NIG}(k)$ and $\bar{S}_N(k)$, three different approximations of the score of an LSEW portfolio whose return is k times the return of the optimal portfolio, in the market of the 10 Datastream world sectorial indices.