When Does The 1/N Rule Work?

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Abstract

We propose a "1/N favorability index" to measure how favorable a market is to holding a 1/N portfolio. This index reflects the extent of difficulty for an optimized portfolio to outperform the 1/N portfolio in a specific market. A single-factor model predicts that bull markets are accompanied by a high 1/N favorability index and vice versa. We validate the model implication that the 1/N portfolio is more difficult to beat in bull markets using stock return datasets from a number of countries as well as the classic datasets used by DeMiguel et al. [2009]. Our results imply that the reported good performance of the 1/N portfolio in the US equity market can be partially attributed to the long-run bullish trend in the market which gives rise to the high favorability of the market to the 1/N portfolio.

JEL classification: G11; G12

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1 Introduction

The 1/N rule is a portfolio allocation scheme which allocates an equal share of wealth to each of N available assets on each portfolio rebalancing date. This 1/N rule is seen to be at odds with the more explicit portfolio optimization theory of Markowitz [1952], since this rule neither requires any estimation of the inputs nor involves any explicit optimization process and is therefore deemed as a passive investment strategy. However, there is evidence (see Benartzi and Thaler [2001]) that many market participants use this simple heuristic when allocating their wealth across different asset classes. Further, DeMiguel et al. [2009] published a seminal study on the performance of the 1/N rule. After conducting an extensive evaluation of the performance of various portfolio methods, including the 1/N rule and a few other methods designed to mitigate the effect of estimation errors, the authors reported that "of the 14 portfolio methods evaluated across seven empirical datasets from the US market, none is consistently better than the 1/N portfolio in terms of the Sharpe ratio, certainty-equivalent return, or turnover". To complement the paper of DeMiguel et al. [2009], Jacobs et al. [2014] focused on the effect of global diversification in the stock market from the perspective of a Eurozone investor and found that none of the optimized portfolios is able to consistently outperform the 1/N rule out-of-sample. However, when Fletcher [2011] replicated the experiment of DeMiguel et al. [2009] using data

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from the UK market, he found that a number of optimal portfolio strategies significantly outperform the 1/N portfolio, even after an adjustment for a trading cost. Thus, Fletcher's paper provides support in defence of the optimized portfolios.

It appears that previous studies reach different conclusions about the performance of the 1/N portfolio in different countries. Grinold [1992] also noticed that the benchmark portfolio[1] is not equally efficient in different countries. Therefore, it is of interest to examine this issue. We hypothesize that there are some underlying features that make it profitable to follow the 1/N rule. A simple thought experiment in support of our conjecture is that, during a crisis period when the prices of most stocks fall, there is little chance that the 1/N will yield a positive return, but it is still possible for a well-managed long-short portfolio to earn a profit. This paper attempts to identify market-specific characteristics which make a market favorable to the 1/N rule.

We begin by considering the following question: what features of a market make it most desirable to hold the 1/N portfolio? A natural answer would be "if the market parameters (i.e., the expected returns and the covariance matrix) are such that the Sharpe ratio maximizing portfolio in the market happens to be the 1/N portfolio". This condition, if translated into mathematical terms, will be shown to be $\boldsymbol{\mu} = c\boldsymbol{\Sigma}\mathbf{e}$ for some $c > 0$, where $\boldsymbol{\mu}$ is an $N \times 1$ vector of the expected asset returns in excess of the riskless rate, $\boldsymbol{\Sigma}$ is the $N \times N$ covariance matrix, and $\mathbf{e}$ is an $N \times 1$ vector of ones. (Hereafter, we use "return" to refer to "return in excess of the riskless rate" for convenience.) Similarly, we are also able to identity a condition which makes the 1/N portfolio the least desirable portfolio to hold, i.e., $\boldsymbol{\mu} = c\boldsymbol{\Sigma}\mathbf{e}$ for some $c < 0$.[2]

Therefore, in line with the above analysis, a market is more favorable to the 1/N portfolio if the positive proportionality between the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}\mathbf{e}$ holds more strongly. Specifically, we devise a measure of how favorable a market is to the 1/N rule. This measure is given by $\cos(\boldsymbol{\mu}, \boldsymbol{\Sigma}\mathbf{e})$, and we describe it as the "1/N favorability index" of a given market. This index measures how close the 1/N portfolio is to optimality and thus is able to reflect how difficult it is for other portfolios to outperform the 1/N rule.

We then study properties of this new index under the single-factor model for cross-sectional asset returns (Sharpe [1963]). The single-factor model is both parsimonious in specification and also reasonably rich in implication. Since the market return is the dominating factor that drives the cross-section of equity returns, we view it as a proper model to use when deriving theoretical properties of the 1/N favorability index. An interesting finding from our analysis is that, conditional on the factor returns during a certain period, the (theoretical) 1/N favorability index has upper and lower bounds which depend on the sign of the factor return over the period. If the factor return is positive, the conditional 1/N favorability index falls in an interval with strictly positive bounds; otherwise, the index falls in one with strictly negative bounds. It is important to emphasize that the finding that bull markets are more favorable to the 1/N portfolio is not as straightforward as it may seem. Admittedly, the 1/N portfolio, as a long-only strategy, is unsurprisingly favored more in bull markets than in bear markets in terms of its *expected return*. However, in our context the notion of favorability is represented by how difficult it is for other portfolios to outperform the 1/N portfolio. In bull markets, all portfolios are exposed to high returns and low volatilities yet optimized portfolios have chances to buy relative winners (in a risk-adjusted sense) and sell relative losers to outperform the 1/N portfolio. Further, we show the consistency of the sample-based estimator, which we call the observed 1/N favorability index, of the conditional 1/N favorability index under high-dimensional asymptotics. The implication is that we should be able to see a positive relationship between the observed 1/N favorability index over a certain period and the market average return during the same period. This implication of the single-factor model is tested and supported by both simulation and empirical studies.

The observed 1/N favorability index over a certain period provides an indication of the extent of the

[1] In Grinold [1992], the value-weighted portfolio, instead of the 1/N portfolio, is viewed as the benchmark.

[2] In this case, the vertex of the efficient frontier lies below the x-axis, and the first order condition for finding the tangency portfolio will give rise to the minimum Sharpe ratio portfolio. We refer to Figure 1 of Kim et al. [2016] for an illustration on this point.

difficulty for an optimized portfolio to outperform the $1/N$ portfolio over the same period. If we imagine the extreme case where there are no estimation errors, so that the in-sample optimal portfolio coincides with the evaluation period, or out-of-sample, optimal portfolio, then the $1/N$ portfolio at best achieves the same performance as the optimized portfolio does, which happens when the observed $1/N$ favorability index is equal to 1. However, in the presence of estimation errors, there is a huge discount in the out-of-sample performance of an in-sample optimal portfolio. Therefore, the $1/N$ portfolio more likely outperforms the contaminated optimized portfolio if it is closer to out-of-sample optimality. This intuitive notion of closeness is measured by the observed $1/N$ favorability index.

We use actual asset returns to study whether the model implication that "bull markets are usually accompanied by a high $1/N$ favorability index" can explain the varying extent of difficulty in outperforming the $1/N$ portfolio in different markets and over different sample periods. A number of one-year buy-and-hold $1/N$ portfolios are constructed in seven different markets and non-overlapping holding years. The performance of a $1/N$ portfolio is quantified according to its Sharpe ratio relative to that of a carefully selected benchmark portfolio with the same constituents and holding period. We perform a logistic regression to examine how the market average return over a year determines the probability that the $1/N$ portfolio outperforms the benchmark portfolio over the same period. We find a significantly positive relationship between the average return and the performance of the $1/N$ portfolio. This relationship can be observed both when portfolios are composed of individual stocks and when portfolios are formed on the datasets used in DeMiguel et al. [2009]. Our result indicates that the single-factor model, despite its simplicity, is able to capture the behavior of the $1/N$ portfolio in different market conditions.

To complement the previous set of results, we also examine the performance of the monthly rebalanced $1/N$ portfolio in the equity market of different countries. Based on our proposed criterion for determining whether or not a market is conducive to outperformance of the $1/N$ portfolio, we identify three markets which are favorable to the $1/N$ portfolio. These are markets composed of the Euronext 100 stocks, the Nikkei 225 stocks, and the S&P 500 stocks. In addition, we find that it is not difficult to outperform the $1/N$ portfolio in the markets composed of the DAX stocks, the S&P ASX 200 stocks, and the S&P TSX stocks. A comparison of the holding period average returns in these markets shows that if the Nikkei 225 market is ignored, there is a clear distinction in terms of the average return between the markets favorable and unfavorable to the $1/N$ portfolio.

This paper makes the following contributions. First, we provide a useful insight into the performance of the $1/N$ portfolio. In particular, we point out that the well-documented outperformance of the $1/N$ portfolio in the US market is not entirely due to the poor quality of the contaminated "optimal" portfolios, but more to the fact that the $1/N$ portfolio is innately closer to mean-variance optimality. In other words, the $1/N$ rule is not as naive as it has been perceived to be. Second, we propose a measure of how favorable a market is to the $1/N$ portfolio. This quantity effectively measures whether a market is suitable for holding a $1/N$ portfolio, and its sample-based estimator is a consistent one under high-dimensional asymptotics. Lastly, we identify the conditions under which the $1/N$ rule works best. We use both a statistical model and a set of numerical studies to show that the $1/N$ rule works better in bull markets. This finding provides guidance to passive investors who pay extra attention to timing of investment and to international investment. However, providing investors guidance on improving their market timing decisions is beyond the scope of our paper.

1.1 Literature Review

This paper is related to several strands of the portfolio management literature. Since the publication of the DeMiguel et al. [2009] paper, the $1/N$ portfolio has become the benchmark for evaluating a portfolio construction method. Previous papers on the $1/N$ portfolio have tried to explain the reported outperformance of this simple heuristic rule. For instance, Windcliff and Boyle [2004] and DeMiguel et al. [2009] attributed the outperformance of the $1/N$ portfolio to the contamination of "optimal" portfolios by the estimation errors in the input components. Other researchers have argued that the reported outperformance of the 1/N rule is due to specific research designs which are advantageous to the naive diversification. For instance, Kirby and Ostdiek [2012] pointed out that the DeMiguel et al. [2009] research design makes no attempt to match the risk characteristics of the optimized portfolios with those of the 1/N portfolio and is therefore skewed in favor of the 1/N rule. Kan et al. [2016] attributed the inferior performance of the optimized portfolios to their exclusion of the riskless asset. Fugazza et al. [2015] found that a longer investment horizon allows for optimizing strategies to exploit linear predictability in returns and therefore to improve the performance of those strategies. An alternative strand of literature has explained the outperformance of the 1/N rule from the perspective of the merits of the simple rule itself. For example, Plyakha et al. [2012] claimed that an investor needs to buy the previous period losers and sell the previous period winners to maintain equal weights and that it is the contrarian nature of the strategy that ensures a good performance of the 1/N portfolio. In a more recent study, Hwang et al. [2018] proposed that the 1/N rule increases a portfolio's tail risk and results in more concave portfolio returns and the outperformance of the 1/N rule acts as a compensation for the tail risk and concavity.

The rest of this paper is organized as follows. Section 2 introduces the 1/N favorability index and explains the rationale behind it. Section 3 discusses the properties of the 1/N favorability index under a single-factor model and the statistical properties of the observed index. The findings in Section 3 show that bull markets will give rise to a high 1/N favorability index. Section 4 tests the proposition that it is more difficult to outperform the 1/N portfolio in bull markets. Section 5 concludes the paper.

2 The 1/N favorability index

The failure of optimizing strategies to consistently outperform the simple 1/N rule, is often attributed to estimation errors which can have perverse interactions with optimization algorithms leading to extreme positions. Other reasons include over-fitting in-sample data. However, is it possible that the markets where the optimizing strategies perform relatively poorly contain features that are favorable the 1/N rule? In this section, we explore such a possibility. In particular, we seek market-specific features which make a market a favorable environment to hold a 1/N portfolio.

Let us consider two buy-and-hold portfolios, a mean-variance optimal one, which maximizes the in-sample Sharpe ratio, and a 1/N portfolio. We hold them for a common out-of-sample period and calculate the holding period Sharpe ratios of both portfolios. Next we consider the following question: what features of the market[3] would make the 1/N portfolio hard to beat? A natural answer is that, if the 1/N portfolio happens to achieve mean-variance efficiency in terms of the out-of-sample period market parameters, then it would be extremely difficult for the optimized portfolio to outperform the 1/N rule. This is, in the first place, due to a lack of stationarity in the time series of asset returns (Merton [1980]), which implies that there is often a discrepancy between the in-sample period optimal portfolio and the out-of-sample one. Moreover, even if the time series of asset returns is stationary, the optimized portfolio will be to some extent, depending on the portfolio size, contaminated by the presence of estimation errors. Various error-mitigating techniques, e.g., Jagannathan and Ma [2003] and Ledoit and Wolf [2004], can be used to reduce but never eliminate those errors.

We now formally state the market-related condition which makes the 1/N portfolio attractive. Let N represent the number of risky assets in the market, $\mathbf{e}$ the $N \times 1$ vector of ones, $\boldsymbol{\mu}$ the $N \times 1$ vector of expected rates of return over the out-of-sample period, and $\boldsymbol{\Sigma}$ the $N \times N$ covariance matrix of asset returns over the same period. Then, the 1/N rule produces a *Sharpe ratio maximizing portfolio* over the out-of-sample period if and only if $\boldsymbol{\mu} = c\boldsymbol{\Sigma}\mathbf{e}$ for some $c > 0$. This relationship is easily obtained by equating the Sharpe ratio

[3]We use the terminology "market-specific feature" to refer to a quantity that characterizes the market as a whole, rather then depicting the profile of individual assets.

maximizing portfolio with the equally weighted portfolio and checking the condition for the Sharpe ratio maximizing portfolio to exist. Similarly, we can show that the $1/N$ rule leads to the *Sharpe ratio minimizing portfolio* if and only if $\boldsymbol{\mu} = c\boldsymbol{\Sigma}\mathbf{e}$ for some $c < 0$. Note that the ith element in the column vector $\boldsymbol{\Sigma}\mathbf{e}$ represents the summation of covariances between asset i and each of the N risky assets. We hereafter refer to $\boldsymbol{\Sigma}\mathbf{e}$ as the "vector of aggregate covariances". The $1/N$ rule dominates other portfolio strategies if the vector of expected returns is equal to a positive multiple of the vector of aggregate covariances. Conversely, if there is a negative proportionality between the vector of expected returns and the vector of aggregate covariances, no other portfolio can be worse-off than the $1/N$ portfolio. An equivalent statement is that it is desirable to hold a $1/N$ portfolio in a market if taking risks (measured by the aggregate covariance) is rewarded in the market.

Actually, DeMiguel et al. [2009] have already hinted at the same arguments in Section 1.1 of their paper where they pointed out that the $1/N$ portfolio can be viewed as a strategy that does estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ but imposes the restriction that $\boldsymbol{\mu} \propto \boldsymbol{\Sigma}\mathbf{e}$. However, we formalize this idea by including the other extreme case where the $1/N$ portfolio is the least desirable one. Later it will be seen that the two extreme cases correspond to the upper and lower bounds respectively of our proposed measure of a market's favorability to the $1/N$ portfolio. Another aspect that distinguishes our discussion from DeMiguel et al. [2009] is our focus on the variation in characteristics of different (both across time and across countries) markets, instead of a single market only.

Following the same line of thought as in our earlier discussion, the stronger the positive (negative) proportional relationship between the vector of expected returns and the vector of aggregate covariances is, the more (less) favorable the market is to the $1/N$ portfolio. Therefore, the extent to which the proportional relationship holds can be viewed as a measure of how favorable a market is to holding a $1/N$ portfolio. This motivates us to construct a cosine measure expressed as:

$$\cos(\boldsymbol{\mu}, \boldsymbol{\Sigma}\mathbf{e}) = \frac{\langle \boldsymbol{\mu}, \boldsymbol{\Sigma}\mathbf{e} \rangle}{\|\boldsymbol{\mu}\| \|\boldsymbol{\Sigma}\mathbf{e}\|}, \tag{1}$$

to quantify how favorable a market is to the $1/N$ portfolio. In eq. (1), $\|\cdot\|$ represents the Euclidean norm of the vector. Hereafter we refer to this cosine value as a market's "$1/N$ favorability index" over the time period which $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ correspond to. A higher value of this index indicates that the market is more favorable to the $1/N$ portfolio over the specific time period. If $\boldsymbol{\mu}$ is perfectly positively proportional to $\boldsymbol{\Sigma}\mathbf{e}$, there are 0 degrees between the two vectors, resulting in the favorability index being valued at 1. According to our earlier discussion, in this case the $1/N$ portfolio maximizes the Sharpe ratio and can not be outperformed by any other portfolio. Otherwise if $\boldsymbol{\mu}$ is perfectly negatively proportional to $\boldsymbol{\Sigma}\mathbf{e}$, the favorability index becomes -1, and the $1/N$ portfolio would perform extremely poorly. The smaller the index, the larger is the angle between the two vectors, and the less favorable is the $1/N$ portfolio. Although, the $1/N$ favorability index is defined in terms of vector angles, but it can also be loosely viewed as a correlation by viewing the two vectors as paired samples. The resulting index can then be interpreted similarly to a Pearson correlation. Indeed Pearson correlation is simply the $1/N$ favorability index defined in terms of the mean-centered vectors. It most useful to think of the index as a relative measure of how proportional the two vectors are.

We note that $\cos(\boldsymbol{\mu}, \boldsymbol{\Sigma}\mathbf{e})$ is not the only possible version of a $1/N$ favorability index. Actually, the relationship $\boldsymbol{\mu} \propto \boldsymbol{\Sigma}\mathbf{e}$ gives rise to a series of possible measures, i.e., $\cos(\boldsymbol{\Sigma}^{-\alpha}\boldsymbol{\mu}, \boldsymbol{\Sigma}^{1-\alpha}\mathbf{e})$, $\alpha \in \mathbb{R}$. However, the version we adopt does not involve inverting the covariance matrix. This advantage makes it convenient when it comes to calculating the $1/N$ favorability index from realized asset returns, because we do not need to worry about the issue of singularity caused by the number of assets being greater than the sample size. In addition, we will show in the next section that the sample-based version of $\cos(\boldsymbol{\mu}, \boldsymbol{\Sigma}\mathbf{e})$ has the important statistical property of consistency under the high-dimensional asymptotics when both the number of assets and the sample size go to infinity.

It is possible to confuse the relationship between $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}\mathbf{e}$ that makes the $1/N$ portfolio favorable with the implications of some models in the asset pricing literature, since most asset pricing models address the problem of how different types of risk are priced in terms of their risk premia. However, a key difference is that asset pricing models are usually derived with an equilibrium condition imposed, while in this paper, we do not require a market to be in an equilibrium status. We believe that the market parameters, i.e., the expected returns and the covariance matrix, may change from time to time, making some periods more favorable to the $1/N$ portfolio and other time periods less so. Our goal is, as indicated by the title of this paper, to investigate the question of when the $1/N$ rule works well.

3 The $1/N$ favorability index in bull and bear markets

This section will show, by assuming a single-factor model for the cross-section of asset returns, that a market featured by a high $1/N$ favorability index is usually accompanied by a contemporaneous bullish trend. This model implication will be validated by using both a synthetic asset return dataset and some real world asset return datasets. The finding of the relationship between the $1/N$ favorability index and the overall market performance will lead to an intuitive proposition that the $1/N$ portfolio tends to outperform in bull markets.

3.1 Single-factor model

A statistical model in which asset returns are generated by one factor has often been employed in finance (Sharpe [1963], MacKinlay and Pastor [2000]). In this sub-section, we show that this simple model yields a pleasingly testable implication on the relationship between the level of cross-sectional average return and the $1/N$ favorability index in a market over a finite period of time. Since the $1/N$ favorability index reflects the extent to which assets are compensated for taking risks, the relationship we discover has an alternative interpretation that the risk-return relationship varies in bull and bear markets. We should emphasize that we use the single-factor model only as a convenient device to obtain simple closed form expressions for the bounds of our proposed cosine measure. Although multi-factor models may be more realistic, the market factor usually plays a dominant role compared with the remaining factors in terms of its contribution to the total variance of stocks. Therefore the single-factor model seems to be a proper proxy for the actual factor structure in the market.

We assume that the cross-section of asset returns is driven by a single factor whose return over the period from $t-1$ to t is denoted by R_{mt}. The $N \times 1$ vector $\mathbf{R}_t$ contains the return of the N risky assets over the period from $t-1$ to t. The individual asset returns are determined by the factor return through an $N \times 1$ vector of factor loadings $\boldsymbol{\beta}$, as well as by a residual term which is independent of the factor return, i.e.,

$$\mathbf{R}_t = \boldsymbol{\beta} R_{mt} + \boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \sigma^2 \mathbf{I}), \tag{2}$$

where $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. A general consensus about a single-factor model for cross-sectional asset returns is that the factor represents the market, thus the loading measures the extent to which each single asset co-moves with the market.

We are interested in determining the key factors that affect the $1/N$ favorability index over a time period of length T. Define $\overline{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{R}_t$. Let

$$\boldsymbol{\mu}_T = E[\overline{\mathbf{R}} | R_{m1}, \ldots, R_{mT}]$$

and

$$\boldsymbol{\Sigma}_T = E\left[\frac{1}{T}\sum_{t=1}^{T} (\mathbf{R}_t - \overline{\mathbf{R}})(\mathbf{R}_t - \overline{\mathbf{R}})^{\intercal} | R_{m1}, \ldots, R_{mT}\right]$$

denote the conditional expectation of the sample mean and that of the sample covariance matrix of T consecutive observations of asset returns respectively. Note that the expectation is taken conditionally on the factor returns, that is, taken over the randomness in the residuals only. Then, according to the single-factor model in eq. (2), we obtain:

$$\boldsymbol{\mu}_T = \boldsymbol{\beta}\overline{R}_m$$

and

$$\boldsymbol{\Sigma}_T = \frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2\boldsymbol{\beta}\boldsymbol{\beta}^{\mathsf{T}} + \sigma^2\mathbf{I},$$

where $\overline{R}_m = \frac{1}{T}\sum_{t=1}^{T} R_{mt}$. We label the cosine value of the angle between $\boldsymbol{\mu}_T$ and $\boldsymbol{\Sigma}_T\mathbf{e}$ the "conditional" $1/N$ favorability index:

$$\cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e}) = \frac{\langle \boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e}\rangle}{\|\boldsymbol{\mu}_T\|\|\boldsymbol{\Sigma}_T\mathbf{e}\|},$$

because $\boldsymbol{\mu}_T$ and $\boldsymbol{\Sigma}_T$ are conditional expectations of the sample mean and the covariance matrix. Next we expand the expression for this index by plugging in the expression for $\boldsymbol{\mu}_T$ and $\boldsymbol{\Sigma}_T$:

$$\cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e}) = \frac{(\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\overline{R}_m\left[\|\boldsymbol{\beta}\|^2\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2+\sigma^2\right]}{\|\boldsymbol{\mu}_T\|\|\boldsymbol{\Sigma}_T\mathbf{e}\|}.$$

To analyze this expression, we first seek an explicit expressions for the two terms in the denominator. According to the single-factor model, we have:

$$\|\boldsymbol{\mu}_T\| = \|\boldsymbol{\beta}\||\overline{R}_m|$$

and

$$\begin{aligned}\|\boldsymbol{\Sigma}_T\mathbf{e}\| &= \left\|(\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2\boldsymbol{\beta}+\sigma^2\mathbf{e}\right\| \\ &\leq (\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\|\boldsymbol{\beta}\|\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2+\sigma^2\sqrt{N}.\end{aligned}$$

The inequality comes from expanding the norm and then applying the Cauchy-Schwartz inequality $\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e} \leq \|\boldsymbol{\beta}\|\|\mathbf{e}\|$. Consequently, we obtain an upper bound for $\|\boldsymbol{\mu}_T\|\|\boldsymbol{\Sigma}_T\mathbf{e}\|$:

$$\begin{aligned}\|\boldsymbol{\mu}_T\|\|\boldsymbol{\Sigma}_T\mathbf{e}\| &\leq (\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\|\boldsymbol{\beta}\|^2|\overline{R}_m|\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2+\sigma^2\sqrt{N}\|\boldsymbol{\beta}\||\overline{R}_m| \\ &\leq \sqrt{N}\|\boldsymbol{\beta}\||\overline{R}_m|\left[\|\boldsymbol{\beta}\|^2\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2+\sigma^2\right].\end{aligned}$$

The second inequality again comes from applying the Cauchy-Schwartz inequality. It follows that the conditional $1/N$ favorability index has a lower bound when $\overline{R}_m > 0$:

$$\cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e}) \geq \frac{(\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\overline{R}_m\left[\|\boldsymbol{\beta}\|^2\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2+\sigma^2\right]}{\sqrt{N}\|\boldsymbol{\beta}\||\overline{R}_m|\left[\|\boldsymbol{\beta}\|^2\frac{1}{T}\sum_{t=1}^{T}(R_{mt}-\overline{R}_m)^2+\sigma^2\right]} = \cos(\boldsymbol{\beta},\mathbf{e}).$$

Similarly, we can show that when $\overline{R}_m < 0$, there is an upper bound for the index, i.e., $\cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e}) \leq -\cos(\boldsymbol{\beta},\mathbf{e})$. Therefore, we obtain the following range for the conditional $1/N$ favorability index:

$$\cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e}) \in \begin{cases}[\cos(\boldsymbol{\beta},\mathbf{e}),1], & \overline{R}_m > 0\\ \{0\}, & \overline{R}_m = 0\,.\\ [-1,-\cos(\boldsymbol{\beta},\mathbf{e})], & \overline{R}_m < 0\end{cases} \tag{3}$$

The range for the conditional $1/N$ favorability index indicates that there is a clear distinction between markets which make the $1/N$ portfolio favorable with those which are not suitable for holding the $1/N$ portfolio. To be specific, if the cross-section of asset returns is driven by a single factor, whether the market is in a bullish or a bearish period is a critical factor of whether holding a $1/N$ portfolio is a wise choice.

Now, let us consider a $1/N$ portfolio strategy that rebalances, or restores to equal weight, after each T periods until the end of the Mth period of length T. In the extreme case where the single (market) factor earns a positive average return in all of these M sub-periods of length T, the M-period average $1/N$ favorability index will be an average of M values in the interval $[\cos(\boldsymbol{\beta},\mathbf{e}),1]$ and therefore be in the same interval. In the other extreme case where the single factor earns a negative average return in all the M sub-periods, the M-period average $1/N$ favorability index will be in the interval $[-1,-\cos(\boldsymbol{\beta},\mathbf{e})]$. But the more probable case is that, some of the M sub-periods are featured with a positive average factor return, and others are accompanied with a negative one. If this is the case, the average $1/N$ favorability index is a result of averaging positive and negative sub-period indices. The more of the M sub-periods are featured by a positive factor return, the more likely that the M-period average $1/N$ favorability index is positive. Thus, an important implication of the single-factor model is that the $1/N$ portfolio is more difficult to outperform in a bullish market.

Our analysis above covers two different $1/N$ portfolio strategies. The first is to build a buy-and-hold $1/N$ portfolio and track its performance for a relatively short investment horizon of length T. The second is to regularly rebalance the portfolio so that it is always restored to an equal weight portfolio on rebalancing dates. We assess the regularly rebalanced portfolio based on its performance over a longer investment horizon of length MT. Our analysis based on the single-factor model shows that both portfolio strategies work better in bullish markets. We do not assess a buy-and-hold $1/N$ portfolio over a long investment horizon, because the weight of a buy-and-hold $1/N$ portfolio will evolve towards that of a value-weighted portfolio. Later, we will examine the empirical performance of both strategies to confirm the implication of the single-factor model.

If $\boldsymbol{\beta}$ happens to be positively proportional to the vector of ones, then the conditional $1/N$ favorability index is coerced to be 1 when the market factor earns a positive return. The implication of many asset pricing models that the market factor earns a positive risk premium may lead people to believe that a $\boldsymbol{\beta}$ vector closer to a positive multiple of $\mathbf{e}$ makes the market favorable to the $1/N$ portfolio. However, another side of the story is that the market factor, although is associated with a positive risk premium, usually oscillates between bullish and bearish periods. This means that the gain from the closeness of the $1/N$ portfolio and the Sharpe ratio maximizing portfolio during a bullish period is compromised by the loss from the closeness of the $1/N$ portfolio and the Sharpe ratio minimizing portfolio during a bearish period. Therefore, whether $\boldsymbol{\beta}$ is more proportional to $\mathbf{e}$ or less so is not as an important factor as the trend in the market, in answering the question of when the $1/N$ rule works.

3.2 Observed 1/*N* favorability index

Up to this point, we have only discussed properties of the "theoretical" version of the $1/N$ favorability index, in the sense that the two vectors used for calculating the cosine measure are conditional expectations of the vector of realized returns and the vector of realized aggregate covariances. But there is a discrepancy between this theoretical $1/N$ favorability index and the observed $1/N$ favorability index which is calculated directly based on realized asset returns. To discuss this issue, we adopt the same notations as the previous sub-section and in addition let

$$\widehat{\boldsymbol{\mu}}_T = \overline{\mathbf{R}}$$

and

$$\widehat{\boldsymbol{\Sigma}}_T = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{R}_t - \overline{\mathbf{R}})(\mathbf{R}_t - \overline{\mathbf{R}})^\intercal$$

denote the sample mean and sample covariance matrix of T realized returns respectively. Then, the observed 1/N favorability index is:

$$\cos(\widehat{\boldsymbol{\mu}}_T, \widehat{\boldsymbol{\Sigma}}_T \mathbf{e}) = \frac{\langle \widehat{\boldsymbol{\mu}}_T, \widehat{\boldsymbol{\Sigma}}_T \mathbf{e} \rangle}{\|\widehat{\boldsymbol{\mu}}_T\| \|\widehat{\boldsymbol{\Sigma}}_T \mathbf{e}\|}.$$

The following proposition (proved in the Appendix) shows that the observed, or sample-based, 1/N favorability index is a consistent estimator for the conditional 1/N favorability index.

Proposition 3.1. *Suppose that the parameter of the single-factor model in eq.* (2) *satisfies the following assumptions:*

A1. $\|\boldsymbol{\beta}\| = O(\sqrt{N})$ *as* $N \to \infty$.

A2. There exists a fixed $\delta > 0$*, such that* $\cos(\boldsymbol{\beta}, \mathbf{e}) > \delta$ *as* $N \to \infty$.

Under the asymptotics that $N, T \to \infty$ *with relative rate* $\frac{N}{T} = O(1)$*, the observed 1/N favorability index is a consistent estimator for the conditional 1/N favorability index, i.e.,*

$$\cos(\widehat{\boldsymbol{\mu}}_T, \widehat{\boldsymbol{\Sigma}}_T \mathbf{e}) - \cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T \mathbf{e}) \xrightarrow{p} 0. \tag{4}$$

The consistency of the observed 1/N favorability index is a critical property for our paper. Without this property, the positive "1/N favorability index - market condition" relationship implied by the single-factor model cannot be observed from realized asset returns. We show the consistency of the observed 1/N favorability index under so-called high-dimensional asymptotics that both N and T go to infinity to cater for the case of a relatively small T and a potentially large portfolio size. Note that the two assumptions we impose in Proposition 3.1 are relatively mild. In particular, A1 is satisfied as long as the elements in $\boldsymbol{\beta}$ are bounded in probability and A2 prevents the cosine of the angle between $\boldsymbol{\beta}$ and $\mathbf{e}$ from going to zero as the portfolio size increases.

3.3 Regime-switching in factor return

Thus far, we have not discussed the dynamics of factor returns. The dynamics plays an important role in determining whether the 1/N favorability index - market condition relationship implied by the single-factor model could be observed in the real world. If the market return factor stays positive across time, the conditional 1/N favorability index is always in its positive range and we will fail to observe the pattern described by eq. (3). Bae et al. [2014] show the relevance of regime-switching in the market in portfolio optimization problems. In this sub-section, we introduce a relatively parsimonious regime-switching lognormal process (RSLN, Hardy [2001]) for modeling the factor returns. In the next sub-section, we will calibrate model parameters using real world asset returns and then simulate synthetic returns based on these parameters to determine whether there is a positive relationship between the 1/N favorability index and the average return in the market.

We assume that the market oscillates between a bullish regime characterized by a high expected market factor return and a bearish regime featured by a low or even negative expected market factor return. Let ρ_t denote the regime applying over the period from $t-1$ to t, $\rho_t = 1, 2$. The transition matrix $P = \{p_{ij}\}_{i,j=1,2}$ contains the probabilities of switching regimes, i.e.,

$$p_{ij} = \Pr\{\rho_{t+1} = j | \rho_t = i\} \quad i = 1, 2,\ j = 1, 2.$$

The probabilities are all assumed to be independent of t. We assume that the market factor return follows a normal distribution within regime, i.e.,

$$R_{mt} | \rho_t \overset{\text{iid}}{\sim} N(\mu_{\rho_t}, \sigma^2_{\rho_t}). \tag{5}$$

In the next sub-section, we will calibrate the model parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and the regime-switching probabilities from real data.

It is worth emphasizing that the positive relationship between the conditional $1/N$ favorability index and the cross-sectional average return is only manifested when we focus on a relatively short time horizon. This is because as $T \to \infty$, both $\boldsymbol{\mu}_T$ and $\boldsymbol{\Sigma}_T$ converge to their respective limits. In particular, it is easy to show that

$$\lim_{T\to\infty} \boldsymbol{\mu}_T \xrightarrow{p} \boldsymbol{\beta}(\pi_1\mu_1 + \pi_2\mu_2)$$

and

$$\lim_{T\to\infty} \boldsymbol{\Sigma}_T \xrightarrow{p} [\pi_1\sigma_1^2 + \pi_2\sigma_2^2 + \pi_1\pi_2(\mu_1-\mu_2)^2]\boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \sigma^2\mathbf{I},$$

where $\{\pi_1, \pi_2\}$ is the stationary distribution of the regime-switching model and can be solved from the equation $\pi P = \pi$. Therefore, if we consider an infinite horizon, both the conditional $1/N$ favorability index $\cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T\mathbf{e})$ and the average conditional expected return $\frac{1}{N}\mathbf{e}^\intercal\boldsymbol{\mu}_T$ converge to a single point, and thus it is not sensible to study the favorability-average return relationship any longer. When we consider a smaller value of T, there will be more variation in terms of the average market factor return across different samples. Consequently, the positive favorability-average return relationship can be better perceived. Another important reason for considering a small value of T is that, as we have discussed earlier, a $1/N$ portfolio which is held for a long period of time without being rebalanced is no longer a $1/N$ portfolio.

3.4 Simulating the relation between favorability and returns

In this sub-section, we check whether the model implication that there is a positive relationship between a market's $1/N$ favorability index and its average return level can be observed from simulated asset returns. Before simulating the asset returns, we first calibrate the parameters in the regime-switching model and those in the single-factor model by using empirical index returns and individual stock returns.

For the purpose of calibration, we assume a monthly regime-switching frequency. The data used for calibrating the regime-switching model is monthly returns of the S&P 500 index from January 1960 to December 2016. We use this data to calibrate the regime-switching model because the S&P 500 index is a value-weighted index whose return, according to classical asset pricing theories, can be interpreted as the return of the market portfolio. Below are the parameters of the calibrated regime-switching model, including the annualized expected return and standard deviation in each of the two regimes and the transition probability matrix:

$$\begin{matrix} (12\mu_1, \sqrt{12}\sigma_1) = (11.17\%, 8.42\%) \\ (12\mu_2, \sqrt{12}\sigma_2) = (3.14\%, 17.13\%) \end{matrix}, P = \begin{bmatrix} 0.95 & 0.05 \\ 0.03 & 0.97 \end{bmatrix}.$$

In addition, we use monthly returns data of 100 stocks randomly picked from the S&P 500 index constituents to calibrate σ^2 and $\boldsymbol{\beta}$. According to the calibration results, the bound (other than ± 1) for the conditional $1/N$ favorability index (recall eq. (3)) is $\pm\cos(\boldsymbol{\beta}, \mathbf{e}) = \pm 0.96$; the residual standard deviation is $\sigma = 7.55\%$.[4]

Once all of the parameters have been obtained, we simulate a sequence of 25,200 daily market returns based on the calibrated regime-switching model. Note that the data frequency (daily) is different from the regime-switching frequency (monthly), so that there is a potential switch in the regime every 21 days. Simulating daily returns allows us to have a greater T to calculate the observed $1/N$ favorability index for each year. After the market returns are generated, we simulate individual daily returns for 100 hypothetical stocks based on the $\boldsymbol{\beta}$ and σ calibrated from the single-factor model. In each year (252 consecutive hypothetical days) we record the observed $1/N$ favorability index $\cos(\widehat{\boldsymbol{\mu}}_T, \widehat{\boldsymbol{\Sigma}}_T\mathbf{e})$ calculated using the simulated daily stock returns. In addition, we also record the annualized market factor return $\sum_t R_{mt}$ during the year. The panel (a) of Figure 1 shows how the observed $1/N$ favorability index varies when the market return is at different

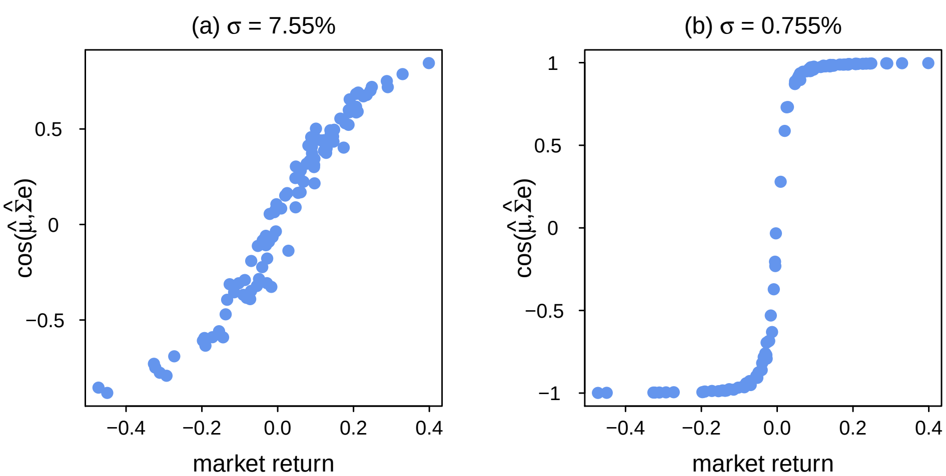
[4]This is a monthly rate.

levels. The pattern is consistent with our conjecture. Specifically, when the market is bullish, the observed $1/N$ favorability index is high and holding a $1/N$ portfolio is highly favorable. When the market is depressed, the observed $1/N$ favorability index is negative and the $1/N$ rule leads to poor performance.

According to our discussions in Section 3.1, the range of the conditional $1/N$ favorability index solely depends on the sign of the annualized market factor return. Therefore, if we create a similar plot as the panel (a) of Figure 1 but use the conditional $1/N$ favorability index to replace the observed index, we expect to see a vertical cliff shape, i.e., the points to the right of the vertical line $x = 0$ lie in the band $[0.96, 1]$, and those to the left of the line $x = 0$ lie in the band $[-1, -0.96]$. According to Proposition 3.1, as the number of assets N and the length of holding period T both increase to infinity at a relative rate of $\frac{N}{T} = O(1)$, the observed $1/N$ favorability index will converge to the conditional $1/N$ favorability index at all levels of the market factor return. Therefore, we expect to see a vertical cliff shape in the observed $1/N$ favorability index - market factor return plot when N and T are sufficiently large.

To illustrate the effects of the residual standard deviation in the single-factor model on the relation between the market return and the $1/N$ favorability index, we simulate another set of individual stock returns based on the same calibrated parameters but only replace the calibrated residual standard deviation σ with $\sigma/10 = 0.755\%$. Since the residual standard deviation has been greatly reduced, the observed and theoretical $1/N$ favorability indices should be much closer. The reason is that the conditional $1/N$ favorability index is defined as the conditional expectation of the observed index, and the conditional expectation is taken only over the randomness in assets' residual returns. As a result, we expect to see a clearer vertical cliff shape in the plot. The panel (b) of Figure 1 shows the market return and the observed $1/N$ favorability index in each hypothetical year. The clear vertical cliff shape in the panel (b) of Figure 1 confirms our result in eq. (3).

Figure 1: Observed $1/N$ favorability index vs. market factor annualized return: simulated data



3.5 Favorability - market condition relationship: empirical results

In this sub-section, we use actual return data to assess whether a market's observed $1/N$ favorability index varies in bull and bear periods, as predicted jointly by the single-factor model introduced in Section 3.1 and the convergence property established in Section 3.2. We investigate the relationship between market condition and the $1/N$ favorability index using both individual stock returns data in seven different countries and six out of the seven real datasets considered in DeMiguel et al. [2009]. It is important to note that in the datasets used in DeMiguel et al. [2009], the portfolio constituents are themselves portfolios or indices instead of individual stocks.

As mentioned earlier, we study the relation between market condition and the $1/N$ favorability index using individual stocks from seven countries. For each country, we pick a representative equity market index and assume the market to be comprised of all constituent stocks of the selected index. Names of the seven indices, as well as the countries, numbers of constituent stocks, and time periods of data, are listed in Table 1. A dynamic list of index constituents and daily price of all stocks that have appeared in an index during the data period are downloaded from the Compustat database. Each dataset covers the time from the index creation date to December 31, 2016. For each country, we compute the observed $1/N$ favorability index for each complete calendar year of the sample period. In this calculation we only use stocks that remain in the index and have complete price records throughout the year. We also compute the average return of the same stocks during the year. In this way, we obtain a number of favorability indices - average return records for each equity market. We repeat the calculation using the six datasets considered in DeMiguel et al. [2009][5]. The abbreviation of the six datasets, the number of assets in each dataset, the data period, as well as a brief description of each dataset are summarized in Table 2. Daily returns of industry and factor portfolios are taken from Kenneth French's website[6]. Monthly returns on MSCI country indices are from Morgan Stanley Capital International (MSCI) website[7]. Only for the MSCI dataset the $1/N$ favorability index is calculated with monthly returns due to limited access to daily data. We consider the same time periods used by DeMiguel et al. [2009] and use the abbreviation in Table 2 of DeMiguel et al. [2009] to refer to the datasets.

Table 1: List of equity market data used in the empirical study

Index name	# of stocks	Data period	Description	Country
DAX	30	Sep 1999 - Dec 2016	A blue chip index consisting of the 30 major German companies trading on the Frankfurt Stock Exchange	Germany
Euronext 100	100	Nov 2006 - Dec 2016	A blue chip index of the pan-European exchange Euronext NV	EU
Nikkei 225	225	Jan 1986 - Dec 2016	A stock market index for the Tokyo Stock Exchange (TSE)	Japan
S&P 500	500	Apr 1964 - Dec 2016	An American stock market index based on the market capitalizations of 500 large companies having common stock listed on the NYSE or NASDAQ	US
S&P ASX 200	200	Feb 2001 - Dec 2016	A market-capitalization weighted and float-adjusted stock market index of stocks listed on the Australian Securities Exchange	Australia
S&P TSX	250	Feb 1982 - Dec 2016	A benchmark Canadian index representing roughly 70% of the total market capitalization on the Toronto Stock Exchange (TSX)	Canada
S&P United Kingdom	86	Jan 1988 - Dec 2016	A sub-index of the S&P Europe 350 - includes all UK-domiciled stocks from the parent index	UK

Table 2: List of DeMiguel et al. (2009) portfolio data used in the empirical study

Abbreviation	# of assets	Data period	Description
Industry	11	Jul 1963 - Nov 2004	Ten industry portfolios and the US equity market portfolio
International	9	Jan 1970 - Jul 2001	Eight MSCI country indices and the MSCI World Index
MKT/SMB/HML	3	Jul 1963 - Nov 2004	SMB and HML portfolios and the US equity market portfolio
FF-1-factor	21	Jul 1963 - Nov 2004	Twenty size- and book-to-market portfolios and the US equity market portfolio
FF-3-factor	23	Jul 1963 - Nov 2004	Twenty size- and book-to-market portfolios and the MKT, SMB, and HML portfolios
FF-4-factor	24	Jul 1963 - Nov 2004	Twenty size- and book-to-market portfolios and the MKT, SMB, HML, and UMD portfolios

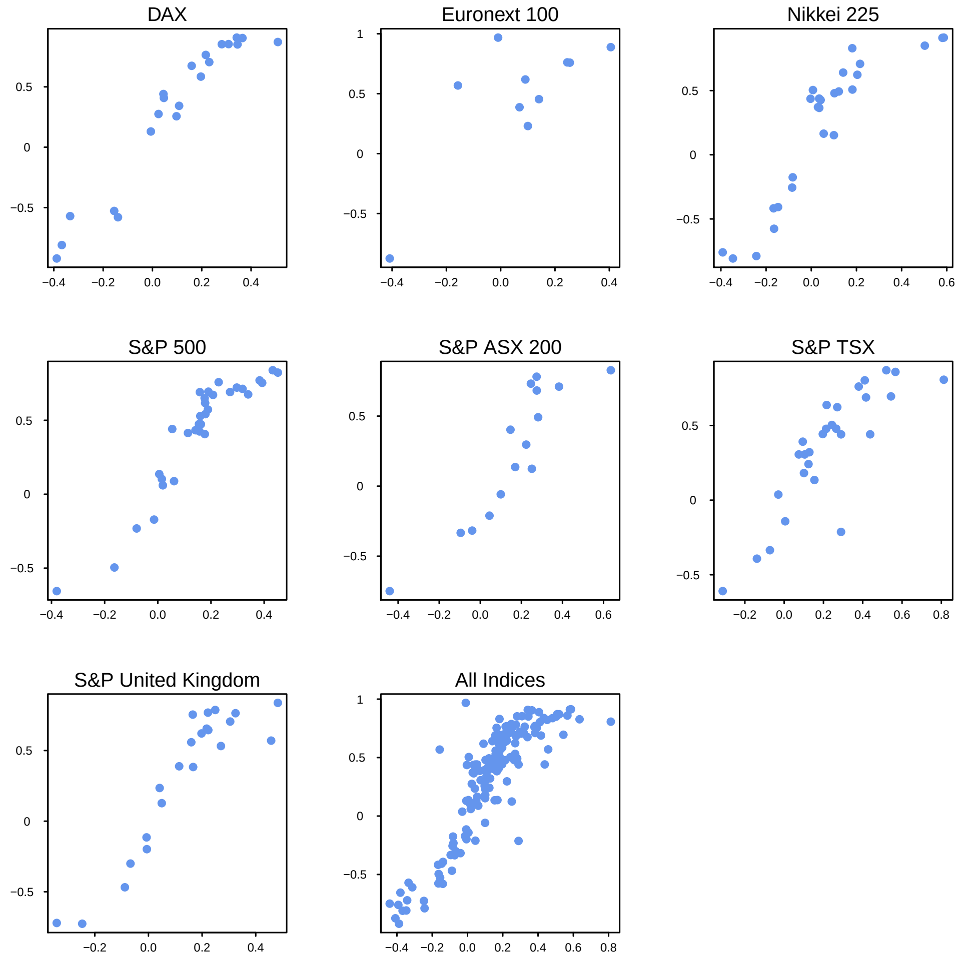
Figure 2 presents the relationship between the observed $1/N$ favorability index and the market average return in each country. Each of the first seven panels shows the result obtained from a particular country.

[5] DeMiguel et al. [2009] consider seven real world datasets. We do not have access to the S&P 500 Sector data and therefore only use the remaining six datasets.

[6] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[7] https://www.msci.com/end-of-day-data-country

Figure 2: Observed $1/N$ favorability index vs. market average return: empirical data

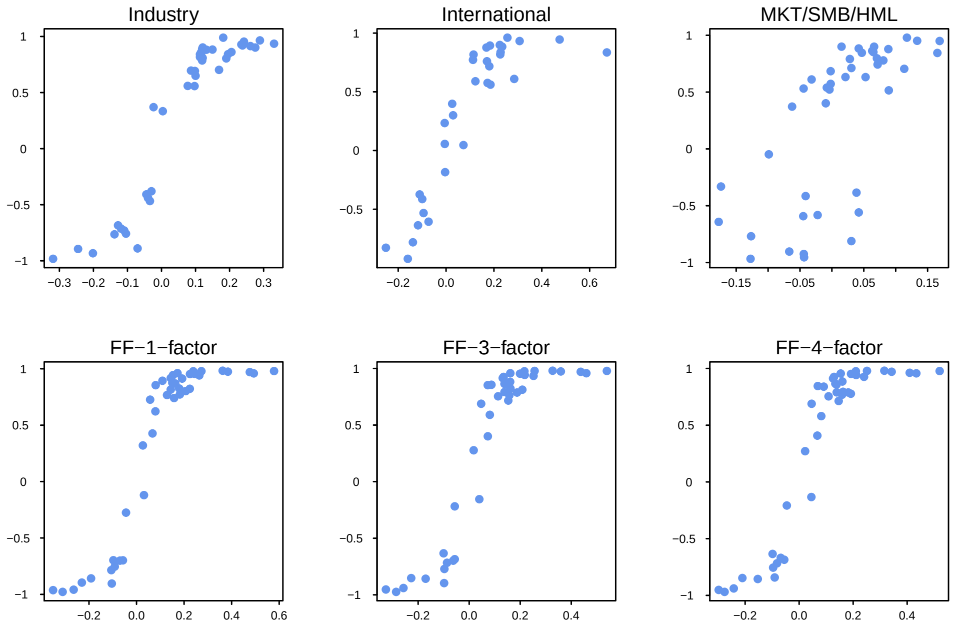


Each point corresponds to a year of the data period. The x-coordinate represents the annualized return of the $1/N$ portfolio. The y-coordinate measures the observed $1/N$ favorability index.

The last panel displays a mix of all points collected from the seven markets. In each of the first seven panels, we observe an apparently upward trend, which means that a bull market tends to be accompanied by a high observed $1/N$ favorability index. This is consistent with what our theoretical analysis suggests and with the simulation study in the previous sub-section. According to the last panel, when pooled together, the points from different countries roughly lie on a common upward sloping curve. Later, it will be seen that this pattern in pooled records from different countries is critical for us to be able to observe the cross-country phenomenon that the $1/N$ portfolio usually performs well in bullish countries, instead of just during bullish periods. Suppose that we see an upward trend in the panel corresponding to each country but cannot see a clear single upward trend when the points from all countries are pooled (note that we may see a few curves lying next to each other). If this happens, a bull market (in terms of its average return in comparison with that in other countries) does not necessarily present a favorable condition for the $1/N$ portfolios because the $1/N$ favorability index is not necessarily high.

Figure 3 presents the relationship between the market return and the $1/N$ favorability index when each of the DeMiguel et al. [2009] datasets is taken as constituting the whole market. As in Figure 2, a clear positive

Figure 3: Observed 1/N favorability index vs. market average return: portfolio of factor portfolios



pattern is observed in all panels. It is worth mentioning that except for the panel for the “MKT/SMB/HML” dataset, the vertical cliff shape is more obvious in the remaining panels of Figure 3 than in Figure 2. A possible explanation of this phenomenon is that the residual volatility in portfolio returns is smaller than that in individual stock returns (recall Figure 1). It is worth pointing out that the “MKT/SMB/HML” dataset contains the Fama-French SMB and HML portfolios as well as the US equity market portfolio. Unlike the other datasets where all portfolio components are affected by at least the market portfolio as a common factor, the SMB and HML portfolios have much less exposure to the market factor due to the way they are constructed. The lack of an explicit factor structure explains why the pattern in the “MKT/SMB/HML” panel is weaker. The three panels in the second row look similar because these three datasets are different only by the factor portfolios included.

We tried to seek for an asset pricing model that would predict a positive relationship between the 1/N favorability index and the average return level. However, most classical asset pricing models, e.g., the CAPM, take an equilibrium perspective and as a result do not admit a negative expected return of the market portfolio, thus making it difficult to justify the negative risk-return relationship in bear markets. Moreover, according to either the classical form of the CAPM or many of its extensions, there is some idiosyncratic-type risk that cannot be rewarded in the form of a higher expected return. Therefore, it is hardly possible to connect the risk measured by aggregate covariance to an extant asset pricing model. Since our focus is more on an ex-post (or realized), instead of an ex-ante (or expected), risk-return relationship in different market conditions, we resorted to a statistical model to account for our observation. Since it is commonly agreed that the market return is the dominating factor that drives the cross-section of equity returns, the single-factor model unsurprisingly provides a reasonable explanation for the empirical fact that bull markets are usually accompanied by a high 1/N favorability index.

4 Empirical performance of the 1/N portfolio

Section 3 shows using both theoretical justification and numerical evidence that bull markets are usually accompanied by a high 1/N favorability index. Therefore, if the 1/N favorability index is a good measure of the difficulty for an optimized portfolio to outperform the 1/N portfolio in terms of the Sharpe ratio, the 1/N portfolio should be more difficult to beat in bull markets. In this section, we conduct an empirical study to check whether the difficulty for an optimized portfolio to outperform a 1/N portfolio indeed depends on the market average return. If the answer is affirmative, it means that we have been adopting an overly demanding benchmark for evaluating portfolio optimization methods in bull markets and that the difficulty for an optimized portfolio to outperform the 1/N rule in such markets is partly due to the fact that the 1/N portfolio is close to being optimal.

Before proceeding with the analysis, we first discuss how to quantify whether or not it is difficult for an optimized portfolio to outperform the 1/N portfolio in a particular market.

4.1 Assessment of the 1/N portfolio

Due to variations in the market environment, it is not desirable to use an absolute measure, such as the Sharpe ratio, to evaluate the performance of the 1/N portfolio in different markets. Kim and Lee [2016] propose a novel portfolio performance measure which assesses the goodness of a portfolio based on the probability for a uniformly distributed random portfolio to outperform it. We use a similar portfolio comparison idea here. To quantify how well the 1/N portfolio performs in a particular market, we select a widely-recognized well-performing optimized portfolio as a benchmark. If the 1/N portfolio yields a higher Sharpe ratio than the benchmark portfolio does, we deem the 1/N portfolio as being hard to beat in the market of interest. Otherwise, we view this particular market as favorable to the 1/N portfolio. The choice of the benchmark portfolio will be discussed in the next section.

As has been stated in the introduction, although we will carry out a systematic comparison between the 1/N portfolio and an optimized portfolio in the subsequent sections, we do not take a specific stand with regard to our preference for any of the portfolios; instead a comparison is made only for the purpose of providing a relative measure of how well the 1/N portfolio performs. That is, we are more interested in identifying the common features of the markets which favor the 1/N portfolio.

4.2 Selection of the portfolio benchmark

As mentioned earlier, an ideal benchmark for assessing the performance of the 1/N portfolio would be a widely-acknowledged well-performing optimized portfolio. This guideline directs us to consider a minimum-variance (MV) type portfolio as the benchmark. The reason for choosing an MV type portfolio is that it implicitly exploits risk-based pricing anomalies (Scherer [2010]) and is widely recognized (Jagannathan and Ma [2003], Clarke et al. [2006], DeMiguel et al. [2009]) for its surprisingly high returns and low realized volatility. This portfolio has the added advantage that its construction does not require estimates of the expected return vector. Furthermore, techniques have been developed in the literature to refine the sample-based MV portfolio to make it less vulnerable to estimation errors. We equip our selected benchmark portfolio with such techniques, so that the benchmark portfolio is more likely to yield a good out-of-sample performance, making it more sensible to assess the difficulty of outperforming the 1/N portfolio based on a comparison with the benchmark portfolio.

We choose two variations of the sample-based MV portfolio as the benchmark for assessing the performance of the 1/N portfolio. The first one is the MV portfolio based on the "shrinkage towards identity" covariance matrix estimator (see Ledoit and Wolf [2004]). We hereafter refer to this portfolio as a "shrinkage-based MV portfolio". The weight vector for this portfolio is given by:

$$\widehat{\mathbf{w}}_{s-MV} = \arg\min_{\mathbf{w}} \mathbf{w}^\intercal \widehat{\mathbf{\Sigma}}_s \mathbf{w} \quad \text{s.t. } \mathbf{w}^\intercal \mathbf{e} = 1,$$

where $\widehat{\mathbf{\Sigma}}_s$ is the shrinkage estimator for the covariance matrix. In our empirical study, this estimator is obtained from the past 60 months' return history. The second optimized portfolio we use as a benchmark is a short-sale-constrained MV portfolio, whose weight vector is given by:

$$\widehat{\mathbf{w}}_{c-MV} = \arg\min_{\mathbf{w}} \mathbf{w}^\intercal \widehat{\mathbf{\Sigma}} \mathbf{w} \quad \text{s.t. } \mathbf{w}^\intercal \mathbf{e} = 1 \text{ and } \mathbf{w} \geq 0,$$

where $\widehat{\mathbf{\Sigma}}$ is the sample covariance matrix estimator. Again, in our analysis, the sample covariance matrix is estimated from the past 60 months' return history. Compared with the sample-based MV portfolio, the shrinkage-based MV portfolio is less vulnerable to estimation errors, because the shrinkage estimator for covariance matrix corrects the over-dispersed sample eigenvalues and thus is closer to the population covariance matrix in the sense of the Frobenius norm. Imposing the no-short-sale constraints is an alternative way of mitigating the issue of estimation errors. Jagannathan and Ma [2003] point out that constraining portfolio weights to be non-negative can reduce the risk in estimated optimal portfolios even when the constraints turn out to be wrong by showing that a short-sale constraint on the MV portfolio is equivalent to shrinking the elements of the sample covariance matrix. Another reason for using the short-sale-constrained MV as a benchmark is the finding reported by DeMiguel et al. [2009] that "of all the optimizing models studied here, the minimum-variance portfolio with constraints studied in Jagannathan and Ma [2003] performs the best in terms of Sharpe ratio".

Admittedly, the way in which we have quantified the superiority of the $1/N$ portfolio makes it credible to assert that the $1/N$ portfolio is not hard to beat when a benchmark portfolio outperforms but it less so to state that the $1/N$ portfolio is hard to beat when the $1/N$ portfolio outperforms, because the benchmark portfolio is not necessarily always the best-performing one among all the optimized portfolios. However, since the MV type portfolios are known to have good performance and we also have applied effective estimation error controlling techniques to improve the portfolios' out-of-sample performance, our method of quantifying the performance of the $1/N$ portfolio cannot be easily dismissed.

4.3 Cross-sample performance of buy-and-hold 1/N portfolios

In this sub-section, we construct and evaluate portfolios by using a number of datasets that include both equity returns in the US market and those in some non-US markets. To construct stock portfolios, we select the equity market indices listed in Table 1 and use their respective constituent stocks to construct a number of buy-and-hold portfolios. We also build portfolios of synthetic assets by using a few of the DeMiguel et al. [2009] datasets listed in Table 2. We are interested in whether the $1/N$ portfolio more likely outperforms a benchmark portfolio in bull markets, as suggested by the implication of the single-factor model.

The tools we use for studying the relation between the relative performance of portfolios and the market condition are buy-and-hold portfolios with a holding period of one year. For each equity index listed in Table 1, we construct one-year buy-and-hold portfolios of the index constituents starting from the beginning of the sixth year. On each portfolio construction date, the stocks that enter the portfolios are those that 1) are an index constituent on the construction date and 2) have at least five years' price history before the construction date. For the two benchmark portfolios, i.e., the shrinkage-based and the short-sale-constrained MV portfolios, we use the monthly returns in the past five years to estimate the portfolio weights. All portfolios are held for one year without rebalancing before being liquidated. Thus, we do not need to account for any transaction cost when reporting performance measures. We compute the Sharpe ratio of each portfolio based on its monthly returns during the holding period. Meanwhile, we record the return of each $1/N$ portfolio and use it to represent the average return on the market during the year over which the portfolio is held.

We only use four out of the six DeMiguel et al. [2009] datasets listed in Table 2 to construct portfolios of synthetic assets. The four datasets are "Industry", "International", "MKT/SMB/HML", and "FF-4-factor". The datasets "FF-1-factor" and "FF-3-factor" are omitted because they are different from the "FF-4-factor" dataset only in two or three individual assets. The portfolio construction procedure and the performance measures reported are the same as those for the stock portfolios.

Pooling the yearly records from the seven countries together and those from the four DeMiguel et al. [2009] datasets together, we obtain 169 and 138 records respectively. Each record consists of the return of the $1/N$ portfolio, as well as the Sharpe ratios of the $1/N$ portfolio, the shrinkage-based MV portfolio, and the short-sale-constrained MV portfolio. For the sake of robustness, we first treat the shrinkage-based MV portfolio as the benchmark to assess the performance of the $1/N$ portfolio and then repeat the same analysis with the short-sale-constrained portfolio as the benchmark. Therefore two sets of parallel results are reported. In some of the records, both the $1/N$ portfolio and the benchmark portfolio yield a negative Sharpe ratio. We discard those records because negative Sharpe ratios are generally more difficult to interpret and not amenable to a meaningful comparison.[8] We label the remaining records with either 1 or 0, according to whether the $1/N$ portfolio or the benchmark portfolio yields a higher Sharpe ratio. We refer to the binary label as the "$1/N$ superiority indicator". Note that if one portfolio yields a positive Sharpe ratio and another yields a negative Sharpe ratio, we view the former as the more superior portfolio. We code the relative performance of the $1/N$ portfolio into a binary variable instead of simply taking the difference of the two Sharpe ratios because, on one hand, when any one of the Sharpe ratios is negative, the difference becomes hard to interpret, and on the other hand, the maximum attainable Sharpe ratio in different markets and sub-periods varies, making the level of difference in Sharpe ratios quite volatile across samples.

Then we perform a logistic regression, where the explanatory variable is the market average return during a sub-period (r_i) and the dependent variable is the $1/N$ superiority indicator assigned to each record (s_i). Recall that $s_i = 1$ corresponds to the case where the $1/N$ portfolio yields a higher Sharpe ratio compared with the benchmark portfolio, and $s_i = 0$ otherwise. Let p_i denote the probability that the $1/N$ portfolio outperforms in a market with an average return of r_i, i.e., $p_i = \Pr\{s_i = 1|r_i\}$. The logistic regression model is given by the following equation:

$$\log \frac{p_i}{1-p_i} = \beta_1 + \beta_2 r_i.$$

Table 3: Estimation and hypothesis testing results for logistic regression of the $1/N$ superiority indicator on the market average return: portfolio of stocks

(a) Benchmark: shrinkage-based MV				(b) Benchmark: short-sale-constrained MV			
	Estimate	Std. Error	p-value		Estimate	Std. Error	p-value
$\hat{\beta}_1$	-1.39	0.31	8.04e-06 ***	$\hat{\beta}_1$	-1.22	0.30	4.89e-05 ***
$\hat{\beta}_2$	8.61	1.77	1.12e-06 ***	$\hat{\beta}_2$	6.96	1.61	1.50e-05 ***
model	–	–	8.85e-10 ***	model	–	–	3.02e-07 ***

Significance codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
In panel (a), the shrinkage-based MV portfolio is used as the benchmark portfolio for generating the $1/N$ favorability index. In panel (b), the short-sale-constrained MV portfolio is used as the benchmark.

The logistic regression model is fitted with the pooled yearly records from the equity portfolios and those from the portfolios of synthetic assets separately. The model is estimated by using the maximum

[8] For example, if Portfolios A and B both achieve an annual return of -5% and standard deviations of 5% and 20% respectively, then Portfolio A has a more negative Sharpe ratio. However, Portfolio A may be the more desirable one, because it is much less volatile compared with Portfolio B. Therefore, a direct comparison between two negative Sharpe ratios can be misleading.

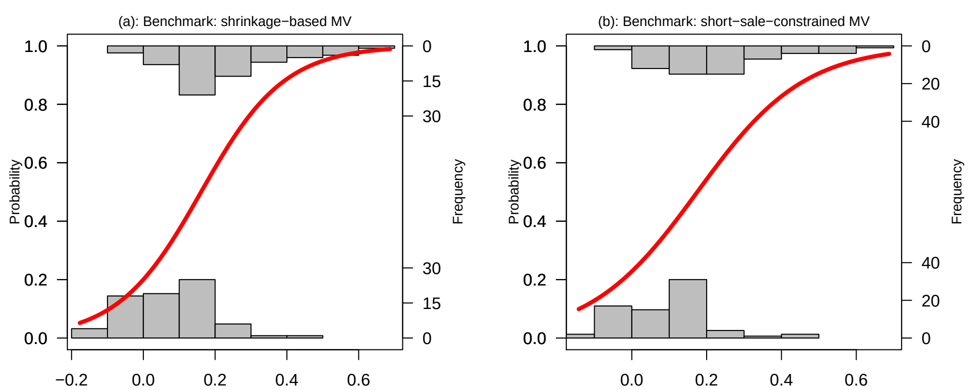
Table 4: Estimation and hypothesis testing results for logistic regression of the $1/N$ superiority indicator on the market average return: portfolio of synthetic assets

(a) Benchmark: shrinkage-based MV				(b) Benchmark: short-sale-constrained MV			
	Estimate	Std. Error	p-value		Estimate	Std. Error	p-value
$\hat{\beta}_1$	-0.76	0.28	0.0058 **	$\hat{\beta}_1$	-0.83	0.30	0.0052 **
$\hat{\beta}_2$	3.95	1.71	0.0205 *	$\hat{\beta}_2$	5.98	1.94	0.0020 **
model	–	–	0.0139 **	model	–	–	0.0005 ***

Significance codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
In panel (a), the shrinkage-based MV portfolio is used as the benchmark portfolio for generating the $1/N$ favorability index. In panel (b), the short-sale-constrained MV portfolio is used as the benchmark.

likelihood method. The estimates of the parameters and the estimated standard errors for $\hat{\beta}_1$ and $\hat{\beta}_2$, the respective p-values based on a Wald test, as well as the p-value for the joint significance of $\hat{\beta}_1$ and $\hat{\beta}_2$ based on a likelihood ratio test are reported. Table 3 summarizes the model fitting results with the stock portfolio records. Table 4 reports the model fitting results with the records from portfolios formed on the DeMiguel et al. [2009] datasets. In panel (a) of Tables 3 and 4, the shrinkage-based MV portfolio is used as the benchmark portfolio for generating the $1/N$ favorability index; in panel (b), the short-sale-constrained MV portfolio is used as the performance benchmark. According to the hypothesis testing results, there is always a significantly positive relationship between the average return in the market and the superior performance of the $1/N$ portfolio regardless of the choice between the two MV portfolios as the benchmark. In addition, this relationship is observed both in stock portfolios and portfolios of synthetic assets.

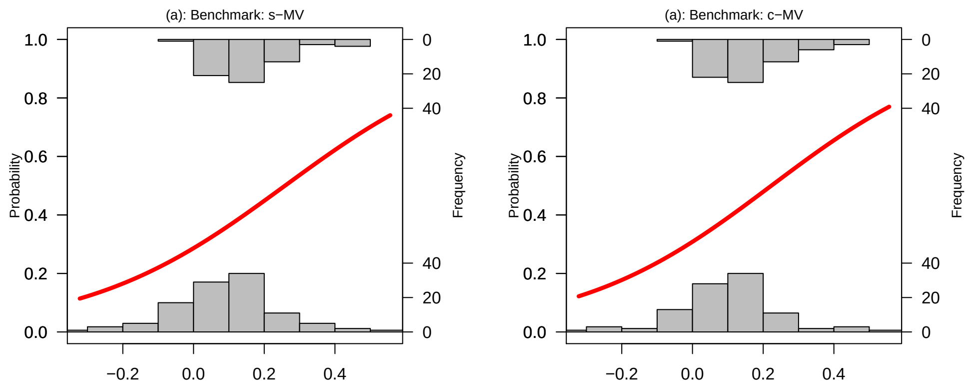
Figure 4: Logistic regression of the $1/N$ superiority indicator on the market average return: portfolio of stocks



The x-axis measures average return in a market over the holding year. The grey barplots are histograms of the average returns grouped by the $1/N$ superiority indicator. The histogram at the top corresponds to the group in which the $1/N$ portfolio outperforms. The histogram at the bottom corresponds to the group in which the benchmark portfolio outperforms. The red curve represents the fitted probability that the $1/N$ portfolio is superior to the benchmark portfolio. In panel (a), the shrinkage-based MV portfolio is used as the benchmark for generating the $1/N$ superiority indicator. In panel (b), the shortsale-constrained MV portfolio is used as the benchmark.

Figures 4 and 5 show the fitted probability that the $1/N$ portfolio outperforms the benchmark portfolio as well as the histograms of holding-year average market returns grouped by the $1/N$ superiority indicator. Figure 4 is created based on the records from the stock portfolios, and Figure 5 based on those from the portfolios formed on DeMiguel et al. [2009] datasets. In both figures, panel (a) presents the results with the

Figure 5: Logistic regression of the $1/N$ superiority indicator on the market average return: portfolio of synthetic assets



The x-axis measures average return in a market over the holding year. The grey barplots are histograms of the average returns grouped by the $1/N$ superiority indicator. The histogram at the top corresponds to the group in which the $1/N$ portfolio outperforms. The histogram at the bottom corresponds to the group in which the benchmark portfolio outperforms. The red curve represents the fitted probability that the $1/N$ portfolio is superior to the benchmark portfolio. In panel (a), the shrinkage-based MV portfolio is used as the benchmark for generating the $1/N$ superiority indicator. In panel (b), the shortsale-constrained MV portfolio is used as the benchmark.

shrinkage-based MV portfolio treated as the benchmark portfolio for generating the $1/N$ superiority indicator; panel (b) presents parallel results when the short-sale-constrained portfolio is used as the benchmark. The two histograms in each panel depict the distribution of market average returns grouped by the $1/N$ superiority indicator. The histogram at the top corresponds to the group where the $1/N$ portfolio outperforms the benchmark portfolio; the histogram at the bottom corresponds to the group where the $1/N$ portfolio underperforms. According to both figures, the higher the market average return is, the more probable that the $1/N$ portfolio will outperform the benchmark portfolio, regardless of the choice between the two benchmark portfolios. In addition, the relative (horizontal) location of the two histograms in each panel of Figures 4 and 5 supports our conclusion.

Comparing the results in Table 3 and Figure 4 with those in Table 4 and Figure 5, we notice that the positive relationship between the market average return and the superior performance of the $1/N$ portfolio is stronger when yearly records from the stock portfolios are used for model fitting. The primary reason is that the MV portfolios formed on DeMiguel et al. [2009] datasets in general have better performance compared with those formed on individual stocks. For stock portfolios, in 45.59% of the index-year records the $1/N$ portfolio outperforms the shrinkage-based MV portfolio; the percentage is 46.97% when the short-sale-constrained MV portfolio is used as benchmark. For portfolios formed on DeMiguel et al. [2009] datasets, the $1/N$ portfolio outperforms the shrinkage-based MV portfolio in 41.96% of the records; the percentage becomes 42.34% when the short-sale-constrained MV portfolio is used as benchmark. One possible reason for the performance discrepancy of MV portfolios when different datasets are used is that the size of portfolios formed on DeMiguel et al. [2009] datasets is in general smaller than the size of stock portfolios, rendering less estimation errors in the MV portfolios. An additional possible reason is that there may be better predictability in the return time series of the synthetic assets, which will also lead to better performance of the MV portfolios.

4.4 Cross-country performance of rebalanced 1/*N* portfolios

In the previous Section, we analyzed the relative performance of one-year buy-and-hold portfolios in different countries and over different time periods. However, real-world investors, especially mutual fund managers, usually adjust their portfolio on a monthly basis and are concerned with the performance of their portfolio over a longer investment horizon. Therefore, in this sub-section, we use the equity return datasets listed in Table 1 to construct monthly rebalanced portfolios with a long investment horizon. We compare the holding period Sharpe ratios of the 1/*N* portfolio and the benchmark portfolios to check whether the outperformance of the 1/*N* portfolio is associated with a bullish market.

Here are the details about the portfolio construction procedure. All of the three portfolios of interest, i.e., the 1/*N* portfolio and the two MV portfolios, are rebalanced at the end of each month. On each portfolio rebalancing date, we use the same set of stocks to construct the three updated portfolios. The stocks that enter the updated portfolios are those that 1) are an index constituent on the rebalancing date and that 2) have at least five years' price history before the rebalancing date. We always use the monthly returns data over the past five years to estimate the weight vector of the MV portfolios. All portfolios are held for a month and then rebalanced again. This portfolio construction procedure leads to a lengthy holding period of the data period minus five years. In addition to building portfolios within each market separately, we also construct an international diversification portfolio which is composed of all available stocks in the seven stock indices we consider. The construction date of the international diversification portfolio is the same as that of the S&P 500 portfolio.

Table 5: Holding period performance (adjusted for transaction costs) of the 1/*N*, shrinkage-based MV, and shortsale-constrained MV portfolios

	(a) Monthly rebalance, no trans. cost				(b) Monthly rebalance, with trans. cost			
	1/*N* return	1/*N* Sharpe	s-MV Sharpe	c-MV Sharpe	1/*N* return	1/*N* Sharpe	s-MV Sharpe	c-MV Sharpe
DAX	3.97%	0.09	0.11	0.11	3.86%	0.08	0.09	0.10
			(0.51)	(0.46)			(0.88)	(0.71)
Euronext 100	12.54%	0.27	0.23	0.17	12.30%	0.27	0.16	0.15
			(0.82)	(0.40)			(0.52)	(0.35)
Nikkei 225	0.76%	0.04	0.03	0.04	0.70%	0.04	-0.05	0.01
			(0.80)	(0.84)			(0.07)	(0.38)
S&P 500	6.93%	0.14	0.11	0.16	6.81%	0.14	0.05	0.13
			(0.49)	(0.58)			(0.03)	(0.72)
S&P ASX 200	5.20%	0.11	0.25	0.22	5.00%	0.11	0.20	0.20
			(0.17)	(0.09)			(0.32)	(0.16)
S&P TSX	6.62%	0.12	0.19	0.17	6.40%	0.12	0.16	0.15
			(0.08)	(0.08)			(0.16)	(0.18)
S&P United Kingdom	7.39%	0.15	0.23	0.17	7.27%	0.15	0.17	0.15
			(0.18)	(0.40)			(0.68)	(0.84)
International	5.48%	0.12	0.11	0.14	5.37%	0.12	0.04	0.10
			(0.71)	(0.46)			(0.05)	(0.69)

Portfolio "s-MV" stands for the shrinkage-based MV portfolio. Portfolio "c-MV" stands for the shortsale-constrained MV portfolio. In panel (a), the performance measures are reported as if any transaction cost is ignored. In panel (b), the performance measures are adjusted for a proportional transaction cost of 50 basis points.

Table 5 summarizes the holding period Sharpe ratios of the 1/*N* and the two benchmark portfolios, as well as the p-values for testing whether the 1/*N* portfolio yields a Sharpe ratio significantly different from that of the benchmark portfolio. The p-value for each Sharpe ratio difference test is reported below the Sharpe ratio of the benchmark portfolio and is shown in parentheses. We adopt the Sharpe ratio difference test proposed in Ledoit and Wolf [2008] when reporting the p-values. In panel (a) of Table 5, Sharpe ratios calculated without considering any transaction cost are reported; in panel (b), we report parallel results after taking transaction costs into account. We set the proportional transaction costs equal to 50 basis points per transaction as assumed in Balduzzi and Lynch [1999] and in DeMiguel et al. [2009]. If we denote by c the proportional transaction cost, then the evolution of wealth for a portfolio strategy k is

$$W_{k,t+1} = W_{k,t}(1+R_{k,t+1})\left(1 - c\sum_{j=1}^{N} |\hat{w}_{k,j,t+1} - \hat{w}_{k,j,t}|\right),$$

where $R_{k,t+1}$ is the portfolio return under strategy k during the period from t to $t+1$, and $\hat{w}_{k,j,t}$ is the weight of the jth asset at time t according to strategy k. In addition to the Sharpe ratio, we also record the holding period annualized returns (adjusted for transaction costs) of the $1/N$ portfolio, which is a reflection of the average return of each equity market during the holding period. This quantity is recorded because we are interested in whether or not the difficulty in outperforming a $1/N$ portfolio depends on the average return in the market.

According to panel (a) of Table 5, if we ignore the transaction costs, the only equity market that makes the $1/N$ portfolio difficult to beat is the market comprised of the Euronext 100 constituents, in the sense that the $1/N$ portfolio yields a holding period Sharpe ratio higher than that of both MV portfolios. Notably, the $1/N$ portfolio of the Euronext 100 stocks yields an annual return of 12.54% over the holding period, unmatched by any other equity market included in our empirical analysis. This confirms our claim that the $1/N$ portfolio is more difficult to be outperformed in bull markets. In the equity markets comprised of DAX, S&P ASX 200, S&P TSX, and S&P United Kingdom stocks, the $1/N$ portfolio is outperformed by both MV portfolios in terms of the holding period Sharpe ratio. In these markets, the annual return of the $1/N$ portfolio is much lower than that in the Euronext 100 market.

When transaction costs are taken into account, the $1/N$ portfolios become more difficult to be outperformed, due to their low turnover and thus low transaction costs incurred. In three out of the seven equity markets, i.e., the Euronext 100 market, the Nikkei 225 market, and the S&P 500 market, the $1/N$ portfolio outperforms both MV portfolios. The annualized holding period return of the $1/N$ portfolio in these three markets is 12.30%, 0.70%, and 6.81% respectively. In another three equity markets, i.e., the DAX market, the S&P ASX 200 market, and the S&P TSX market, the $1/N$ portfolio underperforms both MV portfolios. The annualized holding period return of the $1/N$ portfolio in these three markets is 3.86%, 5.00%, and 6.40% respectively. If we ignore the Nikkei 225 market, then there is a clear distinction in terms of the average return between the markets favorable and unfavorable to the $1/N$ portfolio. When international diversification is allowed, the $1/N$ portfolio has a higher Sharpe ratio than both MV portfolios. This is consistent with the finding in Tu and Zhou [2011] that "the $1/N$ rule remains hard to beat in the international portfolios".

It is useful to clarify the rationale for presenting the results in panel (b), since in this paper, we do not attribute the outperformance of a $1/N$ portfolio to practical reasons such as low transaction costs. The reason for including panel (b) is that, solely based on panel (a), we can only find one equity market which is favorable to the $1/N$ portfolio in the sense that the $1/N$ portfolio outperforms both MV portfolios. But this does not imply that the $1/N$ portfolio performs equally poorly in other markets. When we take transaction costs into consideration, the performance of the MV portfolios deteriorates more than the $1/N$ portfolio does. As a result, the variation across markets in terms of the difficulty to outperform the $1/N$ portfolio is manifested, and we are able to find three markets which are more favorable to the $1/N$ portfolio and then investigate whether those markets are featured by a high average return compared with those not favorable to the $1/N$ portfolio.

In a nutshell, our empirical investigation shows that when the monthly rebalancing scheme is adopted and when the transaction costs are taken into account, the superior performance of the $1/N$ portfolio is prevalent: it outperforms both MV portfolios in terms of the holding period Sharpe ratio in the European, US, and Japanese markets; it underperforms both MV portfolios in the German, Australian, and Canadian markets. Our findings are in accordance to those reported in DeMiguel et al. [2009] which focuses on the US market. The observation that the European and US markets are featured by a high average return and that the German, Australian, and Canadian markets are accompanied by a lower one supports our earlier analysis on the relationship between the $1/N$ favorability index and the market condition. Moreover, our results suggest that the perceived good performance of the $1/N$ portfolio in the US market is partly attributable to the bullish trend in the US market which makes it a favorable environment for holding a $1/N$ portfolio.

It is important to point out that there are two factors that make the results reported in this sub-section weaker than those in Section 4.3. First, the aggregate Sharpe ratio over a long investment horizon may fail to reflect the year-over-year relative performance of two portfolios. Specifically, it is possible that Portfolio A yields a higher Sharpe ratio than Portfolio B does in each individual year but a lower aggregate Sharpe ratio over the whole investment horizon. If this happens, we may not be able to spot the outperformance, in terms of the aggregate Sharpe ratio, of the $1/N$ portfolio even when the market is bullish. Second, since portfolios are rebalanced regularly, there may be other reasons behind the relative performance of portfolios. Plyakha et al. [2012] point out that the contrarian nature of the $1/N$ portfolio exploits the mean-reversion in stock returns. As a result, the $1/N$ portfolio in different markets may benefit to a different extent from its contrarian nature, depending on the return dynamics in the particular market. Because of these two reasons, we view the results in this sub-section to be complementary.

5 Conclusion

We have devised a market-specific measure of how favorable a market is to holding a $1/N$ portfolio and named it the "$1/N$ favorability index". This index measures the deviance between the $1/N$ portfolio and the true optimal portfolio in a market. Consequently, the greater the $1/N$ favorability index is, the more likely that the $1/N$ portfolio outperforms an optimized portfolio contaminated by estimation errors.

We have analyzed the behavior of the $1/N$ favorability index under a single-factor model which explains the cross-section of asset returns using a market factor. The single-factor model predicts a vertical cliff shape of the conditional $1/N$ favorability index vs. average market factor return curve, which means that bull markets tend to be accompanied by a positive $1/N$ favorability index and vice versa. The conditional $1/N$ favorability index can be consistently estimated by its sample estimator, which is called the observed $1/N$ favorability index, even when both the portfolio size and the sample size are large. Therefore, bull markets are expected to be accompanied by a high observed $1/N$ favorability index as well. Since the observed $1/N$ favorability index reflects the difficulty for an optimized portfolio (containing estimation errors) to outperform the $1/N$ portfolio during the period that the observed index corresponds to, a further implication is that the $1/N$ portfolio is more difficult to beat in bull markets.

We have conducted empirical studies to validate the model's implications. By examining the cross-sample performance of a number of one-year buy-and-hold $1/N$ portfolios relative to two carefully selected portfolio benchmarks, we find that the $1/N$ portfolio is more difficult to beat when the average market return during the year is higher. This finding holds for both stock portfolios and portfolios of synthetic assets including factor portfolios and indices. Further, we examine the performance of monthly rebalanced $1/N$ portfolios relative to the two benchmark portfolios in the equity market of different countries. We find that most markets in which the $1/N$ portfolio outperforms the benchmark portfolios have experienced a bullish trend over the holding period.

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Appendix

Proof of Proposition 3.1. Let $\overline{\boldsymbol{\epsilon}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\epsilon}_t$ and $\widehat{\boldsymbol{\Omega}}_T = \frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{\epsilon}_T - \overline{\boldsymbol{\epsilon}})(\boldsymbol{\epsilon}_T - \overline{\boldsymbol{\epsilon}})^\intercal$. We have

$$\widehat{\boldsymbol{\mu}}_T = \overline{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{R}_t = \boldsymbol{\beta}\overline{\mathbf{R}}_m + \overline{\boldsymbol{\epsilon}} = \boldsymbol{\mu}_T + \overline{\boldsymbol{\epsilon}}$$

and

$$\widehat{\boldsymbol{\Sigma}}_T = \frac{1}{T}\sum_{t=1}^{T}(\mathbf{R}_t - \overline{\mathbf{R}})(\mathbf{R}_t - \overline{\mathbf{R}})^\intercal = \boldsymbol{\Sigma}_T + 2\boldsymbol{\beta}\frac{1}{T}\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)(\boldsymbol{\epsilon}_t - \overline{\boldsymbol{\epsilon}})^\intercal + \widehat{\boldsymbol{\Omega}}_T - \sigma^2\mathbf{I}.$$

It follows that

$$\|\widehat{\boldsymbol{\Sigma}}_T\mathbf{e}\| \leq \|\boldsymbol{\Sigma}_T\mathbf{e}\| + 2\frac{1}{T}\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)(\boldsymbol{\epsilon}_t^\intercal\mathbf{e} - \overline{\boldsymbol{\epsilon}}^\intercal\mathbf{e})\|\boldsymbol{\beta}\| + \|(\widehat{\boldsymbol{\Omega}}_T - \sigma^2\mathbf{I})\mathbf{e}\|.$$

We further let the elements in $\boldsymbol{\epsilon}_t$ be $\boldsymbol{\epsilon}_t = (\varepsilon_{t1}, \varepsilon_{t2}, \ldots, \varepsilon_{tN})^\intercal$ and those in $\overline{\boldsymbol{\epsilon}}$ be $\overline{\boldsymbol{\epsilon}} = (\overline{\varepsilon}_1, \overline{\varepsilon}_2, \ldots, \overline{\varepsilon}_N)^\intercal$. To derive the order of $\|\widehat{\boldsymbol{\Sigma}}_T\mathbf{e}\|$, we examine the terms on the RHS of the equation above.

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)(\boldsymbol{\epsilon}_t^\intercal\mathbf{e} - \overline{\boldsymbol{\epsilon}}^\intercal\mathbf{e})\|\boldsymbol{\beta}\| &= \sqrt{N}\|\boldsymbol{\beta}\|\frac{1}{T}\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)\left(\frac{\boldsymbol{\epsilon}_t^\intercal\mathbf{e}}{\sqrt{N}} - \frac{\overline{\boldsymbol{\epsilon}}^\intercal\mathbf{e}}{\sqrt{N}}\right) \\
&= \sqrt{N}\|\boldsymbol{\beta}\|O_p\left(\frac{1}{\sqrt{T}}\right) \\
&= O_p\left(\frac{N}{\sqrt{T}}\right).
\end{aligned}$$

The second last equality in the equation above is a direct application of the result on the convergence rate of sample covariance. The last equality holds because $\|\boldsymbol{\beta}\| = O(\sqrt{N})$.

Since $\boldsymbol{\epsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \sigma^2\mathbf{I})$, for each element $\overline{\varepsilon}_j$, $j = 1, 2, \ldots, N$, we have $\sqrt{T}\overline{\varepsilon}_j = O_p(1)$, i.e., $\overline{\varepsilon}_j = O_p(T^{-\frac{1}{2}})$. Therefore,

$$\|\overline{\boldsymbol{\epsilon}}\| = \sqrt{\sum_{j=1}^{N}\overline{\varepsilon}_j^2} = O_p\left(\sqrt{\frac{N}{T}}\right).$$

$$\begin{aligned}
\|(\widehat{\boldsymbol{\Omega}}_T - \sigma^2\mathbf{I})\mathbf{e}\| &= \sqrt{\sum_{k=1}^{N}\left[\sum_{j=1}^{N}\frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{tk} - \overline{\varepsilon}_k)(\varepsilon_{tj} - \overline{\varepsilon}_j) - \sigma^2\right]^2} \\
&= \sqrt{\sum_{k=1}^{N}\left[\frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{tk} - \overline{\varepsilon}_k)\left(\sum_{j=1}^{N}\varepsilon_{tj} - \sum_{j=1}^{N}\overline{\varepsilon}_j\right) - \sigma^2\right]^2}.
\end{aligned}$$

Let independent random variables X_t and W_t denote $X_t = \varepsilon_{tk}$ and $W_t = \sum_{j\neq k}\varepsilon_{tj}$. Let $Y_t = X_t + W_t = \sum_{j=1}^{N}\varepsilon_{tj}$. It is straightforward to show that $E[X_1Y_1] = \sigma^2$. The variance of X_1Y_1 is:

$$\begin{aligned}
\text{Var}(X_1Y_1) &= E[X_1^2Y_1^2] - \{E[X_1Y_1]\}^2 \\
&= E[X_1^2(X_1 + W_1)^2] - \sigma^4 \\
&= E[X_1^4 + 2X_1^3W_1 + X_1^2W_1^2] - \sigma^4 \\
&= 3\sigma^4 + (N-1)\sigma^4 - \sigma^4 \\
&= (N+1)\sigma^4.
\end{aligned}$$

According to the Central Limit Theorem, we have:

$$\frac{\frac{1}{T}\sum_{t=1}^{T} X_tY_t - \sigma^2}{\sqrt{\frac{(N+1)\sigma^4}{T}}} \xrightarrow{D} Z \sim N(0,1).$$

Therefore, we obtain:

$$\frac{1}{T}\sum_{t=1}^{T} X_tY_t = \sigma^2 + O_p\left(\sqrt{\frac{N}{T}}\right).$$

In addition, $\overline{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$ and $\overline{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t$ have rate $O_p(\frac{1}{\sqrt{T}})$ and $O_p(\sqrt{\frac{N}{T}})$ respectively. According to these results, we have:

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{tk} - \overline{\varepsilon}_k)\left(\sum_{j=1}^{N}\varepsilon_{tj} - \sum_{j=1}^{N}\overline{\varepsilon}_j\right) &= \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{tk}\sum_{j=1}^{N}\varepsilon_{tj} - \overline{\varepsilon_k}\sum_{j=1}^{N}\overline{\varepsilon}_j \\
&= \frac{1}{T}\sum_{t=1}^{T} X_tY_t - \overline{XY} \\
&= \sigma^2 + O_p\left(\sqrt{\frac{N}{T}}\right).
\end{aligned}$$

Plugging this result into the expression for $\|(\widehat{\mathbf{\Omega}}_T - \sigma^2\mathbf{I})\mathbf{e}\|$, we obtain:

$$\|(\widehat{\mathbf{\Omega}}_T - \sigma^2\mathbf{I})\mathbf{e}\| = O_p\left(\frac{N}{\sqrt{T}}\right).$$

Therefore, the denominator of the observed $1/N$ favorability index is:

$$\begin{aligned}
\|\widehat{\boldsymbol{\mu}}_T\|\|\widehat{\mathbf{\Sigma}}_T\mathbf{e}\| &= \left(\|\boldsymbol{\mu}_T\| + O_p\left(\sqrt{\frac{N}{T}}\right)\right)\left(\|\mathbf{\Sigma}_T\mathbf{e}\| + O_p\left(\frac{N}{\sqrt{T}}\right)\right) \\
&= \|\boldsymbol{\mu}_T\|\|\mathbf{\Sigma}_T\mathbf{e}\| + \|\boldsymbol{\mu}_T\|O_p\left(\frac{N}{\sqrt{T}}\right) + \|\mathbf{\Sigma}_T\mathbf{e}\|O_p\left(\sqrt{\frac{N}{T}}\right) + O_p\left(\frac{N^{\frac{3}{2}}}{T}\right). \qquad (6)
\end{aligned}$$

Now, we look at the numerator of the observed $1/N$ favorability index. Since $\|\boldsymbol{\beta}\| = O(\sqrt{N})$ and $\cos(\boldsymbol{\beta},\mathbf{e})$ is bounded away from 0 as $N \to \infty$, we have $\|\boldsymbol{\mu}_T\| = O(\sqrt{N})$ and $\|\mathbf{\Sigma}_T\mathbf{e}\| = O(N^{\frac{3}{2}})$. The second conclusion comes from the relationship

$$(\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\|\boldsymbol{\beta}\|\frac{1}{T}\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)^2 \leq \|\mathbf{\Sigma}_T\mathbf{e}\| \leq (\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e})\|\boldsymbol{\beta}\|\frac{1}{T}\sum_{t=1}^{T}(R_{mt} - \overline{R}_m)^2 + \sigma^2\sqrt{N}$$

and the rate

$$\boldsymbol{\beta}^{\mathsf{T}}\mathbf{e} = \cos(\boldsymbol{\beta},\mathbf{e})\|\boldsymbol{\beta}\|\|\mathbf{e}\| = O(N).$$

These results are used in analyzing the rate of the numerator of the observed $1/N$ favorability index.

$$
\begin{aligned}
\langle \widehat{\boldsymbol{\mu}}_T, \widehat{\boldsymbol{\Sigma}}_T \mathbf{e} \rangle &= (\boldsymbol{\mu}_T + \overline{\boldsymbol{\epsilon}})^{\intercal} \left[\boldsymbol{\Sigma}_T \mathbf{e} + \frac{2\sqrt{N}}{T} \sum_{t=1}^{T} (R_{mt} - \overline{R}_m) \left(\frac{\boldsymbol{\epsilon}_t^{\intercal} \mathbf{e}}{\sqrt{N}} - \frac{\overline{\boldsymbol{\epsilon}}^{\intercal} \mathbf{e}}{\sqrt{N}} \right) \boldsymbol{\beta} + (\widehat{\boldsymbol{\Omega}}_T - \sigma^2 \mathbf{I}) \mathbf{e} \right] \\
&= \boldsymbol{\mu}_T^{\intercal} \boldsymbol{\Sigma}_T \mathbf{e} + \frac{2\overline{R}_m \sqrt{N} \|\boldsymbol{\beta}\|^2}{T} \sum_{t=1}^{T} (R_{mt} - \overline{R}_m) \left(\frac{\boldsymbol{\epsilon}_t^{\intercal} \mathbf{e}}{\sqrt{N}} - \frac{\overline{\boldsymbol{\epsilon}}^{\intercal} \mathbf{e}}{\sqrt{N}} \right) + \boldsymbol{\mu}_T^{\intercal} (\widehat{\boldsymbol{\Omega}}_T - \sigma^2 \mathbf{I}) \mathbf{e} \\
&\quad + \overline{\boldsymbol{\epsilon}}^{\intercal} \boldsymbol{\Sigma}_T \mathbf{e} + \frac{2\sqrt{N} \overline{\boldsymbol{\epsilon}}^{\intercal} \boldsymbol{\beta}}{T} \sum_{t=1}^{T} (R_{mt} - \overline{R}_m) \left(\frac{\boldsymbol{\epsilon}_t^{\intercal} \mathbf{e}}{\sqrt{N}} - \frac{\overline{\boldsymbol{\epsilon}}^{\intercal} \mathbf{e}}{\sqrt{N}} \right) + \overline{\boldsymbol{\epsilon}}^{\intercal} (\widehat{\boldsymbol{\Omega}}_T - \sigma^2 \mathbf{I}) \mathbf{e} \\
&= \boldsymbol{\mu}_T^{\intercal} \boldsymbol{\Sigma}_T \mathbf{e} + O_p\left(\frac{N^{\frac{3}{2}}}{\sqrt{T}} \right) + \|\boldsymbol{\mu}_T\| O_p\left(\frac{N}{\sqrt{T}} \right) \\
&\quad + \|\boldsymbol{\Sigma}_T \mathbf{e}\| O_p\left(\sqrt{\frac{N}{T}} \right) + O_p\left(\frac{N^{\frac{3}{2}}}{T} \right) + O_p\left(\frac{N^{\frac{3}{2}}}{T} \right) \\
&= \boldsymbol{\mu}_T^{\intercal} \boldsymbol{\Sigma}_T \mathbf{e} + O_p\left(\frac{N^2}{\sqrt{T}} \right).
\end{aligned} \tag{7}
$$

Therefore, under the asymptotic that $N, T \to \infty$ with a relative rate of $\frac{N}{T} = O(1)$, the last three terms in eq. (6) are all of a lower order compared with $\|\boldsymbol{\mu}_T\| \|\boldsymbol{\Sigma}_T \mathbf{e}\|$. The absolute value of the first term in the expression for the numerator is $|\boldsymbol{\mu}_T^{\intercal} \boldsymbol{\Sigma}_T \mathbf{e}| \geq \cos(\boldsymbol{\beta}, \mathbf{e}) \|\boldsymbol{\mu}_T\| \|\boldsymbol{\Sigma}_T \mathbf{e}\| = O(N^2)$. As a result, $\boldsymbol{\mu}_T^{\intercal} \boldsymbol{\Sigma}_T \mathbf{e}$ is the dominating term in eq. (7) as T increases to infinity. Thus, we have:

$$
\cos(\widehat{\boldsymbol{\mu}}_T, \widehat{\boldsymbol{\Sigma}}_T \mathbf{e}) - \cos(\boldsymbol{\mu}_T, \boldsymbol{\Sigma}_T \mathbf{e}) \xrightarrow{p} 0
$$

as $N, T \to \infty$ with the relative rate of $\frac{N}{T} = O(1)$. $\square$