Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: http://www.tandfonline.com/loi/rquf20

A uniformly distributed random portfolio

Woo Chang Kim & Yongjae Lee

Quantitative Finance, 2016
Vol. 16, No. 2, 297–307, http://dx.doi.org/10.1080/14697688.2015.1114360

A uniformly distributed random portfolio

WOO CHANG KIM* and YONGJAE LEE

Department of Industrial and Systems Engineering, Korea Advanced Institution of Science and Technology (KAIST), Daejeon, 34141, Republic of Korea

(*Received 24 February 2014; accepted 21 October 2015*)

In this study, we propose a uniformly distributed random portfolio as an alternative benchmark for portfolio performance evaluation. The uniformly distributed random portfolio is analogous to an enumeration of all feasible portfolios without any prior on the market. Therefore, the relative ranking of a portfolio can be evaluated without peer group information. We derive a closed-form expression for the probability distribution of the Sharpe ratio of a uniformly distributed random portfolio, and conduct comparative analysis with US equity mutual funds. We find that the uniformly distributed random portfolio properly captures the historical performance distribution of equity mutual funds. In addition, we evaluate performance of cap-weighted equity portfolios via uniformly distributed random portfolios.

Keywords: Portfolio performance evaluation; Random portfolio; Active investment

1. Introduction

Fabozzi (2009) stated that portfolio performance evaluation takes place to 'determine whether the investment manager (or the portfolio) added value by outperforming the established benchmark'. Thus, one cannot determine whether a portfolio is well performing or not without a reference portfolio, or a benchmark.

It is the norm to employ a cap-weighted index such as the S&P 500 or Russell 1000 as the benchmark portfolio in the US equity market. Its theoretical justification is given by the capital asset pricing model (CAPM) by Sharpe (1964) and Lintner (1965). Namely, the market portfolio is mean–variance optimal under this equilibrium. However, several studies, including Haugen and Baker (1991) and Grinold (1992), suggest the inefficiency of cap-weighted portfolios and many alternative weighting schemes have been proposed (e.g. see Chan *et al.* 1999; Arnott *et al.* 2005; Amenc *et al.* 2011; Amenc *et al.* 2012; Kim and Sra 2014).

In practice, therefore, the representativeness of even 'the established benchmark' might be questionable. Obviously, the practice of performance evaluation relative to the benchmark can only be justified with the assumption that the benchmark is an 'average-performing portfolio', which represents the market properly. Then how can we know if the benchmark is indeed an average portfolio or not? Or equivalently, how do we evaluate the performance of the benchmark portfolio?

*Corresponding author. Email: wkim@kaist.ac.kr

It might become possible to solve this dilemma if we take a different perspective. Suppose that we know the performance of *all feasible portfolios in the market*. Then, one can rank the performance of a portfolio and say that it is at the α-percentile among all portfolios, whereas the relative performance evaluation can only determine whether the performance of a portfolio is better than a benchmark or not. Even the benchmark portfolios can be evaluated under this alternative framework. It is an arguably fairer approach than the traditional one, as it is not affected by the choice of the benchmark.

In fact, this is a commonly employed approach within the active money management business. For instance, prime brokers publish fund performance reports periodically, which typically include the ranking information among active funds. However, this approach has several critical issues when it is used for general portfolio performance evaluations. For example, collecting data requires a significant amount of effort which makes it hard to conduct the analysis routinely. Also, the set of active funds is not necessarily representative of all feasible portfolios in the market.

In order to overcome these issues, Burns (2007) proposed an alternative performance evaluation approach based on random portfolios. More specifically, random portfolios that satisfy predetermined constraints which are generated by a Monte Carlo simulation, and the performance of a specific portfolio is expressed as its relative ranking among the generated random portfolios. However, although this approach does provide a genuine framework for a fair performance evaluation, it might not be always feasible to conduct such an approach due to its excessive computational cost. For instance, if one counts the number of feasible portfolios within the market of S&P 500 stocks by simply considering all the subsets, it becomes 2^{500}, which is larger than the number of electrons in the observable universe. Thus, it is hard to argue that random portfolios generated by a Monte Carlo simulation are representative of all feasible portfolios, and consequently, while this approach could provide useful information, it cannot serve as conclusive evidence on whether a portfolio performance is good or not.

This paper attempts to provide an analytical solution to this problem. In other words, the main objective of this study is to derive the Sharpe ratio distribution of all feasible portfolios. To this end, we propose a uniformly distributed random portfolio (hereafter, UDRP) which is analogous to the enumeration of all feasible portfolios without any prior view on the market. Furthermore, we compare its performance distribution with those of US equity mutual funds, and find that UDRP historically matched the actual performance of the mutual funds in most cases. Also, we provide an example of how to evaluate portfolios via the UDRP. We evaluate performance of cap-weighted portfolios in various data-sets to show that even benchmark portfolios can be evaluated with the UDRP.

This paper is organized as follows. Section 2 develops the methodology. We define the UDRP in Section 2.1 and derive the closed-form expression for the Sharpe ratio distribution of the UDRP in Section 2.2. In Section 2.3, we further extend the UDRP to incorporate the no-short constraint and accordingly derive the Sharpe ratio distribution of the non-negative UDRP. Section 3 empirically demonstrates the validity of UDRP as the full-distributional benchmark for portfolio performance evaluation. Section 4 provides an illustrative example that applies the UDRP to evaluate portfolio performance. Finally, Section 5 concludes the paper.

2. Performance distribution of all feasible portfolios

This section derives analytically the Sharpe ratio distribution of *all feasible portfolios without any prior on the market*. We first introduce the concept of a uniformly distributed random portfolio by showing how all feasible portfolios can be geometrically represented in an n-dimensional Euclidean space when the performance is measured by the Sharpe ratio. Note that the enumeration of all feasible portfolios does not consist only of mean–variance efficient portfolios, but it also includes all portfolios that are dominated by the efficient portfolios. Next, we take advantage of the geometrical representation to arrive to a closed-form expression for the performance distribution of all feasible portfolios and to further develop the URDP to incorporate a no-short constraint.

2.1. Defining a uniformly distributed random portfolio

Consider a market of n risky assets. Let $r \in \mathbb{R}^n$ to denote a random return vector of the n securities that satisfies $\mathbb{E}r - r_f = \mu \in \mathbb{R}^n$ and $\mathbb{V}\mathrm{ar}(r) = \Sigma \in \mathbb{R}^{n \times n}$, where $r_f \in \mathbb{R}$ is the risk-free rate. Throughout the paper, Σ is assumed to be strictly positive definite and, thus, invertible. Note that every covariance matrix is positive-semi definite, and it is strictly positive definite if $\mathrm{rank}(\Sigma) = n$. We also assume that investors are allowed to short the risky assets, and borrow and lend capital at the risk-free rate.

Let $w = (w_1, \cdots, w_n)^T \in \mathbb{R}^n$ be the portfolio weight vector for the risky assets, and $w_f \in \mathbb{R}$ be the weight on the risk-free asset. Then, we have the following budget constraint:

$$\sum_{i=1}^{n} w_i + w_f = 1.$$

Definition 1. Let $w \in \mathbb{R}^n$ be a portfolio on n risky assets, and $(\mu, \Sigma) \in (\mathbb{R}^n, \mathbb{R}^{n \times n})$ be the market expected excess return and the market covariances. Then, for $w \neq \mathbf{0}$, where $\mathbf{0}$ is a vector of zero's, *the Sharpe ratio function* of w given (μ, Σ) is:

$$SR(w|\mu, \Sigma) := \frac{\mu^T w}{\sqrt{w^T \Sigma w}} \qquad \square$$

Notice that the weight on the risk-free asset does not appear in Definition 1 because leveraging does not affect the Sharpe ratio. This bears a significant importance in defining the feasible region for w. While w is constrained by the budget constraint, as w_f does not appear in the Sharpe ratio function, we can treat w as if it was unconstrained.

To see this, let us consider an example of the portfolio construction process by an investor whose only performance measure is the Sharpe ratio, and whose total wealth is 1. First, he decides the (positive or negative) amount of capital to be invested in the risky assets $\left(\sum_{i=1}^{n} w_i\right)$, then the short or long position in the money market will be automatically determined by the budget constraint $\left(\sum_{i=1}^{n} w_i + w_f = 1\right)$. Finally, he determines the risky portfolio so that the sum of the investments to the risky assets can be as set in the first step.

Notice that the amount of capital to be borrowed or lent at the risk-free rate (w_f) does not change his investment performance as long as it does not change the sign of the total investment on the risky assets ($\sum_{i=1}^{n} w_i$). Furthermore, w_f is fully specified by $\sum_{i=1}^{n} w_i$. Consequently, from the perspective of Sharpe ratio, we can take the risky portfolio w as the sole decision variable of the investor and drop the budget constraint from our analysis. Therefore, the feasible region for risky assets consists of all points in $\mathbb{R}^n \backslash \{\mathbf{0}\}$†.

Our goal is to obtain the performance distribution of all feasible portfolios without any prior on the market. In other words, we need the performance distribution of a portfolio w which is randomly chosen from the complete enumeration of all feasible portfolios. More specifically, we want w to be *uniformly distributed* in $\mathbb{R}^n \backslash \{\mathbf{0}\}$. While this assumption makes intuitive sense, it is mathematically wrong as the uniform distribution can be defined only on a set of

†$w = \mathbf{0}$ means that the portfolio consists of only risk-free asset, and thus, the Sharpe ratio cannot be defined. As we are interested in the probability distribution of the Sharpe ratio, excluding a single point in $\mathbb{R}^n$ does not affect our analysis.

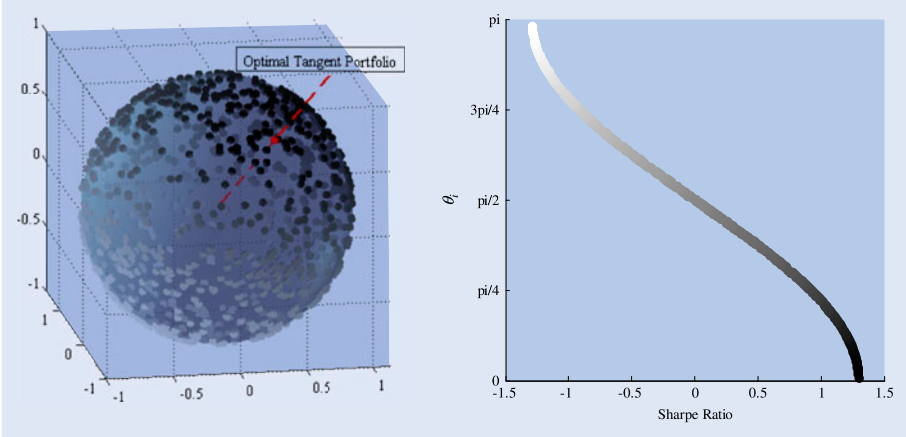

Figure 1. Relationship between θ_i and $SR(w_i|\mu, \Sigma)$ in a three-asset market. Portfolios with higher Sharpe ratio (in dark colour) are closer to the optimal tangent portfolio (red) and have smaller θ_is than portfolios with lower Sharpe ratio (in light colour).

bounded measure. The following proposition allows us to overcome this problem.

Proposition 1. For any $k > 0$, $SR(w|\mu, \Sigma) = SR(kw|\mu, \Sigma)$. In other words, the Sharpe ratio of a portfolio w is *scale-invariant* with respect to w.

Proof

$$SR(w|\mu, \Sigma) = \frac{\mu^T w}{\sqrt{w^T \sum w}} = \frac{k\mu^T w}{k\sqrt{w^T \sum w}} = \frac{\mu^T kw}{\sqrt{(kw)^T \Sigma (kw)}} = SR(kw|\mu, \Sigma),$$

for $k > 0$ □

Proposition 1 implies that, to enumerate all feasible portfolios in $\mathbb{R}^n \backslash \{\mathbf{0}\}$ and to evaluate their Sharpe ratios, we only need to check the ones on the surface of an n-dimensional hypersphere with a fixed radius $k > 0$. Thus, without loss of generality, from the perspective of Sharpe ratio, all portfolios in $\mathbb{R}^n \backslash \{\mathbf{0}\}$ can be fully represented with points on the surface of *the unit hypersphere*. Since the surface of a unit hypersphere is bounded, we can now formally define a uniformly distributed random portfolio as follows.

Definition 2. A *uniformly distributed random portfolio* (UDRP) w^{unif} is a random variable in $\mathbb{R}^n$ that is uniformly distributed on the surface of an n-dimensional unit hypersphere $\{w \mid w \in \mathbb{R}^n \text{ and } ||w||_2 = 1\}$ □

2.2. *Probability distribution of Sharpe ratio of uniformly distributed random portfolio*

Let us first specify the conditions that a portfolio outperforms another.

Proposition 2. Let $w_1, w_2 \in \mathbb{R}^n$ be any two portfolios on n assets, and $w^* \in \mathbb{R}^n$ be the optimal tangent portfolio based on the market expected excess return and the market covariances $(\mu, \Sigma) \in (\mathbb{R}^n, \mathbb{R}^{n\times n})$.

Then,

$$SR(w_1|\mu, \Sigma) \geq SR(w_2|\mu, \Sigma)$$

if and only if

$$\theta_1 \leq \theta_2,$$

where

$$\theta_i := \arccos\left(\frac{(L_\Sigma^T w_i)^T (L_\Sigma^T w^*)}{||L_\Sigma^T w_i||_2 ||L_\Sigma^T w^*||_2}\right) (\text{i.e. angle between } L_\Sigma^T w_i \text{ and } L_\Sigma^T w^*)$$

and L_Σ is the Cholesky decomposition of Σ (i.e. $\Sigma = L_\Sigma L_\Sigma^T$).

Proof. See Appendix A.1. □

When adopting the Sharpe ratio as a performance measure, an interesting geometrical relationship can be established between the angle θ_i and the portfolio performance: the portfolio performance increases as the angle decreases.

Figure 1 graphically illustrates the relationship between the angle and the performance of a portfolio. In a three-asset market with arbitrarily chosen (μ, Σ), we randomly generated 1000 portfolios. In figure 1, portfolios with higher (lower) Sharpe ratios are represented in dark (light) colours, and the optimal tangent portfolio is highlighted in red. Portfolios with higher Sharpe ratio are closer to the optimal tangent portfolio in a L_Σ^T-transformed space (left panel) and have smaller θ_is (right panel) as stated in Proposition 2.

We are now ready to derive the probability distribution of the Sharpe ratio of UDRP. First let us consider an arbitrary portfolio $w_s \in \mathbb{R}^n$ with Sharpe ratio s, and a unit hypersphere† in a L_Σ^T-transformed space. If we let θ_s to be the angle between $L_\Sigma^T w_s$ and $L_\Sigma^T w^*$,

$$\theta_s = \arccos\left(\frac{(L_\Sigma^T w_s)^T (L_\Sigma^T w^*)}{||L_\Sigma^T w_s||_2 ||L_\Sigma^T w^*||_2}\right) = \arccos\left(\frac{SR(w_s|\mu, \Sigma)}{SR(w^*|\mu, \Sigma)}\right) = \arccos\left(\frac{s}{SR(w^*|\mu, \Sigma)}\right)$$

as in the proof of Proposition 2.

As shown in figure 2, the UDRP w^{unif} outperforms w_s when $L_\Sigma^T w^{unif}$ lies on the surface of an n-dimensional unit hyperspherical cap (dark blue) whose axis and colatitude

†A unit hypersphere in the original space does not become a unit hypersphere in a L_Σ^T-transformed space. However, due to the scale-invariance property of the Sharpe ratio, considering a unit hypersphere in a L_Σ^T-transformed space does not affect our analysis.

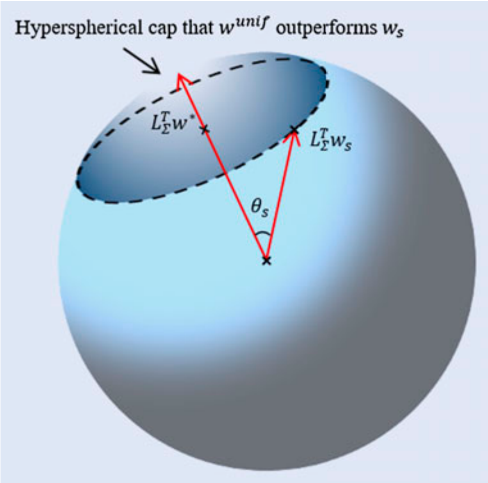

Figure 2. w^{unif} outperforms w_s when $L_\Sigma^T w^{unif}$ lies on the surface of the hyperspherical cap (dark blue).

angle are $L_\Sigma^T w^*$ and θ_s, respectively. Since w^{unif} is uniformly distributed on the hypersphere, the probability for w^{unif} to outperform w_s would be the surface area of the hyperspherical cap divided by the surface area of the unit hypersphere. Consequently, we have the following proposition.

Proposition 3. Let $w_s \in \mathbb{R}^n$ be a portfolio with Sharpe ratio $s \in \left[-\sqrt{\mu^T\Sigma^{-1}\mu}, \sqrt{\mu^T\Sigma^{-1}\mu}\right]$,† and θ_s be the angle between $L_\Sigma^T w_s$ and $L_\Sigma^T w^*$. Then, the probability for the uniformly distributed random portfolio w^{unif} to have the Sharpe ratio less than or equal to s is:

$$\mathbb{P}\left(SR\left(w^{unif}|\mu,\Sigma\right) \leq s\right) = \begin{cases} 1 - \frac{1}{2} I_{\sin^2\theta_s}\left(\frac{n-1}{2}, \frac{1}{2}\right) & \text{if } s \geq 0 \\ \frac{1}{2} I_{\sin^2\theta_s}\left(\frac{n-1}{2}, \frac{1}{2}\right) & \text{else.} \end{cases}$$

where $I_x(a,b)$ is the regularized incomplete beta function.‡

Proof. See Appendix A.2. □

We can now compute the probability distribution of the Sharpe ratio of a UDRP using Proposition 3. Figure 3 illustrates the probability and cumulative density functions of the normalized Sharpe ratio of the UDRP with different numbers of assets, or equivalently, *the performance distributions of all feasible portfolios* in the corresponding markets.

Interestingly, the shape of the probability density function becomes sharper as the number of assets grows. From a portfolio manager's point of view, Proposition 3 indicates that it is harder for a manager to outperform 'average' portfolios with a larger number of assets under management. It is arguably true in practice, as it requires more effort in portfolio management when there are more assets under management.

†Note that the Sharpe ratio of the optimal tangent portfolio calculated in the proof of Proposition 2 is $\sqrt{\mu^T\Sigma^{-1}\mu}$. Thus, if we short the optimal tangent portfolio, the minimum Sharpe ratio would be $-\sqrt{\mu^T\Sigma^{-1}\mu}$.

‡The regularized incomplete beta function is defined as $I_x(a,b) = \frac{B(x;a,b)}{B(a,b)}$, where $B(x;a,b) = \int_0^x u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function and $B(a,b)$ is the beta function.

2.3. *A uniformly distributed random portfolio with no-short constraint*

In this subsection, we illustrate an approach to incorporate constraints to the UDRP. More specifically, we impose a no-short constraint on the UDRP and find its Sharpe ratio distribution.

Imposing a constraint on the UDRP would be equivalent to restricting w^{unif} to reside only on a specific part of a unit hypersphere. Unfortunately, the exact feasible region for the no-short constraint is hardly tractable in a L_Σ^T-transformed space. Hence, we let w^{unif} to be uniformly distributed only on a unit hyperspherical cap that is close to the non-negative part of a unit hypersphere in the L_Σ^T-transformed space.

Note that the exact non-negative part of the surface of a unit hypersphere would be centred at **1**, an n-dimensional vector whose every element is **1,** and would have the surface area equal to $A_n/2^n$, where A_n is the surface area of an n-dimensional unit hypersphere. Consequently, we approximate it as follows.

Definition 3. Let FR^+ be the surface of an n-dimensional unit hyperspherical cap that is centred at $L_\Sigma^T\mathbf{1}$ and has the surface area equal to $A_n/2^n$, where A_n is the surface area of an n-dimensional unit hypersphere. Also, let θ_{FR^+} be the colatitude angle of FR^+. Then,

$$\theta_{FR^+} = \arcsin\sqrt{I^{-1}_{1/2^{n-1}}\left(\frac{n-1}{2}, \frac{1}{2}\right)},$$

since the surface area of FR^+ is:

$$A_n^{\theta_{FR^+}} = \frac{A_n}{2} I_{\sin^2\theta_{FR^+}}\left(\frac{n-1}{2}, \frac{1}{2}\right) = \frac{A_n}{2^n}.$$

We denote FR^+ as *the feasible region corresponding to the no-short constraint* □

Definition 4. A uniformly distributed random portfolio with no-short constraint, or equivalently, *a non-negative uniformly distributed random portfolio* $w_{FR^+}^{unif}$ is a random variable in $\mathbb{R}^n$ such that $L_\Sigma^T w_{FR^+}^{unif}$ is uniformly distributed on FR^+ □

Figure 4 provides a graphical representation of FR^+ (dark blue). Now that we have a reasonable approximation for the feasible region with the no-short constraint, we provide the probability distribution of the Sharpe ratio of the UDRP with the no-short constraint.

Proposition 4. Let $w_s \in \mathbb{R}^n$ be a portfolio with Sharpe ratio $s \in \left[-\sqrt{\mu^T\Sigma^{-1}\mu}, \sqrt{\mu^T\Sigma^{-1}\mu}\right]$, and θ_s and $\theta_v \in [0, \pi]$ be the angle between $L_\Sigma^T w_s$ and $L_\Sigma^T w^*$ and the angle between $L_\Sigma^T\mathbf{1}$ and $L_\Sigma^T w^*$, respectively. Then, the probability for the non-negative UDRP $w_{FR^+}^{unif}$ to have the Sharpe ratio less than or equal to s is§:

$$\mathbb{P}\left(SR\left(w_{FR^+}^{unif}|\mu,\Sigma\right) \leq s\right) = 1 - \frac{A_n^{C\left(L_\Sigma^T\mathbf{1},\theta_{FR^+}\right)\cap C\left(L_\Sigma^T w^*,\theta_s\right)}}{A_n^{C\left(L_\Sigma^T\mathbf{1},\theta_{FR^+}\right)}},$$

§Concise formulas for the surface area of a hyperspherical cap and the intersection of two hyperspherical caps are given in Li (2008) and Lee and Kim (2014), respectively.

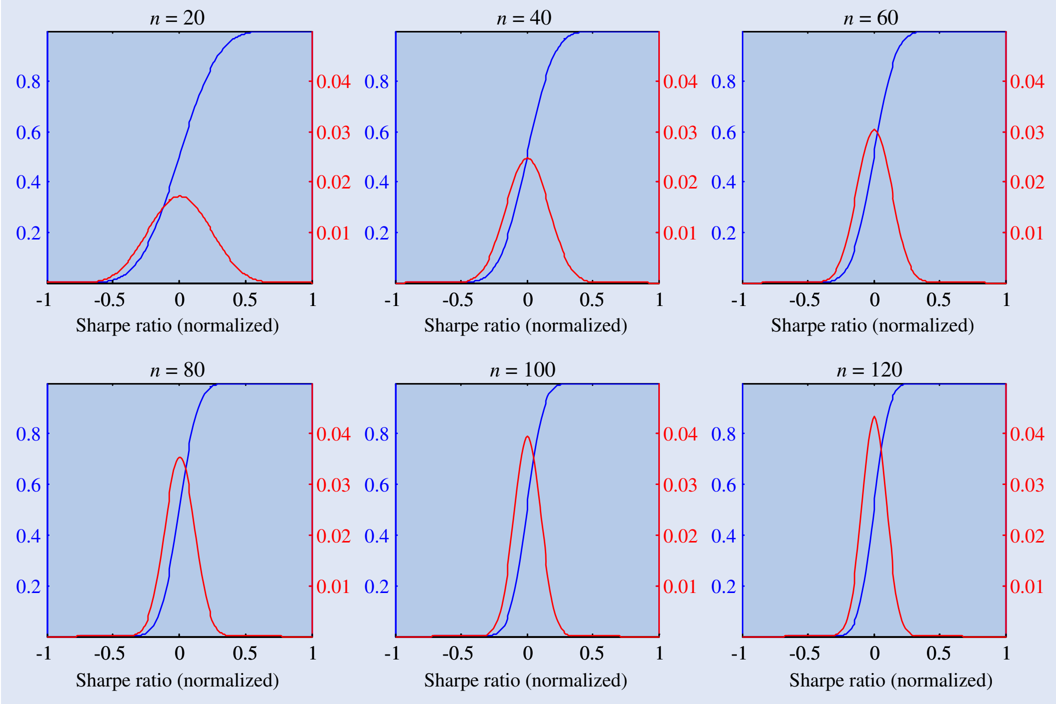

Figure 3. PDF (red) and CDF (blue) of the normalized Sharpe ratio of w^{unif}: Normalized Sharpe ratio means the Sharpe ratio of a specific portfolio divided by that of the optimal tangent portfolio, or by the maximal Sharpe ratio that can be achieved given the market parameters.

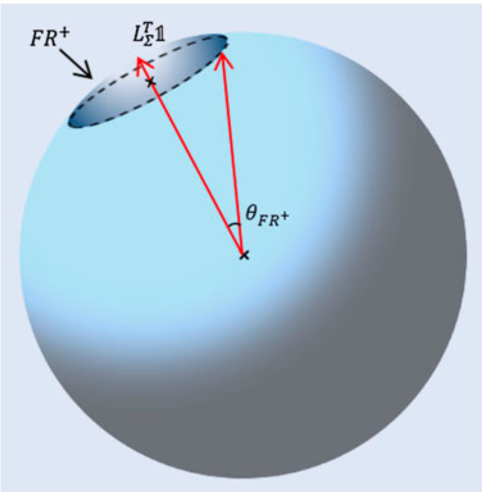

Figure 4. Feasible region corresponding to the no-short constraint (dark blue).

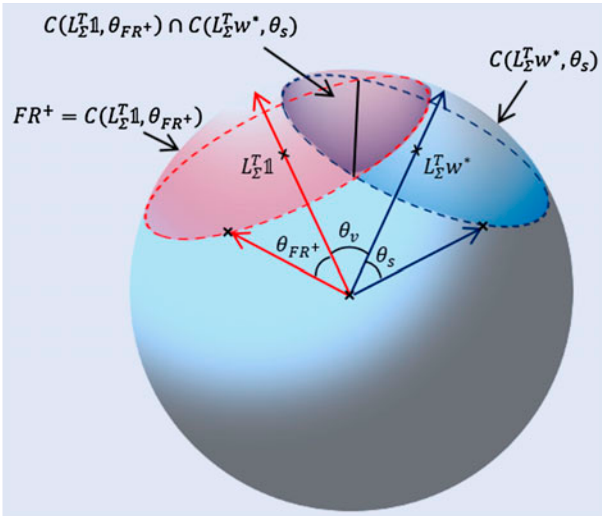

Figure 5. $w_{FR^+}^{unif}$ outperforms w_s when $L_\Sigma^T w_{FR^+}^{unif}$ lies on the surface of $C\left(L_\Sigma^T \mathbf{1}, \theta_{FR^+}\right) \cap C\left(L_\Sigma^T w^*, \theta_s\right)$

where $C(v, \theta)$ denotes the unit hyperspherical cap whose axis and colatitude angle are v and θ, respectively, and $A_n^{C\left(L_\Sigma^T \mathbf{1}, \theta_{FR^+}\right) \cap C\left(L_\Sigma^T w^*, \theta_s\right)}$ and $A_n^{C\left(L_\Sigma^T \mathbf{1}, \theta_{FR^+}\right)}$ are the surface areas of $C\left(L_\Sigma^T \mathbf{1}, \theta_{FR^+}\right) \cap C\left(L_\Sigma^T w^*, \theta_s\right)$ and $C\left(L_\Sigma^T \mathbf{1}, \theta_{FR^+}\right)$, respectively.

Proof. See Appendix A.3. $\square$

Proposition 4 has a nice economic interpretation. Suppose that we enumerate all feasible portfolios so that $L_\Sigma^T w_{FR^+}^{unif}$ is uniformly distributed on FR^+. Then, a portfolio with Sharpe ratio s is ranked at the top

$$100 \times \mathbb{P}\left(SR\left(w_{FR^+}^{unif} | \mu, \Sigma\right) \geq s\right)$$

percentile among all feasible non-negative portfolios. A similar argument on the ordinary UDRP can also be addressed with Proposition 3.

This allows us to evaluate the performance of portfolios in a completely different manner. Conventional approach allows only binary evaluation—better or worse than the benchmark portfolio. However, by employing the uniformly distributed random portfolio, one can obtain a *ranking* of a portfolio without peer group information.

In addition, the proposed approach can incorporate certain types of constraints set by regulators or clients. This subsection illustrates how to impose such constraints to the

UDRP. One may incorporate different kinds of constraints by defining different kinds of feasible regions, and apply Proposition 4 to obtain the Sharpe ratio distributions, as long as the constraints can be geometrically represented on the surface of the hypersphere.

3. Is UDRP representative of all feasible portfolios?

In this section, we conduct empirical tests to demonstrate the effectiveness of the UDRP-based methodology to provide an appropriate performance benchmarking approach. We compare the following three distributions:

(1) Historical Sharpe ratio distribution of US equity mutual funds
(2) Sharpe ratio distribution of UDRP with no-short constraint
(3) Sharpe ratio distribution generated by Monte Carlo simulation.

As for Monte Carlo simulation, we randomly generate 50,000 non-negative portfolio weights for each test period and calculate their Sharpe ratios. While the set of mutual funds is not necessarily representative of all feasible portfolios, as it takes a large portion of all existing portfolios, the use of UDRP can be justified to a certain degree if its performance distribution is similar to that of mutual funds.

For the comparisons tests, we use Kenneth French's 25 style portfolios as a proxy for the US equity market.† There are 25 portfolios that include all NYSE, AMEX and NASDAQ stocks by intersecting five portfolios formed on size (market capitalization) and five portfolios formed on value (book-to-market ratio). We take the sample mean and the sample covariance of the monthly returns of style portfolios as the market expected excess return μ and the market covariances Σ, respectively, for the given time period. Next, by plugging μ and Σ in Proposition 4, we obtain the Sharpe ratio distribution of non-negative UDRP. Finally, the historical Sharpe ratio distribution is assessed from US equity mutual funds data. The mutual fund data is collected from the CRSP database. The test period is 12 years long, ranging from 2003 to 2014. Table 1 shows the number of equity mutual funds available from the CRSP database for each year. Finally, we compare the Sharpe ratio distributions of the UDRP and mutual funds in the US equity market.

The test results are illustrated in figures 6 and 7. Panel A exhibits the comparisons between the theoretical Sharpe ratio distribution of UDRP (blue), the empirical Sharpe ratio distribution of mutual funds (black) and the empirical Sharpe ratio distribution generated by Monte Carlo simulation (red) for each of four-year-long subperiods. Panel B shows the result for the whole 12-year-long period. Clearly, the UDRP captures the actual performance distribution of mutual funds reasonably well.

In order to quantitatively measure the distance between two empirical distributions, the Kolmogorov–Smirnov statistic is employed. For two empirical distribution functions F_1 and F_2, the Kolmogorov–Smirnov statistic (K–S statistic) is defined as

$$D = \sup_x |F_1(x) - F_2(x)|.$$

The distributional distance from mutual funds to UDRP and Monte Carlo simulation are shown in table 2. It can be seen from the table that the theoretical performance distribution (UDRP) provides a better approximation of the actual empirical distribution (MF) than the Monte Carlo simulation, as the K–S statistics of MF-UDRP are always lower by at least 0.3 than those of MF-MC.

We now show more detailed distributional characteristics of the three Sharpe ratio distributions. Table 3 provides several distributional characteristics of the three Sharpe ratio distributions. Mutual funds have lower average Sharpe ratios than UDRP and Monte Carlo. Thus, this observation indicates that a large group of mutual funds may have wrong guesses on the future market direction. For the second moments, the UDRP shows much better approximation of mutual funds than the Monte Carlo. Also for the third moments, the Monte Carlo simulation fails to capture the negative skewness of mutual funds, whereas the UDRP at least captures the direction.‡ Similar observations can be made for $\mathrm{VaR}_{1\%}$ and $\mathrm{VaR}_{5\%}$. However, both UDRP and Monte Carlo do not represent the fat tailedness of mutual funds. Overall, these results show that the UDRP provides a better approximation of the actual empirical Sharpe ratio distribution than the Monte Carlo simulation.

4. Evaluating cap-weighted equity portfolios via UDRP

The conventional portfolio performance evaluation has been conducted relative to the established benchmark. Thus, one could not determine with the conventional framework whether the benchmark is indeed representative of the market or not. In this section, we show how the UDRP allows us to evaluate the performance of even benchmark portfolios. This section could also serve as an example for actual portfolio managers who wish to evaluate their portfolios via the UDRP.

Modern portfolio theory states that the cap-weighted index is the optimal portfolio. Although it is a conditional statement that assumes market equilibrium, it creates a subtle paradox. In some sense, the benchmark portfolio should be representative of the market, thus it is implicitly assumed to be an 'average' portfolio. The studies on the performance of active fund managers such as Malkiel (1995) empirically support this as many of them report that a half of the active equity funds outperform the cap-weighted index, whereas the other half do not. On the other hand, the CAPM suggests that the cap-weighted portfolio should be a well-performing portfolio, if not the best. Thus, there is a gap in the

†Source: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

‡Note that both UDRP and Monte Carlo have no prior on the future market behaviour, and thus, they have almost zero skewness.

Table 1. Number of US equity mutual funds available from the CRSP database for each year from 2003 to 2014.

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Number	9257	9404	10,022	10,643	11,409	13,999	14,211	13,897	14,273	14,470	14,652	13,557

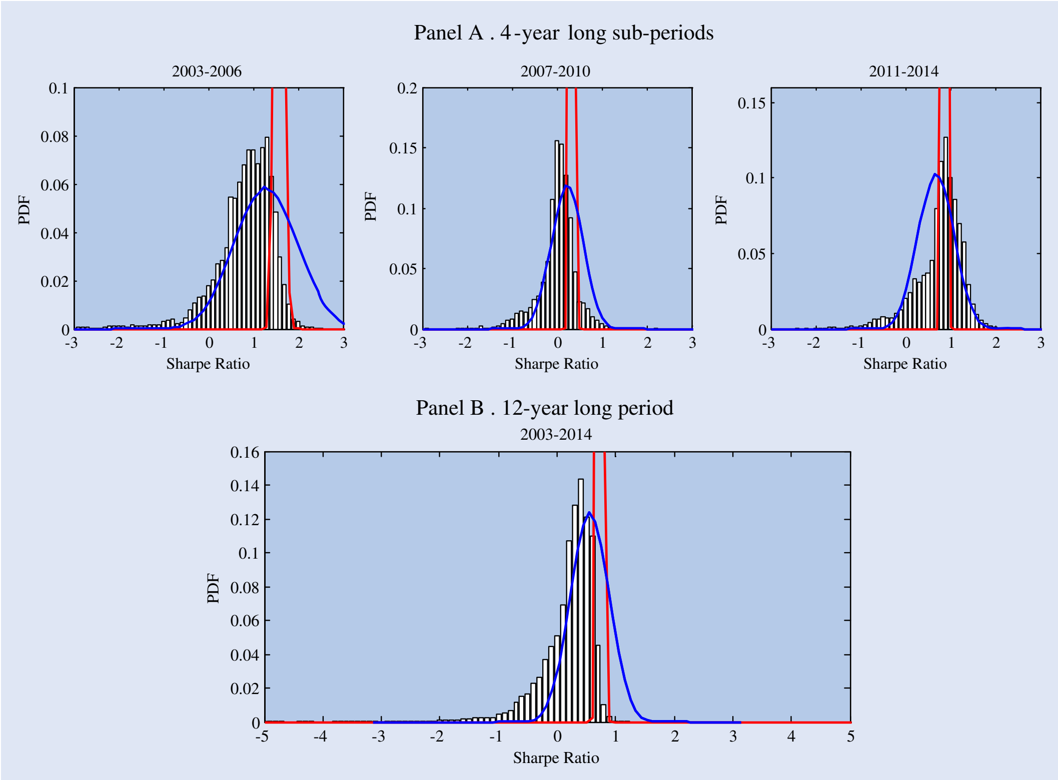

Figure 6. Comparison between the Sharpe ratio distributions of US equity mutual funds (black bar charts), UDRP in US equity market (blue solid lines) and Monte Carlo simulation (red solid lines).

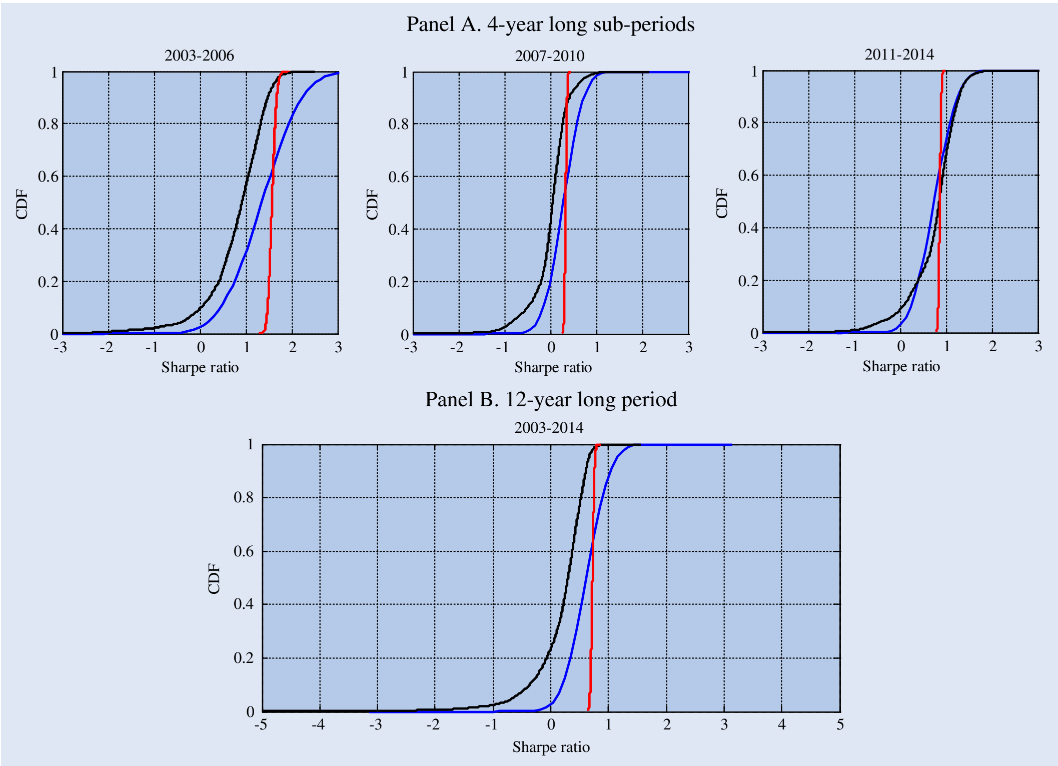

Figure 7. Comparison between the cumulative Sharpe ratio distributions of US equity mutual funds (black), UDRP in US equity market (blue) and Monte Carlo simulation (red).

Table 2. Kolmogorov–Smirnov statistics between the Sharpe ratio distributions of US equity mutual funds (MF), UDRP and Monte Carlo simulation (MC).

Test periods	4-year-long subperiods 2003–2006	2007–2010	2011–2014	12-year-long period 2003–2014
MF-UDRP	0.5105	0.2731	0.0681	0.5521
MF-MC	0.8486	0.8008	0.4280	0.9382

Table 3. Distributional statistics of the Sharpe ratio distributions of US equity mutual funds (MF), UDRP and Monte Carlo simulation (MC).

Test periods		4-year-long subperiods 2003–2006	2007–2010	2011–2014	12-year-long period 2003–2014
Average	MF	0.81	0.00	0.75	0.18
	UDRP	1.66	0.24	0.80	0.73
	MC	1.57	0.34	0.87	0.72
Standard deviation	MF	0.68	0.53	0.53	0.66
	UDRP	0.68	0.35	0.40	0.33
	MC	0.08	0.02	0.03	0.03
Skewness	MF	−2.13	−18.22	−1.71	−32.40
	UDRP	−0.12	−0.05	−0.11	−0.11
	MC	0.19	0.14	0.11	0.14
Kurtosis	MF	14.69	876.18	14.80	2177.58
	UDRP	2.83	2.81	2.83	2.81
	MC	3.01	3.10	2.91	2.94
$VaR_{1\%}$	MF	−1.75	−1.25	−1.05	−1.65
	UDRP	−0.18	−0.49	−0.14	−0.12
	MC	1.35	0.25	0.75	0.65
$VaR_{5\%}$	MF	−0.35	−0.85	−0.35	−0.65
	UDRP	0.24	−0.28	0.07	0.08
	MC	1.45	0.25	0.75	0.65

Table 4. List of data-sets.

#	Data-set and source	n	Rebalancing period	Time period
1	Ten sector portfolios of the S&P 500 Source: Datastream database	10	1	01/1996–12/2013
2	Twenty-five size- and book-to-market portfolios Source: Ken French's website	25	1	07/1990–06/2014
3	US equity large cap (NYSE, NASDAQ, AMEX) Source: Datastream database	100	{1, 1/2}	01/2002–12/2012
4	US equity mid cap (NYSE, NASDAQ, AMEX) Source: Datastream database	100	{1, 1/2}	01/2002–12/2012
5	US equity small cap (NYSE, NASDAQ, AMEX) Source: Datastream database	100	{1, 1/2}	01/2002–12/2012

Notes: This table lists the five data-sets considered in our empirical analysis. n is the total number of risky assets in the corresponding asset universe. The data-sets are analysed based on the specified rebalancing periods, and 1 and 1/2 denote annual and biannual frequencies, respectively.

different perceptions towards the performance of cap-weighted benchmark portfolios.

This issue can be easily resolved, if one can answer the question if the cap-weighted equity index has been performing well or not. However, if one takes the conventional approach, the performance evaluation can only be conducted by employing the comparison principle, and it takes place relative to the benchmark. Therefore, the performance evaluation of the benchmark itself is not straightforward.

An alternative approach is to enumerate all feasible portfolios, and rank the benchmark portfolio among them. As the UDRP is analogous to the all feasible portfolios, it can provide a solution to this. Thus, we set the following null hypothesis for our empirical statistical analysis.

> H_0: Cap-weighted portfolio performs around the average of the non-negative UDRP.

We choose to evaluate the cap-weighted portfolio with the non-negative UDRP instead of the ordinary UDRP, as the former represents a more realistic group of portfolios. Let s_i and (μ_i, Σ_i) be the realized Sharpe ratio of the cap-weighted portfolio and the market parameters in the i th of T rebalancing periods, respectively. Then, under the null hypothesis H_0, the average of the performance ranking of the cap-weighted portfolio should be around the median of the non-negative UDRP.

By Proposition 4, the cap-weighted portfolio outperforms $100 \times \mathbb{P}\left(SR\left(w^{unif}_{FR^+}|\mu_i, \Sigma_i\right) \leq s_i\right)\%$ of all feasible portfolios. That is, $\mathbb{P}\left(SR\left(w^{unif}_{FR^+}|\mu_i, \Sigma_i\right) \leq s_i\right)$ can be regarded as the performance ranking of the cap-weighted portfolio among the enumeration of all feasible non-negative portfolios. If we let

$$\overline{\mathbb{P}\left(SR\left(w^{unif}_{FR^+}|\mu, \Sigma\right) \leq s\right)} = \sum_{i=1}^{T} \frac{\left(SR\left(w^{unif}_{FR^+}|\mu_i, \Sigma_i\right) \leq s_i\right)}{T},$$

then the null hypothesis can be simplified to

$$H_0 : \overline{\mathbb{P}\left(SR\left(w^{unif}_{FR^+}|\mu, \Sigma\right) \leq s\right)} = 0.5.$$

We are now equipped with everything needed for the case study. In this section, we conduct an empirical test, and determine if cap-weighted portfolios have been good portfolios historically. In specific, we conduct statistical tests upon five different data-sets listed in table 4.

The performance evaluation of cap-weighted portfolio via the UDRP takes the following procedure.

For each data-set and for each rebalancing period,

(1) Calculate the expected excess return μ and covariance Σ of the market (all assets in the data-set)
(2) Construct a cap-weighted portfolio and calculate the Sharpe ratio s
(3) Calculate the performance ranking $\mathbb{P}\left(SR\left(w^{unif}_{FR^+}|\mu, \Sigma\right) \leq s\right)$ in Proposition 4

Figure 8 and table 5 summarize the empirical test results.

Overall, the average values of performance rankings of cap-weighted portfolios range from 0.49 to 0.69, indicating that the cap-weighted portfolios outperform around 60% of all feasible non-negative portfolios within the corresponding markets. However, we can see from the figure that the historical rankings fluctuate throughout the test period. So we cannot conclude that cap-weighted portfolios consistently outperform the majority of all feasible non-negative portfolios.

Since it is hard to simply judge the validity of our null hypothesis H_0 due to the fluctuating results, we perform t-test. The statistical test results with the level of significance $\alpha = 0.05$ are summarized in table 6.

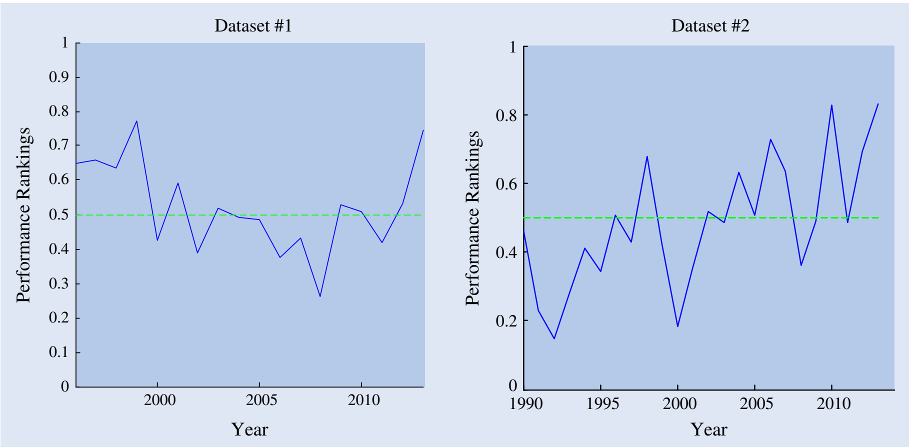

Figure 8a. Historical performance rankings of cap-weighted portfolio among the non-negative UDRP in data-sets #1 and #2.

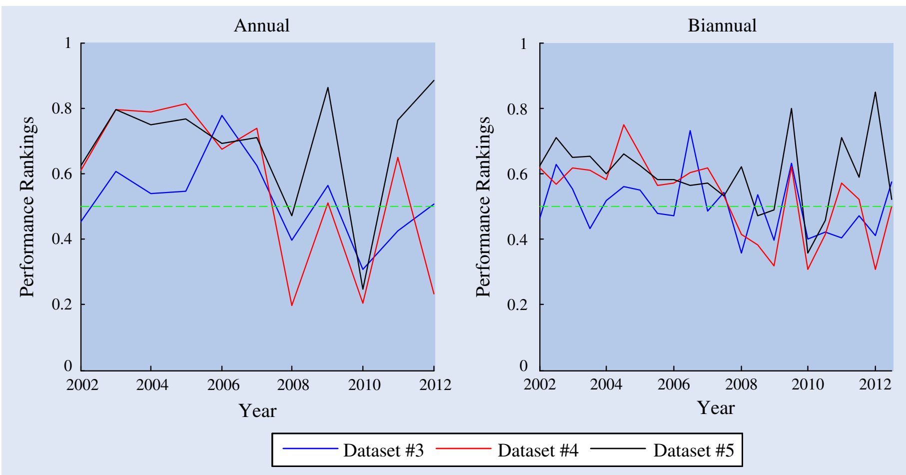

Figure 8b. Historical performance rankings of cap-weighted portfolio among the non-negative UDRP in data-sets #3–5.

Table 5. Averages and standard deviations of historical performance rankings of cap-weighted portfolios.

Data-set	#1	#2	#3		#4		#5	
			1	1/2	1	1/2	1	1/2
Average	0.5233	0.4851	0.5221	0.5004	0.5652	0.5296	0.6880	0.6001
Std dev.	0.1334	0.1871	0.1266	0.0916	0.2442	0.1214	0.1847	0.1104

Table 6. Test of the null hypothesis H_0.

Data-set	#1	#2	#3		#4		#5	
			1	1/2	1	1/2	1	1/2
Reject H_0?	No	No	No	No	No	No	Yes	Yes

In table 6, the null hypothesis H_0 was rejected in two of eight data-sets. That is, cap-weighted portfolios have indeed outperformed more than 50% of the non-negative UDRP in two out of eight cases. However, in the remaining six cases, cap-weighted portfolios were not statistically distinguishable from random portfolios. That is, for most of the cases, cap-weighted portfolios were not so much better than the enumeration of all feasible non-negative portfolios. This accords with previous studies that documented the inefficiency of the cap-weighted index, such as Haugen and Baker (1991), Grinold (1992), Amenc *et al.* (2011) and Amenc *et al*. (2012). Overall, these results indicate that cap-weighted portfolios perform around the average of the non-negative UDRP. Also, the large variability in historical performance ranking suggests the instability of cap-weighted portfolios as benchmark portfolios.

5. Conclusions

Evaluating investment performance is not a trivial task as it involves projecting the results of investment decisions which reside in a high-dimensional space into a low-dimensional space of performance measures. Conventional portfolio performance evaluation takes place relative to a benchmark, so that the results are heavily dependent upon the choice of the benchmark. Besides, the conventional approach allows only binary information of whether a portfolio outperforms the benchmark portfolio or not.

Accordingly, there have been several attempts to rank the performance of a portfolio using the information of its peer group, so that one could evaluate a portfolio without a benchmark. For the peer group information, these approaches use either actual data or data that are randomly generated by Monte Carlo simulations. However, these approaches have several issues critical to being considered as general performance evaluation methods. For instance, the former requires a significant amount of effort on data collection, and the latter is computationally expensive. Furthermore, it is hard to argue that the peer groups of either approach are representative of all feasible portfolios.

In this regard, this paper provides an alternative framework for portfolio performance evaluation. We propose the concept of a uniformly distributed random portfolio that is analogous to the *enumeration of all feasible portfolios without any prior view on the market*, and analytically derive its Sharpe ratio distribution. By employing the uniformly distributed random portfolio, the *ranking* of a portfolio can be obtained even without the information of its peer group. Thus, the uniformly distributed random portfolio allows the performance evaluation of a portfolio without either a benchmark portfolio or peer group information.

Acknowledgements

The authors would like to thank the editor and two anonymous reviewers for their valuable comments.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2015R1D1A1A01059339) and the KAIST High Risk High Return Project (HRHRP).

References

Amenc, N., Golts, F., Lodh, A. and Martellini, L., Diversifying the diversifiers and tracking the tracking error: Outperforming cap-weighted indices with limited risk of underperformance. *J. Portfolio Manage.*, 2012, **38**(3), 72–88.

Amenc, N., Goltz, F., Martellini, L. and Retkowsky, P., Efficient indexation: An alternative to cap-weighted indices. *J. Portfolio Manage.*, 2011, **9**(4), 1–23.

Arnott, R., Hsu, J. and Moor, J., Fundamental indexation. *Financ. Anal. J.*, 2005, **60**(2), 83–99.

Burns, P., Random portfolios for performance measurement. In *Optimisation, Econometric, and Financial Analysis*, edited by E. Kontoghiorghes and C. Gatu, pp. 227–249, 2007 (Springer: New York).

Chan, L.K.C., Karceski, J. and Lakonishok, J., On portfolio optimization: Forecasting covariances and choosing the risk model. *Rev. Financ. Stud.*, 1999, **12**(5), 937–974.

Fabozzi, F.J., *Institutional Investment Management: Equity and Bond Portfolio Strategies and Applications*, 2009 (Wiley: Hoboken, NJ).

Grinold, R., Are benchmark portfolios efficient? *J. Portfolio Manage.*, 1992, **19**(1), 34–40.
Haugen, R. and Baker, N., The efficient market inefficiency of capitalization–weighted stock portfolios. *J. Portfolio Manage.*, 1991, **17**(3), 35–40.
Kim, N. and Sra, S., Portfolio optimization with groupwise selection. *Ind. Eng. Manage. Syst.*, 2014, **13**(4), 442–448.
Lee, Y., and Kim, W.C., Concise formulas for the surface area of the intersection of two hyperspherical caps. KAIST Technical Report, 2014, IE-TR-2014-01.
Li, S., Concise formulas for the area and volume of a hyperspherical cap. *Asian J. Math. Stat.*, 2008, **4**(1), 66–70.
Lintner, J., The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Rev. Econ. Stat.*, 1965, **47**(1), 13–37.
Malkiel, B.G., Returns from investing in equity mutual funds 1971 to 1991. *J. Finance*, 1995, **50**(2), 549–572.
Sharpe, W.F., Capital asset prices: A theory of market equilibrium under conditions of risk. *J. Finance*, 1964, **19**(3), 425–442.

Appendix A.1. Proof of proposition 2

First note that the optimal tangent portfolio

$$w^* = \underset{w\in\mathbb{R}^n}{\operatorname{argmax}}\, SR(w|\mu,\Sigma) = \Sigma^{-1}\mu,$$

and the maximum Sharpe ratio

$$\begin{aligned} SR(w^*|\mu,\Sigma) &= \frac{\mu^T w^*}{\sqrt{w^{*T}\Sigma w^*}} = \frac{\mu^T\left(\Sigma^{-1}\mu\right)}{\sqrt{\mu^T\Sigma^{-1}\Sigma\Sigma^{-1}\mu}} = \frac{\mu^T\Sigma^{-1}\mu}{\sqrt{\mu^T\Sigma^{-1}\mu}} \\ &= \sqrt{\mu^T\Sigma^{-1}\mu}. \end{aligned}$$

Then,

$$\begin{aligned} \theta_1 &= \arccos\left(\frac{\left(L_\Sigma^T w_1\right)^T\left(L_\Sigma^T w^*\right)}{||L_\Sigma^T w_1||_2||L_\Sigma^T w^*||_2}\right) \\ &= \arccos\left(\frac{w_1^T L_\Sigma L_\Sigma^T w^*}{\sqrt{\left(L_\Sigma^T w_1\right)^T L_\Sigma^T w_1}\sqrt{\left(L_\Sigma^T w^*\right)^T L_\Sigma^T w^*}}\right) \\ &= \arccos\left(\frac{w_1^T\Sigma w^*}{\sqrt{w_1^T\Sigma w_1}\sqrt{w^{*T}\Sigma w^*}}\right) \\ &= \arccos\left(\frac{w_1^T\mu}{\sqrt{w_1^T\Sigma w_1}}\frac{1}{\sqrt{\mu^T\Sigma^{-1}\mu}}\right) \\ &= \arccos\left(\frac{SR(w_1|\mu,\Sigma)}{SR(w^*|\mu,\Sigma)}\right), \end{aligned}$$

and similarly

$$\theta_2 = \arccos\left(\frac{SR(w_2|\mu,\Sigma)}{SR(w^*|\mu,\Sigma)}\right).$$

Since $\arccos(x)$ is a decreasing function of x, $\theta_1 \le \theta_2$ if and only if $SR(w_1|\mu,\Sigma) \ge SR(w_2|\mu,\Sigma)$ □

Appendix A.2. Proof of proposition 3

By Proposition 2, the condition that w^{unif} to have the Sharpe ratio greater or equal to s is equivalent to the angle between $L_\Sigma^T w^{unif}$ and $L_\Sigma^T w^*$ being smaller than θ_s. Therefore, w^{unif} outperforms w_s when $L_\Sigma^T w^{unif}$ is located on the hyperspherical cap whose axis and colatitude angle are $L_\Sigma^T w^*$ and θ_s, respectively. Li (2008) has shown that the surface area of the unit hyperspherical cap with the colatitude angle θ_s is:

$$A_n^{\theta_s} = \begin{cases} \frac{A_n}{2} I_{\sin^2\theta_s}\left(\frac{n-1}{2},\frac{1}{2}\right) & \theta_s \in [0,\pi/2] \\ A_n\left(1-\frac{1}{2}I_{\sin^2\theta_s}\left(\frac{n-1}{2},\frac{1}{2}\right)\right) & \theta_s\in(\pi/2,\pi] \end{cases}$$

where $A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the surface area of the n-dimensional unit hypersphere. Note that $\theta_s \in [0, \pi/2]$ if and only if $s \ge 0$ and $\theta_s \in (\pi/2, \pi]$ if and only if $s < 0$.

Since $L_\Sigma^T w^{unif}$ is uniformly distributed on the unit hypersphere, the probability for $L_\Sigma^T w^{unif}$ to be on the hyperspherical cap, or equivalently, the probability for w^{unif} to outperform w_s is:

$$\begin{aligned} \mathbb{P}\left(SR\left(w^{unif}|\mu,\Sigma\right)\ge s\right) &= \frac{A_n^{\theta_s}}{A_n} \\ &= \begin{cases} \frac{1}{2}I_{\sin^2\theta_s}\left(\frac{n-1}{2},\frac{1}{2}\right) & \text{if } s\ge 0 \\ 1-\frac{1}{2}I_{\sin^2\theta_s}\left(\frac{n-1}{2},\frac{1}{2}\right) & \text{else} \end{cases} \end{aligned}$$

Consequently,

$$\mathbb{P}(SR(w^{unif}|\mu,\Sigma)\le s) = 1-\mathbb{P}(SR(w^{unif}|\mu,\Sigma)\ge s)$$ □

Appendix A.3. Proof of proposition 4

By Proposition 2, $w_{FR^+}^{unif}$ outperforms w_s when $L_\Sigma^T w_{FR^+}^{unif}$ is located on the hyperspherical cap $C\left(L_\Sigma^T w^*,\theta_s\right)$. However, since $w_{FR^+}^{unif}$ is conditioned on $FR^+ = C\left(L_\Sigma^T\mathbf{1},\theta_{FR^+}\right)$, it follows that:

$$\begin{aligned} \mathbb{P}\left(SR\left(w_{FR^+}^{unif}|\mu,\Sigma\right)\le s\right) &= 1-\mathbb{P}\left(SR\left(w_{FR^+}^{unif}|\mu,\Sigma\right)\ge s\right) \\ &= 1-\frac{A_n^{C\left(L_\Sigma^T\mathbf{1},\theta_{FR^+}\right)\cap C\left(L_\Sigma^T w^*,\theta_s\right)}}{A_n^{C\left(L_\Sigma^T\mathbf{1},\theta_{FR^+}\right)}} \end{aligned}$$ □