Eigenmath Manual

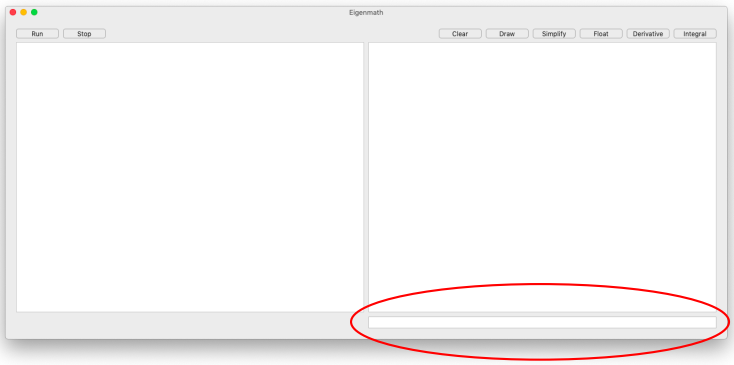
George Weigt

March 17, 2019

Contents

1 Introduction

The field at the bottom of the Eigenmath window is for entering calculations that get evaluated right away.



For example, let us check the following arithmetic from Vladimir Nabokov's autobiography *Speak, Memory.*

> A foolish tutor had explained logarithms to me much too early, and I had read (in a British publication, the *Boy's Own Paper*, I believe) about a certain Hindu calculator who in exactly two seconds could find the seventeenth root of, say, 3529471145760275132301897342055866171392 (I am not sure I have got this right; anyway the root was 212).

We can check Nabokov's arithmetic by entering the following calculation.

```
212^17
```

After pressing the return key, Eigenmath displays the following result.

3529471145760275132301897342055866171392

So Nabokov did get it right after all. Now let us see if Eigenmath can find the seventeenth root of this number, like the Hindu calculator could.

```
N = 212^17
N^(1/17)
```

Eigenmath displays the following result.

212

When a symbol is assigned a value, such as N above, no result is printed. To see the value of a symbol, just evaluate it.

```
N
```

$N = 3529471145760275132301897342055866171392$

The previous example shows a convention that will be used throughout this manual. That is, the color blue indicates something that the user should type. The computer response is shown in black.

1.1 Arithmetic

Normally Eigenmath uses integer and rational number arithmetic.

```
1/2+1/3
```

$$\frac{5}{6}$$

A floating point value causes Eigenmath to switch to floating point arithmetic.

```
1/2+1/3.0
```

0.833333

An integer or rational number result can be converted to a floating point value by entering *float*.

```
212^17
```

3529471145760275132301897342055866171392

```
float
```

3.52947×10^{39}

The following example shows how to enter a floating point value using scientific notation.

```
epsilon = 1.0*10^(-6)
epsilon
```

$\varepsilon = 1.0 \times 10^{-6}$

1.2 Exponents

Eigenmath requires parentheses around negative exponents. For example,

```
10^(-3)
```

instead of

```
10^-3
```

The reason for this is that the binding of the negative sign is not always obvious. For example, consider

```
x^-1/2
```

It is not clear whether the exponent should be -1 or $-1/2$. So Eigenmath requires

```
x^(-1/2)
```

which is unambiguous. In general, parentheses are always required when the exponent is an expression. For example, `x^1/2` is evaluated as $(x^1)/2$ which is probably not the desired result.

```
x^1/2
```

$$\frac{1}{2}x$$

Using `x^(1/2)` yields the desired result.

```
x^(1/2)
```

$x^{1/2}$

1.3 Symbols

As we saw earlier, symbols are defined using an equals sign.

```
N = 212^17
```

No result is printed when a symbol is defined. To see the value of a symbol, just evaluate it.

```
N
```

$N = 3529471145760275132301897342055866171392$

Symbols can have more that one letter. Everything after the first letter is displayed as a subscript.

```
NA = 6.02214*10^23
NA
```

$N_A = 6.02214 \times 10^{23}$

A symbol can be the name of a Greek letter.

```
xi = 1/2
xi
```

$\xi = \dfrac{1}{2}$

Greek letters can appear in subscripts.

```
Amu = 2.0
Amu
```

$A_\mu = 2.0$

The following example shows how Eigenmath scans the entire symbol to find Greek letters.

```
alphamunu = 1
alphamunu
```

$\alpha_{\mu\nu} = 1$

When a symbolic chain is defined, Eigenmath follows the chain as far as possible. The following example sets $A = B$ followed by $B = C$. Then when A is evaluated, the result is C.

```
A = B
B = C
A
```

$A = C$

Although $A = C$ is printed, inside the program the binding of A is still B, as can be seen with the *binding* function.

```
binding(A)
```

B

The *quote* function returns its argument unevaluated and can be used to clear a symbol. The following example clears A so that its evaluation goes back to being A instead of C.

```
A = quote(A)
A
```

A

1.4 User-defined functions

The following example shows a user-defined function with a single argument.

```
f(x) = sin(x)/x
f(pi/2)
```

$$\frac{2}{\pi}$$

The following example defines a function with two arguments.

```
g(x,y) = abs(x) + abs(y)
g(1,-2)
```

3

User-defined functions can be evaluated without an argument list. The binding of the function name is returned when there is no argument list.

```
f(x) = sin(x)/x
f
```

$$f = \frac{\sin(x)}{x}$$

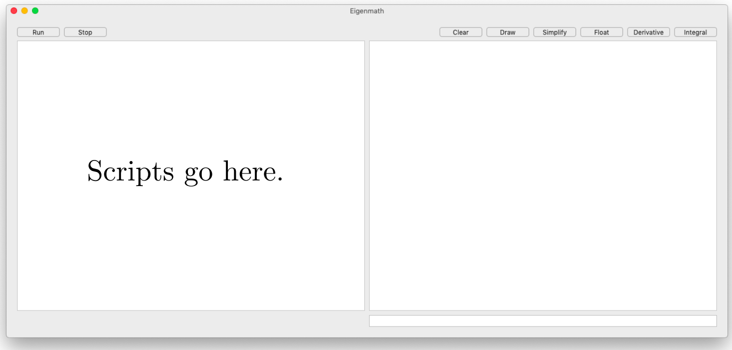
Normally a function body is not evaluated when a function is defined. However, in some cases it is required that the function body be the result of something. The *eval* function is used to accomplish this. For example, the following code causes the function body to be a sixth order Taylor series expansion of $\cos x$.

```
f(x) = eval(taylor(cos(x),x,6))
f
```

$$f = -\frac{1}{720}x^6 + \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

1.5 Scripts

Scripting is a way of automatically running a sequence of calculations. A script is entered in the left-hand field of the Eigenmath window.



To create a script, enter one calculation per line in the script field. Nothing happens until the Run button is clicked. When the Run button is clicked, Eigenmath evaluates the script line by line. After a script runs, all of its symbols are available for immediate mode calculation. Scripts can be saved and loaded using the File menu.

Here is an example script that can be pasted into the script field and then run by clicking the Run button.

```
"Solve for vector X in AX = B"
A = ((1,2),(3,4))
B = (5,6)
X = dot(inv(A),B)
X
```

After clicking the Run button, the following result is displayed.

```
Solve for vector X in AX = B
```

$$X = \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix}$$

A handy debugging aid is to include the line $trace = 1$ in the script. When $trace = 1$ each line of the script is displayed as it is evaluated. For example, here is the previous script with the addition of $trace = 1$.

```
"Solve for vector X in AX = B"
trace = 1
A = ((1,2),(3,4))
B = (5,6)
X = dot(inv(A),B)
X
```

The result is

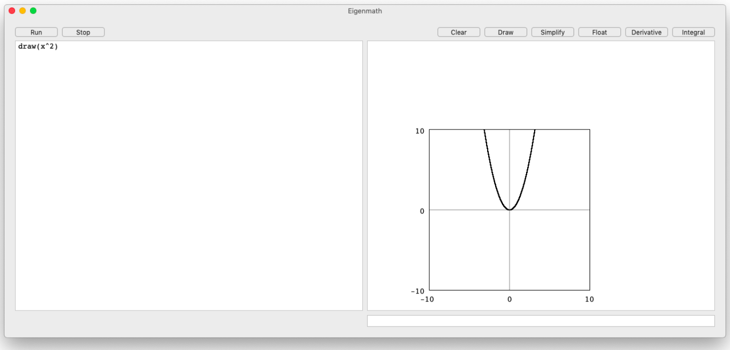
```
Solve for vector X in AX = B
A = ((1,2),(3,4))
B = (5,6)
X = dot(inv(A),B)
X
```

$$X = \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix}$$

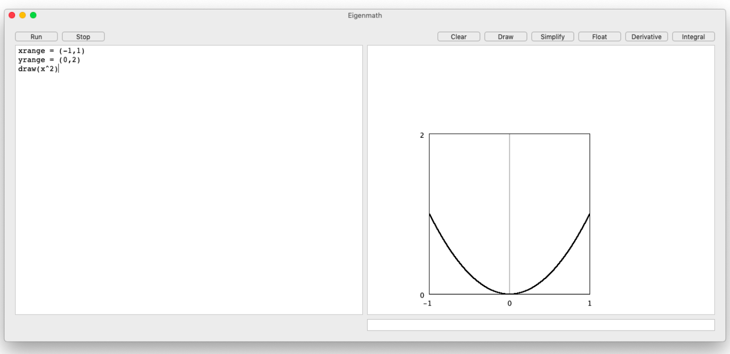
1.6 Draw

draw(f, x) draws a graph of the function f of x. The second argument can be omitted when the dependent variable is literally x or t. The vectors *xrange* and *yrange* control the scale of the graph.

```
draw(x^2)
```

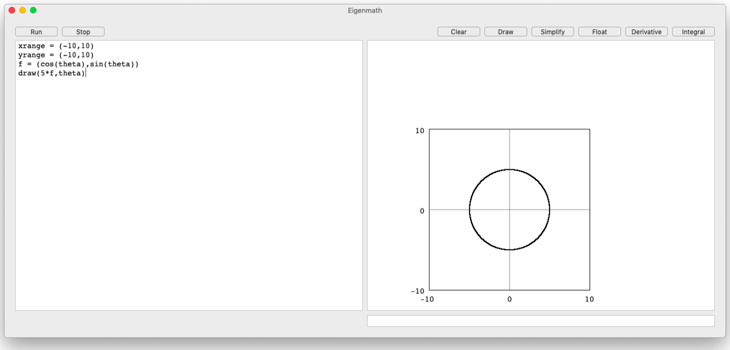



```
xrange = (-1,1)
yrange = (0,2)
draw(x^2)
```



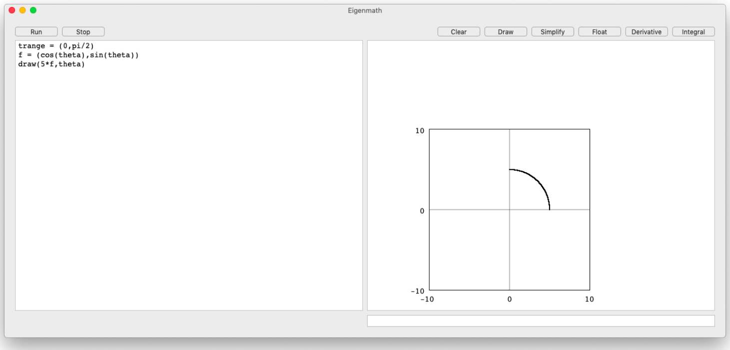
Parametric drawing occurs when a function returns a vector. The vector *trange* controls the parametric range. The default is *trange* $= (-\pi, \pi)$. In the following example, *draw* varies *theta* over the default range $-\pi$ to $+\pi$.

```
xrange = (-10,10)
yrange = (-10,10)
f = (cos(theta),sin(theta))
draw(5*f,theta)
```



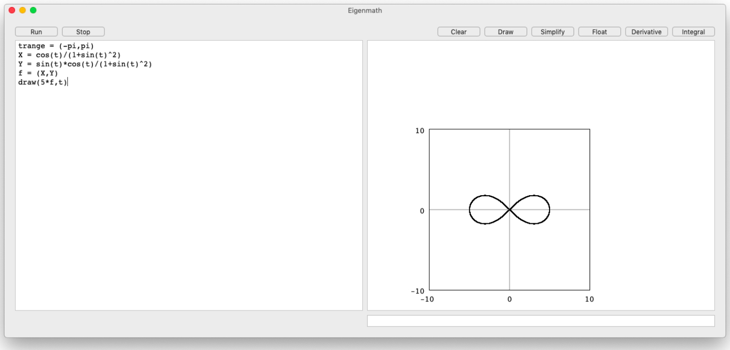
In the following example, *trange* is reduced to draw a quarter circle instead of a full circle.

```
trange = (0,pi/2)
f = (cos(theta),sin(theta))
draw(5*f,theta)
```



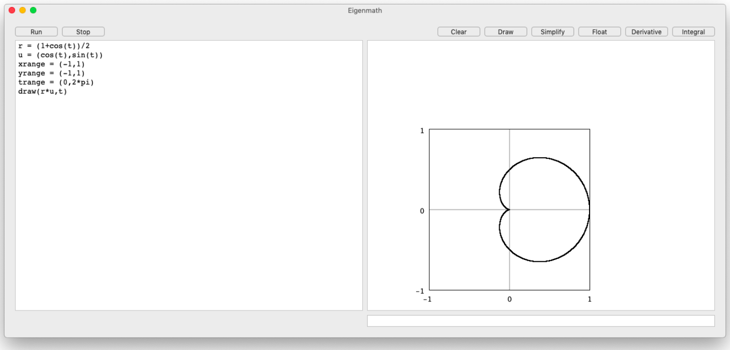
Here are a couple of interesting curves and the code for drawing them. First is a lemniscate.

```
trange = (-pi,pi)
X = cos(t)/(1+sin(t)^2)
Y = sin(t)*cos(t)/(1+sin(t)^2)
f = (X,Y)
draw(5*f,t)
```



Next is a cardioid.

```
r = (1+cos(t))/2
u = (cos(t),sin(t))
xrange = (-1,1)
yrange = (-1,1)
trange = (0,2*pi)
draw(r*u,t)
```



1.7 Complex numbers

When Eigenmath starts up, it defines the symbol i as $i = \sqrt{-1}$. Other than that, there is nothing special about i. It is just a regular symbol that can be redefined and used for some other purpose if need be.

Complex quantities can be entered in either rectangular or polar form.

```
a+i*b
```

$a + ib$

```
exp(i*pi/3)
```

$\exp(\frac{1}{3} i\pi)$

Converting to rectangular or polar coordinates causes simplification of mixed forms.

```
A = 1+i
B = sqrt(2)*exp(i*pi/4)
A-B
```

$1 + i - 2^{1/2} \exp(\frac{1}{4} i\pi)$

```
rect(last)
```

0

Rectangular complex quantities, when raised to a power, are multiplied out.

```
(a+i*b)^2
```

$a^2 - b^2 + 2iab$

When a and b are numerical and the power is negative, the evaluation is done as follows.

$$i(a+ib)^{-n} = \left[\frac{a-ib}{(a+ib)(a-ib)}\right]^n = \left[\frac{a-ib}{a^2+b^2}\right]^n$$

Of course, this causes i to be removed from the denominator. Here are a few examples.

```
1/(2-i)
```

$\frac{2}{5} + \frac{1}{5}i$

```
(-1+3i)/(2-i)
```

$-1+i$

The absolute value of a complex number returns its magnitude.

```
abs(3+4*i)
```

5

Since symbols can have complex values, the absolute value of a symbolic expression is not computed.

```
abs(a+b*i)
```

$\operatorname{abs}(a+ib)$

The result is not $\sqrt{a^2+b^2}$ because that would assume that a and b are real. For example, suppose that $a=0$ and $b=i$. Then

$$|a+ib| = |-1| = 1$$

and

$$\sqrt{a^2+b^2} = \sqrt{-1} = i$$

Hence

$$|a+ib| \neq \sqrt{a^2+b^2} \quad \text{for some } a,b \in \mathbb{C}$$

The *mag* function can be used instead of *abs*. It treats symbols like a and b as real.

```
mag(a+b*i)
```

$(a^2+b^2)^{1/2}$

The imaginary unit can be changed from i to j by defining $j=\sqrt{-1}$.

```
j = sqrt(-1)
sqrt(-4)
```

$2j$

1.8 Linear algebra

The function *dot* is used to multiply vectors and matrices. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The following example computes Ax.

```
A = ((1,2),(3,4))
x = (x1,x2)
dot(A,x)
```

$$\begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

The following example shows how to use *dot* and *inv* to solve for the vector X in $AX = B$.

```
A = ((3,7),(1,-9))
B = (16,-22)
X = dot(inv(A),B)
X
```

$$X = \begin{bmatrix} -\frac{5}{17} \\ \frac{41}{17} \end{bmatrix}$$

The *dot* function can have more than two arguments. For example, $dot(A, B, C)$ can be used for the dot product of three tensors.

Square brackets are used for component access. Index numbering starts with 1.

```
A = ((a,b),(c,d))
A[1,2] = -A[1,1]
A
```

$$\begin{bmatrix} a & -a \\ c & d \end{bmatrix}$$

The following example demonstrates the relation $A^{-1} = \operatorname{adj} A / \det A$.

```
A = ((a,b),(c,d))
inv(A)
```

$$\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

```
adj(A)
```

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

```
det(A)
```

$ad - bc$

```
inv(A) - adj(A)/det(A)
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Sometimes a calculation will be simpler if it can be reorganized to use *adj* instead of *inv*. The main idea is to try to prevent the determinant from appearing as a divisor. For example, suppose for matrices A and B you want to check that

$$A - B^{-1} = 0$$

Depending on the complexity of $\det B$, the software may not be able to find a simplification that yields zero. Should that occur, the following alternative formulation can be tried.

$$(\det B) \cdot A - \operatorname{adj} B = 0$$

The adjunct of a matrix is related to the cofactors as follows.

```
A = ((a,b),(c,d))
C = ((0,0),(0,0))
C[1,1] = cofactor(A,1,1)
C[1,2] = cofactor(A,1,2)
C[2,1] = cofactor(A,2,1)
C[2,2] = cofactor(A,2,2)
C
```

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

```
adj(A) - transpose(C)
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2 Calculus

2.1 Derivative

$d(f, x)$ returns the derivative of f with respect to x. The x can be omitted for expressions in x.

```
d(x^2)
```

$2x$

The following table summarizes the various ways to obtain multiderivatives.

$\dfrac{\partial^2 f}{\partial x^2}$	`d(f,x,x)`	`d(f,x,2)`
$\dfrac{\partial^2 f}{\partial x\, \partial y}$	`d(f,x,y)`	
$\dfrac{\partial^{m+n+\cdots} f}{\partial x^m\, \partial y^n \cdots}$	`d(f,x,...,y,...)`	`d(f,x,m,y,n,...)`

2.2 Gradient

The gradient of f is obtained by using a vector for x in $d(f, x)$.

```
r = sqrt(x^2+y^2)
d(r,(x,y))
```

$$\begin{bmatrix} \frac{x}{(x^2+y^2)^{1/2}} \\ \frac{y}{(x^2+y^2)^{1/2}} \end{bmatrix}$$

The f in $d(f, x)$ can be a tensor function. Gradient raises the rank by one.

```
F = (x+2y,3x+4y)
X = (x,y)
d(F,X)
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

2.3 Template functions

The function f in $d(f)$ does not have to be defined. It can be a template function with just a name and an argument list. Eigenmath checks the argument list to figure out what to do. For example, $d(f(x), x)$ evaluates to itself because f depends on x. However, $d(f(x), y)$ evaluates to zero because f does not depend on y.

```
d(f(x),x)
```

$\mathrm{d}(f(x), x)$

```
d(f(x),y)
```

0

```
d(f(x,y),y)
```

$\mathrm{d}(f(x,y), y)$

```
d(f(),t)
```

$\mathrm{d}(f(), t)$

As the final example shows, an empty argument list causes $d(f)$ to always evaluate to itself, regardless of the second argument.

Template functions are useful for experimenting with differential forms. For example, let us check the identity

$$\mathrm{div}(\mathrm{curl}\, F) = 0$$

for an arbitrary vector function F.

```
F = (F1(x,y,z),F2(x,y,z),F3(x,y,z))
curl(U) = (d(U[3],y)-d(U[2],z),d(U[1],z)-d(U[3],x),d(U[2],x)-d(U[1],y))
div(U) = d(U[1],x)+d(U[2],y)+d(U[3],z)
div(curl(F))
```

0

2.4 Integral

$integral(f, x)$ returns the integral of f with respect to x. The x can be omitted for expressions in x. The argument list can be extended for multiple integrals.

```
integral(x^2)
```

$\frac{1}{3}x^3$

```
integral(x*y,x,y)
```

$\frac{1}{4}x^2y^2$

$defint(f, x, a, b, \ldots)$ computes the definite integral of f with respect to x evaluated from a to b. The argument list can be extended for multiple integrals. The following example computes the integral of $f = x^2$ over the domain of a semicircle. For each x along the abscissa, y ranges from 0 to $\sqrt{1-x^2}$.

```
defint(x^2,y,0,sqrt(1-x^2),x,-1,1)
```

$\frac{1}{8}\pi$

As an alternative, the *eval* function can be used to compute a definite integral step by step.

```
I = integral(x^2,y)
I = eval(I,y,sqrt(1-x^2))-eval(I,y,0)
I = integral(I,x)
eval(I,x,1)-eval(I,x,-1)
```

$\frac{1}{8}\pi$

Here is a useful trick. Difficult integrals involving sine and cosine can often be solved by using exponentials. Trigonometric simplifications involving powers and multiple angles turn into simple algebra in the exponential domain. For example, the definite integral

$$\int_0^{2\pi} \left(\sin^4 t - 2\cos^3(t/2)\sin t\right) dt$$

can be solved as follows.

```
f = sin(t)^4-2*cos(t/2)^3*sin(t)
f = circexp(f)
defint(f,t,0,2*pi)
```

$-\frac{16}{5} + \frac{3}{4}\pi$

Here is a check of the result.

```
g = integral(f,t)
f-d(g,t)
```

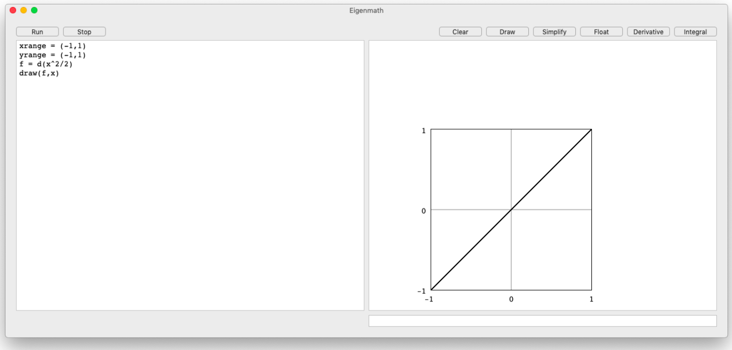
0

The fundamental theorem of calculus is a formal expression of the inverse relation between integrals and derivatives.

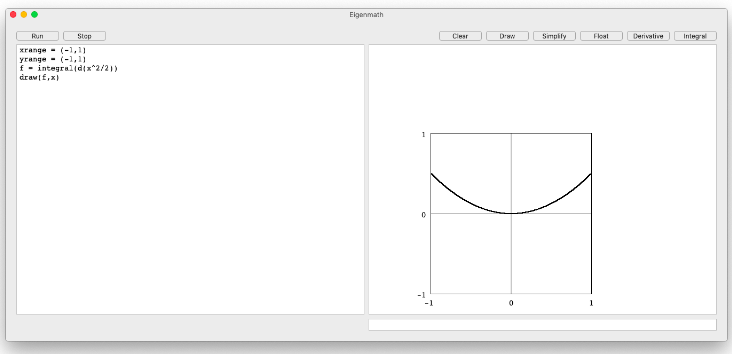
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Here is an Eigenmath demonstration of the fundamental theorem of calculus.

```
xrange = (-1,1)
yrange = (-1,1)
f = d(x^2/2)
draw(f,x)
```




```
xrange = (-1,1)
yrange = (-1,1)
f = integral(d(x^2/2))
draw(f,x)
```



The first graph shows that $f'(x)$ is antisymmetric, therefore the total area under the curve from -1 to 1 sums to zero. The second graph shows that $f(1) = f(-1)$. Hence for $f(x) = \frac{1}{2}x^2$ we have

$$\int_{-1}^{1} f'(x)\,dx = f(1) - f(-1) = 0$$

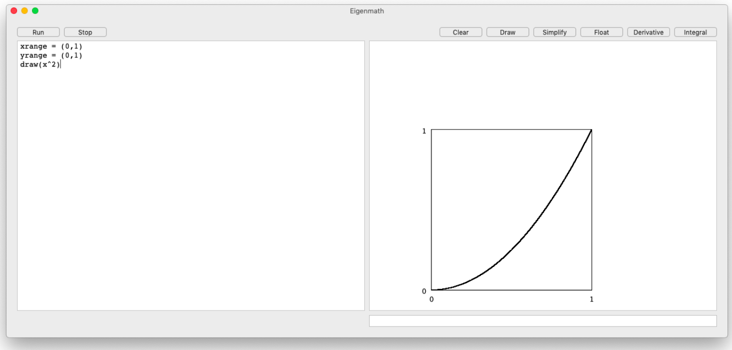
2.5 Arc length

Let $g(t)$ be a function that draws a curve. The arc length from $g(a)$ to $g(b)$ is given by

$$\int_a^b |g'(t)|\,dt$$

where $|g'(t)|$ is the length of the tangent vector at $g(t)$. The integral sums over all of the tangent lengths to arrive at the total length from a to b. For example, let us measure the length of the following curve.

```
xrange = (0,1)
yrange = (0,1)
draw(x^2)
```



A suitable $g(t)$ for the arc is

$$g(t) = (t, t^2), \quad 0 \le t \le 1$$

Hence one Eigenmath solution for computing the arc length is

```
x = t
y = t^2
g = (x,y)
defint(abs(d(g,t)),t,0,1)
```

$$\frac{1}{4}\log(2 + 5^{1/2}) + \frac{1}{2}5^{1/2}$$

```
float
```

1.47894

As expected, the result is greater than $\sqrt{2} \approx 1.414$, the length of the diagonal from $(0,0)$ to $(1,1)$.

The result seems rather complicated given that we started with a simple parabola. Let us inspect $|g'(t)|$ to see why.

```
g
```

$$g = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

```
d(g,t)
```

$$\begin{bmatrix} 1 \\ 2t \end{bmatrix}$$

```
abs(d(g,t))
```

$$(4t^2 + 1)^{1/2}$$

The following script does a discrete computation of the arc length by dividing the curve into 100 pieces.

```
g(t) = (t,t^2)
h(k) = abs(g(k/100.0) - g((k-1)/100.0))
sum(k,1,100,h(k))
```

1.47894

As expected, the discrete result matches the analytic result.

Find the length of the curve $y = x^{3/2}$ from the origin to $x = \frac{4}{3}$.

```
x = t
y = x^(3/2)
g = (x,y)
defint(abs(d(g,x)),x,0,4/3)
```

$\frac{56}{27}$

Because of the way t is substituted for x, the following code yields the same result.

```
g = (t,t^(3/2))
defint(abs(d(g,t)),t,0,4/3)
```

$\frac{56}{27}$

2.6 Line integrals

There are two different kinds of line integrals, one for scalar fields and one for vector fields. The following table shows how both are based on the calculation of arc length.

	Abstract form	Computable form
Arc length	$\int_C ds$	$\int_a^b \lvert g'(t)\rvert \, dt$
Line integral, scalar field	$\int_C f \, ds$	$\int_a^b f(g(t)) \, \lvert g'(t)\rvert \, dt$
Line integral, vector field	$\int_C (F \cdot u) \, ds$	$\int_a^b F(g(t)) \cdot g'(t) \, dt$

For the vector field form, the symbol u is the unit tangent vector

$$u = \frac{g'(t)}{|g'(t)|}$$

The length of the tangent vector cancels with ds as follows.

$$\int_C (F \cdot u) \, ds = \int_a^b \left(F(g(t)) \cdot \frac{g'(t)}{|g'(t)|} \right) \left(|g'(t)| \, dt \right) = \int_a^b F(g(t)) \cdot g'(t) \, dt$$

Evaluate

$$\int_C x \, ds \quad \text{and} \quad \int_C x \, dx$$

where C is a straight line from $(0,0)$ to $(1,1)$.

What a difference the measure makes. The first integral is over a scalar field and the second is over a vector field. This can be understood when we recall that

$$ds = |g'(t)| \, dt$$

Hence for $\int_C x \, ds$ we have

```
x = t
y = t
g = (x,y)
defint(x*abs(d(g,t)),t,0,1)
```

$$\frac{1}{2^{1/2}}$$

For $\int_C x\,dx$ we have

```
x = t
y = t
g = (x,y)
F = (x,0)
defint(dot(F,d(g,t)),t,0,1)
```

$$\frac{1}{2}$$

The following line integral problems are from *Advanced Calculus, Fifth Edition* by Wilfred Kaplan.

Evaluate $\int y^2\,dx$ along the straight line from $(0,0)$ to $(2,2)$.

```
x = 2t
y = 2t
g = (x,y)
F = (y^2,0)
defint(dot(F,d(g,t)),t,0,1)
```

$$\frac{8}{3}$$

Evaluate $\int z\,dx + x\,dy + y\,dz$ along the path $x = 2t+1$, $y = t^2$, $z = 1+t^3$, $0 \le t \le 1$.

```
x = 2t+1
y = t^2
z = 1+t^3
g = (x,y,z)
F = (z,x,y)
defint(dot(F,d(g,t)),t,0,1)
```

$$\frac{163}{30}$$

2.7 Surface area

Let S be a surface parameterized by x and y. That is, let $S = (x,y,z)$ where $z = f(x,y)$. The tangent lines at a point on S form a tiny parallelogram. The area a of the parallelogram is given by the magnitude of the cross product.

$$a = \left|\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right|$$

By summing over all the parallelograms we obtain the total surface area A. Hence

$$A = \iint dA = \iint a\,dx\,dy$$

The following example computes the surface area of a unit disk parallel to the xy plane.

```
z = 2
S = (x,y,z)
a = abs(cross(d(S,x),d(S,y)))
defint(a,y,-sqrt(1-x^2),sqrt(1-x^2),x,-1,1)
```

π

The result is π, the area of a unit circle, which is what we expect. The following example computes the surface area of $z = x^2 + 2y$ over a unit square.

```
z = x^2+2y
S = (x,y,z)
a = abs(cross(d(S,x),d(S,y)))
defint(a,x,0,1,y,0,1)
```

$\frac{3}{2} + \frac{5}{8}\log(5)$

The following exercise is from *Multivariable Mathematics* by Williamson and Trotter, p. 598. Find the area of the spiral ramp defined by

$$S = \begin{bmatrix} u\cos v \\ u\sin v \\ v \end{bmatrix}, \qquad 0 \le u \le 1, \qquad 0 \le v \le 3\pi$$

```
x = u*cos(v)
y = u*sin(v)
z = v
S = (x,y,z)
a = abs(cross(d(S,u),d(S,v)))
defint(a,u,0,1,v,0,3pi)
```

$\frac{3}{2}\pi\log(1 + 2^{1/2}) + \frac{3\pi}{2^{1/2}}$

```
float
```

10.8177

2.8 Surface integrals

A surface integral is like adding up all the wind on a sail. In other words, we want to compute

$$\iint \mathbf{F}\cdot\mathbf{n}\,dA$$

where $\mathbf{F}\cdot\mathbf{n}$ is the amount of wind normal to a tiny parallelogram dA. The integral sums over the entire area of the sail. Let S be the surface of the sail parameterized by x and y. (In this model, the z direction points downwind.) By the properties of the cross product we have the following for the unit normal $\mathbf{n}$ and for dA.

$$\mathbf{n} = \frac{\frac{\partial S}{\partial x}\times\frac{\partial S}{\partial y}}{\left|\frac{\partial S}{\partial x}\times\frac{\partial S}{\partial y}\right|} \qquad dA = \left|\frac{\partial S}{\partial x}\times\frac{\partial S}{\partial y}\right|\,dx\,dy$$

Hence

$$\iint \mathbf{F}\cdot\mathbf{n}\,dA = \iint \mathbf{F}\cdot\left(\frac{\partial S}{\partial x}\times\frac{\partial S}{\partial y}\right)\,dx\,dy$$

The following exercise is from *Advanced Calculus* by Wilfred Kaplan, p. 313. Evaluate the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$

where $\mathbf{F} = xy^2z\mathbf{i} - 2x^3\mathbf{j} + yz^2\mathbf{k}$, S is the surface $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$ and $\mathbf{n}$ is upper.

Note that the surface intersects the xy plane in a circle. By the right hand rule, crossing x into y yields $\mathbf{n}$ pointing upwards hence

$$\mathbf{n}\,d\sigma = \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right) dx\,dy$$

The following Eigenmath code computes the surface integral. The symbols f and h are used as temporary variables.

```
z = 1-x^2-y^2
F = (x*y^2*z,-2*x^3,y*z^2)
S = (x,y,z)
f = dot(F,cross(d(S,x),d(S,y)))
h = sqrt(1-x^2)
defint(f,y,-h,h,x,-1,1)
```

$$\frac{1}{48}\pi$$

2.9 Green's theorem

Green's theorem tells us that

$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

In other words, a line integral and a surface integral can yield the same result.

Example 1. The following exercise is from *Advanced Calculus* by Wilfred Kaplan, p. 287. Evaluate $\oint (2x^3 - y^3)\,dx + (x^3 + y^3)\,dy$ around the circle $x^2 + y^2 = 1$ using Green's theorem.

It turns out that Eigenmath cannot solve the double integral over x and y directly. Polar coordinates are used instead.

```
P = 2x^3-y^3
Q = x^3+y^3
f = d(Q,x)-d(P,y)
x = r*cos(theta)
y = r*sin(theta)
defint(f*r,r,0,1,theta,0,2pi)
```

$$\frac{3}{2}\pi$$

The $defint$ integrand is $f{*}r$ because $r\,dr\,d\theta = dx\,dy$.

Now let us try computing the line integral side of Green's theorem and see if we get the same result. We need to use the trick of converting sine and cosine to exponentials so that Eigenmath can find a solution.

```
x = cos(t)
y = sin(t)
P = 2x^3-y^3
Q = x^3+y^3
f = P*d(x,t)+Q*d(y,t)
f = circexp(f)
defint(f,t,0,2pi)
```

$$\frac{3}{2}\pi$$

Example 2. Compute both sides of Green's theorem for $F = (1 - y, x)$ over the disk $x^2 + y^2 \leq 4$.

First compute the line integral along the boundary of the disk. Note that the radius of the disk is 2.

```
-- Line integral
P = 1-y
Q = x
x = 2*cos(t)
y = 2*sin(t)
defint(P*d(x,t)+Q*d(y,t),t,0,2pi)
```

8π

```
-- Surface integral
x = quote(x) --clear x
y = quote(y) --clear y
h = sqrt(4-x^2)
defint(d(Q,x)-d(P,y),y,-h,h,x,-2,2)
```

8π

```
-- Try computing the surface integral using polar coordinates.
f = d(Q,x)-d(P,y) --do before change of coordinates
x = r*cos(theta)
y = r*sin(theta)
defint(f*r,r,0,2,theta,0,2pi)
```

8π

```
defint(f*r,theta,0,2pi,r,0,2) --try integrating over theta first
```

8π

In this case, Eigenmath solved both forms of the polar integral. However, in cases where Eigenmath fails to solve a double integral, try changing the order of integration.

2.10 Stokes' theorem

Stokes' theorem says that in typical problems a surface integral can be computed using a line integral. (There is some fine print regarding continuity and boundary conditions.) This is a useful theorem because usually the line integral is easier to compute. In rectangular coordinates the equivalence between a line integral on the left and a surface integral on the right is

$$\oint P\,dx + Q\,dy + R\,dz = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\,d\sigma$$

where $\mathbf{F} = (P, Q, R)$. For S parametrized by x and y we have

$$\mathbf{n}\,d\sigma = \left(\frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y}\right) dx\,dy$$

Example: Let $\mathbf{F} = (y, z, x)$ and let S be the part of the paraboloid $z = 4 - x^2 - y^2$ that is above the xy plane. The perimeter of the paraboloid is the circle $x^2 + y^2 = 2$. The following script computes both the line and surface integrals. It turns out that we need to use polar coordinates for the line integral so that *defint* can succeed.

```
-- Surface integral
z = 4-x^2-y^2
F = (y,z,x)
S = (x,y,z)
f = dot(curl(F),cross(d(S,x),d(S,y)))
x = r*cos(theta)
y = r*sin(theta)
defint(f*r,r,0,2,theta,0,2pi)
-- Line integral
x = 2*cos(t)
y = 2*sin(t)
z = 4-x^2-y^2
P = y
Q = z
R = x
f = P*d(x,t)+Q*d(y,t)+R*d(z,t)
f = circexp(f)
defint(f,t,0,2pi)
```

This is the result when the script runs. Both the surface integral and the line integral yield the same result.

-4π

-4π

3 Examples

3.1 François Viète

François Viète was the first to discover an exact formula for π. Here is his formula.

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \cdots$$

Let $a_0 = 0$ and $a_n = \sqrt{2 + a_{n-1}}$. Then we can write

$$\frac{2}{\pi} = \frac{a_1}{2} \times \frac{a_2}{2} \times \frac{a_3}{2} \times \cdots$$

Solving for π we have

$$\pi = 2 \times \frac{2}{a_1} \times \frac{2}{a_2} \times \frac{2}{a_3} \times \cdots = 2 \prod_{k=1}^{\infty} \frac{2}{a_k}$$

Let us now use Eigenmath to compute π according to Viète's formula. Of course, we cannot calculate all the way out to infinity, we have to stop somewhere. It turns out that nine factors are just enough to get six digits of accuracy.

```
a(n)=test(n=0,0,sqrt(2+a(n-1)))
float(2*product(k,1,9,2/a(k)))
```

3.14159

The function $a(n)$ calls itself n times so overall there are 54 calls to $a(n)$. By using a different algorithm with temporary variables, we can get the answer in just nine steps.

```
a = 0
b = 2
for(k,1,9,a=sqrt(2+a),b=b*2/a)
float(b)
```

3.14159

3.2 Curl in tensor form

The curl of a vector function can be expressed in tensor form as

$$\operatorname{curl} \mathbf{F} = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

where ϵ_{ijk} is the Levi-Civita tensor. The following script demonstrates that this formula is equivalent to computing curl the old fashioned way.

```
-- Define epsilon
epsilon = zero(3,3,3)
epsilon[1,2,3] = 1
epsilon[2,3,1] = 1
epsilon[3,1,2] = 1
epsilon[3,2,1] = -1
epsilon[1,3,2] = -1
epsilon[2,1,3] = -1
-- F is a generic vector function
F = (FX(),FY(),FZ())
-- A is the curl of F
A = outer(epsilon,d(F,(x,y,z)))
A = contract(A,3,4) --sum across k
A = contract(A,2,3) --sum across j
-- B is the curl of F computed the old fashioned way
BX = d(F[3],y)-d(F[2],z)
BY = d(F[1],z)-d(F[3],x)
BZ = d(F[2],x)-d(F[1],y)
B = (BX,BY,BZ)
-- Are A and B equal? Subtract to find out.
A-B
```

Here is the result when the script runs.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The following is a variation on the previous script. The product $\epsilon_{ijk}\,\partial F_k/\partial x_j$ is computed in just one line of code. In addition, the outer product and the contraction across k are now computed with a dot product.

```
F = (FX(),FY(),FZ())
epsilon = zero(3,3,3)
epsilon[1,2,3] = 1
epsilon[2,3,1] = 1
epsilon[3,1,2] = 1
epsilon[3,2,1] = -1
epsilon[1,3,2] = -1
epsilon[2,1,3] = -1
A = contract(dot(epsilon,d(F,(x,y,z))),2,3)
BX = d(F[3],y)-d(F[2],z)
BY = d(F[1],z)-d(F[3],x)
BZ = d(F[2],x)-d(F[1],y)
B = (BX,BY,BZ)
-- Are A and B equal? Subtract to find out.
A-B
```

This is the result when the script runs.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3.3 Quantum harmonic oscillator

For total energy E, kinetic energy K and potential energy V we have

$$E = K + V$$

The corresponding formula for a quantum harmonic oscillator is

$$(2n+1)\psi = -\frac{d^2\psi}{dx^2} + x^2\psi$$

where n is an integer and represents the quantization of energy values. The solution to the above equation is

$$\psi_n(x) = \exp(-x^2/2)H_n(x)$$

where $H_n(x)$ is the nth Hermite polynomial in x. The following Eigenmath code checks $E = K + V$ for $n = 7$.

```
n = 7
psi = exp(-x^2/2)*hermite(x,n)
E = (2*n+1)*psi
K = -d(psi,x,x)
V = x^2*psi
E-K-V
```

0

3.4 Hydrogen wavefunctions

Hydrogen wavefunctions ψ are solutions to the differential equation

$$\frac{\psi}{n^2} = \nabla^2\psi + \frac{2\psi}{r}$$

where n is an integer representing the quantization of total energy and r is the radial distance of the electron. The Laplacian operator in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

The general form of ψ is

$$\psi = r^l e^{-r/n} L_{n-l-1}^{2l+1}(2r/n) P_l^{|m|}(\cos\theta) e^{im\phi}$$

where L is a Laguerre polynomial, P is a Legendre polynomial and l and m are integers such that

$$1 \le l \le n-1, \qquad -l \le m \le l$$

The general form can be expressed as the product of a radial wavefunction R and a spherical harmonic Y.

$$\psi = RY, \qquad R = r^l e^{-r/n} L_{n-l-1}^{2l+1}(2r/n), \qquad Y = P_l^{|m|}(\cos\theta)e^{im\phi}$$

The following script checks $E = K + V$ for $n, l, m = 7, 3, 1$.

```
laplacian(f) = 1/r^2*d(r^2*d(f,r),r)+
  1/(r^2*sin(theta))*d(sin(theta)*d(f,theta),theta)+
  1/(r*sin(theta))^2*d(f,phi,phi)
n = 7
l = 3
m = 1
R = r^l*exp(-r/n)*laguerre(2*r/n,n-l-1,2*l+1)
Y = legendre(cos(theta),l,abs(m))*exp(i*m*phi)
psi = R*Y
E = psi/n^2
K = laplacian(psi)
V = 2*psi/r
simplify(E-K-V)
```

This is the result when the script runs.

0

3.5 Space shuttle and Corvette

The space shuttle accelerates from zero to 17,000 miles per hour in 8 minutes. A Corvette accelerates from zero to 60 miles per hour in 4.5 seconds. The following script compares the two.

```
vs = 17000*"mile"/"hr"
ts = 8*"min"/(60*"min"/"hr")
as = vs/ts
as
vc = 60*"mile"/"hr"
tc = 4.5*"sec"/(3600*"sec"/"hr")
ac = vc/tc
ac
"Time for Corvette to reach orbital velocity:"
vs/ac
vs/ac*60*"min"/"hr"
```

Here is the result when the script runs. It turns out that the space shuttle accelerates more than twice as fast as a Corvette.

$$a_s = \frac{127500\ \text{mile}}{(\text{hr})^2}$$

$$a_c = \frac{48000\ \text{mile}}{(\text{hr})^2}$$

```
Time for Corvette to reach orbital velocity:
```

0.354167 hr

21.25 min

3.6 Avogadro's constant

There is a proposal to define Avogadro's constant as exactly 84446886 to the third power. (Fox, Ronald and Theodore Hill. "An Exact Value for Avogadro's Number." *American Scientist* 95 (2007): 104–107.) The proposed number in the article is actually $(84446888)^3$. In a subsequent addendum the authors reduced it to 84446886^3 to make the number divisible by 12. (See `www.physorg.com/news109595312.html`.) This number corresponds to an ideal cube of atoms with 84,446,886 atoms along each edge. Let us check the difference between the proposed value and the measured value of $(6.0221415 \pm 0.0000010) \times 10^{23}$ atoms.

```
A = 84446886^3
B = 6.0221415*10^23
A-B
```

-5.17173×10^{16}

```
0.0000010*10^23
```

1×10^{17}

We see that the proposed value is within the experimental error. Just for the fun of it, let us factor the proposed value.

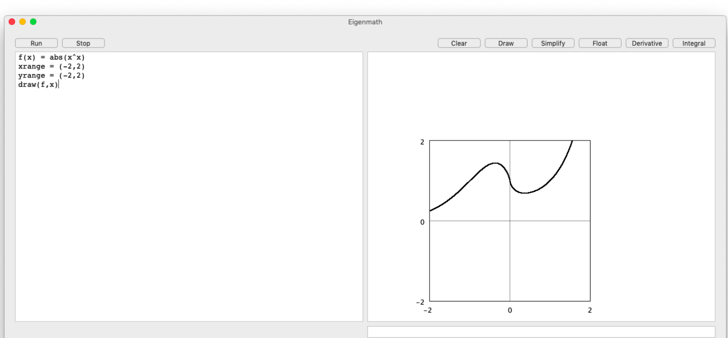
```
factor(A)
```

$2^3 \times 3^3 \times 1667^3 \times 8443^3$

3.7 Zero to the zero power

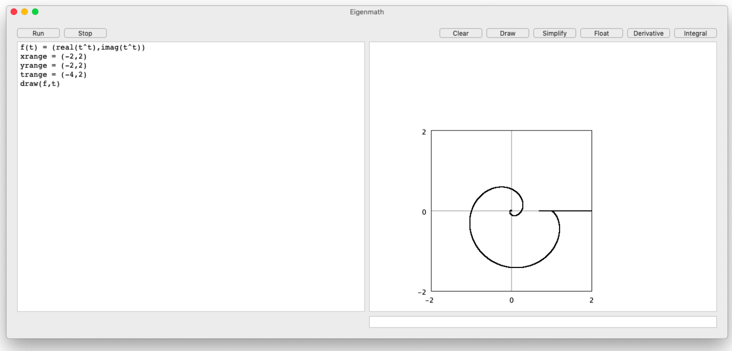
The following example draws a graph of the function $f(x) = |x^x|$. The graph shows why the convention $0^0 = 1$ makes sense.

```
f(x) = abs(x^x)
xrange = (-2,2)
yrange = (-2,2)
draw(f,x)
```



We can see how $0^0 = 1$ results in a continuous line through $x = 0$. Now let us see how x^x behaves in the complex plane.

```
f(t) = (real(t^t),imag(t^t))
xrange = (-2,2)
yrange = (-2,2)
trange = (-4,2)
draw(f,t)
```



3.8 Euler's identity

It is easy to "believe" that $e^{i\pi} = -1$ by looking at Taylor series expansions.

First, consider the Taylor series expansion of e^y.

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \frac{y^6}{6!} + \frac{y^7}{7!} + \cdots$$

Next, substitute ix for y.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots \end{aligned}$$

Next, collect the real and imaginary terms.

$$\begin{aligned} e^{ix} &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

Finally, substitute π for x.

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

The following script checks the identity $e^{ix} = \cos x + i \sin x$ for order n.

```
n = 7
E = taylor(e^y,y,n)
E = eval(E,y,i*x)
C = taylor(cos(x),x,n)
S = taylor(sin(x),x,n)
test(E=C+i*S,"true","false")
```

4 Built-in functions

abs

abs(x) returns the absolute value or vector length of x. The mag function should be used for complex x.

```
P = (x,y)
abs(P)
```

$(x^2 + y^2)^{1/2}$

adj

adj(m) returns the adjunct of matrix m.

and

and($a, b, \ldots$) returns the logical "and" of predicate expressions.

arccos

arccos(x) returns the inverse cosine of x.

arccosh

arccosh(x) returns the inverse hyperbolic cosine of x.

arcsin

arcsin(x) returns the inverse sine of x.

arcsinh

arcsinh(x) returns the inverse hyperbolic sine of x.

arctan

arcttan(x) returns the inverse tangent of x.

arctanh

arctanh(x) returns the inverse hyperbolic tangent of x.

arg

arg(z) returns the angle of complex z.

ceiling

ceiling(x) returns the smallest integer not less than x.

check

check(x) In a script, if the predicate x is true then continue, else stop.

choose

choose(n, k) returns $\binom{n}{k}$

circexp

circexp(x) returns expression x with circular functions converted to exponential forms. Sometimes this will simplify an expression.

coeff

coeff(p, x, n) returns the coefficient of x^n in polynomial p.

cofactor

cofactor(m, i, j) returns of the cofactor of matrix m with respect to row i and column j.

conj

conj(z) returns the complex conjugate of z.

contract

contract(a, i, j) returns tensor a summed over indices i and j. If i and j are omitted then indices 1 and 2 are used. contract(m) is equivalent to the trace of matrix m.

cos

cos(x) returns the cosine of x.

cosh

cosh(x) returns the hyperbolic cosine of x.

cross

cross(u, v) returns the cross product of vectors u and v.

curl

curl(u) returns the curl of vector u.

d

d(f, x) returns the derivative of f with respect to x.

defint

defint$(f, x, a, b, \ldots)$ returns the definite integral of f with respect to x evaluated from a to b. The argument list can be extended for multiple integrals. For example, $d(f, x, a, b, y, c, d)$.

deg

deg(p, x) returns the degree of polynomial p in x.

denominator

denominator(x) returns the denominator of expression x.

det

det(m) returns the determinant of matrix m.

do

do($a, b, \ldots$) evaluates the argument list from left to right. Returns the result of the last argument.

dot

dot($a, b, \ldots$) returns the dot product of tensors.

draw

draw(f, x) draws the function f with respect to x.

erf

erf(x) returns the error function of x.

erfc

erf(x) returns the complementary error function of x.

eval

eval(f, x, n) returns f evaluated at $x = n$.

exp

exp(x) returns e^x.

expand

expand(r, x) returns the partial fraction expansion of the ratio of polynomials r in x.

```
expand(1/(x^3+x^2),x)
```

$$\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$$

expcos

expcos(x) returns the cosine of x in exponential form.

```
expcos(x)
```

$$\frac{1}{2}\exp(-ix)+\frac{1}{2}\exp(ix)$$

expsin

expsin(x) returns the sine of x in exponential form.

```
expsin(x)
```

$$\frac{1}{2}i\exp(-ix)-\frac{1}{2}i\exp(ix)$$

factor

factor(n) factors the integer n.

```
factor(12345)
```

$3\times 5\times 823$

$factor(p,x)$ factors polynomial p in x. The last argument can be omitted for polynomials in x. The argument list can be extended for multivariate polynomials. For example, factor(p,x,y) factors p over x and then over y.

```
factor(125*x^3-1)
```

$(5x-1)(25x^2+5x+1)$

factorial

Example:

```
10!
```

3628800

filter

filter($f,a,b,\ldots$) returns f with terms involving a, b, etc. removed.

```
1/a+1/b+1/c
```

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

```
filter(last,a)
```

$$\frac{1}{b}+\frac{1}{c}$$

float

float(x) converts x to a floating point value.

```
sum(n,0,20,(-1/2)^n)
```

$$\frac{699051}{1048576}$$

```
float(last)
```

0.666667

floor

floor(x) returns the largest integer not greater than x.

for

for($i, j, k, a, b, \ldots$) For i equals j through k evaluate a, b, etc.

```
x = 0
y = 2
for(k,1,9,x=sqrt(2+x),y=2*y/x)
float(y)
```

3.14159

gcd

gcd($a, b, \ldots$) returns the greatest common divisor.

hermite

hermite(x, n) returns the nth Hermite polynomial in x.

hilbert

hilbert(n) returns a Hilbert matrix of order n.

imag

imag(z) returns the imaginary part of complex z.

inner

inner($a, b, \ldots$) returns the inner product of tensors. Same as the dot product.

integral

integral(f, x) returns the integral of f with respect to x.

inv

inv(m) returns the inverse of matrix m.

isprime

isprime(n) returns 1 if n is prime, zero otherwise.

```
isprime(2^53-111)
```

1

laguerre

laguerre(x, n, a) returns the nth Laguerre polynomial in x. If a is omitted then $a = 0$ is used.

lcm

lcm($a, b, \ldots$) returns the least common multiple.

leading

leading(p, x) returns the leading coefficient of polynomial p in x.

```
leading(5x^2+x+1,x)
```

5

legendre

legendre(x, n, m) returns the nth Legendre polynomial in x. If m is omitted then $m = 0$ is used.

log

log(x) returns the natural logarithm of x.

mag

mag(z) returns the magnitude of complex z.

mod

mod(a, b) returns the remainder of a divided by b.

not

not(x) negates the result of predicate expression x.

nroots

nroots(p, x) returns all of the roots, both real and complex, of polynomial p in x. The roots are computed numerically. The coefficients of p can be real or complex.

numerator

numerator(x) returns the numerator of expression x.

or

or($a, b, \ldots$) returns the logical "or" of predicate expressions.

outer

outer($a, b, \ldots$) returns the outer product of tensors.

Example 1.

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

```
outer((1,0),(1,0))
```

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

Example 2. From the identity

$$|A\rangle=\sum_i |i\rangle\langle i|A\rangle$$

it follows that

$$\sum_i |i\rangle\langle i| = \mathbf{I}$$

For a two-state system with basis vectors

$$|1\rangle=\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix},\quad\text{and}\quad|2\rangle=\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$$

the following code computes $\sum|i\rangle\langle i|$.

```
X1 = (1/sqrt(2),i/sqrt(2))
X2 = (1/sqrt(2),-i/sqrt(2))
outer(conj(X1),X1) + outer(conj(X2),X2)
```

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

The following code uses a different approach. First, tensor T is computed such that

$$T=\begin{pmatrix}|1\rangle\langle 1| & |1\rangle\langle 2|\\ |2\rangle\langle 1| & |2\rangle\langle 2|\end{pmatrix}$$

Then contraction is used to sum the diagonal elements.

```
X1 = (1/sqrt(2),i/sqrt(2))
X2 = (1/sqrt(2),-i/sqrt(2))
X = (X1,X2)
T = outer(conj(X),X)
T = transpose(T,2,3)
contract(T)
```

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

```
-- verify components of T
T[1,1] == outer(conj(X1),X1)
T[1,2] == outer(conj(X1),X2)
T[2,1] == outer(conj(X2),X1)
T[2,2] == outer(conj(X2),X2)
```

1
1
1
1

polar

polar(z) converts complex z to polar form.

prime

prime(n) returns the nth prime number, $1 \leq n \leq 10{,}000$.

print

print($a, b, \ldots$) evaluates expressions and prints the results.. Useful for printing from inside a "for" loop.

product

product(i, j, k, f) returns $\displaystyle\prod_{i=j}^{k} f$

quote

quote(x) returns expression x unevaluated.

quotient

quotient(p, q, x) returns the quotient of polynomials in x.

rank

rank(a) returns the number of indices that tensor a has. A scalar has no indices so its rank is zero.

rationalize

rationalize(x) puts everything over a common denominator.

```
rationalize(a/b+b/a)
```

$$\frac{a^2 + b^2}{ab}$$

real

real(z) returns the real part of complex z.

rect

rect(z) returns complex z in rectangular form.

roots

roots(p, x) returns the values of x such that the polynomial $p(x) = 0$. The polynomial should be factorable over integers.

simplify

simplify(x) returns x in a simpler form.

sin

sin(x) returns the sine of x.

sinh

sinh(x) returns the hyperbolic sine of x.

sqrt

sqrt(x) returns the square root of x.

stop

In a script, it does what it says.

subst

subst(a, b, c) substitutes a for b in c and returns the result.

sum

sum(i,j,k,f) returns $\displaystyle\sum_{i=j}^{k} f$

tan

tan(x) returns the tangent of x.

tanh

tanh(x) returns the hyperbolic tangent of x.

taylor

taylor(f,x,n,a) returns the Taylor expansion of f of x at a. The argument n is the degree of the expansion. If a is omitted then $a=0$ is used.

```
taylor(1/cos(x),x,4)
```

$$\frac{5}{24}x^4+\frac{1}{2}x^2+1$$

test

test$(a,b,c,d,\ldots)$ If a is true then b is returned else if c is true then d is returned, etc. If the number of arguments is odd then the last argument is returned when all else fails.

transpose

transpose(a,i,j) returns the transpose of tensor a with respect to indices i and j. If i and j are omitted then 1 and 2 are used. Hence a matrix can be transposed with a single argument.

```
A = ((a,b),(c,d))
transpose(A)
```

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

unit

unit(n) returns an $n\times n$ identity matrix.

```
unit(2)
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

zero

zero$(i, j, \ldots)$ returns a null tensor with dimensions i, j, etc. Useful for creating a tensor and then setting the component values.

5 Syntax

Math	*Eigenmath*	*Alternate form and/or comment*
$-a$	`-a`	
$a+b$	`a+b`	
$a-b$	`a-b`	
ab	`a*b`	`a b` *with a space in between*
$\dfrac{a}{b}$	`a/b`	
$\dfrac{a}{bc}$	`a/b/c`	
a^2	`a^2`	
$\sqrt{a}$	`a^(1/2)`	`sqrt(a)`
$\dfrac{1}{\sqrt{a}}$	`a^(-1/2)`	`1/sqrt(a)`
$a(b+c)$	`a*(b+c)`	`a (b+c)` *with a space in between*
$f(a)$	`f(a)`	
$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$	`(a,b,c)`	
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	`((a,b),(c,d))`	
T^{12}	`T[1,2]`	*tensor component access*
$2\,\mathrm{km}$	`2*"km"`	*units of measure are quoted*

6 Tricks

1. The Eigenmath result field can be copied to the pasteboard by click, drag, then release.
2. The last result is stored in the symbol `last`.
3. In a script, setting `trace=1` causes each line to be printed just before it is evaluated. Useful for debugging.
4. Use `contract(A)` to get the mathematical trace of matrix A.
5. Calculations in a script can span multiple lines. The trick is to arrange things so the parser will keep going. For example, if a calculation ends with a plus sign, the parser will go to the next line to get another term. Also, the parser will keep going when it expects a close parenthesis.
6. Normally a function body is not evaluated when a function is defined. However, in some cases it is required that the function body be the result of something. The trick is to use eval. For example, the following code causes the function body to be a sixth order Taylor series expansion of $cos(x)$.

   ```
   f(x) = eval(taylor(cos(x),x,6))
   ```

7. Use `binding` to see the unevaluated binding of a symbol.

   ```
   binding(f)
   ```

8. This is how to clear a symbol.

   ```
   f = quote(f)
   ```