Model:

Set $\eta = 0$ (assumption 2) and $\tilde{B} = 1$ (just a choice of units)

Production function $$Y = \left[\int_{N-1}^{N} y(i)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}}$$ [Eq 1]

$$\Rightarrow y(i) = \begin{cases} k(i) + \gamma(i)\ell(i) & i \leq I \quad \text{[Eq 3]} \\ \gamma(i)\ell(i) & i > I \quad \text{[Eq 2]} \end{cases}$$

Utility Function $$u(C, L) = \frac{[Ce^{-\nu(L)}]^{1-\theta} - 1}{1-\theta}$$ [Eq 4]

$L = \int_{N-1}^{N} \ell(i)\, di$ $\quad$ L is determined by utility maximization

$K = \int_{N-1}^{N} k(i)\, di$ $\quad$ K is fixed

Threshold Definitions

I: Current technology level (Exogenous)

$\tilde{I}$: level of capital usage if there were no technological limitation (Endogenous, calculated below)

$I^* = \min\{\tilde{I}, I\}$ actual level of capital usage (Endogenous, calculated below)

Deriving the Equations for Static Equilibrium

- Step 1: Find task demand

price of task i, $p(i)$, is the marginal product of the task:

$$p(i) = \frac{\delta Y}{\delta y}(i) \quad \text{(Functional derivative of } Y\text{)}$$

$$= \frac{\sigma}{\sigma-1}\left(Y^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} \underbrace{\frac{\delta}{\delta y}\int_{N-1}^{N} y(i)^{\frac{\sigma-1}{\sigma}}\,di}_{\text{"}\frac{d}{dy}\, y^{\frac{\sigma-1}{\sigma}}\text{"}} = \cancel{\frac{\sigma}{\sigma-1}}\, Y^{1-\frac{\sigma-1}{\sigma}}\, \cancel{\frac{\sigma-1}{\sigma}}\, y(i)^{\frac{\sigma-1}{\sigma}-1}$$

Chain rule

$$= Y^{1/\sigma}\, y(i)^{-1/\sigma}$$

$\Rightarrow$ Demand for task i is $y(i) = Y\, p(i)^{-\sigma}$ [Eq 7]

- <u>Step</u> 2: Find Capital and Labor demand

Quantity demanded of capital for task i $(i < I^*)$ at rental rate R:

$$k(i) = \underbrace{(\text{Quantity of task } i \text{ demanded})}_{Y\,p(i)^{-\sigma}}\underbrace{(\text{amount of capital required per unit of task } i)}_{\frac{dk(i)}{dy(i)} = 1}$$

$$= Y \underbrace{p(i)}_{\substack{\text{price of task } i \\ = (\text{price of capital rental})\cdot \frac{dk(i)}{dy(i)} \\ = R\cdot 1}}^{-\sigma} = Y R^{-\sigma}$$

$$\Rightarrow k(i) = \begin{cases} Y R^{-\sigma} & i \le I^* \\ 0 & i > I^* \end{cases} \quad \text{[Eq 7.5]}$$

Quantity demanded of Labor for $i > I^*$:

$$\ell(i) = Y\underbrace{p(i)}_{w\frac{d\ell(i)}{dy(i)}}{}^{-\sigma}\frac{d\ell(i)}{dy(i)} = YW^{-\sigma}\Bigg(\underbrace{\frac{d\ell(i)}{dy(i)}}_{\frac{1}{\gamma(i)}}\Bigg)^{1-\sigma} = YW^{-\sigma}\gamma(i)^{\sigma-1}$$

$$\Rightarrow \ell(i) = \begin{cases} 0 \\ YW^{-\sigma}\gamma(i)^{\sigma-1} \end{cases} \quad [\text{Eq } 7.5]$$

- Step 3: Market Clearing Conditions (Supply = Demand)
  - Supply of Capital is fixed at K

$$\Rightarrow K = \int_{N-1}^{I^*} k(i)\,di = \int_{N-1}^{I^*} YR^{-\sigma}\,di = YR^{-\sigma}(I^* - N + 1)$$

$$\Rightarrow K = YR^{-\sigma}(I^* - N + 1) \quad [\text{Eq } 8]$$

  - Supply of Labor is determined by household utility maximization.

$$u(c, L) \qquad c = Y \quad \text{and} \quad C = RK + WL$$

$$\Rightarrow \underbrace{L^S(W)}_{\text{Labor supply}} = \operatorname*{argmax}_L u(RK + WL, L) \quad \text{at fixed } R, K$$

$$= \operatorname*{argmax}_L \frac{\left[(RK + WL)e^{-\nu(L)}\right]^{1-\theta} - 1}{1-\theta}$$

First order condition:

$$\frac{\partial}{\partial L} \frac{\left[(RK+wL)e^{-\nu(L)}\right]^{1-\theta}-1}{1-\theta} = 0 \quad \text{at} \quad L=L^s$$

$$= \left[(RK+wL)e^{-\nu L}\right]^{-\theta}\left[we^{-\nu(L)} - (RK+wL)\nu'(L)e^{-\nu(L)}\right] = 0$$

$$\Rightarrow \quad w = \underbrace{(RK+wL)}_{C}\nu'(L) \quad \text{at} \quad L=L^s$$

$$\Rightarrow \quad \nu'(L) = \frac{w}{C} \quad [\text{Eq } 10.5]$$

$$\Rightarrow \quad L^s(w) \text{ satisfies } \quad 1 = \left(\frac{RK}{w} + L^s\right)\nu'(L^s)$$

$$\Rightarrow \quad L^s \text{ is a function of only } \frac{w}{RK}, \quad L^s\left(\frac{w}{RK}\right) \quad [\text{Eq } 11]$$

Setting Labor Supply = Labor demand

$$L^s\left(\frac{w}{RK}\right) = \int_{I^*}^{N} \ell(i) = \int_{I^*}^{N} Y w^{-\sigma}\gamma(i)^{\sigma-1}\,di$$

$$\Rightarrow L = L^s\left(\frac{w}{RK}\right) = Y w^{-\sigma}\int_{I^*}^{N}\gamma(i)^{\sigma-1} \quad [\text{Eq } 9]$$

- Step 4: Firms Maximize Profit
  - The "Ideal Price Index Condition" (Eq 10) is the result of solving this maximization, as derived below.

- Note that the production function is homogeneous $\Rightarrow$ Production can be divided between arbitrarily many identical firms (which compete with one another). For fixed Y, the number of firms has no effect.

  Therefore, we can just assume a "representative firm" which produces all output (not assuming this would just add one extra step in the calculation)

- Express profit maximization as a cost minimization problem for fixed output:

$$\min_{y(i)} \int p(i)\, y(i)\, di \quad \text{such that} \quad \left[\int y(i)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} = Y$$

Note: All integrals run from $N-1$ to N

Equivalently:

$$\min_{y(i)} \int p(i)\, y(i)\, di \quad \text{such that} \quad \int y(i)^{\frac{\sigma-1}{\sigma}} di = Y^{\frac{\sigma-1}{\sigma}}$$

Lagrangian
$$\mathcal{L}[y(i)] = \int p(i)\, y(i)\, di - \lambda\left[\int y(i)^{\frac{\sigma-1}{\sigma}} di - Y^{\frac{\sigma-1}{\sigma}}\right]$$
$$= \int \left[p(i)\, y(i) - \lambda\, y(i)^{\frac{\sigma-1}{\sigma}}\right] di + \lambda\, Y^{\frac{\sigma-1}{\sigma}}$$

$$\Rightarrow \quad \frac{\delta \mathcal{L}}{\delta y} = p(i) - \lambda \frac{\sigma-1}{\sigma}\, y(i)^{\frac{\sigma-1}{\sigma}-1} = 0$$

$$\Rightarrow \quad p(i) = \lambda \frac{\sigma-1}{\sigma}\, y(i)^{-1/\sigma} \quad \Rightarrow \quad y(i) = \left[\frac{p(i)}{\lambda}\frac{\sigma}{\sigma-1}\right]^{-\sigma}$$

Note that the representative firm's costs are $\int p(i)\, y(i)\, di = RK + LW = Y$

$$\Rightarrow \int p^{1-\sigma}\left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{-\sigma} di = Y \quad \Rightarrow \left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{-\sigma} \int p(i)^{1-\sigma} di = Y$$

$$\Rightarrow \int p(i)^{1-\sigma} di = Y\left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{\sigma} \quad \text{☆}$$

From constraint: $\int y(i)^{\frac{\sigma-1}{\sigma}} di = Y^{\frac{\sigma-1}{\sigma}}$

$$\Rightarrow Y^{\frac{\sigma-1}{\sigma}} = \int \left[\frac{p}{\lambda} \frac{\sigma}{\sigma-1}\right]^{1-\sigma} di = \left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{1-\sigma} \int p(i)^{1-\sigma} di$$

$$\Rightarrow \int p(i)^{1-\sigma} di = Y^{\frac{\sigma-1}{\sigma}} \left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{\sigma-1} \quad ☆☆$$

Setting ☆ = ☆☆:

$$Y^{\frac{\sigma-1}{\sigma}} \left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{\cancel{\sigma}-1} = Y\left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{\cancel{\sigma}}$$

$$\Rightarrow Y^{\cancel{1}-\frac{1}{\sigma}} \left[\frac{\sigma}{\lambda(\sigma-1)}\right]^{-1} = \cancel{Y} \Rightarrow \frac{\sigma}{\lambda(\sigma-1)} = Y^{-1/\sigma}$$

$\Rightarrow$ ☆ implies that $\int p(i)^{1-\sigma} di = Y\left[Y^{-1/\sigma}\right]^{\sigma} = \frac{Y}{Y} = 1$

Now, we know that $\int_{N-1}^{N} p(i)^{1-\sigma} di = 1$, we can plug

in prices for tasks done by capital and labor, in terms

of R and W.

For Capital $(i \le I^*)$ $\quad p(i) = R$

For Labor $(i > I^*)$ $\quad p(i) = \frac{W}{\gamma(i)}$

[Eq 5]

these are $\frac{P_{factor}}{MP}$ where

$P_{factor} = $ R for capital, W for Labor

and MP (Marginal product for performing task) is

$\frac{\partial y}{\partial k} = 1$ for capital

$\frac{\partial y}{\partial l} = \gamma(i)$ for labor

$$\Rightarrow \int_{N-1}^{I^*} R^{1-\sigma}\, di + \int_{I^*}^{N} \left(\frac{W}{\gamma(i)}\right)^{1-\sigma} di = 1$$

$$\Rightarrow (I^* - N + 1)\, R^{1-\sigma} + \int_{I^*}^{N} \left(\frac{W}{\gamma(i)}\right)^{1-\sigma} di = 1 \quad [\text{Eq } 10]$$

This is the Ideal Price Index Condition, which captures Firm's profit maximization.

- <u>Step 5</u> : Finding output in terms of K and L (Eq. 12)

  - Rearrange Eq 8 and Eq 9 to get equations for R and W then insert these into Eq 10 :

- From Eq 8 : $R = \left[\dfrac{K}{Y(I^* - N + 1)}\right]^{-1/\sigma}$

- From Eq 9 : $W^{-\sigma}\, Y \int_{I^*}^{N} \gamma(i)^{\sigma-1}\, di = L$

$$\Rightarrow W = \left[\frac{L}{Y \int_{I^*}^{N} \gamma(i)^{\sigma-1}\, di}\right]^{-1/\sigma}$$

- Inserting into Eq 10:

Eq 10 is $(I^* - N + 1)\, R^{1-\sigma} + W^{1-\sigma} \int_{I^*}^{N} \left(\frac{1}{\gamma(i)}\right)^{1-\sigma} di = 1$

$$\Rightarrow (I^* - N + 1)\left[\frac{K}{Y(I^* - N + 1)}\right]^{\frac{-1}{\sigma}(1-\sigma)} + \left[\frac{L}{Y \int_{I^*}^{N} \gamma(i)^{\sigma-1}\, di}\right]^{\frac{-1}{\sigma}(1-\sigma)} \int_{I^*}^{N} \left(\frac{1}{\gamma(i)}\right)^{1-\sigma} di = 1$$

$$\Rightarrow \quad (I^*-N+1)\left[\frac{K}{Y(I^*-N+1)}\right]^{\frac{\sigma-1}{\sigma}} + \left(\frac{L}{Y}\right)^{\frac{\sigma-1}{\sigma}} \left[\int_{I^*}^{N}\left(\frac{1}{\gamma(i)}\right)^{1-\sigma} di\right]^{1-\frac{\sigma-1}{\sigma}} = 1$$

Move Y to otherside:

$$\Rightarrow \quad (I^*-N+1)\left[\frac{K}{(I^*-N+1)}\right]^{\frac{\sigma-1}{\sigma}} + L^{\frac{\sigma-1}{\sigma}} \left[\int_{I^*}^{N}\left(\frac{1}{\gamma(i)}\right)^{1-\sigma} di\right]^{1-\frac{\sigma-1}{\sigma}} = Y^{\frac{\sigma-1}{\sigma}}$$

Simplify exponents, $1-\frac{\sigma-1}{\sigma}=\frac{1}{\sigma}$

$$\Rightarrow \quad (I^*-N+1)^{1/\sigma} K^{\frac{\sigma-1}{\sigma}} + \left[\int_{I^*}^{N}\left(\frac{1}{\gamma(i)}\right)^{1-\sigma} di\right]^{1/\sigma} L^{\frac{\sigma-1}{\sigma}} = Y^{\frac{\sigma-1}{\sigma}}$$

$$\Rightarrow \quad Y = \left[(I^*-N+1)^{1/\sigma} K^{\frac{\sigma-1}{\sigma}} + \left[\int_{I^*}^{N}\left(\frac{1}{\gamma(i)}\right)^{1-\sigma} di\right]^{1/\sigma} L^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \quad [\text{Eq } 12]$$

Computing Equilibrium quantities

In utility function. Note that θ has no effect.

Given exogenous parameters σ, $\gamma(i)$, $\nu(L)$, I, K, N

Find endogenous quantities W, R, L, Y

Steps

1) Find I^* and $\omega \equiv \frac{W}{RK}$ using method described in Figure 3.

2) $L = L^s(\omega)$ The L^s function must be computed from Eq 9

3) Y computed with Eq 12 (using L and I^* found above)

4) R computed with Eq 8
W computed with Eq 9
(using L, Y, I^* found above)

(My python code computes these for the case that $\gamma(i) = a + bi$ and $v(L) = \frac{1}{2}L^2$)

Alternative Framing: maximize profits, aka minimize costs given fixed output

(note that $C = Y$ because profit is 0 for efficient markets)

$$\min_{a(i)} \int p(i) a(i)\, di \quad \text{s.t.} \left[\int a(i)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} = A$$

$$y(i) = \alpha a(i)$$

$$\Rightarrow Y = \alpha A = \left[\int y(i)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}} \quad \checkmark$$

Tasks