On the complexity of regular languages

Corentin Barloy

Supervised by: Charles Paperman, Michaël Cadilhac and Sylvain Salvati

Université de Lille

Optimisation of sequential computations

Sequential vs Parallel

Does the string contain aa:

b	a	b	a	a	b

Sequential vs Parallel

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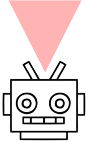
Sequential: one operation at a time

Sequential vs Parallel

Does the string contain *aa*:

b	*a*	*b*	*a*	*a*	*b*



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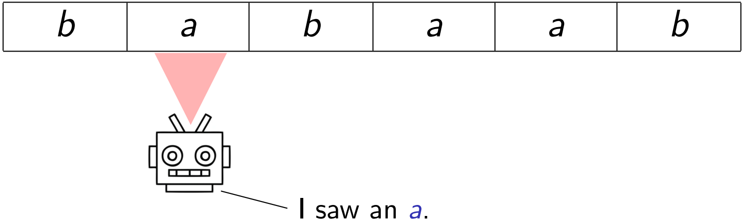

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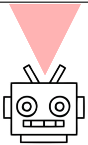

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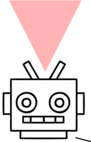

I saw an a.

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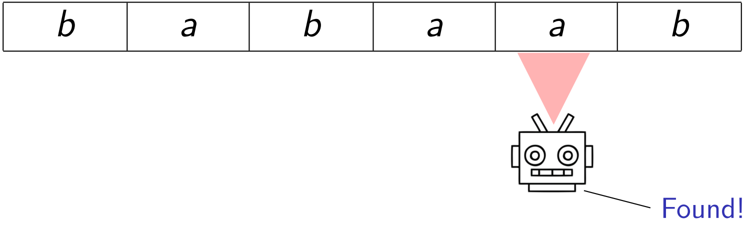

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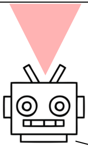

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Parallel: several operations at the same time

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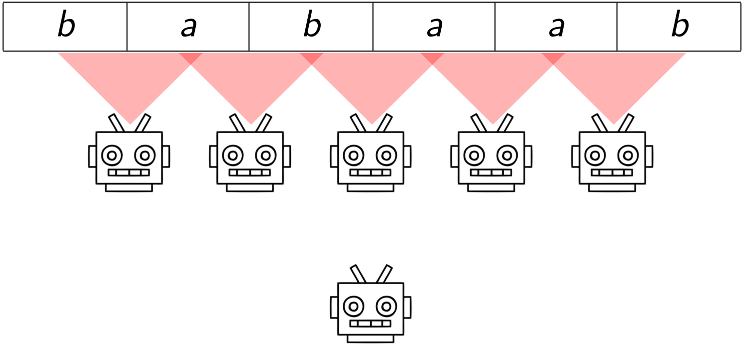

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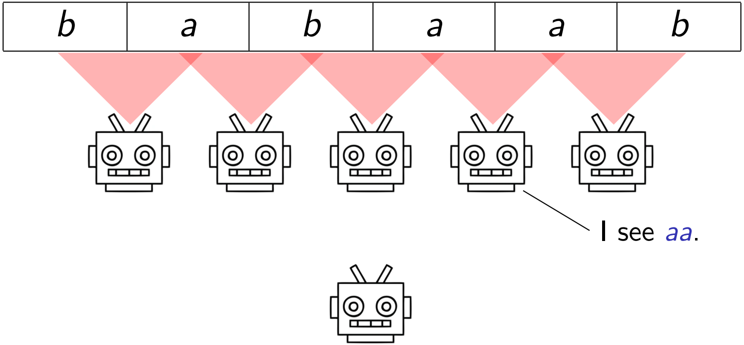

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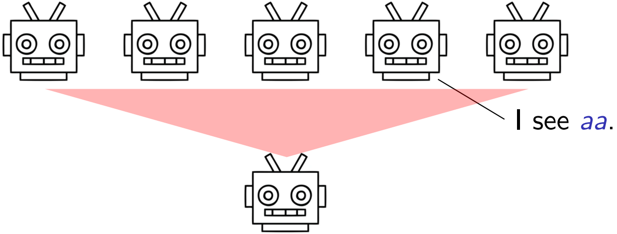

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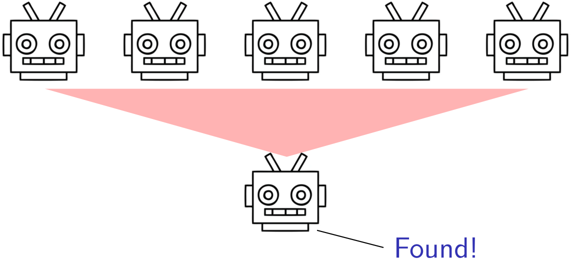

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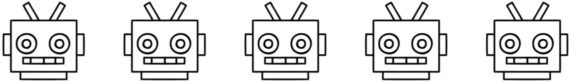

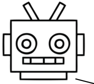

Found!

Sequential: one operation at a time $\rightarrow$ Finite Automata

Parallel: several operations at the same time $\rightarrow$ Boolean Circuit

Finite automata

Finite automata

A simple sequential model:

Finite automata

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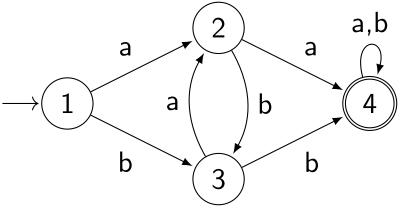
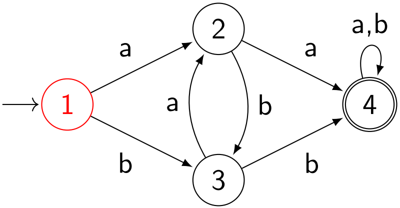


Finite automata

A simple sequential model:

w=babaab



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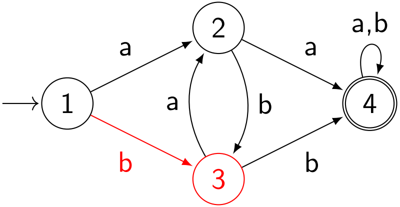

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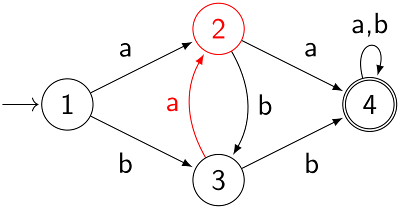

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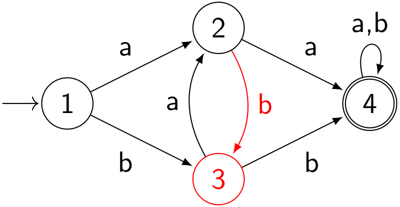

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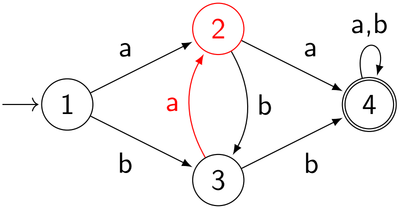

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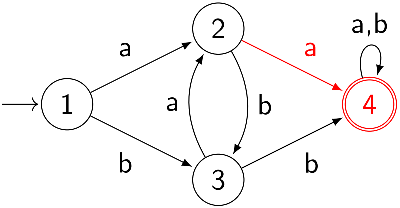

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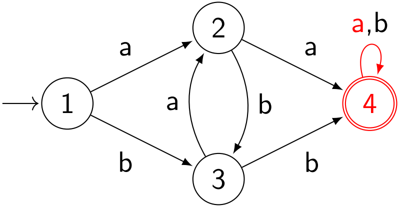

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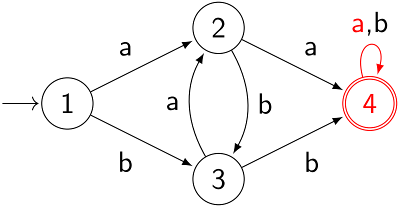

Computes Twice = contains *aa* or *bb*

Finite automata

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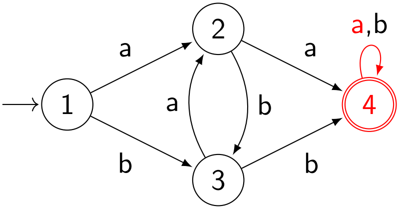

Computes Twice = contains *aa* or *bb*

Ubiquitous:

- Data extraction
- Compilers
- Text editors
- Linguistics
- Bioinformatic
- Security
- ...

Algebra

Many classes of regular languages enjoy several closure properties: they form varieties

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$\rightarrow$ their languages can be understood algebraically

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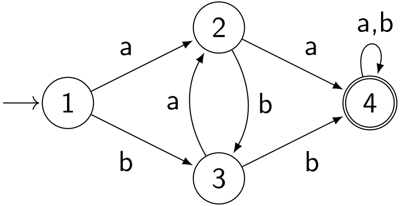
Minimal automaton



Syntactic monoid

	1	2	3	4

Algebra

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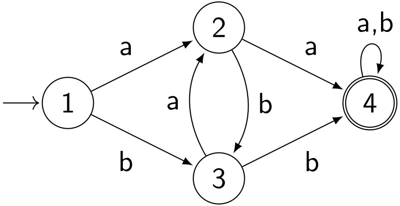

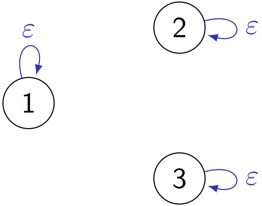

Syntactic monoid

	1	2	3	4
f_ε	1	2	3	4

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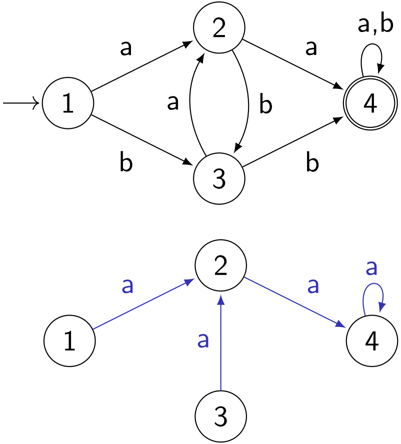

Syntactic monoid

	1	2	3	4
f_ε	1	2	3	4
f_a	2	4	2	4

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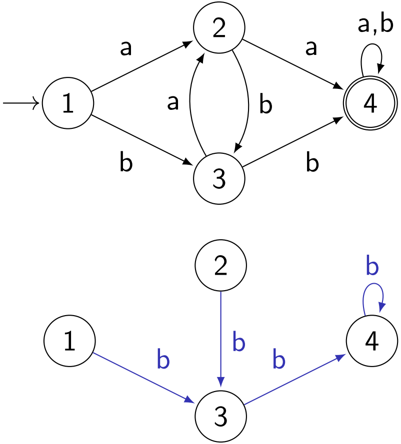

Syntactic monoid

	1	2	3	4
f_ε	1	2	3	4
f_a	2	4	2	4
f_b	3	3	4	4

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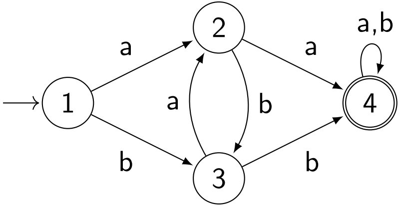

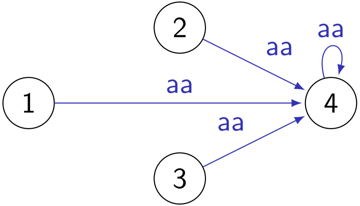

Syntactic monoid

	1	2	3	4
f_ε	1	2	3	4
f_a	2	4	2	4
f_b	3	3	4	4
f_{aa}	4	4	4	4

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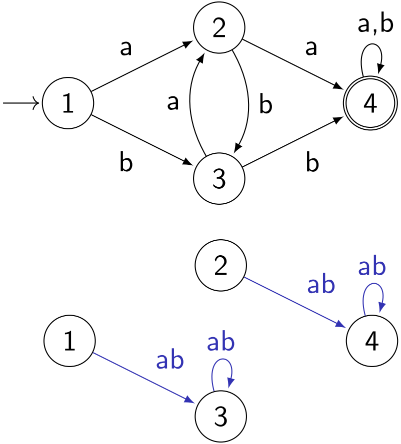

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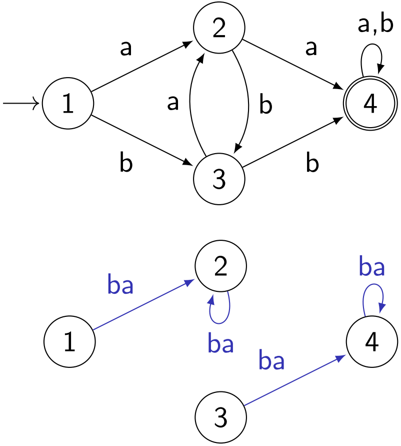

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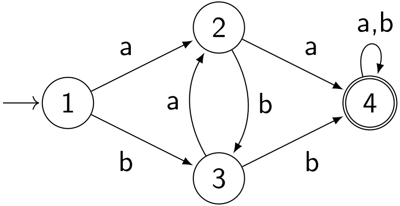

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$$\begin{aligned} f_{bb} &= f_{aa} \\ f_{aba} &= f_a \\ f_{bab} &= f_b \end{aligned}$$

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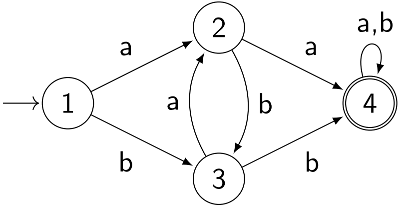

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$$f_{bb} = f_{aa}$$
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f_{ababab}

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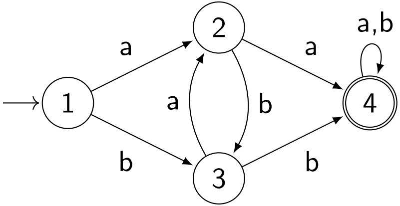

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$$f_a \circ f_{bab} \circ f_{ab}$$

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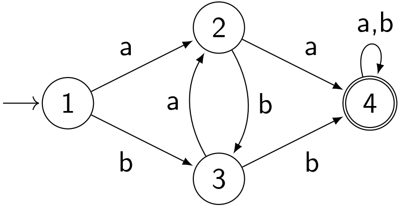

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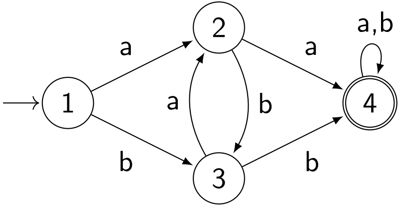

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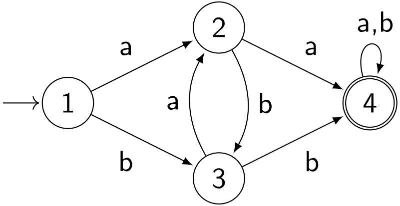

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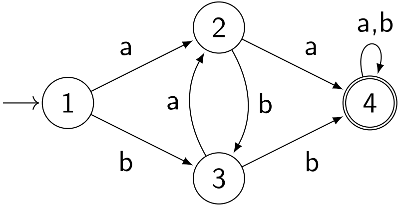

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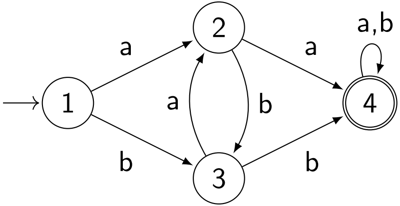

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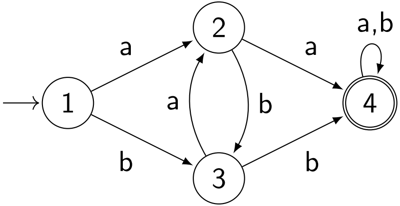

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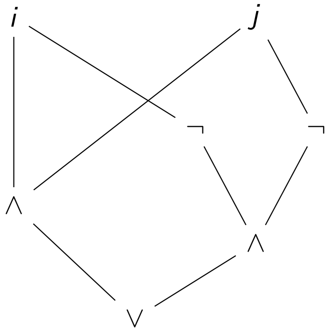
f_{ab}

We can look at algebraic properties instead of combinatorial ones
Ex: idempotents

Boolean circuits

Circuits



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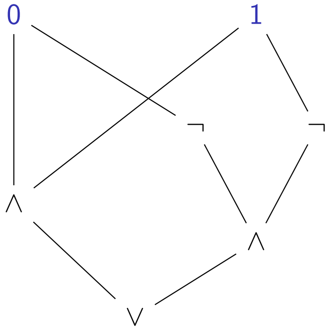

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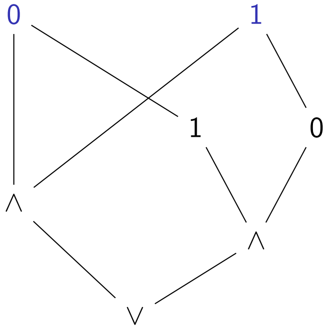

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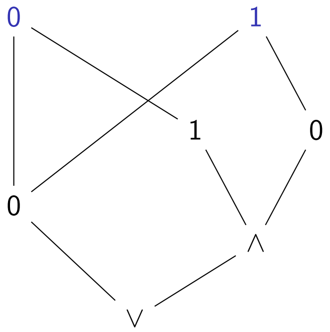

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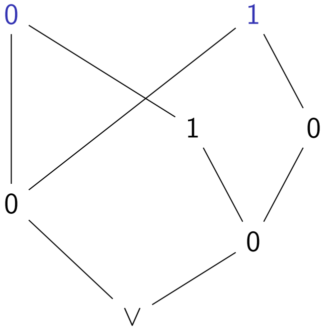

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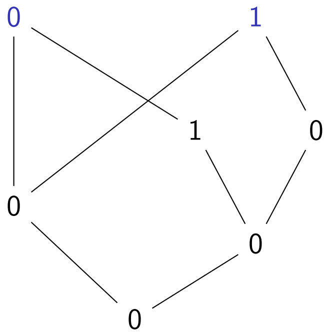
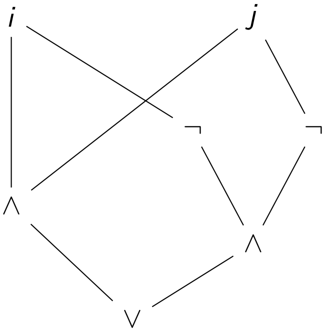
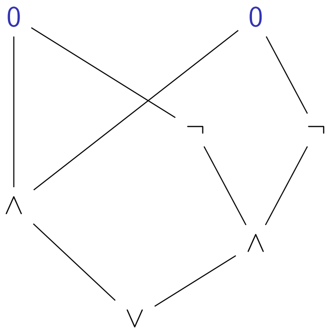


Circuits



Circuits



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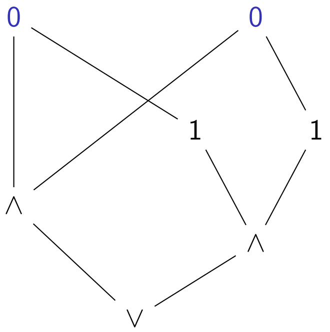

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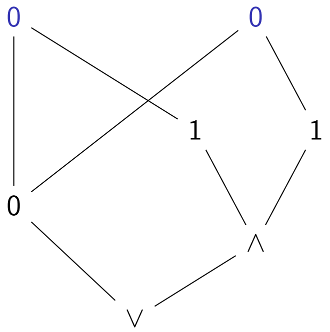

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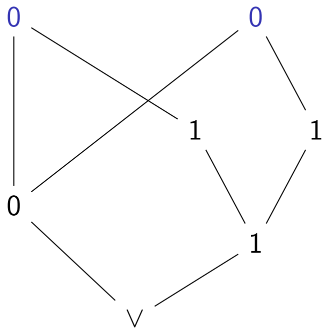

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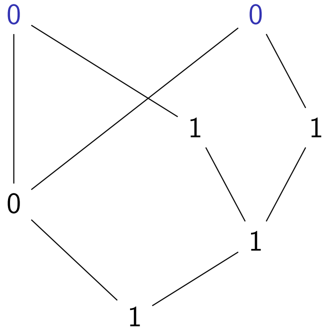

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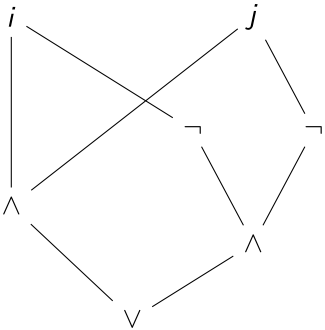
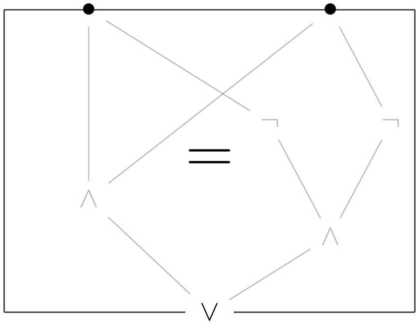


outputs $1 \Leftrightarrow i = j$

Circuits



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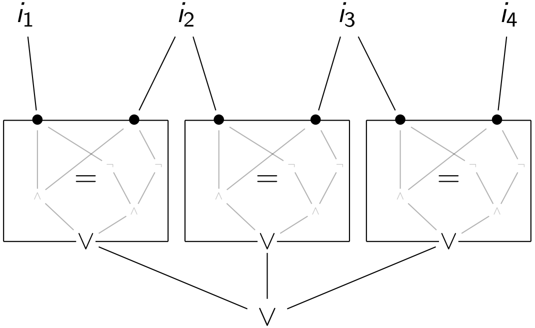

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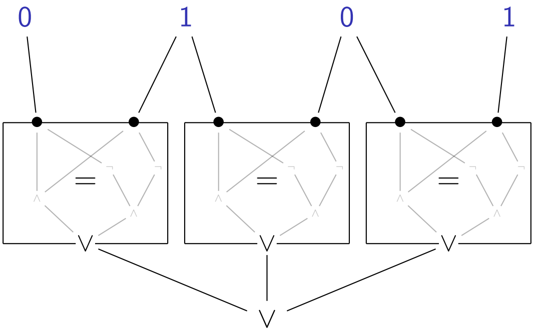

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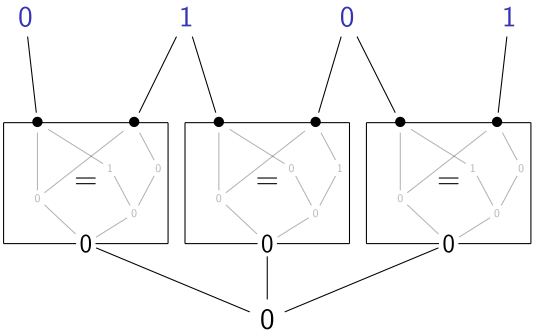

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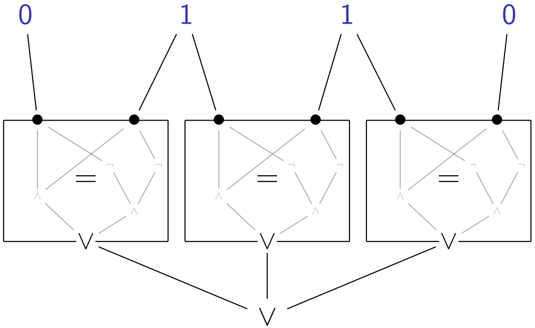

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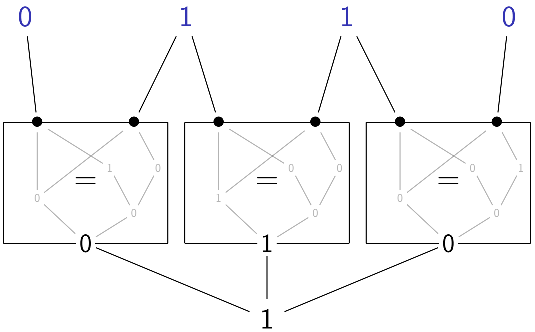

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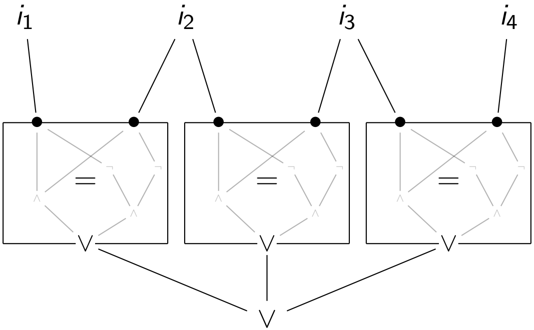
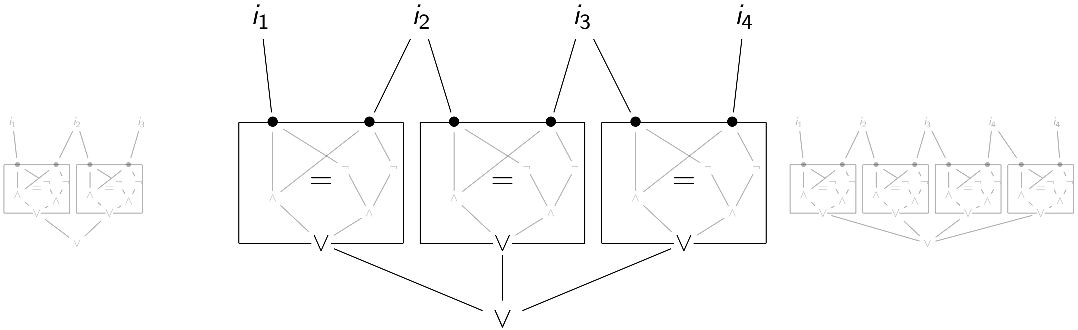


Computes Twice

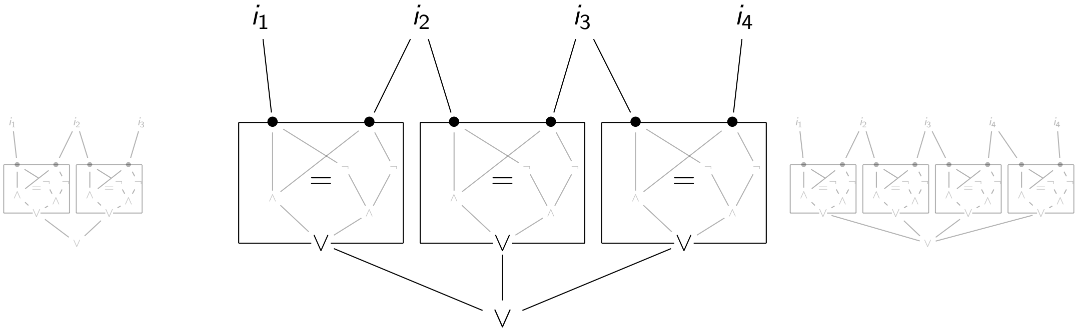
Circuits



Computes Twice

Circuits

$$\text{size}(n) = \text{number of gates}$$
$$\text{depth}(n) = \text{maximal length of a path}$$
$$\text{fan-in}(n) = \text{maximal in-degree of a gate}$$



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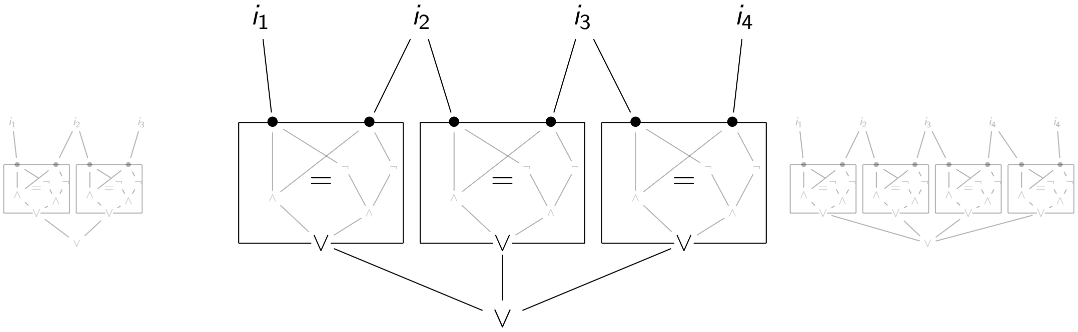

Computes Twice

$\text{size}(n) = 5 \cdot (n-1) + 1$ $\text{depth}(n) = 4$ $\text{fan-in}(n) = n - 1$

Regular languages in circuit classes

Which regular languages can be parallelized efficiently?

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$\rightarrow$ Study regular languages in circuit classes

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poly size
constant depth
unbounded fan-in
$+$ mod counting

poly size
constant depth
unbounded fan-in

poly size
log depth
bounded fan-in

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Even $\#$1s
Furst, Saxe, Sipser (1984)

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same languages
iff
same regular languages
Barrington (1992)

$\subsetneq$

$\subseteq$

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poly size constant depth unbounded fan-in	$\subsetneq$	poly size constant depth unbounded fan-in + mod counting	$\subseteq$	poly size log depth bounded fan-in

$\subseteq$

NL

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$\not\supseteq$

NEXP

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$\subseteq$

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log depth
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$\subseteq$

NL

Many more (bounded depth classes, addition, ...)

Logic

x, y two positions in a string:

$$\mathrm{eq}(x, y) = (a(x) \wedge a(y)) \vee (b(x) \wedge b(y))$$

Logic

x, y two positions in a string:

Letter predicate

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Formulas are interpreted over strings:

$$\exists x \; \exists y \; (x = y + 1) \wedge \mathrm{eq}(x, y)$$

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Quantification over a position

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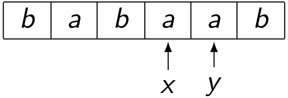

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b	a	b	a	a	b
			$\uparrow$ x	$\uparrow$ y	

Describes Twice

Logic

Fragments

Numerical predicates

Logic

Fragments

- First-order logic: FO

Numerical predicates

Logic

Fragments

- First-order logic: FO
- Bounded quantifier alternation: Σ_k, Π_k

Numerical predicates

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- Many more (restricting the number of variables, adding modular quantifiers,...)

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Fragments

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- Many more (restricting the number of variables, adding modular quantifiers,...)

Numerical predicates

- The order: $x < y$
- The successor predicate: $x + 1 = y$

Logic

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$$\exists x \exists y\ (x = y + 1) \wedge \mathrm{eq}(x, y) \in \Sigma_1[+1]$$

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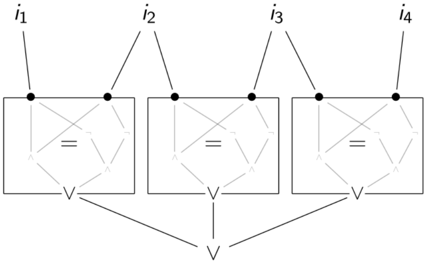
Arbitrary predicates: ARB

$\forall x,$  $(x) \Rightarrow a(x)$

$\in \Pi_1[\mathrm{ARB}]$

Logic and circuits



$\longleftrightarrow$

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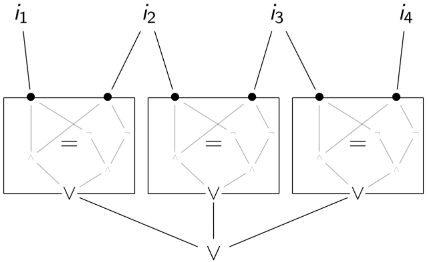
Logic and circuits



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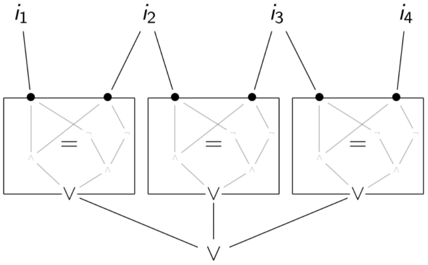
Constant depth circuits $\longleftrightarrow$ FO[ARB]

Logic and circuits



$\longleftrightarrow$ $\exists x \exists y\ (x = y + 1) \wedge \mathrm{eq}(x, y)$

Constant depth circuits $\longleftrightarrow$ FO[ARB]

depth $\longleftrightarrow$ quantifiers alternation

Straubing's properties

For $\mathcal{L}$ a logical fragment,

What are the regular languages of $\mathcal{L}[ARB]$ *?*

Straubing's properties

For $\mathcal{L}$ a logical fragment,

Are they decidable?

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True

- Σ_1
- FO
- FO with prime modular quantifiers

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constant depth + mod counting

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True	False
▶ Σ_1	▶ FO + S_5
▶ FO ←→ constant depth	▶ FO ∘ MOD
▶ FO with prime modular quantifiers	

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True	False	Open
► Σ_1 ► FO ←→ constant depth ► FO with prime modular quantifiers	► FO + S_5 ► FO ∘ MOD	► FO with composite modular quantifiers ► FO with two variables ► Σ_k, $k \geq 3$

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Circuits with
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(AND with MOD_6 gates)

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Linear size circuits (complexity of addition)

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FO ⟷ constant depth	FO ∘ MOD	FO with two variables
FO with prime modular quantifiers		Σ_k, $k \geq 3$

constant depth + mod counting

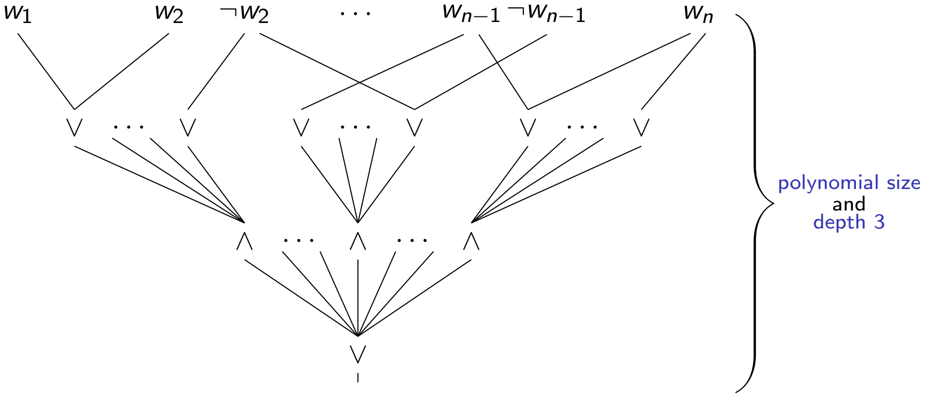
Chap 5: $\Sigma_2[\mathrm{ARB}] \cap \mathrm{Reg} = \Sigma_2[\mathrm{REG}]$

Linear size circuits (complexity of addition)

Straubing property for Σ_2

The circuit class Σ_2

$w = w_1 w_2 \cdots w_{n-1} w_n$



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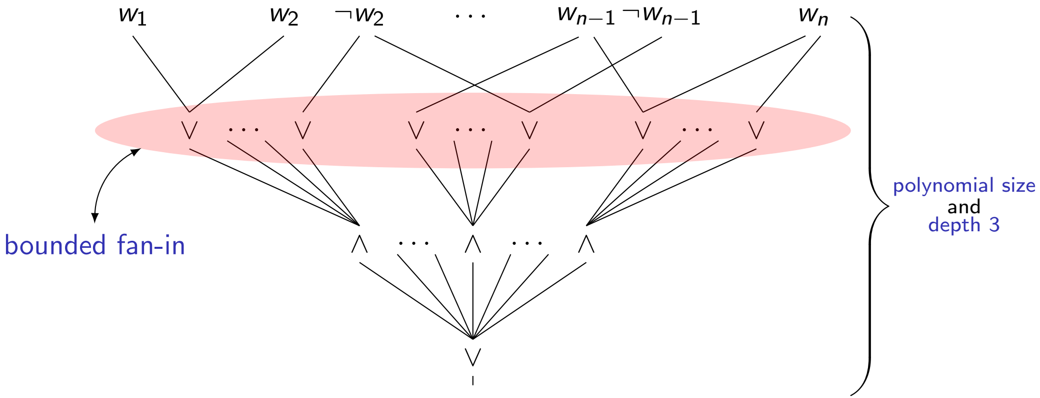

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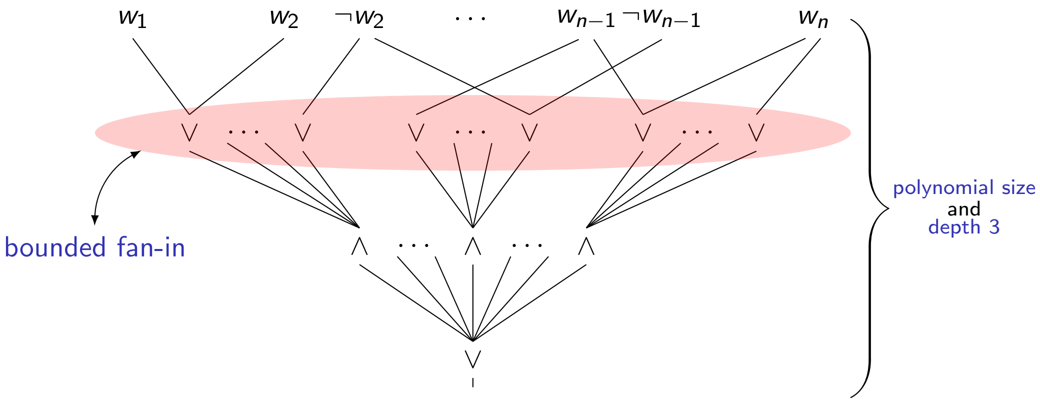

It is equivalent to Σ_2[ARB].

Theorem 5.34: (*Barloy*, Cadilhac, Paperman, Zeume)

$$\Sigma_2[\mathrm{ARB}] \cap \mathrm{Reg} = \Sigma_2[\mathrm{REG}]$$

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- $\supseteq$: Immediate
- $\subseteq$: Take a regular language not in $\Sigma_2[\mathrm{REG}]$, show that it is not in $\Sigma_2[\mathrm{ARB}]$

Theorem 5.34: (*Barloy*, Cadilhac, Paperman, Zeume)

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- $\supseteq$: Immediate
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- $\supseteq$: Immediate
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Proof sketch

Algebra | Lower bound

Proof sketch

Theorem (Pin, Weil)

$\mathcal{L}$ a language and M its syntactic monoid

Proof sketch

Theorem (Pin, Weil)

$\mathcal{L}$ a language and M its syntactic monoid

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iff

$$\forall x, y, z \in M,\ (xyz)^{|M|!} = (xyz)^{|M|!} y (xyz)^{|M|!}$$

Proof sketch

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$$\forall x, y, z \in M,\ (xyz)^{|M|!} = (xyz)^{|M|!} y (xyz)^{|M|!}$$

Decidable!

Proof sketch

Prop: (Håstad, Jukna, Pudlák)
With C a Σ_2 circuit for $\mathcal{L}$, top fan-in k
If for all large $A \subseteq \mathcal{L}$ there exists $u \notin \mathcal{L}$ s.t.

For any k pos of u, there is a string in A that matches u on these

Then C accepts a word outside of $\mathcal{L}$

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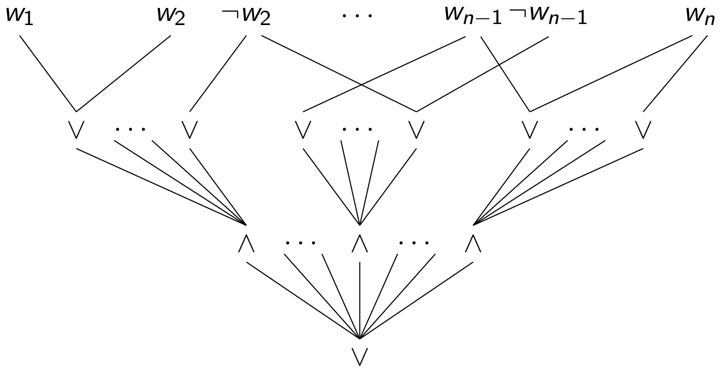

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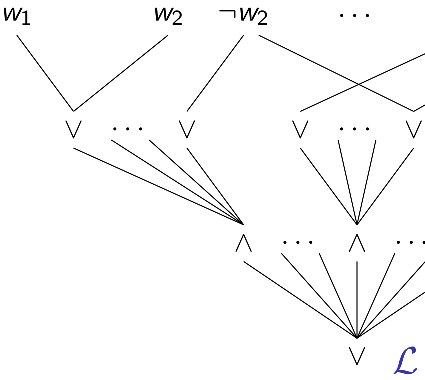

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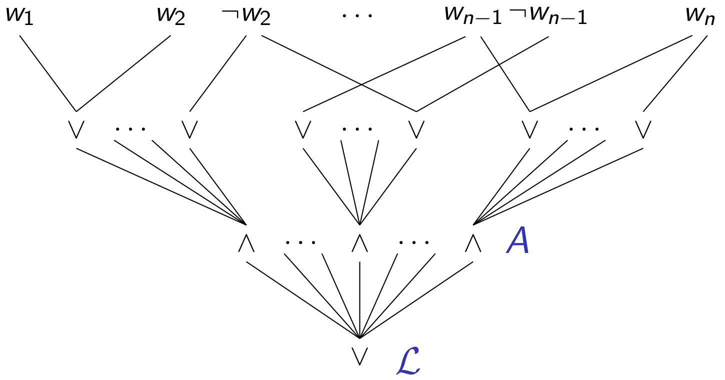

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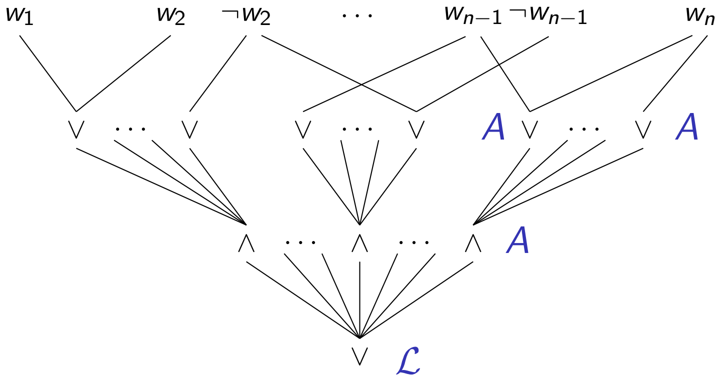

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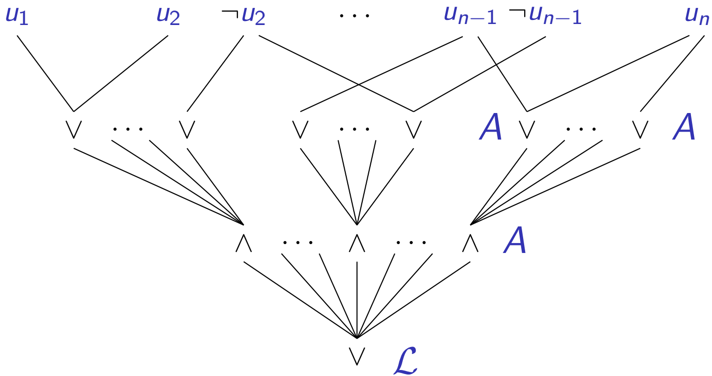

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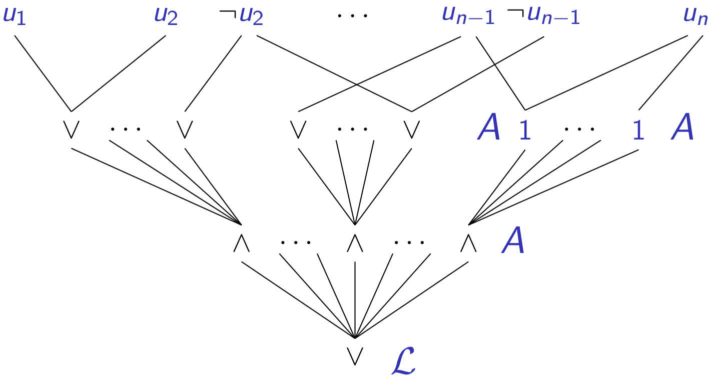

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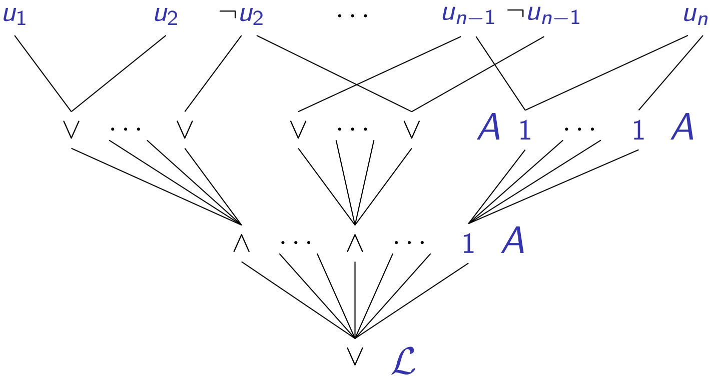

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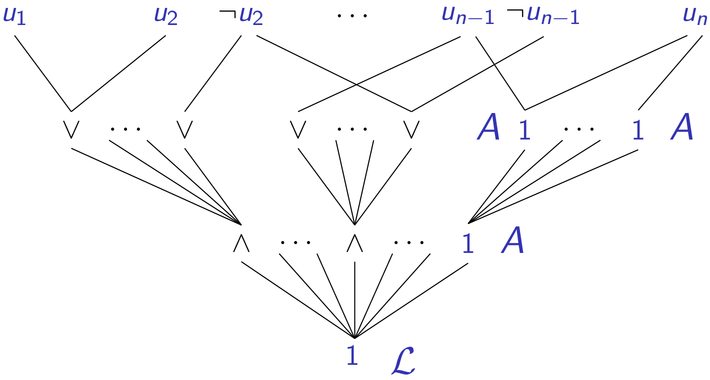

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Find it!

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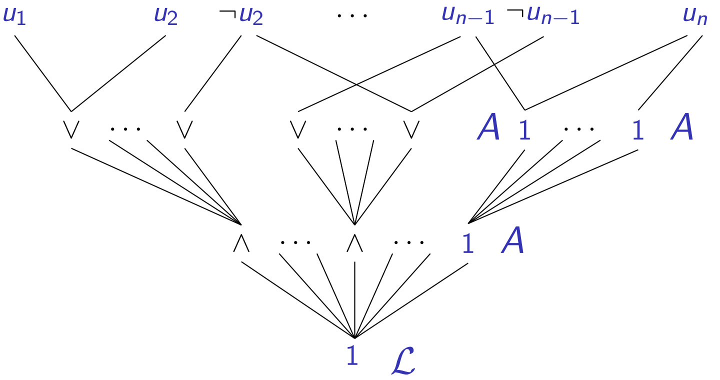

Other optimisations

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XML complexity: Process regular properties of large serialized trees

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$\rightarrow$ space trichotomy for path languages:

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Coauthors

Chap 6:
Murlak
Paperman

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Chap 7:
Tschirbs
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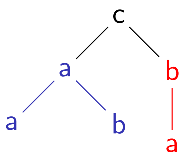
Thanks!

Appendix

XML complexity

XML encoding of trees:



$\langle c \rangle$
$\quad \langle a \rangle$
$\qquad \langle a \rangle \langle / a \rangle$
$\qquad \langle b \rangle \langle / b \rangle$
$\quad \langle / a \rangle$
$\quad \langle b \rangle$
$\qquad \langle a \rangle \langle / a \rangle$
$\quad \langle / b \rangle$
$\langle / c \rangle$

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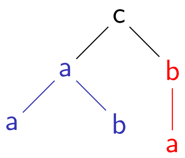

$\langle c\rangle$
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RPQs: the path from the root belongs to a given regular language.

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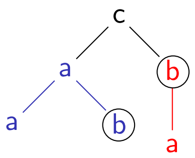

$\langle c\rangle$
$\langle a\rangle$
$\langle a\rangle\langle/a\rangle$
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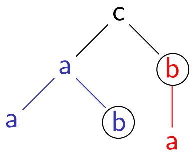

$\langle c \rangle$
$\langle a \rangle$
$\langle a \rangle \langle / a \rangle$
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Thm (B., Murlak, Paperman): Let $\mathcal{L}$ be a regular language. Its RPQ can be executed either in

- Constant memory.
- Logarithmic memory, but not less.
- Linear memory, but not less.

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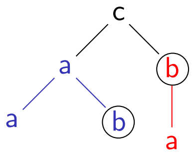

$\langle c \rangle$
$\quad \langle a \rangle$
$\quad\quad \langle a \rangle \langle / a \rangle$
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} Algebraic characterisations

Incremental complexity

a	b	b	a	b	a	a	b

$\in (a+b)^*aa(a+b)^*$

Incremental complexity

a	b	b	a	b	a	b	b

$\notin (a+b)^* aa (a+b)^*$

Incremental complexity

a	b	b	a	b	a	b	a

$\notin (a+b)^*aa(a+b)^*$

Incremental complexity

a	b	a	a	b	a	b	a

$\in (a+b)^* aa (a+b)^*$

Incremental complexity

a	b	a	a	b	a	b	a	$\in (a+b)^* aa (a+b)^*$

$\rightarrow$ use auxiliary data structures

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a	b	a	a	b	a	b	a

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RAM model

FO formulas

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RAM model (Linear preprocessing)	FO formulas

Incremental complexity

a	b	a	a	b	a	b	a

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NEW: characterization of this class for regular languages of trees

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Binary	Unary
Prop $\leftrightarrow$ Reg	

Incremental complexity

$$\boxed{a \mid b \mid a \mid a \mid b \mid a \mid b \mid a} \in (a+b)^* aa (a+b)^*$$

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Prop $\leftrightarrow$ Reg	Prop $\leftrightarrow$ Groups

Incremental complexity

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	FO^2 $\leftrightarrow$ Reg