Universality of unambiguous register automata

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Joint work with Lorenzo Clemente - MIMUW (Warsaw)

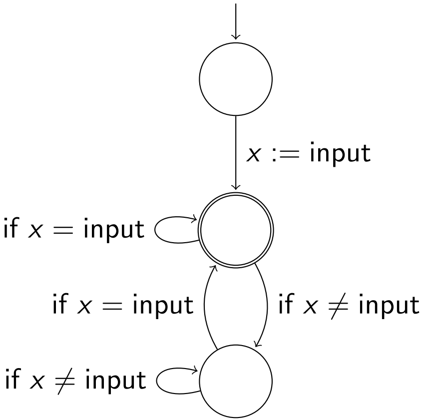
STACS 2021

1. Unambiguous register automata
2. Solving the universality problem

Unambiguous register automata

A first (deterministic) register automaton

x is a register that stores values from $\mathbb{N}$



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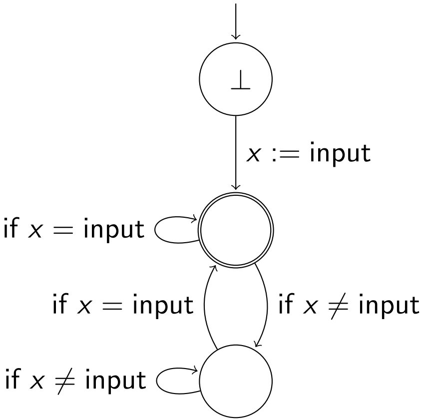

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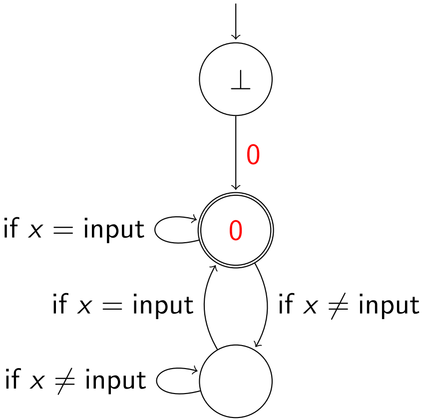

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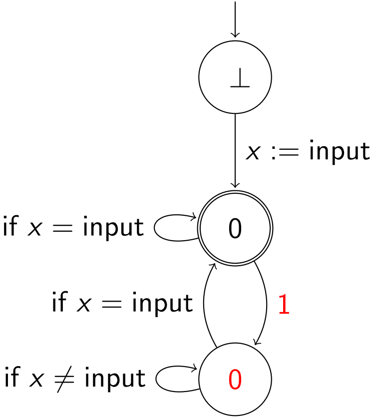

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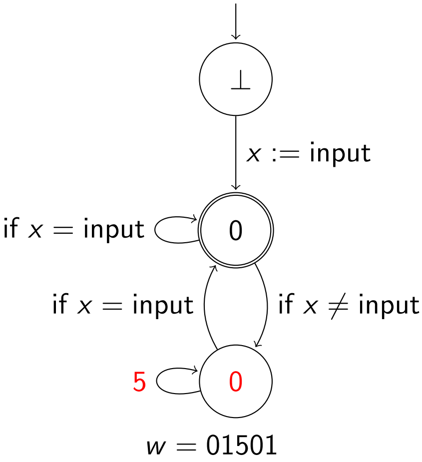

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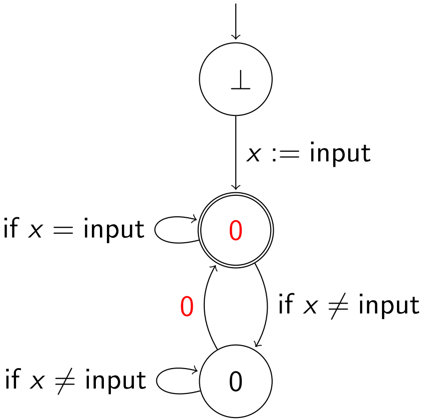

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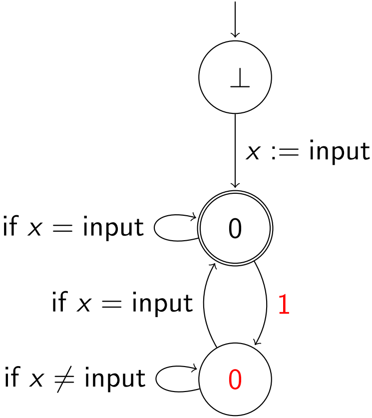

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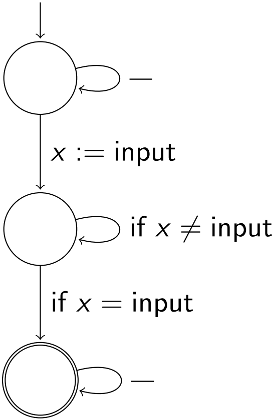
A second (non-deterministic) one



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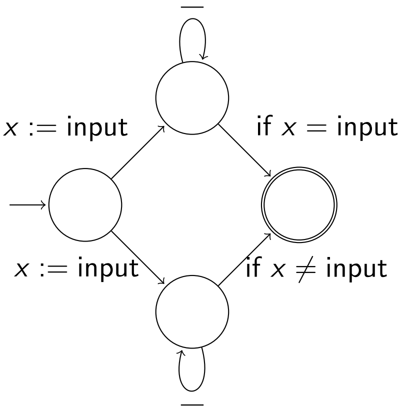

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This is quite surprising: for DCFG, Universality is decidable but not Inclusion!

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Theorem [Mottet, Quaas STACS'19]

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- Undecidable in general.
- Extend Schutzenberger counting approach.
- There is hope for generalisations to other data domains.

Solving the universality problem

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$\mathbb{N}^k$ has B_k many orbits; where B_k is the k^{th} Bell number.

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Ryll-Nardzewski function

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$F_s(n, \ ) = |\{o(r) \mid r \text{ is a run ending in } s, \text{ of length } n \qquad\qquad \}| \ .$

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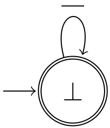
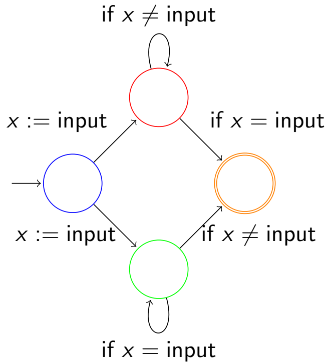


$$F(n+1,k+1) = F(n,k) + (k+1)\cdot F(n,k+1)$$

(Stirling numbers of the second kind)

Counting orbits



n = length of the word, k = number of distinct data values

$$
\begin{cases}
f_{\color{blue}\bullet}(n+1,k+1) = & 0 \\
f_{\color{red}\bullet}(n+1,k+1) = & f_{\color{blue}\bullet}(n,k) + (k+1)\cdot f_{\color{blue}\bullet}(n,k+1) + f_{\color{red}\bullet}(n,k) + k\cdot f_{\color{red}\bullet}(n,k+1) \\
f_{\color{green}\bullet}(n+1,k+1) = & f_{\color{blue}\bullet}(n,k) + (k+1)\cdot f_{\color{blue}\bullet}(n,k+1) + f_{\color{green}\bullet}(n,k+1) \\
f_{\color{orange}\bullet}(n+1,k+1) = & f_{\color{red}\bullet}(n,k+1) + f_{\color{green}\bullet}(n,k) + k\cdot f_{\color{green}\bullet}(n,k+1)
\end{cases}
$$

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- Linrec is closed by substraction.

Modelling linrec

$$f(n+1,k+1) - f(n+1,k) - f(n,k) - (k+1)\cdot f(n,k+1) = 0$$
$$\downarrow$$
$$\partial_1\partial_2 \cdot f - \partial_1 \cdot f - f - (k+1)\partial_2 \cdot f = 0$$
$$\downarrow$$
$$(\partial_1\partial_2 - \partial_1 - (k+1)\partial_2 - 1)\cdot f = 0$$

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Note that those operators do not commute:

$$\partial_1 p(n,k) = p(n+1,k)\partial_1 \neq p(n,k)\partial_1 \quad .$$

Modelling linrec

A system of equations with variables f_i, S and g, with coefficients that are operators.

$$
\left\{
\begin{array}{rrrrr}
(\partial_1\partial_2)\cdot f_{\color{blue}\bullet} & & & & = 0 \\
-\partial_2\cdot f_{\color{blue}\bullet} & +(\partial_1\partial_2-(k+1)\partial_2-1)\cdot f_{\color{red}\bullet} & & & = 0 \\
-\partial_2\cdot f_{\color{blue}\bullet} & & + \quad (\partial_1\partial_2-\partial_2)\cdot f_{\color{green}\bullet} & & = 0 \\
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 & (\partial_1\partial_2-(k+1)\partial_2-1)\cdot S & & & = 0 \\
 & g-S+f_{\color{orange}\bullet} & & & = 0
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\end{cases}
$$

Goal: find an operator L such that $L \cdot g = 0$.

Euclidean properties

Let L_1 and L_2 be two operators.

- It is possible to perform an Euclidean division of L_1 by L_2.
- There exists a common left multiple of L_1 and L_2. (Trivial for commutative ring!)

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With $L_1 = \partial_2 - (k+1)$ and $L_2 = \partial_2 - 1$:

$$\begin{aligned}
\mathrm{CLM}(L_1, L_2) &= (k\partial_2 - (k+1)) \cdot L_1 \\
&= (k\partial_2 - (k+1)^2) \cdot L_2 \\
&= k\partial_2^2 - (k^2 + 3k + 1)\partial_2 + (k+1)^2
\end{aligned}$$

The elimination process

$$\left\{\begin{array}{lllll}
(\partial_1\partial_2)\cdot f_{\textcolor{blue}{\bullet}} & & & & = 0 \\
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 & -\partial_2\cdot f_{\textcolor{red}{\bullet}} & +\quad(-(k-1)\partial_2-1)\cdot f_{\textcolor{green}{\bullet}} & +(\partial_1\partial_2)\cdot f_{\textcolor{orange}{\bullet}} & = 0 \\
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$$\Downarrow \text{ multiply lines 2 and 3 by } \partial_1$$

$$\left\{\begin{array}{lllll}
(\partial_1\partial_2)\cdot f_{\textcolor{blue}{\bullet}} & & & & = 0 \\
-\textcolor{red}{\partial_1}\partial_2\cdot f_{\textcolor{blue}{\bullet}} & +(\partial_1^2\partial_2-(k+1)\partial_1\partial_2-\partial_1)\cdot f_{\textcolor{red}{\bullet}} & & & = 0 \\
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$$\left\{\begin{array}{lclcll}
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 & + & (\partial_1^2\partial_2 - \partial_1\partial_2)\cdot f_{\color{green}\bullet} & & & = 0\\
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$\Uparrow$ add first line to second and third lines

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$$((k^2 - 5k + 6)\partial_1^3\partial_2^3 + (2k + 2)\partial_1^3\partial_2^2 - 2\partial_1^2\partial_2 + \partial_2^2 - (3k + 3)) \cdot g = 0$$

Deciding zeroness of g

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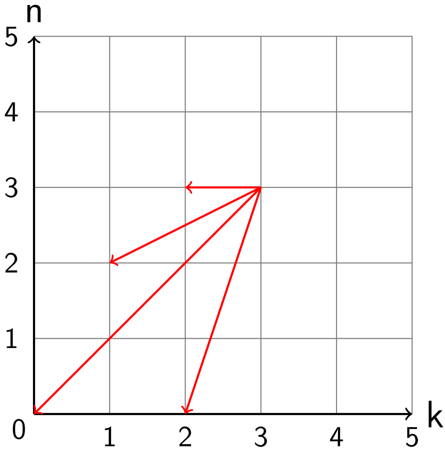

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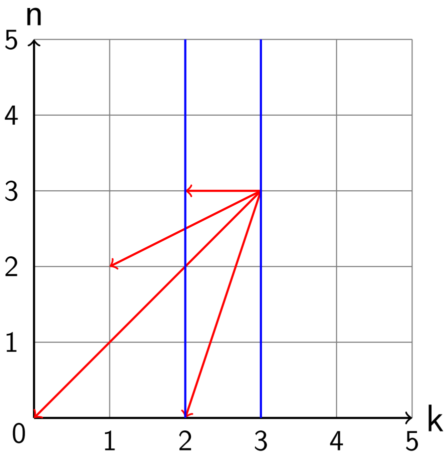

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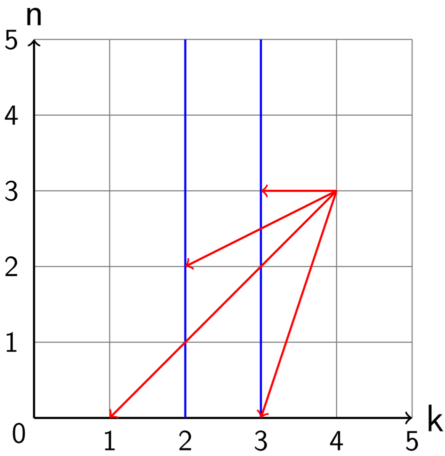

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$$2^{2^{2^{2^n}}}$$

Complexity

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$$2^{2^{2^{2^{n}}}}$$

- Lagrange bound on zero of polynomials.
- Zeroness of sections (dimension 1).

Hermite forms

Idea: Instead of removing one variable at a time, invert the matrix using (non-commutative) linear algebra.

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- Put matrices of operators in a triangular form (Hermite form):

$$\begin{pmatrix} (\partial_1 - 1)\partial_2 & -\partial_2 \\ -k\partial_2 - 1 & \partial_1\partial_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & (\frac{k}{\partial_1 - 1} - \partial_1)\partial_2 \\ 0 & \partial_2^2 - \frac{1}{\partial_1^2 - \partial_1 - (k+1)}\partial_2 \end{pmatrix}$$

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- Obtain a bound on the length of a short witness of non-universality.

Main theorems

Theorem [B.,Clemente 20]

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(The complexity becomes EXP-TIME and PTIME if the monicity conjecture holds)

Conclusion

- Improving the complexity. Monicity conjecture: monic cancelling relations suffice:

$$(\cancel{(k^2-5k+6)}\partial_1^3\partial_2^3 + (2k+2)\partial_1^3\partial_2^2 - 2\partial_1^2\partial_2 + \partial_2^2 - (3k+3)) \cdot g = 0$$

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- Extend to weighted automata.

M. Giesbrecht and M. S. Kim.
Computing the Hermite form of a matrix of Ore polynomials.
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