MATH 3400 - Group and Ring Theory
Assignment 8

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1 Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, and $H = \{(0,0),(2,0),(0,2),(2,2)\}$. Is G/H isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$?

We know that $|G/H| = |G|/|H| = \frac{4*4}{4} = 4$. G/H has 4 elements consisting of H, as follows,

$$
\begin{aligned}
(0,0) + H = H &= \{(0,0),(2,0),(0,2),(2,2)\} \\
(0,1) + H &= \{(0,1),(2,1),(0,3),(2,3)\} \\
(1,0) + H &= \{(1,0),(3,0),(1,2),(3,2)\} \\
(1,1) + H &= \{(1,1),(3,1),(1,3),(3,3)\}
\end{aligned}
$$

Each of these have order 2, so G/H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2 Let H be a normal subgroup of G and let a belong to G. If the element aH has order 4 in the group G/H and $|H| = 12$, what are the possibilities for the order of a in G?

Let $|aH| = n$ and $|a| = m$. Then, we have $(aH)^m = a^m H = eH = H$, and so n divides m. Also, there exists some integer t, such that $m = |a| = nt = |aH|t$, and so $|a^t| = \frac{m}{gcd(m,t)} = \frac{m}{t} = n$. Since we are given that $n = 4$, then $m = nt = 4t = |a|$. So we know that the possibilities for $|a|$ are orders that are multiples of 4 and divisors of $|H| = 12$. That is, $|a| = 4$.

3 Prove that there is no homomorphism from $\mathbb{Z}_8 \times \mathbb{Z}_2$ onto $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Let $G = \mathbb{Z}_8 \times \mathbb{Z}_2$, and $H = \mathbb{Z}_4 \times \mathbb{Z}_4$. We know that $\mathbb{Z}_8$ is a cyclic group of order 8, and that 1 generates the group. Also, $\mathbb{Z}_2$ is a cyclic group of order 2, and that 1 generates the group. So we have an element $(1,1) \in G$ with order $|(1,1)| = lcm(8,2) = 8$. For H, note that there is no element in H that has order more than 4. But we have just shown that there is an element in G with order 8. Then the mapping cannot be onto, thus G is not be isomorphic to H, and there is no homomorphism from G onto H.

4 How many homomorphisms are there from $\mathbb{Z}_{30}$ onto $\mathbb{Z}_{12}$? How many are there to $\mathbb{Z}_{12}$?

4.1 How many homomorphisms are there from $\mathbb{Z}_{30}$ onto $\mathbb{Z}_{12}$?

There are none, because 12 does not divide 30. (Long answer) Suppose (towards a contradiction) there exists homomorphism ϕ from $\mathbb{Z}_{30}$ onto $\mathbb{Z}_{12}$, then since ϕ is onto, there must exist some element $g \in \mathbb{Z}_{30}$ such that $\phi(g) = 1 \in \mathbb{Z}_{12}$ and $|\phi(g)| = |1|$ divides $|g|$. $\mathbb{Z}_{12}$ is cyclic, and 1 is a generator. For $\mathbb{Z}_{12} = \langle\ 1\ \rangle$, we get $|1| = 12$. This implies that 12 divides $|g|$, but that also means that $|g|$ divides $|\mathbb{Z}_{30}| = 30$ (that is, 12 divides 30; which is false), which is a contradiction. Therefore, ϕ must not be onto, and there are no homomorphisms from $\mathbb{Z}_{30}$ onto $\mathbb{Z}_{12}$.

4.2 How many are there to $\mathbb{Z}_{12}$?

Let $\phi : \mathbb{Z}_{30} \longrightarrow \mathbb{Z}_{12}$ be a homomorphism; it is completely defined by $\phi(1)$ since $\mathbb{Z}_{30}$ is cyclic. Then $|\phi(1)|$ divides both 30 and 12. Now, we need to find all possible orders for $\phi(1)$ in $\mathbb{Z}_{30}$. $|1| = 30$ in $\mathbb{Z}_{30}$, so $|\phi(1)|$ divides 30. Since elements of $\mathbb{Z}_{12}$ have order 1, 2, 3, 4, 6, or 12, $|\phi(1)|$ must be in $\{1, 2, 3, 6\}$.

Now, we find all elements in $\mathbb{Z}_{12}$ with orders in $\{1, 2, 3, 6\}$. They are:

$$\begin{aligned} |0| &= 1 \\ |2| &= 6 \\ |4| &= 3 \\ |6| &= 2 \\ |8| &= 3 \\ |10| &= 6 \end{aligned}$$

So, the possibles images for 1 in $\mathbb{Z}_{30}$ are in $\{0, 2, 4, 6, 8, 10\}$; there are 6 of them, (that is, 6 choices for $\phi(1)$; what we map 1 to). Therefore, there are 6 homomorphisms from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{12}$.