MATH 3400 - Group and Ring Theory
Assignment 7

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1 Suppose that $G_1 \cong G_2$ and $H_1 \cong H_2$. Prove that $G_1 \times H_1 \cong G_2 \times H_2$.

Let $\alpha : G_1 \longrightarrow G_2$ and $\beta : H_1 \longrightarrow H_2$ be isomorphisms (as given). We wish to show that $G_1 \times H_1 \cong G_2 \times H_2$. We proceed with the 4 step isomorphism test:

Step 1

$$\begin{aligned} \alpha \times \beta : G_1 \times H_1 &\longrightarrow G_2 \times H_2 \\ (g, h) &\longrightarrow (\alpha(g), \beta(h)) \end{aligned}$$

Step 2 We need to prove $\alpha \times \beta$ is one-to-one. Suppose $\alpha \times \beta(a, b) = \alpha \times \beta(g, h)$. Then, $(\alpha(a), \beta(b)) = (\alpha(g), \beta(h))$ and $\alpha(g) = \alpha(h)$, and also $\beta(b) = \beta(h)$. Since α and β are isomorphisms, and thus are both one-to-one, we have $a = g$ and $b = h$. Then, $(a, b) = (g, h)$, so $\alpha \times \beta$ is one-to-one.

Step 3 We need to prove $\alpha \times \beta$ is onto. Let $(g, h) \in G_2 \times H_2$ be arbitrary. This means, $g \in G_2, h \in H_2$. Then, $\exists a \in G_1$ and $\exists b \in H_1$ such that $\alpha(a) = g$ and $\beta(b) = h$ (since α and β are isomorphisms, and thus are both onto). Then, $(a, b) \in G_1 \times H_1$ and $\alpha \times \beta(a, b) = (\alpha(a), \beta(b)) = (g, h)$, thus $\alpha \times \beta$ is onto.

Step 4 We need to show that $\alpha \times \beta$ is operation preserving. Let $(a, b), (g, h) \in G_1 \times H_1$ be arbitrary, then since both α and β are operation preserving, we have,

$$\begin{aligned} \alpha \times \beta((a, b)(g, h)) &= \alpha \times \beta(ag, bh) \\ &= (\alpha(ag), \beta(bh)) \\ &= (\alpha(a)\alpha(b), \beta(b)\beta(h)) \\ &= ((\alpha(a), \beta(b)), (\alpha(g), \beta(h))) \\ &= (\alpha \times \beta(a, b), \alpha \times \beta(g, h)) \end{aligned}$$

So $\alpha \times \beta$ is operation preserving as well. Thus be our 4-step isomorphism test, we can conclude that if $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \times H_1 \cong G_2 \times H_2$, as desired.

2 How many elements of order 25 does $\mathbb{Z}_5 \times \mathbb{Z}_{25}$ have?

We have $\mathbb{Z}_5 \times Z_{25}$. Using the fact: $\varphi(p^n) = p^n - p^{n-1}$ for any odd prime p, we know there are $\varphi(25) + \varphi(5) = (25 - 5) + (5 - 1) = 24$ elements of order 5, and we know there are $5 * 25 = 125$ elements in total (one of which is the identity) in $\mathbb{Z}_5 \times \mathbb{Z}_{25}$, then the number of elements of order 25 must be $\varphi(25) + \varphi(5) * \varphi(25) = 20 + 4 * 20 = 100$.

3 How many cyclic subgroups of order 4 does $\mathbb{Z}_4 \times \mathbb{Z}_2$ have?

First, note that (1) $\mathbb{Z}_4 \times \mathbb{Z}_2$ has order 8, (2) every subgroup of order 2 must be cyclic, (3) the only subgroup order 1 is that containing the identity element, (4) the only subgroup order 8 is the whole group, and (5) the only other subgroups have order 4, since 4 divides the order of the group. We are interested in knowing the number of cyclic subgroups of order 4. Since each subgroup of order 4 has 2 elements of order 4, and two cyclic subgroups of order 4 have an element of order 4 in common, we know there are $\frac{4}{2} = 2$ cyclic subgroups order 4.

In particular, for $\mathbb{Z}_4 \times \mathbb{Z}_2$, there are 3 subgroups of order 4, but only two are cyclic. Take the subgroup: $\{(2, 0), (0, 1), (2, 1), (0, 0)\}$, which is not cyclic, which supports our claim above there are 2 cyclic subgroups of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_2$.

4 Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup is also normal in G.

We have $N = \langle\ n\ \rangle, n \in G,\ N \trianglelefteq G$. Let $H \leq N$ be an arbitrary subgroup. We wish to show that if N is cyclic, then $H \trianglelefteq G$ also. We start with $N \trianglelefteq G \implies \forall g \in gng^{-1} = n^m$ for some integer m. That is, all elements of $n \in N$ have the form n^m. We have some arbitrary $H \leq N = \langle\ n\ \rangle$, so $H = \langle\ n^k\ \rangle$ for some integer k. That is, all elements of $H \leq N$ have the form n^k for some integer k. It is easy to see that when $|n| = 1$, $H = N = \{1\}$, and $H \trianglelefteq G$.

Note when $|n| = \infty$, we have the desired $H \trianglelefteq G$, so we will omit this and prove for finite $|n|$. For $1 < |n| < \infty$, since $N \trianglelefteq G$, $\forall g \in G$, we have $gNg^{-1} = N = \langle\ gng^{-1}\ \rangle$, such that $\forall g \in G, \langle\ gng^{-1}\ \rangle = \langle\ n\ \rangle$. So gng^{-1} is a generator of $\langle\ n\ \rangle, \forall g \in G$.

Now, $\forall g \in G, \langle\ gng^{-1}\ \rangle = \langle\ n\ \rangle \implies gng^{-1} = n^j$ for some integer j such that $gcd(|n|, j) = 1$. Then, $gng^{-1} = n^j \implies gn^kg^{-1} = n^{jk}$, and also

$g\langle\ n^k\ \rangle g^{-1} = \langle\ n^{jk}\ \rangle$.

We have $\langle\ n^k\ \rangle = \langle\ n^{jk}\ \rangle, \langle\ n^k\ \rangle \supseteq \langle\ n^{jk}\ \rangle$, we wish to show $\langle\ n^k\ \rangle \subseteq \langle\ n^{jk}\ \rangle$ is also true. We have $gcd(|n|, j) = 1$, so $\exists s, t \in \mathbb{Z}$ such that $1 = s|n| + tj$. Equivalently, $k = ks|n| + ktj$.

$$\begin{aligned}
n^k &= n^{ks|n|} n^{ktj} \\
&= (n^{|n|})^{ks} n^{ktj} \\
&= n^{ktj} \quad \text{(by } n^{|n|} = 1\text{)} \\
&= (n^k)^{tj}
\end{aligned}$$

So we have $n^k = (n^k)^{tj} \implies \langle\ n^k\ \rangle \subseteq \langle\ n^{jk}\ \rangle$, as desired. So $\langle\ n^k\ \rangle = \langle\ n^{jk}\ \rangle$.

Thus, $\forall g \in G, g\langle\ n^k\ \rangle g^{-1} = \langle\ n^k\ \rangle$ and $gNg^{-1} = H$ so $H \trianglelefteq G$.