Mystery Hats

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Problem

A group of 100 individuals are playing a game. Each individual simultaneously rolls a regular six-sided die. The participants are sitting in a circle in such a way so that each of them cannot see their own die roll, but can see the rolls of all ninety-nine of the other individuals. After all the dice rolls, each of the hundred individual writes down a number from one through six. The participants may choose their number based on the dice rolls of everyone else, but are not allowed to communicate in any way or to see what the others are writing. The hundred individual are then simultaneously examined. The group wins if every individual correctly guessed the number of his or her own die; if even one individual writes down a number that does not match his or her die roll, the group loses. What strategy maximizes the group's chances of winning?

A Lower Bound

It should be clear that the probability that any one individual guesses his or her roll correctly is $\frac{1}{6}$. In general, if the probability of event A happening is p and the probability of event B happening is q, then the probability of events A and B happening is pq. Therefore, the probability that every individual in the group of 100 guesses randomly and they all happen to guess correctly is $\left(\frac{1}{6}\right)^{100}$ which is essentially 0.

In what follows, it'll be easier to consider the 2 case, where there are 2 individuals instead of 100. Here, the probability that each of them guesses randomly and correctly is $\frac{1}{36}$. Our question becomes how to better their chances.

Playing the Odds

It might seem that we can do no better than $\frac{1}{36}$. After all, if you and I are playing, then your roll should have no effect on my roll, so seeing what you roll should have no affect on my guess. It is true that there is no causal relationship between your roll and my roll, but that does not mean that your roll does not afford me some information about what my roll is likely to be. For consider the sums of our rolls. Combined, our rolls will sum to some number between 2 and 12. In particular, of the 36 possible combinations (the enumeration here is left as an exercise to the reader), there is only 1 way to make 2 and 1 way to make 12. But there are multiple ways for our sum to be 3 (I roll a 1 and you roll at 2 or I roll a 2 and you roll a 1). As it turns out, 7 is the most likely sum, and there are exactly 6 ways our rolls can sum to 7.

Therefore even before we roll, if someone should ask me what I think the sum of our dice will be, my best bet would be to say 7. But it follows that whatever your roll is, I should still expect the sum to be 7, so if you roll an n I should guess my roll to be $7-n$. Similarly, if I roll an m, you should guess your roll to be $7-m$. This will increase our chances of being successful from $\frac{1}{36}$ to $\frac{1}{6}$.

The General Case

How does this generalize? We must first calculate the expected roll of any one individual. On an $n-$sided die, the expected value is the sum of the value of the sides divided by the number of sides. For us, it's 3.5. The expected value for the sum of the 100 rolls is $100 * 3.5 = 350$. Our strategy, then, is to add up the rolls of the other 99 individuals and guess a number between 1 and 6 such that the total sum is as close to 350 as possible. A full justification of this is left to the reader.

A Note About the Two Case

It was pointed out to me by a colleague of mine that for the 2 case, another strategy matches our proposed strategy. The strategy is to guess the same value as the value you see your partner roll. If you and I are playing and you see that I roll a 3 then you guess 3 as well. Why does this yield a $\frac{1}{6}$ success rate? Because there are 6 possible ways our rolls are equal, and therefore we have a $\frac{6}{36}$ or $\frac{1}{6}$ chances of both guessing correctly. This strategy, however, obviously does not generalize in the same way ours does.