Mystery Hats

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Problem

Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by (an independent) coin toss. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats (but not their own), the players must simultaneously guess the color of their own hats or pass. What strategy maximizes the probability that at least one person guesses correctly and no-one guesses incorrectly?

A Lower Bound

Here is a strategy that provides a lower bound which any successful strategy should hope to surpass. Call the three players in the game A, B, and C. The strategy is for A and B to always skip and for C to guess randomly between a red hat or a blue hat. Because A and B are always skipping, they will never guess incorrectly. C will (assuming C can in fact produce random guesses) guess correctly 50% of the time and incorrectly 50% of the time. Our lower bound is therefore 50%, so our strategy should aim to better this.

A Winning Strategy Through Enumeration

Let's enumerate the set of possibilities respectively for A, B, and C, where r is red and b is blue.

1. r, r, r
2. r, r, b
3. r, b, r
4. r, b, b
5. b, r, r
6. b, r, b
7. b, b, r
8. b, b, b

Here is the strategy: each person looks at what the other two are wearing. If the other two are wearing the same colored hat, then guess the opposite color; otherwise, (if the other two players are wearing different colored hats) skip. For example, if A sees that B and C are both wearing r then she guesses b. If A sees that B is wearing r but C is wearing b then she skips. Notice that scenario (1) will fail, since each individual will guess r. Similarly, scenario (8) will fail, since each individual will guess b. But in every other case, the group will succeed. Consider scenario (2). Here, A and B will both skip, since A sees B and C who are wearing different colored hats, and B sees A and C who are wearing different colored hats. But C will see A and B who are both wearing r, so C will guess b successfully. This holds for the other 5 cases. In short then, $\frac{6}{8}$ scenarios will be successful, so the overall rate of success will be $\frac{3}{4}$, or 75%.