Bertrand's Box Paradox

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Problem

You are given three boxes:

1. A box with two gold coins
2. A box with two silver coins
3. A box with one gold coin and one silver coin

Suppose a box is chosen at random and a coin is chosen at random from that box. Given that the selected coin is gold, what is the probability that the other coin in the box is also gold?

An Intuitive Answer

Given that we have three boxes, gold-gold (GG), silver-silver (SS), and gold-silver (GS), and given that we know the selected coin is gold, we can rule out SS. Therefore because the gold coin must have come from either GG or GS and the box which was selected was randomly selected, the probability that it came from GG is the same as the probability that it came from GS, so the probability that the other coin in the box is gold (that is, the probability that the box is GG) is simply $\frac{1}{2}$.

Conditional Probability

In fact, the reasoning in the above section is incorrect. In particular, the probability that the box is GG is not the same as the probability that the box is GS, given that we know the selected coin is gold.

Let us think, for a second, about the following question: given one of the three boxes, what is the probability that the coin I draw at random is gold? For SS, this probability is 0. Since there is no gold coin in SS, I cannot possibly draw a gold coin from that box. For GG, this probability is 1. Since both coins are gold in GG, any coin I draw from that box must be gold. For GS, this probability is $\frac{1}{2}$. Since there is one gold coin and one silver coin and I am drawing randomly, the probability of me picking the gold coin is the same as the probability of me picking the silver coin. From these three facts, I know the following (I will write "C_G" to mean "choosing a gold coin" and I will use "|" to mean "given"):

1. $P(C_G \mid SS) = 0$
2. $P(C_G \mid GG) = 1$
3. $P(C_G \mid GS) = \frac{1}{2}$

The probability that the box we have chosen is GG given that the randomly selected coin from that box is gold is given by:

$$\frac{P(C_G \mid GG)}{P(C_G \mid SS) + P(C_G \mid GG) + P(C_G \mid GS)} = \frac{1}{0 + 1 + \frac{1}{2}} = \frac{2}{3}$$

We conclude that the probability that the box we have randomly selected is GG given that the coin we randomly selected from it is gold, is $\frac{2}{3}$.

An Intuitive Explanation

Another way of thinking about this problem is as follows. Let us consider the number of ways we can draw a gold coin from a randomly selected box. Clearly, we rule out SS. For GS, there is only one way I can do this. For GG, however, there are two ways I can do this. I can first select coin one (which is obviously gold), or I can selected coin two (which is also obviously gold). Therefore there are three possible ways I can select a gold coin given a random box. Two of those possible ways yield GG, so the probability that I have selected GG given that selected coin is gold is $\frac{2}{3}$.