Assignment 2, Problem 3

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Problem

Solve this Frog Puzzle: using what you have learnt about Markov Processes so far.

Solution

There are n steps to the bank, which is located at position n. Let the state of the frog at time t be represented by S_t where $S_t = \{0, 1, ..., n\}$ is the current position of the frog. At each time step, the frog hops with uniform probability to a new state, such that $S_{t+1} \in \{S_t + 1, S_t + 2, ..., n-1, n\}$. Then this can be described as a Markov Process since:

$$(Pr(S_{t+1} = s|S_0 = s_0, S1 = s_1, ..., S_t = s_t) = Pr(S_{t+1} = s|S_t = s_t) = \frac{1}{n - s_t}$$

$$\forall s \in \{s_t + 1, s_t + 2, ..., n-1, n\}$$

Define the reward function for a given future state s as :

$$R(s) = E[R_{t+1}|S_t = s_t] = 1$$

Thus, after each step taken the reward function increases by one. We can then define the value function as the expected sum of all rewards (no discount) starting from state s before termination at time T when $S_T = n$:

$$V(s) = 1 + \sum_{s'=s+1}^{n} P(s, s') \cdot V(s')$$

$$\forall s \in \{0, 1, ..., n-1\}$$

which is simply the number of steps required to reach the bank. Note that $V(n) = 0$, since the process immediately terminates. With this knowledge, one could solve recursively by hand from $V(n-1)$ to $V(n-2)$ to $V(0)$. Alternatively, we can write the value function in the form of a linear algebra equation and solve directly for $V(s)$:

$$V = (I - P)^{-1} \cdot R$$

where P is the $(n-1)$ by $(n-1)$ transition matrix given by:

$$P = \begin{pmatrix} 0 & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ 0 & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 0 & 0 & \frac{1}{n-2} & \cdots & \frac{1}{n-2} \\ \vdots & & & \ddots & & \vdots \\ \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Solving this yields (for $n = 10$):

$$V(s) = \begin{pmatrix} 2.92896825 \\ 2.82896825 \\ 2.71785714 \\ 2.59285714 \\ 2.45 \\ 2.28333333 \\ 2.08333333 \\ 1.83333333 \\ 1.5 \\ 1 \\ 0 \end{pmatrix}$$