On stratification diagrams, algorithmic spectrum estimates and vector-like pairs in F-theory

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With M. Cvetič, R. Donagi, L. Lin, M. Liu, F. Rühle – 2020.06***

Motivation

Obtain (MS)SM from String theory construction . . .

- $E_8 \times E_8$ [Candelas Horowitz Strominger Witten '85], [Greene Kirklin Miron Ross '86], [Braun He Ovrut Pantev '05], [Bouchard Donagi '05], [Anderson Gray He Lukas '10], . . .
- Type II [Berkooz Douglas Leigh '96], [Aldazabal Franco Ibanez Rabadan Uranga '00], [Ibanez Marchesano Rabadan '00], [Blumenhagen Kors Lust Ott '01], [Cvetič Shiu Uranga '01],
- F-theory [Krause Mayrhofer Weigand '12], [Cvetič Klevers Pena Oehlmann Reuter '15], [Lin Weigand '16], [Cvetič Lin Liu Oehlmann '18], [Cvetič Halverson Lin Liu Tian '19], . . .

. . . including vector-like spectra

- Why vector-like spectra? Higgs fields matter & are characteristic feature of QFTs
- $E_8 \times E_8$: [Bouchard Donagi '05], [Braun He Ovrut Pantev '05], [Anderson Gray Lukas Palti '10 & '11], . . .
- F-theory: [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

Outline

In this talk

- Recent progress to understand *vector-like spectra* in F-theory
- Based on
  - Machine learning (c.f. L. Lin at *String pheno 2020*)
  - Analytic insights (Brill Noether theory, stratifications . . .)
- Today: Focus on analytics

Outline

1. Revision: How to count vector-like spectra in F-theory?
2. Analytics of jumps

Vector-like spectra in F-theory [M.B. Mayrhofer Pehle Weigand '14], [M.B. Mayrhofer Weigand '17], [M.B. '18]

- Gauge degrees **localized** on 7-branes $S \subset \mathcal{B}_3$
- Zero modes **localized** on matter curves $C_{\mathbf{R}} \subset S$
- G_4-flux and matter surface $S_{\mathbf{R}}$ define line bundle $\mathcal{L}_{\mathbf{R}}$ on $C_{\mathbf{R}}$
- Vector-like pairs:

$$\text{massless chiral modes} \leftrightarrow h^0(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$$
$$\text{massless anti-chiral modes} \leftrightarrow h^1(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$$

- Typically, $h^i(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}})$ hard to determine:
  - By definition – non-topological data
  - Oftentimes, $\mathcal{L}_{\mathbf{R}}$ not pullback from $\mathcal{B}_3$

    Coherent sheaves on $\mathcal{B}_3$ $\leftrightarrow$ Freyd categories [S. Posur '17], [M.B., S. Posur '19]
  - Deformation $C_{\mathbf{R}} \to C'_{\mathbf{R}}$ can lead to jumps

$$h^i(C_{\mathbf{R}}, \mathcal{L}_{\mathbf{R}}) = (h^0, h^1) \to h^i(C'_{\mathbf{R}}, \mathcal{L}'_{\mathbf{R}}) = (h^0 + a, h^1 + a)$$

Strategy

Geometric setup

- Realistic F-theory geometries computationally too involved

$\Rightarrow$ Learn from simpler geometries first

- Choice of geometry:

$$\text{Curve} \leftrightarrow C(\mathbf{c}) = V(P(\mathbf{c})) \ \text{hypersurface in } dP_3$$

$$\text{Line bundle} \leftrightarrow \mathcal{L}(\mathbf{c}) = \mathcal{O}_{dP_3}(D_L)|_{C(\mathbf{c})}$$

Challenge

Find $h^0(C(\mathbf{c}), \mathcal{L}(\mathbf{c})) \equiv h^0(\mathbf{c})$ as function of the complex structure $\mathbf{c}$

How to find $h^0\left(C(\mathbf{c}), \mathcal{L}\right) \equiv h^0(\mathbf{c})$?

1. Pullback line bundle admits Koszul resolution:

$$0 \to \mathcal{O}_{dP_3}(D_L - D_C) \xrightarrow{P(\mathbf{c})} \mathcal{O}_{dP_3}(D_L) \to \mathcal{L} \to 0$$

2. Obtain long exact sequence in sheaf cohomology:

$$\begin{array}{l} 0 \longrightarrow H^0\left(D_L - D_C\right) \longrightarrow H^0\left(D_L\right) \longrightarrow H^0\left(\mathcal{L}\right) \\ \hookrightarrow H^1\left(D_L - D_C\right) \longrightarrow H^1\left(D_L\right) \longrightarrow H^1\left(\mathcal{L}\right) \\ \hookrightarrow H^2\left(D_L - D_C\right) \longrightarrow H^2\left(D_L\right) \longrightarrow 0 \longrightarrow 0 \end{array}$$

3. Sometimes: $0 \to H^0(\mathcal{L}) \to H^1(D_L - D_C) \xrightarrow{M_\varphi(\mathbf{c})} H^1(D_L) \to H^1(\mathcal{L}) \to 0$
4. By exactness: $h^0(\mathcal{L}) = \ker(M_\varphi(\mathbf{c}))$

$\Rightarrow$ Study $\ker\left(M_\varphi(\mathbf{c})\right)$ as function of complex structure $\mathbf{c}$

Example: $g = 3$, $\chi = 1$ ($d = 3$)

- $C(\mathbf{c}) = V(P(\mathbf{c}))$ and $P(\mathbf{c}) = c_1 x_1^3 x_2^3 x_3^2 x_4 + \cdots + c_{12} x_3^2 x_4 x_5^3 x_6^3$
- For $D_L = H + 2E_1 - 2E_2 - E_3$ find

$$0 \to H^0(\mathcal{L}) \to \mathbb{C}^3 \xrightarrow{M_\varphi(\mathbf{c})} \mathbb{C}^2 \to H^1(\mathcal{L}) \to 0 \,, \quad M_\varphi = \begin{pmatrix} c_3 & c_2 & c_1 \\ 0 & c_{12} & c_{11} \end{pmatrix}$$

- $h^0(\mathcal{L}) = 3 - \mathrm{rk}(M_\varphi(\mathbf{c}))$ & stratification of curve geometries:

$\mathrm{rk}(M_\varphi)$	explicit condition	curve splitting
2	$(c_3 c_{11}, c_3 c_{12}, c_2 c_{11} - c_1 c_{12}) \neq \mathbf{0}$	C^1
1	$c_3 = 0,\ c_2 c_{11} - c_1 c_{12} = 0$	C^2
1	$c_1 = c_2 = c_3 = 0$	$B_2 \cup \mathbb{P}^1_b$
1	$c_{11} = c_{12} = 0$	$\mathbb{P}^1_a \cup B_1$
0	$c_1 = c_2 = c_3 = c_{11} = c_{12} = 0$	$\mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b$

Stratification diagram

$$
\begin{tikzcd}
h^0 = 1 & & \boxed{C^1} \arrow[dl] \arrow[d] \arrow[dr] & \\
h^0 = 2 & \boxed{\mathbb{P}^1_a \cup B_2} \arrow[dr] & \boxed{C^2} \arrow[d] & \boxed{B_1 \cup \mathbb{P}^1_b} \arrow[dl] \\
h^0 = 3 & & \boxed{\mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b} &
\end{tikzcd}
$$

Types of jumps

- Brill-Noether theory: C^2 smooth, irreducible but line bundle divisor **special**
- Curve splittings: Factoring off $\mathbb{P}^1_a$, $\mathbb{P}^1_b$ leads to jump

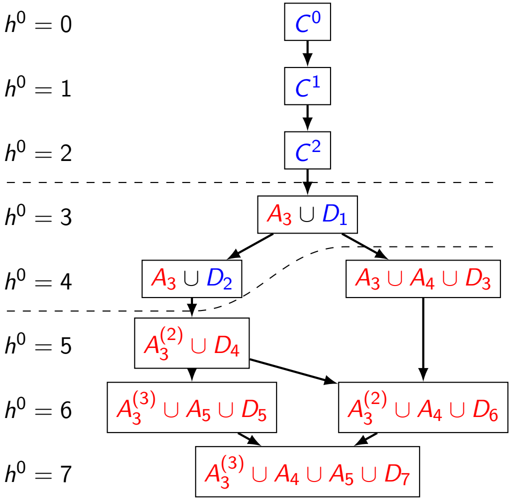
Example 2: $g = 5$, $\chi = 0$ $(d = 4)$

- $P(\mathbf{c}) = c_1 x_1^3 x_2^4 x_3^2 x_4^2 + \cdots + c_{16} x_3^3 x_4 x_5^4 x_6^3$
- $D_L = H + E_1 - 4E_2 + E_3$
- Koszul resolution gives

$$h^0(\mathcal{L}) = 7 - \mathrm{rk}(M_\varphi(\mathbf{c}))$$

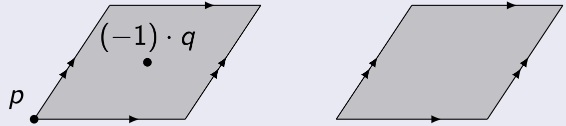
$$M_\varphi = \begin{pmatrix} c_{15} & c_{11} & c_7 & 0 & 0 & 0 & 0 \\ 0 & c_{10} & c_6 & c_3 & c_{11} & c_7 & 0 \\ c_{12} & c_6 & c_3 & 0 & c_7 & 0 & 0 \\ 0 & c_5 & c_2 & 0 & c_6 & c_3 & c_7 \\ c_8 & c_2 & 0 & 0 & c_3 & 0 & 0 \\ 0 & c_{14} & c_{11} & c_7 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & c_3 \end{pmatrix}$$

$\Rightarrow$ Study $\mathrm{rk}(M_\varphi(\mathbf{c}))$ as function of $\mathbf{c}$



Brill-Noether theory [1874 Brill, Noether] – more modern exposition in [Mumford '75], [Griffiths, Harris '94] ...

Example on torus $C_1 \cong \mathbb{C}/\Lambda = \mathrm{Jac}(C_1)$



$$h^0\left(\mathcal{O}_{C_1}(p-q)\right) = 0 \quad \to \quad h^0\left(\mathcal{O}_{C_1}(0)\right) = 1$$

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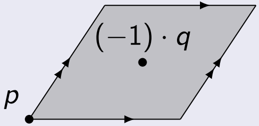

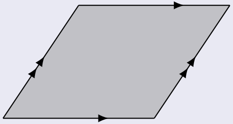

$$G_0^0 = \{\mathcal{L} \, , \; d = n = 0\} \cong \{q \in \mathbb{C}/\Lambda \, , \; q \neq 0\}$$

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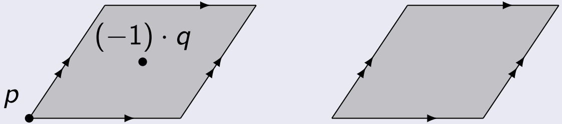

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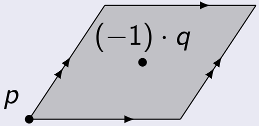

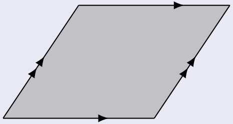

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General picture

- Abel-Jacobi map gives $\varphi_d \colon \mathrm{Div}_d(C) \to \mathrm{Jac}(C) \cong \mathbb{C}^g/\Lambda$
- $G_d^n = \{\varphi_d(\mathcal{L})\,,\; h^0(C,\mathcal{L}) = n\} \subseteq \mathrm{Jac}(C)$
- $\dim G_d^n \geq \rho(d,n,g) = g - n \cdot (n + \chi)$
- $\dim G_d^n = \rho$ for **generic** curves [1980 Griffiths, Harris]

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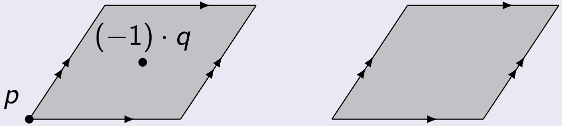

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h^0	h^1	ρ
0	0	1
1	1	0
2	2	-3

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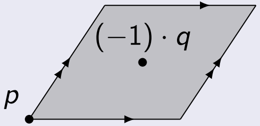

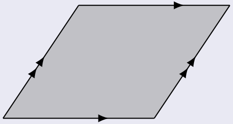

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- $\dim G_d^n = \rho$ for **generic** curves [1980 Griffiths, Harris]

$\Rightarrow$ Upper bound for h^0 on generic curves [Watari, 16]

h^0	h^1	ρ
0	0	1
1	1	0
2	2	-3

Gluing *local* sections

$$\begin{tikzcd}
h^0 = 1 & & C^1 \arrow[dl] \arrow[d] \arrow[dr] & \\
h^0 = 2 & \mathbb{P}^1_a \cup B_2 \arrow[dr] & C^2 \arrow[d] & B_1 \cup \mathbb{P}^1_b \arrow[dl] \\
h^0 = 3 & & \mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b &
\end{tikzcd}$$

Gluing *local* sections

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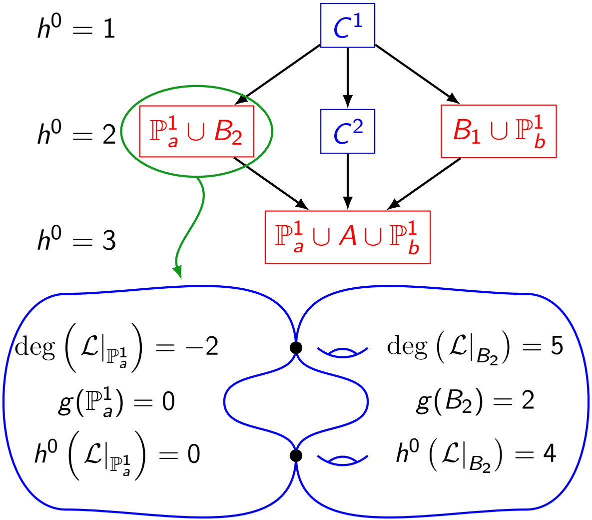
$$

Gluing *local* sections



Gluing *local* sections II

$$
\begin{tikzcd}
h^0 = 1 & & \boxed{C^1} \arrow[dl] \arrow[d] \arrow[dr] & \\
h^0 = 2 & \boxed{\mathbb{P}^1_a \cup B_2} \arrow[dr] & \boxed{C^2} \arrow[d] & \boxed{B_1 \cup \mathbb{P}^1_b} \arrow[dl] \\
h^0 = 3 & & \boxed{\mathbb{P}^1_a \cup A \cup \mathbb{P}^1_b} &
\end{tikzcd}
$$

Gluing *local* sections II

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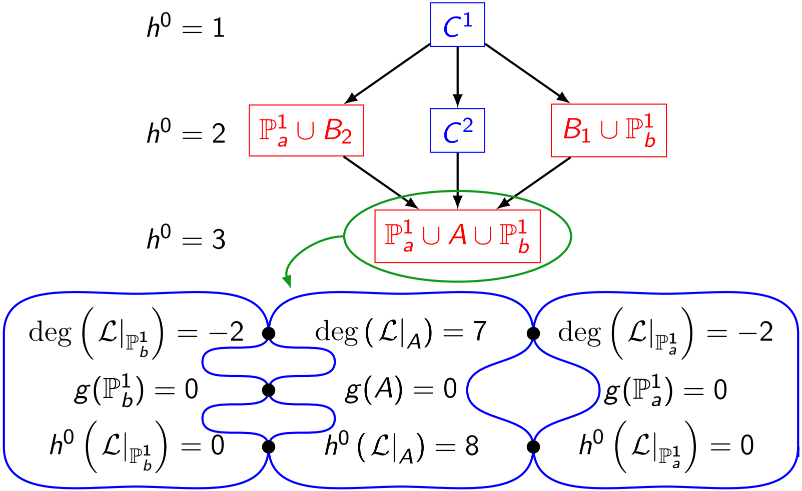


Quality assessment of counting procedure

- Quick: Uses only topological data (genus, chiral index)
- **But:** Relative position of bundle divisor and intersections of curve components matters [Cayley 1889, Bacharach 1886]

$\Rightarrow$ Systematically **over**estimates # of independent conditions

$\Rightarrow$ Obtain **under**estimate # of global sections

- Application to our data base:
  - 83 pairs (D_C, D_L) with complex structure deformations: $\sim 1.8 \times 10^6$ data sets
  - Counting procedure can be applied to $\sim 38\%$
  - Accuracy $\sim 98.5\%$
- Lead-offs:
  1. Sufficient conditions for jump
  2. Algorithmic h^0-spectrum estimate

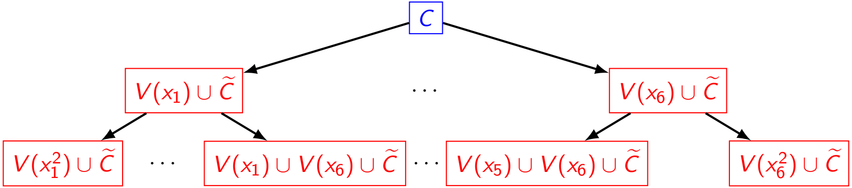
Algorithmic estimate for h^0-spectrum

$\boxed{C}$

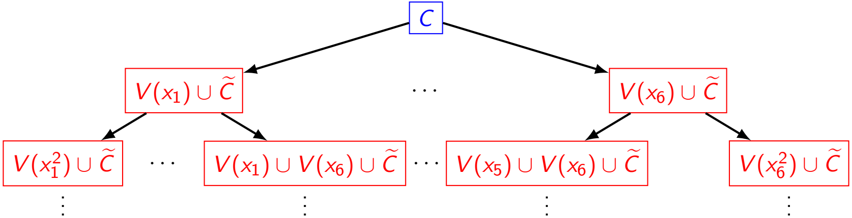
Algorithmic estimate for h^0-spectrum

$$
\begin{tikzcd}
 & \boxed{C} \arrow[dl] \arrow[dr] & \\
\boxed{V(x_1) \cup \widetilde{C}} & \cdots & \boxed{V(x_6) \cup \widetilde{C}}
\end{tikzcd}
$$

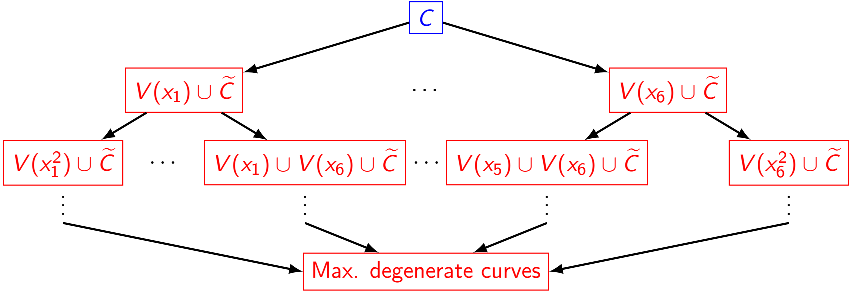
Algorithmic estimate for h^0-spectrum



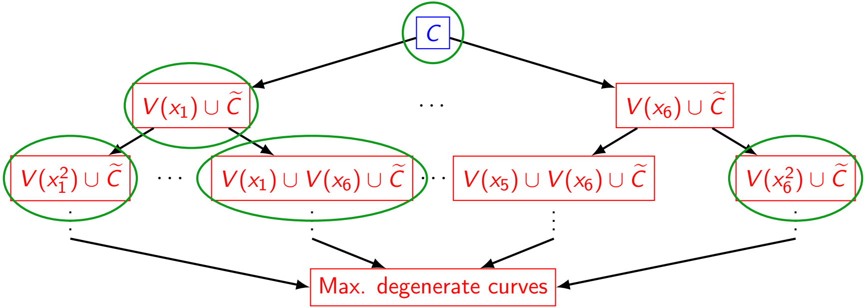
Algorithmic estimate for h^0-spectrum



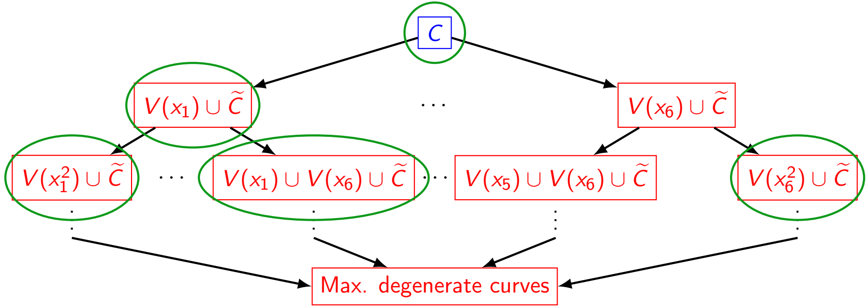
Algorithmic estimate for h^0-spectrum



Algorithmic estimate for h^0-spectrum



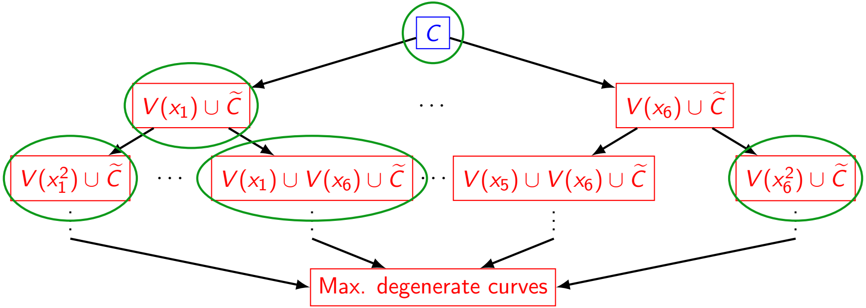
Algorithmic estimate for h^0-spectrum



https://github.com/homalg-project/SheafCohomologyOnToricVarieties

- Estimate h^0-spectrum from lower bounds at **subset of nodes**
- Implemented in package *H0Approximator*

Algorithmic estimate for h^0-spectrum



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- Estimate h^0-spectrum from lower bounds at **subset of nodes**
- Implemented in package *H0Approximator*
- Caveat: Check that $\widetilde{C}$ is irreducible

Summary

- Computing vector-like spectra in global F-theory models is hard
- We study how vector-like spectrum changes over moduli space of curve ($\leftrightarrow$ qualitatively different from prevous bundle cohomology studies)
- Insights from interplay between
  - Machine learning techniques (decision trees)
  - Analytic insights (Brill-Noether theory, stratification diagrams)
- Finding in dP_3: Factor off (rigid) $\mathbb{P}^1$s $\leftrightarrow$ jumps
- Results:
  1. Formulate sufficient condition for jump
  2. Implement quick (mostly based on topological data) h^0-spectrum approximator

     H0Approximator: https://github.com/homalg-project/SheafCohomologyOnToricVarieties/

Outlook

- Technical extensions:
  - non-pullback bundle and “fractional” bundles
  - stratification for several curves in one global F-theory model
- Conceptual:
  - Vector-like spectra for pseudo-real representations
  - Non-vertical G_4 (flux moduli dependence!)
  - (Geometric) symmetries protecting vector-like pairs
- Practical:
  - model building
  - (S)CFTs
  - swampland program

Thank you for your attention!