Open string methods and Gopakumar-Vafa invariants

Andrea Sangiovanni
University of Trieste
May 25th 2021

Seminar series on String Phenomenology

based on *A. Collinucci, AS, R. Valandro [21]*
A. Collinucci, M. De Marco, AS, R. Valandro [21]

Context

Punchline:

Get insight and concretely handle computations of physically relevant mathematical objects using string theory techniques

Outline and motivation

- What are the Gopakumar-Vafa topological invariants (a.k.a **GV invariants**)?

Outline and motivation

- What are the Gopakumar-Vafa topological invariants (a.k.a **GV invariants**)?
- Why is it interesting for physicists to count them?

Outline and motivation

- What are the Gopakumar-Vafa topological invariants (a.k.a **GV invariants**)?
- Why is it interesting for physicists to count them?
- Idea: find an easy and physics-based way of **computing (genus zero) GV invariants for a class of non-toric CY**

Outline and motivation

- What are the Gopakumar-Vafa topological invariants (a.k.a **GV invariants**)?

- Why is it interesting for physicists to count them?

- Idea: find an easy and physics-based way of **computing (genus zero) GV invariants for a class of non-toric CY**

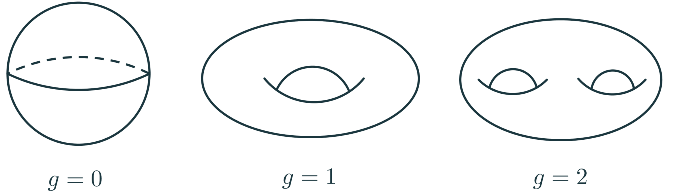
  Strategy:

  - Use M-theory/type IIA duality to reduce the problem to a more familiar setting
  - Tachyon condensation formalism to compute open string spectrum
  - **Open string spectrum $\Longleftrightarrow$ GV invariants**

What are GV invariants?

Topological invariants: properties of an object that are invariant under homeomorphism

Trivial example: **genus**



For applications in string theory: interested in topological invariants of **Calabi-Yau manifolds**

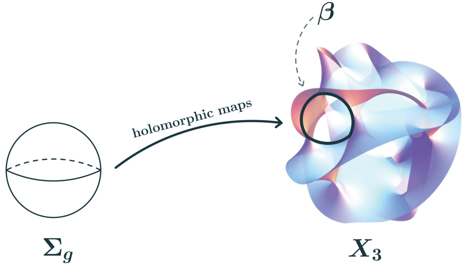
GV invariants

"Mathematical" definition of GV invariant

Consider:

- a Riemann surface Σ_g of genus g
- a Calabi-Yau threefold X_3 (i.e. 3 complex-dimensional)
- a 2-cycle β inside the CY ($\beta \in H_2(X_3, \mathbb{Z})$)

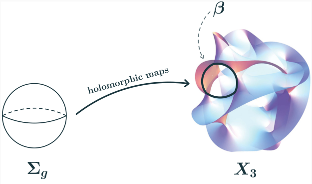
How many **holomorphic** maps from Σ_g to $\beta \in X_3$?



GV invariants

- How many **holomorphic** maps from Σ_g to $\beta \in X_3$?

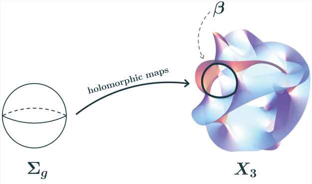
They are counted by the
Gromov-Witten (GW) invariants



GV invariants

- How many **holomorphic** maps from Σ_g to $\beta \in X_3$?

They are counted by the
Gromov-Witten (GW) invariants



GW invariants are related (*Gopakumar, Vafa* [98]) to the:

GV invariants n^g_β

We are interested in genus 0 GV invariants n^0_β (i.e. the Riemann surface is just a sphere)

GV invariants

- Why do physicists care about them?

$\boldsymbol{n_\beta^g}$ gives the number of BPS states of M2-branes wrapping the curve β

- $\boldsymbol{n_{d\beta}^g}$ counts bound states of d M2-branes wrapping the β class
- $\boldsymbol{n_\beta^0}$ gives instantonic corrections to the Kähler potential in type II theories (*Gopakumar-Vafa* [98]):

$$K = \underbrace{P(moduli)}_{perturbative} + \underbrace{\sum_\beta P'(n_\beta^0, moduli)}_{instanton\ correction}$$

GV invariants

How to compute the GV invariants?

- For toric CY threefolds there are known techniques: e.g. the topological vertex (*Aganagic, Klemm, Marino, Vafa* [03])
- For some non-toric cases there are available methods (*Katz* [06], *Donovan, Wemyss* [13], *Toda* [14])

How to compute the GV invariants?

- For toric CY threefolds there are known techniques: e.g. the topological vertex (*Aganagic, Klemm, Marino, Vafa* [03])
- For some non-toric cases there are available methods (*Katz* [06], *Donovan, Wemyss* [13], *Toda* [14])

We are interested in singular non-toric CY threefolds that arise as one-parameter families of deformations of ADE singularities

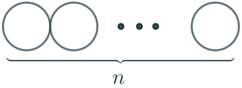
- They often admit a $\mathbb{C}^*$-fibration (with or without orientifolding) and this will be crucial in tackling the problem of computing their GV invariants

ADE singularities

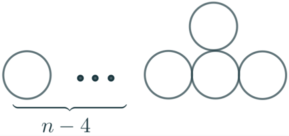
- ADE surfaces are singular spaces classified in terms of the exceptional divisors pattern in their resolution

Example:

$$A_n : x^2 + y^2 = z^{n+1}$$



$$D_n : x^2 + zy^2 + z^{n-1} = 0$$



We consider singular threefolds where only 1 $\mathbb{P}^1$ can be blown up (simple flops)

ADE singularities

- Deforming ADE singularities with terms depending on a single parameter w gives a threefold.

Example:

$$\underbrace{x^2 + y^2 = z^2}_{A_1 \text{singularity}} - \underbrace{w}_{\text{def}} \qquad \text{family is NOT SINGULAR}$$

Conifold

- Weyl theory dictates how we can choose deformation parameters in order to achieve a singularity and a specific resolution pattern:

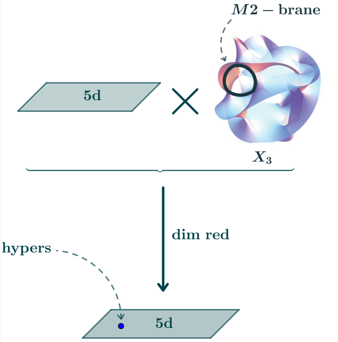
$$\underbrace{x^2 + y^2 = z^2}_{A_1 \text{singularity}} - \underbrace{w^2}_{\text{def}} \qquad \text{family is SINGULAR}$$

M-theory/IIA duality

Consider **M-theory** on a singular threefold arising from a one-parameter deformed ADE singularity

The EFT in 5d contains **hypermultiplets** descending from M2-branes wrapped on holomorphic curves (*Witten* [96])

- These are the states that we want to count



M-theory/IIA duality

- Write the threefold in the form of a $\mathbb{C}^*$-fibration:

$$uv = \det T(z,w)$$

$\mathbb{C}^*$-action: $\quad (u,v,z,w) \to (\mu u, \mu^{-1} v, z, w)$

- The S^1 in the $\mathbb{C}^*$-action is the M-theory circle
- Example (**conifold**): $\quad x^2 + y^2 = z^2 - w^2 \;\;\Rightarrow\;\; uv = z^2 - w^2$

$$\begin{array}{ccc} \textbf{M-theory} & & \textbf{type IIA} \\ \updownarrow & \Longleftrightarrow & \updownarrow \\ uv = \det T(z,w) & & \text{D6-branes on } \det T(z,w) = 0 \end{array}$$

- $\det T(z,w) = 0$ is the locus where the $\mathbb{C}^*$-fiber degenerates

M-theory/IIA duality

- Write the threefold in the form of a $\mathbb{C}^*$-fibration:

$$uv = \det T(z,w)$$

$\mathbb{C}^*$-action: $\quad (u,v,z,w) \to (\mu u, \mu^{-1} v, z, w)$

- The S^1 in the $\mathbb{C}^*$-action is the M-theory circle
- Example (**conifold**): $\quad x^2 + y^2 = z^2 - w^2 \;\; \Rightarrow \;\; uv = z^2 - w^2$

M-theory		**type IIA**
$\updownarrow$	$\Longleftrightarrow$	$\updownarrow$
$uv = \det T(z,w)$		D6-branes on $\det T(z,w) = 0$

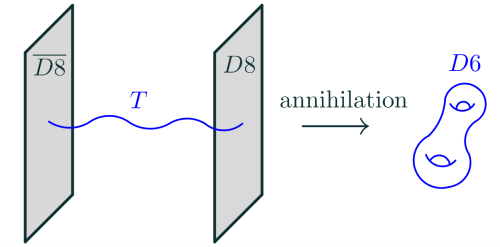
- $\det T(z,w) = 0$ is the locus where the $\mathbb{C}^*$-fiber degenerates

Tachyon condensation

- What is $T(z, w)$?

Tachyon condensation formalism

(*Sen* [99])



$$0 \longrightarrow \tilde{E} \xrightarrow{\quad\quad T \quad\quad} E \longrightarrow \mathrm{coker}(T) \to 0$$

- $\tilde{E}$ and E are vector bundles on $\overline{D8}$ and $D8$ respectively
- $\mathrm{coker}(T) = E/\mathrm{Im}T$ is a sheaf localized on the locus $\det T = 0$

Tachyon condensation

- Gauge symmetry of $D8$ and $\overline{D8}$ acts on the tachyon:

$$\begin{tikzcd} \tilde{E} \arrow[r, "T"] \arrow[loop below, "G_{\overline{\mathrm{D8}}}"] & E \arrow[loop below, "G_{\mathrm{D8}}"] \end{tikzcd} \qquad \Longrightarrow \qquad T \longrightarrow G_{\mathrm{D8}} \cdot T \cdot G^{-1}_{\overline{\mathrm{D8}}}$$

Objective: compute **open string states** in the D6 brane system left over after $D8 - \overline{D8}$ partial annihilation

$$\begin{tikzcd} & \tilde{E} \arrow[r, "T"] \arrow[d, "\delta\phi"] & E \\ \tilde{E} \arrow[r, "T"] & E & \end{tikzcd}$$

The maps $\delta\phi$ are the **open string states**

Tachyon condensation

- $\delta\phi$ must be modded out by gauge equivalences:

$$\begin{tikzcd} & \tilde{E} \arrow[r, "T"] \arrow[dl, "g_{\overline{D8}}"'] \arrow[d, "\delta\phi"] & E \arrow[dl, "g_{D8}"] \\ \tilde{E} \arrow[r, "T"] & E & \end{tikzcd}$$

In the Lie algebra:

$$\delta\phi \sim \delta\phi + g_{D8} \cdot T + T \cdot g_{\overline{D8}}$$

Half-way recap

Strategy to compute the GV invariants:

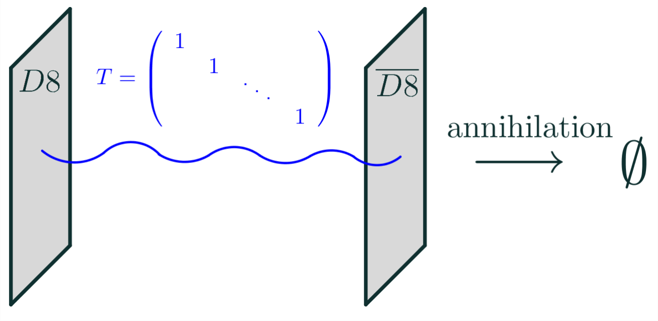
- Consider CY threefolds defined as **one-parameter families of ADE deformations**
- Use **M-theory/IIA duality** to give a description of the corresponding D6 brane system
- Build the **tachyon** that describes the D6 brane system
- Compute the **fluctuations of the tachyon** that correspond to the **open string modes**
- Identify the open string modes with the **GV invariants**

Let's see this in concrete examples!

Non-example

Take $T = \mathbb{1}_{n\times n}$: stack of n $D8$ branes and n $\overline{D8}$ anti-branes

- $\implies$ coker$(T) = 0$ $\implies$ there is total annihilation, i.e. nothing to compute



Reid's pagodas

Reid's pagodas are a natural generalization of the conifold:

- They can be written as deformations of A_{2k-1} singularities fibered over the complex plane w:

$$\underbrace{x^2 + y^2 = z^{2k}}_{A_{2k-1}} - \underbrace{w^2}_{\text{def}} \qquad \text{with } k \in \mathbb{Z}$$

admitting a single $\mathbb{P}^1$ in the resolution

Reid's pagodas

Reid's pagodas are a natural generalization of the conifold:

- They can be written as deformations of A_{2k-1} singularities fibered over the complex plane w:

$$\underbrace{x^2 + y^2 = z^{2k}}_{A_{2k-1}} - \underbrace{w^2}_{\text{def}} \qquad \text{with } k \in \mathbb{Z}$$

  admitting a single $\mathbb{P}^1$ in the resolution
- Changing variables they can be immediately rearranged as a $\mathbb{C}^*$-fibration:

$$uv = (z^k + w)(z^k - w)$$

- We must find the tachyon, i.e. a matrix T in the algebra of A_{2k-1} (or one of its subalgebras) such that $\det T = (z^k + w)(z^k - w)$

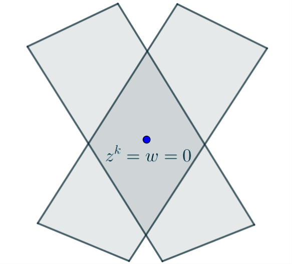
Reid's pagodas

- Consider a stack of 2 $D8$ and 2 $\overline{D8}$

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \xrightarrow{\quad T \quad} \mathcal{O}^{\oplus 2} \longrightarrow \mathrm{coker}(T) \to 0$$

- The minimal tachyon that does the work is:

$$T = \begin{pmatrix} z^k + w & 0 \\ 0 & z^k - w \end{pmatrix} \qquad \det T = (z^k + w)(z^k - w)$$



Reid's pagodas

- To find the open string modes we must consider fluctuations $\delta\phi$ of the tachyon and mod them by gauge equivalences:

$$\delta\phi \sim \delta\phi + g_{D8} \cdot T + T \cdot g_{\overline{D8}}$$

  where $(g_{D8}, g_{\overline{D8}}) \in \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

Reid's pagodas

- To find the open string modes we must consider fluctuations $\delta\phi$ of the tachyon and mod them by gauge equivalences:

$$\delta\phi \sim \delta\phi + g_{D8} \cdot T + T \cdot g_{\overline{D8}}$$

  where $(g_{D8}, g_{\overline{D8}}) \in \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

- There is a choice of $g_{D8}, g_{\overline{D8}}$ that preserves the tachyon:

$$g_{D8} = \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix} \qquad g_{\overline{D8}} = -g_{D8}$$

  This is a $u(1)_{\mathbb{C}} \subset \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

Reid's pagodas

- We are then left with:

$$\delta\phi \sim \delta\phi + g_{D8} \cdot T + T \cdot g_{\overline{D8}}$$

where:

$$g_{\overline{D8}} = \begin{pmatrix} \frac{1}{2}g & a_+ \\ a_- & -\frac{1}{2}g \end{pmatrix} \qquad g_{D8} = \begin{pmatrix} \frac{1}{2}g & b_+ \\ b_- & -\frac{1}{2}g \end{pmatrix}$$

Reid's pagodas

- We are then left with:

$$\delta\phi \sim \delta\phi + g_{D8} \cdot T + T \cdot g_{\overline{D8}}$$

where:

$$g_{\overline{D8}} = \begin{pmatrix} \frac{1}{2}g & a_+ \\ a_- & -\frac{1}{2}g \end{pmatrix} \qquad g_{D8} = \begin{pmatrix} \frac{1}{2}g & b_+ \\ b_- & -\frac{1}{2}g \end{pmatrix}$$

- We get:

$$\delta\phi \sim \underbrace{\begin{pmatrix} \delta\phi_0 & \delta\phi_+ \\ \delta\phi_- & -\delta\phi_0 \end{pmatrix}}_{\delta\phi} + \underbrace{\begin{pmatrix} g\left(z^k - w\right) & (a_+ + b_+)\, z^k + (a_+ - b_+)\, w \\ (a_- + b_-)\, z^k + (b_- - a_-)\, w & -g\left(z^k + w\right) \end{pmatrix}}_{g_{D8} \cdot T + T \cdot g_{\overline{D8}}}$$

- We want to understand if there is any **localized fluctuation**

Reid's pagodas

$$\delta\phi \sim \underbrace{\begin{pmatrix} \delta\phi_0 & \delta\phi_+ \\ \delta\phi_- & -\delta\phi_0 \end{pmatrix}}_{\delta\phi} + \underbrace{\begin{pmatrix} g\left(z^k - w\right) & (a_+ + b_+)\, z^k + (a_+ - b_+)\, w \\ (a_- + b_-)\, z^k + (b_- - a_-)\, w & -g\left(z^k + w\right) \end{pmatrix}}_{g_{D8}\cdot T + T\cdot g_{\overline{D8}}}$$

- $\delta\phi_0$ is localized on the ideal $\left(z^k \pm w\right)$ $\Rightarrow$ not dynamical in 5d
- $\delta\phi_+$ is localized on the locus $\left(z^k, w\right)$
  $\Rightarrow \delta\phi_+ \in \mathbb{C}[z,w]/(z^k, w) \cong \mathbb{C}^k \Rightarrow \boldsymbol{k}$ **modes**
- same for $\delta\phi_- \Rightarrow \boldsymbol{k}$ **modes**

Reid's pagodas

- We find a total of $2k$ modes, corresponding to $\boldsymbol{k}$ **hypermultiplets in 5d**
- On the other hand the GV invariants are: $\boldsymbol{n_{\beta}^{g=0} = n^{0}_{[\mathbb{P}^1]} = k}$ and all the others vanish

Reid's pagodas

- We find a total of $2k$ modes, corresponding to $\boldsymbol{k}$ **hypermultiplets in 5d**
- On the other hand the GV invariants are: $\boldsymbol{n_{\beta}^{g=0} = n_{[\mathbb{P}^1]}^{0} = k}$ and all the others vanish

AGREEMENT!

Reid's pagodas

- We find a total of $2k$ modes, corresponding to $\boldsymbol{k}$ **hypermultiplets in 5d**
- On the other hand the GV invariants are: $\boldsymbol{n_{\beta}^{g=0} = n_{[\mathbb{P}^1]}^{0} = k}$ and all the others vanish

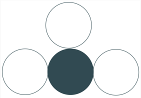
AGREEMENT!

- All the hypers have the same charge with respect to the $U(1)$ commuting with the tachyon:

$$U(1) \sim \begin{pmatrix} g & 0 \\ 0 & -g \end{pmatrix}$$

Let's give a taste of a more complicated example: **Laufer's flop**

- It arises as a deformation of a D_{2k+3} singularity, with equation:

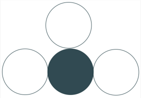


$$x^2 - zy^2 - w(z^{2k+1} - w^2) = 0$$

admitting a single $\mathbb{P}^1$ in the resolution

Let's give a taste of a more complicated example: **Laufer's flop**

- It arises as a deformation of a D_{2k+3} singularity, with equation:



$$x^2 - zy^2 - w(z^{2k+1} - w^2) = 0$$

admitting a single $\mathbb{P}^1$ in the resolution

- In order to realize it in type IIA we must introduce an $O6^-$ plane along with $D6$ branes, with orientifold action:

$$\xi \longrightarrow -\xi \quad \text{and } z = \xi^2$$

Laufer's flop

- Then it can be written as a $\mathbb{C}^*$-fibration, with brane locus:

$$\text{det } T = w\xi^2(w+\xi^{2k+1})(w-\xi^{2k+1})$$

- The tachyon is a map in the exact sequence:

$$0 \longrightarrow \mathcal{O}^{\oplus 4} \xrightarrow{\quad T \quad} \mathcal{O}^{\oplus 4} \longrightarrow \text{coker}(T) \to 0$$

and explicitly it reads:

$$T = \begin{pmatrix} 0 & \xi^{2k+1}+w & 0 & 0 \\ \xi^{2k+1}-w & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & w\xi \end{pmatrix}$$

Laufer's flop

- Due to the orientifold projection the gauge equivalence for the fluctuations of the tachyon is modified (*Collinucci, Denef, Esole* [08]):

$$\delta\phi \sim \delta\phi + g \cdot T + T \cdot \sigma^* g^t$$

where σ^* is the pull-back of the involution

Laufer's flop

- Due to the orientifold projection the gauge equivalence for the fluctuations of the tachyon is modified (*Collinucci, Denef, Esole* [08]):

$$\delta\phi \sim \delta\phi + g \cdot T + T \cdot \sigma^* g^t$$

where σ^* is the pull-back of the involution

$$\Downarrow$$

- There is a $U(1)$ commuting with the tachyon:

$$U(1) \sim \begin{pmatrix} g & 0 & 0 & 0 \\ 0 & -g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Laufer's flop

Performing the explicit computations we find:

- $\boldsymbol{2k+3}$ hypers of charge 1
- $\boldsymbol{k}$ hypers of charge 2

The GV invariants are (*Toda* [14]):

- $\boldsymbol{n^0_{[\mathbb{P}^1]} = 2k+3}$
- $\boldsymbol{n^0_{2[\mathbb{P}^1]} = k}$

Analogous agreeing results for other D_n cases (*Brown, Wemyss* [17])

Non-resolvable singularities

There are ADE fibrations that **do not** admit any Kähler resolution, e.g.:

$$\underbrace{uv = z^3}_{A_2} + \underbrace{w^2}_{\text{def}}$$

- What can we say about such cases?

Non-resolvable singularities

There are ADE fibrations that **do not** admit any Kähler resolution, e.g.:

$$\underbrace{uv = z^3}_{A_2} + \underbrace{w^2}_{\text{def}}$$

- What can we say about such cases?

$$\text{Tachyon:} \quad \begin{pmatrix} z & w & 0 \\ 0 & z & w \\ 1 & 0 & z \end{pmatrix} \quad \Rightarrow \quad \textbf{1 uncharged hyper}$$

...and so on for many other examples

The moral seems to be: if you can come up with a tachyon describing the D6 brane setup, you can compute open string states, and so genus 0 GV invariants!

Recap

We have seen that the **tachyon** is a multipurpose tool, that allows us to:

- compute **open string states** between D6 branes (and GV invariants)
- Find the preserved **gauge group** after tachyon condensation
- Investigate the structure of the **Higgs branch** of the 5d theory (detecting also discrete groups acting on the moduli space) allowing a check with existing results (*Closset, Schafer-Nameki, Wang* [20])
- Detect the **flavour group** of the 5d theories

Conclusions

- There is a physics-based way to compute genus 0 GV invariants for a wide class of non-toric CY
- It applies both in orientifold and non-orientifold D6 branes setups
- It furnishes info also about non-resolvable singularities

Conclusions

- There is a physics-based way to compute genus 0 GV invariants for a wide class of non-toric CY
- It applies both in orientifold and non-orientifold D6 branes setups
- It furnishes info also about non-resolvable singularities

Outlook

- multiple flops singularities
- switching on T-brane entries
- apply to other more involved single-flop and non-resolvable singularities (connecting with existing literature)