Computing Yukawa Textures Using Spectral Data

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❖ Introduction

- Low energy limit of the Heterotic string is a 10 dimensional supergravity with $E_8 \times \mathrm{E}_8$ or $Spin(32)/Z_2$ gauge groups.

- The field content is,

Gravity Multiplet:	$B_{\mu\nu},\ g_{\mu\nu},\ \phi,\ \psi_\mu,\ \lambda$
Gauge Multiplet:	$A_\mu,\ \chi$

- In particular if $H = 0$ and $d\phi = 0$, $(H = dB + \omega_L - \omega_Y)$, we need to have a covariantly constant spinor on the 6d (or 4d) internal manifold, $D\eta = 0 \Rightarrow$ Calabi-Yau threefold (twofold) X.

- The solution for the $\delta\chi = 0$ gives

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \qquad g^{\bar{a}b} F_{\bar{a}b} = 0.$$

- These are satisfied by a holomorphic and stable vector bundle V.

❖ Yukawa Couplings

- Goven a $4d$ Heterotic $E_8 \times E_8$ model (X, V), the (holomorphic) Yukawa couplings in EFT is given by

$$\int_X \Omega \wedge A \wedge A \wedge A \to \quad \Phi_1 \Phi_2 \Phi_3$$

$$A \wedge A \wedge A \to \bar{\Omega}$$

- Therefore, in terms of cohomologies,

$SU(3)$ Bundle $V \quad \to \quad H^1(V) \otimes H^1(V) \otimes H^1(V) \to H^3(\Lambda^3 V) = H^{3,0}(X) = \mathbb{C}$

$\mathbf{27}\ \mathbf{27}\ \mathbf{27} \quad E_6$

$SU(4)$ Bundle $V \quad \to \quad H^1(V) \otimes H^1(V) \otimes H^1(\Lambda^2 V) \to H^3(\Lambda^4 V) = H^{3,0}(X) = \mathbb{C}$

$\mathbf{16}\ \mathbf{16}\ \mathbf{10} \quad SO(10)$

$SU(5)$ Bundle $V \quad \to \quad H^1(V) \otimes H^1(V) \otimes H^1(\Lambda^3 V) \to H^3(\Lambda^5 V) = H^{3,0}(X) = \mathbb{C}$

$\mathbf{10}\ \mathbf{10}\ \mathbf{5} \quad SU(5)$

$SU(5)$ Bundle $V \quad \to \quad H^1(V) \otimes H^1(\Lambda^2 V) \otimes H^1(\Lambda^2 V) \to H^3(\Lambda^5 V) = H^{3,0}(X) = \mathbb{C}$

$\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}} \quad SU(5)$

❖ Goal

- Direct Computation of these cohomologies can be very hard.

Anderson et.al

- The relation with the Yukawa coupling in other string theory models is obscure.

- Since most of Calabi-Yau threefolds constructed so far are elliptically fibered, we use another approach.

Anderson et. al,
Taylor et. Al,

❖ Spectral Data

- Let $\pi: X \to B_2$ be a Weierstrass elliptically fibered CY3. V Stable, Holomorphic, Degree Zero bundle over X.

- Instead of constructing the bundle directly one can use the Fourier-Mukai transform,

$$\begin{tikzcd} & X\times_B X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

$$\Phi(V) \coloneqq R\pi_{2*}(\pi_1^* V \otimes \mathcal{P})$$
$$\mathcal{P} \coloneqq \mathcal{I}_\Delta \otimes \pi_1^* \mathcal{O}(\sigma) \otimes \pi_2^* \mathcal{O}(\sigma) \otimes K_B^{-1}$$

- Given V as above with rank n, its Fourier transform is a Torsion sheaf which is supported over finite cover of B_2:

$$\Phi(V) = i_{S_n *} \mathcal{L}\,[-1]$$

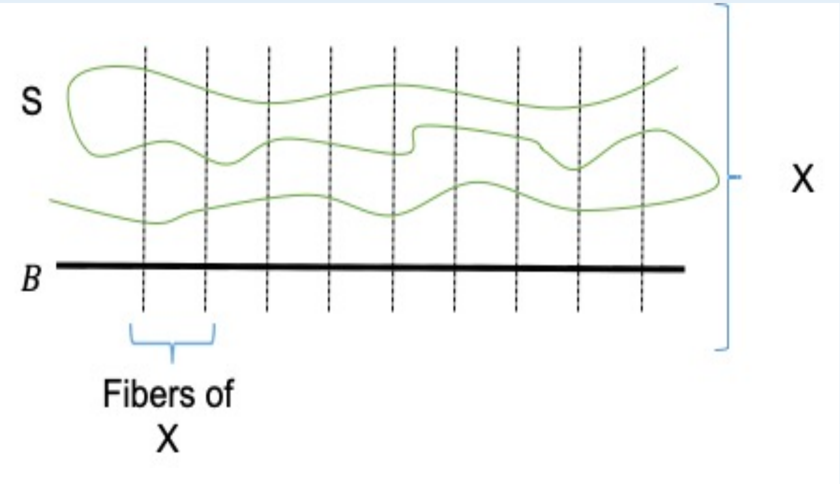
❖ Spectral Data

$$\Phi(V) = i_{S_3 *}\mathcal{L}\,[-1]$$

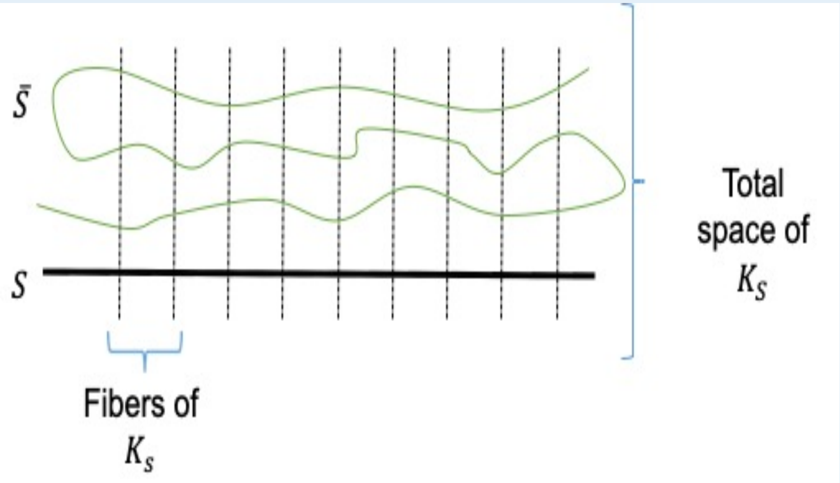
$$i_{S_3}: S_3 \hookrightarrow X$$

$\mathcal{L}$: A coherent rank one sheaf over S_3.

S_3: A finite cover of B_2 of degree 3.



- F-th, 7-brane local data is given by the Hitchin system (E, ϕ) equivalent to the spectral data $(\bar{S}, \bar{\mathcal{L}})$.
- The neighborhood of the zero-section of X is mapped to $Tot(K_S)$.



Beasley, Heckman, Vafa 2008
Donagi, Wijnholt 2008

❖ Pontrjagin Product

E: Smooth elliptic curve.

$$m(p,q) \coloneqq p \boxplus q$$

$$\begin{tikzcd} & E \times E \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "m"] & & E \\ E & & E & \end{tikzcd}$$

$$\mathcal{F} \star \mathcal{G} \coloneqq Rm_*(\pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{G})$$

$$\Phi(\mathcal{F} \otimes \mathcal{G}) = \Phi(\mathcal{F}) \star \Phi(\mathcal{G})[+1]$$

- Naively for an elliptic fibration this formula generalizes to,

$$\Phi(\mathcal{F} \otimes \mathcal{G}) = \Phi(\mathcal{F}) \star \Phi(\mathcal{G}) \otimes K_B^{-1}[+1]$$

❖Cohomologies

- Pontjagin product cannot be used over singular fibers of X. More precisely the group law is only valid for the smooth points of the singular elliptic curve.

- This is not too bad,
  1. We only need anti-symmetrized product.
  2. Only the restriction of the spectral sheaf over the zero section σ contributes to the Cohomology of the bundles.

$$E_2^{p,q} = H^p(B_2, R^q\pi_*V) \Rightarrow H^{p+q}(X,V)$$

$$H^0(X,V) = H^0(\pi_*V)$$

$$0 \to H^1(\pi_*V) \to H^1(V) \to H^0(R^1\pi_*V) \to H^2(\pi_*V) \to H^2(V) \to H^1(R^1\pi_*V) \to 0$$

$$H^3(V) = H^2(R^1\pi_*V)$$

$$R^*\pi_*V = Li_\sigma^*\Phi(V) = Li_\sigma^* i_{S_2*}\mathcal{L}[-1]$$

$$i_\sigma : \sigma \hookrightarrow X$$

❖Cohomologies

$$\Phi(\Lambda^2 V) = \left(i_{S_2*}\mathcal{L}\right) \star_A \left(i_{S_2*}\mathcal{L}\right) \otimes K_B^{-1}[-1]$$

$$R\pi_*\Lambda^2 V = Li_\sigma^*\left(i_{S_2*}\mathcal{L}\right) \star_A \left(i_{S_2*}\mathcal{L}\right) \otimes K_B^{-1}[-1]$$

$$E_2^{p,q} = H^p(B_2, R^q\pi_*\Lambda^2 V) \Rightarrow H^{p+q}(X, \Lambda^2 V)$$

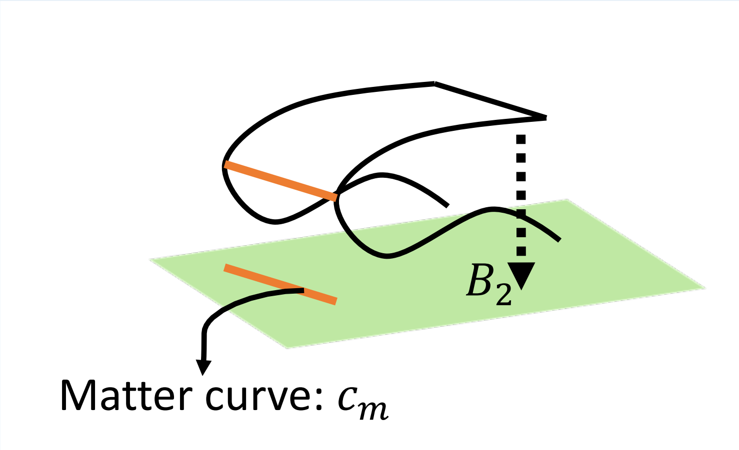
- The idea is to compute $H^1(V)$ and $H^i(\Lambda^2 V)$ in terms of the spectral data using the Leray spectral sequence and the pushforward formulas.

❖ Smooth Spectral Cover

- Consider the E_6 model ($SU(3)$ bundle V),

$$S_3 = a_3 Y + a_2 XZ + a_0 Z^3$$

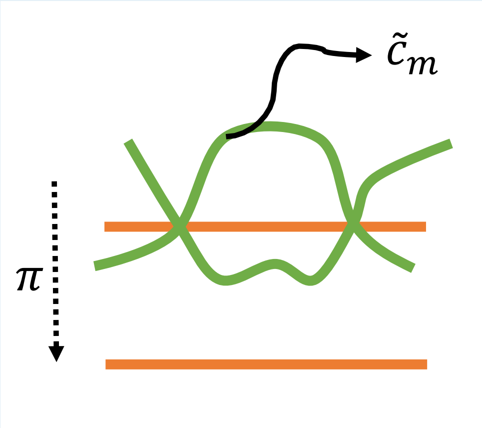
$$\Phi(V) = i_{S_3 *}\mathcal{L}[-1]$$



$$R^1\pi_* V = i_{S_3 \cap \sigma *}\mathcal{L}$$

- If S_3 intersects a generic fiber at P_1, P_2, P_3 S.T $P_1 + P_2 + P_3 = 0$, then $S_{\Lambda^2 V}$ intersect at,

$$P_1 + P_2, \quad P_1 + P_3, \quad P_2 + P_3. \qquad P_1 + P_2 = 0 \Rightarrow P_3 = 0$$



$$R^1\pi_*\Lambda^2 V = Det\left(\pi_*\mathcal{L}\Big|_{\tilde{c}_m}\right) \otimes K_B^{-1}$$

$$S_{\Lambda^2 V} = -a_3 Y + a_2 X\, Z + a_0 Z^3$$

❖ Smooth Spectral Cover

$H^1(V) \otimes H^1(V) \otimes H^1(V) \to \mathbb{C}$ equivalently $H^1(V) \otimes H^1(V) \to H^2(\Lambda^2 V)$

This map cannot be non-zero ⇒ Vanishing Yukawa coupling $\mathbf{27\ 27\ 27}$

- Similarly for $SU(4)$ bundle one can do a similar analysis to show there are non-vanishing $\mathbf{16\ 16\ 10}$ Couplings on the intersection of the curves

$$S_4 = a_4 X^2 + a_3 YZ + a_2 XZ^2 + a_0 Z^4 \qquad\qquad a_4 = a_3 = 0$$

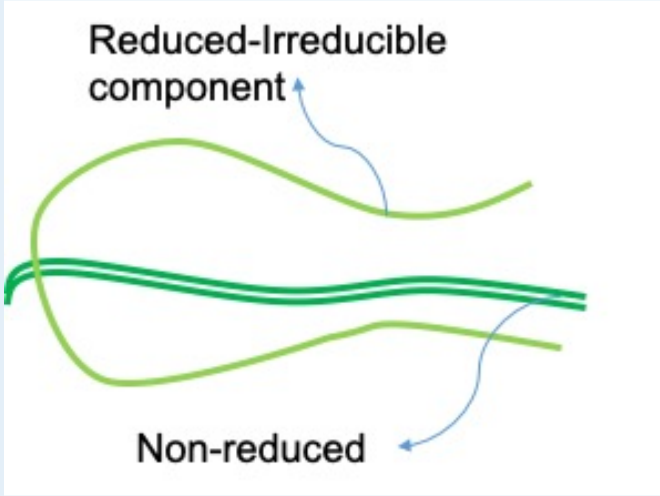
- For $SU(5)$ models $\mathbf{10\ 10\ 5}$ Yukawa couplings are zero for similar reasons as in $SU(3)$ but $\mathbf{10\ \overline{5}\ \overline{5}}$ will be non-zero over the intersections of the following curves

$$S_5 = a_5 XY + a_4 XZ + a_3 YZ^2 + a_2 XZ^3 + a_0 Z^5 \qquad\qquad a_5 = a_4 a_3^2 - a_2 a_3 a_5 + a_0 a_5^2 = 0$$

❖Singular Spectral Covers

- S_n can be reducible and even have non-reduced components.

- In such cases we can always express the spectral sheaf as an extension of simpler parts.



$$S_n = Z\, S_{n-1}$$

$$0 \to i_{\sigma *}\mathcal{L}_1 \to i_{S_n *}\mathcal{L}_n \to i_{S_{n-1} *}\mathcal{L}_{n-1} \to 0$$

$$0 \to i_{\sigma *}\mathcal{L}_1 \star i_{S_{n-1} *}\mathcal{L}_{n-1} \to i_{S_n *}\mathcal{L}_n \star_A i_{S_n *}\mathcal{L}_n \to i_{S_{n-1} *}\mathcal{L}_{n-1} \star_A i_{S_{n-1} *}\mathcal{L}_{n-1} \to 0.$$

- The idea is to construct all spectral data iteratively using such relations.

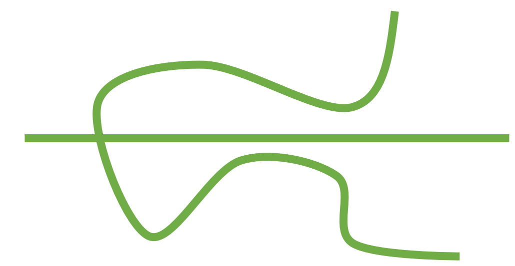
- At the intersection of the components of S_n there can be singularities, generically they do not contribute in the Yukawa couplings, therefore we assume from now on the spectral sheaves over each components is a smooth line bundle.

❖ Singular $SU(3)$ Spectral Data

- Reducible S_3 with Reduced components:

$$0 \to \mathcal{L}_1 \to \mathcal{L}_3 \to \mathcal{L}_2 \to 0$$

$$0 \to \pi^*\pi_*\mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_3 \star_A \mathcal{L}_3 \to i_{\sigma *}\, Det(\pi_*\mathcal{L}_2) \to 0$$

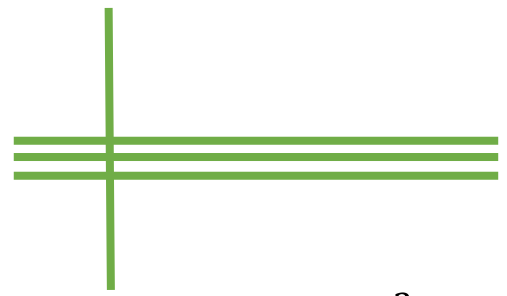


$S_3 = Z(a_2X + a_0Z^2)$

- In this case there are *possible* **27 27 27** couplings from interaction of Bulk and local zero-modes localized on $a_2 = 0$:

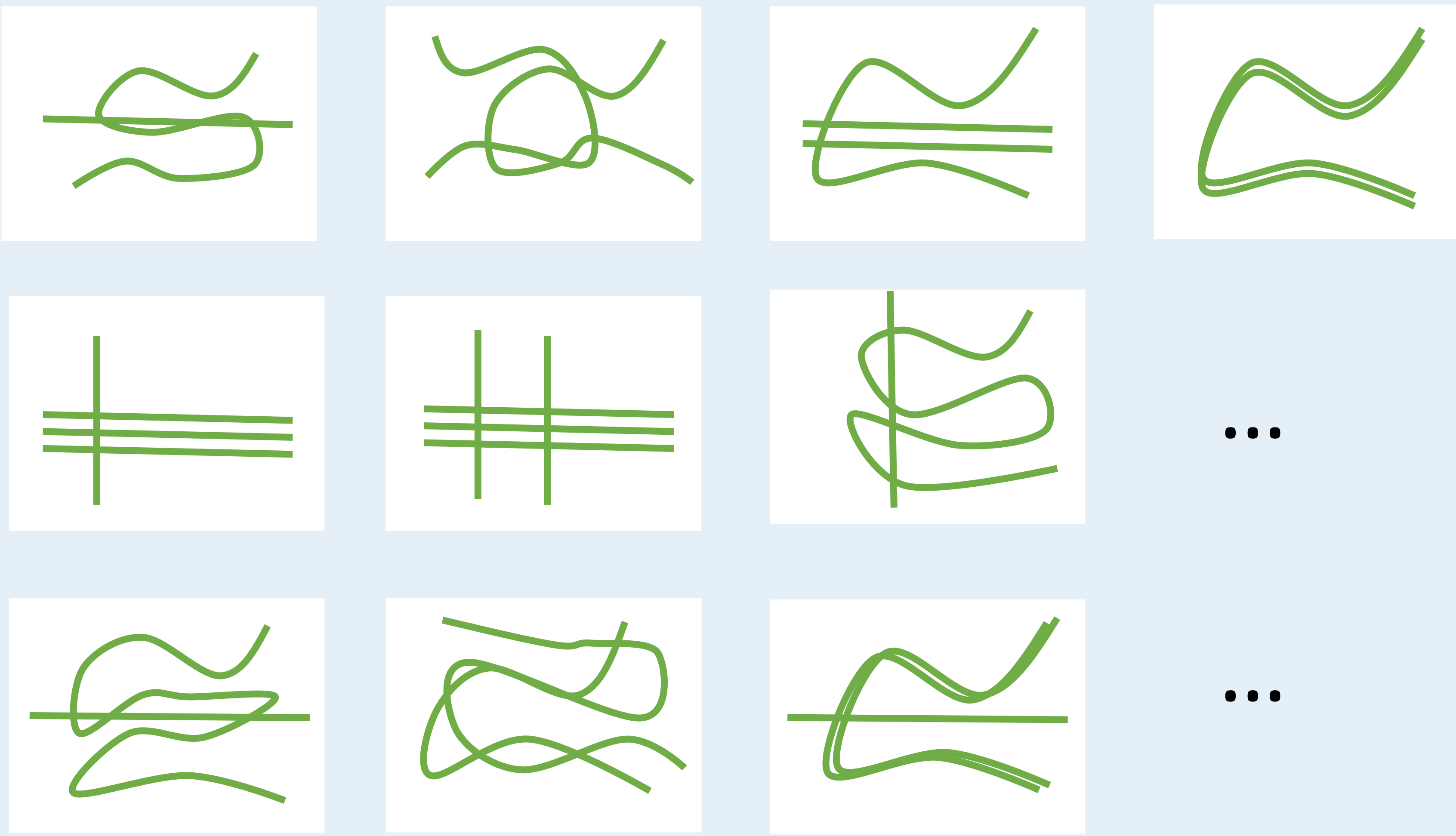
$$H^1(B_2, \mathcal{L}_1 \otimes K_B^{-1}) \otimes H^0\left(B_2, i_{\{a_2=0\}*}\,\mathcal{L}_2\right) \to H^1(B_2, i_{\{a_2=0\}*}\,\mathcal{L}_2 \otimes \mathcal{L}_1 \otimes K_B^{-1})$$

- For non reduced spectral cover it is possible to have couplings coming from the bulk zero modes.



$S_3 = a_0Z^3$

❖ $SU(4)/SU(5)$ Spectral Data



❖ Caveats

- The possible non-vanishing couplings means one should compute whether the corresponding Cohomologies are non-zero and whether they contribute in $H^1(V)$ and/or $H^i(\Lambda^2 V)$.

- The analysis so far is restricted to CY3 manifold with at least a holomorphic section.

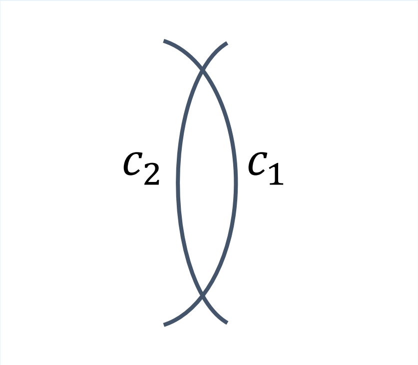
❖ Elliptic CY3 w/ Non-Holomorphic sections.

- Non-holomorphic section wrap around rational curve(s) in the fiber.

- Over such fibers the restriction Poincare' sheaf doesn't parametrize the (semi)-stable.

- To cure this one can modify the kernel of the integral transform,

$$\bar{\mathcal{P}} \coloneqq \mathcal{P} \otimes \pi_1^* \Lambda^2 N_{c_1/X}$$



- The effect of this twist is that the restriction of $\bar{\mathcal{P}}$ on the I_2 fiber is a well defined Poincare' sheaf.

❖ Elliptic CY3 w/ Non-Holomorphic sections.

- The Fourier transform of V with this new kernel will not be sheaf in general,

$$\overline{\Phi}(V) \coloneqq R\pi_{2*}(\pi_1^* V \otimes \bar{\mathcal{P}}) = \mathcal{L}^{\cdot}$$

$$\mathcal{L}^{\cdot}: \; 0 \to \mathcal{L}_1 \to \mathcal{L}_2 \to 0$$

- Where $\mathcal{L}_1$ can wrap the components of the reducible fibers.