$\mathbb{Z}_5$ Symmetries in F-theory, Homological Projective Duality and Modular Forms

Seminar Series on String Phenomenology

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universität wien

FWF

Topological strings on torus fibered Calabi-Yau 3-folds

- Genus-one fibrations w/ N-sections for $N \leq 4$
  $\rightarrow Z_{top.}$ *can be expanded in* $\Gamma_1(N)$ *weak Jacobi forms*
  [Huang,Katz,Klemm'15],**[DelZotto,Gu,Huang,Kashani-Poor,Klemm,Lockhart'17]**,**[Lee,Lerche,Weigand'18]**,**[Cota,Klemm,T.S.'19]**
- Result of monodromies in stringy Kähler moduli space
  [T.S.'19],**[Cota,Klemm,T.S.'19]**

Torus fibered Calabi-Yau are abundant

[Huang,Taylor'19]: 99.99994% of 4d reflexive polytopes have structure indicating T^2 fibration

(see also e.g. **[Anderson,Gao,Gray,Lee'16]**)

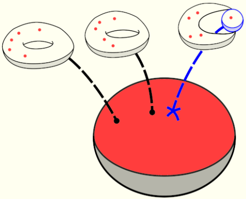
Moreover, they encode F-theory vacua and play pivotal role in network of string dualities!

This talk: $N = 5 \rightarrow$ Rich in novel phenomena

published soon [Johanna Knapp, Emanuel Scheidegger, T.S.'21]

Higgs transitions in M-/F-theory

Algebraic Geometry



Genus-one fibered CY with 5-sections

Arithmetic Geometry

$\Gamma_1(5)$ Jacobi forms

Extremal transitions toric $\rightarrow$ non-toric CY

Derived Equivalences, Homological Projective Duality

What is the geometry of torus fibered CY 3-folds?

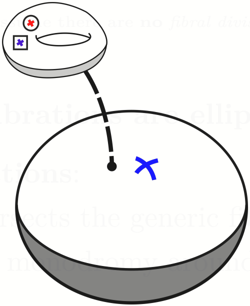
*Avoid unnecessary details:** Assume there are **no** *fibral divisors, non-flat fibers, multiple fibers*

Not all genus-one fibrations are elliptic fibrations!

Some only have N-**sections**:

- An N-section intersects the generic fiber N times
- Points experience monodromy around loops in the base

Example: 2-section



Intersections experience monodromy!

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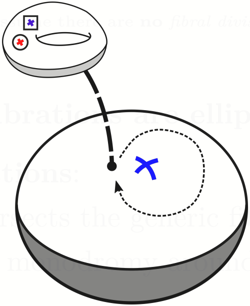

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Then have **three types of curves**
on a genus 1 fibered Calabi-Yau:

1. Curves in the base

2. The generic fiber

3. Isolated components
   of reducible fibers

   lead to charged
   hypermultiplets in F-theory

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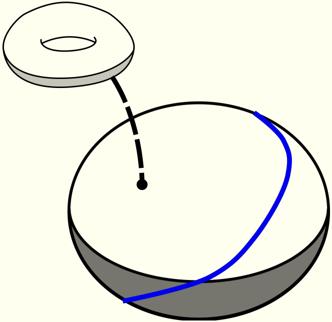

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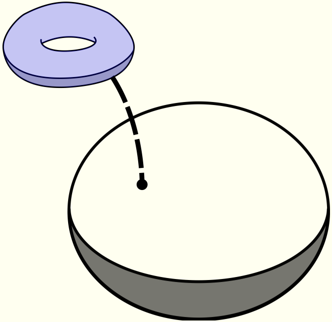

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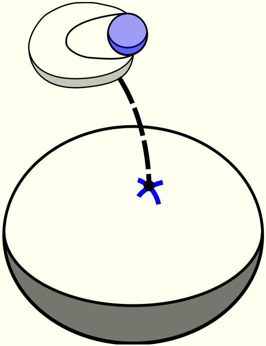

Topological strings and weak Jacobi forms

Conjecture: For genus-one fibered Calabi-Yau with N-sections

[Huang,Katz,Klemm'15],**[DelZotto,Gu,Huang,Kashani-Poor,Klemm,Lockhart'17]**,**[Lee,Lerche,Weigand'18]**,**[Cota,Klemm,T.S.'19]**

$$Z_{\text{top}}(\underline{t},\lambda) = Z_0(\tau,\lambda)\left(1 + \sum_{\beta\in H_2(B,\mathbb{Z})} Z_\beta(\tau,\underline{m},\lambda)\,Q^\beta\right)$$

$Z_\beta(\tau,\underline{m},\lambda)$ are $\Gamma_1(N)$ lattice Jacobi forms

$\rightarrow$ *All-genus results for compact CY 3-folds!*

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$$Q^\beta = \exp\left(2\pi i \beta_j t^j\right)$$

t^j: (shifted) volumes of base curves

$$Z_{\text{top}}(\underline{t},\lambda) = Z_0(\tau,\lambda)\left(1 + \sum_{\beta\in H_2(B,\mathbb{Z})} Z_\beta(\tau,\underline{m},\lambda)\,Q^\beta\right)$$

τ: $\frac{1}{N}\times$ fiber volume

$\underline{m}$: Volumes of fiber components

$Z_\beta(\tau,\underline{m},\lambda)$ … obi forms

$\rightarrow$ *All-genus results for compact CY 3-folds!*

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A **lattice Jacobi form** $\phi(\tau, z_1, ..., z_n)$
of weight k and index matrix M

Admits a Fourier expansion and transforms as

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z_1}{c\tau+d}, ..., \frac{z_n}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i \frac{-cM_{ij}z^i z^j}{c\tau+d}} \phi(\tau, z_1, ..., z_n)$$

Given k, M the Jacobi form ϕ can be fixed with finite data!

$$Z_{\text{top}}(\underline{t}, \lambda) = Z_0(\tau, \lambda)\left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})} Z_\beta(\tau, \underline{m}, \lambda) Q^\beta\right)$$

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Modularity from brane monodromies

Interpret cplx. volumes $\tau, \underline{m}, \underline{t}$ as 2-brane charges

$\rightarrow$ *For $N \leq 4$ auto-equivalences of brane category generate $\Gamma_1(N)$-action*

- B-field shifts act as $\tau \mapsto \tau + 1$, $m \to m + 1$, ...
- Moreover, the **fiberwise conifold transformation** acts as

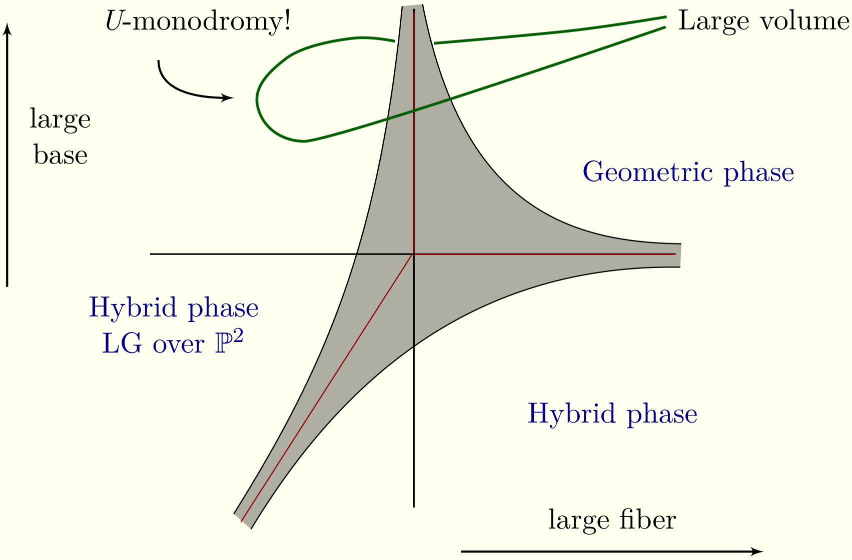
$$U: \begin{cases} \tau & \mapsto & \tau/(1+N\tau) \\ m_i & \mapsto & m_i/(1+N\tau)\,, \quad i = 1, ..., \mathrm{rk}(G) \\ Q_i & \mapsto & (-1)^{a_i} \exp\left(-\frac{N}{1+N\tau} \cdot \frac{1}{2} m^a m^b C^i_{ab} + \mathcal{O}(Q_i)\right) Q_i \end{cases}$$

[T.S.'19], **[Cota,Klemm,T.S.'19]**

Together leads to modular properties of $Z_{top.}$!

Moduli space of genus-1 fibration with $N \leq 4$-sections

(Example: genus one fibration w/ 2-sections over $\mathbb{P}^2$)



Puzzle 1:

For $N > 4$ the group $\Gamma_1(N)$ has at least 3 generators!

$\rightarrow$ Let's consider $N = 5$ to see how it works.

But how do we obtain genus-one fibrations with 5*-sections?*

Normal forms for genus-one curves

- **N=1:** Weierstraß form in $\mathbb{P}_{123}$
  $y^2 = x^3 + f \cdot xz^4 + g \cdot z^6$
- **N=2:** Quartic curve in $\mathbb{P}_{112}$
  $z^2 + z \cdot (c_1 x^2 + c_2 xy + c_3 y^2) + c_4 x^4 + \cdots + c_8 y^4 = 0$
- **N=3:** Cubic curve in $\mathbb{P}^2$
  $x^3 + c_1 x^2 y + c_2 x^2 z + \cdots + c_9 z^3 = 0$
- **N=4:** Intersection of two quadrics in $\mathbb{P}^3$

$$\begin{aligned} x^2 + c_1 xy + c_2 xz + c_3 xw + \cdots + c_9 w^2 &= 0 \\ c_{10} x^2 + c_{11} xy + c_{12} xz + c_{13} xw + \cdots + c_{19} w^2 &= 0 \end{aligned}$$

Note: All are complete intersections in toric varieties

What about $N = 5$?

Normal forms for genus-one curves w/ 5-sections

Can be mapped into **Pfaffian curve** in $\mathbb{P}^4$

- Consider anti-symmetric 5×5 matrix A with entries linear in $[x_1 : \cdots : x_5] \in \mathbb{P}^4$
- $\{\mathrm{rk}(A) \leq 2\} \subset \mathbb{P}^4$ is genus-one curve with "5-point"

  $\rightarrow$ *Vanishing locus of 4×4 Pfaffians*

Dual description as **complete intersection in Grassmanian!**

Plücker embedding $Gr(2,5) \rightarrow \mathbb{P}^9$ also defined by 4×4 Pfaffians

$\rightarrow$ Can also represent curve as codim 5 CI in $\mathrm{Gr}(2,5)$

see e.g. **[Fisher'06]**

Example of homological projective duality! **[Kuznetsov'05]**

1-fold analogon of Rødland Calabi-Yau **[Rødland'98]**

GLSM for Pfaffian curve constructed in **[Hori,Knapp'14]**

Gauge group $G = U(2)$, fields $p_{1,\dots,5}$ in $\det^{-1}$, $x_{1,\dots,5}$ in $\square$

Periods via localization using

[Benini,Cremonesi'12],**[Doroud,Gomis,Le Floch,Lee'12]**

[Jockers,Kumar,Lapan,Morrison,Romo'13]

Moduli space:

Pfaffian curve

Grassmanian curve

Conifold points

(we use numerical analytic continuation)

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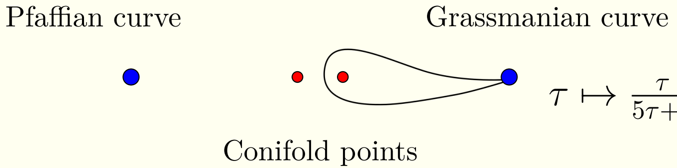

$$\tau \mapsto \frac{\tau}{5\tau+1}$$

(we use numerical analytic continuation)

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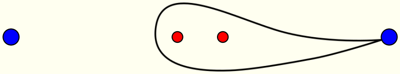

$\tau \mapsto \frac{-9\tau+5}{-20\tau+11}$

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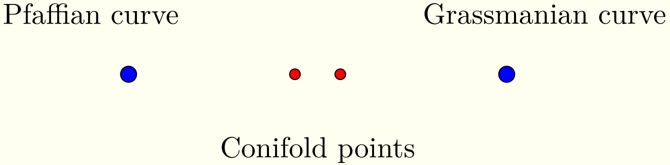

Together monodromies generate $\Gamma_1(5)$*!*

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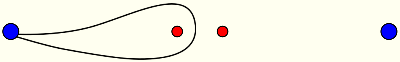
Pfaffian curve   Grassmanian curve



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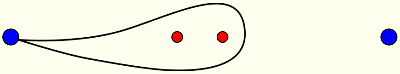

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Conifold points

Together monodromies generate $\Gamma_1(5)$*!*

Let's move on to CY 3-folds!

The monodromies of fiber embed into moduli space of fibration.
$\rightarrow$ *Can try to apply the modular bootstrap!*

But how do we get the such 3-folds?

Strategy: Promote coefficients to sections of bundles on base and construct associated **non-Abelian GLSM**, use **localization** to obtain quantum periods *(loc. w/ techniques from* **[Gerhardus,Jockers'15]***)*

But which bundles lead to smooth fibrations?

$+$ **Puzzle 2:**

What do the two geometric phases mean in context of fibrations?

30 second introduction to F-theory

F-theory on CY 3-fold $\leftrightarrow$ 6d supergravity

Fibral divisors

$k-1$ independent N-sections

$$G = \frac{G_{\text{non-Abelian}}}{\mathbb{Z}_n} \times U(1)^k \times \mathbb{Z}_N$$

n-torsional sections

No section but only N-sections *or torsion 3-forms*

How do Higgs transitions work in M- and F-theory?

$U(1) \to \mathbb{Z}_N$ relates elliptic and genus-one fibrations!

Extremal transition from elliptic to genus-one fibration

see e.g. **[Morrison'99]**, **[Morrison,Braun'14]**, **[Morrison,Taylor'14]**
[Anderson,Garcia-Extebarria,Grimm,Keitel'14],
[Klevers,Mayorga Pena,Oehlmann,Piragua,Reuter'14],
[Mayrhofer,Palti,Till,Weigand'14],
[Cvetic,Donagi,Klevers,Piragua,Poretschkin'15],
[Oehlmann,T.S.'19]

Interlude:

A genus one fibration X has associated elliptic fibration
the **Jacobian fibration** $J(X)$.

F-theory only depends on $J(X)$, M-theory differs on X and $J(X)$!

Genus one fibrations with same Jacobian fibration form
Tate-Shafarevich group Ш

Elements are Geometry X + Action of Jacobian fibration on X
((X, a) and (X, a^{-1}) in general different in $\text{Ш}(X)$)

Example: $U(1) \to \mathbb{Z}_2$

F-Theory **M-Theory**

on elliptic CY with 2 independent sections

$$\begin{tikzcd}
U(1) \arrow[d, "q=2"'] \\
\mathbb{Z}_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & U(1) \times U(1)_{KK} \arrow[dl, bend right, "\substack{q=2 \\ q_{KK}=0}"'] \arrow[dr, bend left, "\substack{q=2 \\ q_{KK}=1}"] & \\
\mathbb{Z}_2 \times U(1)_{KK} & & \widetilde{U(1)}_{KK}
\end{tikzcd}$$

$X^{(0)}, X^{(1)}$

$X^{(0)}$

Jacobian of $X^{(1)}$
(singular, elliptic)

$X^{(1)}$

Genus-one fibered
with 2-sections

We engineer elliptic fibers as CICY in Toric varieties

- Use GV-Spectroscopy to obtain generic F-theory spectrum **[Oehlmann,T.S.19]**

  $\rightarrow$ *Base-independent multiplicities of hypermultiplets*

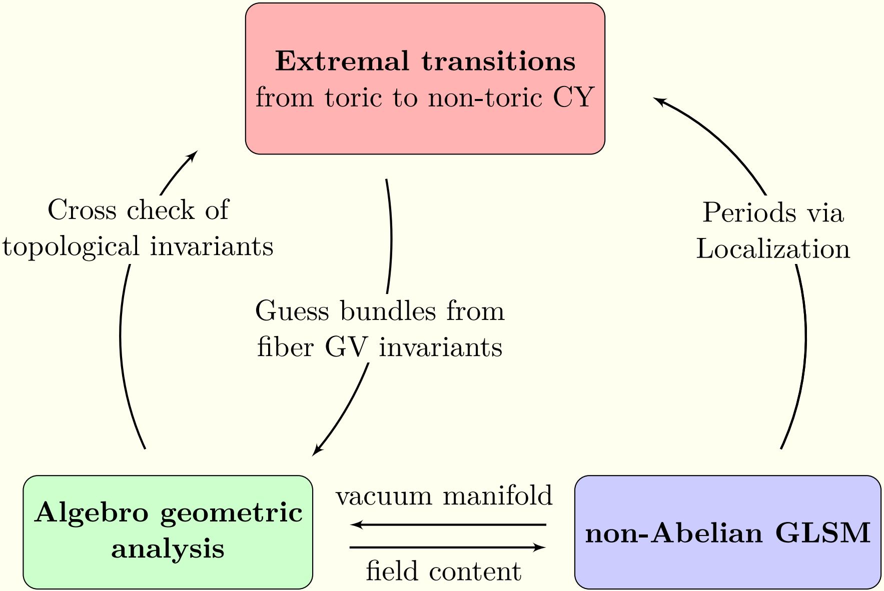
  $\rightarrow$ *F-theory allows Higgs transition to* $\mathbb{Z}_5$

- Can construct all smooth fibrations over e.g. $\mathbb{P}^2$

  $\rightarrow$ **Get Picard-Fuchs systems for 5-sect. fibrations**

  (all topological invariants can be deduced from GV-inv.)

Bonus: First F-theory vacua w/ charge 5 matter



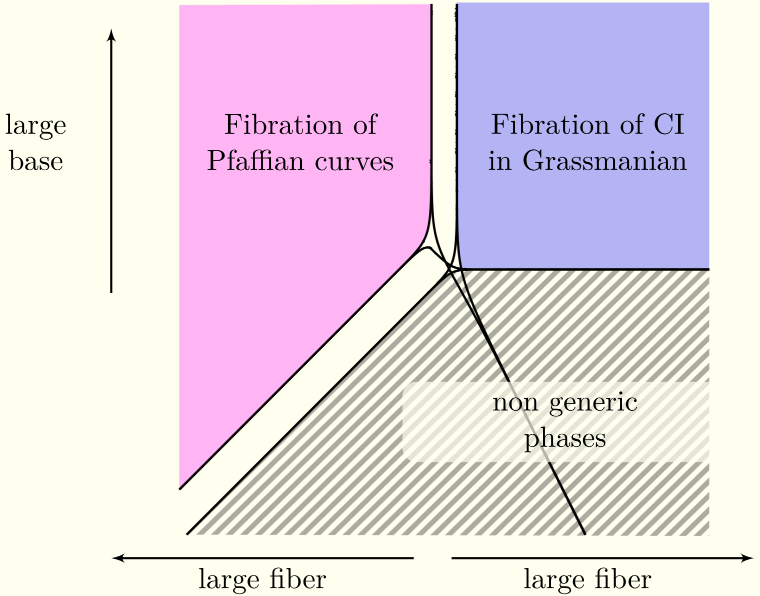
Extremal transitions
from toric to non-toric CY
Cross check of
topological invariants
Guess bundles from
fiber GV invariants
Periods via
Localization
Algebro geometric
analysis
vacuum manifold
field content
non-Abelian GLSM

Let's come back to

The Puzzle of the 2 Geometric Phases

Moduli space of genus-1 fibration with 5-sections

(Example: genus one fibration w/ 5-sections over $\mathbb{P}^2$)



Let us perform modular bootstrap!

Note: Weight 1 Eisenstein series $E_{1,1}$, $E_{1,2}$ generate $M_*(\Gamma_1(5))$

Also need (explained later)

$$\Delta_{10} = E_{1,1}^6 E_{1,2}^4\,, \quad \Delta'_{10} = E_{1,1}^4 E_{1,2}^6$$

Consider an example over $\mathbb{P}^2$:

- **Grassmanian phase**

$$\chi = -104\,, \quad n_{\pm 1} = 108\,, \quad n_{\pm 2} = 110$$

$$Z_B(\tau,\lambda) = \frac{\Delta_{10}^{\frac{2}{5}} E_{1,1}^4 E_{1,2}^5 (22E_{1,1}^3 + 507E_{1,1}^2 E_{1,2} - 386E_{1,1}E_{1,2}^2 + E_{1,2}^3)}{\eta(5\tau)^{36}\phi_{-2,1}(5\tau,\lambda)}$$

- **Pfaffian phase**

$$\chi = -104\,, \quad n_{\pm 1} = 110\,, \quad n_{\pm 2} = 108$$

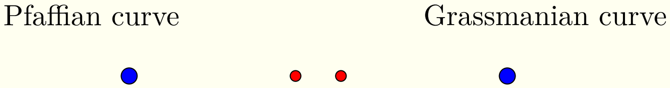
$$Z_B(\tau,\lambda) = \frac{{\Delta'_{10}}^{\frac{2}{5}} E_{1,1}^5 E_{1,2}^4 (22E_{1,2}^3 - 507E_{1,2}^2 E_{1,1} - 386E_{1,2}E_{1,1}^2 - E_{1,1}^3)}{\eta(5\tau)^{36}\phi_{-2,1}(5\tau,\lambda)}$$

Note:

$n_{\pm 1}$ exchanged with $n_{\pm 2}$

Z_B related via $E_{1,1} \to E_{1,2}$, $E_{1,2} \to -E_{1,1}$!

Remember Moduli space of degree 5 curve:



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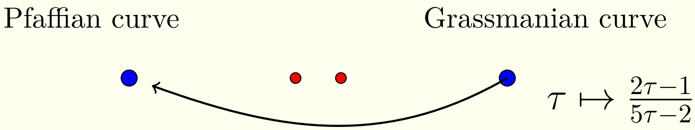

Transfer matrix is element of $\Gamma_0(5)$

$E_{1,1}$, $E_{1,2}$ *transform as vector valued MF!*

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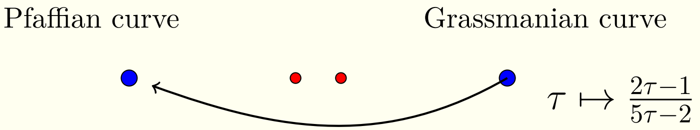

Transfer matrix is element of $\Gamma_0(5)$

$E_{1,1}$, $E_{1,2}$ *transform as vector valued MF!*

But there is actually more to this story...

Where do $\Delta_{10}, \Delta'_{10}$ come from?

Consider base degree 0 $Z_{\text{top.}}$ on elliptic fibration

$$Z_0(\tau, m, \lambda) = 1 + \frac{1}{\lambda^2} \sum_{q_{KK}=0}^{\infty} \sum_{q_{U(1)}=-5}^{5} n_{q_{KK}, q_{U(1)}} \zeta^{q_{U(1)}} q^{q_{KK}} + \mathcal{O}(\lambda^0)$$

How do we get massless Higgs w/ $q_{U(1)} = -5$, $q_{KK} = 1$?

$\rightarrow$ Set $\tau \rightarrow 5\tau$, $m \rightarrow \tau$

$$q^{\frac{1}{5}} \phi_{-2,1}(5\tau, \tau) = \Delta_{10}^{-\frac{1}{5}}\,, \quad q^{\frac{1}{5}} \phi_{0,1}(5\tau, \tau) = \Delta_{10}^{-\frac{1}{5}} E_2^{(5)}$$

Analogous relation for $N \leq 4$ found in **[Cota,Klemm,T.S.'19]**

But what happens for other KK charges?

Let's consider KK-charges $1, \ldots, 4$ that lead to smooth genus-one fibrations:

$$q^{\frac{k^2}{5}} \phi_{-2,1}(5\tau, k\tau) = \begin{cases} (\Delta_{10})^{-\frac{1}{5}} & k = 1, 4\,, \\ (\Delta'_{10})^{-\frac{1}{5}} & k = 2, 3\,, \end{cases}$$

$$q^{\frac{k^2}{5}} \phi_{0,1}(5\tau, k\tau) = \begin{cases} E_2^{(5)}(\Delta_{10})^{-\frac{1}{5}} & k = 1, 4\,, \\ E_2^{(5')}(\Delta'_{10})^{-\frac{1}{5}} & k = 2, 3\,, \end{cases}$$

- $k = 1, 4$ and $k = 2, 3$ transform into each other!
- 2 phases from 2 genus-one fibrations w/ same Jacobian

  $\rightarrow$ *All 4 non-trivial elements of Tate-Shafarevich group* $\mathbb{Z}_5$
- These elements are *expected* to be derived equivalent **[Caldararu'00]**

Conjecture: $Z_{\text{top.}}$ on elements of TS group
are related by modular transformations.
(actually a bit more subtle)

(soon published [**Knapp,Scheidegger,T.S.'21**], [**T.S.'21**])

Summary

- We perform detailed study of T^2 fibered CY with 5-sections
  23 examples from extr. transitions + non-Abelian GLSMs

  published soon [Knapp,Scheidegger,T.S.'21]

- Fibrations of Pfaffian curves and Grassmanian curves derived equivalent and share Jacobian
  $\rightarrow$ *get all 4 non-elliptic geometries in $\mathbb{Z}_5$ TS-group*

- The derived equivalence is example of (relative) homological projective duality

- $Z_{\text{top.}}$ have expansion in $\Gamma_1(5)$ Jacobi forms

- $Z_{\text{top.}}$ on Pf. and Gr. fibration related via $\Gamma_0(5)$ action
  $\rightarrow$ *consequence of Higgs transition in M-/F-theory*

More general story ***published soon*** **[T.S.'21]**

Thank you for your attention!