Numerical Calabi-Yau Metrics from Holomorphic Networks

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Seminar Seires on String Phenomenology

February 16, 2021

Based on arxiv:2012.04797 with Michael R. Douglas and Subramanian Lakshminarasimhan

Introduction

Calabi-Yau manifolds were proven to admit Ricci flat metrics and they play an important role in string compactification.

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Works on numerical approximations:

Headrick and Wiseman [hep-th/0506129]
Donaldson [math.DG/0512625]
Douglas, Karp, Lukic, Reinbacher (DKLR) [hep-th/0606261, hep-th/0612075]
Braun, Ovrut *et al* [0712.3563, 0805.3689]
Anderson *et al* [0904.2186, 1004.4399, 1103.3041]
Headrick and Nassar [0908.2635]
Cui and Gray [1912.11068]

The Embedding Method

A simple example of Calabi-Yau manifolds is the quintics hypersurface in $\mathbb{CP}^4$:

$$0 = f(Z^1, Z^2, Z^3, Z^4, Z^5) = \sum_{i=1}^{5} (Z^i)^5 + \psi Z^1 Z^2 Z^3 Z^4 Z^5$$

One can get a parameterized family of metrics by pulling back the Fubini-Study metrics, which is defined by the Kähler potential:

$$K = \log \sum_{i,\bar{j}=1}^{d+1} h_{i\bar{j}} Z^i \bar{Z}^{\bar{j}},$$

where h is a positive definite hermitian matrix.

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This can be generalized by replacing the Zs with a basis of sections s^I of a line bundle $\mathcal{L}^k$, which are the homogeneous polynomials of degree k in Z^i:

$$K = \log \sum_{I,\bar{J}} h_{I,\bar{J}} s^I \bar{s}^{\bar{J}}$$

Geometric quantities

A CY manifold admits two fundamental differential forms:

- The Kähler form:

$$\omega = \partial_i \partial_{\bar{j}} K \, dZ^i \wedge d\bar{Z}^{\bar{j}},$$

  which can be used to write the volume form:

$$d\mu_g \equiv \det \omega_g = \det_{i,\bar{j}} \partial_i \partial_{\bar{j}} K$$

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For a Ricci flat metric, the ratio $\eta = \frac{d\mu_g}{d\mu_\Omega}$ can be set to 1 by choice of normalization.

Geometric quantities

With the embedding method

$$K = \log \sum_{I,\bar{J}} h_{I,\bar{J}} s^I \bar{s}^{\bar{J}},$$

one can minimize an energy function: [Headrick, Nassar, arxiv:0908.2635]

$$E = \int_M d\mu_{ref} \left(\eta - 1\right)^2$$

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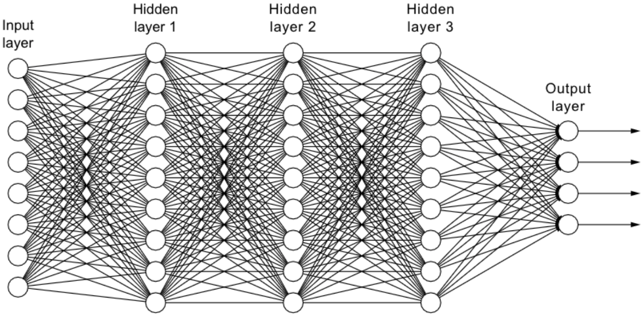
However, the number of parameters in h also increases in $\mathcal{O}(k^6)$, which demands a lot of computational resource when k goes higher.

Most of the previous work only consider the symmetric hypersurfaces to reduce the number of parameters.

Feed-forward Networks



$$F_w = W^{(d)} \circ \theta \circ W^{(d-1)} \circ \ldots \circ \theta \circ W^{(1)} \circ \theta \circ W^{(0)}$$

- $W^{(i)}$: Weight matrix
- θ: Non-linear activation function

Supervised Learning

Feed-forward network:

- A parameterized function: $F_w : \mathcal{X} \to \mathcal{Y}$
- The universal approximation theorem shows that feed-forward networks can approximate arbitrary functions [Cybenko, 1989; Hornik, 1991]

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Loss function:

- Mean squared error (MSE):

$$\mathcal{E} = \mathbb{E}_{\mathcal{P}} \left[\left(f_w(x) - y \right)^2 \right]$$

- Mean absolute percentage error (MAPE):

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Training:

- One simple algorithm to minimize the loss is the gradient descent:

$$w(t+1) = w(t) - \epsilon(t) \frac{\partial \mathcal{E}_{train}}{\partial w} \bigg|_{w=w(t)}$$

Supervised Learning

Compare the supervised learning with the CY metrics problem:

Dataset:

- Input x: the points on the hypersurface
- Output y: the holomorphic volume form $d\mu_\Omega \equiv \mathcal{N}_\Omega \Omega \wedge \bar{\Omega}$

Parameterized function F_w: mapping the points to the metric volume form $d\mu_g$

Loss function: The Monte Carlo integration of

$$\mathcal{E} = \int_M d\mu_{ref} \, (\eta - 1)^2$$

Training algorithm

Holomorphic embeddings

The first way to construct F_w is by what we called **holomorphic networks**:

We replace the homogeneous polynomials in the Kähler potential

$$K = \log \sum_{I,\bar{J}} h_{I,\bar{J}} s^I \bar{s}^{\bar{J}}$$

by a complex feed-forward network which takes the coordinates as the input:

$$F_w = W^{(d)} \circ \theta \circ W^{(d-1)} \circ \ldots \circ \theta \circ W^{(1)} \circ \theta \circ W^{(0)}$$

We choose the activation functions to be:

$$\theta(z) = z^2$$

so the output of F_w is a vector of sections of $\mathcal{L}^k$ with $k = 2^d$, and the Kähler potential can be represented by

$$K_w = \log |F_w(Z)|^2$$

Bihomogeneous embeddings

The performance for the holomorphic networks is not so good in our experiments.

Alternatively, we can also construct F_w by what we called **bihomogeneous networks**, which takes the real and imaginary parts of the $Z^i \bar{Z}^{\bar{j}}$ as inputs.

So F_w will be a network with real weights and a single output

$$F_w = W^{(d)} \circ \theta \circ W^{(d-1)} \circ \ldots \circ \theta \circ W^{(1)} \circ \theta \circ W^{(0)}$$

and the Kähler potential is

$$K_w = \log F_w(\mathrm{Re} Z^i \bar{Z}^{\bar{j}}, \mathrm{Im} Z^i \bar{Z}^{\bar{j}})$$

Training Algorithm

The activation function $\theta(z) = z^2$ is actually not a usual choice in a typical machine learning problem:

- The weights in the front layers have higher degrees than the rest
- The gradient is easy to blow up or vanish during the optimization

We used a two-phase training method:

- Train with the Adam algorithm [Kingma, Ba, 2015] which computes an individual adaptive learning rate for each parameter
- Then train with L-BFGS near the minima

Supervised Learning

Dataset:

- Input x: the points on the hypersurface $\rightarrow Z^i \bar{Z}^{\bar{j}}$
- Output y: the holomorphic volume form $d\mu_\Omega \equiv \mathcal{N}_\Omega \Omega \wedge \bar{\Omega}$

Parameterized function: $\det \partial\bar{\partial} \log F_w[x_i]$

Loss function: The Monte Carlo integration of

$$\mathcal{E} = \int_M d\mu_{ref} \, (\eta - 1)^2$$

Training algorithm: Adam and L-BFGS

Why machine learning?

- Hardware (GPU) and software (Tensorflow/Keras) optimized for tensor operations
- Automatic differentiation techniques to speed up the training process

Implementation

Our code on Github: `http://github.com/yidiq7/MLGeometry`

```
class twolayers(tf.keras.Model):

    def __init__(self, n_units):
        super(twolayers, self).__init__()
        self.bihomogeneous = bnn.Bihomogeneous()
        self.layer1 = bnn.Dense(25, n_units[0], activation=tf.square)
        self.layer2 = bnn.Dense(n_units[0], n_units[1], activation=tf.square)
        self.layer3 = bnn.Dense(n_units[1], 1)

    def call(self, inputs):
        x = self.bihomogeneous(inputs)
        x = self.layer1(x)
        x = self.layer2(x)
        x = self.layer3(x)
        x = tf.math.log(x)
        return x
```

Experiments and Results

We scanned through CY manifolds with different symmetries:

- The Dwork quintics $f = f_1$ below with $\phi = 0$
- A two parameter family with less symmetry:

$$f_1 = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \psi z_0 z_1 z_2 z_3 z_4 + \phi(z_3 z_4^4 + z_3^2 z_4^3 + z_3^3 z_4^2 + z_3^4 z_4)$$

- Another two parameter family with no symmetry:

$$\begin{aligned} f_2 = f_1|_{\phi=0} + \alpha\Big(& z_2 z_0^4 + z_0 z_4 z_1^3 + z_0 z_2 z_3 z_4^2 + z_3^2 z_1^3 + z_4 z_1^2 z_2^2 + z_0 z_1 z_2 z_3^2 + \\ & z_2 z_4 z_3^3 + z_0 z_1^4 + z_0 z_4^2 z_2^2 + z_4^3 z_1^2 + z_0 z_2 z_3^3 + z_3 z_4 z_0^3 + z_1^3 z_4^2 + \\ & z_0 z_2 z_4 z_1^2 + z_1^2 z_3^3 + z_1 z_4^4 + z_1 z_2 z_0^3 + z_2^2 z_4^3 + z_4 z_2^4 + z_1 z_3^4 \Big) \end{aligned}$$

Experiments and Results

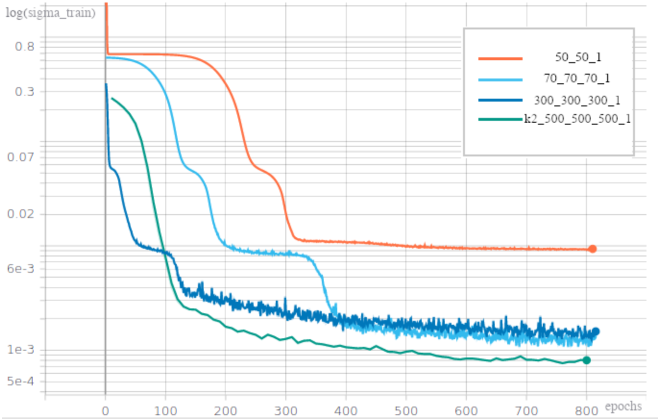


Figure: The training curves for the Dwork quintic with $\psi = 0.5$, trained with Adam optimizer and MAPE loss

Experiments and Results

We studied the dependence of the accuracy on different parameters:

For the geometry of CY:

- The shortest length scale $\sim$ distance in moduli space to a singualr CY $f_s = \nabla f_s = 0$:

$$\sin\theta \propto d \equiv \min_{Z_0 \in M} \frac{|\partial_i f(Z_0)|_H}{||f||_H |Z_0|^{n-1}}.$$

- The complexity of the CY

For the network:

- The depth d of the network, with $k = 2^d$
- The total number of parameters

Experiments and Results

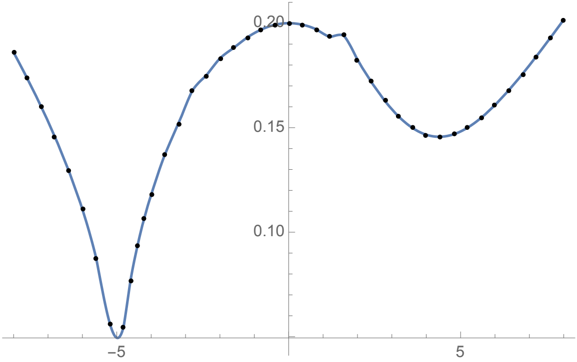


Figure: Distance to a singular CY for the Dwork quintics f_0, X axis is ψ, Y axis is the distance.

Experiments and Results

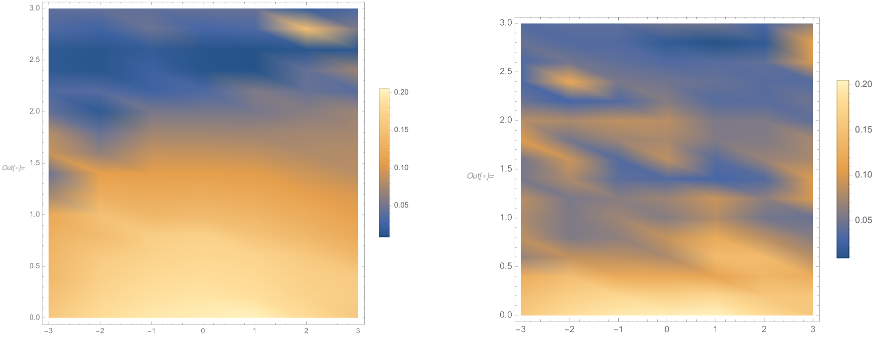
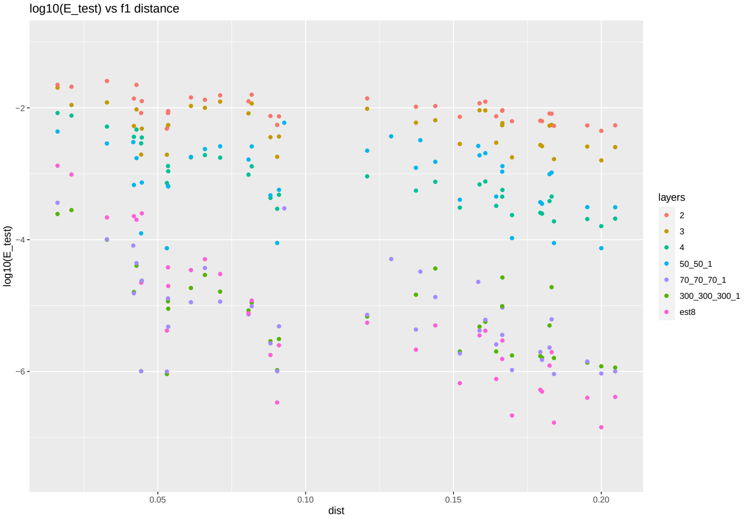


Figure: Distance to singular CY as function of ψ, ϕ in f_1 and ψ, α in f_2

Experiments and Results



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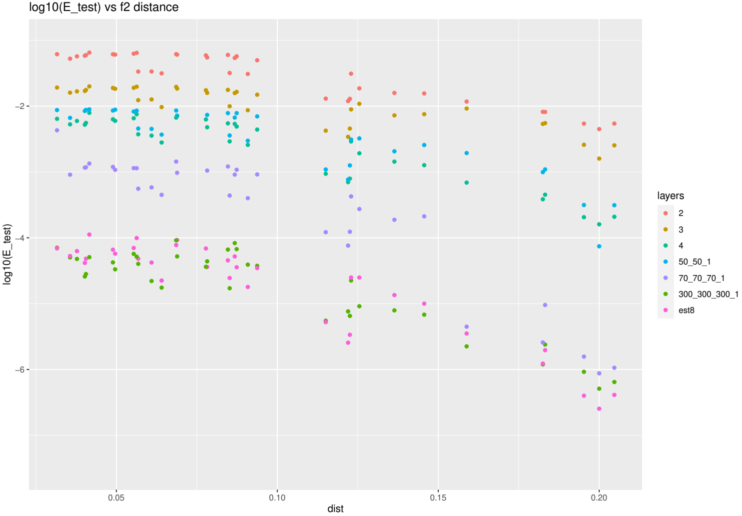

Summary

- We developed a Tensorflow/Keras package to approximate numerical CY metrics with high accuracy
- Other interesting topics in geometry: SYZ special Lagrangian torus fibrations
- Questions in ML: hyperparameters tuning, over-parameterization, etc
- Other recent work using ML:

  Ashmore, He and Ovrut [1910.08605]
  Ashmore [2011.13929]
  Anderson, Gerdes, Gray, Krippendorf,Raghuram and Ruehle [2012.04656]
  Jejjala, Pena, Mishra [2012.15821]