Modelling General Asymptotic Calabi-Yau Periods

Based on [2105.02232] with T. Grimm and D. van de Heisteeg

String Pheno seminar series

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Motivation

Calabi-Yau three-fold period vector is an important ingredient in string compactifications,

Unique (3,0)-form

$$\Pi(t) = \int_\gamma \Omega$$

Basis of three-cycles

CS moduli

Kinetic term : $\mathscr{L}_{\text{kin}} \sim K_{i\bar{j}} dt^i \wedge *_4 d\bar{t}^{\bar{j}}$

e.g. 4d $\mathcal{N} = 1{,}2$ type IIB

Kähler metric

Superpotential: $W = \int_{Y_3} G_3 \wedge \Omega = G_3^T \eta \Pi$

Problem: Period vector is in general a complicated function of the moduli

Goal: Understand the asymptotic structure and construct models using principles from Hodge theory

Outline

1. Background information:

   Boundaries in CS moduli space, Variation of Hodge structure

2. Results from asymptotic Hodge theory:

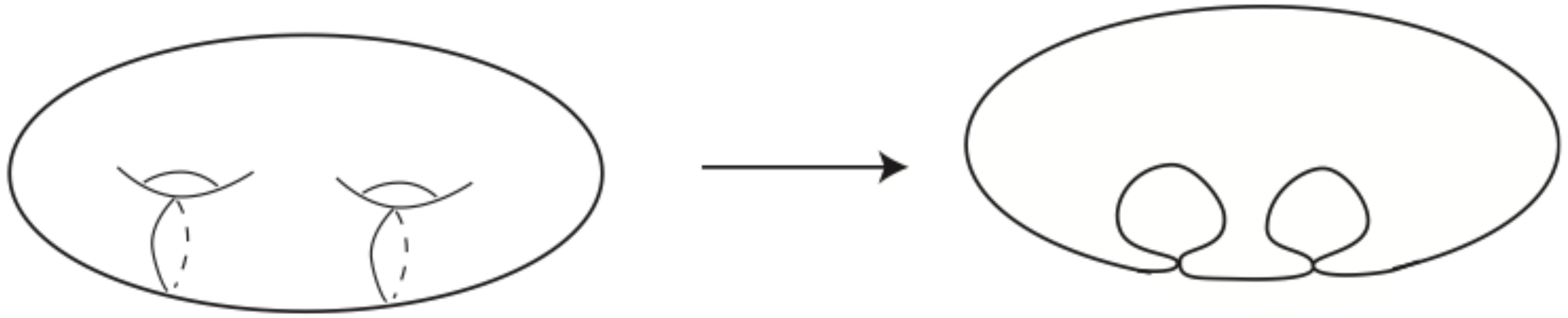
   Near boundary expansion of the period vector, Classification, Boundary data

3. Reverse engineering and specific example

Background Information

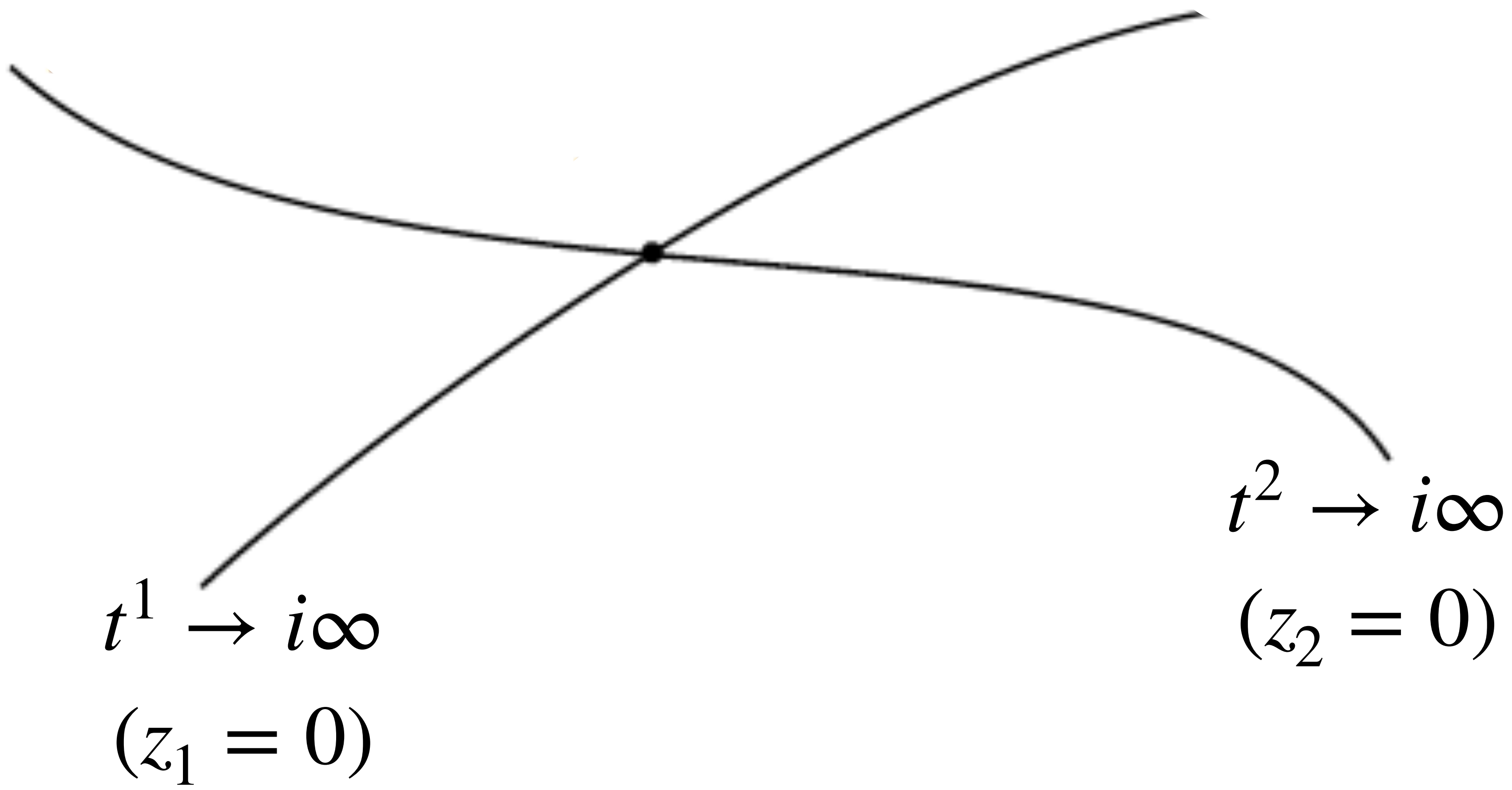
Boundaries in CS Moduli space

In $\mathscr{M}_{\text{cs}}$, there are divisors locally defined by $z_i = 0$ where the Calabi-Yau develops singularities



We can always locally describe the intersection of n loci by normal crossing divisors [Hironaka]

$$z_1 = z_2 = \ldots = z_n = 0$$



(We frequently use the coordinates $t^i = \dfrac{1}{2\pi i} \log z_i$)

We refer to these loci as co-dimension n boundaries in $\mathscr{M}_{\text{cs}}$

Most famous example of co-dimension $h^{2,1}$ boundary : Large Complex Structure (LCS) point

Variation of Hodge structure

Moving through $\mathscr{M}_{\rm cs}$ changes what we call holomorphic $\longrightarrow$ Hodge decomposition on the middle cohomology varies

$$H^3(Y_3,\mathbb{C}) = H^{3,0}\oplus H^{2,1}\oplus H^{1,2}\oplus H^{0,3}$$

e.g. orientation of the complex line spanned by Ω.

This variation is not arbitrary and in terms of the Hodge filtration F^p

[Griffiths]

$$\frac{\partial F^p}{\partial z}\subset F^{p-1} \longrightarrow \text{Horizontality (More familiar: } \partial_i\Omega \in H^{3,0}\oplus H^{2,1}\text{)}$$

Hodge filtration:

$$F^3 = H^{3,0}$$
$$F^2 = H^{3,0}\oplus H^{2,1}$$
$$F^1 = H^{3,0}\oplus H^{2,1}\oplus H^{1,2}$$
$$F^0 = H^3(Y_3,\mathbb{C})$$

We say that the variation is polarised if

$$\langle F^p, F^{4-p}\rangle = 0$$

$$i^{p-q}\langle \omega,\bar\omega\rangle > 0\,,\quad \omega\in H^{p,q}\ \text{ and }\ \omega\neq 0$$

(Hodge-Riemann relations)

Bilinear Pairing:

$$\langle u, v\rangle = \int_{Y_3} u\wedge v$$

Important fact : In a CY threefold, we can span $H^3(Y_3,\mathbb{C})$ by taking derivatives of Ω

Asymptotic Hodge theory

Behaviour near the boundary

The period vector captures the VHS and is particularly constrained in asymptotic regions

As we circle a single boundary component ($z_k = 0$), the period vector undergoes monodromy

$$\mathbf{\Pi}(z_k e^{2\pi i}) = T_k \mathbf{\Pi}(z_k) \quad , \quad T_k \in Sp(2h^{2,1} + 2, \mathbb{R})$$

Monodromies can always be made unipotent, i.e. $(T_k - \mathbb{I})^s = 0$

We can define the nilpotent log-monodromy matrices : $N_k = \log T_k$

Co-dimension n boundary $\longrightarrow$ n log-monodromy matrices N_k

$$\boxed{t = \frac{\log z}{2\pi i}}$$

Near the co-dimension n boundary the period vector can be expanded as [Schmid '73]

$$\mathbf{\Pi}(t) = e^{t^i N_i}(\mathbf{a}_0 + \mathbf{a}_{1,r} e^{2\pi i t^r} + \ldots) = e^{t^i N_i} e^{\Gamma(z)} \mathbf{a}_0$$

- $\mathbf{a}_0$ is an element of a Mixed Hodge structure $\longrightarrow$ captures splitting of $H^3(Y_3, \mathbb{C})$ at boundary
- $e^{t^i N_i}$ generates polynomial behaviour
- $\Gamma(z)$ satisfies $\Gamma(0) = 0$ and generates the exponential behaviour $\longrightarrow$ instanton map (Mirror instantons at LCS)

The nilpotent orbit theorem by Schmid guarantees that the polynomial approximation

$$\Pi_{\text{poly}}(t) = e^{t^i N_i} \mathbf{a}_0$$

still defines a proper variation of polarised Hodge structure $\longrightarrow$ Hodge-Riemann satisfied

These results hold for all elements $\omega \in F^p \longrightarrow$ Basis for the splitting of $H^3(Y_3, \mathbb{C})$ at boundary

In practice, we use the derivatives of $\mathbf{\Pi} \longrightarrow$ Limiting filtration F_0^p

The possible splittings are classified !

Codimension n boundary

$\downarrow$

Nilpotent orbit data:

(N_i, F_0^p)

Working with the polynomial approximation $\mathbf{\Pi}_{\text{poly}}$ has led to promising results in physics

[Grimm, Heisteeg, Li, Palti, Valenzuela...]

Where is the problem then?

The mathematics results are for all the elements of $H^3(Y_3, \mathbb{C})$!

In physics, we only work with the period vector (representative of $\Omega \in H^{3,0}$) :

$$\text{e.g. Kähler metric : } K_{i\bar{j}} = -\, e^{-K} \partial_i \Pi^T \eta \overline{\partial_j \Pi} + e^{-2K} \left(\partial_i \Pi^T \eta \bar{\Pi}\right) \left(\Pi^T \eta \overline{\partial_j \Pi}\right)$$

Using Π_{poly} gives a simpler expression but might result in $\det K_{i\bar{j}} = 0$!

Not guaranteed that we can span the entire $H^3(Y_3, \mathbb{C})$ from derivatives of the polynomial approximation:

$$\partial_i \Pi_{\text{poly}} = e^{t^r N_r} N_i \mathbf{a}_0$$
$$\partial_i \partial_j \Pi_{\text{poly}} = e^{t^r N_r} N_i N_j \mathbf{a}_0$$
$$\partial_i \partial_j \partial_k \Pi_{\text{poly}} = e^{t^r N_r} N_i N_j N_k \mathbf{a}_0$$

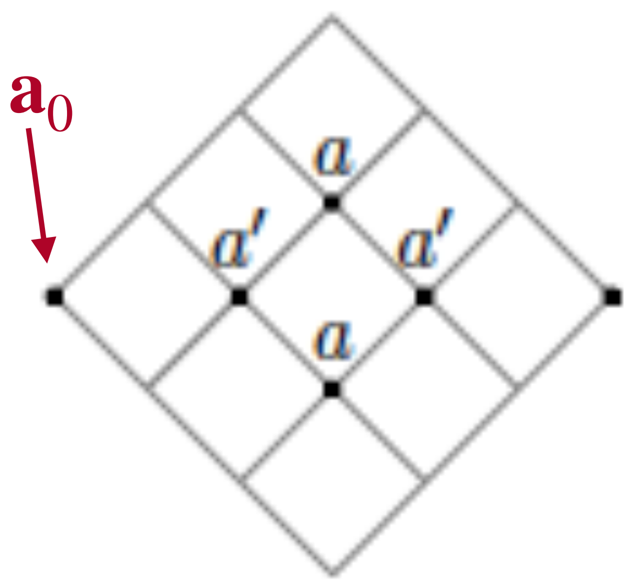
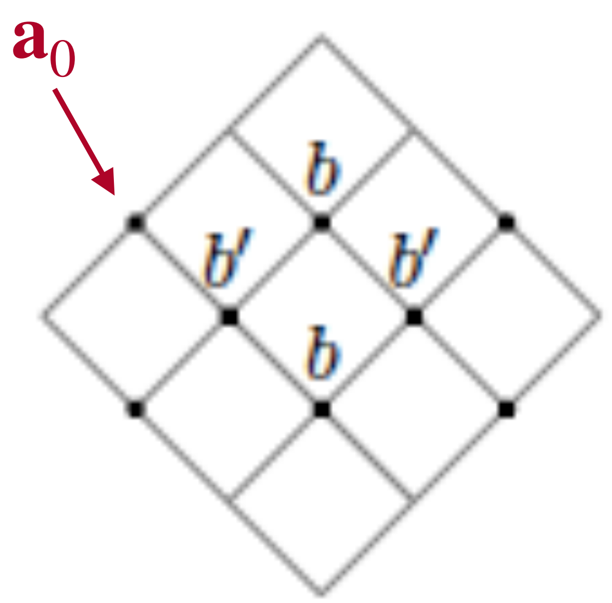
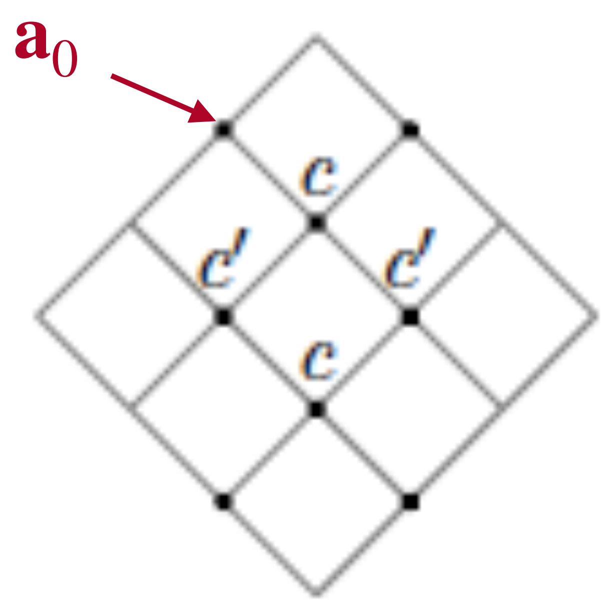
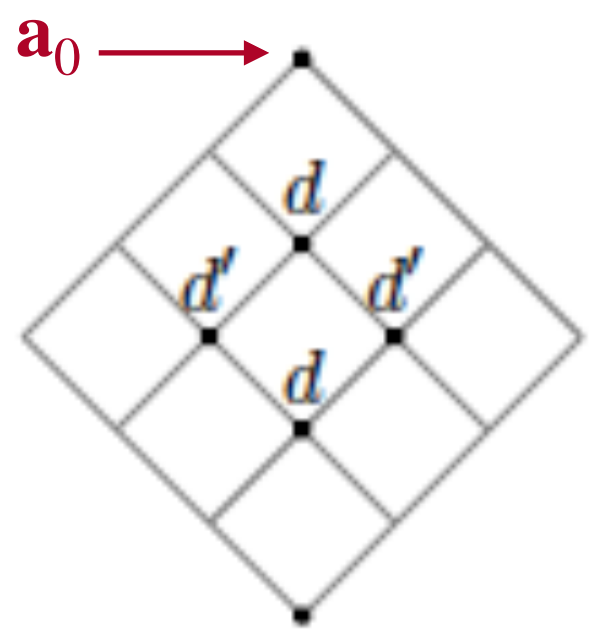
Extreme cases:

- Conifold type : $\quad N_i \mathbf{a}_0 = 0\,, \quad \forall i$
- LCS type : $\quad N_i N_j N_k \mathbf{a}_0 \neq 0\,, \quad \forall i, j, k \longrightarrow$ All exponential terms capture instantons for the mirror

Most cases require exponential corrections to the polynomial part ! $\longrightarrow$ Instanton map $\Gamma(e^{2\pi i t})$

What is the action of N_i and Γ on $\mathbf{a}_0$?

The possible splitting of $H^3(Y_3, \mathbb{C})$ can be represented diagrammatically by Hodge-Deligne diamonds and are classified as follows

singularity	I_a	II_b	III_c	IV_d
HD diamond				
index	$a + a' = h^{2,1}$ $0 \leq a \leq h^{2,1}$	$b + b' = h^{2,1} - 1$ $0 \leq b \leq h^{2,1} - 1$	$c + c' = h^{2,1} - 1$ $0 \leq c \leq h^{2,1} - 2$	$d + d' = h^{2,1}$ $1 \leq d \leq h^{2,1}$

The N_i act by mapping vertically down $\longrightarrow$ Away from LCS, $\mathbf{a}_0$ is not sufficient to span everything, we need some of the $\mathbf{a}_i$

Lower bound on number of terms: $\mathrm{I}_a : 2h^{2,1} - a + 1\,,\quad \mathrm{II}_b : 2h^{2,1} - b - 1\,,\quad \mathrm{III}_c : c + 1\,,\quad 2(h^{2,1} - d)$

$$\boxed{\mathbf{\Pi}(t) = e^{t^i N_i}(\mathbf{a}_0 + \mathbf{a}_{1,r} e^{2\pi i t^r} + \ldots) = e^{t^i N_i} e^{\Gamma(z)} \mathbf{a}_0}$$

We need to rely on $\Gamma(z)$ to generate some correction terms !

For a fixed number of moduli, we can simply list the possible cases!

For one modulus ($h^{2,1} = 1$) only three possibilities :

1. Conifold singularity I_1
2. Tyurin degeneration II_0
3. Large complex structure IV_1

For two moduli ($h^{2,1} = 2$) there are four classes : [Kerr, Pearlstein, Robles '19]

$$
\begin{aligned}
\mathrm{I}_2 \text{ class :} \quad & \langle \mathrm{I}_1|\mathrm{I}_2|\mathrm{I}_1\rangle\,,\ \langle \mathrm{I}_2|\mathrm{I}_2|\mathrm{I}_1\rangle\,,\ \langle \mathrm{I}_2|\mathrm{I}_2|\mathrm{I}_2\rangle\,,\\
\text{Coni-LCS class :} \quad & \langle \mathrm{I}_1|\mathrm{IV}_2|\mathrm{IV}_1\rangle\,,\ \langle \mathrm{I}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,
\end{aligned}
$$

[Alvarez-Garcia, Blumenhagen et al. '20] [Demirtas et al. '20]

$$
\mathrm{II}_1 \text{ class :} \quad \langle \mathrm{II}_0|\mathrm{II}_1|\mathrm{I}_1\rangle\,,\ \langle \mathrm{II}_1|\mathrm{II}_1|\mathrm{I}_1\rangle\,,\ \langle \mathrm{II}_0|\mathrm{II}_1|\mathrm{II}_1\rangle\,,\ \langle \mathrm{II}_1|\mathrm{II}_1|\mathrm{II}_1\rangle\,,
$$

[Kachru, Klemm et al. '96]

$$
\begin{aligned}
\text{LCS class :} \quad & \langle \mathrm{II}_1|\mathrm{IV}_2|\mathrm{III}_0\rangle\,,\ \langle \mathrm{II}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,\ \langle \mathrm{III}_0|\mathrm{IV}_2|\mathrm{III}_0\rangle\,,\ \langle \mathrm{III}_0|\mathrm{IV}_2|\mathrm{IV}_1\rangle\,,\\
& \langle \mathrm{III}_0|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,\ \langle \mathrm{IV}_1|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,\ \langle \mathrm{IV}_2|\mathrm{IV}_2|\mathrm{IV}_2\rangle\,,
\end{aligned}
$$

Simplified boundary data

What does the boundary data look like for a given type?

There is a further approximation, the <u>sl(2)-orbit theorem</u> which provides an algorithm to reduced the nilpotent orbit data to : [Cattani, Kaplan, Schmid '86]

Rotation operator

1. Distinguished basis $\tilde{F}_0^p = e^{-i\delta} F_0^p$
2. Sets of sl(2)-triples (N_i^-, N_i^+, N_i^0)

Standard sl(2):
$[N_i^+, N_1^-] = 2N_i^0$
$[N_i^\pm, N_i^0] = \pm N_i^\pm$

Co-dimension n boundary $\longrightarrow$ Nilpotent orbit data (N_i, F_0^p) $\longrightarrow$ sl(2)-data $(\tilde{F}_0^p, N_i^\pm, N_i^0)$

This sl(2) data is easily constructed from representation theory and the starting point of our analysis !

Reverse engineering

Reverse-engineering

We start from sl(2)-data and work back to get the essential terms of the period vector:

$$\Pi = e^{t^i N_i} e^{\Gamma(z)} e^{i\delta} \tilde{\mathbf{a}}_0$$

$$\boxed{\begin{array}{c} sl(2)^n\text{-splitting} \\ \tilde{F}_0^p,\ (N_i^{\pm}, N_i^0) \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{nilpotent orbit} \\ F_0^p,\ N_i \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{periods w/ instantons} \\ F_0^3,\ \Gamma(z),\ N_i \end{array}}$$

Coni-LCS : $\langle \mathrm{I}_1 \,|\, \mathrm{IV}_2 \,|\, \mathrm{IV}_1 \rangle$ and $\langle \mathrm{I}_1 \,|\, \mathrm{IV}_2 \,|\, \mathrm{IV}_2 \rangle$

Collecting the ingredients we have

$$\tilde{\mathbf{a}}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -n & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad \delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a(z) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ f(z) & e(z) & d(z) & 0 & -a(z) & 0 \\ e(z) & 0 & b(z) & 0 & 0 & 0 \\ d(z) & b(z) & c(z) & 0 & 0 & 0 \end{pmatrix}$$

$+$ Constraints from horizontality on Γ

Can construct the general period for the coni-LCS class (constraints not imposed)

$$\Pi(z) = \begin{pmatrix} 1 \\ a(z) \\ \frac{\log[z_2]}{2\pi i} \\ i\delta_2 + i\delta_1 a(z) + f(z) - \frac{(\frac{1}{2}a(z)c(z) + d(z))\log[z_2]}{2\pi i} - \frac{\log[z_2]^3}{6(2\pi i)^3} \\ i\delta_1 + e(z) - \frac{a(z)(\log[z_1] + n\log[z_2])}{2\pi i} \\ \frac{1}{2}a(z)c(z) + d(z) - \frac{\log[z_2]^2}{2(2\pi i)^2} \end{pmatrix}$$

- No $\log[z_1] = t^1$ dependence at leading order $\longrightarrow$ Characteristic of conifold type boundary
- $\log[z_2]^3 = (t^2)^3$ dependence $\longrightarrow$ Similar to LCS type boundary

Upon picking a simple ansatz we can solve the constraints and write down the model

$$\Pi = \begin{pmatrix} 1 \\ az_1 \\ \frac{\log[z_2]}{2\pi i} \\ -\frac{i\log[z_2]^3}{48\pi^3} - \frac{ia^2nz_1^2\log[z_2]}{4\pi} + \frac{a^2}{4\pi i}z_1^2 + i\delta_2 + i\delta_1 az_1 \\ -az_1\frac{\log[z_1]+n\log[z_2]}{2\pi i} + i\delta_1 \\ -\frac{\log[z_2]^2}{8\pi^2} - \frac{1}{2}a^2nz_1^2 \end{pmatrix}$$

$\langle \mathrm{I}_1 \vert \mathrm{IV}_2 \vert \mathrm{IV}_1 \rangle$	$\langle \mathrm{I}_1 \vert \mathrm{IV}_2 \vert \mathrm{IV}_2 \rangle$
$n=0$	$n \in \mathbb{Q}_{>0}$
$a \in \mathbb{R} - \{0\}$	
$\delta_1, \delta_2 \in \mathbb{R}$	

We give the construction for all non-LCS $h^{2,1} = 1$ and $h^{2,1} = 2$ cases in the paper !

Outlook

- Run the program for more than 2 moduli $\longrightarrow$ No type III_c boundary for $h^{2,1} < 3$
- Generalize to Calabi-Yau fourfolds
- Use models for computations in the Swampland programme
- Relate to Picard-Fuchs approach [Kerr '20] , [Bloch, Vlasenko '19]

Thank you !