Elliptic Calabi-Yau manifolds with the largest $h^{1,1}$

Yi-Nan Wang

University of Oxford

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Papers Mentioned

1. YNW, On the Elliptic Calabi-Yau Fourfold with Maximal $h^{1,1}$, (2001.07258)
2. Tian, YNW, Elliptic Calabi-Yau fivefolds and 2d (0,2) F-theory landscape, (2009.10668)

Calabi-Yau manifolds with extremal $h^{1,1}$

• Mathematical question: given any compact irreducible d-dimensional Calabi-Yau manifold X_d ($d \geq 3$), what is the value of

$$H_d = \max(h^{1,1}(X_d)) \tag{1}$$

• In physics, consider various string/M-theory on X_d with the maximal $h^{1,1}(X_d)$, give rise to the (currently known) largest number of fields coupled to supergravity in a specific dimension.

(1) M-theory on the CY3 with maximal $h^{1,1}$: largest number of vector multiplets in 5d $\mathcal{N} = 1$ SUGRA landscape

(2) F-theory on the elliptic CY3 with maximal $h^{1,1}$: largest number of tensor and vector multiplets in 6d (1,0) SUGRA landscape

(3) F-theory on the elliptic CY4 with maximal $h^{1,1}$: largest number of axion scalars and vector multiplets in 4d $\mathcal{N} = 1$ SUGRA landscape

Elliptic Calabi-Yau manifolds

• It is conjectured that most of Calabi-Yau manifolds with a moderately large $h^{1,1}$ is elliptic.

• CY3 case

(1) CICY: over 99% has elliptic structure (Anderson, Gao, Gray, Lee 16' 17')(Anderson, Gray, Hammack 18')

(2) Batyrev construction from reflexive polytopes: (Huang Taylor 19')
All CY3s with $h^{1,1} \geq 195$ or $h^{2,1} \geq 228$ has an elliptic fibration
The fraction of polytopes without a toric fibration structure scales as $0.1 \times 2^{5-h^{1,1}}$ for $h^{1,1} \lesssim 20$

• Maximal $h^{1,1}$ for elliptic CY = maximal $h^{1,1}$ for CY?

• Not much is known about CY4 or higher cases. Nonetheless, has interesting F-theory applications if an elliptic structure exists on X_d.

Calabi-Yau manifolds with extremal Hodge numbers

- The maximal known $h^{1,1}$ for X_d: Klemm-Lian-Roan-Yau 97'.
- Define a sequence of integers:

$$m_0 = 1 \ , \ m_{k+1} = m_k(m_k + 1) \, . \tag{2}$$

- The first a few m_i are

$$m_1 = 2 \ , \ m_2 = 6 \ , \ m_3 = 42 \ , \ m_4 = 1\,806 \ , \ m_5 = 3\,263\,442 \, . \tag{3}$$

- We define a reflexive polytope Δ^*_{d+1} associated to a $(d+1)$-dimensional weighted projective space $\mathbb{P}^{1,1,d_1,d_2,\dots,d_d}$.
- The weights are computed as:

$$d_1 = 2 \cdot m_{d-1} \ , \ d_2 = (2 + d_1) \cdot m_{d-2} \ , \ d_{k+1} = \left(2 + \sum_{i=1}^{k} d_i \right) \cdot m_{d-k-1} \, . \tag{4}$$

Calabi-Yau manifolds with extremal Hodge numbers

- After an $SL(d+1,\mathbb{Z})$ rotation, the ambient space has the vertices

$$\begin{aligned}&(\vec{0}_{d-1},0,1)\\&(\vec{0}_{d-1},1,0)\\&(v_i^*,-2,-3)\quad(i=1,\ldots,d)\end{aligned}\tag{5}$$

- A $\mathbb{P}^{2,3,1}$ fibration over $(d-1)$-dimensional base B_{d-1}^* with vertices v_i^*.
- The Calabi-Yau hypersurface X_d^* is an elliptic fibration over B_{d-1}^*, with extremal $h^{d-1,1}$.

(1) $d=3$: $(h^{1,1},h^{2,1})=(11,491)$
(2) $d=4$: $(h^{1,1},h^{2,1},h^{3,1})=(252,0,303148)$
(3) $d=5$:
$(h^{1,1},h^{2,1},h^{3,1},h^{4,1},h^{2,2})=(151701,0,0,247538602581,758522)$(Tian YNW 20')

- Hodge numbers can be computed by Landau-Ginzburg methods. (Vafa 89')

Calabi-Yau manifolds with extremal Hodge numbers

• As Δ^*_{d+1} is reflexive, the Calabi-Yau hypersurface X_d of its dual polytope Δ_{d+1} gives the extremal $h^{1,1}$ (mirror symmetry)

$$\begin{aligned} H_3 &= 491 \\ H_4 &= 303148 \\ H_5 &= 247538602581\,. \end{aligned} \tag{6}$$

• Δ_{d+1} can be rotated into the form of $\mathbb{P}^{2,3,1}$ fibration as well:

$$\begin{aligned} &(\vec{0}_{d-1}, 0, 1) \\ &(\vec{0}_{d-1}, 1, 0) \\ &(v_i, -2, -3) \quad (i = 1, \ldots, d) \end{aligned} \tag{7}$$

• Hence X_d is an elliptic fibration over the base manifold B_{d-1} with toric vertices v_i.

• The monomials in the Tate model correspond to points in Δ^*_{d+1}, and we can read off the F-theory gauge group over each v_i.

CY3 with largest $h^{1,1}$

- $X_3 : (h^{1,1}, h^{2,1}) = (491, 11)$
- A generic elliptic fibration over base B_2 (Morrison, Taylor 12')(Taylor 12')
- The construction of B_2

(1) start from a toric base with cyclic representation

$$(-12//-11//(-12//)^{13}, -11//-12, 0)\,. \tag{8}$$

$$// \equiv -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1\,. \tag{9}$$

(2) Blow up the two (-11)-curves at generic point to remove cod-2 (4,6) points. B_2 is no longer toric.

- In the known 6d (1,0) landscape, this F-theory model has the largest

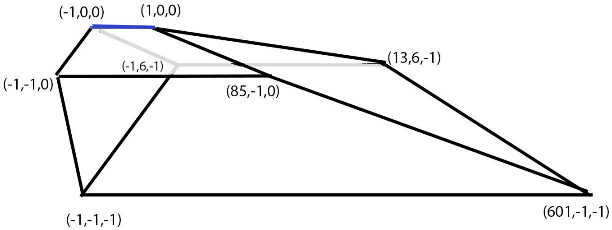
(1) $T = 193$

(2) $\text{rk}(G) = 296$, $G = E_8^{17} \times F_4^{16} \times (G_2 \times SU(2))^{32}$.

CY4 with largest $h^{1,1}$

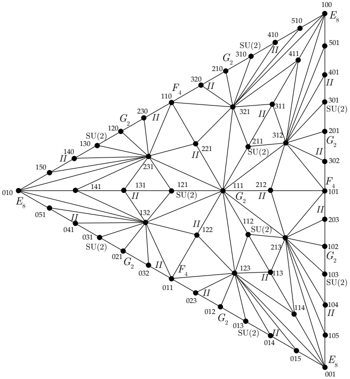
- $X_4 : (h^{1,1}, h^{2,1}, h^{3,1}) = (303148, 0, 252)$
- A generic elliptic fibration over base B_3 (Wang 20')
- The construction of B_3

(1) construct a non-compact toric threefold B_{E_8} given by the following 3d polytope:



- After triangulation, the toric fan has 5016 3d cones, 7576 2d cones and 2561 rays
- Tune an E_8 on each of these rays!

(2) blow up the 5016 (E_8, E_8, E_8) non-minimal loci and 7576 (E_8, E_8) non-minimal loci.



(3) Blow up 619 non-toric cod-2 (4,6) curves on divisors that support E_8

(4) add two more rays into B_{E_8} to make it compact

CY4 with largest $h^{1,1}$

- In the known 4d $\mathcal{N}=1$ landscape, the F-theory model has the largest

(1) $G = E_8^{2561} \times F_4^{7576} \times G_2^{20168} \times SU(2)^{30200}$ (Candelas, Perevalov, Rajesh 97')

(2) $N_{\mathrm{axion}} = h^{1,1}(B_3) + 1 = 181820$

- Comments on moduli stabilization:

$$\frac{\chi(X_4)}{24} \approx 75000 \gg h^{3,1}(X_4) = 252\,, \tag{10}$$

exceeds the bound in (Bena, Blåbäck, Graña, Lüst 20')

- Complex structure moduli can be stabilized
- Kähler moduli stabilization? As most of the divisors come from blow-ups, one expects most of the Kähler moduli can be stabilized (Halverson, Plesser, Ruehle, Tian 19')
- Construct models with a standard model sector in this regime, such that all the moduli can be stabilized?

CY5 and 2d F-theory

- F-theory on elliptic CY5 $\rightarrow$ 2d (0,2) SUGRA (Weigand, Schafer-Nameki 16')(Lawrie, Schafer-Nameki, Weigand 16')(Weigand, Xu 18')
- M-theory on CY5 $\rightarrow$ 1d $\mathcal{N} = 2$ SQM (Haupt, Lukas, Stelle 09')
- Why study 2d F-theory?

(1) Gravitational anomaly can be checked

(2) Potential relation to other 2d theories like GLSM

- The supermultiplet structure of 2d (0,2) theory:

(1) Vector multiplet: (A_μ, η_-, D)

(2) Chiral multiplet: (φ, χ_+)

(3) Fermi multiplet: (ρ_-, G)

(4) Tensor multiplet: (ϕ, ψ_-)

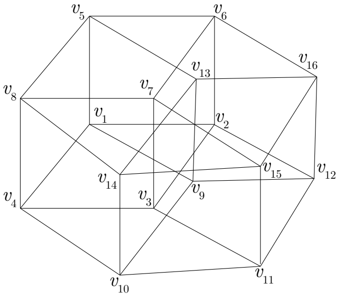
(5) Supergravity multiplet

- Rank of gauge group $= h^{1,1}(X_5) - h^{1,1}(B_4) - 1$.

CY5 with largest $h^{1,1}$

- $(h^{1,1}, h^{2,1}, h^{3,1}, h^{4,1}) = (247538602581, 0, 0, 151701)$
- Generic fibration over B_4, constructed as:

(1) Start with a (non-compact) toric fourfold, triangulate it

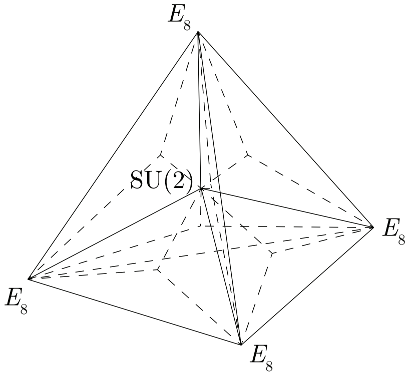


(2) Tune an E_8 on each toric divisor

(3) Blow up all the (E_8, E_8, E_8, E_8), (E_8, E_8, E_8), (E_8, E_8) collisions

CY5 with largest $h^{1,1}$

- New cod-4 blow up:



(4) Blow up 167873112 non-toric cod-2 (4,6) loci on E_8 divisors

(5) Add three toric rays back in to make B_4 compact.

- $h^{1,1}(B_4) = 181\,299\,558\,192$
- $G = E_8^{482\,632\,421} \times F_4^{3\,224\,195\,728} \times G_2^{11\,927\,989\,964} \times SU(2)^{25\,625\,222\,180}$

A list of elliptic CY5s

• We studied the CY5 hypersurfaces of reflexive $\mathbb{P}^{1,d_1,d_2,d_3,d_4,d_5,d_6}$ with a $\mathbb{P}^{2,3,1}$ fibration structure, with degree

$$1 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 \leq 150 \tag{11}$$

• A sublist of (Kreuzer, Riegler, Sahakyan 01')

• In total 140 cases, including

(1) Smooth bases, e. g. $\mathbb{P}^4$

(2) Singular bases, e. g. $\mathbb{P}^{1,1,1,1,2}$

• Checked the 2d gravitational anomaly cancellation for a number of examples without gauge group or with non-Higgsable gauge groups.

2d Gravitational anomaly cancellations

- In 2d, the anomaly polynomial is a 4-form: I_4
- While Green-Schwarz mechanism can be used to cancel the gauge anomaly, the pure gravitational anomaly needs to be cancelled by itself

(Lawrie, Schafer-Nameki, Weigand 16')(Weigand, Xu 18')

$$I_{4,\mathrm{grav}} = \frac{1}{24} p_1(T)(\mathcal{A}_{\mathrm{moduli}} + \mathcal{A}_{\mathrm{universal}} + \mathcal{A}_{3-7} + \mathcal{A}_{7-7}) \qquad (12)$$

(1) $\mathcal{A}_{\mathrm{moduli}}$: bulk moduli fields

2d multiplet	Multiplicity
Vector	$h^{1,1}(X_5) - h^{1,1}(B_4) - 1$
Chiral	$h^{2,1}(X_5) + h^{4,1}(X_5) - (-h^{1,1}(B_4) + h^{2,1}(B_4) - h^{3,1}(B_4)) - 1$
Fermi	$h^{2,1}(B_4) - h^{3,1}(B_4) + h^{3,1}(X_5)$
Tensor	$\tau(B_4)$

- Chiral multiplet has $+1$ contribution
- Vector, Fermi and tensor multiplets have (-1) contribution

$$\mathcal{A}_{\mathrm{moduli}} = -\tau(B_4) + \chi_1(X_5) - 2\chi_1(B_4) \qquad (13)$$

2d Gravitational anomaly cancellations

(2) $\mathcal{A}_{\mathrm{universal}} = 24$: contribution from the single gravity mutiplet

(3) $\mathcal{A}_{3-7}$: contribution from supermultiplets in the 3-7 sector:

In 2d F-theory, there are forced D3-branes wrapping 2-cycle C on B_4:

$$C = \frac{1}{24}\pi_* c_4(X_5) - \frac{1}{2}\pi_*(G_4 \cdot G_4) \tag{14}$$

Multiplet	Multiplicity
Chiral	$h^0(C, N_{C/B_4}) + g(C) - 1 + c_1(B_4) \cdot C$
Fermi	$h^0(C, N_{C/B_4}) + g(C) - 1 + 7c_1(B_4) \cdot C$

$$\mathcal{A}_{3-7} = -6c_1(B_4) \cdot C|_{B_4} \tag{15}$$

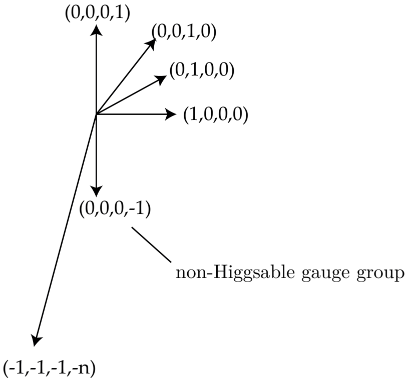
(4) $\mathcal{A}_{7-7}$: contribution from the 7-7 sector

$$\mathcal{A}_{7-7} = \sum_{\mathrm{R}} \dim(\mathrm{R})\chi(\mathrm{R}) - \mathrm{rk}(G)\chi(\mathrm{adj}) \tag{16}$$

- For non-Higgsable gauge groups, $\chi(\mathrm{adj}) = -\frac{1}{24}\int_S c_1(S)c_2(S)$.

Examples

- 6d ambient space: $\mathbb{P}^{1,1,1,1,n,2n+8,3n+12}$
- For $n \geq 4$, the base is "generalized Hirzebruch fourfold" $B_{n,4}$, analogous to Hirzebruch surfaces $\mathbb{F}_n$ in 6d F-theory



Examples

Base	G_{nH}	$\chi_1(X_5)$	$\xi_1(B_4)$	$\tau(B_4)$	$\mathcal{A}_{\text{grav}\|3\text{-}7}$	$\mathcal{A}_{\text{grav}\|7\text{-}7}$
$B_{4,4}$	-	93 188	-2	0	$-93\,216$	0
$B_{6,4}$	$SU(3)$	151 466	-2	0	$-151\,488$	-6
$B_{8,4}$	$SO(8)$	235 292	-2	0	$-235\,296$	-24
$B_{12,4}$	E_6	494 924	-2	0	$-494\,880$	-72
$B_{24,4}$	E_8	2 314 868	-2	0	$-2\,314\,656$	-240

- Gravitational anomaly is cancelled, independent of G_4 flux choice!
- In fact, these holds for any vertical flux $G_4 \in H_V^{2,2}(X_5)$ that

(1) Satisfies transversality conditions:

$$\int G_4 \wedge S_0 \wedge \omega_4 = 0,\ \int G_4 \wedge \omega_6 = 0,\ \forall \omega_4 \in H^4(B_4),\ \omega_6 \in H^6(B_4). \tag{17}$$

(2) Does not break gauge groups

$$\int G_4 \wedge E_i \wedge \omega_4 = 0,\ \forall \omega_4 \in H^4(B_4)\,. \tag{18}$$

Singular bases

- Consider $B_4 = \mathbb{P}^{1,1,1,1,n}$, has a $\mathbb{C}^4/\mathbb{Z}_n$ orbifold singularity.
- The hyperplane class H has the self-intersection number

$$H^4 = \frac{1}{n}\,. \tag{19}$$

The various Chern classes of $\mathbb{P}^{1,1,1,1,n}$ are

$$\begin{aligned} c_1 &= (n+4)H \\ c_2 &= (4n+6)H^2 \\ c_3 &= (6n+4)H^3 \\ c_4 &= 4 + \frac{1}{n}\,. \end{aligned} \tag{20}$$

- In this case, $\tau(B_4)$ and $\chi_1(B_4)$ are still well-defined: (Maxim, Schürmann 15')

$$\begin{aligned} \tau(\mathbb{P}^{1,1,1,1,n}) &= 1 \\ \chi_1(\mathbb{P}^{1,1,1,1,n}) &= -1\,. \end{aligned} \tag{21}$$

Singular bases

- The total gravitational anomaly I_4 does not cancel, needs to add a new contribution from the base singularity:

$\mathbb{Z}_n$	$\mathcal{A}_{\text{orbifold}}$
$\mathbb{Z}_2$	-1
$\mathbb{Z}_3$	1
$\mathbb{Z}_4$	3
$\mathbb{Z}_6$	9
$\mathbb{Z}_8$	15
$\mathbb{Z}_{12}$	27
$\mathbb{Z}_{24}$	63

- What is the physical origin? localized 2d (0,2) SCFT?