# New Aspects of Line Bundle Cohomology and Applications to String Phenomenology


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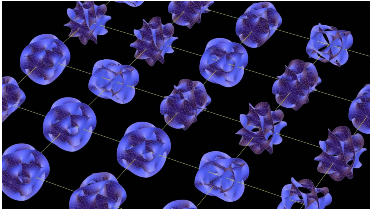
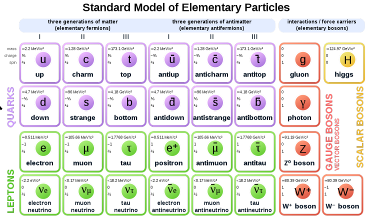


Top-down approach

Bottom-up approach



# Model building in the $E_8 \times E_8$ heterotic string theory

The effective theory can be specified in terms of 2 pieces of geometrical data:

- a Calabi-Yau threefold $X$
- a slope-zero, polystable, holom. vector bundle $V$ with structure group $G$

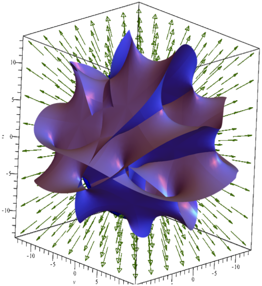
Result: a four-dimensional model with $N = 1$ supersymmetry, a gauge group given by the commutant of $G$ in $E_8$ and chiral matter.

The simplest class of vector bundles are abelian bundles, i.e. sums of line bundles. Example: $V = \bigoplus_{i=1}^{5} L_i$, resulting in an $SU(5)$ GUT.

| multiplet | bundle | total number | required |
|---|---|---|---|
| $\mathbf{10}$ | $V$ | $\sum_i h^1(X, L_i)$ | 3 |
| $\overline{\mathbf{10}}$ | $V^*$ | $\sum_i h^1(X, L_i^*)$ | 0 |
| $\overline{\mathbf{5}}$ | $\wedge^2 V$ | $\sum_{i<j} h^1(X, L_i \otimes L_j)$ | $3 + n_H$ |
| $\mathbf{5}$ | $\wedge^2 V^*$ | $\sum_{i<j} h^1(X, L_i^* \otimes L_j^*)$ | $n_H$ |
| $\mathbf{1}$ | $V \otimes V^*$ | $\sum_{i,j} h^1(X, L_i \otimes L_j^*)$ | |

# Line bundle cohomology formulae

topological data of $(X, V)$



global data:

cohomology groups

$h^\bullet(X, V)$

local data

# The Euler characteristic

- The Hirzebruch-Riemann-Roch theorem gives

$$\chi(X, V) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, V) = \int_X \mathrm{ch}(V) \cdot \mathrm{td}(X)$$

Main question: is there anything like $h^i(X, V) = \int_X \text{topological inv}(X, V)$?

- Nice bundle: all higher cohomologies vanish, then $h^0(X, V) = \chi(X, V)$.
- Example: line bundles on $\mathbb{P}^n$, the Bott formula

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{k+n}{n} = \frac{1}{n!}(1+k)\ldots(n+k)\ ,\ \text{if } k \geq 0, \text{ and 0 otherwise.}$$

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0\ ,\ \text{if } 0 < i < n\ .$$

$$h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{-k-1}{-n-k-1} = \frac{1}{n!}(-n-k)\ldots(-1-k)\ ,\ \text{if } k \leq -n-1,$$

and 0 otherwise.

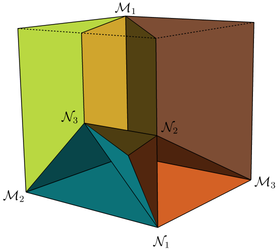
- Notation: $\mathcal{O}_{\mathbb{P}^n}(k)$ is the line bundle whose curvature 2-form is $kJ$, where $J$ is the standard Kähler form on $\mathbb{P}^n$.

# Formulae on complex surfaces

# Line bundle cohomology on surfaces: generalities

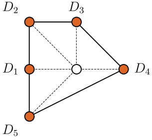
- We studied: compact toric surfaces, (generalised) del Pezzo surfaces and K3 surfaces.
- General feature: the Picard lattice splits into a number of regions (cones), in each of which the dimension of the zeroth cohomology of $L$ can be computed as the Euler characteristic of $L$ or of some other line bundle $\tilde{L}$
- Information needed to write down the general formula: the intersection form and the generators of the Mori cone; these give both the regions and the map $L \to \tilde{L}$ in each region.
- In general, finding the intersection form and the generators of the Mori cone is difficult.

# Complex Surfaces: a toric example




- Zariski decomposition: $D = P + N$
  $P = D - N$
  $D$ effective, integral; $P$ nef, rational
- $h^0(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S(\lceil P \rceil)) = h^0(S, \mathcal{O}_S(\lfloor P \rfloor))$
- vanishing theorem for $\mathcal{O}_S(\lfloor P \rfloor))$ or $\mathcal{O}_S(\lceil P \rceil))$
- $h^0\big(S, \mathcal{O}_S(D)\big) = \chi\big(S, \mathcal{O}_S(\lfloor P \rfloor)\big)$

| $\Sigma$ | $h^0(F_6, \mathcal{O}_{F_6}(D))$ |
|---|---|
| $\Sigma_{\mathrm{nef}}$ | $\chi(F_6, \mathcal{O}_{F_6}(D))$ |
| $\Sigma_1$ | $\chi\Big(F_6, \mathcal{O}_{F_6}\big(D - \big\lceil -\frac{1}{2} D \cdot \mathcal{M}_1 \big\rceil \mathcal{M}_1\big)\Big)$ |
| $\Sigma_2$ | $\chi\Big(F_6, \mathcal{O}_{F_6}\big(D - (-D \cdot \mathcal{M}_2)\mathcal{M}_2\big)\Big)$ |
| $\Sigma_3$ | $\chi\Big(F_6, \mathcal{O}_{F_6}\big(D - (-D \cdot \mathcal{M}_3)\mathcal{M}_3\big)\Big)$ |
| $\Sigma_{1,2}$ | $\chi\Big(F_6, \mathcal{O}_{F_6}\big(D - (-D \cdot (\mathcal{M}_1 + \mathcal{M}_2))\mathcal{M}_1 - (-D \cdot (\mathcal{M}_1 + 2\mathcal{M}_2))\mathcal{M}_2\big)\Big)$ |
| $\Sigma_{1,3}$ | $\chi\Big(F_6, \mathcal{O}_{F_6}\big(D - \big\lceil -\frac{1}{2} D \cdot \mathcal{M}_1 \big\rceil \mathcal{M}_1 - (-D \cdot \mathcal{M}_3)\mathcal{M}_3\big)\Big)$ |



| $\Sigma$ | $h^0\big(F_6, \mathcal{O}_{F_6}(k_1\mathcal{M}_1 + k_2\mathcal{M}_2 + k_3\mathcal{M}_3)\big)$ |
|---|---|
| $\Sigma_{\mathrm{nef}}$ | $1 - k_1^2 + \frac{1}{2}k_2 + k_1k_2 - \frac{1}{2}k_2^2 + \frac{1}{2}k_3 + k_2k_3 - \frac{1}{2}k_3^2$ |
| $\Sigma_1$ | $1 + \frac{1}{2}k_2 - \frac{1}{2}k_2^2 + \frac{1}{2}k_3 + k_2k_3 - \frac{1}{2}k_3^2 + k_2\left\lfloor\frac{1}{2}k_2\right\rfloor - \left\lfloor\frac{1}{2}k_2\right\rfloor^2$ |
| $\Sigma_2$ | $1 + \frac{1}{2}k_1 - \frac{1}{2}k_1^2 + k_3 + k_1k_3$ |
| $\Sigma_3$ | $1 - k_1^2 + k_2 + k_1k_2$ |
| $\Sigma_{1,2}$ | $1 + \frac{3}{2}k_3 + \frac{1}{2}k_3^2$ |
| $\Sigma_{1,3}$ | $1 + k_2 + k_2\left\lfloor\frac{1}{2}k_2\right\rfloor - \left\lfloor\frac{1}{2}k_2\right\rfloor^2$ |

## Theorem: line bundle cohomology formula for toric surfaces

*Let $S$ be a smooth projective toric surface, and $D$ an effective integral divisor with Zariski decomposition $D = P + N$. Then*

$$h^0\big(S, \mathcal{O}_S(D)\big) = \chi\big(S, \mathcal{O}_S(\lfloor P \rfloor)\big)\,.$$

*Explicitly, if $D$ lies in the Zariski chamber $\Sigma_{i_1,\ldots i_n}$, obtained by translating a codimension $n$ face $F$ of the nef cone along the set of dual Mori cone generators $\{\mathcal{M}_{i_1}, \mathcal{M}_{i_2}, \ldots \mathcal{M}_{i_n}\}$ orthogonal (with respect to the intersection form) to the face $F$, then*

$$h^0\big(S, \mathcal{O}_S(D)\big) = \chi\left(S, \mathcal{O}_S\left(D - \sum_{k=1}^{n} \left\lceil -D \cdot \mathcal{M}^{\vee}_{i_k,\{i_1,\ldots,i_n\}} \right\rceil \mathcal{M}_{i_k}\right)\right)\,.$$

Similar theorems for generalised del Pezzo surfaces and K3 surfaces, more details in [Brodie, AC, 2009.01275].

# Line Bundle Cohomology on CY3

## General features

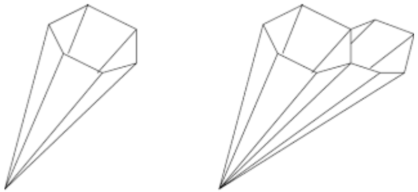
- We studied: CICY three-folds, smooth quotients thereof by freely acting discrete symmetries, (hypersurfaces) in toric varieties.
- We know empirically that analytic formulae exist for all cohomology groups. By Serre duality, it is enough to understand the zeroth and the first cohomologies. So far: the zeroth cohomology (global sections).
- The Picard group splits into various cones, in each of which the zeroth cohomology can be computed as an index.
- In the Kähler cone $\mathcal{K}(X)$, due to Kodaira's vanishing theorem

$$h^0(X, L) = \chi(X, L)$$

where the Euler characteristic of $L = \mathcal{O}_X(D)$, on a Calabi-Yau 3-fold is

$$\begin{aligned}\chi(X, \mathcal{O}_X(D)) &= \frac{1}{6}D^3 + \frac{1}{12}c_2(X)\cdot D \\ &= \frac{1}{6}d_{ijk}k^i k^j k^k + \frac{1}{12}d_{ijk}c_2(X)^{ij}k^k\ , \quad D = \sum_{i=1}^{h^{1,1}(X)} k^i D_i\end{aligned}$$

# General features



(a) (b)

- We use line bundle cohomology to infer the existence of the flops and to read-off Gromov-Witten invariants.
- Neighbouring $\mathcal{K}(X)$, there are Kähler cones of flopped manifolds $X'$, which we detect by fitting the zeroth cohomology data to the Euler characteristic in these regions

$$h^0(X, L) = h^0(X', L') = \chi(X', L')$$

- We read off the triple intersection numbers on $X'$ as well as the $c_2(X')$ form. The way in which these differ from the topological data on $X$ is related to Gromov-Witten invariants, which we are able to read off
- In certain cases the effective cone contains other subcones that are not Kähler cones of flopped manifolds.

# Flops

- Flops are co-dimension two surgery operations and isomorphisms in co-dimension one.
- On a three-fold, a flop contracts (cuts out) a number of isolated $\mathbb{P}^1$-curves (rational curves) and replaces them with others

$$\begin{tikzcd} X \arrow[rr, "\text{flop}"] \arrow[dr, "\text{contr. }1"'] & & X' \arrow[dl, "\text{contr. }2"] \\ & X_{\mathrm{sing}} & \end{tikzcd}$$

- The flopped manifold $X'$ is Calabi-Yau and has the same Hodge numbers as $X$. The triple intersection numbers and the $c_2$-form change in the following way:

$$D'^3 = D^3 - \sum_i (D \cdot \mathcal{C}_i)^3$$

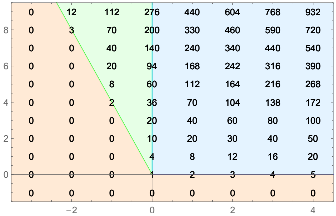
$$c_2(X') \cdot D' = c_2(X) \cdot D + 2 \sum_i D \cdot \mathcal{C}_i \ ,$$

where $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N$ are the isolated exceptional $\mathbb{P}^1$ curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ contracted in the flop.

# Examples

# The manifold 7887, favourable embedding

$$X = \begin{matrix}\mathbb{P}^1 \\ \mathbb{P}^3\end{matrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}^{2,86} \qquad L = \mathcal{O}_X(k_1 D_1 + k_2 D_2)$$




- The positive quadrant: $\mathcal{K}(X)$.
- Here $h^0(X, L) = \chi(X, L)$, the Euler characteristic being computed with the following topological data:

| $d_{111}$ | $d_{112}$ | $d_{122}$ | $d_{222}$ | $c_2 \cdot D_1$ | $c_2 \cdot D_2$ |
|---|---|---|---|---|---|
| 0 | 0 | 4 | 2 | 24 | 44 |

where $d_{ijk} = D_i \cdot D_j \cdot D_k$.

- The other cone is $\mathcal{K}(X')$, and has generators $\{\tilde{D}'_1 = -D'_1 + 4D_2, \tilde{D}'_2 = D'_2\}$. The topological data for $X'$ is given by

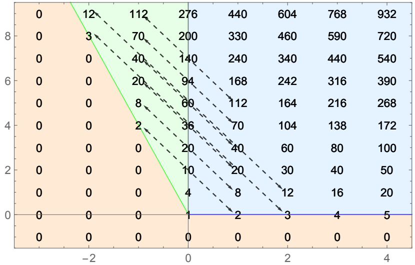
| $d'_{111}$ | $d'_{112}$ | $d'_{122}$ | $d'_{222}$ | $c'_2 \cdot D'_1$ | $c'_2 \cdot D'_2$ |
|---|---|---|---|---|---|
| $-64$ | 0 | 4 | 2 | 152 | 44 |

- In the basis $\{\tilde{D}'_1, \tilde{D}'_2\}$, this data is identical to that for $X$. Hence $X' \cong X$.
- The number of isolated curves in the class dual to $D_1$ is 64. This is the correct Gromov-Witten invariant.

# The manifold 7887, continued

Since $X' \cong X$, it is not surprising that the zeroth coh. displays a $\mathbb{Z}_2$ symmetry

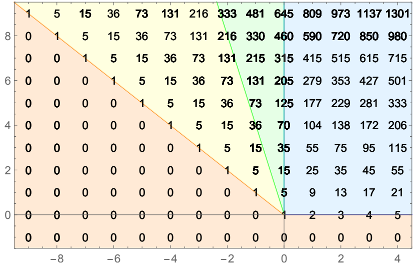
$$h^0(X, \mathcal{O}_X(\mathbf{k})) = h^0(X, \mathcal{O}_X(M\mathbf{k})) \text{ with } M = \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix}, \ \mathbf{k} = (k_1, k_2)^T$$




| region in eff. cone | $h^0(X, L = \mathcal{O}_X(k_1 D_1 + k_2 D_2))$ |
|---|---|
| $\mathcal{K}(X)$ | $\chi(X, L)$ |
| $\mathcal{K}(X')$ | $\chi(X', L')$ |
| $k_1 = 0,\ k_2 > 0$ | $\chi(X, L) = \chi(X', L')$ |
| $k_1 \leq 0,\ k_2 = -4k_1$ | $\chi(\mathbb{P}^1, -k_1 H_{\mathbb{P}^1})$ |
| $k_1 \geq 0,\ k_2 = 0$ | $\chi(\mathbb{P}^1, k_1 H_{\mathbb{P}^1})$ |

# The manifold 7885

$$X = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \left[ \begin{array}{cc} 1 & 1 \\ 4 & 1 \end{array} \right]^{2,86} \qquad L = \mathcal{O}_X(k_1 D_1 + k_2 D_2)$$

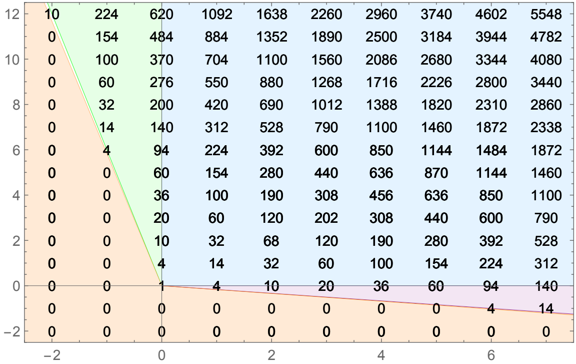



New feature: the presence of a Zariski chamber.

| region in eff. cone | $h^0(X, L = \mathcal{O}_X(D = k_1 D_1 + k_2 D_2))$ |
|---|---|
| $\mathcal{K}(X)$ | $\chi(X, \mathcal{O}_X(D))$ |
| $\overline{\mathcal{K}}(X') \setminus \{\mathcal{O}_X\}$ | $\chi(X', \mathcal{O}_{X'}(D'))$ |
| $\overline{\Sigma}$ | $\chi\left(X', \mathcal{O}_{X'}\left(D' - \left\lceil \frac{D' \cdot \tilde{C}_2'}{\Gamma' \cdot \tilde{C}_2'} \right\rceil \Gamma'\right)\right)$ |
| $k_1 \geq 0,\ k_2 = 0$ | $\chi(\mathbb{P}^1, (D \cdot C_1) H_{\mathbb{P}^1})$ |

# The manifold 7863

$$X = \begin{array}{c} \mathbb{P}^3 \\ \mathbb{P}^3 \end{array} \left[ \begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 1 \end{array} \right]^{2,66} \qquad L = \mathcal{O}_X(k_1 D_1 + k_2 D_2)$$




New features: infinitely many Kähler cones. The effective cone (in this case the extended Kähler cone) turns out to be irrational.

# Heterotic Flop Transitions

# Topological transitions in the heterotic string: an open problem

- Topological transitions in the heterotic string context: difficult, due to the presence of internal gauge flux. What happens with the bundle when the manifold undergoes a topological transition?
- We have a proposal for carrying line bundles over flop transitions, understood in terms of the divisor - line bundle correspondence and the observation that flops are co-dimension two surgery operations and isomorphisms in co-dimension one.
- Divisor classes, and hence line bundles, can unambiguously be matched across the transition.
- Things to understand: the anomaly cancellation condition and the changes in the massless spectrum.
- Slope stability of the the bundle (N=1 supersymmetry) may prohibit access to the flopping locus. For enough Kähler moduli, this is not a problem.

## Summary and Outlook

## An overview of the work to date

- formula on the tetra-quadric CY 3-fold (hypersurface of degree (2,2,2,2) in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$) [AC DPhil thesis '13; Buchbinder, AC, Lukas 1311.1941]
- formulae on several Picard number 2, 3 and 4 CICY threefolds and smooth quotients thereof by discrete symmetries [AC, Lukas, 1808.09992]
- (hypersurfaces) in toric varieties [Klaewer, Schlechter, 1809.02547]
- more Picard number 2 CICY 3-folds [Larfors, Schnedeir, 1906.00392]
- formulae on del Pezzo and Hirzebruch surfaces; first mathematical proofs [Brodie, AC, Deen, Lukas, 1906.08769, 1906.08363]
- machine learning implementation [Brodie, AC, Deen, Lukas, 1906.08730]
- certain classes of surfaces understood: theorems for toric surfaces, generalised del Pezzo surfaces and K3 surafaces; simple elliptic fibrations over such surfaces [Brodie, AC, 2009.01275]
- zeroth cohomology on CICY threefolds understood [Brodie, AC, Lukas 2010.06597]
- ...more to come

Case-by-case results computed algorithmically with

- the CICY package [Anderson, Gray, He, Lee, Lukas]
- cohomCalg package [Blumenhagen, Jurke, Rahn, Thorsten, Roschy]
- pyCICY - a Python CICY toolkit [Larfors, Schneider]

There are a number of questions that we would like to address in the near future:

- work out cohomology formulae for manifolds with higher Picard number; automatise / machine learn
- understand the structures present in the higher cohomologies (on threefolds: the first cohomology)
- cohomology formulae for non-abelian bundles: monads, extensions
- implement a bottom-up approach to model building; reinforcement learning of the string landscape
- explore the complex structure dependence of the cohomology formulae; moduli stabilisation