Analytic Conifold Periods for Small Superpotentials

arXiv:2009.03325

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October 6, 2020

Why Periods?

- Setting: Type II string theory on CY 3-fold with fluxes
- $\mathcal{N} = 1$ theory described by Kahler- (K) and superpotential (W)
- These depend on the fluxes and periods Π

$$K = -\log(-i\,\bar{\Pi} \cdot \Sigma \cdot \Pi) - \log(S + \bar{S}) \tag{1}$$

$$W = (F - SH) \cdot \Sigma \cdot \Pi \tag{2}$$

- For KKLT small W_0 required

Idea

- Known mechanism at the LCS point
  [Demirtas, Kim, McAllister, Moritz 19']
- Idea: Generalize the same mechanism to the conifold
- This Talk: How to compute the periods close to the conifold
- Second talk today by Max: How to generalize the method of Demirtas, Kim, McAllister and Moritz to the conifold
- Closely related independent work [Demirtas, Kim, McAllister, Moritz 20']

Periods

Let X be a CY hypersurface or CICY in $\mathbb{WCP}$.

Choose symplectic basis $\gamma^\alpha \in H_3(X, \mathbb{Z}) \qquad \alpha = 0, \ldots, 2h^{2,1}+1.$

$$\Pi^\alpha(x) = \int_{\gamma^\alpha} \Omega(x) \tag{3}$$

The x denote the moduli of the CY, Ω is the unique holomorphic 3-form.

Charge Vectors

- CY hypersurfaces or CICYs can be constructed using polyhedra.
- Linear dependences between the vertices of these polyhedra give the charge vectors l.
- Closely related to the generators of the Mori cone.
- Listed in literature and can be computed automatically.

How to compute the periods?

The periods fulfill a system of differential equations, the Picard-Fuchs equations.

$$\mathcal{D}_l = \prod_{l_i>0} \left(\frac{\partial}{\partial_{a_i}}\right)^{l_i} - \prod_{l_i<0} \left(\frac{\partial}{\partial_{a_i}}\right)^{-l_i} , \qquad l \in \{l_i\} , \tag{4}$$

$$x_k = (-1)^{l_0^{(k)}} a_0^{l_0^{(k)}} \dots a_s^{l_s^{(k)}} \tag{5}$$

$$\mathcal{D}_l \, \omega = 0 \tag{6}$$

How to compute?

Periods in a symplectic basis hard to compute. Solving the PF equations in any local basis is easy.

$$\Pi = m \cdot \omega \tag{7}$$

$\rightarrow$ Compute local solutions and compute transition matrices m to the symplectic basis.

- At the LCS monodromies fix the transition matrix uniquely.
- At the conifold monodromies are not enough.

$\rightarrow$ Compute the symplectic basis at the LCS point and continue numerically to other points in the moduli space.

Bases and their relations

$$
\begin{tikzcd}
\Pi \arrow[d, leftrightarrow, "\text{monodromy}"', "m_1"] & & \Pi \arrow[d, leftrightarrow, "m_2 = m \cdot m_1"] \\
\omega \arrow[r, leftrightarrow, "\text{analytic continuation}"'] & \omega \arrow[r, leftrightarrow, "\text{m}", "\text{coefficient matching}"'] & \omega_c \\
\text{LCS} & & \text{conifold}
\end{tikzcd}
$$

Figure: The different bases involved in the computation and the relations in between them.

Local Solutions

The Picard-Fuchs equations

$$\mathcal{D}_I \omega = 0 \tag{8}$$

can be solved locally by the power series Ansatz

$$\omega_j = \sum_k \sum_i c_{i,j,k}\, x^{i+\beta_j} \,(\log x)^{\alpha_{j,k}} \;. \tag{9}$$

The resulting equation are solved order by order fixing the $c_{i,j,k}$

Local Solutions at LCS

At the LCS a 'closed' form is given by [Hosono, Klemm, Theisen, Yau 93']

$$\omega_0 = \sum_{n_i=0 i=1,\ldots,h^{2,1}}^{\infty} \left(\prod_{i=1}^{h^{2,1}} x_i^{n_i+\rho_i} \right) \frac{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(0)}(n_k+\rho_k)\right]}{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(0)}\rho_k\right]} \cdot \prod_{j=1}^{p} \frac{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(j)}\rho_k\right]}{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(j)}(n_k+\rho_k)\right]}\,.$$

Depends on the charge vectors l, the moduli x_i and the indices ρ_i

Local Solutions at LCS

$$
\begin{aligned}
D_{1,i} &= \frac{1}{2\pi i}\partial_{\rho_i} \, , \\
D_{2,i} &= \frac{1}{2}\frac{K_{ijk}}{(2\pi i)^2}\partial_{\rho_j}\partial_{\rho_k} \, , \\
D_3 &= -\frac{1}{6}\frac{K_{ijk}}{(2\pi i)^3}\partial_{\rho_i}\partial_{\rho_j}\partial_{\rho_k} \, ,
\end{aligned} \tag{10}
$$

$$
\omega = \left.\begin{pmatrix} \omega_0 \\ D_{1,i}\,\omega_0 \\ D_{2,i}\,\omega_0 \\ D_3\,\omega_0 \end{pmatrix}\right|_{\rho_i=0} \tag{11}
$$

How to analytically continue

- Numerically

  [Curio, Klemm, Lüst, Theisen 00', Huang, Klemm, Quackenbush 09', Alim, Scheiddeger 14', Bizet, Loaiza-Brito, Zavala 16', Blumenhagen, Herrschmann, Wolf 16', Joshi, Klemm 19' ...]
- Norlunds method (recurrence of infinite series)

  [Scheiddeger 16', Knapp, Romo, Scheidegger 16']
- Use polylog structure of the prepotential

  [Demirtas, Kim, McAllister, Moritz 20'].
- ϵ expansion of hypergeometric functions

  [Álvarez-García, Blumenhagen, Brinkmann, LS 20']
- KSZ-Conjecture

  [Klemm, Scheiddeger, Zagier 20'?]

Numerical method

- Compute local bases around the LCS and the conifold.
- Expand both bases an overlapping region of convergence.
- Compare expansion coefficients $\rightarrow$ equations for m.

- Easy to use, but only numerical results.
- Enough for most applications.
- Results depend on the chosen point.
- We need certain rationality properties. These cannot be seen in numerical solutions.

Norlunds method

- Analytical continuation of hypergeometric ${}_pF_q$ to argument 1.
- Works only in the 1 parameter case
- Works recursively, results in infinite sums of ${}_pF_q$s
- Useful for stable numerical results (no point dependence)
- Analytic results for 1- and 2D CYs

Analytic continuation of the prepotential(DKMM)

$$\mathcal{F} = -\frac{1}{6}K_{ijk}t_i t_j t_k + \frac{1}{2}a_{ij}t_i t_j + b_i t_i + \chi\frac{\zeta(3)}{(2\pi i)^3} + \mathcal{F}_{\text{inst}}\,. \qquad (12)$$

$$\mathcal{F}_{\text{inst}} = -\frac{1}{(2\pi i)^2}\sum_{\mathcal{C}\neq\mathcal{C}_\nu} n_0^{\mathcal{C}}\beta_i^{\mathcal{C}} Li_2(q^{\mathcal{C}}) \qquad (13)$$

$q = exp(2\pi i t)$

- Eulers reflection formula relates $Li_2(x)$ with $Li_2(1-x)$
- Allows evaluation of the prepotential for shrinking curves
- Does not allow to compute the period dual to the fundamental period (related to the volume, all curves contribute)
- Gives the superpotential (if one does not put fluxes on F_0), but not the full scalar potential

ϵ expansion of hypergeometric functions

One can rewrite the fundamental period at the LCS as a hypergeometric function:

$$\omega_0 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \overline{x}^{n_1+\rho_1} \overline{y}^{n_2+\rho_2} \overline{z}^{\rho_z} f(n_1, n_2, \rho_1, \rho_2, \rho_3) \; {}_pF_q(\vec{a}, \vec{b}, \overline{z}) \; , \tag{14}$$

The periods are given by up to third derivatives with respect to the indices ρ. These appear in the parameters of the hypergeometric function.

$\rightarrow$ Need to expand the ${}_pF_q$ around its parameters (to order 3) and evaluate at 1.

- Well studied in the amplitudes community (2003-2013)

  [Weinzierl 04', Kalmykov, Kniehl 10', Greynat, Sesma 13'. . .]

- Recently much progress in the math community (2014-2020)

  [Wan, Zucker 14', Aiblinger 15', Campbell, D'Aurizio, Sondow 17', Cantarini, D'Aurizio 18', Zhao 19',Zhao 20'. . .]

Hypergeometric functions

$$ {}_pF_q(\vec{a},\vec{b},x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{x^n}{n!} $$

- Defined by 2 parameter vectors $\vec{a}, \vec{b}$ of length p and q
- $(a)_n$ Pochhammer symbols
- For CYs $p = q + 1$
- The larger the p, the harder is the evaluation

Hypergeometric functions

- ${}_1F_0(a,,x) = (1-x)^{-a}$
- ${}_2F_1(\vec{a},b,x))$ closed form available for all possible a and b.
- ${}_3F_2(\vec{a},\vec{b},x))$ closed form available for integer and half-integer parameters.
- ${}_pF_{p-1}(\vec{a},\vec{b},x))$ closed form available for integer parameters and special values.
- Up to the ${}_3F_2$ case already implemented in the HypExp2 Mathematica package

Charge Vectors

- The parameters of the hypergeometric function depend inversly on the charge vector l of the CY.
- If there is a row with entries ≤ 2 this results in half-integer parameters.
- $l = (0, \ldots, 0, 1, 1, -2)$ describes a $\mathbb{P}^1$ and appears commonly.
- $l = (0, \ldots, 0, 1, 1, -1, -1)$ also works
- $l = (0, \ldots, 0, 1, 1, 1, -3)$ would give parameters $\frac{1}{3}$ and $\frac{2}{3}$, hard!
- $l = (0, \ldots, 0, 1, 1, 1, 1, -4)$ is easier

Example $\mathbb{P}_{1,1,2,8,12}[24]$

$$
\begin{aligned}
l_1 &= (-6, 3, 2, 0, 0, 0, 1, 0), \\
l_2 &= (0, 0, 0, 0, 1, 1, 0, -2), \\
l_3 &= (0, 0, 0, 1, 0, 0, -2, 1).
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\mathcal{D}_1 &= \Theta_x(\Theta_x - 2\Theta_z) - 12x(6\Theta_x + 5)(6\Theta_x + 1), \\
\mathcal{D}_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_z + 1)(2\Theta_y - \Theta_z), \\
\mathcal{D}_3 &= \Theta_z(\Theta_z - 2\Theta_y) - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x),
\end{aligned} \tag{16}
$$

$\Theta_i = x_i \partial_{x_i}$

Example $\mathbb{P}_{1,1,2,8,12}[24]$

If we sum over the third coordinate in the fundamental period we get

$$\omega_0 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \overline{x}^{n_1+\rho_1} \overline{y}^{n_2+\rho_2} \overline{z}^{\rho_z} f(n_1, n_2, \rho_1, \rho_2, \rho_3)\ {}_pF_q(\vec{a}, \vec{b}, \overline{z})\ ,$$

$${}_3F_2\left(\left\{1, \rho_z - \tfrac{1}{2}\rho_1 - \tfrac{1}{2}n_1, \tfrac{1}{2} + \rho_z - \tfrac{1}{2}\rho_1 - \tfrac{1}{2}n_1\right\}, \left\{1+\rho_z, 1+\rho_z - 2\rho_2 - 2n_2\right\}, \overline{z}\right)\ .$$

- ${}_3F_2$ with only half integer parameters!
  $\rightarrow$ HypExp2 can deal with these for any n_i .
- Results in harmonic polylogarithms.
- Derivatives of harmonic polylogarithms known.
- Can be expanded into a power series around $z = 1$.

Example $\mathbb{P}_{1,1,2,8,12}[24]$

local solution around conifold:

$$
\begin{aligned}
\omega_{c,1} &= \tilde{w}_1 \,, \\
\omega_{c,2} &= \tilde{w}_2 + \tilde{w}_1 \log(x_1) \,, \\
\omega_{c,3} &= \tilde{w}_3 + \frac{1}{2}\tilde{w}_1 \log(x_2) + \tilde{w}_1 \log(x_3) \,, \\
\omega_{c,4} &= \tilde{w}_4 \,, \\
\omega_{c,5} &= \tilde{w}_5 + \tilde{w}_4 \log(x_2) \,, \\
\omega_{c,6} &= \tilde{w}_6 + \tilde{w}_1 \log^2(x_1) + 2\tilde{w}_2 \log(x_1) \,, \\
\omega_{c,7} &= \tilde{w}_7 + \frac{1}{2}\tilde{w}_1 \log(x_1)\log(x_2) + \tilde{w}_1 \log(x_1)\log(x_3) + \tilde{w}_1 \log^2(x_1) \\
&\quad + (2\tilde{w}_2 + \tilde{w}_3)\log(x_1) + \frac{1}{2}\tilde{w}_2 \log(x_2) + \tilde{w}_2 \log(x_3) \,, \\
\omega_{c,8} &= \tilde{w}_8 + \frac{3}{4}\tilde{w}_1 \log^2(x_1)\log(x_2) + \frac{3}{2}\tilde{w}_1 \log^2(x_1)\log(x_3) + \tilde{w}_1 \log^3(x_1) \\
&\quad + \left(3\tilde{w}_2 + \frac{3\tilde{w}_3}{2}\right)\log^2(x_1) + \frac{3}{2}\tilde{w}_2 \log(x_1)\log(x_2) + 3\tilde{w}_2 \log(x_1)\log(x_3) \\
&\quad + \frac{3}{4}\tilde{w}_6 \log(x_2) + \frac{3}{2}\tilde{w}_6 \log(x_3) + 3\tilde{w}_7 \log(x_1) \,,
\end{aligned}
$$

Example $\mathbb{P}_{1,1,2,8,12}[24]$

$$m_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{id}{2\pi} & -\frac{i}{2\pi} & 0 & -\frac{i}{2\pi} & 0 & 0 & 0 & 0 \\ \frac{i\log(2)}{\pi} & 0 & -\frac{i}{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\pi} & 0 & 0 & 0 & 0 \\ \frac{a_7}{2} & \frac{-11\log(2)-6\log(3)}{2\pi^2} & -\frac{d}{2\pi^2} & \frac{1-3\log(2)}{2\pi^2} & \frac{1}{4\pi^2} & 0 & \frac{1}{2\pi^2} & 0 \\ a_6 & -\frac{d}{2\pi^2} & 0 & 0 & 0 & \frac{1}{4\pi^2} & 0 & 0 \\ a_7 & \frac{-11\log(2)-6\log(3)}{\pi^2} & -\frac{d}{\pi^2} & 0 & 0 & 0 & \frac{1}{\pi^2} & 0 \\ a_8 & b & c & 0 & 0 & -\frac{i\log(2)}{4\pi^3} & -\frac{id}{2\pi^3} & \frac{i}{6\pi^3} \end{pmatrix}, \tag{17}$$

where

$$a_6 = \frac{4\pi^2 + 25\log^2(2) + 9\log^2(3) + 30\log(2)\log(3)}{4\pi^2},$$

$$a_7 = \frac{23\pi^2 + 180\log^2(2) + 54\log^2(3) + 198\log(2)\log(3)}{6\pi^2},$$

$$a_8 = \frac{i\left(726\zeta(3) - 325\log^3(2) - 54\log^3(3) - 540\log^2(2)\log(3)\right)}{12\pi^3}$$

$$+ i\frac{\left(-297\log(2)\log^2(3) + 127\pi^2\log(2) + 69\pi^2\log(3)\right)}{12\pi^3}.$$

Example $\mathbb{P}_{1,1,2,8,12}[24]$

numerical results at 2 points:

$$m_2 = \begin{pmatrix} 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.08i & -0.159i & 0 & -0.159i & 0 & 0 & 0 & 0 \\ 0.238i & 0 & -0.318i & -0.0194i & 0 & 0 & 0 & 0 \\ -0.00776i & 0 & 0 & 0.327i & 0 & 0 & 0 & 0 \\ 4.45 & -0.724 & -0.350 & -0.0527 & 0.0246 & 0.00150 & 0.0492 & 0 \\ 2.14 & -0.343 & 0 & 0.0179 & 0 & 0.0253 & 0 & 0 \\ 8.93 & -1.44 & -0.685 & 0.0103 & 0 & 0 & 0.101 & 0 \\ 4.80i & 0.204i & 0.0523i & -0.0196i & 0.000191i & -0.00601i & -0.109i & 0.00538i \end{pmatrix}$$

$$m_2 = \begin{pmatrix} 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.07i & -0.159i & 0 & -0.158i & 0 & 0 & 0 & 0 \\ 0.220i & 0 & -0.318i & 0 & 0 & 0 & 0 & 0 \\ 0.00292i & 0 & 0 & 0.315i & 0 & 0 & 0 & 0 \\ 4.49 & -0.720 & -0.343 & -0.0802 & 0.0256 & 0 & 0.0507 & 0 \\ 2.16 & -0.343 & 0 & 0.000337 & 0 & 0.0253 & 0 & 0 \\ 8.94 & -1.44 & -0.685 & -0.000187 & 0 & 0 & 0.101 & 0 \\ 4.77i & 0.203i & 0.0505i & 0.00721i & -0.0000747i & -0.00558i & -0.109i & 0.00538i \end{pmatrix}$$

Within the errors these agree perfectly with the analytic result!

Example $\mathbb{P}_{1,1,2,8,12}[24]$

- With the analytic transition matrix we can compute the periods at the conifold fast to very high order (30+) as well as
- The mirror maps,
- Kahler-, super-, pre- and scalar potential.

$$
\begin{aligned}
\mathcal{F} = &-\frac{4}{3}(U^1)^3 - U^2(U^1)^2 + \frac{23}{6}U^1 + U^2 - \frac{120i}{\pi^3}e^{2i\pi U^1} - \frac{35496i}{\pi^3}e^{4\pi i U^1} \\
&- \frac{Z^3}{4} - 2(U^1)^2 Z - U^2 U^1 Z - U^1 Z^2 + \frac{23}{12}Z + \frac{120}{\pi^2}e^{2i\pi U^1} Z \\
&+ Z^2\left(\frac{i\log(2\pi Z)}{2\pi} - \frac{3i}{4\pi} + \frac{1}{4}\right) + \frac{121 i\zeta(3)}{4\pi^3} + \text{higher order}\,.
\end{aligned}
$$

Most non-rational factors cancel out!
Application to moduli stabilization $\to$ See Max Talk

Outlook

So far: ${}_3F_2$ with half-integer parameters. What is needed for more general geometries?

- General shrinking curve: ${}_3F_2$ with general parameters
- Shrinking divisors/CYs: higher ${}_pF_q$s with $p > 3$
- ϵ-expansion in all cases algorithmically known! [Greynat, Sesma 13']
- Results either in higher ${}_pF_q$s or infinite harmonic sums$\rightarrow$ number theoretic problem
- ${}_3F_2(a, b, 1)$ results in weight 4 colored multiple zeta values (CMZV).
- ${}_4F_3(a, b, 1)$ leads to critical L-values of Hecke eigenforms.

$$\omega_0(x) = {}_4F_3\left(\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\},\{1,1,1\},x\right) \tag{18}$$

Value of this function at 1 [Rogers,Wan,Zucker 13']

$$\omega_0(1) = \frac{16}{\pi^2} L(f,2) \approx 1.118636\ldots \tag{19}$$

where

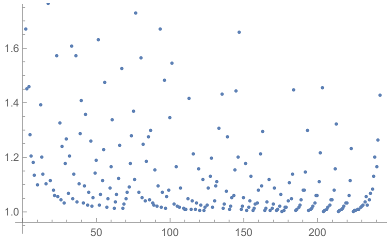
$$f = \eta(2\tau)^4\eta(4\tau)^4. \tag{20}$$

Agrees with KSZ conjecture. Ratios of L-values behave nicely!

$$\frac{L(g,2k)}{L(g,2k-2)} = \text{algebraic number} \cdot \pi^s \tag{21}$$

- Only 76(288) different values of ${}_4F_3$ with parameters $\{1, 1/2, 1/3, 1/4(, 1/5)\}$ out of 16384(268000) functions
- Large dataset, many structural relations, high dimensional parameter space, real number results

$\rightarrow$ nicely suited for deep NNs.



- In the ${}_3F_2$ case: Bases of CMZV known
- Use integer relations algorithms to identify closed forms of numerical values!

Absurd Identities

$$
\begin{aligned}
&{}_{12}F_{11}\left(\{1\}_8,\{\tfrac{3}{2}\}_4;\{2\}_6,\{\tfrac{5}{2}\}_5;1\right) = \\
&1.00082108253015483737894447386267231509413998277901388225147983 = \\
&-31104S_1+31104S_2+4860\zeta(\bar{5},1)-972\zeta(\bar{5},1,1)-972\log(2)\zeta(\bar{5},1)- \\
&2612736C-1741824\Im\left(\mathrm{Li}_3\left(\tfrac{1}{2}+\tfrac{i}{2}\right)\right)-870912\Im\left(\mathrm{Li}_4\left(\tfrac{1}{2}+\tfrac{i}{2}\right)\right)- \\
&248832\Im\left(\mathrm{Li}_5\left(\tfrac{1}{2}+\tfrac{i}{2}\right)\right)+435456\beta(4)-272160\mathrm{Li}_4\left(\tfrac{1}{2}\right)-116640\mathrm{Li}_5\left(\tfrac{1}{2}\right)- \\
&38880\mathrm{Li}_6\left(\tfrac{1}{2}\right)-7776\mathrm{Li}_7\left(\tfrac{1}{2}\right)+2430\zeta(3)^2+\tfrac{135\pi^4\zeta(3)}{8}+7290\pi^2\zeta(3)- \\
&476280\zeta(3)+\tfrac{1053\pi^2\zeta(5)}{2}+\tfrac{112995\zeta(5)}{4}+\tfrac{33291\zeta(7)}{64}+243\pi^2\zeta(3)\log^2(2)+ \\
&\tfrac{7533}{8}\zeta(5)\log^2(2)-486\zeta(3)^2\log(2)-2430\pi^2\zeta(3)\log(2)-\tfrac{37665}{4}\zeta(5)\log(2)+ \\
&\tfrac{4059\pi^6}{112}+\tfrac{2835\pi^5}{8}+\tfrac{3591\pi^4}{2}+13608\pi^3-244944\pi^2+5132160+\tfrac{54\log^7(2)}{35}- \\
&54\log^6(2)+\tfrac{54}{5}\pi^2\log^5(2)+972\log^5(2)-270\pi^2\log^4(2)+162\pi\log^4(2)- \\
&11340\log^4(2)-\tfrac{171}{20}\pi^4\log^3(2)+3240\pi^2\log^3(2)-4536\pi\log^3(2)+ \\
&\tfrac{513}{4}\pi^4\log^2(2)+243\pi^3\log^2(2)-22680\pi^2\log^2(2)+54432\pi\log^2(2)- \\
&\tfrac{4059}{560}\pi^6\log(2)-\tfrac{1539}{2}\pi^4\log(2)-3402\pi^3\log(2)+136080\pi^2\log(2)
\end{aligned}
$$

Summary

- We can compute the periods at the conifold analytically using closed forms of ${}_pF_q$s.
- For shrinking curves already doable.
- In principle computable for any manifold.
- Generalization requires rather deep number theory.
- Machine Learning could help find the hidden structures in the hypergeometric functions.

The End

Thank You

Given the q-series expansion of a weight k modular function f

$$f(\tau) = \sum_{n\geq 0} a_n\, q^n \ , \tag{22}$$

where $q = e^{2\pi i\tau}$, its corresponding L-function is defined as

$$L(f, x) = \sum_{n\geq 0} \frac{a_n}{n^x} \ . \tag{23}$$

A value $L(f, j)$ is called a critical L-value if $j \in \{1, 2, \ldots, k-1\}$. The Hecke operators T_m are defined by their action on a modular form as

$$T_m f(\tau) = m^{k-1} \sum_{d|m} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{m\tau + bd}{d^2}\right) \ . \tag{24}$$

A modular form which is an eigenfunction of all Hecke operators is called a Hecke eigenform, i.e.

$$T_m f(\tau) = \lambda_m f(\tau) \ . \tag{25}$$