VANISHING ORDERS AND U(1) CHARGES IN F-THEORY

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Based on upcoming work with w/ Andrew Turner

GENERAL MOTIVATIONS

BROAD QUESTIONS

Which massless spectra can occur in F-theory vacua?
- What types of gauge groups?
- Which representations of light charged matter?

How are these spectra realized in F-theory?

Important for a few reasons:

PHYSICS
- A good way to explore the F-theory landscape and swampland
  - Which consistent supergravity spectra can be realized in F-theory?
  - Could lead to new constraints, string constructions ...

MATHEMATICS
- Many physical properties of F-theory models are encoded in geometry
- Can teach us about mathematical properties of elliptic fibrations, Calabi–Yau manifolds, etc.

U(1)'S IN F-THEORY

Many interesting aspects of this to explore for U(1)'s.

- What U(1) charges can massless matter have in F-theory?
- In 6D, an infinite swampland of charge spectra **[Taylor, Turner '18]**
  - Infinite families of models with massless matter having arbitrarily large charges that satisfy anomaly cancellation conditions
  - Not known which of these models can be realized in F-theory

It's worth better understanding massless U(1) charges in F-theory

- In particular, what are the geometric features of F-theory models realizing different charges?

TODAY'S QUESTIONS

Can we make general statements about how different charges are realized in F-theory?

Can U(1) charges be determined prior to resolution?

OVERVIEW OF F-THEORY

Describe a model using an elliptically-fibered CY manifold

Nonabelian Gauge Algebras

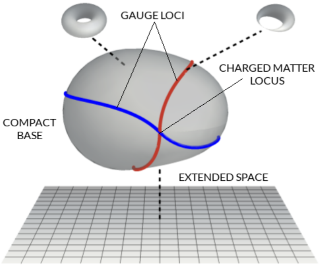
Divisors in base with singular fibers

- Singularity types ↔ gauge algebra
- Charged matter at codim-two loci where singularity type enhances
- Enhanced singularity type determines representation

U(1) Algebras

Extra rational sections of fibration

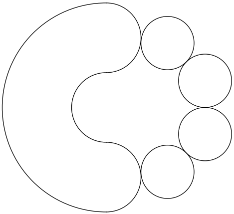
- Charged matter still at codim-two loci with enhanced singularity types
- U(1) charge determined by behavior of section



NONABELIAN GAUGE ALGEBRAS

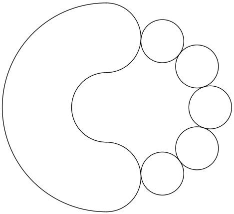
1. After resolving singularities, elliptic fibers at certain loci may take shape of affine ADE diagrams

CODIMENSION ONE



$\hat{A}_4$ *(SU(5) Gauge Algebra)*

CODIMENSION TWO



$\hat{A}_5$ *(SU(5) Fundamental Matter)*

Determine gauge algebra, matter representations from resolution

- At codimension-one, wrapping M2 branes on components gives roots of gauge algebra (in dual M-theory picture)
  - For non-simply-laced algebras, monodromy identifies components
- Wrapping M2 branes on extra components at codimension-two gives weights of charged matter

NONABELIAN GAUGE ALGEBRAS

2. For a model in Weierstrass form

$$y^2 = x^3 + fxz^4 + gz^6 \qquad \Delta = 4f^3 + 27g^2$$

the Kodaira table relates singularity types to vanishing orders of f, g, Δ

Singularity Type		Algebra	ord(f)	ord(g)	ord(Δ)
I_1	—	—	0	0	1
I_n	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor \frac{n}{2} \rfloor)$	0	0	n
II	—	—	1	1	2
III	A_1	$\mathfrak{su}(2)$	1	2	3
IV	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$	2	2	4
I_0^*	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$	2	3	6
I_n^*	D_{n+4}	$\mathfrak{so}(2n+8)$ or $\mathfrak{so}(2n+7)$	2	3	$n+6$
IV^*	E_6	$\mathfrak{e}_6$ or $\mathfrak{f}_4$	3	4	8
III^*	E_7	$\mathfrak{e}_7$	3	5	9
II^*	E_8	$\mathfrak{e}_8$	4	5	10

Nonabelian gauge algebra can often be read off from codimension-one orders of vanishing of f, g, Δ.

- There are also simple rules to test for monodromy

NONABELIAN GAUGE ALGEBRAS

For nonabelian charged matter, representation can be found using Katz–Vafa method **[Katz, Vafa '96]**

- Determine gauge algebras associated with codimension-one singularities (G) and codimension-two singularities (H)
- Break adjoint of H into reps of G.
- Charged matter rep can be read off branching pattern

You can often determine G, H from the orders of vanishing of f, g, Δ

- Strictly speaking, Kodaira classification only holds in codimension-one
- But you can often get away with using it at codimension-two

You can often determine nonabelian matter representations without resolution, at least heuristically.

U(1)'S IN F-THEORY

U(1)'s comes from extra rational sections of the elliptic fibration

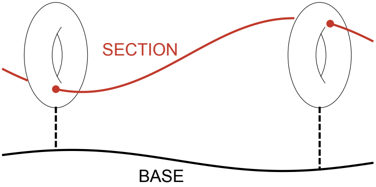
- We'll always assume there's at least one section, the zero section
- There may be more than one section (even an infinite number)
- Sections form a finitely-generated group under elliptic curve addition:

$$\text{Mordell–Weil Group: } \mathbb{Z}^r \oplus \mathcal{G}$$

  - $\mathcal{G}$ is the finite torsion subgroup, which is unimportant for today
  - r is the Mordell–Weil Rank

The resulting abelian gauge algebra is U(1)^r

- Roughly, each U(1) is associated with a generating section



U(1) CHARGED MATTER

Matter still occurs at codim-two loci with enhanced singularity type

- Wrapping M2 branes on extra components still gives matter

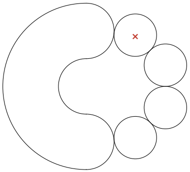
But the U(1) charge is determined by the behavior of the section

$$q = \sigma(\hat{s}) \cdot c$$

- $\hat{s}$ The generating section
- $\sigma(\hat{s})$ The Shioda map, a homomorphism from MW group to the Neron-Severi group
- c An extra component of the fiber

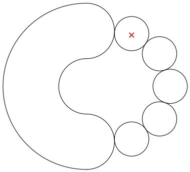
The charge can be determined by examining the resolved geometry

CODIMENSION ONE

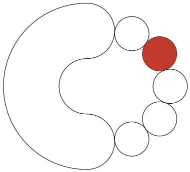


$SU(5) \times U(1)$ Gauge Algebra

CODIMENSION TWO



$\mathbf{5}_{1/5}$ *Matter*



$\mathbf{5}_{-4/5}$ *Matter*

ABELIAN WEIERSTRASS MODELS

Suppose our elliptic fibration is global Weierstrass form

$$y^2 = x^3 + f\,x\,z^4 + g\,z^6$$
$$[x : y : z] \equiv [\lambda^2 x : \lambda^3 y : \lambda z]$$

Suppose we want a model with a U(1) gauge algebra (MW group $\mathbb{Z}$)

- Need a generating section $\hat{s}$ that generates $\mathbb{Z}$
- $\hat{s}$ described by components $[\hat{x} : \hat{y} : \hat{z}]$ solving Weierstrass equations
- $\hat{x}$, $\hat{y}$, $\hat{z}$ are holomorphic sections of line bundles on base
  - In principle, they can be rational
  - Rescale to remove denominators, remove shared factors if possible
- It's also convenient to define

$$\hat{w} = 3\hat{x}^2 + f\hat{z}^4$$

Q: Is there something like the Katz–Vafa method for U(1) charges?

Not using only orders of vanishing of f, g, Δ

- Example: Singlets occur at codim-two loci where (f, g, Δ) vanish to orders $(0, 0, 2)$, regardless of charge

But what if we look at section components?

SINGLETS

Previously observed that, for charged singlets, the orders of vanishing of the section components are correlated with charge

- The algebraic structure of the F-theory models is closely linked to these orders of vanishing **[NR '17]**

[Morrison, Park '12]

[NR '17]

Charge	$\hat{z}$	$\hat{x}$	$\hat{y}$	$\hat{w}$
$q=0$	0	0	0	0
$q=1$	0	0	1	1
$q=2$	1	2	3	4
$q=3$	2	4	7	9
$q=4$	4	8	12	16

[Klevers, Mayorga-Pena, Oehlmann, Piragua, Reuter '14]

Is this part of a more general pattern?

- If so, maybe we could use this pattern to read off U(1) charges?

CHALLENGE Tough to construct F-theory models with larger charges

CONJECTURING ON HIGHER CHARGES

[Morrison, Park '12]

To guess how sections admitting larger charges behave:

1. Start with a U(1) model with only charge $q = 1$ matter:

$$y^2 - f_9^2 = (x - f_6)\left(x^2 + f_6 x + \hat{f}_{12} - f_6^2\right) \qquad \hat{s} = [f_6 : f_9 : 1]$$

   - At $\hat{f}_{12} = f_9 = 0$, there's an I_2 fiber with an extra component c
   - The generating section $\hat{s}$ satisfies

$$\sigma(\hat{s}) \cdot c = 1,$$

   where σ is the Shioda map

2. There are also sections $n\hat{s}$ generated from $\hat{s}$ by elliptic curve addition for any integer n
3. Since σ is a homomorphism, we also have

$$\sigma(n\hat{s}) \cdot c = n\sigma(\hat{s}) \cdot c = n$$

4. The section $n\hat{s}$ behaves in a way that looks like it admits charge n

SINGLETS

For the $n\hat{s}$ sections, look at the orders of vanishing at $\hat{f}_{12} = f_9 = 0$:

n	1	2	3	4	5	6	7	8
$\hat{z}$	0	1	2	4	6	9	12	16
$\hat{x}$	0	2	4	8	12	18	24	32
$\hat{y}$	1	3	7	12	19	27	37	48
$\hat{w}$	1	4	9	16	25	36	49	64

Agree with previous models

PATTERN

The components of the $n\hat{s}$ sections vanish to orders

$$\operatorname{ord}(\hat{z}) = \frac{1}{2}\left(\frac{n^2}{2} - \frac{(n \bmod 2)}{2}\right)$$

$$\operatorname{ord}(\hat{x}, \hat{y}, \hat{w}) = (2,3,4) \times \operatorname{ord}(\hat{z}) + (0,1,1) \times (n \bmod 2)$$

Components for a generating section should vanish to the same orders at a genuine $q = n$ locus.

[NR '17]

SINGLETS

PATTERN

$$\operatorname{ord}(\hat{z}) = \frac{1}{2}\left(\frac{q^2}{2} - \frac{(q \bmod 2)}{2}\right)$$

$$\operatorname{ord}((\hat{x}, \hat{y}, \hat{w}) = (2, 3, 4) \times \operatorname{ord}(\hat{z}) + (0, 1, 1) \times (q \bmod 2)$$

Why this pattern?

- Similar numbers appear for valuations of elliptic divisibility sequences corresponding to p-adic elliptic curves **[Stange '11]**

Do similar patterns hold in other situations?

- What if matter is charged under both a $U(1)$ and a nonabelian gauge algebra?

IDEA **Use higher sections in a model with $\mathfrak{g} \oplus \mathfrak{u}(1)$ gauge algebra**

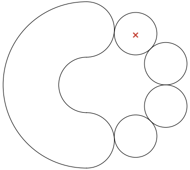
U(1) CHARGES & NONABELIAN MATTER

For a Weierstrass model w/ a $\mathfrak{g} \oplus \mathfrak{u}(1)$ algebra:

$$q = \sigma(\hat{s}) \cdot c = \mathcal{S} \cdot c + \sum_{I,J} (\mathcal{S} \cdot \alpha_I) \mathcal{C}^{-1}_{IJ} (\mathcal{T}_J \cdot c)$$

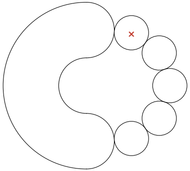
- $\mathcal{S}$ Homology class of section
- c Extra curve in codim-2 fiber supporting a weight of matter
- α_I Curve in singular fibers supporting simple root of $\mathfrak{g}$
- $\mathcal{T}_J$ Divisor found by fibering α_J over codim-1 locus
- $\mathcal{C}^{-1}$ Inverse Cartan matrix for $\mathfrak{g}$

CODIMENSION-ONE



SU(5) Gauge Algebra

CODIMENSION-TWO



$\mathbf{5}_{1/5}$ *Matter*

U(1) CHARGES & NONABELIAN MATTER

$$q = \sigma(\hat{s}) \cdot c = \mathcal{S} \cdot c + \sum_{I,J} (\mathcal{S} \cdot \alpha_I) \mathcal{C}^{-1}_{IJ} (\mathcal{T}_J \cdot c)$$

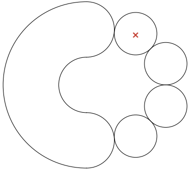
Matter charged under $\mathfrak{g}$ can have fractional $\mathfrak{u}(1)$ charges

- Due to the inverse Cartan matrix
- Non-trivial contribution when section hits one of the α_I at codimension-one
- Singlets still have integer charges

Allowed fractional charges controlled by $\mathcal{S} \cdot \alpha_I$

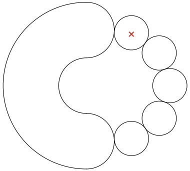
- Which α_I is hit by the section?
- Codimension-one phenomenon
- Related to global structure of gauge group [Cvetic, Lin '17]

CODIMENSION-ONE



SU(5) Gauge Algebra

CODIMENSION-TWO



$\mathbf{5}_{1/5}$ *Matter*

U(1) CHARGES & NONABELIAN MATTER

IDEA **Use higher sections in a model with $\mathfrak{g} \oplus \mathfrak{u}(1)$ gauge algebra**

- As before, if generating section satisfies

$$\sigma(\hat{s}) \cdot c = q,$$

  at a codimension-two locus, the $n\hat{s}$ section satisfies

$$\sigma(n\hat{s}) \cdot c = nq,$$

- Find orders of vanishing for the $n\hat{s}$ sections at this locus
- In models with genuine charge nq matter, generating section should vanish to same orders

We've done this exercise for simply-laced $\mathfrak{g}$ with generic reps:

$\mathfrak{su}(n)$	Fundamental and Antisymmetric
$\mathfrak{so}(n)$	Vector, Spinors up through $\mathfrak{so}(14)$
$\mathfrak{e}_6$	**27** Representation
$\mathfrak{e}_7$	**56** Representation

In total, around 550 sets of orders of vanishing... and there's a pattern

PROPOSAL

Consider a model with a $\mathfrak{g} \oplus \mathfrak{u}(1)$ algebra, where $\mathfrak{g}$ is simply-laced

- Codim-one locus with singularity type G (the universal cover of $\mathfrak{g}$)
  - For singlets, take G to be "SU(1)"
- Matter at a codim-two locus where singularity type enhances to H

Codimension-One Orders of Vanishing

$$\mathrm{ord}_{(1)}(\hat{z}) = 0 \qquad \left(\mathrm{ord}_{(1)}(\hat{x}), \mathrm{ord}_{(1)}(\hat{y}), \mathrm{ord}_{(1)}(\hat{w})\right) = \vec{\tau}_G(\mathcal{I})$$

Codimension-Two Orders of Vanishing

$$\mathrm{ord}_{(2)}(\hat{z}) = \frac{1}{2}\left(\frac{d_G}{d_H}q^2 + \left(\mathcal{C}_G^{-1}\right)_{\mathcal{I}\mathcal{I}} - \left(\mathcal{C}_H^{-1}\right)_{\mathcal{J}\mathcal{J}}\right)$$

$$\left(\mathrm{ord}_{(2)}(\hat{x}), \mathrm{ord}_{(2)}(\hat{y}), \mathrm{ord}_{(2)}(\hat{w})\right) = (2,3,4) \times \mathrm{ord}_{(2)}(\hat{z}) + \vec{\tau}_H(\mathcal{J})$$

$\mathcal{I}$ (or $\mathcal{J}$)	An integer ranging from 0 to rank(G) (or rank(H)) Roughly, component of resolved fiber hit by section
$\vec{\tau}_G(\mathcal{I})$	Triplet of integers (given by particular expressions)
d_G	Number of elements in the center of G
$(\mathcal{C}_G^{-1})_{\mathcal{I}\mathcal{I}}$	$\mathcal{I}$'th diagonal entry of inverse Cartan matrix (or 0 if $\mathcal{I} = 0$)

PROPOSAL, CONT.

Codimension-One Orders of Vanishing

$$\mathrm{ord}_{(1)}(\hat{z}) = 0 \qquad \left(\mathrm{ord}_{(1)}(\hat{x}), \mathrm{ord}_{(1)}(\hat{y}), \mathrm{ord}_{(1)}(\hat{w})\right) = \vec{\tau}_G(\mathcal{I})$$

Codimension-Two Orders of Vanishing

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$$\left(\mathrm{ord}_{(2)}(\hat{x}), \mathrm{ord}_{(2)}(\hat{y}), \mathrm{ord}_{(2)}(\hat{w})\right) = (2,3,4) \times \mathrm{ord}_{(2)}(\hat{z}) + \vec{\tau}_H(\mathcal{J})$$

For G = SU(N), $\mathcal{I}$ ranges from 0 to N − 1:

$$\vec{\tau}_{SU(N)}(\mathcal{I}) = (0, u_N(\mathcal{I}), u_N(\mathcal{I})) \qquad u_N(\mathcal{I}) = \min(\mathcal{I}, N - \mathcal{I})$$

$$\left(\mathcal{C}_{SU(N)}^{-1}\right)_{\mathcal{I}\mathcal{I}} = \frac{\mathcal{I}(N - \mathcal{I})}{N} \qquad d_{SU(N)} = N$$

For SU(N) fundamentals, H is SU(N+1)

PROPOSAL, CONT.

The formulas seem to work in a variety of models in the literature:

- U(1) model with $q = 1, 2$ matter in **[Morrison, Park '12]**
- Toric hypersurface models in **[Klevers, Mayorga-Pena, Oehlmann, Piragua, Reuter '14]**
- U(1) model with $q = 1, 2, 3, 4$ matter in **[NR '17]**
- SU(5), SO(10), E_6 and E_7 models in **[Küntzler, Schäfer-Nameki '14]**
- The SU(5)$\times U(1)^2$ and SU(4)$\times U(1)^2$ models from **[Borchmann, Mayrhofer, Palti, Weigand '13]**

Whenever we've used these formulas to derive the charge spectrum of a model, the results have agreed with anomaly cancellation

POTENTIAL BENEFITS

- Gives a way to read off charges (at least up to sign) without resolving
- This is a formula for general charges
  - Could be used for exploring the F-theory landscape/swampland

SU(5) EXAMPLE

from **[Küntzler, Schäfer-Nameki '14]**

Consider a Weierstrass model $y^2 = x^3 + fxz^4 + gz^6$ with

$$
\begin{aligned}
f =& -\frac{1}{48}\left(b_{1,0}^2 + 4\sigma c_{2,1}\right)^2 + \frac{1}{2}\sigma^2 b_{0,0} b_{1,0} c_{1,2} + \sigma^3 \left(c_{1,2}c_{3,1} - \sigma b_{0,0}^2 c_{0,4}\right) \\
g =& -\frac{1}{1728}\left(b_{1,0}^2 + 4\sigma c_{2,1}\right)^3 - \frac{1}{12} f \left(b_{1,0}^2 + 4\sigma c_{2,1}\right) + \frac{1}{4}\sigma^4 b_{0,0}^2 c_{1,2}^2 \\
& - \sigma^5 c_{0,4}\left(b_{0,0}^2 c_{2,1} - b_{1,0} b_{0,0} c_{3,1} - \sigma c_{3,1}^2\right)
\end{aligned}
$$

This model has an SU(5) on $\{\sigma = 0\}$ and a U(1) with generating section

$$
\begin{aligned}
\hat{x} =& \frac{1}{12} b_{0,0}^2 \left(b_{1,0}^2 - 8\sigma c_{2,1}\right) + \sigma b_{1,0} b_{0,0} c_{3,1} + \sigma^2 c_{3,1}^2 \\
\hat{y} =& \frac{1}{2}\sigma \Big[b_{0,0}^2 b_{1,0} \left(b_{0,0} c_{2,1} - b_{1,0} c_{3,1}\right) \\
& - \sigma b_{0,0} \left(b_{0,0}^3 c_{1,2} - 2 b_{0,0} c_{2,1} c_{3,1} + 3 b_{1,0} c_{3,1}^2\right) - 2\sigma^2 c_{3,1}^3 \Big] \\
\hat{z} =& b_{0,0}
\end{aligned}
$$

There is $\mathbf{5}_{6/5}$ matter at $\{\sigma = b_{0,0} = 0\}$ with SU(5)→SU(6) enhancement

$$
\underset{\sigma=0}{\mathrm{ord}}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = (0, 1, 0, 1) \qquad \underset{\sigma=b_{0,0}=0}{\mathrm{ord}}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = (2, 3, 1, 4)
$$

SU(5) EXAMPLE, CONTINUED

$$\underset{\sigma=0}{\mathrm{ord}}(\hat{x},\hat{y},\hat{z},\hat{w}) = (0,1,0,1) \qquad \underset{\sigma,b_{0,0}=0}{\mathrm{ord}} = (2,3,1,4)$$

$$\vec{\tau}_{SU(N)}(\mathcal{I}) = (0,1,1)\times \min(\mathcal{I}, N-\mathcal{I}) \qquad \left(C^{-1}_{SU(N)}\right)_{\mathcal{I}\mathcal{I}} = \frac{\mathcal{I}(N-\mathcal{I})}{N}$$

AT CODIMENSION ONE The singularity type is SU(5)

$$\begin{aligned}\underset{\sigma=0}{\mathrm{ord}}(\hat{x},\hat{y},\hat{w}) = \vec{\tau}_{SU(5)}(\mathcal{I}) &= (0,1,1)\times \min(\mathcal{I}, 5-\mathcal{I}) \\ &= (0,1,1)\end{aligned}$$

Therefore, $\mathcal{I}$ is 1 or 4, and $(C^{-1}_{SU(5)})_{\mathcal{I}\mathcal{I}} = \frac{4}{5}$

SU(5) EXAMPLE, CONTINUED

$$\underset{\sigma=0}{\mathrm{ord}}(\hat{x},\hat{y},\hat{z},\hat{w}) = (0,1,0,1) \qquad \underset{\sigma,b_{0,0}=0}{\mathrm{ord}} = (2,3,1,4)$$

$$\vec{\tau}_{SU(N)}(\mathcal{I}) = (0,1,1)\times \min(\mathcal{I}, N-\mathcal{I}) \qquad \left(C^{-1}_{SU(N)}\right)_{\mathcal{I}\mathcal{I}} = \frac{\mathcal{I}(N-\mathcal{I})}{N}$$

AT CODIMENSION TWO The singularity type enhances to SU(6).

$$\mathrm{ord}_{(2)}(\hat{x},\hat{y},\hat{w}) - (2,3,4)\times \mathrm{ord}_{(2)}(\hat{z}) = \vec{\tau}_{SU(6)}(\mathcal{J}) = (0,0,0)$$

Therefore, $\mathcal{J}$ is 0, and we take $(C^{-1}_{SU(6)})_{\mathcal{J}\mathcal{J}}$ to be 0.

Since $d_{SU(N)} = N$ and $(\mathcal{C}^{-1}_{SU(5)})_{\mathcal{I}\mathcal{I}} = \frac{4}{5}$, we have

$$\begin{aligned} \mathrm{ord}_{(2)}(\hat{z}) &= \frac{1}{2}\left(\frac{d_{SU(5)}}{d_{SU(6)}}q^2 + \left(\mathcal{C}^{-1}_{SU(5)}\right)_{\mathcal{I}\mathcal{I}} - \left(\mathcal{C}^{-1}_{SU(6)}\right)_{\mathcal{J}\mathcal{J}}\right) \\ &= \frac{5}{12}q^2 + \frac{4}{10} = 1 \end{aligned}$$

Therefore, $|q| = \frac{6}{5}$, as expected!

OTHER ASPECTS

- Formulas seem to work for other simply-laced gauge algebras and representations
- Extends naturally if there are multiple U(1)'s
- Slight generalization of formulas seems to work for bifundamentals
- Similar types of numbers/expressions appear for p-adic valuations of elliptic divisibility sequences **[Stange '11]**

CONCLUSIONS

In summary, information about U(1) charges seems to be encoded in orders of vanishing of the section components.

FUTURE DIRECTIONS

- More rigorous understanding/confirmation of these patterns, either mathematically or physically
- Extension to non-simply laced algebras
- More exotic matter reps, such as symmetric rep of SU(N)
- Superconformal matter
- Are similar formulas possible for other descriptions of elliptic fibers (e.g. Tate form, cubic in $\mathbb{P}^2$, ...)

Thank you!