Supersymmetric Flux Vacua and Calabi-Yau Modularity

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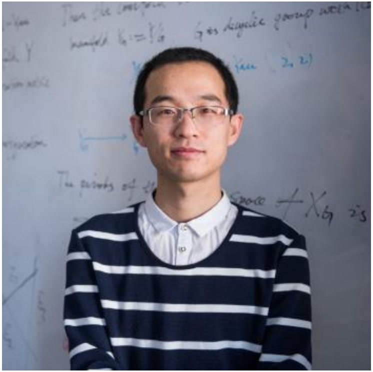
Based on 2001.06022 + 2010.07285 w/ S. Kachru & W. Yang



Overview

- In string theory, we sometimes take six of the ten dimensions to be a Calabi-Yau threefold (CY3), i.e. a compact, Ricci-flat, Kahler manifold
- Analyzing the complex geometry has taught a great deal about string vacua
  - String dualities, strongly coupled field theories, etc
- On the other hand, mathematicians are currently interested in arithmetic geometry, i.e. the geometry of varieties over $\mathbb{Q}$
  - Langlands conjectures, Fermat's last theorem, etc
- Natural question: does this arithmetic point of view have anything to do with physics?
- This talk: Yes! Supersymmetric flux compactifications have rich arithmetic properties.
- A fascinating link between physics and an exciting area of math
- Related work: Moore '98, '04; Candelas et al '19; Yang '19, '20

Flux Compactifications

- Consider IIB string theory compactified on a CY3 X, with $h^{2,1}$-d cpx structure moduli space $\mathcal{M}$
  - $H^3(X) = H^{(3,0)}(X_\phi) \oplus H^{(2,1)}(X_\phi) \oplus H^{(1,2)}(X_\phi) \oplus H^{(0,3)}(X_\phi)$
- IIB string theory has two-form gauge fields C and B with field strengths F, H, as well as axiodilaton τ, packaged as $G_3 \equiv F - \tau H$
- There exist string compactifications where F, H have integral fluxes $f, h \in H^3(X, \mathbb{Z})$ through X [Becker-Becker '96, GKP, ...]
- Dynamics governed by a superpotential [Gukov-Vafa-Witten '99]

$$W = \int G_3 \wedge \Omega$$

  - Find flux vacua by solving $D_I W = 0$, where $I = \tau$, cpx structure moduli
- Fluxes f, h define a flux vacuum at a point $X_* \in \mathcal{M}$ with dilaton τ if

$$G_3 \in H^{2,1}(X_*) \oplus H^{0,3}(X_*)$$

Supersymmetric Flux Compactifications

- When $W = 0$, supersymmetry is preserved
  - "Supersymmetric flux vacua"
  - [GKP, DeWolfe-Giryavets-Kachru-Taylor '04, DeWolfe '05, ...]
- $W = \int G_3 \wedge \Omega \implies$ susy flux $\iff G_3 \in H^{2,1}(X_*)$

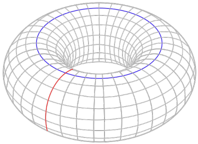
Susy flux constraint implies $f, h \in H^{2,1}(X_*) \oplus H^{1,2}(X_*)$

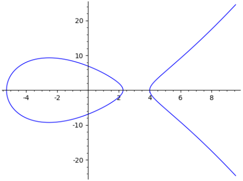
- Very rare: need to solve an overconstrained system of equations
  - Nevertheless, can come in isolated points or continuous families
- Somewhat analogous to the attractor equation [Moore '04]
  - A subset of attractors, "rank-two attractors", are known to have rich arithmetic properties [Moore '98, Candelas '19]
- This talk:

Do susy flux vacua enjoy special arithmetic properties?

Arithmetic Geometry

- So far, we've viewed the CY3 as a complex geometric object
- To study its arithmetic properties, need arithmetic geometry
- View some complex manifolds as "varieties" over $\mathbb{Q}$
- Simplest example: Tori $\implies$ elliptic curves over $\mathbb{Q}$





- All elliptic curves over $\mathbb{Q}$ can be written in Weierstrass form:

$$y^2 = x^3 + ax + b \text{ where } a, b \in \mathbb{Z},\ \ x, y \in \mathbb{Q}$$

- Invariants: $\Delta = -16(4a^3 + 27b^2)$, $j = 6912a^3/(4a^3 + 27b^2)$
- Curve is nonsingular unless $\Delta = 0$

Point Counts & Modularity

- We can also reduce a curve over $\mathbb{Q}$ mod a finite field $\mathbb{F}_p$
- The resulting curve will be nonsingular for all p other than the prime divisors of Δ
  - Good vs. bad primes
- Once we've reduced to $\mathbb{F}_p$, we can ask how many points there are on curve
  - Reduced equation has a finite number of solutions $\#(E, p)$
  - Expected number of solutions $= p + 1 \sim |\mathbb{F}_p| +$ pt at infinity
  - "Scatter coefficient" $a_p = p + 1 - \#(E, p)$
- The point counts a_p are the basic objects in arithmetic geometry
- Modularity theorem: [Wiles, Wiles-Taylor,...] for each elliptic curve E over Q, there exists a modular form $f_E(\tau) = \sum b_n e^{2\pi i n \tau}$ s.t. $a_p = b_p$ for all good p

Point counts give the Fourier coefficients of modular forms!

- More precisely, f_E is a weight-2 cusp form for some $\Gamma_0(N)$

Point Counts & Modularity

- Let's quickly do an example: $y^2 = x^3 - 4x$
- $j = 1728, \Delta = 4096 \implies$ only bad prime is $p = 2$
- Reduce over $\mathbb{F}_3$ to get $y^2 = x^3 + 2x$
  - 4 points: $(x, y) = (0, 0), (1, 0), (2, 0)$ + pt at infinity
  - $a_3 = 3 + 1 - 4 = 0$
- Similarly, reduce over $\mathbb{F}_5$ to get $y^2 = x^3 + x$
  - Again, 4 points: $(x, y) = (0, 0), (2, 0), (3, 0)$ + pt at infinity
  - $a_5 = 5 + 1 - 4 = 2$
- Consider weight-2 eigenform for $\Gamma_0(64)$, $f(\tau) = q + 2q^5 - 3q^9 + \cdots$
  - $b_3 = 0 = a_3$
  - $b_5 = 2 = a_5$
- Can check that this pattern continues for all primes p!

Rational Models

- So far we've skipped an important step: the choice of a rational model
- Over $\mathbb{C}$, tori are entirely classified by j, but over $\mathbb{Q}$ there are infinitely many arithmetically inequivalent EC with the same j
- An explicit example: $\mathcal{E} : y^2 = x^3 - 4$ and $\mathcal{E}' : y^2 = x^3 + 4$
  - Both have $j = 0$, so they are isomorphic over $\mathbb{C}$
  - Related by $y \to iy$, so they are not isomorphic over $\mathbb{Q}$
  - Called different "rational models" for the same torus
  - Over $\mathbb{F}_7$, $\mathcal{E} \to y^2 = x^3 + 3$ and $\mathcal{E}' \to y^2 = x^3 + 4$
  - These have $a_7 = -5, 5$, respectively

> Different rational models are associated to different eigenforms!

- In ordinary string theory, we never make this choice- work with CY3s defined up to $\mathbb{C}$-isomorphism!

A More Abstract Point of View

- This a rather concrete formulation of a beautiful abstract story
- The middle cohomology of a $2n$-dimensional variety over $\mathbb{Q}$ furnishes a b^n-dim'l rep of the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of pointwise automorphisms of $\mathbb{Q}$
  - For an elliptic curve E, this gives us a 2d rep $\rho_2(\mathrm{E})$
- Similarly, cusp forms f are associated to 2d reps $\rho_2(f)$ of the same group
- A more abstract statement of the modularity theorem is that for all E there exists an f s.t.

$$\rho_2(E) = \rho_2(f)$$

- This language is how mathematicians usually think about the problem, and will be used later to help us relate this to string theory

Modularity of Threefolds

- Tori are CY 1-folds, so its natural to ask whether this can be generalized to higher-dim'l CYs
- Modularity for threefolds over $\mathbb{Q}$ is an active area of research
  - See e.g. [Meyer '05, Yui '12] for reviews
- What is a threefold over $\mathbb{Q}$?
  - E.g. a projective hypersurface with all coefficients rational
  - $x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 - 8\psi x_1 x_2 x_3 x_4 x_5 - 2\phi x_1^4 x_2^4 = 0 \subset \mathbb{P}(1,1,2,2,2)$ with $\psi, \phi \in \mathbb{Q}$
- Technically a much harder problem
  - Computing the a_p numerically is computationally difficult
  - The associated Galois representations $\rho_{2+2h^{2,1}}$ are high-dimensional $\iff$ hard to work with
- The best known results are for rigid threefolds (i.e. $h^{2,1} = 0$) over $\mathbb{Q}$
  - $2 + 2h^{2,1} = 2 \implies$ can use results from elliptic curves
- Theorem [Gouvea-Yui '09]: rigid threefolds over $\mathbb{Q}$ are weight-4 modular

Reducible Representations

- There's one other way we can get a 2d Galois rep out of a threefold: $\rho_{2+2h^{2,1}}$ can become reducible!
- If $\rho_{2+2h^{2,1}} = \rho_2 \oplus \rho_{2h^{2,1}}$, we can study ρ_2 directly
- ρ_2 can sometimes be related to the rep associated to an elliptic curve [Gouvea-Yui '09]
- In this case, the threefold is modular, and associated to a weight-two cusp form [Wiles, ...]
- When does this happen? Subject to some technical assumptions,

> Threefolds are weight-two modular whenever
> a 2d subspace of $H^3(X, \mathbb{Q})$ has Hodge type 2,1 + 1,2

Flux Modularity

- Supersymmetric flux vacua always have this type of split!
- Remember we had $f, h \in H^{2,1} \oplus H^{1,2}$. Then define

$$H_{\text{flux}} = \mathbb{Q}f + \mathbb{Q}h \subset H^3(X_*, \mathbb{Q})$$

- Thus, we have our main result: [Kachru-RAN-Yang '20]

> Supersymmetric flux compactifications over $\mathbb{Q}$ are (conjecturally) modular, and associated to weight-two cusp forms!

- A similar argument shows that rank-two attractors over $\mathbb{Q}$ are associated to weight-four cusp forms [Candelas et al '19]

An Example: The CY3 in $\mathbb{P}(1,1,2,2,2)$

- We can check this conjecture in a rich example: CY3 in $\mathbb{P}(1,1,2,2,2)$

$$x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 - 8\psi x_1 x_2 x_3 x_4 x_5 - 2\phi x_1^4 x_2^4 = 0$$

- Susy flux vacua everywhere on the $\psi = 0$ locus in moduli space [DeWolfe '05]
- For each rational ϕ, we we get a susy flux vacuum over $\mathbb{Q}$
  - Defining polynomial gives a rational model for each ϕ
- For each ϕ, we get a modular form for different $\Gamma_0(N)$
- Infinite number of examples to check!
- This can be done using point counts computed in [Kadir '04]

An Example: The CY3 in $\mathbb{P}(1,1,2,2,2)$

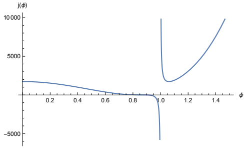
- Explicit modular forms for numerical values of ϕ

ϕ	Bad primes	Modular form
0	2	64.2.a.a
1/2	2, 3	24.2.a.a
3/5	2, 5	400.2.a.e
11/8	2, 3, 19	57.2.a.c
2	2, 3	192.2.a.a
3	2	32.2.a.a
7	2, 3	48.2.a.a
9	2, 5	40.2.a.a

- Forms are listed by their LMFDB labels- N indicated
- Strong evidence for flux modularity!

Flux Modularity and the Dilaton

- Obvious question: is there a physical interpretation?
- No obvious physical motivation for point counts, but can we at least get the elliptic curve?
- In addition to f, h, to define a susy flux vacuum we need a dilaton τ
- $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$, so you can think about it as the τ of a torus
  - This is the key insight underlying F-theory
- The dilaton is the τ of the elliptic curve associated to the modular form! [Candelas et al '19, Kachru-RAN-Yang '20]
- Ex: $\mathbb{P}(1,1,2,2,2)$
  - $j(\phi) = \frac{64\left(4\phi^2-3\right)^3}{\phi^2-1} \implies$ for each rational ϕ we get a rational value of j



F Theory and Sen's Weierstrass Model

- F-theory gives an explicit Weierstrass form corresponding to τ [Sen '09]
- For a susy flux, takes the form

$$y^2 = x^3 + (C-3)\,x + (C-2)$$

- $j(C) = 1728\frac{4(C-3)^3}{C^2(4C-9)}$. Set $j(C) = j(\phi) \implies C(\phi) = \frac{9}{4\phi^2}$

$$y^2 = x^3 + \left(\frac{9}{4\phi^2} - 3\right)x + \left(\frac{9}{4\phi^2} - 2\right)$$

- For each rational ϕ, we get an elliptic curve over $\mathbb{Q}$ with the right j!
  - Can also be checked in several other examples
- These aren't the rational models that correspond to the eigenforms in the table we saw before. Can we do better?

F Theory and Rational Models

$$y^2 = x^3 + \left(\frac{9}{4\phi^2} - 3\right) x + \left(\frac{9}{4\phi^2} - 2\right)$$

- As $\phi \to 0$, we get a bad equation for our elliptic curve
- Resolve coordinate singularity by redefining $y \to \sqrt{a} y$

$$y^2 = x^3 + \left(\frac{9}{4\phi^2} - 3\right) a^2 x + \left(\frac{9}{4\phi^2} - 2\right) a^3$$

- On the other hand, $\phi = 0$ corresponds to the eigenform 64.2.a.a
  - $y^2 = x^3 + x$
- Pick a s.t. $\lim_{\phi \to 0} \left(\frac{9}{4\phi^2} - 3\right) a^2 = 1, \lim_{\phi \to 0} \left(\frac{9}{4\phi^2} - 2\right) a^3 = 0.$
  - $a(\phi) = \frac{2}{3}\phi$ works nicely!

$$y^2 = x^3 + \left(1 - \frac{4\phi^2}{3}\right) x + \left(\frac{2\phi}{3} - \frac{16\phi^3}{27}\right)$$

F Theory and Rational Models

ϕ	LMFDB label	Weierstrass Coefficients	Associated form
0	64.a4	$\{1, 0\}$	64.2.a.a
1/2	24.a5	$\{2/3, 7/27\}$	24.2.a.a
3/5	400.e4	$\{13/25, 34/125\}$	400.2.a.e
11/8	57.c3	$\{-73/48, -539/864\}$	57.2.a.c
2	192.a3	$\{-13/3, -92/27\}$	192.2.a.a
3	32.a1	$\{-11, -14\}$	32.2.a.a
7	48.a2	$\{-193/3, -5362/27\}$	48.2.a.a
9	40.a1	$\{-107, -426\}$	40.2.a.a

- By fixing choice of rational model at $\phi = 0$, we get out the rational models for all other ϕ for free!
- This is fairly surprising- a priori, F-theory doesnt know anything about rational models!

Conclusion

- String theory is usually framed in complex geometry, but in many cases arithmetic geometry is more exciting. Can we related this to physics?
- Yes! We have argued that supersymmetric flux compactifications over $\mathbb{Q}$ are modular
- Physical origins still unclear, but F theory provides a great hint
- An early sign of a beautiful relationship between string theory and arithmetic geometry?
- Dream: physical interpretation of Fourier coefficients a_p

Thanks for listening!