Cyclic Free Quotients of Favorable CICY 3-Folds

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Motivation of Our Work

Free quotients of CY are important in string model building and other context. They provide more possibilities for getting exactly the MSSM spectrum and symmetry group and has been widely used, for example Greene et al, ‘86, Braun et al, ‘05,’12, Bouchard et al, ‘05, Blumenhagen et al, ‘06, Anderson et al, ‘11 and many others.

For CICYs, Their free quotients can descend from the automorphism of the ambient space. Part of this kind of free quotients were manually found(Candelas et al,’08), and a full classification of them was done by Braun(Braun,’10) by using the method of character valued index to help judge the fixed point freeness.

There are also works on classification of free quotients of CY of other types, for example, classification of free quotients of CY as hypersurface in ambient toric four fold(Andreas Braun et al,’17).

In our work, we want to classify all the cyclic free quotient of CICYs, where the symmetry action involved descends from the ambient space of the recently found favorable configurations. Character valued index method are very useful here since the favorable configurations are usually very big because of the splitting process.

CICY 3-Fold

$$X = \left[\begin{array}{c|cccc} \mathbb{P}^{n_1} & q_1^1 & q_2^1 & \cdots & q_k^1 \\ \mathbb{P}^{n_2} & q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & q_1^m & q_2^m & \cdots & q_k^m \end{array}\right]$$

$$n_i + 1 = \sum_{j=1}^{k} q_j^i \iff c_1 = 0$$

$$\sum_{j=1}^{m} n_j = k + 3 \iff \text{3-fold condition}$$

Example:

$$\begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 1 \end{array}\right] & \begin{array}{c} (x_{1,1}, x_{1,2}) \\ \vdots \\ \\ \vdots \end{array} \end{array}$$

$$\begin{aligned} f_1 = & c_{1,1}x_{1,1}x_{5,1}x_{6,1} + c_{1,5}x_{1,2}x_{5,1}x_{6,1} + c_{1,3}x_{1,1}x_{5,2}x_{6,1} + \\ & c_{1,7}x_{1,2}x_{5,2}x_{6,1} + c_{1,2}x_{1,1}x_{5,1}x_{6,2} + c_{1,6}x_{1,2}x_{5,1}x_{6,2} + \\ & c_{1,4}x_{1,1}x_{5,2}x_{6,2} + c_{1,8}x_{1,2}x_{5,2}x_{6,2} \end{aligned}$$

The CICY-3 folds were classified in 1987(Candelas et. al, ’87), there are 7890 of them in total

Favorable CICY 3-Fold

Favorable CICY is a CICY with the following property: the second cohomology of it can be fully expanded by its ambient space's Kahler forms restricted on it and this will bring lots of nice properties in calculation.

There are 4896 favorable CICYs in the original 7890 CICYs. A recent work(Anderson et. al, '17) discovered that, in the remaining 2994 CICYs, 2946 of them can actually have a favorable configuration through the following (ineffective) splitting process(Candelas et. al, '88) :

$$X = \left[\begin{array}{c|cc} \mathcal{A} & c & C \end{array} \right] \Longrightarrow X = \left[\begin{array}{c|cccccc} \mathbb{P}^n & 1 & 1 & \dots & 1 & 0 \\ \mathcal{A} & c_1 & c_2 & \dots & c_{n+1} & C \end{array} \right] \qquad c = \sum_{i=1}^{n+1} c_i$$

Example:

$$\left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 0 & 3 \end{array} \right] \Longrightarrow \left[\begin{array}{c|cccc} \mathbb{P}^2 & 1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 0 & 1 \\ \mathbb{P}^2 & 0 & 0 & 0 & 3 \end{array} \right]$$

An example of freely acting symmetry $\mathbf{Z_4}$

$$
\left[\begin{array}{c|ccc}
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1
\end{array}\right]
\xrightarrow[(13)(26)(45)]{\text{Act on row}}
\left[\begin{array}{c|ccc}
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1
\end{array}\right]
\xrightarrow[(12)]{\text{Act on column}}
\left[\begin{array}{c|ccc}
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 0 & 1 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1 \\
\mathbb{P}^1 & 1 & 0 & 1
\end{array}\right]
$$

Coordinate action

$$
\begin{pmatrix} x_{1,1} \\ x_{1,2} \\ x_{2,1} \\ x_{2,2} \\ x_{3,1} \\ x_{3,2} \\ x_{4,1} \\ x_{4,2} \\ x_{5,1} \\ x_{5,2} \\ x_{6,1} \\ x_{6,2} \end{pmatrix}
\Longrightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} x_{1,1} \\ x_{1,2} \\ x_{2,1} \\ x_{2,2} \\ x_{3,1} \\ x_{3,2} \\ x_{4,1} \\ x_{4,2} \\ x_{5,1} \\ x_{5,2} \\ x_{6,1} \\ x_{6,2} \end{pmatrix}
$$

Polynomial action

$$
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}
\Longrightarrow
\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}
$$

Defining polys are invariant under these actions

Symmetries classified by Braun have the following properties:

Have linear actions on the homogeneous coordinates of the ambient projective space

Can permute the rows and columns of the CICY matrix, can also have non-trivial linear actions on the defining polynomials

This classification actually depends on the CICY configuration we are using! Actually, the possible symmetry orders are decided by indices whose calculation rely on CICY configuration matrix, so indeed if we change the configuration, we may have symmetry with a new group order.

But in the process of getting the favorable configuration of CICY, we have used splitting process which WILL CHANGE the configuration matrix. In this case, we can possibly get new symmetries which descend from ambient space action.

We tried to investigate this possibility by first looking at cyclic symmetries on CICYs in their newly found favorable configuration matrix

Character Valued Index, How and Why

Traditional way to get smooth free quotient

Find all row and column permutations of configuration matrix

Could be done on a desktop

Get all possible linear actions on homogeneous coordinates and defining polynomials

Basically enumerate all the matrices which are reps of the symmetry group, which is computationally heavy already

Get invariant polynomials under the above action

Check explicitly if fixed points of the group action are on the variety defined by the above invariant polynomials

Check smoothness for fixed point free cases

Involve Grobner basis calculation

Traditional way to get smooth free quotient

Find all row and column permutations of configuration matrix

Get all possible linear actions on homogeneous coordinates and defining polynomials

Get invariant polynomials under the above action

Getting explicit action matrices and invariant polys is a heavy work for all the possibilities(order of 10^10 in normal CICY)

Check explicitly if fixed points of the group action are on the variety defined by the above invariant polynomials

Will be worse if we do Grobner basis calculation here

Is there a way out?

Check smoothness for fixed point free cases

Character valued index

For a given finite group G with group element g, suppose g's action on a CY is $\gamma(g)$

For a G-invariant line bundle $\mathcal{L}$, there is an induced action: $\gamma(g)^{\star}\mathcal{L} \to \mathcal{L}$

Character valued index is defined as(Braun,'10):

$$\chi(\mathcal{L})(\mathrm{g}) = \sum_{\mathrm{i}} (-1)^{\mathrm{i}} \mathrm{Tr}_{\mathrm{H}^{\mathrm{i}}(\mathcal{L})}(\gamma(\mathrm{g})^{\star})$$

Clearly for g=1, this character valued index is the usual Euler characteristic.

For g not the identity element, by using the generalization of the Lefshetz fixed point theorem to holomorphic vector bundles,

g's action is fixed point free $\Longrightarrow$ **character valued indices of invariant line bundles vanish**

Then how do we use this fact? $\Longrightarrow$

Braun’s Idea

Find all row and column permutations of configuration matrix

Get all possible linear actions on homogeneous coordinates and defining polynomials

…..

Check explicitly if fixed points of the group action are on the variety defined by the above invariant polynomials

…..

Equivalent to find representations of the symmetry group

Braun devised a way to get character valued index from character !

…..

Could be done by enumerate all the characters of the symmetry group

Now we can use character valued index to help in checking fixed point freeness

…..

Improved Scanning Process

Find all row and column permutations of configuration matrix

Enumerate all the possible actions on homogeneous coordinates and defining polynomials by using characters

Avoid explicit reps in matrices and generating invariant polynomials

Check possible actions are fixed points free or not by calculate character valued index of sample invariant line bundles, remember this is only necessary check

No Grobner basis calculation involved at this stage. Only elementary manipulations of integers

We do a sufficient check for fixed point freeness and finally for what left check smoothness

Results

Some statistics:

Out of 2946 manifolds with new favorable matrix, 1393 could have a symmetry allowed by indices.

We got 257 of possible fixed points free quotients on 99 manifolds by using character valued index calculation.

After explicitly constructing their representations and plug in fixed points, we found actually 230 of them are really fixed point free.

So actually to use character valued index to judge fixed point freeness is very effective!

As we know, symmetries classified by Braun depends on configuration matrix, and what found explicitly confirmed this.

In the original CICY list, there are lots of symmetries without row and column permutations, basically only toric actions on homogeneous coordinates. For CICY in their newly found favorable configuration, there is no symmetries of this kind. **So there are symmetries in Braun' list that can not descend from the newly found favorable configuration!**

But for 54 of the fixed points free quotients, their normal configuration partner even don't have a symmetry with the same group order. So if those 54 new quotients are smooth, they are brand new symmetries of CICY. Up to now, we found 33 of them are smooth. **Therefore we found new symmetries that are not in Braun's list as well!**

Example:

$$\left(\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 1 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^2 & 0 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{c|ccccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \mathbb{P}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Row action:

$$(24)(58)$$

Column action:

$$(12)(36)(45)$$

Normal CICY does not have any free quotient.

Favorable CICY has an order 2 symmetry

There are also symmetries that can descend to normal CICY:

$$\left(\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 0 & 3 \end{array}\right)$$

Normal CICY has an order 3 action with row permutations: (123)

$$\left(\begin{array}{c|cccc} \mathbb{P}^2 & 1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 1 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 0 & 1 \\ \mathbb{P}^2 & 0 & 0 & 0 & 3 \end{array}\right)$$

Favorable CICY also has an order 3 symmetry with permutations:

row: (234) column: (132)

Take the first 3 defining polys of favorable CICY, take homogeneous coordinates of the projective 2 space as variables, take the coefficient matrix and take its determinant, we get exactly the first defining poly of the normal CICY and the symmetry action on this determinant is exactly the same as the symmetry action of the normal CICY!

Conclusion and Future Direction

1. Braun's method is very effective in checking fixed point freeness.

2. As we can see in the result, symmetries of CICYs that descend from actions of their ambient space are related to their configuration matrix, and indeed we find new symmetries of some CICYs in their favorable configuration.

3. We will be releasing an easy to use database for the final symmetries in a simple form, which can be used for any desired application.

4. We can extend to non-cyclic groups in the next step.

5. We can study the free quotients of gCICY(Anderson et al, '15) by using the same method.

Thanks!