Sum of products versus product of sums

We usually think of multiple conjunctural causation in terms of “sum of products” expressions. For example, the equation:

$$Y = AB + CD$$

states simply that there are two ways to produce Y (which are linked via “logical or” in the equation), each involving a combination of conditions (i.e., products). Sometimes it is more convenient to express causal relations in terms of product of sums, as in:

$$Y = (A + B)(C + D)$$

(This is not logically equivalent to the first equation!) In this formulation, there are two basic causal conditions. The first, which we’ll call **X** can be satisfied by the presence of either A or B; the second, which we’ll call **Z** can be satisfied by either C or D. In effect, the equation is:

$$Y = \mathbf{XZ}$$

In this way of seeing the equation, there is only one causal combination (or recipe) and two essential conditions, but there are two ways to satisfy each of these general conditions, **X** can be satisfied with A or B, and **Z** can be satisfied with C or D.

It is important to understand that ANY sum of products expression can be converted to a product of sums expression and vice versa. One way of representing logical relations is not inherently superior to the other. For example, it is simple to multiply (A + B)(C + D) to get AC + AD + BC + BD. The two expressions are equivalent.

In product of sums form, solutions can be interpreted in terms of broader conditions that can be satisfied in more than one way. This is what Gary Goertz calls a two-level theory in his book *Social Science Concepts: A User's Guide*. However, it is important to understand that while it is tempting to interpret these broader concepts as essential ingredients and thus as a specification of substitutable necessary conditions, the key relationship is still one of sufficiency. That is, the product-of-sums equation specifies the same exact subset of the outcome that the sum-of-products equation specifies. The "leap" to necessary conditions can be made (from the product-of-sums equation) *only if coverage approaches 1.0*.

More generally, the claim the evidence indicates a necessary conditions relationship is valid only when the researcher demonstrates that the causal condition (or combination of conditions) is a superset of the outcome.

How to derive the product-of-sums version of a sum-of-products equation

Most of the time we are interested in sum-of-products equations. However, it is often useful to examine the product-of-sums expression, especially if you suspect **substitutable** conditions, as in the example just described.

Suppose you conducted a truth table analysis and derived a logical equation for the presence of the outcome and found:

$$Y \geq AB + AC + bC + Bc$$

Applying De Morgan's Law yields:

$$y \leq (a + b)(a + c)(B + c)(b + C)$$

which simplifies to:

$$y \leq aBC + bc$$

It is possible to use the equation for y (the absence of Y) to derive the product of sums version of the equation for the presence of Y. After all, the original equation for Y looks like it could be factored in interesting ways, but it is not clear how to do it.

Here are the remaining steps

Use the equation for y (the absence of Y), just derived:

$$y = aBC + bc$$

Applying De Morgan's Law to this equation yields:

$$Y = (A + b + c)(B + C)$$

You now have the product of sums version of the equation for Y, which in fact is simpler than the sum of products version (in this example). The question now is: Do A, b, and c make sense are substitutable conditions? What about B and C?

Summary:

1. Derive the sum-of-products solution for outcome Y.

2. Apply De Morgan's Law (which gives the product-of-sums equation for y, the negation of Y).

3. Simplify the results algebraically (which gives the sum-of-products equation for y, the negation of Y).

4. Re-apply De Morgan's Law (which gives the product-of-sums solution for Y).