# Some Basic Principles of Boolean Algebra

1. Set union, logical "or,” represented with addition

The union of sets A and B is "A + B", which means all cases in set A or in set B (or in both sets).

2. Set intersection, logical "and", represented with multiplication

The intersection of sets A and B is "A·B", which means all cases residing in both sets simultaneously. Intersection is also represented without a symbol, as in AB.

3. A + B = B + A and A·B = B·A (Order does not matter. If order does matter, it must be represented separately.)

4. The negation of a set is all instances of its absence.

Mathematically, the calculation is: 1 - A = a, or membership in the set "Absence of A" equals 1-membership in set A.

There are three ways to represent absence: lower case letters, a negation sign (~), and 0s (where specific location in a logical expression matters). There are three different notational systems for the same algebra. When using 1's and 0's as the notational system, the dash (-) is used as a placeholder to indicate irrelevance.

There are two central issues in negation: (1) the definition of the domain, and (2) "true" dichotomies (clearly defined negated set) versus dichotomies that are only *logical* (the negation of a set may be quite heterogeneous).

5. When order matters (see #3), it is often only the order of conditions that are *present* that is the focus.

For example, if it matters which occurs first, A or B, then both must have occurred for order to be an issue. Consider: ab, aB, Ab, AB.

6. Logical expressions can be factored for the sake of clarity. For example, Ac + AB can be factored to show the common term: A(c + B)

7. $A + A \cdot B = A$

Joining a subset of a set and the original set using logical "or" yields the original set. This is sometimes called "inclusion" or "containment."

8. A· (A·B) = A·B

Intersecting a subset of a set with the original set using logical "and" yields the intersection.

9. A + a = 1

The union of a set with its negation is all inclusive.  This principle does not hold in fuzzy algebra.

10. A·a = {empty set} or {Ø}

The intersection of a set with its negation is an empty set.  An empty set is also expressed as {0}.  This principle does not hold in fuzzy algebra.

11. A·{Ø} = {Ø}; A + {Ø} = A

The intersection of a set with an empty set is an empty set.  The union of a set with an empty set is the original set.

12. A·1 = A; A + 1 = 1

The intersection of a set with totality is the original set. The union of a set with totality is totality.

13. Truth tables can be used to express the links between conditions and outcomes.

Example:

| A | B | Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

14. De Morgan's Law: In order to find the negation of a statement, change all presence to absence and vice versa, and all multiplication to addition, and vice versa. In equations, change “less than or equal to” to “greater than or equal to” and vice versa.

Example 1: $Y = AB + cd$

$y = (a + b) \cdot (C + D)$ (De Morgan’s applied)

$= aC + aD + bC + bD$ (expanded)

Example 2: $Y \geq a + BC$

$y \leq A\cdot (b+ c)$ (De Morgan's applied)

$y \leq Ab + Ac$ (expanded)

15. There are two major forms of equations, "sums of products" and "products of sums":

$Y = Ac + aB$ (sums of products)

$Y = (A + B)(a + c)(B + c)$ (products of sums)

16. Using De Morgan's it is possible to convert from one to the other:

$Y = Ac + aB$

$y = (a + C)(A + b)$ (De Morgan's applied)

$y = ab + AC + bC$ (expanded)

$Y = (A + B)(a + c)(B + c)$ (De Morgan's re-applied)

The last equation is equivalent to the first; only the form differs.

## More on De Morgan's Law

There is a simple way to reverse whole equations using Boolean algebra. Suppose, for example, that we find that whenever austerity measures are dictated by the International Monetary Fund (AUSTERITY) in situations of high levels of urbanization (URBAN) *or* high levels of foreign investment (FORINV) in Third World countries, protest erupts (RIOTS). The equation is:

AUSTERITY·URBAN + AUSTERITY·FORINV = RIOTS

To find out the conditions for the absence of protest, as indicated by the negation of this equation, simply reverse code presence to absence, absence to presence, "logical or" to "logical and," and "logical and" to "logical or," as follows:

(austerity + urban)*(austerity + forinv) = riots

which simplifies to:

austerity + urban*forinv = riots

If the equation is an inequality, then it is also necessary to switch the direction of the inequality, as in:

AUSTERITY ≥ RIOTS

austerity ≤ riots

## Key Differences Between Fuzzy Algebra and Boolean Algebra

There are two operations that are valid in Boolean algebra, but not in fuzzy algebra:

1. A + ~A = 1 (Boolean algebra only)

In fuzzy algebra, logical *or* (union) is the same as taking the maximum value. Thus, if A = .4, then ~A = .6 and the expression A + ~A is equal to .6, the greater of the two values.

2. A(~A) = {Ø} (Boolean algebra only)

In fuzzy algebra, logical *and* (intersection) is the same as taking the minimum value. Thus, if A = .4, then ~A = .6 and the expression A(~A) is equal to .4, the smaller of the two values.

This limitation of fuzzy algebra has important implications for analysis. One of the most basic rules for simplifying truth tables is to combine rows that differ on only one causal condition. For example, assume that both ABC and ABc are linked to outcome Y:

$$ABC + ABc = AB(C + c) = AB(1) = AB$$

This elimination of causes is (technically) not possible in fuzzy algebra because $C + c \neq 1$, but instead equals a value that can be anywhere in the interval from .5 to 1.0.

However, the inclusion rule still holds in fuzzy algebra, which is that $AB \leq A$. This rule is useful because it means that if A is sufficient for Y (i.e., $A \leq Y$), then so is AB, along with A combined with any other set or sets (e.g., ACD).