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Chapter 2

The Algebra of Sets

2.1 SETS

By a *set* we mean any collection of objects.† For example, we may speak of the set of all living Americans, the set of all letters of the English alphabet, or the set of all real numbers less than 4. In most cases, sets will be defined by means of a characteristic property of the objects belonging to the set. In the examples above, we used the properties of being a living American, a letter of the English alphabet, or a real number less than 4.

Notation: For a given property $P(x)$, let $\{x : P(x)\}$ denote the set of all objects x such that $P(x)$ is true.

Example 2.1.

The set of all real roots of the equation $x^4 - 2x^2 - 3 = 0$ is denoted by

$$\{x : x \text{ is a real number } \& \ x^4 - 2x^2 - 3 = 0\}$$

Sometimes we shall define a set merely by listing its elements within braces: $\{a, b, c, \ldots, h\}$. In particular, $\{b\}$ is the set having b as its only member. Such a set $\{b\}$ is called a *singleton*. The set $\{b, c\}$ contains b and c as its only members, and, if $b \neq c$, then $\{b, c\}$ is called an *unordered pair*. Notice that $\{b, c\} = \{c, b\}$.

Example 2.2.

The set of integers strictly between 1 and 5 is equal to $\{2, 3, 4\}$.

Example 2.3.

The set of all real roots of the equation $x^4 - 2x^2 - 3 = 0$ is equal to the set $\{\sqrt{3}, -\sqrt{3}\}$.

We shall extend this method of denoting sets by listing a few elements of the set, followed by dots, in such a way as to indicate the characteristic property of the elements of the set.

Example 2.4.

$\{1, 2, 3, 4, \ldots\}$ is intended to represent the set of positive integers. $\{1, 4, 9, 16, 25, \ldots, n^2, \ldots\}$ is the set of squares of positive integers. {Washington, Adams, Jefferson, Madison, ...} is the set of Presidents of the United States.

Definition: An object x belonging to a set A is said to be a *member* or *element* of A. We shall write $x \in A$ to indicate that x is a member of A. The denial of $x \in A$ will be written $x \notin A$.

Example 2.5. $6 \in \{x : x \text{ is an even integer}\}$, $1 \notin \{x : x \text{ is an even integer}\}$

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†Synonyms for *set* are *totality*, *family*, and *class*.

2.2 EQUALITY AND INCLUSION OF SETS. SUBSETS

The sets A and B are equal when and only when A and B have the same members. Equality of A and B is designated in the usual way by $A = B$, and denial of this equality by $A \neq B$.

Example 2.6. $\{x : x^2 = 1 \text{ and } x \text{ is a real number}\} = \{x : x = 1 \text{ or } x = -1\}$

We say that A is a *subset* of B if and only if every member of A is also a member of B. We write $A \subseteq B$ as an abbreviation for: *A is a subset of B*. Sometimes, instead of saying that A is a subset of B, one says that A is *included* in B. The denial of $A \subseteq B$ is written $A \nsubseteq B$.

Example 2.7. $\{1, 3\} \subseteq \{1, 2, 3, 6\}$; $\{b, a\} \subseteq \{c, a, b\}$; $\{1, 2, 4\} \nsubseteq \{1, 2, 5\}$

Obvious properties of the inclusion relation are

Incl (i) $A \subseteq A$ (Reflexivity).

Incl (ii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (Transitivity).

Incl (iii) $A = B$ if and only if $(A \subseteq B \;\&\; B \subseteq A)$.

It is convenient to introduce a special sign for the relation of *proper inclusion*. We shall use $A \subset B$ as an abbreviation for $A \subseteq B \;\&\; A \neq B$. Thus $A \subset B$ if and only if every member of A is a member of B but there is a member of B which is not a member of A. If $A \subset B$, we say that A is a *proper subset* of B. Hence the only subset of B which is not a proper subset of B is B itself. The denial of $A \subset B$ is written $A \not\subset B$.

Some basic properties of proper inclusion are:

PI(i) $A \not\subset A$.

PI(ii) If $A \subset B \;\&\; B \subseteq C$, then $A \subset C$.

PI(iii) If $A \subseteq B \;\&\; B \subset C$, then $A \subset C$.

PI(iv) If $A \subset B$, then $B \not\subset A$.

Example 2.8. $\{1, 3\} \subset \{1, 2, 3\}$; $\{1, 3\} \not\subset \{1, 3\}$; $\{1, 4\} \not\subset \{1, 3\}$

2.3 NULL SET. NUMBER OF SUBSETS

Whenever $P(x)$ is a property satisfied by no objects at all, then $\{x : P(x)\}$ is a set having no members. For example, $\{x : x \neq x\}$ is a set with no members. We shall use $\emptyset$ to denote a set with no members. The set $\emptyset$ is called the *null set* or *empty set*. There is precisely one null set, since any two null sets would contain the same members (namely, none at all) and therefore must be equal. The null set is included in every set: $\emptyset \subseteq A$ for all A.

Example 2.9.

The only subset of $\emptyset$ is $\emptyset$ itself.

Example 2.10.

The subsets of $\{x\}$ are $\emptyset$ and $\{x\}$. Thus a singleton has two subsets.

Example 2.11.

If $x \neq y$, the subsets of the unordered pair $\{x, y\}$ are $\emptyset$, $\{x\}$, $\{y\}$ and $\{x, y\}$. Thus a two-element set has four subsets.

Example 2.12.

If x, y and z are three distinct objects, then the subsets of $\{x, y, z\}$ are $\emptyset$, $\{x\}$, $\{y\}$, $\{z\}$, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$ and $\{x, y, z\}$. Thus there are eight subsets of a three-element set.

Let $\mathcal{P}(A)$ denote the set of all subsets of A. Then $\mathcal{P}(A) = \{B : B \subseteq A\}$. Examples 2.9-2.12 suggest the following result.

Theorem 2.1. For any non-negative integer n, if a set A has n elements, then the set $\mathcal{P}(A)$ of all subsets of A has 2^n elements.

First proof: The result is clear when $n = 0$ (Example 2.9). Assume a set A has n elements, where $n > 0$. In choosing an arbitrary subset C of A, there are two possibilities for each element x of A: $x \in C$ or $x \notin C$. Whether one element x is in the subset C is independent of whether any other element y is in C. Hence there are 2^n ways of choosing a subset of A.

Second proof: By induction on n. The case for $n = 0$ is clear (Example 2.9). Assume that the result is true for $n = k$, and assume that A is a set with $k+1$ elements, i.e. $A = \{a_1, \ldots, a_k, a_{k+1}\}$. We must prove that A has 2^{k+1} subsets. Let $B = \{a_1, \ldots, a_k\}$. Since B has k elements, then by inductive hypothesis B has 2^k subsets. Every subset C of B can be thought of as determining two distinct subsets of A, i.e. C itself and C together with the element a_{k+1}. In addition, every subset D of A is determined in this way by precisely one subset C of B, i.e. C is obtained by removing a_{k+1} from D (where, if $a_{k+1} \notin D$, then C is identical with D). Thus the number of subsets of A is twice the number of subsets of B. But since B has 2^k subsets, A has 2^{k+1} subsets. ▶

2.4 UNION

Given sets A and B, their *union* $A \cup B$ consists of all elements of A or B or both. Thus $A \cup B = \{x : x \in A \vee x \in B\}$. Remember that $\vee$ stands for the inclusive "or", i.e. for any sentences **A**, **B**, **A** $\vee$ **B** means **A** or **B** or both.

Example 2.13.

$$\{1, 2, 3\} \cup \{1, 3, 4, 6\} = \{1, 2, 3, 4, 6\}$$

$$\{a\} \cup \{b\} = \{a, b\}$$

$$\{0, 2, 4, 6, 8, \ldots\} \cup \{1, 3, 5, 7, 9, \ldots\} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

If we represent the elements of A and B by points within two circles, then their union consists of all points lying within either of the two circles (see Fig. 2-1). The union operation on sets has the obvious properties:

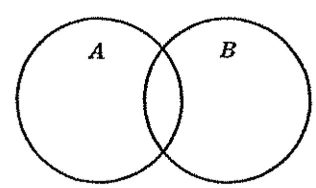


Fig. 2-1

U(i) $A \cup A = A$ (Idempotence)

U(ii) $A \cup B = B \cup A$ (Commutativity)

U(iii) $A \cup \emptyset = A$

U(iv) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity)

U(v) $A \cup B = B$ if and only if $A \subseteq B$

U(vi) $A \subseteq A \cup B$ & $B \subseteq A \cup B$

2.5 INTERSECTION

Given sets A and B, their *intersection* $A \cap B$ consists of all objects which are in both A and B. Thus,

$$A \cap B = \{x : x \in A \ \& \ x \in B\}$$

Example 2.14.

$$\{1, 2, 4\} \cap \{1, 3, 4\} = \{1, 4\}$$
$$\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$$
$$\{1, 3, 5\} \cap \{0, 2\} = \emptyset$$
$$\{0, 1, 2\} \cap \{0, 3\} = \{0\}$$

Pictorially, we can visualize the intersection as consisting of the shaded area of Fig. 2-2.

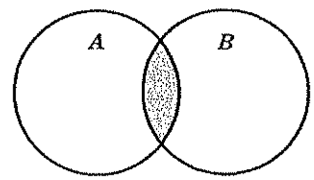


Fig. 2-2

Two sets A and B such that $A \cap B = \emptyset$ are said to be *disjoint*.

The following properties of the intersection operation are evident.

Int (i) $A \cap A = A$ (Idempotence).

Int (ii) $A \cap B = B \cap A$ (Commutativity).

Int (iii) $A \cap \emptyset = \emptyset$.

Int (iv) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity).

Int (v) $A \cap B = A$ if and only if $A \subseteq B$.

Int (vi) $A \cap B \subseteq A$ & $A \cap B \subseteq B$.

The associative laws for unions and intersections allow us to omit parentheses in writing unions or intersections of three or more sets. Thus we write $A \cap B \cap C$ to stand for either $(A \cap B) \cap C$ or $A \cap (B \cap C)$, since these sets are equal. Similarly $A \cap B \cap C \cap D$ has a unique meaning, since any of the five ways of inserting parentheses yields the same result.

Important relations between unions and intersections are given by the *distributive laws*:

Dist (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Dist (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Property Dist (i) can be verified directly from the definitions by logical manipulations. Thus,

$$\begin{aligned} A \cap (B \cup C) &= \{x : \ x \in A \ \& \ x \in B \cup C\} \\ &= \{x : \ x \in A \ \& \ (x \in B \vee x \in C)\} \\ &= \{x : \ (x \in A \ \& \ x \in B) \vee (x \in A \ \& \ x \in C)\} \\ &= \{x : \ x \in A \cap B \ \vee \ x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

We also can check Dist (i) pictorially. In Fig. 2-3 below, we have vertical lines for $B \cup C$ and horizontal lines for A. Hence $A \cap (B \cup C)$ is represented by the cross-hatched area. In Fig. 2-4 below, the vertical lines indicate $A \cap B$ and the horizontal lines $A \cap C$. The combined area represents $(A \cap B) \cup (A \cap C)$ and is seen to be identical with the cross-hatched area of Fig. 2-3. Dist (ii) may be handled in a similar manner (see Problem 2.3).

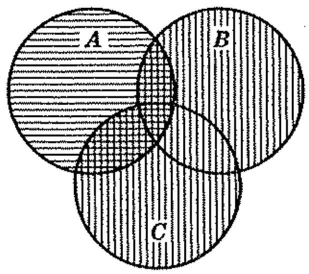


Fig. 2-3

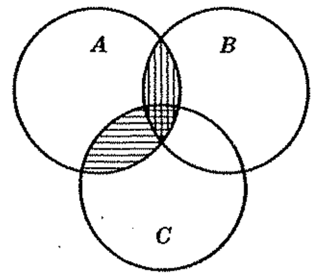


Fig. 2-4

Diagrams as shown in Fig. 2-3 and 2-4 are called *Venn diagrams*. They are useful for verifying identities involving operations on sets, but should not be considered tools of rigorous mathematical proof. Similar pictorial methods can be given for four or more sets (see [112]† and J. F. Randolph, *American Mathematical Monthly*, 1965, pp. 117-127), but this does not seem fruitful enough to warrant consideration here.

Example 2.15. Show that $A \cap (A \cup B) = A$.

By U(vi), $A \subseteq A \cup B$. Hence, by Int (v), $A \cap (A \cup B) = A$.

Example 2.16. Show that $A \cup (A \cap B) = A$.

$$A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B) = A$$

The first equality is justified by Dist (ii), the second by U(i), and the third by Example 2.15.

The distributive laws have the obvious generalizations:

Dist (i′) $\quad A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$

Dist (ii′) $\quad A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$

These can be proved directly, using mathematical induction.

2.6 DIFFERENCE AND SYMMETRIC DIFFERENCE

By the *difference* $B \sim A$ of B and A we mean the set of all those objects in B which are not in A (the shaded area of Fig. 2-5). Thus,

$$B \sim A = \{x : x \in B \ \& \ x \notin A\}$$

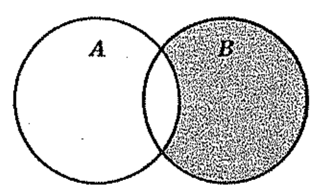


Fig. 2-5

Clearly,

D(i) $\quad B \sim B = \emptyset$

D(ii) $\quad B \sim \emptyset = B$

D(iii) $\quad \emptyset \sim A = \emptyset$

D(iv) $\quad (A \sim B) \sim C = A \sim (B \cup C) = (A \sim C) \sim B$

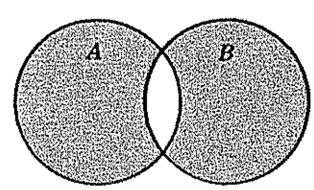


Fig. 2-6

The *symmetric difference* $A \Delta B$ of sets A and B is $(A \sim B) \cup (B \sim A)$ (the shaded area of Fig. 2-6). Fig. 2-6 makes it clear that $A \Delta B = (A \cup B) \sim (A \cap B)$.

†Throughout this book numbers in brackets refer to Bibliography, pages 201-208.

The following properties are easily verified.

SD(i) $A \triangle A = \emptyset$

SD(ii) $A \triangle B = B \triangle A$

SD(iii) $A \triangle \emptyset = A$

Example 2.17.

Let $A = \{0,1,2,3,5\}$, $B = \{0,1,2,3\}$, $C = \{0,1,4,5\}$. Then $B \subset A$, $C \not\subseteq A$, $A \cap C = \{0,1,5\}$, $B \cap C = \{0,1\}$, $A \sim B = \{5\}$, $A \sim C = \{2,3\}$, $B \sim C = \{2,3\}$, $C \sim B = \{4,5\}$, $A \triangle B = \{5\}$, $A \triangle C = \{2,3,4\}$, $B \triangle C = \{2,3,4,5\}$.

2.7 UNIVERSAL SET. COMPLEMENT

We shall often find it useful to confine our attention to the subsets of some given set X. In such a case, X is called the *universal set* or the *universe* (of discourse).

The union, intersection, difference, and symmetric difference of subsets of X are again subsets of X. When we restrict ourselves to subsets of X, there is still another operation which can be introduced. If $A \subseteq X$, then the *complement* $\bar{A}$ of A is defined to be $X \sim A$. Thus, $\bar{A} = \{x : x \in X \;\&\; x \notin A\}$. *Whenever we use complements, it is assumed that we are dealing only with subsets of some fixed universe* X.

The following assertions are easily verified.

C(i) $\bar{\bar{A}} = A$

C(ii) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ } De Morgan's Laws

C(iii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

C(iv) $A \cap \bar{A} = \emptyset$

C(v) $A \cup \bar{A} = X$

C(vi) $\bar{\emptyset} = X$

C(vii) $\bar{X} = \emptyset$

C(viii) $A \subseteq B$ if and only if $\bar{B} \subseteq \bar{A}$

C(ix) $A = B$ if and only if $\bar{A} = \bar{B}$

C(x) $A \sim B = A \cap \bar{B}$

C(xi) $A \triangle B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$

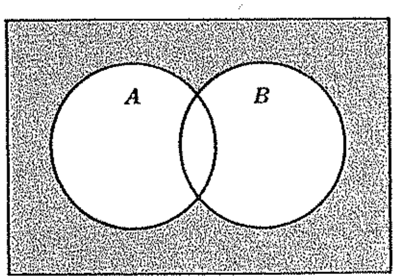
From C(x) and C(xi) we see that difference and symmetric difference are dispensable in the presence of union, intersection and complement.

Example 2.18.

Let us check C(ii) using definitions and logical transformations.

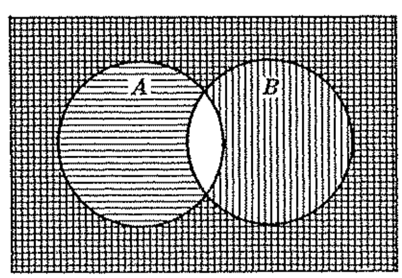
$$\begin{aligned}\overline{A \cup B} &= \{x : x \in X \;\&\; x \notin A \cup B\} = \{x : x \in X \;\&\; \neg(x \in A \vee x \in B)\} \\ &= \{x : x \in X \;\&\; (x \notin A \;\&\; x \notin B)\} = \{x : (x \in X \;\&\; x \notin A) \;\&\; (x \in X \;\&\; x \notin B)\} \\ &= \{x : x \in X \;\&\; x \notin A\} \cap \{x : x \in X \;\&\; x \notin B\} = \bar{A} \cap \bar{B}\end{aligned}$$

We also may use Venn diagrams to verify the validity of C(ii). Compare Fig. 2-7 and 2-8.



$\overline{A \cup B}$ is the shaded area.

Fig. 2-7



$\bar{A} \cap \bar{B}$ is the cross-hatched area.

Fig. 2-8

De Morgan's Laws C(ii)-C(iii) have the obvious generalizations:

C(ii′) $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n$

C(iii′) $\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_n$

Example 2.19.

$A \subseteq B$ if and only if $A \cap \bar{B} = \emptyset$. In Fig. 2-9, the cross-hatched area is $A \cap \bar{B}$. To say that this is $\emptyset$ is equivalent to saying that A is entirely within B.

Logical proof:

$$A = A \cap X = A \cap (B \cup \bar{B}) = (A \cap B) \cup (A \cap \bar{B})$$

Hence if $A \cap \bar{B} = \emptyset$, then $A = A \cap B$; therefore, by Int (v), $A \subseteq B$. On the other hand, if $A \subseteq B$, then by Int (v), $A = A \cap B$ and therefore

$$A \cap \bar{B} = (A \cap B) \cap \bar{B} = A \cap (B \cap \bar{B}) = A \cap \emptyset = \emptyset$$

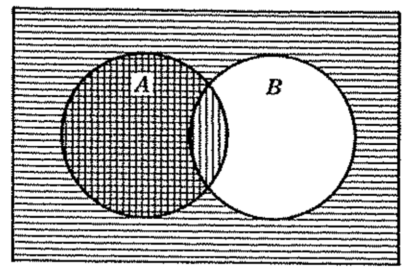


Fig. 2-9

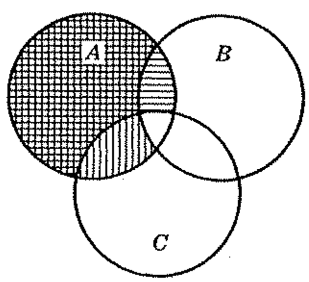
2.8 DERIVATIONS OF RELATIONS AMONG SETS

We have seen two ways of verifying propositions about sets: by means of analogous logical laws, or by pictorial methods (usually Venn diagrams). The first method is the only rigorous one, but the use of diagrams is sometimes quicker and more intuitive.

Example 2.20.

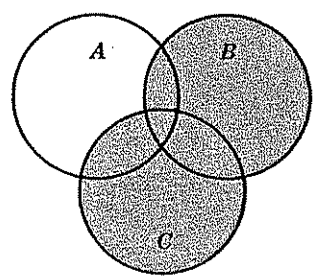
Prove $A \sim (B \cup C) = (A \sim B) \cap (A \sim C)$.

This is clear from Figs. 2-10 and 2-11.



Cross-hatched: $(A \sim B) \cap (A \sim C)$

Fig. 2-10



Unshaded: $A \sim (B \cup C)$

Fig. 2-11

More rigorously,

$$\begin{aligned} A \sim (B \cup C) &= \{x : x \in A \ \& \ x \notin (B \cup C)\} \\ &= \{x : x \in A \ \& \ (x \notin B \ \& \ x \notin C)\} \\ &= \{x : (x \in A \ \& \ x \notin B) \ \& \ (x \in A \ \& \ x \notin C)\} \\ &= \{x : x \in A \ \& \ x \notin B\} \cap \{x : x \in A \ \& \ x \notin C\} \\ &= (A \sim B) \cap (A \sim C) \end{aligned}$$

Example 2.21.

Prove: $(A \cup B) \cap \bar{B} = A$ if and only if $A \cap B = \emptyset$.

In Fig. 2-12 below, the cross-hatched part represents $(A \cup B) \cap \bar{B}$ and lies entirely within A. The rest of A is the lens-shaped intersection $A \cap B$. Hence to say that $(A \cup B) \cap \bar{B}$ is identical with A is equivalent to saying that $A \cap B = \emptyset$.

Logical proof:

$$(A \cup B) \cap \bar{B} = (A \cap \bar{B}) \cup (B \cap \bar{B}) \quad \text{(by Dist (ii))}$$
$$= (A \cap \bar{B}) \cup \emptyset \quad \text{(by C(iv))}$$
$$= A \cap \bar{B} \quad \text{(by U(iii))}$$

Hence $(A \cup B) \cap \bar{B} = A$ if and only if $A \cap \bar{B} = A$. But $A \cap \bar{B} = A$ if and only if $A \subseteq \bar{B}$ (Int (v)), which holds if and only if $A \cap B = \emptyset$ (by Example 2.19 and C(i)).

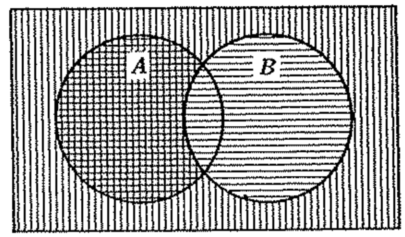


Fig. 2-12

Example 2.22.

Simplify $\overline{A \cap \bar{B}} \cup (B \cap C)$.

$$\overline{A \cap \bar{B}} \cup (B \cap C) = (\bar{A} \cup \bar{\bar{B}}) \cup (B \cap C) \quad \text{(by C(iii))}$$
$$= (\bar{A} \cup B) \cup (B \cap C) \quad \text{(by C(i))}$$
$$= \bar{A} \cup (B \cup (B \cap C)) \quad \text{(by U(iv))}$$
$$= \bar{A} \cup B \quad \text{(by Example 2.16)}$$

In simplifying expressions, we make frequent use of De Morgan's Laws C(ii) and C(iii) for distributing complement bars over smaller expressions, C(i) for eliminating double complements, Examples 2.15 and 2.16, and the distributive laws Dist (i) and Dist (ii).

Example 2.23.

Simplify $(A \cup B \cup C) \cap (\overline{A \cap \bar{B} \cap \bar{C}}) \cap \bar{C}$.

$$(A \cup B \cup C) \cap (\overline{A \cap \bar{B} \cap \bar{C}}) \cap \bar{C} = (A \cup B \cup C) \cap (\bar{A} \cup B \cup C) \cap \bar{C} \quad \text{(De Morgan)}$$
$$= [(A \cup B \cup C) \cap (\bar{A} \cup B \cup C)] \cap \bar{C} \quad \text{(Associativity of } \cap\text{)}$$
$$= [(A \cap \bar{A}) \cup (B \cup C)] \cap \bar{C} \quad \text{(Dist (ii))}$$
$$= [B \cup C] \cap \bar{C} = (B \cap \bar{C}) \cup (C \cap \bar{C}) = (B \cap \bar{C}) \cup \emptyset \quad \text{(C(iv), U(iii))}$$
$$= B \cap \bar{C}$$

2.9 PROPOSITIONAL LOGIC AND THE ALGEBRA OF SETS

Every truth-functional operation determines a corresponding operation on sets. For example, denial determines complementation: $\bar{A} = \{x : \neg(x \in A)\}$; conjunction determines the intersection operation: $A \cap B = \{x : x \in A \ \& \ x \in B\}$; and disjunction determines the union operation: $A \cup B = \{x : x \in A \vee x \in B\}$. In general, if # is a connective corresponding to a truth function $f(x_1, \ldots, x_n)$, then we define a corresponding operation @ on sets by $@(A_1, \ldots, A_n) = \{x : \#(x \in A_1, \ldots, x \in A_n)\}$. Thus the set-theoretic operation of symmetric difference corresponds to the exclusive usage of "or".

Example 2.24.

The operation of alternative denial determines the set-theoretic operation $\overline{A \cap B}$, while joint denial determines the operation $\bar{A} \cap \bar{B}$.

In general, a uniform way of determining the set-theoretic operation corresponding to a given truth function is to express the latter in terms of $\neg$, &, $\vee$, and then replace $\neg$, &, $\vee$ by $^-$, $\cap$, $\cup$ respectively. The statement letters need not be replaced since they can serve as set variables in the new expression.

2.10 ORDERED PAIRS. FUNCTIONS

If $x \neq y$, then $\{x, y\}$ was called the *unordered pair* of x and y. We say "unordered" because $\{x, y\} = \{y, x\}$. Let us define an *ordered pair* $\langle x, y\rangle$, which is determined by x and y, in that order. By this we mean that the following proposition holds: if $\langle x, y\rangle = \langle u, v\rangle$, then $x = u$ and $y = v$.

Theorem 2.2. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ is an adequate definition of an ordered pair.

Proof. Assume $\langle x, y\rangle = \langle u, v\rangle$. We must prove that $x = u$ and $y = v$. We have

$$\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\} \tag{2.1}$$

Since $\{x\}$ is a member of the left side of equation (*2.1*), it must also be a member of the right side. Hence

$$\{x\} = \{u\} \quad \text{or} \quad \{x\} = \{u, v\}$$

Therefore $x = u$ or $x = u = v$. In either case, $x = u$. Now by (*2.1*),

$$\{x, y\} = \{u\} \quad \text{or} \quad \{x, y\} = \{u, v\}$$

If $\{x, y\} = \{u, v\}$, then $\{x, y\} = \{x, v\}$ since $x = u$. Hence $y = x$ or $y = v$. If $y = x$, then $\{y\} = \{y, v\}$ and $y = v$. In all cases, $y = v$. If $\{x, y\} \neq \{u, v\}$, then $\{x, y\} = \{u\}$ and so $x = y = u$. By (*2.1*),

$$\{u, v\} = \{x\} \quad \text{or} \quad \{u, v\} = \{x, y\}$$

Since $\{u, v\} \neq \{x, y\}$, $\{u, v\} = \{x\}$ and so $u = v = x$. Therefore $y = v$. ▸

Let us recall the definition of a function. A *function* f from A into B is a way of associating an element of B to each element of A. The phrase "way of associating" may be replaced by a more precise notion:

(1) f is a set of ordered pairs such that, if $\langle x, y\rangle \in f$ and $\langle x, z\rangle \in f$, then $y = z$;

(2) for every x in A there exists some y in B such that $\langle x, y\rangle \in f$. (Such an object y must be unique, by virtue of (1); it is denoted in the standard way by $f(x)$.)

We say that f is a function from A *onto* B if f is a function from A into B and every element of B is a value $f(x)$ for some x in A.

Example 2.25.

The function f such that $f(x) = x^2$ for every x in the set A of all integers is a function from A into (but not onto) A. On the other hand, f is a function from A onto the set B of all squares.

A function f is said to be *one-one* if it assigns different values to different arguments, i.e. $f(x) = f(y)$ implies $x = y$.

Example 2.26.

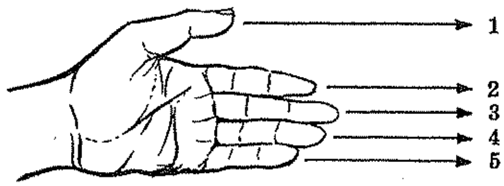
The function f in Example 2.25 is not one-one, since $f(-n) = n^2 = f(n)$ for all integers n. On the other hand, the function g such that $g(x) = x^2$ for all non-negative integers x is a one-one function, since $u^2 = v^2$ implies $u = v$ for all non-negative integers u and v.

A one-one function from A onto B is called a *one-one correspondence* between A and B. For example, the function h such that $h(x) = x + 1$ for all odd integers x is a one-one correspondence between the set of all odd integers and the set of all even integers.

2.11 FINITE, INFINITE, DENUMERABLE, AND COUNTABLE SETS

A *finite* set is a set which is either empty or can be enumerated by the positive integers from 1 up to some integer n. More precisely, A is finite if there is a positive integer n such that there is a one-one correspondence between A and the set of all positive integers less than n. (When $n = 1$, A must be the empty set.)

For example, to justify the assertion that the set of fingers on a hand is finite we set up the correspondence



It is clear that a subset of a finite set is finite (and hence that the intersection of any set with a finite set is finite). Also obvious is the fact that the union of two finite sets is finite.

A set is said to be *infinite* if it is not finite. Examples are the set of positive integers, the set of rational numbers, and the set of real numbers. Clearly any set containing an infinite set must also be infinite, and therefore the union of an infinite set with any other set is infinite. However, the intersection of two infinite sets need not be infinite. For example, the set of even integers and the set of odd integers have an empty intersection.

A set A is said to be *denumerable* (or *countably infinite*) if and only if A can be enumerated by the set P of all positive integers, i.e. if there is a one-one correspondence between P and A.

Example 2.27.

(1) The set of positive even integers is denumerable. Here the one-one correspondence is given by $f(n) = 2n$. (2) The set of all integers is denumerable. Here the enumeration is given by $0, 1, -1, 2, -2, 3, -3, \ldots$. The one-one correspondence is $g(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd} \end{cases}$.

Clearly the union of a finite set and a denumerable set is also denumerable. (Just enumerate the finite set first and continue with the enumeration of the denumerable set, omitting repetitions.) The union of two denumerable sets is again denumerable. (For, if $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$, then $A \cup B = \{a_1, b_1, a_2, b_2, \ldots\}$, where in the latter enumeration we omit any repeated objects.) If we remove a finite number of elements from a denumerable set, the remaining set is still denumerable.

A set is said to be *countable* if and only if it is either finite or denumerable. Obviously, any subset B of a countable set A is also countable. (For, in an enumeration of A, we omit all objects which are not in B. The resulting enumeration of B does or does not terminate. If it does, B is finite. If it does not, B is denumerable.) The union of two countable sets is a countable set. This follows from what has been said above about finite and denumerable sets.

2.12 FIELDS OF SETS

By a *field of sets on X* we mean a non-empty collection $\mathcal{F}$ of subsets of X such that, for any members A and B in $\mathcal{F}$, the sets $A \cup B$, $A \cap B$, and $\bar{A}$ are also in $\mathcal{F}$. Another way of expressing this is to say that $\mathcal{F}$ is closed under the operations of union, intersection and complementation. Since $A \cup B = \overline{\bar{A} \cap \bar{B}}$ and $A \cap B = \overline{\bar{A} \cup \bar{B}}$, it suffices to verify closure under complementation and either union or intersection.

Examples of fields of sets are:

(1) the set $\mathcal{P}(X)$ of all subsets of X;

(2) the set of all finite subsets of X and their complements;

(3) $\{\emptyset, X\}$.

Notice that any field $\mathcal{F}$ of subsets of X must contain both $\emptyset$ and X. For, if $B \in \mathcal{F}$, then $\bar{B} \in \mathcal{F}$ and hence $\emptyset = B \cap \bar{B} \in \mathcal{F}$. Therefore $X = \bar{\emptyset} \in \mathcal{F}$.

2.13 NUMBER OF ELEMENTS IN A FINITE SET

Let $\#(A)$ stand for the number of elements in a finite set A. Clearly

$$\#(A_1 \cup A_2) = \#(A_1) + \#(A_2) - \#(A_1 \cap A_2)$$

For three sets, we have

$$\begin{aligned}\#(A_1 \cup A_2 \cup A_3) = {} & \#(A_1) + \#(A_2) + \#(A_3) \\ & - \#(A_1 \cap A_2) - \#(A_1 \cap A_3) - \#(A_2 \cap A_3) \\ & + \#(A_1 \cap A_2 \cap A_3)\end{aligned}$$

and, for four sets,

$$\begin{aligned}\#(A_1 \cup A_2 \cup A_3 \cup A_4) = {} & \#(A_1) + \#(A_2) + \#(A_3) + \#(A_4) \\ & - \#(A_1 \cap A_2) - \#(A_1 \cap A_3) - \#(A_1 \cap A_4) - \#(A_2 \cap A_3) - \#(A_2 \cap A_4) - \#(A_3 \cap A_4) \\ & + \#(A_1 \cap A_2 \cap A_3) + \#(A_1 \cap A_2 \cap A_4) + \#(A_1 \cap A_3 \cap A_4) + \#(A_2 \cap A_3 \cap A_4) \\ & - \#(A_1 \cap A_2 \cap A_3 \cap A_4)\end{aligned}$$

The general formula for n sets should be clear from the examples for $n = 2, 3, 4$.

Example 2.28.

In a two-party election district consisting of 135 voters, 67 people voted for at least one Democrat and 84 people voted for at least one Republican. How many people voted for candidates of both parties?

$$\#(R \cap D) = \#(R) + \#(D) - \#(R \cup D) = 84 + 67 - 135 = 16$$

Here R is the set of people who voted for at least one Republican and D the set of people who voted for at least one Democratic candidate. Hence $R \cup D$ is the set of all voters and $R \cap D$ is the set of all people who split their ballots.

Example 2.29.

A government committee reported that, among the students using marijuana, LSD or heroin at a certain university, 90% used marijuana, 6% used LSD and 7% heroin, while 4% took marijuana and LSD, 5% marijuana and heroin, 2% heroin and LSD, and 1% took all three. Are the committee's figures consistent?

Note that, if there are n students taking at least one of the drugs, and if H is a set of students, then the percentage in H is $\#(H)/n$. Hence if we let A, B, C be the sets of students taking marijuana, LSD and heroin respectively, and we divide the equation for $\#(A \cup B \cup C)$ by n to obtain the percentages,

$$\%(A \cup B \cup C) = \%(A) + \%(B) + \%(C) - \%(A \cap B) - \%(A \cap C) - \%(B \cap C) + \%(A \cap B \cap C)$$

$$100 = 90 + 6 + 7 - 4 - 5 - 2 + 1 = 93$$

which is impossible. Hence the figures are not consistent.

Solved Problems

2.1. Show that the cancellation law

$$\text{if} \quad A \cup B = A \cup C \quad \text{then} \quad B = C$$

is false by giving a counterexample.

Solution:

$A = C = \{a\}$. $B = \emptyset$.

2.2. Show that parentheses are necessary for writing expressions involving more than one of the operations $\cap$ and $\cup$.

Solution:

Consider $A \cap B \cup C$. This is either $A \cap (B \cup C)$ or $(A \cap B) \cup C$. But these two sets are not necessarily equal. Take $A = \emptyset$ and $B = C \neq \emptyset$. Then $A \cap (B \cup C) = \emptyset$, but $(A \cap B) \cup C = \emptyset \cup C = C$.

2.3. Prove the distributive law Dist (ii), page 33: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution:

Logical Proof.

$$\begin{aligned} A \cup (B \cap C) &= \{x : x \in A \vee x \in (B \cap C)\} \\ &= \{x : x \in A \vee (x \in B \;\&\; x \in C)\} = \{x : (x \in A \vee x \in B) \;\&\; (x \in A \vee x \in C)\} \\ &= \{x : x \in A \vee x \in B\} \cap \{x : x \in A \vee x \in C\} = (A \cup B) \cap (A \cup C) \end{aligned}$$

Pictorial Proof. In Fig. 2-13, the vertical lines indicate $B \cap C$ and the shaded area is A. In Fig. 2-14, the vertical lines indicate $A \cup B$, the horizontal lines $A \cup C$, and the cross-hatched area $(A \cup B) \cap (A \cup C)$ is identical with the marked area of Fig. 2-13.

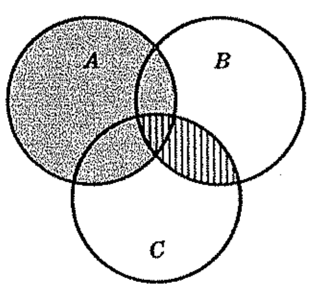


Fig. 2-13

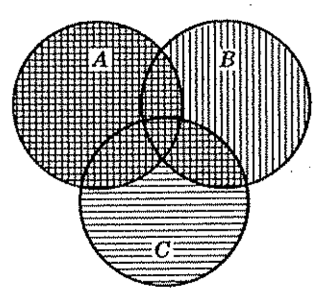


Fig. 2-14

2.4. Prove the generalized distributive law Dist (ii′), page 34:

$$A \cup (B_1 \cap \cdots \cap B_n) = (A \cup B_1) \cap \cdots \cap (A \cup B_n)$$

Solution:

For $n = 1$, the assertion is obvious and the case $n = 2$ is the distributive law Dist (ii). Now using mathematical induction, we assume the result true for $n = k$. Then for $n = k + 1$,

$$\begin{aligned} A \cup (B_1 \cap \cdots \cap B_k \cap B_{k+1}) &= A \cup ((B_1 \cap \cdots \cap B_k) \cap B_{k+1}) \\ &= [A \cup (B_1 \cap \cdots \cap B_k)] \cap [A \cup B_{k+1}] && \text{(by Dist(ii))} \\ &= [(A \cup B_1) \cap \cdots \cap (A \cup B_k)] \cap (A \cup B_{k+1}) && \text{(by the inductive hypothesis)} \\ &= (A \cup B_1) \cap \cdots \cap (A \cup B_k) \cap (A \cup B_{k+1}) \end{aligned}$$

DIFFERENCE AND SYMMETRIC DIFFERENCE

2.5. Using a Venn diagram, determine whether the following conditions are compatible.

(i) $A \cap B = \emptyset$ (iii) $(C \cap A) \sim B = \emptyset$

(ii) $(C \cap B) \sim A = \emptyset$ (iv) $(C \cap A) \cup (C \cap B) \cup (A \cap B) \neq \emptyset$

Solution:

In Fig. 2-15, (iv) says that $E \cup F \cup G \cup H$ is non-empty. (i) says that $E \cup F$ is empty. Hence $G \cup H$ is non-empty. (iii) says that G is empty and (ii) says that H is empty. Hence $G \cup H$ is empty. Therefore conditions (i)-(iv) are inconsistent.

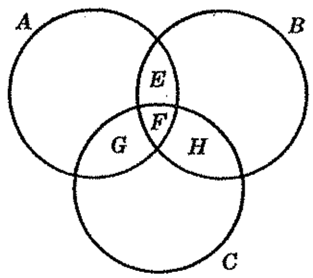


Fig. 2-15

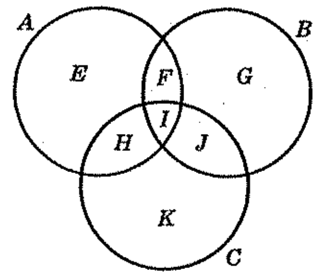


Fig. 2-16

2.6. Show that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

Solution:

In Fig. 2-16, $A \triangle B = E \cup H \cup G \cup J$ and $C = H \cup I \cup J \cup K$, and so $(A \triangle B) \triangle C = E \cup G \cup I \cup K$. $B \triangle C = F \cup G \cup H \cup K$ and $A = E \cup F \cup H \cup I$, and so $A \triangle (B \triangle C) = G \cup K \cup I \cup E$. Thus $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Observe that

$$(A \triangle B) \triangle C = \overbrace{(A \cap \bar{B} \cap \bar{C})}^{E} \cup \overbrace{(\bar{A} \cap B \cap \bar{C})}^{G} \cup \overbrace{(\bar{A} \cap \bar{B} \cap C)}^{K} \cup \overbrace{(A \cap B \cap C)}^{I}$$

A logical derivation of this result is rather tedious and is left to the reader. It is easiest to prove by showing $(A \triangle B) \triangle C \subseteq A \triangle (B \triangle C)$ and $A \triangle (B \triangle C) \subseteq (A \triangle B) \triangle C$.

2.7. Show that $A \triangle B = \emptyset$ if and only if $A = B$.

Solution:

$A \triangle B = \emptyset$ if and only if $(A \sim B) \cup (B \sim A) = \emptyset$,

if and only if $A \sim B = \emptyset$ and $B \sim A = \emptyset$,

if and only if $A \subseteq B$ and $B \subseteq A$,

if and only if $A = B$.

Note: By C(xi), page 35, this result can be restated as

$$A = B \quad \text{if and only if} \quad (A \cap \bar{B}) \cup (\bar{A} \cap B) = \emptyset$$

2.8. Prove the cancellation law: If $A \triangle B = A \triangle C$, then $B = C$.

Solution:

Assume $A \triangle B = A \triangle C$. Then $A \triangle A \triangle B = A \triangle A \triangle C$ (parentheses can be omitted by virtue of Problem 2.6). Since $A \triangle A = \emptyset$, we obtain: $\emptyset \triangle B = \emptyset \triangle C$. But $\emptyset \triangle D = D$ for any D. Hence $B = C$.

2.9. Prove the distributive law: $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

Solution:

$$\begin{aligned} A \cap (B \Delta C) &= A \cap ((B \sim C) \cup (C \sim B)) \\ &= (A \cap (B \sim C)) \cup (A \cap (C \sim B)) \;=\; ((A \cap B) \sim C) \cup ((A \cap C) \sim B) \\ &= ((A \cap B) \sim (A \cap C)) \cup ((A \cap C) \sim (A \cap B)) \;=\; (A \cap B) \Delta (A \cap C) \end{aligned}$$

The problem can also be handled by means of a Venn diagram, as in Problem 2.6.

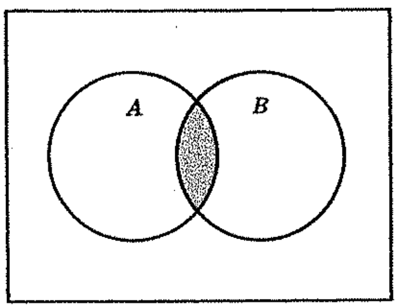
2.10. Prove C(iii): $\overline{A \cap B} = \bar{A} \cup \bar{B}$, logically and pictorially.

Solution:

Logical Proof.

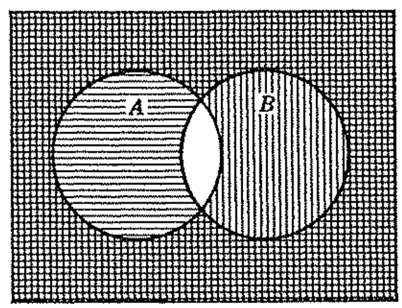
$$\begin{aligned} \overline{A \cap B} &= \{x : x \in X \;\&\; x \notin (A \cap B)\} \\ &= \{x : x \in X \;\&\; (x \notin A \vee x \notin B)\} \\ &= \{x : (x \in X \;\&\; x \notin A) \vee (x \in X \;\&\; x \notin B)\} \\ &= \{x : x \in X \;\&\; x \notin A\} \cup \{x : x \in X \;\&\; x \notin B\} \;=\; \bar{A} \cup \bar{B} \end{aligned}$$

Pictorial Proof. See Fig. 2-17 and 2-18.



Unshaded region: $\overline{A \cap B}$

Fig. 2-17



$\bar{A}$: vertical, $\bar{B}$: horizontal

Marked region: $\bar{A} \cup \bar{B}$

Fig. 2-18

2.11. Prove C(viii): $A \subseteq B$ if and only if $\bar{B} \subseteq \bar{A}$.

Solution:

Recall that A and B are subsets of some fixed universe X. Then

$A \subseteq B$ if and only if, for any x in X, if $x \in A$, then $x \in B$,
if and only if, for any x in X, if $x \notin B$, then $x \notin A$,†
if and only if, for any x in X, if $x \in \bar{B}$, then $x \in \bar{A}$,
if and only if $\bar{B} \subseteq \bar{A}$.

2.12. Using mathematical induction prove the generalized De Morgan Law C(iii′):

$$\overline{A_1 \cap \cdots \cap A_n} = \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_n$$

Solution:

It is obvious for $n = 1$. The case $n = 2$ is simply C(iii). Assume the result true for $n = k$. Then for $n = k+1$,

$$\begin{aligned} \overline{A_1 \cap \cdots \cap A_k \cap A_{k+1}} &= \overline{(A_1 \cap \cdots \cap A_k) \cap A_{k+1}} \\ &= \overline{A_1 \cap \cdots \cap A_k} \cup \bar{A}_{k+1} && \text{(by C(iii))} \\ &= (\bar{A}_1 \cup \cdots \cup \bar{A}_k) \cup \bar{A}_{k+1} && \text{(by inductive hypothesis)} \\ &= \bar{A}_1 \cup \cdots \cup \bar{A}_k \cup \bar{A}_{k+1} \end{aligned}$$

†We have used here the logical law of contraposition: $P \to Q$ is logically equivalent to $\neg Q \to \neg P$.

2.13. Prove: $A \cup B = \overline{\bar{A} \cap \bar{B}}$ and $A \cap B = \overline{\bar{A} \cup \bar{B}}$.

Solution:

By De Morgan's law C(ii), $\overline{A \cup B} = \bar{A} \cap \bar{B}$. Hence $\overline{\overline{A \cup B}} = \overline{\bar{A} \cap \bar{B}}$. But $\overline{\overline{A \cup B}} = A \cup B$, by C(i). Likewise, by De Morgan's law C(iii), $\overline{A \cap B} = \bar{A} \cup \bar{B}$. Hence $A \cap B = \overline{\overline{A \cap B}} = \overline{\bar{A} \cup \bar{B}}$.

2.14. Prove: (1) $(A \cup B) \cap \bar{A} = B \cap \bar{A}$ (3) $(A \cap B) \cup B = B$

(2) $(A \cap B) \cup \bar{A} = B \cup \bar{A}$ (4) $(A \cup B) \cap B = B$

Solution:

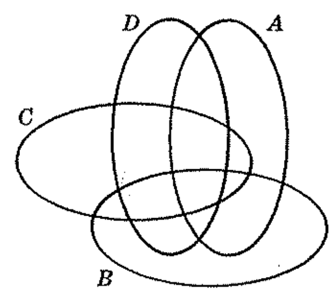
(1) $(A \cup B) \cap \bar{A} = (A \cap \bar{A}) \cup (B \cap \bar{A}) = \emptyset \cup (B \cap \bar{A}) = B \cap \bar{A}$.

(2) $(A \cap B) \cup \bar{A} = (A \cup \bar{A}) \cap (B \cup \bar{A}) = X \cap (B \cup \bar{A}) = B \cup \bar{A}$.

(3) $(A \cap B) \cup B \subseteq B \cup B = B$. Also, $B \subseteq (A \cap B) \cup B$. Hence $(A \cap B) \cup B = B$.

(4) $(A \cup B) \cap B = (A \cap B) \cup (B \cap B) = (A \cap B) \cup B = B$ (by (3)).

2.15. (a) Show that the four ellipses in the diagram below form an appropriate Venn diagram for four sets.



(b) Using the diagram of part (a), what conclusion can you draw from the following assumptions?

(i) $C \subseteq (B \cap \bar{D}) \cup (D \cap \bar{B})$.

(ii) Everything in both A and C is either in both B and D or in neither B nor D.

(iii) Everything in both B and C is either A or D.

(iv) Everything in both C and D is either in A or B.

Solution:

(a) Show that the fifteen regions of the diagram cover all possible cases:

$$A \cap B \cap C \cap \bar{D},\ A \cap B \cap \bar{C} \cap D,\ A \cap \bar{B} \cap C \cap D,\ \bar{A} \cap B \cap C \cap D,\ A \cap B \cap C \cap D,$$
$$A \cap B \cap \bar{C} \cap \bar{D},\ A \cap \bar{B} \cap C \cap \bar{D},\ \bar{A} \cap B \cap C \cap \bar{D},\ A \cap \bar{B} \cap \bar{C} \cap D,\ \bar{A} \cap B \cap \bar{C} \cap D,$$
$$\bar{A} \cap \bar{B} \cap C \cap D,\ A \cap \bar{B} \cap \bar{C} \cap \bar{D},\ \bar{A} \cap B \cap \bar{C} \cap \bar{D},\ \bar{A} \cap \bar{B} \cap C \cap \bar{D},\ \bar{A} \cap \bar{B} \cap \bar{C} \cap D.$$

(b) $C = \emptyset$.

2.16. Algebra of Sets and Algebra of Logic. Given a statement form **C** in $\neg$, &, $\vee$, let $S(\mathbf{C})$ be the expression obtained from **C** by substituting $^{-}$, $\cap$, $\cup$ for $\neg$, &, $\vee$ respectively. Example:

$$S((A \vee B) \,\&\, \neg C) = (A \cup B) \cap \bar{C}$$

(a) Prove: **A** is logically equivalent to **B** if and only if $S(\mathbf{A}) = S(\mathbf{B})$ holds for all sets (where the statement letters of a statement form **C** are interpreted in $S(\mathbf{C})$ as set variables ranging over all subsets of a fixed universe).

(*b*) Prove that **A** logically implies **B** if and only if $S(\mathbf{A}) \subseteq S(\mathbf{B})$ holds for all sets.

Solution:

(*a*) If we replace each statement letter $\mathbf{L}$ in **A** by $x \in \mathbf{L}$, then the resulting sentence is equivalent to $x \in S(\mathbf{A})$ (since $x \in W_1 \cap W_2$ if and only if $x \in W_1$ & $x \in W_2$; $x \in W_1 \cup W_2$ if and only if $x \in W_1 \vee x \in W_2$; and $x \in \bar{W}$ if and only if $\neg(x \in W)$). Hence if **A** is logically equivalent to **B**, then $x \in S(\mathbf{A})$ if and only if $x \in S(\mathbf{B})$, which implies that $S(\mathbf{A}) = S(\mathbf{B})$. Conversely, assume that **A** is not logically equivalent to **B**. In general if we are given a truth assignment to the statement letters in an arbitrary statement form **C**, and if we replace statement letters which are T by $\{\emptyset\}$ and statement letters which are F by $\emptyset$, then, under this substitution of sets for statement letters, $S(\mathbf{C}) = \{\emptyset\}$ if **C** is T under the given assignment, and $S(\mathbf{C}) = \emptyset$ if **C** is F under the given assignment. This holds because, under the correspondence associating $\{\emptyset\}$ with T and $\emptyset$ with F, the truth-functional operations correspond to the set-theoretic operations (where sets are restricted to subsets of the universe $\{\emptyset\}$).

$$\begin{array}{rcl} \neg T = F & \qquad & \overline{\{\emptyset\}} = \emptyset \\ \neg F = T & & \bar{\emptyset} = \{\emptyset\} \\ T \,\&\, T = T & & \{\emptyset\} \cap \{\emptyset\} = \{\emptyset\} \\ T \,\&\, F = F \,\&\, T = F & & \{\emptyset\} \cap \emptyset = \emptyset \cap \{\emptyset\} = \emptyset \\ F \,\&\, F = F & & \emptyset \cap \emptyset = \emptyset \\ T \vee T = T \vee F = F \vee T = T & & \{\emptyset\} \cup \{\emptyset\} = \{\emptyset\} \cup \emptyset = \emptyset \cup \{\emptyset\} = \{\emptyset\} \\ F \vee F = F & & \emptyset \cup \emptyset = \emptyset \end{array}$$

Since **A** is not logically equivalent to **B**, there is a truth assignment making one of them T and the other F, say **A** is T and **B** is F. Then under the substitution of $\{\emptyset\}$ for the true statement letters and of $\emptyset$ for the false statement letters, $S(\mathbf{A}) = \{\emptyset\}$ and $S(\mathbf{B}) = \emptyset$. Hence $S(\mathbf{A}) = S(\mathbf{B})$ does not always hold.

Remark: Lurking behind this rather long-winded discussion is what in mathematics is called an "isomorphism" between the structures

$$\langle\{T, F\}, \neg, \&, \vee\rangle \qquad \text{and} \qquad \langle\{\{\emptyset\}, \emptyset\}, {}^{-}, \cap, \cup\rangle$$

Note that we also have shown that an equation $S(\mathbf{A}) = S(\mathbf{B})$ holds for all sets if and only if it holds in the domain $\{\{\emptyset\}, \emptyset\}$ of all subsets of $\{\emptyset\}$.

(*b*) **A** logically implies **B** if and only if **A** & **B** is logically equivalent to **A**. By part (*a*), the latter holds if and only if $S(\mathbf{A} \,\&\, \mathbf{B}) = S(\mathbf{A})$ always holds. But $S(\mathbf{A} \,\&\, \mathbf{B}) = S(\mathbf{A}) \cap S(\mathbf{B})$, and $S(\mathbf{A}) \cap S(\mathbf{B}) = S(\mathbf{A})$ if and only if $S(\mathbf{A}) \subseteq S(\mathbf{B})$.

2.17. Define ordered n-tuples (for $n \geqq 3$) by induction:

$$\langle x_1, x_2, \ldots, x_n\rangle = \langle\langle x_1, x_2, \ldots, x_{n-1}\rangle, x_n\rangle$$

Thus $\langle x_1, x_2, x_3\rangle = \langle\langle x_1, x_2\rangle, x_3\rangle$ and $\langle x_1, x_2, x_3, x_4\rangle = \langle\langle\langle x_1, x_2\rangle, x_3\rangle, x_4\rangle$. Prove that if $\langle x_1, x_2, \ldots, x_n\rangle = \langle u_1, u_2, \ldots, u_n\rangle$, then $x_1 = u_1,\ x_2 = u_2,\ \ldots,\ x_n = u_n$.

Solution:

We already have proved this result for $n = 2$. Now assume it is true for $n = k \geqq 2$, and we shall prove it must then hold for $k+1$. We have, by assumption,

$$\langle x_1, x_2, \ldots, x_k, x_{k+1}\rangle = \langle u_1, u_2, \ldots, u_k, u_{k+1}\rangle$$

Hence by definition,

$$\langle\langle x_1, x_2, \ldots, x_k\rangle, x_{k+1}\rangle = \langle\langle u_1, u_2, \ldots, u_k\rangle, u_{k+1}\rangle$$

By the result for $n = 2$, we conclude $x_{k+1} = u_{k+1}$ and $\langle x_1, x_2, \ldots, x_k\rangle = \langle u_1, u_2, \ldots, u_k\rangle$. But the latter equation, by virtue of the inductive hypothesis, implies $x_1 = u_1,\ x_2 = u_2,\ \ldots,\ x_k = u_k$.

FINITE, INFINITE, DENUMERABLE, AND COUNTABLE SETS

2.18. Prove that the set W of all ordered pairs of non-negative integers is denumerable.

Solution:

Arrange W in the following infinite array:

$$\begin{array}{cccccc}
\langle 0,0\rangle & \langle 0,1\rangle & \langle 0,2\rangle & \langle 0,3\rangle & \langle 0,4\rangle & \cdots \\
\langle 1,0\rangle & \langle 1,1\rangle & \langle 1,2\rangle & \langle 1,3\rangle & \cdots & \\
\langle 2,0\rangle & \langle 2,1\rangle & \langle 2,2\rangle & \langle 2,3\rangle & \cdots & \\
\langle 3,0\rangle & \langle 3,1\rangle & \langle 3,2\rangle & \langle 3,3\rangle & \cdots & \\
\langle 4,0\rangle & \langle 4,1\rangle & \cdots & \cdots & & \\
\vdots & \vdots & \vdots & \vdots & &
\end{array}$$

Enumerate the ordered pairs as indicated by the arrows, going up each diagonal from left to right. Notice that the pair $\langle i,j\rangle$ appears in the $[((i+j)(i+j+1)/2)+(j+1)]$th place in the enumeration. This can be seen as follows: all pairs in the same diagonal have the same sum. Adding up all pairs in diagonals preceding the one containing $\langle i,j\rangle$, we obtain

$$1+2+\cdots+(i+j) \;=\; (i+j)(i+j+1)/2$$

There are j pairs in the same diagonal as $\langle i,j\rangle$ and preceding $\langle i,j\rangle$.

2.19. Prove that the set of all non-negative rational numbers is denumerable.

Solution:

Every non-negative rational number corresponds to a fraction m/n, where (i) m and n are non-negative integers, (ii) $n \neq 0$, and (iii) m and n have no common integral factors other than ± 1. We can associate the ordered pair $\langle m,n\rangle$ with m/n, and use the enumeration given in Problem 2.18, merely omitting those pairs $\langle m,n\rangle$ which do not satisfy conditions (i)-(iii).

2.20. The set A of all real roots of all nonzero polynomials with integral coefficients (such roots are called *real algebraic numbers*) is denumerable.

Solution:

Any nonzero polynomial has only a finite number of roots. First list the finite set of all real roots of all polynomials of degree at most one whose coefficients are in magnitude ≤ 1 (i.e. whose coefficients are either $0, 1,$ or -1). Then list the finite set of all real roots of all polynomials of degree ≤ 2 whose coefficients are in magnitude ≤ 2, etc. In general, at the nth step we list the finite set of all real roots of all polynomials of degree $\leq n$ whose coefficients are in magnitude $\leq n$. Of course, we omit repetitions. In this way, we obtain an enumeration of all real algebraic numbers. That the set A is not finite follows from the fact that all integers belong to A.

2.21. Show that the set of all real numbers is not countable.

Solution:

Let R_1 be the set of all real numbers x such that $0 \leq x < 1$. It suffices to show that R_1 is not countable, since any subset of a countable set is countable. Every x in R_1 is representable as a unique infinite decimal

$$x \;=\; .a_1a_2a_3\ldots$$

where the infinite decimal does not end with an infinite string of 9's. (Thus although a decimal such as $.1362000\ldots$ is also representable as $.1361999\ldots$, we shall use only the first representation.) Assume now that R_1 can be enumerated:

$$\begin{aligned} x_1 &= .a_{11}a_{12}a_{13}\ldots \\ x_2 &= .a_{21}a_{22}a_{23}\ldots \\ &\ldots\ldots\ldots\ldots \\ x_k &= .a_{k1}a_{k2}a_{k3}\ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Construct a decimal $y = .b_1b_2b_3\ldots$ as follows:

$$b_i = \begin{cases} 0 & \text{if } a_{ii} \neq 0 \\ 1 & \text{if } a_{ii} = 0 \end{cases}$$

Thus, for all i, $a_{ii} \neq b_i$. But then, y is in R_1 and is different from all of the numbers $x_1, x_2, \ldots$ (since the decimal representation of y differs from that of x_i in the ith place). This contradicts the assumption that the sequence $x_1, x_2, \ldots$ exhausts R_1.

2.22. Given two sets A and B. We say that A *has the same cardinality as* B if there is a one-one correspondence between A and B. We say that A *has smaller cardinality than* B if there is a one-one correspondence between A and a subset of B but A does not have the same cardinality as B.

Prove that, for any set A, A has smaller cardinality than the set $\mathcal{P}(A)$ of all subsets of A (Cantor's Theorem).

Solution:

(1) There is a one-one correspondence between A and a subset of $\mathcal{P}(A)$. Namely, to each element x of A associate the set $\{x\}$ in $\mathcal{P}(A)$. Clearly if x and y are distinct elements of A, $\{x\} \neq \{y\}$.

(2) We must show that there is no one-one correspondence f between A and $\mathcal{P}(A)$. Assume, on the contrary, that there is such a one-one correspondence f. Let $C = \{x : x \in A \;\&\; x \notin f(x)\}$. Thus C consists of all elements x of A such that x is not a member of the corresponding subset $f(x)$ of A. But $C \subseteq A$. Hence $C \in \mathcal{P}(A)$. So there must be an element y in A such that $f(y) = C$. Then by definition of C, $y \in C$ if and only if $y \notin f(y)$. Since $f(y) = C$, it follows that $y \in C$ if and only if $y \notin C$. But either $y \in C$ or $y \notin C$. Hence $y \in C \;\&\; y \notin C$, which is a contradiction.

FIELD OF SETS

2.23. Prove that the collection $\mathcal{F}$ of all subsets B of X such that either B or $\bar{B}$ is countable is a field of sets.

Solution:

Assume $B \in \mathcal{F}$. Then either $\bar{\bar{B}}$ or $\bar{B}$ is countable. Hence $\bar{B} \in \mathcal{F}$. Assume now that A is also in $\mathcal{F}$.

Case 1: B is countable. Then $A \cap B$ is countable. Hence $A \cap B \in \mathcal{F}$.

Case 2: A is countable. Then $A \cap B$ is countable. Hence $A \cap B \in \mathcal{F}$.

Case 3: $\bar{B}$ is countable and $\bar{A}$ is countable. Hence $\bar{A} \cup \bar{B}$ is countable. But $\overline{A \cap B} = \bar{A} \cup \bar{B}$. Therefore $A \cap B \in \mathcal{F}$.

2.24. Let X be the set of all integers, and let k be a fixed integer. Let $\mathcal{G}$ be the collection of all subsets B of X such that, for any u in B, both $u+k$ and $u-k$ are also in B. (This means that a shift of k units does not alter B.) Show that $\mathcal{G}$ is a field of sets.

Solution:

Let $B \in \mathcal{G}$. Assume $u \in \bar{B}$. Hence $u \notin B$. So $u - k \notin B$. (For, if $u - k \in B$, then $u = (u-k) + k \in B$.) Also, $u + k \in \bar{B}$. (For, if $u + k \in B$, then $u = (u+k) - k \in B$.) Thus

$\bar{B} \in \mathcal{G}$. Assume now that A and B are in $\mathcal{G}$. Let us consider $A \cap B$. Assume $u \in A \cap B$. Then $u \in A$ & $u \in B$. Hence $u \pm k \in A$ & $u \pm k \in B$. Therefore $u \pm k \in A \cap B$. Thus, $A \cap B \in \mathcal{G}$.

Additional question: How many elements does $\mathcal{G}$ have?

NUMBER OF ELEMENTS IN A FINITE SET

2.25. Derive the equality

$$\begin{aligned}\#(A \cup B \cup C) &= \#(A) + \#(B) + \#(C) \\ &\quad - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) \\ &\quad + \#(A \cap B \cap C)\end{aligned} \tag{1}$$

for arbitrary finite sets A, B and C.

Solution:

Take any element x in $A \cup B \cup C$. If x is in precisely one of the sets A, B, C, then x is counted once on the right side of (*1*). (For example, if $x \in B \cap \bar{A} \cap \bar{C}$, then x is counted only in $\#(B)$.) If x belongs to precisely two of the sets A, B, C, then x will be counted twice in the positive sense on the right side and once in the negative sense. (For example, if $x \in A \cap \bar{B} \cap C$, then x is counted twice in the positive sense in $\#(A)$ and $\#(C)$, and x is subtracted once in $\#(A \cap C)$.) Lastly, if x belongs to $A \cap B \cap C$, then x is counted in every term on the right side, four times in the positive sense and three times in the negative sense. Thus the net effect of the right side of (*1*) is to count the number of elements in $A \cup B \cup C$.

2.26. If a boating club of 75 members admitted only owners of sailboats or powerboats, and if 48 members owned sailboats and 33 members owned powerboats, how many members owned both sailboats and powerboats?

Solution:

Let A = the set of all members owning sailboats, and B = the set of all members owning powerboats.

$$\#(A \cup B) = \#(A) + \#(B) - \#(A \cap B)$$

$$75 = 48 + 33 - \#(A \cap B)$$

Hence $\#(A \cap B) = 6$.

2.27. Among 50 students taking examinations in mathematics, physics and chemistry, 37 passed mathematics, 24 physics and 43 chemistry; at most 19 passed mathematics and physics, at most 29 mathematics and chemistry, and at most 20 physics and chemistry. What is the largest possible number that could have passed all three?

Solution:

Let M, P, C stand for the collections of students passing mathematics, physics and chemistry, respectively.

$$\#(M \cup P \cup C) = \#(M) + \#(P) + \#(C) - \#(M \cap P) - \#(M \cap C) - \#(P \cap C) + \#(M \cap P \cap C)$$

$$50 \geq 37 + 24 + 43 - \#(M \cap P) - \#(M \cap C) - \#(P \cap C) + \#(M \cap P \cap C)$$

Hence

$$\begin{aligned}\#(M \cap P \cap C) &\leq \#(M \cap P) + \#(M \cap C) + \#(P \cap C) - 54 \\ &\leq 19 + 29 + 20 - 54 = 14\end{aligned}$$