Appendix: Proofs of Theorems and Metarules

NOTE: The items below numbered as (n), ($n.m$), or ($n.m.a$) refer to numbered items in Part II. So, for example, references to item (9.1) do *not* refer to Chapter 9.1, but rather to item (9.1), which occurs in Part II, Chapter 7.

(46.1) (Exercise)

(46.2) (Exercise)

(47.1) We establish the rule only for $\vdash$. If φ is an element of $\mathbf{\Lambda}$, then the one element sequence φ is a proof of φ, by (43.2). ⋈

(47.2) We establish the rule only for $\vdash$. If φ is an element of Γ, then the one element sequence φ is a derivation of φ from Γ, by (43.1). ⋈

(47.3) We establish the rule only for $\vdash$. If $\vdash \varphi$, then by definition (43.2), there is a sequence of formulas every element of which is either a member of $\mathbf{\Lambda}$ or a direct consequence of some of the preceding members of the sequence by virtue of MP. Since $\mathbf{\Lambda} \subseteq \mathbf{\Lambda} \cup \Gamma$, there is a sequence of formulas every element of which is either a member of $\mathbf{\Lambda} \cup \Gamma$ or a direct consequence from some of the preceding members of the sequence by virtue of MP. Hence, by definition (43.1), $\Gamma \vdash \varphi$.

(47.4) (Exercise)

(47.5) We establish the rule only for $\vdash$. Assume $\Gamma \vdash \varphi$ and $\Gamma \vdash (\varphi \to \psi)$. Then there is a sequence $\chi_1, \ldots, \chi_{n-1}, \varphi$ (= S_1) that is a derivation of φ from Γ and there is a sequence $\theta_1, \ldots, \theta_{m-1}, \varphi \to \psi$ (= S_2) that is a derivation of $\varphi \to \psi$ from Γ. So the consider the sequence:

$$\chi_1, \ldots, \chi_{n-1}, \varphi, \theta_1, \ldots, \theta_{m-1}, \varphi \to \psi, \psi \qquad (S_3)$$

Since every element of S_1 and S_2 is either an element of $\mathbf{\Lambda} \cup \Gamma$ or follows from preceding members by MP, the same holds for every member of the initial segment of S_3 up to and including $\varphi \to \psi$. Since the last member of S_3 follows from previous members by MP, we know that every element of S_3 is either an element of $\mathbf{\Lambda} \cup \Gamma$ or follows from preceding members by MP. Hence, S_3 is a derivation of ψ from Γ. ⋈

(47.6) (Exercise)

(47.7) We establish the rule only for $\vdash$. Assume $\Gamma \vdash \varphi$ and that $\Gamma \subseteq \Delta$. Then by the former, there is a sequence S ending in φ such that every member of the sequence is either in $\mathbf{\Lambda} \cup \Gamma$ or follows from previous members by MP. But since $\Gamma \subseteq \Delta$, it follows that every member of S is either in $\mathbf{\Lambda} \cup \Delta$ or follows from previous members by MP, i.e., $\Delta \vdash \varphi$. ⋈

(47.8) We establish the rule only for $\vdash$. Assume $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$. By the former, there is a sequence $S_1 = \chi_1, \ldots, \chi_{n-1}, \varphi$ such that every member of the sequence is either in $\mathbf{\Lambda} \cup \Gamma$ or follows from previous members by MP. By the latter, there is a sequence $S_2 = \theta_1, \ldots, \theta_{m-1}, \psi$ such that every member of the sequence is either in $\mathbf{\Lambda} \cup \{\varphi\}$ or follows from previous members by MP. So consider, then, the following sequence:

$$\chi_1, \ldots, \chi_{n-1}, \theta_1, \ldots, \theta_{m-1}, \psi \tag{S_3}$$

This sequence is the concatenation of the first $n-1$ members of S_1 with the entire sequence S_2. Since S_1 and S_2 are derivations, we know that all the χ_i $(1 \leq i \leq n-1)$ and θ_j $(1 \leq j \leq m-1)$ are either elements of $\mathbf{\Lambda} \cup \Gamma$ or follow from two of the preceding members of the sequence by MP. The only potential exceptions are possible occurrences of φ among the θ_js. But note that φ follows by MP from two members of $\chi_1, \ldots, \chi_{n-1}$. Thus S_3 is a derivation of ψ from Γ and, hence, $\Gamma \vdash \psi$. ⋈

(47.9) We establish the rule only for $\vdash$. Suppose $\Gamma \vdash \varphi$. Since the instances of (21.1) are axioms, we know by (47.1) that $\vdash \varphi \to (\psi \to \varphi)$, where ψ is any formula. So by (47.3), we have $\Gamma \vdash \varphi \to (\psi \to \varphi)$. From our initial hypothesis and this last result, it follows by an instance of (47.5) (i.e., an instance in which we set ψ in (47.5) to $\psi \to \varphi$) that $\Gamma \vdash (\psi \to \varphi)$, where ψ is any formula. ⋈

(47.10) We establish the rule only for $\vdash$. Suppose $\Gamma \vdash (\varphi \to \psi)$. Since $\Gamma \subseteq \Gamma \cup \{\varphi\}$, it follows from (47.7) that:

(ϑ) $\Gamma \cup \{\varphi\} \vdash \varphi \to \psi$

But since $\varphi \in \Gamma \cup \{\varphi\}$, it follows by (47.2) that:

(ξ) $\Gamma \cup \{\varphi\} \vdash \varphi$

So from (ϑ) and (ξ), it follows by (47.5) that $\Gamma \cup \{\varphi\} \vdash \psi$, i.e., $\Gamma, \varphi \vdash \psi$. ⋈

(50) Suppose (a) $\Gamma \vdash \varphi$, and (b) α doesn't occur free in any formula in Γ. We show by induction on the length of the derivation of φ from Γ that $\Gamma \vdash \forall\alpha\varphi$.

Base case. The derivation of φ from Γ is a one-element sequence, in which case the sequence must be φ itself since a derivation of φ from Γ must end with φ. Then by the definition of *derivation from*, (43.1), $\varphi \in \mathbf{\Lambda} \cup \Gamma$. So we have two cases: (A) φ is an element of $\mathbf{\Lambda}$, i.e., φ is one of the axioms asserted in Chapter 8, or (B) φ is an element of Γ.

Case A. $\varphi \in \mathbf{\Lambda}$. Then $\forall\alpha\varphi \in \mathbf{\Lambda}$, since the generalizations of all axioms are axioms.[257] So, $\forall\alpha\varphi \in \mathbf{\Lambda} \cup \Gamma$, and so by (47.4), it follows that $\Gamma \vdash \forall\alpha\varphi$.

Case B. $\varphi \in \Gamma$. Then, by hypothesis, α doesn't occur free in φ. Consequently, $\varphi \to \forall\alpha\varphi$ is an instance of axiom (29.4) meeting the condition that α doesn't occur free in φ. So by (43.1), the sequence $\varphi, \varphi \to \forall\alpha\varphi, \forall\alpha\varphi$ is a witness to $\Gamma \vdash \forall\alpha\varphi$, since every member of the sequence is either a member of $\mathbf{\Lambda} \cup \Gamma$ or is a direct consequence of two previous members by MP.

Inductive Case. Suppose that the derivation of φ from Γ is a sequence of length n, where $n > 1$. Then either $\varphi \in \mathbf{\Lambda} \cup \Gamma$ or φ follows from two previous members of the sequence, namely, $\psi \to \varphi$ and ψ, by MP. If $\varphi \in \mathbf{\Lambda} \cup \Gamma$, then using the same reasoning as in the base case, $\Gamma \vdash \forall\alpha\varphi$. If φ follows from previous members $\psi \to \varphi$ and ψ by MP, then by the definition of derivation, we know that $\Gamma \vdash \psi \to \varphi$ and $\Gamma \vdash \psi$, where these are derivations of length less than n. Since our IH is that the theorem holds for all derivations of formulas from Γ of length less than n, it follows that $\Gamma \vdash \forall\alpha(\psi \to \varphi)$ and $\Gamma \vdash \forall\alpha\psi$. So there is a sequence $S_1 = \chi_1, \ldots, \chi_i$, where $\chi_i = \forall\alpha(\psi \to \varphi)$, that is a witness to the former and a sequence $S_2 = \theta_1, \ldots, \theta_j$, where $\theta_j = \forall\alpha\psi$, that is a witness to the latter. Now by using an instance of axiom (29.3), we may construct the following sequence:

$$\chi_1, \ldots, \chi_i, \theta_1, \ldots, \theta_j, \forall\alpha(\psi \to \varphi) \to (\forall\alpha\psi \to \forall\alpha\varphi), \forall\alpha\psi \to \forall\alpha\varphi, \forall\alpha\varphi \qquad (S_3)$$

The antepenultimate member of S_3 is an instance of axiom (29.3), and so an element of $\mathbf{\Lambda}$ and hence of $\mathbf{\Lambda} \cup \Gamma$. The penultimate member of S_3 follows from previous members (namely, the antepenultimate member and χ_i) by MP, and the last member of S_3 follows from previous members (namely, the penultimate member and θ_j) by MP. Hence, every element of S_3 is either in $\mathbf{\Lambda} \cup \Gamma$ or follows from previous members by MP. So $\Gamma \vdash \forall\alpha\varphi$. ⋈

(52) Suppose $\Gamma \vdash_\Box \varphi$, i.e., that there is a modally-strict derivation of φ from Γ. We show by induction on the length of the derivation that $\Box\Gamma \vdash \Box\varphi$, i.e., that there is a derivation of $\Box\varphi$ from $\Box\Gamma$.

Base Case. If $n = 1$, then the modally-strict derivation of φ from Γ consists of a single formula, namely, φ itself. So by the definition of $\Gamma \vdash_\Box \varphi$, φ must

[257] We did not, however, assert the *necessitations* of all the axiomatic formulas asserted as axioms; e.g., we did not assert the necessitations of axiom (30)★. But we did assert their $\Box$-free closures. So, indeed, if $\varphi \in \mathbf{\Lambda}$, then $\forall\alpha\varphi \in \mathbf{\Lambda}$.

be in $\mathbf{\Lambda}_\Box \cup \Gamma$. So either (a) φ is in $\mathbf{\Lambda}_\Box$ or (b) φ is in Γ. If (a), then φ must be a necessary axiom and so its necessitation $\Box\varphi$ is an axiom. So $\vdash \Box\varphi$ by (47.1) and $\Box\Gamma \vdash \Box\varphi$ by (47.3).[258] If (b), then $\Box\varphi$ is in $\Box\Gamma$, by the definition of $\Box\Gamma$. Hence by (47.2), it follows that $\Box\Gamma \vdash_\Box \Box\varphi$. But then by (46.1), it follows that $\Box\Gamma \vdash \Box\varphi$.

Inductive Case. Suppose that the modally-strict derivation of φ from Γ is a sequence S of length n, where $n > 1$. Then either $\varphi \in \mathbf{\Lambda}_\Box \cup \Gamma$ or φ follows by MP from two previous members of the sequence, namely, $\psi \to \varphi$ and ψ. If $\varphi \in \mathbf{\Lambda}_\Box \cup \Gamma$, then using the reasoning in the base case, it follows that $\Box\Gamma \vdash \Box\varphi$. If φ follows from previous members $\psi \to \varphi$ and ψ by MP, then by the definition of a modally-strict derivation, we know both that $\Gamma \vdash_\Box \psi \to \varphi$ and $\Gamma \vdash_\Box \psi$. Consequently, since our IH is that the theorem holds for all such derivations of length less than n, it implies:

(a) $\Box\Gamma \vdash \Box(\psi \to \varphi)$

(b) $\Box\Gamma \vdash \Box\psi$

Now since instances of the K schema (32.1) are axioms, we know:

$$\vdash \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$$

So by (47.3), it follows that:

$$\Box\Gamma \vdash \Box(\psi \to \varphi) \to (\Box\psi \to \Box\varphi)$$

So by (47.5), it follows from this and (a) that:

$$\Box\Gamma \vdash \Box\psi \to \Box\varphi$$

And again by (47.5), it follows from this and (b) that:

$$\Box\Gamma \vdash \Box\varphi$$ ⋈

(53) Axiom (21.2) asserts:

$$\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

If we let φ in the above be φ, let ψ in the above be $(\varphi \to \varphi)$, and let χ in the above be φ, then we obtain the following instance of (21.2):

$$(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$$

But the following is an instance of (21.1):

$$\varphi \to ((\varphi \to \varphi) \to \varphi)$$

[258] Note that (47.3) says that if $\vdash \varphi$, then $\Gamma \vdash \varphi$, for *any* Γ. So, in this case, we've substituted $\Box\Gamma$ for Γ in (47.3). The clash of variables is not an egregious one.

Since this latter is the antecedent of the former, we may apply MP to obtain:

$$(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$$

But now the following is also an instance of (21.1):

$$\varphi \to (\varphi \to \varphi)$$

By applying MP to our last two results we obtain:

$$\varphi \to \varphi$$ ⋈

(54) Suppose $\Gamma, \varphi \vdash \psi$. We show by induction on the length of a derivation of ψ from $\Gamma \cup \{\varphi\}$ that $\Gamma \vdash (\varphi \to \psi)$.

Base case. The derivation of ψ from $\Gamma \cup \{\varphi\}$ is a one-element sequence, namely, ψ itself. Then by the definition of *derivation from*, (43.1), $\psi \in \mathbf{\Lambda} \cup \Gamma \cup \{\varphi\}$. So we have two cases: (*A*) ψ is an element of $\mathbf{\Lambda} \cup \Gamma$, i.e., ψ is one of the axioms asserted in Chapter 8 or an element of Γ, or (*B*) $\psi = \varphi$.

Case A. $\psi \in \mathbf{\Lambda} \cup \Gamma$. Then by (43.1), $\Gamma \vdash \psi$. Since the instances of (21.1) are axioms governing conditionals, we know $\vdash (\psi \to (\varphi \to \psi))$, by (47.1). So, by (47.3), it follows that $\Gamma \vdash (\psi \to (\varphi \to \psi))$. Hence by (47.5), it follows that $\Gamma \vdash (\varphi \to \psi)$.

Case B. $\psi = \varphi$. Then by (53), we know $\vdash (\psi \to \psi)$. So, $\vdash (\varphi \to \psi)$, and hence, by (47.3), it follows that $\Gamma \vdash (\varphi \to \psi)$.

Inductive Case. The derivation of ψ from $\Gamma \cup \{\varphi\}$ is a sequence of length n, where $n > 1$. Then either $\psi \in \mathbf{\Lambda} \cup \Gamma \cup \{\varphi\}$ or ψ follows from two previous members of the sequence, namely, $\chi \to \psi$ and χ, by MP. If $\psi \in \mathbf{\Lambda} \cup \Gamma \cup \{\varphi\}$, then using the same reasoning as in the base case, $\Gamma \vdash (\varphi \to \psi)$. If ψ follows from previous members $\chi \to \psi$ and χ by MP, then since our IH is that the theorem holds for all derivations of formulas from Γ of length less than n, it implies both:

(a) $\Gamma \vdash (\varphi \to \chi)$

(b) $\Gamma \vdash (\varphi \to (\chi \to \psi))$

Now since the instances of (21.2) are axioms governing conditionals, we know, by (47.1):

$$\vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$$

So, by (47.3), it follows that:

$$\Gamma \vdash (\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$$

From this and (b), it follows by (47.5) that:

$$\Gamma \vdash (\varphi \to \chi) \to (\varphi \to \psi)$$

And from this last conclusion and (a), it follows that:

$$\Gamma \vdash (\varphi \to \psi) \qquad \bowtie$$

(55.1) Assume:

(a) $\Gamma_1 \vdash \varphi \to \psi$

(b) $\Gamma_2 \vdash \psi \to \chi$

So, by definition (43.1), there is a sequence, say S_1, that is a witness to (a) and a sequence, say S_2, that is a witness to (b). Then consider the sequence S_3 consisting of the members of S_1, followed by the members of S_2, followed by $\varphi \to \psi$, $\psi \to \chi$, φ, ψ, and ending in χ. It is not hard to show that this is a witness to:

(ϑ) $\Gamma_1, \Gamma_2, \varphi \to \psi, \psi \to \chi, \varphi \vdash \chi$

since every element of S_3 either: (a) is an element of Γ_1, or (b) is an element of Γ_2, or (c) is just the formula $\varphi \to \psi$, $\psi \to \chi$, or φ, or (d) follows from previous members of the sequence by MP. By an application of the Deduction Theorem to (ϑ), it follows that:

$$\Gamma_1, \Gamma_2, \varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$$

By an application of the Deduction Theorem to the above, and another application to the result, we obtain:

(ξ) $\Gamma_1, \Gamma_2, \vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

But from (a) and (b), respectively, it follows by (47.7) that:

(c) $\Gamma_1, \Gamma_2 \vdash \varphi \to \psi$

(d) $\Gamma_1, \Gamma_2 \vdash \psi \to \chi$

So from (ξ) and (c) it follows by (47.5) that:

(ζ) $\Gamma_1, \Gamma_2, \vdash (\psi \to \chi) \to (\varphi \to \chi)$

And from (ζ) and (d) it again follows by (47.5) that $\Gamma_1, \Gamma_2 \vdash \varphi \to \chi$. $\bowtie$

(55.2) (Exercise)

(55.3) Consider the premise set $\Gamma = \{\varphi \to \psi,\ \psi \to \chi,\ \varphi\}$. From the first and third members of Γ, we obtain ψ by MP. From ψ and the second member of Γ, we obtain χ by MP. Hence, the sequence consisting of the members of Γ

followed by ψ and χ constitute a witness to $\varphi \to \psi$, $\psi \to \chi$, $\varphi \vdash \chi$. So by the Deduction Theorem (54), $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$. ⋈

(55.4) Consider the premise set $\Gamma = \{\varphi \to (\psi \to \chi), \psi, \varphi\}$. Then from the first and third members of Γ, we obtain $\psi \to \chi$ by MP, and from this and the second member of Γ we obtain χ by MP. Hence the sequence consisting of the members of Γ followed by $\psi \to \chi$ and χ constitute a witness to $\varphi \to (\psi \to \chi), \psi, \varphi \vdash \chi$. So by the Deduction Theorem (54), it follows that $\varphi \to (\psi \to \chi), \psi \vdash \varphi \to \chi$. ⋈

(58.1) As an instance of (21.3), we have: $(\neg\varphi \to \neg\neg\varphi) \to ((\neg\varphi \to \neg\varphi) \to \varphi)$. Moreover, by (53), we have $\neg\varphi \to \neg\varphi$. Then by (56.2), it follows that $(\neg\varphi \to \neg\neg\varphi) \to \varphi$. But $(\neg\neg\varphi \to (\neg\varphi \to \neg\neg\varphi))$ is an instance of (21.1). So it follows that $\neg\neg\varphi \to \varphi$, by (55.1). ⋈

(58.2) As an instance of (21.3), we have: $(\neg\neg\neg\varphi \to \neg\varphi) \to ((\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi)$. Moreover, as an instance of (58.1), we know: $\neg\neg\neg\varphi \to \neg\varphi$. So by MP, it follows that $(\neg\neg\neg\varphi \to \varphi) \to \neg\neg\varphi$. But as an instance of (21.1), we know: $\varphi \to (\neg\neg\neg\varphi \to \varphi)$. So by (55.1), it follows that $\varphi \to \neg\neg\varphi$. ⋈

(58.3) Assume $\neg\varphi$ for conditional proof. Now assume φ for a conditional proof nested within our conditional proof. Then since $\varphi \to (\neg\psi \to \varphi)$ is an instance of axiom (21.1), it follows from this and our second assumption that:

(a) $\neg\psi \to \varphi$

Moroever, since $\neg\varphi \to (\neg\psi \to \neg\varphi)$ is an instance of axiom (21.1), it follows from this and our first assumption that:

(b) $\neg\psi \to \neg\varphi$

But as an instance of axiom (21.3), we know:

(c) $(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$

From (b) and (c), it follows that $(\neg\psi \to \varphi) \to \psi$. And from this and (a), it follows that ψ. So, discharging the premise of our nested conditional proof, it follows that $\varphi \to \psi$. Hence, discharging the premise of our original conditional proof, it follows that $\neg\varphi \to (\varphi \to \psi)$. ⋈

(58.4) We establish $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$ by conditional proof. Assume $\neg\psi \to \neg\varphi$. Then as an instance of (21.3), we know: $(\neg\psi \to \neg\varphi) \to ((\neg\psi \to \varphi) \to \psi)$. So it follows that $(\neg\psi \to \varphi) \to \psi$. But as an instance of (21.1), we know: $\varphi \to (\neg\psi \to \varphi)$. So by hypothetical syllogism (55.1) from from our last two results, it follows that $\varphi \to \psi$. So, by conditional proof (CP), it follows that $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$. ⋈

(58.5) We establish $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ by conditional proof. Assume $\varphi \to \psi$. We know by (58.1) that $\neg\neg\varphi \to \varphi$. So it follows by hypothetical syllogism (56.1 that:

(a) $\neg\neg\varphi \to \psi$

But by (58.2), we know:

(b) $\psi \to \neg\neg\psi$

So it follows from (a) and (b) by hypothetical syllogism (56.1 that $\neg\neg\varphi \to \neg\neg\psi$. But as an instance of (58.4), we know: $(\neg\neg\varphi \to \neg\neg\psi) \to (\neg\psi \to \neg\varphi)$. Hence it follows that $\neg\psi \to \neg\varphi$. So, by conditional proof (CP), it follows that $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$. ⋈

(58.6) Assume $\varphi \to \neg\psi$, to show $\psi \to \neg\varphi$ by conditional proof. Now assume ψ for a conditional proof nested within our conditional proof. From ψ it follows by (58.1) that $\neg\neg\psi$. Then from $\varphi \to \neg\psi$ and $\neg\neg\psi$, it follows by Modus Tollens that $\neg\varphi$. So discharging the premise of our nested conditional proof, we have $\psi \to \neg\varphi$. And discharging the premise of our original conditional proof, it follows that $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$. ⋈

(58.7) (Exercise)

(58.8) – **(58.9)** Follow the proofs in Mendelson 1997, Lemma 1.11(f) – (g), pp. 39–40. ⋈

(58.10) (Exercise)

(59.1) Assume $\Gamma_1 \vdash (\varphi \to \psi)$ and $\Gamma_2 \vdash \neg\psi$. Since $\Gamma_1 \subseteq \Gamma_1 \cup \Gamma_2$, it follows from the first assumption by (47.7) that:

(a) $\Gamma_1, \Gamma_2 \vdash (\varphi \to \psi)$

Since $\Gamma_2 \subseteq \Gamma_1 \cup \Gamma_2$, it follows from the second assumption by (47.7) that:

(b) $\Gamma_1, \Gamma_2 \vdash \neg\psi$

Now as an instance of (58.5), we know:

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

and hence by (47.3) that:

$$\Gamma_1, \Gamma_2 \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

So by applying (47.5) to this last result and (a), we have: $\Gamma_1, \Gamma_2 \vdash \neg\psi \to \neg\varphi$. And by applying (47.5) to this result and (b), we have $\Gamma_1, \Gamma_2 \vdash \neg\varphi$. ⋈

(59.2) (Exercise)

(60.1) ($\to$) Assume:

(ϑ) $\Gamma \vdash \varphi \to \psi$

But given (58.5), we know:

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

and hence by (47.3) that:

$$\Gamma \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$$

Hence by applying (47.5) to this last result and (ϑ), we obtain $\Gamma \vdash (\neg\psi \to \neg\varphi)$. ($\leftarrow$) By symmetrical reasoning, but using (58.4). ⋈

(60.2) ($\to$) Assume:

(ϑ) $\Gamma \vdash \varphi \to \neg\psi$

But given (58.6), we know:

$$\vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$$

and hence by (47.3) that:

$$\Gamma \vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$$

Hence by applying (47.5) to this last result and (ϑ), we obtain $\Gamma \vdash (\psi \to \neg\varphi)$. ($\leftarrow$) By symmetrical reasoning, but using (58.7) ⋈

(61.1) Assume $\Gamma_1, \neg\varphi \vdash \neg\psi$ and $\Gamma_2, \neg\varphi \vdash \psi$. By analogy with the first step of the reasoning in (59.1), it follows by (47.7) that both:

(a) $\Gamma_1, \Gamma_2, \neg\varphi \vdash \neg\psi$

(b) $\Gamma_1, \Gamma_2, \neg\varphi \vdash \psi$

Now, by the Deduction Theorem (54), it follows from (a) and (b), respectively, that:

(ϑ) $\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \neg\psi)$

(ζ) $\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \psi)$

But the instances of (21.3) are axioms and hence theorems, by (47.1). So we know:

$$\vdash (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$$

From this result it follows by (47.3) that:

(ξ) $\Gamma_1, \Gamma_2 \vdash (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

But by apply (47.5) to (ϑ) and (ξ), and to the result and (ζ), apply (47.5) again. It follows that $\Gamma_1, \Gamma_2 \vdash \varphi$. ⋈

(61.2) (Exercise)

(62.1) (Exercise)

(62.2) (Exercise)

(63.1.a) – (63.10.e) (Exercises)

(64.1) – (64.8) (Exercises)

(66.1) If we let $\tau = \alpha$, then as an instance of axiom (29.2), we have $\exists\beta(\beta = \alpha)$ (clearly, the variable β is not free in the variable α, and so the condition of the axiom is met). Since this is a theorem, we may apply GEN to obtain $\forall\alpha\exists\beta(\beta = \alpha)$. ⋈

(66.2) If we let $\tau = \alpha$, then as a *bona fide* instance of axiom (29.2), we have $\exists\beta(\beta = \alpha)$. Since this is a $\Box$-theorem (recall that in item (48)★ we defined $\Box$-theorems as those theorems having modally-strict proofs), it follows by RN that $\Box\exists\beta(\beta = \alpha)$. ⋈

(66.3) Since (66.1) is a $\Box$-theorem, we may apply RN to obtain $\Box\forall\alpha\exists\beta(\beta = \alpha)$. ⋈

(66.4) From (66.2), by GEN. ⋈

(66.5) From the $\Box$-theorem (66.4), by RN. ⋈

(67.1) As an instance of (63.4.a), we have $xF^1 \equiv xF^1$. Since this is a theorem, we may apply GEN (50) to obtain $\forall x(xF^1 \equiv xF^1)$. Since this is a $\Box$-theorem, it follows by RN (52) that $\Box\forall x(xF^1 \equiv xF^1)$. Thus, by the definition of property identity (16.1), $F^1 = F^1$. ⋈

(67.2) Let φ be $F^1 = F^1$ and φ' be $G^1 = F^1$. Since we've taken the closures of the substitution of identicals (25) as axioms, the following universal generalization is an axiom:

(ϑ) $\forall F^1 \forall G^1(F^1 = G^1 \to (F^1 = F^1 \to G^1 = F^1))$

Since $\exists H^1(H^1 = F^1)$ and $\exists H^1(H^1 = G^1)$ are *bona fide* instances of axiom (29.2), we may formulate the relevant instances of axiom (29.1) that allow us to instantiate F^1 for $\forall F^1$ and G^1 for $\forall G^1$ in (ϑ) so as to conclude:

$$F^1 = G^1 \to (F^1 = F^1 \to G^1 = F^1)$$

Now assume $F^1 = G^1$, for conditional proof. Then it follows by MP that $F^1 = F^1 \to G^1 = F^1$. And from this last result and the theorem that $F^1 = F^1$ (67.1), it follows by MP that $G^1 = F^1$. So by conditional proof, $F^1 = G^1 \to G^1 = F^1$. ⋈

(67.3) By (67.2), we know that:

(a) $F^1 = G^1 \rightarrow G^1 = F^1$

Now let φ be $G^1 = H^1$ and let φ' be $F^1 = H^1$. Then, instantiating appropriately into a universal generalization of (25), we also know:

(b) $G^1 = F^1 \rightarrow (G^1 = H^1 \rightarrow F^1 = H^1)$

From (a) and (b), it now follows by hypothetical syllogism (56.1 that:

$$F^1 = G^1 \rightarrow (G^1 = H^1 \rightarrow F^1 = H^1)$$

So by Importation (63.8.b), it follows that:

$$F^1 = G^1 \,\&\, G^1 = H^1 \rightarrow F^1 = H^1 \qquad \bowtie$$

(67.4) Let F^n be an arbitrary n-place relation, $n \geq 2$. So, as instances of (67.1), we have the following property identities of the form $G^1 = G^1$:

- $[\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, F^n y x_1 \ldots x_{n-1}]$
- $[\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}]$

  $\vdots$

- $[\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, F^n x_1 \ldots x_{n-1} y]$

Hence, by &I (64.1), we may conjoin all of these to form the claim:

$$\begin{aligned}&[\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, F^n y x_1 \ldots x_{n-1}] \,\&\\ &\quad [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] \,\&\ldots\&\\ &\quad [\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, F^n x_1 \ldots x_{n-1} y]\end{aligned}$$

Since this is a theorem, we may apply GEN (50) $n-1$ times, universally generalizing on each of the variables $x_1, \ldots, x_{n-1}$, to obtain:

$$\begin{aligned}&\forall x_1 \ldots \forall x_{n-1}([\lambda y\, F^n y x_1 \ldots x_{n-1}] = [\lambda y\, F^n y x_1 \ldots x_{n-1}] \,\&\\ &\quad [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] = [\lambda y\, F^n x_1 y x_2 \ldots x_{n-1}] \,\&\ldots\&\\ &\quad [\lambda y\, F^n x_1 \ldots x_{n-1} y] = [\lambda y\, F^n x_1 \ldots x_{n-1} y])\end{aligned}$$

Hence, by the definition of relation identity (16.2), it follows that $F^n = F^n$. $\bowtie$

(67.5) From (67.4) using reasoning analogous to (67.2). $\bowtie$

(67.6) From (67.5) using reasoning analogous to (67.3). $\bowtie$

(67.7) As an instance of (67.1), we have $[\lambda y\, p] = [\lambda y\, p]$. So $p = p$, by (16.3). $\bowtie$

(67.8) From (67.7), using reasoning analogous to (67.2). $\bowtie$

(67.9) From (67.8), using reasoning analogous to (67.3). $\bowtie$

(68) Let (a) τ be any complex n-place relation term ($n \geq 0$), (b) τ' be an alphabetic variant of τ, (c) α be the variable F with same arity as τ and τ', (c) τ and τ' both be substitutable for F in φ, and (d) φ' be the result of replacing zero or more occurrences of τ in φ^{τ}_{F} with occurrences of τ'. Note independently that the n-place relation case of the axiom for the substitution of identicals (25) has the following universal generalization with respect to φ, where the superscripts on F^n and G^n are suppressed:

(ϑ) $\forall F \forall G(F = G \rightarrow (\varphi \rightarrow \varphi''))$, where φ'' is the result of replacing zero or more occurrences of F in φ with occurrences of G.

Since τ and τ' are both n-place relation terms, we know that for some variable β that doesn't occur free in τ, the following are instances of axiom (29.2):

$$\exists\beta(\beta = \tau)$$
$$\exists\beta(\beta = \tau')$$

So by an appropriate instance of axiom (29.1), we may first instantiate τ for $\forall F$ in (ϑ) and then instantiate τ' for $\forall G$ in the result, to obtain:

$$\tau = \tau' \rightarrow (\varphi^{\tau}_{F} \rightarrow \varphi'),$$

where φ' is defined by hypothesis (d), i.e., as the result of replacing zero or more occurrences of τ in φ^{τ}_{F} with occurrences of τ'. Now we know independently that the following is an instance of α-Conversion (36.1), given that τ and τ' are alphabetically-variant n-place relation terms:

$$\tau = \tau'$$

It follows from our last two displayed results by MP that:

$$\varphi^{\tau}_{F} \rightarrow \varphi'$$

Hence, by (47.10), we have established:

(ξ) $\varphi^{\tau}_{F} \vdash \varphi'$

Now assume that $\Gamma \vdash \varphi^{\tau}_{F}$. Then from this and ($\xi$) it follows by (47.8) that $\Gamma \vdash \varphi'$. ⋈

(69.1) Since the universal generalizations of β-Conversion (36.2) are axioms, the following is an axiom:

$$\forall x_1 \forall x_2([\lambda y_1 y_2\, O!y_1 \,\&\, O!y_2 \,\&\, \Box\forall F(Fy_1 \equiv Fy_1)]x_1 x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$$

Since $[\lambda xy\, O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]$ is an alphabetic variant of $[\lambda y_1 y_2\, O!y_1 \,\&\, O!y_2 \,\&\, \Box\forall F(Fy_1 \equiv Fy_1)]$, it follows by rule (68) that:

(ϑ) $\forall x_1 \forall x_2([\lambda xy\, O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)]x_1 x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$

By definition (12), (ϑ) becomes:

$$\forall x_1 \forall x_2 (=_E x_1 x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$$

and by definition (13), this last formula becomes:

$$(\xi)\quad \forall x_1 \forall x_2 (x_1 =_E x_2 \equiv (O!x_1 \,\&\, O!x_2 \,\&\, \Box\forall F(Fx_1 \equiv Fx_2)))$$

Now since $\exists z(z = x)$ and $\exists z(z = y)$ are *bona fide* instances of axiom (29.2), an appropriate instance of (29.1) allows us to instantiate x for $\forall x_1$ in (ϑ) and then instantiate y for $\forall x_2$ in the result, to conclude:

$$x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \qquad \bowtie$$

(69.2) Assume $x =_E y$, for conditional proof. Then it follows by $\vee$Introduction (64.3.a) that:

$$x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

But this is, by definition (15), $x = y$. ⋈

(69.3) By $\equiv$I (64.5), it suffices to show both directions of the biconditional. ($\rightarrow$) Assume $x = y$, for conditional proof. Then, by definition (15), this is equivalent to:

$$\text{(a)}\quad x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

But by theorem (69.1), we also know the following about the left disjunct of (a):

$$\text{(b)}\quad x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$$

So by appealing to a disjunctive syllogism of the form (64.4.e), it follows from (a), (b) and the tautology $\psi \equiv \psi$, where ψ is the right disjunct of (a), that:

$$(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

($\leftarrow$) Reverse the reasoning, but first apply the tautology (63.3.h), i.e., $(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)$ to (69.1) to obtain the commuted form of (b). ⋈

(70.1) By theorem (69.3) and biconditional syllogism (64.6.b), it suffices to show:

$$(\vartheta)\quad (O!x \,\&\, O!x \,\&\, \Box\forall F(Fx \equiv Fx)) \vee (A!x \,\&\, A!x \,\&\, \Box\forall F(xF \equiv xF))$$

Now to establish (ϑ), we shall be reasoning by a disjunctive syllogism (64.4.d), as follows:

From:

(a) $\Diamond E!x \vee \neg\Diamond E!x$

(b) $\Diamond E!x \to (O!x \,\&\, O!x \,\&\, \Box\forall F(Fx \equiv Fx))$

(c) $\neg\Diamond E!x \to (A!x \,\&\, A!x \,\&\, \Box\forall F(xF \equiv xF))$

we may conclude (ϑ).

So if we can establish (a), (b), and (c), we're done. Now (a) is trivial, as it is an instance of Excluded Middle (63.2). We now show (b) and (c) in turn:

(b) Assume $\Diamond E!x$, for CP. Note that $[\lambda y\, \Diamond E!y]x \equiv \Diamond E!x$ is an instance of β-Conversion (36.2). So, by a biconditional syllogism $\equiv$E (64.6.b) from this instance and our assumption, it follows that $[\lambda y\, \Diamond E!y]x$. By our Rule of Substitution of Alphabetically-Variant Relation Terms (68), it follows that $[\lambda x\, \Diamond E!x]x$. Hence, by definition of $O!$ (11.1), it follows that $O!x$. By the idempotency of & (63.3.a), it follows that $O!x \,\&\, O!x$. Note, independently of our conditional proof, that as an instance of (63.4.a), we have $Fx \equiv Fx$. Since this is a theorem, we may apply GEN (50) to obtain $\forall x(Fx \equiv Fx)$. Since this is a $\Box$-theorem, it follows by RN (52) that $\Box\forall x(Fx \equiv Fx)$. Returning to our conditional proof, we may conjoin this last result by &I (64.1) with what we have established so far, to conclude $O!x \,\&\, O!x \,\&\, \Box\forall x(Fx \equiv Fx)$. Hence, by conditional proof, $\Diamond E!x \to (O!x \,\&\, O!x \,\&\, \Box\forall x(Fx \equiv Fx))$.

(c) Assume $\neg\Diamond E!x$, for CP. Note that $[\lambda y\, \neg\Diamond E!y]x \equiv \neg\Diamond E!x$ is an instance of β-Conversion (36.2). So, by a biconditional syllogism $\equiv$E (64.6.b) from this instance and our assumption, it follows that $[\lambda y\, \neg\Diamond E!y]x$. By our Rule of Substitution of Alphabetically-Variant Relation Terms (68), it follows that $[\lambda x\, \neg\Diamond E!x]x$. Hence, by definition of $A!$ (11.2), it follows that $A!x$. By the idempotency of & (63.3.a), it follows that $A!x \,\&\, A!x$. Note, independently of our conditional proof, that as an instance of (63.4.a), we have $xF \equiv xF$. Since this is a theorem, we may apply GEN to obtain $\forall x(xF \equiv xF)$. Since this is a $\Box$-theorem, it follows by RN that $\Box\forall x(xF \equiv xF)$. Returning to our conditional proof, we may conjoin this last result by &I (64.1) with what we have established so far, to conclude $A!x \,\&\, A!x \,\&\, \Box\forall x(xF \equiv xF)$. By conditional proof, $\neg\Diamond E!x \to (A!x \,\&\, A!x \,\&\, \Box\forall x(xF \equiv xF))$. ⋈

(70.2) From (70.1), using reasoning analogous to (67.2). ⋈

(70.3) From (70.2), using reasoning analogous to (67.3). ⋈

(71.1) By cases (67.1), (67.4), (67.7) and (70.1). ⋈

(71.2) By cases (67.2), (67.5), (67.8) and (70.2). ⋈

(71.3) By cases (67.3), (67.6), (67.9) and (70.3). ⋈

(72.1) Apply GEN to theorem (71.1). Since the result is a $\Box$-theorem, apply RN. ⋈

(72.2) Since (71.1) is a $\Box$-theorem, we may apply RN to obtain $\Box(\alpha = \alpha)$ as a theorem. Hence, by GEN, we have: $\forall\alpha\Box(\alpha = \alpha)$. ⋈

(73.1) Proof by cases, where the cases are: (A) τ is any term other than a description, and (B) τ is a definite description.

Case A. If τ is any term other than a description, then by axiom (29.2), we know $\exists\beta(\beta = \tau)$, where β is a variable of the same type as τ that doesn't occur free in τ. Hence, by axiom (21.1), it follows that $\tau = \tau' \rightarrow \exists\beta(\beta = \tau)$.

Case B. Suppose τ is the description $\imath x\chi$, and for conditional proof, assume $\tau = \tau'$, to show $\exists\nu(\nu = \tau)$, where ν is an individual variable not free in τ. Then by definition (15), it follows that:

$$\tau =_E \tau' \vee (A!\tau \,\&\, A!\tau' \,\&\, \Box\forall F(\tau F \equiv \tau' F))$$

Our proof strategy is to reason by cases (64.4.a) to establish $\exists\nu(\nu = \tau)$. If we let φ be $\tau =_E \tau'$ and let ψ be $A!\tau \,\&\, A!\tau' \,\&\, \Box\forall F(\tau F \equiv \tau' F))$, then we can reach the conclusion (d) by the reasoning:

(a) $\varphi \vee \psi$
(b) $\varphi \rightarrow \exists\nu(\nu = \tau)$
(c) $\psi \rightarrow \exists\nu(\nu = \tau)$
(d) $\exists\nu(\nu = \tau)$

We have already established (a).

To show (b), assume φ, i.e., $\tau =_E \tau'$. Then by definition (13), it follows that $=_E \tau\tau'$, which is an exemplification formula in which τ occurs as one of the individual terms. Now consider the formula $=_E z\tau'$, where z doesn't appear free in τ'. Call this formula ψ, so that $=_E \tau\tau'$ is $\psi_z^{\imath x\chi}$. Then where ν is an individual variable that doesn't occur free in τ, the conditions of (29.5) are met and we know $\psi_z^{\imath x\chi} \rightarrow \exists\nu(\nu = \tau)$ is an axiom. Since we've established the antecedent, it follows that $\exists\nu(\nu = \tau)$.

To show (c), assume ψ, i.e., $A!\tau \,\&\, A!\tau' \,\&\, \Box\forall F(\tau F \equiv \tau' F)$. Then by an application of &E (64.2.a), it follows that:

$$A!\tau$$

Since this is exemplification formula in which τ is a description appearing as an argument, the conditions of (29.5) are met and so the following is an instance of that axiom:

$A!\tau \rightarrow \exists\nu(\nu = \tau)$, where ν doesn't occur free in τ

Hence, from our last two displayed results, we have, by MP, $\exists\nu(\nu = \tau)$.

Hence by (a), (b), and (c), it follows that (d). ⋈

(73.2) By reasoning analogous to (73.1). ⋈

(74.1) Since (71.1) is a theorem, we know:

$$\vdash \alpha = \alpha$$

Hence by GEN, we know:

(a) $\vdash \forall\alpha(\alpha = \alpha)$

Since we know that the instances of (29.1) are axioms, we also know, where β is some variable that doesn't occur free in τ:

(b) $\vdash \forall\alpha(\alpha = \alpha) \rightarrow (\exists\beta(\beta = \tau) \rightarrow \tau = \tau)$

So from (a) and (b), it follows by (47.6) that:

(c) $\vdash \exists\beta(\beta = \tau) \rightarrow \tau = \tau$

By hypothesis, τ is any term other than a description and so the instances of (29.2), given our choice of β, are axioms, yielding:

(d) $\vdash \exists\beta(\beta = \tau)$

Hence, from (c) and (d) it follows by (47.6) that $\vdash \tau = \tau$. ⋈

(74.2) Assume $\Gamma_1 \vdash \varphi^{\tau}_{\alpha}$ and $\Gamma_2 \vdash \tau = \tau'$. So by (47.7), where $\Delta = \Gamma_1 \cup \Gamma_2$, we know:

(ξ) $\Gamma_1, \Gamma_2 \vdash \varphi^{\tau}_{\alpha}$

(ϑ) $\Gamma_1, \Gamma_2 \vdash \tau = \tau'$

We have to show $\Gamma_1, \Gamma_2 \vdash \varphi'$. Now by applying (47.10) to theorems (73.1) and (73.2), we obtain, for some variable γ that doesn't occur free in τ:

(a) $\tau = \tau' \vdash \exists\gamma(\gamma = \tau)$

(b) $\tau = \tau' \vdash \exists\gamma(\gamma = \tau')$

Hence by applying (47.8) first to the pair (ϑ) and (a) and then to the pair (ϑ) and (b), we obtain, respectively:

(c) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau)$

(d) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau')$

Since generalizations of the axiom schema (25) for the substitution of identicals are axioms, we know by (47.1) that:

$\vdash \forall\alpha\forall\beta(\alpha = \beta \to (\varphi \to \varphi'))$, whenever β is substitutable for α in φ and φ' is the result of replacing zero or more occurrences of α in φ with occurrences of β.

It follows from this last result by (47.3) that:

(e) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\forall\beta(\alpha = \beta \to (\varphi \to \varphi'))$, whenever β is substitutable for α in φ and φ' is the result of replacing zero or more occurrences of α in φ with occurrences of β.

Now let ψ be $\forall\beta(\alpha = \beta \to (\varphi \to \varphi'))$ so that we may abbreviate (e) as:

(e) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\psi$

Since instances of (29.1) are axioms, we also know, for some variable γ not free in τ, that $\vdash \forall\alpha\psi \to (\exists\gamma(\gamma = \tau) \to \psi^\tau_\alpha)$, by (47.1). By (47.3) this yields:

(f) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\psi \to (\exists\gamma(\gamma = \tau) \to \psi^\tau_\alpha)$

From (e) and (f) it follows by (47.5) that:

(g) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau) \to \psi^\tau_\alpha$

From (g) and (c) it follows by (47.5) that $\Gamma_1, \Gamma_2 \vdash \psi^\tau_\alpha$, i.e.,

(h) $\Gamma_1, \Gamma_2 \vdash \forall\beta(\tau = \beta \to (\varphi^\tau_\alpha \to \varphi'))$, where φ' is the result of replacing zero or more occurrences of τ in φ^τ_α by occurrences of β

which we may abbreviate as:

(h) $\Gamma_1, \Gamma_2 \vdash \forall\beta\chi$

Note separately that since we know by (29.1) that $\vdash \forall\beta\chi \to (\exists\gamma(\gamma = \tau') \to \chi^{\tau'}_\beta)$, we have by now familiar reasoning:

(i) $\Gamma_1, \Gamma_2 \vdash \forall\beta\chi \to (\exists\gamma(\gamma = \tau') \to \chi^{\tau'}_\beta)$

From (i) and (h) it follows by familiar reasoning:

(j) $\Gamma_1, \Gamma_2 \vdash \exists\gamma(\gamma = \tau') \to \chi^{\tau'}_\beta$

From (j) and (d) it follows that $\Gamma_1, \Gamma_2 \vdash \chi^{\tau'}_\alpha$, i.e.,

(k) $\Gamma_1, \Gamma_2 \vdash \tau = \tau' \to (\varphi^\tau_\alpha \to \varphi')$, where φ' is the result of replacing zero or more occurrences of τ in φ^τ_α by occurrences of τ'

Hence, from (ϑ) and (k), it follows by familiar reasoning that $\Gamma_1, \Gamma_2 \vdash \varphi^\tau_\alpha \to \varphi'$. From this last result and (ξ), it follows that $\Gamma_1, \Gamma_2 \vdash \varphi'$, which is what we had to show. $\bowtie$

(74.3) (Exercise)

(75) By $\equiv$I (64.5), it suffices to show both directions of the biconditional. ($\rightarrow$) Note that we have established $\alpha = \alpha$ (71.1) as a $\Box$-theorem. So we may apply RN to this theorem to conclude $\Box\alpha = \alpha$. Now assume $\alpha = \beta$, for conditional proof. Then by Rule SubId (74.2), it follows that $\Box\alpha = \beta$. ($\leftarrow$) This is an instance of the T schema (32.2). ⋈

(76) ($\rightarrow$) By taking the individual variables z, w as instances of the previous theorem and applying GEN twice, we know:

(ϑ) $\forall z \forall w(z = w \equiv \Box z = w)$

Now assume $\imath x\varphi = \imath y\psi$, for conditional proof. Then by (73.1) and (73.2), it follows both that:

(a) $\exists z(z = \imath x\varphi)$

(b) $\exists z(z = \imath y\psi)$

provided z doesn't occur free in φ or ψ. Note however that the following is an instance of (29.1):

$$(\vartheta) \rightarrow (\exists z(z = \imath x\varphi) \rightarrow \forall w(\imath x\varphi = w \rightarrow \Box \imath x\varphi = w))$$

From this result, (ϑ) and (a), it follows by two applications of MP that:

(ξ) $\forall w(\imath x\varphi = w \rightarrow \Box \imath x\varphi = w)$

But now we have the following instance of (29.1):

$$(\xi) \rightarrow (\exists z(z = \imath y\psi) \rightarrow (\imath x\varphi = \imath y\psi \rightarrow \Box \imath x\varphi = \imath y\psi))$$

From this result, (ξ) and (b), it follows by two applications of MP that:

$$\imath x\varphi = \imath y\psi \rightarrow \Box \imath x\varphi = \imath y\psi$$

($\leftarrow$) Assume $\Box(\imath x\varphi = \imath y\psi)$. Then by the T schema, $\imath x\varphi = \imath y\psi$. ⋈

(77.1) Assume $\Gamma_1 \vdash \forall\alpha\varphi$ and $\Gamma_2 \vdash \exists\beta(\beta = \tau)$. Assume further that τ is substitutable for α in φ. By (47.7), it follows both that:

(a) $\Gamma_1, \Gamma_2 \vdash \forall\alpha\varphi$

(b) $\Gamma_1, \Gamma_2 \vdash \exists\beta(\beta = \tau)$

Since we've taken instances of (21.1) as axioms, it follows by (47.1) that:

$$\vdash \forall\alpha\varphi \rightarrow (\exists\beta(\beta = \tau) \rightarrow \varphi^\tau_\alpha)$$

Hence by (47.3), we know:

$$\Gamma_1, \Gamma_2 \vdash \forall\alpha\varphi \to (\exists\beta(\beta = \tau) \to \varphi^\tau_\alpha)$$

From this, (a) and (47.5), it follows that:

$$\Gamma_1, \Gamma_2 \vdash \exists\beta(\beta = \tau) \to \varphi^\tau_\alpha$$

But from this, (b) and (47.5), it follows that $\Gamma_1, \Gamma_2 \vdash \varphi^\tau_\alpha$. ⋈

(77.1) [Proof of the Variant form of the rule.] Assume $\forall\alpha\varphi$ and $\exists\beta(\beta = \tau)$, where τ is substitutable for α in φ. Then by (64.1), it follows that:

$$\forall\alpha\varphi \,\&\, \exists\beta(\beta = \tau)$$

However, by applying an appropriate instance of Exportation (63.8.b) to our first quantifier axiom (29.1), it follows that:

$$(\forall\alpha\varphi \,\&\, \exists\beta(\beta = \tau)) \to \varphi^\tau_\alpha$$

So by MP, φ^τ_α. Thus, we've established $\forall\alpha\varphi, \exists\beta(\beta = \tau) \vdash \varphi^\tau_\alpha$. ⋈

(77.2) Assume that $\Gamma \vdash \forall\alpha\varphi$, that τ is substitutable for α in φ, and that τ is not a description. Since instances of (21.2) are axioms, we know by (47.1) that $\vdash \exists\beta(\beta = \tau)$, and hence, $\Gamma \vdash \exists\beta(\beta = \tau)$. So by (77.1) (with $\Gamma = \Gamma_1 = \Gamma_2$), it follows that $\Gamma \vdash \varphi^\tau_\alpha$. ⋈

(77.2) [Proof of the Variant form of the rule.] Assume $\forall\alpha\varphi$, where τ is substitutable for α in φ and τ is not a description. Then we have, as an instance of the second axiom of quantification theory (29.2), that $\exists\beta(\beta = \tau)$, for some variable β not free in τ. Once we conjoin $\forall\alpha\varphi$ and $\exists\beta(\beta = \tau)$ by &I, it follows by reasoning used in the proof of the preceding theorem that φ^τ_α. Hence we've established that $\forall\alpha\varphi \vdash \varphi^\tau_\alpha$ ⋈

(79.1) Assume $\forall\alpha\varphi$ and that τ is substitutable for α in φ and is not a description. Then by Rule ∀E (77.2), it follows that φ^τ_α. So by conditional proof (CP), $\forall\alpha\varphi \to \varphi^\tau_\alpha$. ⋈

(79.2) Assume $\forall\alpha(\varphi \to \psi)$, where α is not free in φ. Now assume φ for a nested conditional proof. By (79.1), we know $\forall\alpha(\varphi \to \psi) \to (\varphi \to \psi)$. So from this and our first assumption, we have $\varphi \to \psi$ by MP. From this and our second assumption, we have ψ by MP. Since we've derived ψ from two premises in which α isn't free, we may apply GEN to ψ to conclude $\forall\alpha\psi$. Hence we may discharge the premise of our nested conditional proof to conclude $\varphi \to \forall\alpha\psi$, and then discharge the premise of our initial conditional proof to conclude that: $\forall\alpha(\varphi \to \psi) \to (\varphi \to \forall\alpha\psi)$. ⋈

(80) Follow the proof in Enderton 1972, Theorem 24F (p. 116). (In the second edition, Enderton 2001, the proof of Theorem 24F occurs on pp. 123–124.) All of the results needed for the proof are in place. [Warning: Note that in the proof of this theorem, Enderton uses φ^c_y to be the result of replacing every occurrence of the constant c in φ with an occurrence of the variable y. By contrast, we use φ^α_τ to be the result of replacing every occurrence of the constant τ in φ with an occurrence of the variable α.] ⋈

(81.1) We need not prove this lemma by induction; the recursive definitions of φ^τ_α and *substitutable for* carry the following reasoning through all the inductive cases. Assume β is substitutable for α in φ and β doesn't occur free in φ. There are two cases. In the case where φ has no free occurrences of α, then by the definition of *substitutable for*, β is trivially substitutable for α in φ. In that case, however, φ^β_α just is φ and so $(\varphi^\beta_\alpha)^\alpha_\beta = \varphi^\alpha_\beta$. Moreover by hypothesis, β doesn't occur free in φ. So by an analogous fact, $\varphi^\alpha_\beta = \varphi$. Hence $(\varphi^\beta_\alpha)^\alpha_\beta = \varphi$.

In the case where α has at least one free occurrence in φ, then without loss of generality, consider any free occurrence of α in φ. Since β is substitutable for α in φ (by hypothesis), we know that β will not be bound when substituted for α at this occurrence. Thus, β will be free at this occurrence in φ^β_α. And since φ has no free occurrences of β (by hypothesis), we know that every free occurrence of β in φ^β_α replaced a free occurrence of α in φ. Thus no free occurrence of β in φ^β_α falls under the scope of a variable binding operator that binds α. Hence, α is substitutable for β in φ^β_α.

Now we must show that $(\varphi^\beta_\alpha)^\alpha_\beta = \varphi$. Suppose, for reductio, that $(\varphi^\beta_\alpha)^\alpha_\beta \neq \varphi$. Since substitution only changes the substituted variables, then there must be some occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$ that is not in φ or some occurrence of α in φ that is not in $(\varphi^\beta_\alpha)^\alpha_\beta$. But both cases lead to contradiction:

> Case 1. Suppose there is an occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$ that is not in φ. Then there was an occurrence of β in φ that remained an occurrence of β in φ^β_α but was replaced by an occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$. But if an occurrence of β in φ^β_α was replaced by an occurrence of α in $(\varphi^\beta_\alpha)^\alpha_\beta$, then that occurrence of β in φ^β_α had to be a free occurrence. But that occurrence of β must have been a free occurrence in φ (had it been a bound occurrence, it would have remained a bound occurrence in φ^β_α). And this contradicts the hypothesis that β doesn't occur free in φ.
>
> Case 2. Suppose there is an occurrence of α in φ that is not in $(\varphi^\beta_\alpha)^\alpha_\beta$. Then there was an occurrence of α in φ that was replaced by an occurrence of β in φ^β_α which, in turn, remained an occurrence of β in $(\varphi^\beta_\alpha)^\alpha_\beta$. But if that occurrence of β remained an occurrence of β in the re-replacement, then it must be bound by a variable-binding operator binding β in φ^β_α. But this contradicts the hypothesis that β is substitutable for α in φ, which

requires that β must remain free at every occurrence of α in φ that it replaces in φ^{β}_{α}. ⋈

(81.2) Assume τ is a constant symbol that doesn't occur in φ. If α has no free occurrences in φ, then τ is trivially substitutable for α in φ and $\varphi^{\tau}_{\alpha} = \varphi$. In that case, both $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau} = \varphi$, and $\varphi^{\beta}_{\alpha} = \varphi$. Hence $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau} = \varphi^{\beta}_{\alpha}$. So we consider only the case where α has at least one free occurrence in φ.

Suppose, for reductio, that $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau} \neq \varphi^{\beta}_{\alpha}$. Then there must be some occurrence of β in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$ that is not in φ^{β}_{α} or some occurrence of β in φ^{β}_{α} that is not in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$. But both cases lead to contradiction:

Case 1. Suppose there is an occurrence of β in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$ that is not in φ^{β}_{α}. Since by hypothesis, there are no occurrences of τ in φ, then there must be an occurrence of α in φ that remained an occurrence of α in φ^{τ}_{α} but became an occurrence of β in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$. But if an occurrence of α in φ^{τ}_{α} became an occurrence of β in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$, then that occurrence of α in φ^{τ}_{α} had to be a free occurrence. But that contradicts the fact that φ^{τ}_{α} is, by definition, the result of replacing every free occurrence of α by τ in φ.

Case 2. Suppose there is an occurrence of β in φ^{β}_{α} that is not in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$. Then there must be an occurrence of α in φ that became an occurrence of τ in φ^{τ}_{α} which remained an occurrence of τ in $(\varphi^{\tau}_{\alpha})^{\beta}_{\tau}$. But that contradicts the definition of ψ^{β}_{τ}, which signifies the result of replacing every occurrence of τ in ψ by an occurrence of β. ⋈

(81.3) (Exercise)

(83.1) ($\rightarrow$) Assume $\forall\alpha\forall\beta\varphi$, for conditional proof, to show $\forall\beta\forall\alpha\varphi$. Then by Rule $\forall$E (77.2), it follows that $\forall\beta\varphi$. By a second application of this same rule it follows that φ. Since α is not free in our hypothesis $\forall\alpha\forall\beta\varphi$, we may infer $\forall\alpha\varphi$ from φ, by GEN. Since β is not free in our hypothesis, we may infer $\forall\beta\forall\alpha\varphi$ from $\forall\alpha\varphi$, by GEN. Hence, by conditional proof, $\forall\alpha\forall\beta\varphi \rightarrow \forall\beta\forall\alpha\varphi$. ($\leftarrow$) By symmetrical reasoning. ⋈

(83.2) (Exercise)

(83.3) Assume $\forall\alpha(\varphi \equiv \psi)$ and apply Rule $\forall$E (77.2) to obtain $\varphi \equiv \psi$. By $\equiv$I (64.5), it suffices to establish both directions of $\forall\alpha\varphi \equiv \forall\alpha\psi$. ($\rightarrow$) Assume $\forall\alpha\varphi$. By applying Rule $\forall$E (77.2) to our second assumption, we have φ. So by a biconditional syllogism (64.6.a), it follows that ψ. Since α is not free in either of our premises, it follows that $\forall\alpha\psi$, by GEN. Discharging our second assumption, we've established $\forall\alpha\varphi \rightarrow \forall\alpha\psi$. ($\leftarrow$) Assume $\forall\alpha\psi$. The conclusion is then reached by analogous reasoning, but by biconditional syllogism (64.6.b). ⋈

(83.4) ($\rightarrow$) Assume, for conditional proof, that $\forall\alpha(\varphi \,\&\, \psi)$. Then by Rule $\forall$E (77.2), we have $\varphi \,\&\, \psi$. From this we have both φ and ψ, by (64.2.a) and (64.2.b), respectively. Since α isn't free in our assumption, we may apply GEN to both conclusions to obtain $\forall\alpha\varphi$ and $\forall\alpha\psi$. Hence by (64.1), it follows that $\forall\alpha\varphi \,\&\, \forall\alpha\psi$. [We here omit the last step of assembling the conditional to be proved, since it is now obvious.] ($\leftarrow$) Assume $\forall\alpha\varphi \,\&\, \forall\alpha\psi$, for conditional proof. It follows by (64.2.a) and (64.2.b) that $\forall\alpha\varphi$ and $\forall\alpha\psi$. Hence, by applying Rule $\forall$E (77.2) to both, we obtain both φ and ψ. So by &I, we have $\varphi \,\&\psi$. Since α isn't free in our assumption, we may apply GEN to obtain $\forall\alpha(\varphi \,\&\, \psi)$. $\bowtie$

(83.5) Assume, for conditional proof, that $\forall\alpha_1 \ldots \forall\alpha_n\varphi$. By the special case of Rule $\forall$E (77.2), it follows that $\forall\alpha_2 \ldots \forall\alpha_n\varphi$. By analogous reasoning, we can strip off the quantifier $\forall\alpha_2$. Once we have legitimately stripped off the outermost quantifier in this way a total of n times, it follows that φ. $\bowtie$

(83.6) ($\rightarrow$) This direction is a special case of theorem (79.1). ($\leftarrow$) Assume $\forall\alpha\varphi$. Then since α isn't free in our assumption, we may apply GEN to obtain $\forall\alpha\forall\alpha\varphi$. So by conditional proof, $\forall\alpha\varphi \rightarrow \forall\alpha\forall\alpha\varphi$. $\bowtie$

(83.7) By hypothesis, α isn't free in φ. By $\equiv$I (64.5), it suffices to prove both directions of the biconditional. ($\rightarrow$) Assume $\varphi \rightarrow \forall\alpha\psi$. Now for a secondary conditional proof, assume φ. Then by MP, it follows that $\forall\alpha\psi$ and by Rule $\forall$E (77.2), it follows that ψ. Discharging the premise of our secondary conditional proof, it follows that $\varphi \rightarrow \psi$. Since α isn't free in φ, it isn't free in our remaining (original) assumption. So the conditions of GEN are met and we may conclude that $\forall\alpha(\varphi \rightarrow \psi)$. ($\leftarrow$) Assume $\forall\alpha(\varphi \rightarrow \psi)$, for conditional proof. For a secondary conditional proof, assume φ. By applying Rule $\forall$E (77.2) to our initial assumption, it follows that $\varphi \rightarrow \psi$, and hence ψ, by MP. Since α isn't free in φ, it isn't free in either premise and we may apply GEN to establish $\forall\alpha\psi$. Discharging our premise of the secondary conditional proof, it follows that $\varphi \rightarrow \forall\alpha\psi$, which is what we had to show to complete this direction of the proof. $\bowtie$

(83.8) (Exercise)

(83.9) (Exercise)

(83.10) Assume $\forall\alpha(\varphi \equiv \psi) \,\&\, \forall\alpha(\psi \equiv \chi)$. By &E, this yields $\forall\alpha(\varphi \equiv \psi)$ and $\forall\alpha(\psi \equiv \chi)$. By Rule $\forall$E (79.2), it follows, respectively, that $\varphi \equiv \psi$ and $\psi \equiv \chi$. By the transitivity of $\equiv$ (64.6.e), it follows that $\varphi \equiv \chi$. Since α isn't free in our assumption, we may apply GEN to conclude $\forall\alpha(\varphi \equiv \chi)$. $\bowtie$

(83.11) (Exercise)

(83.12) Suppose β is substitutable for α in φ and doesn't occur free in φ. Then there are two cases. *Case 1*. β just is the variable α. Then our theorem becomes:

$\forall\alpha\varphi \equiv \forall\alpha\varphi^{\alpha}_{\alpha}$. But $\varphi^{\alpha}_{\alpha}$ just is φ by definition. So our theorem becomes $\forall\alpha\varphi \equiv \forall\alpha\varphi$, which is an instance of a tautology. *Case 2.* β is distinct from α. ($\rightarrow$) Assume $\forall\alpha\varphi$. Since β is substitutable for α in φ, it follows by Rule $\forall$E (77.2) that φ^{β}_{α}. Furthermore, since β doesn't occur free in φ, β doesn't occur free in our assumption $\forall\alpha\varphi$. So we may apply GEN to obtain $\forall\beta\varphi^{\beta}_{\alpha}$. ($\leftarrow$) Assume $\forall\beta\varphi^{\beta}_{\alpha}$. Since β is substitutable for α in φ and doesn't occur free in φ, it follows, by the re-replacement lemma (81.1), both (a) that α is substitutable for β in φ^{β}_{α} and (b) that $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta} = \varphi$. From (a) and the fact that α is not a description, it follows from our assumption that $(\varphi^{\beta}_{\alpha})^{\alpha}_{\beta}$, by Rule $\forall$E (77.2). But by (b), this is just φ, and since α isn't free in our assumption, it follows by GEN that $\forall\alpha\varphi$. ⋈

(84.1) (Exercise)

(84.2) (Exercise)

(85) Follow the proof in Enderton 1972, Corollary 24H (p. 117). (In the second edition, Enderton 2001, the proof of Corollary 24H occurs on p. 124.) The warning in the proof of (80), about how our notation differs from Enderton's, still apply. ⋈

(86.1) Assume $\forall\alpha\varphi$. Then by Rule $\forall$E (77.2), it follows that φ. Hence, by the rule $\exists$I (84.2), it follows that $\exists\alpha\varphi$. ⋈

(86.2) By $\equiv$I (64.5), it suffices to prove both directions of the biconditional. ($\rightarrow$) Assume $\neg\forall\alpha\varphi$. We want to show $\exists\alpha\neg\varphi$. So by the definition of $\exists$, we have to show: $\neg\forall\alpha\neg\neg\varphi$. For reductio, assume $\forall\alpha\neg\neg\varphi$. Then it follows that $\neg\neg\varphi$, by Rule $\forall$E (77.2). Hence, by double negation elimination (64.8), we may infer φ. Since α isn't free in any of our assumptions, it follows by GEN that $\forall\alpha\varphi$. Since we've reached a contradiction, we may discharge our reductio assumption and conclude, by a version of RAA (62.2), that $\neg\forall\alpha\neg\neg\varphi$, i.e., by definition, $\exists\alpha\neg\varphi$. ($\leftarrow$) Assume $\exists\alpha\neg\varphi$, for conditional proof. Assume that τ is an arbitrary such object, i.e., that $\neg\varphi^{\tau}_{\alpha}$. Assume, for reductio, that $\forall\alpha\varphi$. Then by Rule $\forall$E (77.2), we have φ^{τ}_{α}. Since we've reached a contradiction, we can discharge our reductio assumption by a version of RAA (62.2) and conclude $\neg\forall\alpha\varphi$. Since τ doesn't appear in φ or $\neg\forall\alpha\varphi$, we may apply Rule $\exists$E (85) to discharge the assumption that $\neg\varphi^{\tau}_{\alpha}$ and conclude $\neg\forall\alpha\varphi$. ⋈

(86.3) By $\equiv$I (64.5), it suffices to prove both directions of the biconditional. [Henceforth, we omit mention of this proof strategy for biconditionals.] ($\rightarrow$) Assume $\forall\alpha\varphi$. We want to show $\neg\exists\alpha\neg\varphi$. For reductio, assume $\exists\alpha\neg\varphi$. From this and (86.2), it follows that $\neg\forall\alpha\varphi$, by a biconditional syllogism (64.6.b). Since we've reached a contradiction, we may discharge our reductio assumption and conclude by a version of RAA (62.1) that $\neg\exists\alpha\neg\varphi$. ($\leftarrow$) Assume $\neg\exists\alpha\neg\varphi$, for conditional proof. Assume, for reductio, that $\neg\forall\alpha\varphi$. From this and by

(86.2), it follows that $\exists\alpha\neg\varphi$, by a biconditional syllogism (64.6.a). Since we've reached a contradiction, we can discharge our reductio assumption using a version of RAA (62.2) and conclude $\forall\alpha\varphi$. ⋈

(86.4) (Exercise)

(86.5) (Exercise)

(86.6) (Exercise)

(86.7) (Exercise)

(86.8) Suppose β is substitutable for α in φ and doesn't occur free in φ. Then there are two cases: *Case 1*. β just is α. Then our theorem states $\varphi \equiv \exists\alpha(\alpha{=}\alpha \ \& \ \varphi^{\alpha}_{\alpha})$. By hypothesis, β, i.e., α, doesn't occur free in φ and so $\varphi^{\alpha}_{\alpha} = \varphi$. So our theorem states $\varphi \equiv \exists\alpha(\alpha{=}\alpha \ \& \ \varphi)$. We leave the remainder of the proof as an exercise. *Case 2*. β and α are distinct variables of the same type. ($\rightarrow$) Assume φ, for conditional proof. Then, by definition of substitutions, we know $\varphi^{\alpha}_{\alpha}$. Since it is a theorem (71.1) that $\alpha = \alpha$, we have, by &I, that $\alpha = \alpha \ \& \ \varphi^{\alpha}_{\alpha}$. Hence, by $\exists$I, it follows that $\exists\beta(\beta = \alpha \ \& \ \varphi^{\beta}_{\alpha})$. ($\leftarrow$) Assume, for conditional proof:

(ϑ) $\exists\beta(\beta = \alpha \ \& \ \varphi^{\beta}_{\alpha})$,

Assume τ is an arbitrary such entity; formally, the metavariable τ denotes a new constant that doesn't appear in (ϑ), that has the same type as the variable β, and that is therefore substitutable for β in the matrix of (ζ). So we know:

(ζ) $\tau = \alpha \ \& \ (\varphi^{\beta}_{\alpha})^{\tau}_{\beta}$

We may detach the two conjuncts of (ζ) by &E. Since β is substitutable for α in φ and doesn't occur free in φ, and τ is substitutable for α in φ, the Re-replacement theorem (81.3) tells us that $(\varphi^{\beta}_{\alpha})^{\tau}_{\beta} = \varphi^{\tau}_{\alpha}$. So it follows from the right conjunct of (ζ) that φ^{τ}_{α}. But from this latter conclusion and the left conjunct of (ζ), it follows by Rule SubId Special Case (74.2) that we may substitute α for every occurrence of τ in φ^{τ}_{α}, to obtain $(\varphi^{\tau}_{\alpha})^{\alpha}_{\tau}$. But, by Re-replacement lemma (81.2), since τ is a constant that doesn't appear in φ, this is just $\varphi^{\alpha}_{\alpha}$, i.e., φ. By $\exists$E (85), we can discharge (ζ) and conclude φ. ⋈

(86.9) Let τ be any term other than a description and is substitutable for α in φ. Now choose any variable of the same type as α, say β, that is substitutable for α in φ and that doesn't occur free in φ. Then by applying GEN to (86.8), we know the following applies to φ:

(ϑ) $\forall\alpha(\varphi \equiv \exists\beta(\beta{=}\alpha \ \& \ \varphi^{\beta}_{\alpha}))$

Since τ is substitutable for α in φ, it is substitutable for α in the matrix of (ϑ). So we may instantiate τ into $\forall\alpha$ in (ϑ) by $\forall$E, to obtain:

$$\varphi_\alpha^\tau \equiv \exists\beta(\beta{=}\tau \;\&\; \varphi_\alpha^\beta))$$

But given our choice of β, we know that by commuting an appropriate instance of (86.7), the following applies to φ:

$$\exists\beta(\beta{=}\tau \;\&\; \varphi_\alpha^\beta) \equiv \exists\alpha(\alpha = \tau \;\&\; \varphi)$$

So, by the transitivity of the biconditional, it follows that

$$\varphi_\alpha^\tau \equiv \exists\alpha(\alpha = \tau \;\&\; \varphi) \qquad \bowtie$$

(86.10) Suppose α, β are distinct variables and consider any formula φ in which β is substitutable for α and doesn't occur free. ($\rightarrow$) Assume:

(ζ) $\varphi \;\&\; \forall\beta(\varphi_\alpha^\beta \rightarrow \beta = \alpha)$

to show $\forall\beta(\varphi_\alpha^\beta \equiv \beta = \alpha)$. By (83.2), it suffices to show $\forall\beta(\varphi_\alpha^\beta \rightarrow \beta = \alpha) \;\&\; \forall\beta(\beta = \alpha \rightarrow \varphi_\alpha^\beta)$. By applying &E (64.2.b) to (ζ), we already have $\forall\beta(\varphi_\alpha^\beta \rightarrow \beta = \alpha)$. So by &I, it remains to show $\forall\beta(\beta = \alpha \rightarrow \varphi_\alpha^\beta)$. By hypothesis, β doesn't occur free in φ and, hence, doesn't occur free in our assumption (ζ). So by GEN, it remains to show $\beta = \alpha \rightarrow \varphi_\alpha^\beta$. So assume $\beta = \alpha$, which by the symmetry of identity (71.2), yields $\alpha = \beta$. Now by applying &E to (ζ), we know φ. Since β is, by hypothesis, substitutable for α in φ, it follows by Rule SubId Special Case (74.2) that φ_α^β. ($\leftarrow$) Assume:

(ϑ) $\forall\beta(\varphi_\alpha^\beta \equiv \beta = \alpha)$

for conditional proof. If we can show:

(a) φ

(b) $\forall\beta(\varphi_\alpha^\beta \rightarrow \beta = \alpha)$

then by &I we are done. (a) Since, by hypothesis, β is substitutable for α in φ and isn't free in φ, it follows by the Re-replacement lemma (81.1) that α is substitutable for β in φ_α^β. Hence α is substitutable for β in $\varphi_\alpha^\beta \equiv \beta = \alpha$. So we may, by Rule $\forall$E (77.2), instantiate $\forall\beta$ in (ϑ) to α, and thereby obtain:

$$(\varphi_\alpha^\beta \equiv \beta = \alpha)_\beta^\alpha$$

By the definition of $\equiv$ and the definition of ψ_β^α (23), this becomes:

(ξ) $(\varphi_\alpha^\beta)_\beta^\alpha \equiv (\beta = \alpha)_\beta^\alpha$

Since the Re-replacement lemma is operative, the left condition of (ξ) is just φ. By definition, the right condition of (ξ) is $\alpha = \alpha$ (this is obvious, but we leave the strict proof, by way of the cases in the definition of $=$, as an exercise). Hence, (ξ) resolves to:

$$\varphi \equiv \alpha = \alpha$$

But since we know $\alpha = \alpha$ by (71.1), it follows that φ, by biconditional syllogism. (b) By (83.2), it follows from (ϑ) that:

$$\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha) \,\&\, \forall\beta(\beta = \alpha \rightarrow \varphi^{\beta}_{\alpha})$$

So $\forall\beta(\varphi^{\beta}_{\alpha} \rightarrow \beta = \alpha)$ follows by &E. ⋈

(86.11) (Exercise)

(86.12) Assume $\neg\exists\alpha\varphi \,\&\, \neg\exists\alpha\psi$. From the first conjunct and (86.4) it follows that $\forall\alpha\neg\varphi$, and from the second conjunct and the same theorem it follows that $\forall\alpha\neg\psi$. Hence, by Rule $\forall$E, it follows, respectively, that $\neg\varphi$ and $\neg\psi$, which by &I gives us $\neg\varphi \,\&\, \neg\psi$. By $\vee$I (64.3.b), we may infer $(\varphi \,\&\, \psi) \vee (\neg\varphi \,\&\, \neg\psi)$. So by (63.5.i), we know $\varphi \equiv \psi$. Since α isn't free in our assumption, it follows by GEN that $\forall\alpha(\varphi \equiv \psi)$. ⋈

(86.13) (Exercise)

(88) Suppose $\Gamma \vdash \varphi$, i.e., that there is a derivation of φ from Γ. We show by induction on the length of the derivation that $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$, i.e., that there is a derivation of $\mathcal{A}\varphi$ from $\mathcal{A}\Gamma$.

Base Case. If $n = 1$, then the derivation of φ from Γ consists of a single formula, namely, φ itself. So, by the definition of $\Gamma \vdash \varphi$, φ must be in $\mathbf{\Lambda} \cup \Gamma$. So either (a) φ is in $\mathbf{\Lambda}$ or (b) φ is in Γ. If (a), then φ is an axiom and so its actualization closure $\mathcal{A}\varphi$ is an axiom. So $\vdash \mathcal{A}\varphi$ by (47.1) and $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$ by (47.3). If (b), then $\mathcal{A}\varphi$ is in $\mathcal{A}\Gamma$, by the definition of $\mathcal{A}\Gamma$. Hence by (47.2), it follows that $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$.

Inductive Case. Suppose that the derivation of φ from Γ is a sequence S of length n, where $n > 1$. Then either $\varphi \in \mathbf{\Lambda} \cup \Gamma$ or φ follows by MP from two previous members of the sequence, namely, $\psi \rightarrow \varphi$ and ψ. If $\varphi \in \mathbf{\Lambda} \cup \Gamma$, then using the reasoning in the base case, it follows that $\mathcal{A}\Gamma \vdash \mathcal{A}\varphi$. If φ follows from previous members $\psi \rightarrow \varphi$ and ψ by MP, then by the definition of a derivation, we know both that $\Gamma \vdash \psi \rightarrow \varphi$ and $\Gamma \vdash \psi$, where these are sequences of less than n. Since our IH is that the theorem holds for all such derivations of length less than n, it follows that:

(a) $\mathcal{A}\Gamma \vdash \mathcal{A}(\psi \rightarrow \varphi)$

(b) $\mathcal{A}\Gamma \vdash \mathcal{A}\psi$

Now since it is axiomatic that actuality distributes over conditionals and vice versa (31.2), we know:

$$\vdash \mathcal{A}(\psi \rightarrow \varphi) \equiv (\mathcal{A}\psi \rightarrow \mathcal{A}\varphi)$$

By definition (7.4.c), this is just:

$$\vdash (\mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi)) \,\&\, ((\mathcal{A}\psi \to \mathcal{A}\varphi) \to \mathcal{A}(\psi \to \varphi))$$

So by &E (64.2.a), it follows that:

$$\vdash \mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi)$$

It follows from this by (47.3) that:

$$\mathcal{A}\Gamma \vdash \mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi)$$

So by (47.5), it follows from this and (a) that:

$$\mathcal{A}\Gamma \vdash \mathcal{A}\psi \to \mathcal{A}\varphi$$

And again by (47.5), it follows from this and (b) that:

$$\mathcal{A}\Gamma \vdash \mathcal{A}\varphi \qquad \bowtie$$

(90.1)★ Since $\mathcal{A}\varphi \equiv \varphi$ is an instance of (30)★, it follows by a tautology for biconditionals (63.5.d) that $\neg\mathcal{A}\varphi \equiv \neg\varphi$. ⋈

(90.2)★ If we substitute $\neg\varphi$ for φ on both sides of (90.1)★, we obtain the following instance: $\neg\mathcal{A}\neg\varphi \equiv \neg\neg\varphi$. But it is a tautology that $\neg\neg\varphi \equiv \varphi$, by (63.4.b) and (63.3.h). So by the transitivity of the biconditional (64.6.e), it follows that $\neg\mathcal{A}\neg\varphi \equiv \varphi$. ⋈

(91.1) The T schema $\Box\varphi \to \varphi$ is an axiom, and hence so are its closures. Hence, $\mathcal{A}(\Box\varphi \to \varphi)$ is an axiom. Since actuality distributes over a conditional by the left-to-right direction of (31.2), it follows that $\mathcal{A}\Box\varphi \to \mathcal{A}\varphi$. However, axiom (33.2) is $\Box\varphi \equiv \mathcal{A}\Box\varphi$, from which it follows *a fortiori* that $\Box\varphi \to \mathcal{A}\Box\varphi$. So by hypothetical syllogism, $\Box\varphi \to \mathcal{A}\varphi$. ⋈

(91.2) ($\to$) A tautology of conjunction simplification (63.9.a) is $(\varphi \,\&\, \psi) \to \varphi$. By the Rule of Actualization (88), it follows that $\mathcal{A}((\varphi \,\&\, \psi) \to \varphi)$ and by the left-to-right direction of (31.2), it follows that:

(a) $\mathcal{A}(\varphi \,\&\, \psi) \to \mathcal{A}\varphi$

By analogous reasoning from the other tautology of conjunction simplification (63.9.b), i.e., $(\varphi \,\&\, \psi) \to \psi$, we may similarly infer:

(b) $\mathcal{A}(\varphi \,\&\, \psi) \to \mathcal{A}\psi$

Now for conditional proof, assume $\mathcal{A}(\varphi \,\&\, \psi)$. Then by (a) and (b), respectively, we may conclude both $\mathcal{A}\varphi$ and $\mathcal{A}\psi$. So by &I, $\mathcal{A}\varphi \,\&\, \mathcal{A}\psi$. ($\leftarrow$) The principle of Adjunction (63.10.a) is $\varphi \to (\psi \to (\varphi \,\&\, \psi))$. Since this is a $\Box$-theorem, we may apply RN to obtain:

(c) $\Box(\varphi \to (\psi \to (\varphi \,\&\, \psi)))$

Now theorem (91.1) is that $\Box\chi \to \mathcal{A}\chi$, so it follows from (c) that:

(d) $\mathcal{A}(\varphi \to (\psi \to (\varphi \,\&\, \psi)))$

Distributing actuality over a conditional by the left-to-right direction of (31.2) again, (d) implies:

(e) $\mathcal{A}\varphi \to \mathcal{A}(\psi \to (\varphi \,\&\, \psi))$

By applying the same distribution law to the consequent of (e) we obtain:

(f) $\mathcal{A}(\psi \to (\varphi \,\&\, \psi)) \to (\mathcal{A}\psi \to \mathcal{A}(\varphi \,\&\, \psi))$

It now follows from (e) and (f) by hypothetical syllogism that:

(g) $\mathcal{A}\varphi \to (\mathcal{A}\psi \to \mathcal{A}(\varphi \,\&\, \psi))$

From (g), it follows by Importation (63.8.b) that $(\mathcal{A}\varphi \,\&\, \mathcal{A}\psi) \to \mathcal{A}(\varphi \,\&\, \psi)$. ⋈

(91.3) As an instance of (91.2), we have:

$$\mathcal{A}((\varphi \to \psi) \,\&\, (\psi \to \varphi)) \equiv (\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi))$$

But since $\varphi \equiv \psi$ is, by definition (7.4.c), $(\varphi \to \psi) \,\&\, (\psi \to \varphi)$, we may rewrite the left-side of the biconditional and conclude:

$$\mathcal{A}(\varphi \equiv \psi) \equiv (\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi))$$ ⋈

(91.4) ($\to$) The left-to-right direction of (31.2) is a theorem of which the following are both instances:

(a) $\mathcal{A}(\varphi \to \psi) \to (\mathcal{A}\varphi \to \mathcal{A}\psi)$

(b) $\mathcal{A}(\psi \to \varphi) \to (\mathcal{A}\psi \to \mathcal{A}\varphi)$

So by Double Composition (63.10.e), we may conjoin the antecedents of (a) and (b) into a single conjunctive antecedent and conjoin the consequents of (a) and (b) into a single conjunctive consequent, to obtain:

$$(\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi)) \to ((\mathcal{A}\varphi \to \mathcal{A}\psi) \,\&\, (\mathcal{A}\psi \to \mathcal{A}\varphi))$$

By applying the definition of $\equiv$ to the consequent, the previous claim just is:

$$(\mathcal{A}(\varphi \to \psi) \,\&\, \mathcal{A}(\psi \to \varphi)) \to (\mathcal{A}\varphi \equiv \mathcal{A}\psi)$$

(←) Assume $\mathcal{A}\varphi \equiv \mathcal{A}\psi$, for conditional proof. Then, by definition of $\equiv$ and &E, it follows that $\mathcal{A}\varphi \to \mathcal{A}\psi$ and $\mathcal{A}\psi \to \mathcal{A}\varphi$. But by the right-to-left direction of axiom (31.2), the first implies $\mathcal{A}(\varphi \to \psi)$ and the second implies $\mathcal{A}(\psi \to \varphi)$. So by &I, we are done. ⋈

(91.5) (→) By biconditional syllogism (64.6.e) from theorems (91.3) and (91.4). ⋈

(91.6) As an instance of axiom (33.2), we know $\Box\neg\varphi \equiv \mathcal{A}\Box\neg\varphi$. By a classical tautology (63.5.d), it follows that $\neg\Box\neg\varphi \equiv \neg\mathcal{A}\Box\neg\varphi$. Now as an instance of axiom (31.1), we know independently that $\neg\mathcal{A}\Box\neg\varphi \equiv \mathcal{A}\neg\Box\neg\varphi$. So by the transitivity of the biconditional (64.6.e) it follows that $\neg\Box\neg\varphi \equiv \mathcal{A}\neg\Box\neg\varphi$. Applying the definition of $\Diamond$ to both sides, we obtain $\Diamond\varphi \equiv \mathcal{A}\Diamond\varphi$. ⋈

(91.7) (→) This direction is just axiom (33.1). (←) This direction is an instance of the T schema. ⋈

(91.8) Assume $\mathcal{A}\Box\varphi$. Then by the right-to-left direction of axiom (33.2), it follows that $\Box\varphi$. So by (91.1), it follows that $\mathcal{A}\varphi$. But then by (33.1), it follows that $\Box\mathcal{A}\varphi$. ⋈

(91.9) Assume $\Box\varphi$. From this and axiom (33.2), it follows by biconditional syllogism that $\mathcal{A}\Box\varphi$. From this latter and theorem (91.8), it follows that $\Box\mathcal{A}\varphi$. ⋈

(91.10) (→) Assume $\mathcal{A}(\varphi \vee \psi)$, for conditional proof. But now assume, for reductio, $\neg(\mathcal{A}\varphi \vee \mathcal{A}\psi)$. Then by a De Morgan's Law (63.6.d), it follows that $\neg\mathcal{A}\varphi \& \neg\mathcal{A}\psi$. By &E, we have both $\neg\mathcal{A}\varphi$ and $\neg\mathcal{A}\psi$. These imply, respectively, by axiom (31.1) and biconditional syllogism (64.6.b), that $\mathcal{A}\neg\varphi$ and $\mathcal{A}\neg\psi$. We may conjoin these by &I to produce $\mathcal{A}\neg\varphi \,\&\, \mathcal{A}\neg\psi$, and by an appropriate instance of theorem (91.2), namely, $\mathcal{A}(\neg\varphi \,\&\, \neg\psi) \equiv \mathcal{A}\neg\varphi \,\&\, \mathcal{A}\neg\psi$, it follows by biconditional syllogism that:

(a) $\mathcal{A}(\neg\varphi \,\&\, \neg\psi)$

Now, independently, by the commutativity of $\equiv$ (63.3.h), we may transform an instance of De Morgan's law (63.6.d) to obtain $(\neg\varphi \,\&\, \neg\psi) \equiv \neg(\varphi \vee \psi)$ as a theorem. So we may apply the Rule of Actualization to this instance to obtain:

(b) $\mathcal{A}((\neg\varphi \,\&\, \neg\psi) \equiv \neg(\varphi \vee \psi))$

Hence, from (b) it follows by an appropriate instance of (91.5) that:

(c) $\mathcal{A}(\neg\varphi \,\&\, \neg\psi) \equiv \mathcal{A}\neg(\varphi \vee \psi))$

From (a) and (c), it follows by biconditional syllogism that $\mathcal{A}\neg(\varphi \vee \psi)$. But by axiom (31.1), it follows that $\neg\mathcal{A}(\varphi \vee \psi)$, which contradicts our initial assumption. Hence, we may conclude by reductio (RAA) version (62.1) that $\mathcal{A}\varphi \vee \mathcal{A}\psi$. (←) Exercise. ⋈

(91.11) As an instance of axiom (31.3), we have:

(a) $\mathcal{A}\forall\alpha\neg\varphi \equiv \forall\alpha\mathcal{A}\neg\varphi$

By the tautology $(\psi \equiv \chi) \equiv (\neg\psi \equiv \neg\chi)$, it follows from (a) that:

(b) $\neg\mathcal{A}\forall\alpha\neg\varphi \equiv \neg\forall\alpha\mathcal{A}\neg\varphi$

Now independently, as an instance of axiom (31.1), we know:

(c) $\mathcal{A}\neg\forall\alpha\neg\varphi \equiv \neg\mathcal{A}\forall\alpha\neg\varphi$

So from (c) and (b), it follows by transitivity of the biconditional that:

(d) $\mathcal{A}\neg\forall\alpha\neg\varphi \equiv \neg\forall\alpha\mathcal{A}\neg\varphi$

Now, independently, since $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$ is a $\Box$-theorem (31.1), we may apply GEN to obtain: $\forall\alpha(\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi)$. By a theorem of quantification theory (83.3), it follows that $\forall\alpha\mathcal{A}\neg\varphi \equiv \forall\alpha\neg\mathcal{A}\varphi$. So by our tautology $(\psi \equiv \chi) \equiv (\neg\psi \equiv \neg\chi)$, we have:

(e) $\neg\forall\alpha\mathcal{A}\neg\varphi \equiv \neg\forall\alpha\neg\mathcal{A}\varphi$

So from (d) and (e) it follows by the transitivity of the biconditional that:

(f) $\mathcal{A}\neg\forall\alpha\neg\varphi \equiv \neg\forall\alpha\neg\mathcal{A}\varphi$

But applying the definition of $\exists$ to both sides of (f) yields:

(g) $\mathcal{A}\exists\alpha\varphi \equiv \exists\alpha\mathcal{A}\varphi$ ⋈

(92)★ Let φ be any formula in which z is substitutable for x and doesn't occur free. Before we prove our theorem, we first establish some simple facts. By axiom (30)★, we know $\mathcal{A}\varphi \equiv \varphi$, and by the commutativity of the biconditional (63.3.h), that $\varphi \equiv \mathcal{A}\varphi$. By GEN, it follows, respectively, that $\forall x(\mathcal{A}\varphi \equiv \varphi)$ and $\forall x(\varphi \equiv \mathcal{A}\varphi)$. Given our hypothesis about z, it follows, respectively, by (83.12), that $[\forall z(\mathcal{A}\varphi \equiv \varphi)]^z_x$ and $[\forall z(\varphi \equiv \mathcal{A}\varphi)]^z_x$. From these two claims, it follows, respectively, by the definition of substitution (23) that:

(ξ) $\forall z(\mathcal{A}\varphi^z_x \equiv \varphi^z_x)$

(ζ) $\forall z(\varphi^z_x \equiv \mathcal{A}\varphi^z_x)$

With these last two facts our theorem follows simply: ($\rightarrow$) Assume $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. From ($\zeta$) and our assumption, it follows by (83.10) that $\forall z(\varphi^z_x \equiv z = x)$. ($\leftarrow$) Assume $\forall z(\varphi^z_x \equiv z = x)$. From ($\xi$) and our assumption, it follows, again appealing to (83.10), that $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. ⋈

(93)★ Consider any φ in which z is substitutable for x and doesn't occur free. By axiom (34), we know $x = \imath x\varphi \equiv \forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. But by our previous theorem

(92)★, we know that $\forall z(\mathcal{A}\varphi^z_x \equiv z = x) \equiv \forall z(\varphi^z_x \equiv z = x)$. Hence, by the transitivity of $\equiv$ (64.6.e), it follows that $x = \imath x\varphi \;\equiv\; \forall z(\varphi^z_x \equiv z = x)$. ⋈

(94)★ Consider any φ in which z is substitutable for x and doesn't occur free. Then by the fundamental theorem (93)★ for descriptions, we know:

(a) $x = \imath x\varphi \;\equiv\; \forall z(\varphi^z_x \equiv z = x)$

But, given our hypothesis that z is substitutable for x in φ and doesn't occur free in φ, we have as an instance of (86.10) that:

$$(\varphi \,\&\, \forall z(\varphi^z_x \to z = x)) \equiv \forall z(\varphi^z_x \equiv z = x)$$

From this last claim, it follows by the commutativity of $\equiv$ that:

(b) $\forall z(\varphi^z_x \equiv z = x) \equiv (\varphi \,\&\, \forall z(\varphi^z_x \to z = x))$

So by biconditional syllogism from (a) and (b) it follows that:

$$x = \imath x\varphi \equiv (\varphi \,\&\, \forall z(\varphi^z_x \to z = x))$$ ⋈

(95)★ Suppose (a) ψ is either an exemplification formula $\Pi^n\kappa_1 \ldots \kappa_n$ ($n \geq 1$) or an encoding formula $\kappa_1\Pi^1$, (b) x occurs in ψ and only as one or more of the κ_i ($1 \leq i \leq n$), and (c) z is substitutable for x in φ and doesn't appear free in φ. We want to show:

(a) $\psi^{\imath x\varphi}_x \equiv \exists x(\varphi \,\&\, \forall z(\varphi^z_x \to z = x) \,\&\, \psi)$

Our strategy will be to use Hintikka's schema (94)★. Since z is substitutable for x in φ and doesn't appear free in φ, we know that Hintikka's schema applies to φ, so that we have:

$$x = \imath x\varphi \;\equiv\; (\varphi \,\&\, \forall z(\varphi^z_x \to z = y))$$

By GEN, it follows that:

(b) $\forall x(x = \imath x\varphi \equiv (\varphi \,\&\, \forall z(\varphi^z_x \to z = y)))$

(b) will be used in proving both directions of (a). ($\to$) Assume:

(c) $\psi^{\imath x\varphi}_x$

for conditional proof. By hypothesis, ψ is an exemplification or encoding formula and so it follows immediately by axiom (29.5), where y is some variable that doesn't occur free in φ, that:

(d) $\exists y(y = \imath x\varphi)$

Assume that a is an arbitrary such object, so that we know $a = \imath x\varphi$. If we instantiate (b) to a by Rule $\forall$E (77.2), we obtain:

$$a = \imath x\varphi \equiv \varphi^a_x \,\&\, \forall z(\varphi^z_x \rightarrow z = a)$$

It follows that:

(e) $\varphi^a_x \,\&\, \forall z(\varphi^z_x \rightarrow z = a)$

by biconditional syllogism. Note independently that we've established the symmetry of identity for objects (70.2), so that by GEN, we know $\forall x\forall y(x = y \rightarrow y = x)$. In this universal claim, we may instantiate $\forall x$ to a and, given (d) and (29.1), instantiate $\forall y$ to $\imath x\varphi$, thereby inferring from the assumption that $a = \imath x\varphi$ that $\imath x\varphi = a$. From this and (c) it follows by Rule SubId Special Case (74.2) that ψ^a_x (i.e., the result of substituting a for *all* the occurrences of $\imath x\varphi$ in $\psi^{\imath x\varphi}_x$). Conjoining this last result with (e) by &I we obtain:

$$\varphi^a_x \,\&\, \forall z(\varphi^z_x \rightarrow z = a) \,\&\, \psi^a_x$$

Hence, by ∃I, our desired conclusion follows:

$$\exists x(\varphi \,\&\, \forall z(\varphi^z_x \rightarrow z = x) \,\&\, \psi)$$

Since we've inferred this conclusion from the assumption that $a = \imath x\varphi$, where a is arbitrary, we may discharge the supposition to reach our conclusion from (d), by ∃E (85). (←) Assume, for conditional proof:

(f) $\exists x(\varphi \,\&\, \forall z(\varphi^z_x \rightarrow z = x) \,\&\, \psi)$

Assume b is an arbitrary such object, so that we know:

(g) $\varphi^b_x \,\&\, \forall z(\varphi^z_x \rightarrow z = b) \,\&\, \psi^b_x$

By instantiating b into (b), we have:

(h) $b = \imath x\varphi \equiv \varphi^b_x \,\&\, \forall z(\varphi^z_x \rightarrow z = b)$

If we now detach the first two conjuncts of (g) from the third conjunct by one application of &E, we have:

(i) $\varphi^b_x \,\&\, \forall z(\varphi^z_x \rightarrow z = b)$

(j) ψ^b_x

From (i) and (h) it follows by biconditional syllogism that:

(k) $b = \imath x\varphi$

From (k) and (j), it follows by Rule SubId Special Case (74.2) that $\psi^{\imath x\varphi}_x$. Thus, we may discharge (g) to reach this same conclusion from (f) by ∃E (85). ⋈

(96)★ Let φ be any formula in which y doesn't occur free. (→) Assume, for conditional proof:

(a) $\exists y(y = \imath x\varphi)$

Assume further that a is an arbitrarily chosen such object, so that we know:

(b) $a = \imath x\varphi$

Now we want to show $\exists!x\varphi$. By definition (87.1), we have to show, where ν is some individual variable substitutable for x in φ and doesn't occur free in φ, $\exists x(\varphi \,\&\, \forall\nu(\varphi^{\nu}_{x} \to \nu = x))$. Without loss of generality, suppose ν is the variable z. Then by applying GEN to the Hintikka schema (94)★ and instantiating to a, we know:

$$a = \imath x\varphi \equiv (\varphi^{a}_{x} \,\&\, \forall z(\varphi^{z}_{x} \to z = a))$$

From this and (b), it follows by biconditional syllogism that:

$$\varphi^{a}_{x} \,\&\, \forall z(\varphi^{z}_{x} \to z = a)$$

By $\exists$I, it follows that $\exists x(\varphi \,\&\, \forall z(\varphi^{z}_{x} \to z = x))$. By Rule $\exists$E (85), we may discharge the assumption that $a = \imath x\varphi$ and conclude that $\exists x(\varphi \,\&\, \forall z(\varphi^{z}_{x} \to z = x))$ follows from (a) alone. ($\leftarrow$) Reverse the reasoning. ⋈

(97.1)★ Assume $x = \imath x\varphi$. Then by Hintikka's schema (94)★, it follows that $\varphi \,\&\, \forall z(\varphi^{z}_{x} \to z = x)$, provided z is substitutable for x in φ. *A fortiori*, φ. ⋈

(97.2)★ By applying GEN to the previous theorem (97.1)★, we know $\forall x(x = \imath x\varphi \to \varphi)$. Since z is substitutable for x in φ, it is substitutable for x in $x = \imath x\varphi \to \varphi$. So it follows by Rule UE that $[x = \imath x\varphi \to \varphi]^{z}_{x}$. Since the free occurrences of x in φ are bound by $\imath x$ in the description $\imath x\varphi$, the last result resolves to: $z = \imath x\varphi \to \varphi^{z}_{x}$. ⋈

(97.3)★ Let φ be any formula in which y doesn't occur free. Then we may assume, for conditional proof:

(a) $\exists y(y = \imath x\varphi)$

Suppose c is an arbitrary such object, so that we know:

(b) $c = \imath x\varphi$

Since the previous theorem (97.2)★ is generalizable to any z, instantiate the generalization to c, so that we obtain φ^{c}_{x}. But by (b) and Rule SubId, it follows that $\varphi^{\imath x\varphi}_{x}$. So by Rule $\exists$E, we can discharge our hypothesis (b) and conclude that $\varphi^{\imath x\varphi}_{x}$ follows from (a) alone. ⋈

(98) Let φ be any formula in which z is substitutable for x and doesn't occur free. ($\to$) Axiom (31.3) is $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$ and by the commutativity of $\equiv$, $\mathcal{A}\mathcal{A}\varphi \equiv \mathcal{A}\varphi$ is a theorem. By GEN, it follows, respectively, that $\forall x(\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi)$ and $\forall x(\mathcal{A}\mathcal{A}\varphi \equiv \mathcal{A}\varphi)$. Given our hypothesis about z, it follows, respectively, by (83.12), that $[\forall z(\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi)]^{z}_{x}$ and $[\forall z(\mathcal{A}\mathcal{A}\varphi \equiv \mathcal{A}\varphi)]^{z}_{x}$. By the definition of substitution (23), it follows, respectively, that:

(ξ) $\forall z(\mathcal{A}\varphi^z_x \equiv \mathcal{A}\mathcal{A}\varphi^z_x)$

(ζ) $\forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv \mathcal{A}\varphi^z_x)$

($\rightarrow$) Assume $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$. From ($\zeta$) and this assumption, it follows that $\forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv z = x)$, by (83.10). ($\leftarrow$) Assume $\forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv z = x)$. From ($\xi$) and this assumption, it follows that $\forall z(\mathcal{A}\varphi^z_x \equiv z = x)$, also by (83.10). ⋈

(99.1) We may reason as follows:

$$\begin{array}{rcll} x = \imath x\varphi & \equiv & \forall z(\mathcal{A}\varphi^z_x \equiv z = x) & \text{by axiom (34)} \\ & \equiv & \forall z(\mathcal{A}\mathcal{A}\varphi^z_x \equiv z = x) & \text{by theorem (98)} \\ & \equiv & x = \imath x\mathcal{A}\varphi & \text{by axiom (34)} \end{array}$$

⋈

(99.2) Assume $\exists y(y = \imath x\varphi)$, where y isn't free in φ. Independently, from the previous theorem (99.1), it follows by GEN that $\forall x(x = \imath x\varphi \;\equiv\; x = \imath x\mathcal{A}\varphi)$. Since $\imath x\varphi$ is substitutable for x in the matrix of this last universal claim, our assumption and axiom (29.1) allows us to instantiate $\forall x$ to $\imath x\varphi$, to obtain $\imath x\varphi = \imath x\varphi \;\equiv\; \imath x\varphi = \imath x\mathcal{A}\varphi$. But for reasons already mentioned, we can also instantiate $\imath x\varphi$ into $\forall x(x = x)$ — the latter being the universal generalization of theorem (70.1) — to obtain $\imath x\varphi = \imath x\varphi$. Hence $\imath x\varphi = \imath x\mathcal{A}\varphi$. ⋈

(100.1) For conditional proof, assume $\psi^{\imath x\varphi}_x$, where ψ is either an exemplification formula $\Pi^n\kappa_1\ldots\kappa_n$ $(n \geq 1)$ or an encoding formula $\kappa_1\Pi^1$, and (b) x occurs in ψ and only as one or more of the κ_i $(1 \leq i \leq n)$. Then where ν is any individual variable that doesn't occur free in φ, it follows by (29.5) that $\exists\nu(\nu = \imath x\varphi)$. Assume a is an arbitrary such object, so that we have $a = \imath x\varphi$. Independently, apply GEN to theorem (99.1) to obtain $\forall x(x = \imath x\varphi \equiv x = \imath x\mathcal{A}\varphi)$. If we instantiate this to a, we may conclude $a = \imath x\varphi \equiv a = \imath x\mathcal{A}\varphi$. Hence $a = \imath x\mathcal{A}\varphi$, by biconditional syllogism, and so we may conclude $\exists\nu(\nu = \imath x\mathcal{A}\varphi)$. So we've reached this conclusion once we discharge our assumption about a, by Rule $\exists$E. ⋈

(100.2) By (29.5) and (99.2). ⋈

(101.1) For both the left-to-right and the right-to-left directions, follow the proof of (94)★, but instead of appealing to the ★-theorem (93)★, appeal to the necessary axiom (34) and reason with respect to $\mathcal{A}\varphi$ instead of φ. ⋈

(101.2) (Exercise)

(102) Assume $\mathcal{A}(\varphi \equiv \psi)$ and that y doesn't occur free in φ or ψ. By GEN, it suffices to show $x = \imath x\varphi \equiv x = \imath x\psi$. ($\rightarrow$) Assume $x = \imath x\varphi$. Then by the strict version of Hintikka's Schema (101.1) it follows that:

(ϑ) $\mathcal{A}\varphi \;\&\; \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x)$

Note that by the right-to-left direction of the strict Hintikka's Schema, to show $x = \imath x\psi$ we have to show:

(ξ) $\mathcal{A}\psi \,\&\, \forall z(\mathcal{A}\psi^z_x \rightarrow z = x)$

By &I, it suffices to show the conjuncts separately. Before we begin, note the following series of consequences that we may infer from our first assumption $\vdash (\varphi \equiv \psi)$:

(a)	$\vdash \mathcal{A}\varphi \equiv \mathcal{A}\psi$	from our initial assumption, by (91.5)
(b)	$\vdash \forall x(\mathcal{A}\varphi \equiv \mathcal{A}\psi)$	from (a) by GEN
(c)	$\vdash \mathcal{A}\varphi^z_x \equiv \mathcal{A}\psi^z_x$	also from (b), by $\forall$E

Now, from (a) and the first conjunct of (ϑ) we obtain the first conjunct of (ξ), i.e., $\mathcal{A}\psi$, by biconditional elimination ($\equiv$E). To show the second conjunct of (ξ), it suffices, by GEN, to show $\mathcal{A}\psi^z_x \rightarrow z = x$. So assume $\mathcal{A}\psi^z_x$, to show $z = x$ by conditional proof. Then by (c) and $\equiv$E, we may infer $\mathcal{A}\varphi^z_x$. But by the second conjunct of (ϑ), it follows that $z = x$. ($\leftarrow$) Exercise. ⋈

(103.1) For both directions, follow the proof of (95)★, but instead of appealing to (94)★, appeal to the modally-strict version of Hintikka's schema (101.1) and reason with respect to $\mathcal{A}\varphi$ instead of φ. ⋈

(103.2) (Exercise)

(104.1) Suppose y doesn't occur free in φ. ($\rightarrow$) Assume $\exists y(y = \imath x\varphi)$. Assume that a is an arbitrary such object, so that we have $a = \imath x\varphi$. Now we have to show $\exists!x\mathcal{A}\varphi$. By definition (87.1), we have to show $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x))$, where z is an individual variable that is substitutable for x in φ and that doesn't occur free in φ. So suppose, without loss of generality, that z is such a variable. Then by applying GEN to theorem (101.1) and instantiating to a, we know:

$$a = \imath x\varphi \equiv \mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = a)$$

So it follows by biconditional syllogism that $\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = a)$. Hence, by $\exists$I, it follows that $\exists x(\mathcal{A}\varphi \,\&\, \forall z(\mathcal{A}\varphi^z_x \rightarrow z = x))$. This last conclusion remains once we discharge our assumption about a by $\exists$E. ($\leftarrow$) Use analogous reasoning in reverse. ⋈

(104.2) Suppose y doesn't occur free in φ. ($\rightarrow$) Assume $\exists y(y = \imath x\mathcal{A}\varphi)$. Assume that b is an arbitrary such object, so that we have $b = \imath x\mathcal{A}\varphi$. Then by theorem (99.1), it follows that $b = \imath x\varphi$. But this now reduces our theorem to the previous case (104.1). ($\leftarrow$) Exercise. Use the previous theorem (104.1) and theorem (99.1). ⋈

(104.3) Suppose y doesn't occur free in φ. Assume $\exists y(y = \imath x\varphi)$. Assume that b is an arbitrary such object, so that we have $b = \imath x\varphi$. Without loss of generality, assume that z is substitutable for x in φ and doesn't occur free in φ. (We can say this becase if z fails these conditions with respect to φ, we know we can pick some other variable and appeal in what follows to a universally-generalized alphabetic-variant of the modally-strict version of Hintikka's schema, which we know now how to prove.) Then by the modally-strict version of Hintikka's schema (101.1) and &E, it follows that $\mathcal{A}\varphi^b_x$. From this and our second assumption it follows by Rule SubId Special Case that $\mathcal{A}\varphi^{\imath x\varphi}_x$. But then this follows from our first assumption by $\exists$E. ⋈

(104.4) (Exercise)

(106.1) Assume $\Gamma \vdash_{\Box} \varphi \to \psi$, i.e., that there is a modally-strict derivation of $\varphi \to \psi$ from Γ. Then the conditions of RN are met and we may apply RN to conclude $\Box\Gamma \vdash \Box(\varphi \to \psi)$. Since instances of the K schema are axioms, we know $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, by (47.1). So $\Box\Gamma \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, by (47.3), and by (47.6), it follows that $\Box\Gamma \vdash \Box\varphi \to \Box\psi$. ⋈

(106.2) Assume $\Gamma \vdash_{\Box} \varphi \to \psi$, i.e., that there is a modally-strict derivation of $\varphi \to \psi$ from Γ. Since the metarules of contraposition (60) apply generally to all derivations, they apply to modally-strict derivations, and so it follows by (60.1) that $\Gamma \vdash_{\Box} \neg\psi \to \neg\varphi$. Hence by RM (106.1), it follows that $\Box\Gamma \vdash \Box\neg\psi \to \Box\neg\varphi$. So, again by our metarule of contraposition (60.1), it follows that $\Box\Gamma \vdash \neg\Box\neg\varphi \to \neg\Box\neg\psi$. But by definition of $\Diamond$, this last result implies: $\Box\Gamma \vdash \Diamond\varphi \to \Diamond\psi$. ⋈

(107.1) By the first axiom (21.1) governing conditionals, we have $\varphi \to (\psi \to \varphi)$. Since this is a $\Box$-theorem, we may apply RM to conclude $\Box\varphi \to \Box(\psi \to \varphi)$. ⋈

(107.2) Since we have the tautology (58.3), i.e., $\neg\varphi \to (\varphi \to \psi)$, as a non-contingent theorem, it follows by RM that $\Box\neg\varphi \to \Box(\varphi \to \psi)$. ⋈

(107.3) ($\to$) A tautology of conjunction simplification (63.9.a) is $(\varphi \,\&\, \psi) \to \varphi$. Since this is a $\Box$-theorem, we may apply RM (106) to obtain:

(a) $\Box(\varphi \,\&\, \psi) \to \Box\varphi$

By analogous reasoning from $(\varphi \,\&\, \psi) \to \psi$ (63.9.b), we obtain:

(b) $\Box(\varphi \,\&\, \psi) \to \Box\psi$

Assume $\Box(\varphi \,\&\, \psi)$ for conditional proof. Then from (a) and (b), respectively, we may infer $\Box\varphi$ and $\Box\psi$. Hence by &I, $\Box\varphi \,\&\, \Box\psi$. ($\leftarrow$) The principle of Adjunction (63.10.a) is $\varphi \to (\psi \to (\varphi \,\&\, \psi))$. Since this is a $\Box$-theorem, we may apply RM to obtain:

(c) $\Box\varphi \to \Box(\psi \to (\varphi \,\&\, \psi))$

The consequent of (c) can be used to form an instance of K (32.1):

(d) $\Box(\psi \rightarrow (\varphi \,\&\, \psi)) \rightarrow (\Box\psi \rightarrow \Box(\varphi \,\&\, \psi))$

By hypothetical syllogism (56.1), if follows from (c) and (d) that $\Box\varphi \rightarrow (\Box\psi \rightarrow \Box(\varphi \,\&\, \psi))$. Then by Importation (63.8.b), it follows that $(\Box\varphi \,\&\, \Box\psi) \rightarrow \Box(\varphi \,\&\, \psi)$. ⋈

(107.4) As an instance of (107.3), we have:

$$\Box((\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)) \equiv (\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi))$$

But, then, by applying the definition of $\equiv$ to the antecedent, we have:

$$\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi))$$ ⋈

(107.5) The following are both instances of the K axiom (32.1):

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

$$\Box(\psi \rightarrow \varphi) \rightarrow (\Box\psi \rightarrow \Box\varphi)$$

So by Double Composition (63.10.e), we may conjoin the two antecedents into a single conjunctive antecedent and conjoin the two consequents into a single conjunctive consequent, to obtain:

$$(\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi)) \rightarrow ((\Box\varphi \rightarrow \Box\psi) \,\&\, (\Box\psi \rightarrow \Box\varphi))$$

By applying the definition of $\equiv$ to the consequent, the previous claim just is:

$$(\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi)) \rightarrow (\Box\varphi \equiv \Box\psi)$$ ⋈

(107.6) Theorem (107.4) is $\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi))$. From this, it follows by definition of the main connective $\equiv$ and an application of &E (64.2.a) that $\Box(\varphi \equiv \psi) \rightarrow (\Box(\varphi \rightarrow \psi) \,\&\, \Box(\psi \rightarrow \varphi))$. But from this and (107.5), it follows by hypothetical syllogism (56.1 that $\Box(\varphi \equiv \psi) \rightarrow (\Box\varphi \equiv \Box\psi)$. ⋈

(107.7) Assume, for conditional proof, $\Box\varphi \,\&\, \Box\psi$, which by &E, yields both $\Box\varphi$ and $\Box\psi$. From the latter, it follows by (107.1) that $\Box(\varphi \rightarrow \psi)$. From the former, it follows by (107.1) that $\Box(\psi \rightarrow \varphi)$. Hence, by &I, it follows that $\Box(\varphi \rightarrow \psi) \& \Box(\psi \rightarrow \varphi)$. So by biconditional syllogism, this last conclusion together with (107.4) imply $\Box(\varphi \equiv \psi)$. ⋈

(108.1) Our global assumption is:

(ξ) $\vdash \Box(\psi \equiv \chi)$

Since instances of the T schema (32.2) are axioms, we know $\vdash \Box(\psi \equiv \chi) \rightarrow (\psi \equiv \chi)$, by (47.1). From this and our global assumption, it follows by (47.6) that:

(ϑ) $\vdash \psi \equiv \chi$

We frequently appeal to either (ξ) or (ϑ) in establishing the following cases of the consequent of the rule:

(.a) Show $\vdash \neg\psi \equiv \neg\chi$. Since instances of (63.5.d) are theorems, we know:

$$\vdash \psi \equiv \chi \equiv \neg\psi \equiv \neg\chi$$

From this and (ϑ), it follows by biconditional syllogism (64.6.a) that:

$$\vdash \neg\psi \equiv \neg\chi$$

(.b) Show $\vdash (\psi \to \theta) \equiv (\chi \to \theta)$. Note that since instances of (63.5.e) are theorems, we know:

$$\vdash (\psi \equiv \chi) \to ((\psi \to \theta) \equiv (\chi \to \theta))$$

From this and (ϑ), it follows by (47.6) that $\vdash (\psi \to \theta) \equiv (\chi \to \theta)$.

(.c) Show $\vdash (\theta \to \psi) \equiv (\theta \to \chi)$. By reasoning analogous to the previous case, but starting with an instance of (63.5.f) instead of (63.5.e).

(.d) Show $\vdash \forall\alpha\psi \equiv \forall\alpha\chi$. From ($\vartheta$), it follows by GEN that $\vdash \forall\alpha(\psi \equiv \chi)$. But by our proof of (83.3), we know:

$$\vdash \forall\alpha(\psi \equiv \chi) \to (\forall\alpha\psi \equiv \forall\alpha\chi)$$

Hence it follows by (47.6) that $\vdash \forall\alpha\psi \equiv \forall\alpha\chi$.

(.e) Show $\vdash \mathcal{A}\psi \equiv \mathcal{A}\chi$. Our proof of (91.1) establishes that:

$$\vdash \Box(\psi \equiv \chi) \to \mathcal{A}(\psi \equiv \chi)$$

From this and our global assumption (ξ), it follows that:

(ζ) $\vdash \mathcal{A}(\psi \equiv \chi)$,

by (47.6).[259] But by (91.5), we know that $\mathcal{A}(\psi \equiv \chi) \equiv (\mathcal{A}\psi \equiv \mathcal{A}\chi)$, so that we know:[260]

$$\vdash \mathcal{A}(\psi \equiv \chi) \equiv (\mathcal{A}\psi \equiv \mathcal{A}\chi)$$

Hence it follows that $\vdash \mathcal{A}\psi \equiv \mathcal{A}\chi$, by (64.6.a).

[259] Although we could have established $\vdash \mathcal{A}(\psi \equiv \chi)$ by citing (ϑ) and appealing to the necessitation-averse axiom (30)$\star$, we have refrained from doing so. If we had done so, our Rule of Substitution would have become a non-strict rule, since its proof would depend on an axiom that fails to be necessarily true. Any conclusion drawn using the rule derived in this manner would have been a $\star$-theorem. By appealing to (91.1), we prove this case without an appeal to any $\star$-theorems.

[260] Again, (91.5) could have been proved using an appeal to (30)$\star$, but for the reasons given in footnote 259, we are relying on the proof that makes no appeal to $\star$-theorems.

(.f) Show $\vdash \Box\psi \equiv \Box\chi$. By our proof of (107.6), we know:

$$\vdash \Box(\psi \equiv \chi) \to (\Box\psi \equiv \Box\chi)$$

But it follows from this an our global assumption (ξ) that $\vdash \Box\psi \equiv \Box\chi$, by biconditional syllogism (64.6.a). ⋈

(108.2) [Informal proof; see below for a strict proof] The informal proof appeals to the fact that (108.1) includes all the cases that *ground* the ways ψ can appear as a *proper* subformula of φ (in the case where ψ just is φ, ψ is trivially a subformula of φ). From (108.1), therefore, it follows that if both (a) φ is any formula that can be generated from ψ using $\neg$, $\to$, $\forall\alpha$, $\mathcal{A}$, and $\Box$, and (b) φ' is generated from χ in exactly the same way that φ is generated from ψ, then if $\Box(\psi \equiv \chi)$ is a theorem, so is $\varphi \equiv \varphi'$. But, by the definition of subformula (8), this applies to every formula φ of our language, since every formula of our language can be generated from exemplification and encoding formulas using $\neg$, $\to$, $\forall\alpha$, $\mathcal{A}$, and $\Box$. ⋈

(108.2) [Strict proof] Assume:

(ϑ) $\vdash \Box(\psi \equiv \chi)$

and let φ' be the result of substituting the formula χ for zero or more occurrences of ψ where the latter is a subformula of φ. We then show, by induction on the complexity of φ, that $\vdash \varphi \equiv \varphi'$. Note that if no occurrences of ψ in φ are replaced by χ, then $\varphi' = \varphi$, and we simply have to show $\vdash \varphi \equiv \varphi$. But $\varphi \equiv \varphi$ is theorem (63.4.a). Note also that if ψ is a subformula of φ because $\psi = \varphi$, then $\varphi' = \chi$ and (ϑ) becomes $\vdash \Box(\varphi \equiv \varphi')$. Since instances of the T schema (32.2) are axioms, we know $\vdash \Box(\varphi \equiv \varphi') \to (\varphi \equiv \varphi')$, by (47.1). It follows that $\vdash \varphi \equiv \varphi'$, by (47.6). So, we may assume in what follows that ψ is a proper subformula of φ (i.e., a subformula of φ not identical with φ), and that in φ', χ has been substituted for at least one occurrence of ψ in φ.

Base Case. φ is an exemplification formula of the form $\Pi^n\kappa_1 \ldots \kappa_n$ ($n \geq 0$) or an encoding formula of the form $\kappa\Pi^1$. Then, φ has no proper subformulas, and the theorem is true trivially by failure of the antecedent.

Inductive Case 1. $\varphi = \neg\theta$. Then $\varphi' = \neg\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. Since we've proved instances of the tautology (63.5.d), we know:

$$\vdash (\theta \equiv \theta') \equiv (\neg\theta \equiv \neg\theta')$$

From this and $\vdash \theta \equiv \theta'$, it follows by biconditional syllogism (64.6.a) that $\vdash \neg\theta \equiv \neg\theta'$, i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 2. $\varphi = \theta \to \omega$. Then φ' must be $\theta' \to \omega'$. Since our IHs are:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \omega \equiv \omega'$

it follows from these and (ϑ) that $\vdash \theta \equiv \theta'$ and $\vdash \omega \equiv \omega'$. Hence by &I (64.1), it follows that:

(a) $\vdash (\theta \equiv \theta') \,\&\, (\omega \equiv \omega)'$

But one can prove (as an exercise) the tautology:

$((\theta \equiv \theta') \,\&\, (\omega \equiv \omega')) \rightarrow ((\theta \rightarrow \omega) \equiv (\theta' \rightarrow \omega'))$

It therefore follows from the theoremhood of this tautology and (a), by (47.6), that:

$\vdash (\theta \rightarrow \omega) \equiv (\theta' \rightarrow \omega')$

i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 3. $\varphi = \forall\alpha\theta$. Then $\varphi' = \forall\alpha\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. So by GEN it follows that $\vdash \forall\alpha(\theta \equiv \theta')$. But by our proof of (83.3), we know:

$\vdash \forall\alpha(\theta \equiv \theta') \rightarrow (\forall\alpha\theta \equiv \forall\alpha\theta')$

Hence it follows by biconditional syllogism (64.6.a) that $\vdash \forall\alpha\theta \equiv \forall\alpha\theta'$, i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 4. $\varphi = \mathcal{A}\theta$. Then $\varphi' = \mathcal{A}\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. So by the Rule of Actualization (88), we have $\vdash \mathcal{A}(\theta \equiv \theta')$. But since (91.5) is a theorem, we know $\vdash \mathcal{A}(\theta \equiv \theta') \equiv (\mathcal{A}\theta \equiv \mathcal{A}\theta')$. So by biconditional syllogism (64.6.a), we have $\vdash \mathcal{A}\theta \equiv \mathcal{A}\theta'$, i.e., $\vdash \varphi \equiv \varphi'$.

Inductive Case 5. $\varphi = \Box\theta$. Then $\varphi' = \Box\theta'$. Since our IH implies:

if $\vdash \Box(\psi \equiv \chi)$, then $\vdash \theta \equiv \theta'$,

it follows from this and (ϑ) that $\vdash \theta \equiv \theta'$. Since no $\star$-theorems were cited to draw this conclusion, it follows by the Rule of Necessitation that $\vdash \Box(\theta \equiv \theta')$. But by our proof of (107.6), we know:

$\vdash \Box(\theta \equiv \theta') \rightarrow (\Box\theta \equiv \Box\theta')$

Hence it follows by (47.6) that $\vdash \Box\psi \equiv \Box\chi$, i.e., $\varphi \equiv \varphi'$. ⋈

(109) Assume (a) $\vdash_{\Box} \psi \equiv \chi$, (b) φ' is the result of substituting the formula χ for zero or more occurrences of ψ where the latter is a subformula of φ, and (c) $\Gamma \vdash \varphi$. Then from (a) it follows by RN that $\vdash \Box(\psi \equiv \chi)$. So by (108.2), $\vdash \varphi \equiv \varphi'$. By the definition of $\equiv$ and &E, it follows that $\vdash \varphi \rightarrow \varphi'$. From this last result, it follows by (47.10) that $\varphi \vdash \varphi'$. From this and (c) it follows by (47.8) that $\Gamma \vdash \varphi'$. ⋈

(111) Let φ' be any alphabetic variant of φ. It suffices to prove the (somewhat more easily established) Variant rule $\varphi \dashv\vdash \varphi'$, for by the following argument, we may obtain the stated rule from the Variant:

> $(\rightarrow)$ Assume $\Gamma \vdash \varphi$. From this, and the left-to-right direction of the Variant rule, $\varphi \vdash \varphi'$, it follows by (47.8) that $\Gamma \vdash \varphi'$. $(\leftarrow)$ By analogous reasoning.

So we turn to a proof of the Variant rule by induction on the complexity of φ, with a secondary induction on the complexity of terms τ occurring in φ. Note that it suffices to prove $\varphi \vdash \varphi'$, since alphabetic variance is a symmetric relation. However, sometimes it is useful to deploy the full, bidirectional inductive hypothesis, $\psi \dashv\vdash \psi'$, where ψ is any formula of lesser complexity than φ.

Formula Induction: Base Case. φ is $\Pi^n\kappa_1 \ldots \kappa_n$ $(n \geq 0)$ or $\kappa\Pi^1$. These cases are proved as part of the Term Induction Base Case.

Term Induction: Base Case. The κ_i and Π^n in φ are all simple. Then the only alphabetic variants of φ are φ itself. So $\varphi' = \varphi$, and since $\varphi \vdash \varphi$ by a special case of (47.2), it follows that $\varphi \vdash \varphi'$. This applies not only when φ is $\Pi^n\kappa_1 \ldots \kappa_n$ $(n \geq 0)$ but also when φ is $\kappa\Pi^1$.

Term Induction: Inductive Case 1. *Case A*. φ is $\Pi^n\kappa_1 \ldots \kappa_n$, where Π^n is $[\lambda\nu_1 \ldots \nu_n\, \psi^*]$ and the κ_i are all simple. Then φ' is $[\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$, where $[\lambda\nu_1 \ldots \nu_n\, \psi^*]'$ is some alphabetic variant of Π^n. By α-Conversion (36.1), we know that:

$$[\lambda\nu_1 \ldots \nu_n\, \psi^*] = [\lambda\nu_1 \ldots \nu_n\, \psi^*]'$$

Now assume $[\lambda\nu_1 \ldots \nu_n\psi^*]\kappa_1 \ldots \kappa_n$. By the Rule of Substitution of Alphabetically-Variant Relation Terms (68) it follows that $[\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$. So by conditional proof we have:

$$[\lambda\nu_1 \ldots \nu_n\, \psi^*]\kappa_1 \ldots \kappa_n \rightarrow [\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$$

which by (47.10) yields that:

$$[\lambda\nu_1 \ldots \nu_n\, \psi^*]\kappa_1 \ldots \kappa_n \vdash [\lambda\nu_1 \ldots \nu_n\, \psi^*]'\kappa_1 \ldots \kappa_n$$

i.e., $\varphi \vdash \varphi'$.[261]

Case B. φ is $\kappa\Pi^1$, where Π^1 is $[\lambda\nu\, \psi^*]$ and κ is simple. Then φ' is $\kappa[\lambda\nu\, \psi^*]'$, where $[\lambda\nu\, \psi^*]'$ is an alphabetic variant of Π^1. The conclusion follows by reasoning similar similar to *Case A*, but starting with the α-Conversion instance $[\lambda\nu\, \psi^*] = [\lambda\nu\, \psi^*]'$.

Term Induction: Inductive Case 2. *Case A*. φ is $\Pi^n\kappa_1 \ldots \kappa_n$, where one or more of the κ_i is a description and Π^n is simple. Without loss of generality, suppose κ_1 is the description $\imath x\theta$, so that φ is $\Pi^n\imath x\theta\kappa_2 \ldots \kappa_n$. Then φ' is $\Pi^n(\imath x\theta)'\kappa_2 \ldots \kappa_n$, where $(\imath x\theta)'$ is some alphabetic variant of $\imath x\theta$. So by Metatheorem $\langle 8.3\rangle$, $(\imath x\theta)'$ must have the form $\imath y\theta'^{y}_{x}$, for some variable y that is substitutable for x in θ' and not free in θ'. Then φ' is $\Pi^n\imath y\theta'^{y}_{x}\kappa_2 \ldots \kappa_n$. Note that it is *not* axiomatic that $\imath x\theta = \imath y\theta'^{y}_{x}$, since this identity claim is not valid.[262] To show $\varphi \vdash \varphi'$, assume φ, i.e., $\Pi^n\imath x\theta\kappa_2 \ldots \kappa_n$. Now let ψ be the formula $\Pi^n x\kappa_2 \ldots \kappa_n$, then φ is $\psi^{\imath x\theta}_{x}$ and the following is an instance of the theorem asserting the *necessary* version of Russell's analysis for descriptions (103), where z a variable meeting the requirements of the theorem:[263]

$$\Pi^n\imath x\theta\kappa_2 \ldots \kappa_n \equiv \exists x(\mathcal{A}\theta \;\&\; \forall z(\mathcal{A}\theta^{z}_{x} \rightarrow z = x) \;\&\; \Pi^n x\kappa_2 \ldots \kappa_n)$$

Since we've assumed the left side, we may conclude by biconditional syllogism that:

$$(\xi)\;\; \exists x(\mathcal{A}\theta \;\&\; \forall z(\mathcal{A}\theta^{z}_{x} \rightarrow z = x) \;\&\; \Pi^n x\kappa_2 \ldots \kappa_n)$$

Now since y is substitutable for x in θ and not free in θ, the following alphabetic-variant of (ξ) follows by (83.12):

$$(\zeta)\;\; \exists y(\mathcal{A}\theta^{y}_{x} \;\&\; \forall z(\mathcal{A}\theta^{z}_{x} \rightarrow z = y) \;\&\; \Pi^n y\kappa_2 \ldots \kappa_n)$$

[261] Strictly speaking, to apply the Rule of Substitution of Alphabetically-Variant Relation terms, we have to let α be an n-place relation variable, χ be $\alpha\kappa_1 \ldots \kappa_n$, and $\tau = [\lambda\nu_1 \ldots \nu_n\, \psi^*]$. Then we know:

$$\varphi = \chi^{\tau}_{\alpha}$$

The Rule of Substitution of Alphabetically-Variant Relation Terms (68) then allows us to conclude:

$$\chi^{\tau}_{\alpha} \vdash \chi'$$

where $\tau' = [\lambda\nu_1 \ldots \nu_n\, \psi^*]'$ and χ' is the result of substituting τ' for zero or more occurrences of τ in χ^{τ}_{α}. Since φ' results from φ by substituting τ' for a single occurrence of τ in φ, φ' is such a χ' and it follows that $\varphi \vdash \varphi'$.

[262] As we've remarked upon before, once we expand the definiendum into primitive notation, we can see that the resulting definiens is false in those interpretations where $\imath x\theta$ doesn't have a denotation. For in such interpretations, the exemplification formulas $\imath x\theta =_E \imath y\theta'^{y}_{x}$, $A!\imath x\theta$, and $A!\imath y\theta^{y}_{x}$, all of which appear as conjuncts in the expanded identity claim, are false, making the entire, disjunctive identity claim false.

[263] We cite the necessary version of Russell's analysis to avoid using the $\star$-theorem (95)$\star$ in the proof of this derived rule. Recall the discussion in Remarks (51) and (89) that explains why we've studiously avoided the construction of non-strict rules.

Note independently that our IH implies both that $\theta \vdash \theta'$and $\theta' \vdash \theta$. By the Deduction Theorem, it follows from the former that $\vdash \theta \rightarrow \theta'$ and from the latter that $\vdash \theta' \rightarrow \theta$. Hence, by &I (64.1), it follows that $\vdash (\theta' \rightarrow \theta) \,\&\, (\theta \rightarrow \theta')$, and so, $\vdash \theta \equiv \theta'$. Since this is a $\Box$-theorem, it follows from this and (ζ) by the Rule of Substitution that:

(ζ') $\exists y(\mathcal{A}\theta'^{y}_{x} \,\&\, \forall z(\mathcal{A}\theta'^{z}_{x} \rightarrow z = y) \,\&\, \Pi^n y\kappa_2 \ldots \kappa_n)$

Now if we let ψ be the formula $\Pi^n y\kappa_2 \ldots \kappa_n$, then $\psi^{\imath y\theta'^{y}_{x}}_{y}$ is $\Pi^n \imath y\theta'^{y}_{x}\kappa_2 \ldots \kappa_n$ and the following is also an instance of Russell's analysis (103):

$$\Pi^n \imath y\theta'^{y}_{x}\kappa_2 \ldots \kappa_n \equiv \exists y(\mathcal{A}\theta'^{y}_{x} \,\&\, \forall z(\mathcal{A}\theta'^{z}_{x} \rightarrow z = y) \,\&\, \Pi^n y\kappa_2 \ldots \kappa_n)$$

From this last fact and (ζ'), it follows by biconditional syllogism that:

$$\Pi^n \imath y\theta'^{y}_{x}\kappa_2 \ldots \kappa_n$$

i.e., φ'. So we've established that $\varphi \vdash \varphi'$.

Case B. φ is $\kappa\Pi^1$, where κ is a description, say $\imath x\theta$. Then φ' is $\imath x\theta'\Pi^1$, where $\imath x\theta'$ is some alphabetic variant of $\imath x\theta$. Then $\varphi \vdash \varphi'$ follows by reasoning analogous to *Case A*.

Formula Induction: Inductive Case 1. φ is $\neg\psi$, $\Box\psi$ or $\mathcal{A}\psi$. Then, by Metatheorem ⟨8.3⟩, φ' is either $\neg(\psi')$, $\Box(\psi')$ or $\mathcal{A}(\psi')$, where ψ' is some alphabetic variant of ψ. Our IH implies both that $\psi \vdash \psi'$ and $\psi' \vdash \psi$. By the Deduction Theorem, it follows from the former that $\vdash \psi \rightarrow \psi'$ and from the latter that $\vdash \psi' \rightarrow \psi$. Hence, by &I (64.1), it follows that $\vdash (\psi' \rightarrow \psi) \,\&\, (\psi \rightarrow \psi')$, and so by definition, $\vdash \psi \equiv \psi'$. Since this is a $\Box$-theorem, it follows by RN that $\vdash \Box(\psi \equiv \psi')$, and hence by derived Rules of Necessary Equivalence (108.1.a), (108.1.e), and (108.1.f), that:

$$\vdash \neg\psi \equiv \neg\psi'$$

$$\vdash \mathcal{A}\psi \equiv \mathcal{A}\psi'$$

$$\vdash \Box\psi \equiv \Box\psi'$$

as the case may be. Hence, it follows in each case that:

$$\vdash \varphi \equiv \varphi'$$

By definition, this last result is $\vdash (\varphi \rightarrow \varphi') \,\&\, (\varphi' \rightarrow \varphi)$. So by &E (64.2.a) we have $\vdash \varphi \rightarrow \varphi'$ and by (47.10), that $\varphi \vdash \varphi'$.

Formula Induction: Inductive Case 2. φ is $\psi \rightarrow \chi$. Then in light of Metatheorem ⟨8.3⟩(d), φ' is $\psi' \rightarrow \chi'$, where ψ' and χ' are alphabetic variants of ψ and χ, respectively. Our IH implies both that $\psi \dashv\vdash \psi'$ and that $\chi \dashv\vdash \chi'$. By the reasoning used in previous inductive cases, we know that these latter imply:

(a) $\vdash \Box(\psi \equiv \psi')$

(b) $\vdash \Box(\chi \equiv \chi')$

But from (a), it follows by a Rule of Necessary Equivalence (108.2) that:[264]

(c) $\vdash \psi \to \chi \equiv \psi' \to \chi$

And from (b), it follows by this same Rule of Necessary Equivalence (108.2) that:

(d) $\vdash \psi' \to \chi \equiv \psi' \to \chi'$

Hence from (c) and (d), it follows by biconditional syllogism (64.6.e) that:

$$\vdash (\psi \to \chi) \equiv (\psi' \to \chi')$$

i.e., $\vdash \varphi \equiv \varphi'$. Thus, by the reasoning used in at the end of Inductive Case 1 for Formulas, we have established $\varphi \vdash \varphi'$.

Formula Induction: Inductive Case 3. φ is $\forall\alpha\psi$. Then by Metatheorem $\langle 8.3\rangle$, φ' has the form $\forall\beta(\psi'^{\beta}_{\alpha})$, for some ψ' that is an alphabetic variant of ψ and some variable β substitutable for α in ψ' and not free in ψ'. Although, as noted at the outset, it suffices to prove $\varphi \vdash \varphi'$, we also prove $\varphi' \vdash \varphi$, since this particular direction involves an interesting application of the Re-replacement lemma. ($\to$) By our IH, it follows that $\psi \dashv\vdash \psi'$, and so by reasoning developed in Inductive Case 1 for Formulas, it follows that $\vdash \Box(\psi \equiv \psi')$. Hence by a derived Rule of Necessary Equivalence (108.1.d), it follows that $\vdash \forall\alpha\psi \equiv \forall\alpha\psi'$. This in turn implies $\vdash \forall\alpha\psi \to \forall\alpha\psi'$, which by (47.10), yields:

(a) $\forall\alpha\psi \vdash \forall\alpha\psi'$

Now since β is, by hypothesis, a variable substitutable for α, we have as an instance of Rule $\forall$E (77.2) that $\forall\alpha\psi' \vdash \psi'^{\beta}_{\alpha}$. By hypothesis, β isn't free in ψ' and so isn't free in the premise of this last conclusion. So we may apply GEN to infer:

(b) $\forall\alpha\psi' \vdash \forall\beta(\psi'^{\beta}_{\alpha})$

Hence from (a) and (b), we obtain by (47.8) that:

$$\forall\alpha\psi \vdash \forall\beta(\psi'^{\beta}_{\alpha})$$

i.e., $\varphi \vdash \varphi'$. ($\leftarrow$) Assume $\forall\beta(\psi'^{\beta}_{\alpha})$. Since β is, by hypothesis, substitutable for α and doesn't occur free in ψ', it follows by the Re-replacement lemma (81.1) that α is substitutable for β in ψ'^{β}_{α}. By Rule $\forall$E (77.2), we can instantiate our

[264]The meaning of φ' in the Rule of Necessary Equivalence should not be confused with the meaning of φ' in the present theorem.

assumption to α, to obtain $(\psi'^{\beta}_{\alpha})^{\alpha}_{\beta}$, which by Re-replacement lemma (81.1) is just ψ'. Our IH is $\psi \dashv\vdash \psi'$, and so it follows that ψ. So we have therefore established $\forall\beta(\psi'^{\beta}_{\alpha}) \vdash \psi$. Since α isn't free in the premise, it follows by GEN that $\forall\beta(\psi'^{\beta}_{\alpha}) \vdash \forall\alpha\psi$, i.e., $(\forall\alpha\psi)' \vdash \forall\alpha\psi$, i.e., $\varphi' \vdash \varphi$. ⋈

(112.1) Assume φ. Then by the Rule of Alphabetic Variants (111), it follows that φ'. So $\varphi \to \varphi'$, by conditional proof. By analogous reasoning, it follows that $\varphi' \to \varphi$. Hence by &I (64.1), it follows that $(\varphi \to \varphi')\,\&\,(\varphi' \to \varphi)$. So, by definition of $\equiv$, this is just $\varphi \equiv \varphi$. ⋈

(112.2) Assume $\exists y(y = \imath\nu\varphi)$. Now by applying GEN to theorem (70.1), we know $\forall x(x = x)$. So it follows by Rule $\forall$E that $\imath\nu\varphi = \imath\nu\varphi$. Now by definition (35.4), (35.5) and the ensuing discussion in (35), we know that since $\imath\nu\varphi$ and $(\imath\nu\varphi)'$ are alphabetically-variant terms, the formulas $\imath\nu\varphi = \imath\nu\varphi$ and $\imath\nu\varphi = (\imath\nu\varphi)'$ are alphabetic variants. So by the Rule of Alphabetic Variants (111), it follows that $\imath\nu\varphi = (\imath\nu\varphi)'$. ⋈

(113.1) The tautology $\varphi \equiv \neg\neg\varphi$ (63.4.b) is a $\Box$-theorem. So by RN, we have $\Box(\varphi \equiv \neg\neg\varphi)$. Since φ is a subformula of $\Box\varphi$, we may, by a Rule of Necessary Equivalence (108.2), conclude: $\Box\varphi \equiv \Box\neg\neg\varphi$. ⋈

(113.2) ($\rightarrow$) Assume $\neg\Box\varphi$, for conditional proof. We want to show $\Diamond\neg\varphi$. By definition of $\Diamond$, it remains to show $\neg\Box\neg\neg\varphi$. For reductio, assume $\Box\neg\neg\varphi$. From this and (113.1), it follows by biconditional syllogism that $\Box\varphi$, which contradicts our initial assumption. Hence, we may discharge our reductio assumption and conclude by a version of RAA (62.2) that $\neg\Box\neg\neg\varphi$. ($\leftarrow$) Assume $\Diamond\neg\varphi$, i.e., $\neg\Box\neg\neg\varphi$, for conditional proof. We want to show $\neg\Box\varphi$. So, for reductio, assume $\Box\varphi$. From this and (113.1), it follows by biconditional syllogism that $\Box\neg\neg\varphi$, which contradicts our initial assumption. Hence, we may discharge our reductio assumption by a version of RAA (62.1) and conclude that $\neg\Box\varphi$. ⋈

(113.3) [With Rule of Substitution] From tautologies (58.1) and (58.2) it follows by $\equiv$I (64.5) that $\varphi \equiv \neg\neg\varphi$ is a $\Box$-theorem. As an instance of this last fact, we therefore know:

$$(\vartheta)\quad \Box\varphi \equiv \neg\neg\Box\varphi$$

But since $\varphi \equiv \neg\neg\varphi$ is a $\Box$-theorem, we may use the Rule of Substitution to substitute $\neg\neg\varphi$ for the very last occurrence of φ in (ϑ) to obtain:

$$(\xi)\quad \Box\varphi \equiv \neg\neg\Box\neg\neg\varphi$$

But, by definition of $\Diamond$, this implies $\Box\varphi \equiv \neg\Diamond\neg\varphi$. ⋈

(113.3) [Without Rule of Substitution] ($\rightarrow$) Assume $\Box\varphi$, for conditional proof. We want to show $\neg\Diamond\neg\varphi$. For reductio, assume $\Diamond\neg\varphi$. From this and (113.2), it follows by biconditional syllogism (64.6.b) that $\neg\Box\varphi$, which contradicts our initial assumption. Discharging our reductio assumption by RAA, it follows that $\neg\Diamond\neg\varphi$. ($\leftarrow$) Assume $\neg\Diamond\neg\varphi$, for conditional proof. We want to show $\Box\varphi$. Assume $\neg\Box\varphi$, for reductio. From this and (113.2), it follows by biconditional syllogism (64.6.a) that $\Diamond\neg\varphi$, which contradicts our initial assumption. Discharging our reductio assumption, it follows by RAA that $\Box\varphi$. ⋈

(113.4) (Exercise)

(113.5) By the special case of (47.2), we know $(\varphi \to \psi) \vdash_{\Box} (\varphi \to \psi)$. So by RM$\Diamond$, it follows that $\Box(\varphi \to \psi) \vdash \Diamond\varphi \to \Diamond\psi$. Hence, by conditional proof, it follows that $\Box(\varphi \to \psi) \to \Diamond\varphi \to \Diamond\psi$. ⋈

(113.6) As an instance of (107.3), we have:

$$\Box(\neg\varphi \,\&\, \neg\psi) \equiv (\Box\neg\varphi \,\&\, \Box\neg\psi)$$

If follows from this and an appropriate instance of theorem (113.3), by $\equiv$E (64.6.f) that:

$$(\vartheta)\quad \neg\Diamond\neg(\neg\varphi \,\&\, \neg\psi) \equiv (\Box\neg\varphi \,\&\, \Box\neg\psi)$$

Similarly, since $\Box\neg\varphi \equiv \neg\Diamond\varphi$ and $\Box\neg\psi \equiv \neg\Diamond\psi$ are instances of our non-contingent theorem (113.4), we may use two simultaneous applications of the Rule of Substitution to infer from (ϑ) that:

$$\neg\Diamond\neg(\neg\varphi \,\&\, \neg\psi) \equiv (\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

Since De Morgan's Law (63.6.b) is a non-continent theorem, we may transform our last displayed result using the Rule of Substitution into:

$$\neg\Diamond(\varphi \vee \psi) \equiv (\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

By an appropriate instance of (63.5.d), we can negate both sides of the biconditional to obtain:

$$\neg\neg\Diamond(\varphi \vee \psi) \equiv \neg(\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

But given the equivalence of a formula and its double negation (63.4.b), it is a theorem that $\neg\neg\Diamond(\varphi \vee \psi) \equiv \Diamond(\varphi \vee \psi)$. So we may use the Rule of Substitution to conclude:

$$\Diamond(\varphi \vee \psi) \equiv \neg(\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$$

But De Morgan's Law (63.6.b) is a $\Box$-theorem, and so we may, with the Rule of Substitution, use the instance asserting the equivalence of $\neg(\neg\Diamond\varphi \,\&\, \neg\Diamond\psi)$ and $\Diamond\varphi \vee \Diamond\psi$ to obtain:

$$\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Diamond\psi) \qquad ⋈$$

(113.7) By simple conditional proofs and the rules for $\vee$I (64.3.a) and (64.3.b), we can establish the following $\Box$-theorems:

$$\varphi \to (\varphi \vee \psi)$$

$$\psi \to (\varphi \vee \psi)$$

Hence it follows by RM that:

$$\Box\varphi \to \Box(\varphi \vee \psi)$$

$$\Box\psi \to \Box(\varphi \vee \psi)$$

So by an appropriate instance of the tautology (63.10.d), it follows that $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$. ⋈

(113.8) The tautologies of conjunction simplification (63.9) are $(\varphi \,\&\psi) \to \varphi$ and $(\varphi \,\&\psi) \to \psi$. Since these are non-contintent theorems, it follows by RM$\Diamond$ that $\Diamond(\varphi \,\&\psi) \to \Diamond\varphi$ and $\Diamond(\varphi \,\&\psi) \to \Diamond\psi$. So assume $\Diamond(\varphi \,\&\psi)$, for conditional proof. It follows that both $\Diamond\varphi$ and $\Diamond\psi$. So by &I, we have $\Diamond\varphi \,\&\, \Diamond\psi$. Hence, by conditional proof, $\Diamond(\varphi \,\&\psi) \to (\Diamond\varphi \,\&\, \Diamond\psi)$. ⋈

(113.9) As an instance of (113.6), we have $\Diamond(\neg\varphi \vee \psi) \equiv (\Diamond\neg\varphi \vee \Diamond\psi)$. But $\Box$-theorem (113.2) is $\Diamond\neg\varphi \equiv \neg\Box\varphi$. So by the Rule of Substitution, it follows that $\Diamond(\neg\varphi \vee \psi) \equiv (\neg\Box\varphi \vee \Diamond\psi)$. Since it can be established (exercise) as a $\Box$-theorem that $(\neg\varphi \vee \psi) \equiv (\varphi \to \psi)$, we may use the Rule of Substitution to conclude $\Diamond(\varphi \to \psi) \equiv (\neg\Box\varphi \vee \Diamond\psi)$. And since it is an instance of the $\Box$-theorem just assigned as an exercise that $(\neg\Box\varphi \vee \Diamond\psi) \equiv (\Box\varphi \to \Diamond\psi)$, a final application of the Rule of Substitution yields: $\Diamond(\varphi \to \psi) \equiv (\Box\varphi \to \Diamond\psi)$. ⋈

(113.10) By (113.2), it is a $\Box$-theorem that $\neg\Box\varphi \equiv \Diamond\neg\varphi$. Hence, by RN, it follows that $\Box(\neg\Box\varphi \equiv \Diamond\neg\varphi)$. So by (107.6), it follows that $\Box\neg\Box\varphi \equiv \Box\Diamond\neg\varphi$. By the relevant instance of the biconditional tautology (63.5.d), we can negate both sides to obtain: $\neg\Box\neg\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$. And by the definition of $\Diamond$, it follows that $\Diamond\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$. ⋈

(113.11) ($\to$) Assume $\Diamond\Diamond\varphi$. Then, by the definition of the second occurrence of $\Diamond$, this becomes $\Diamond\neg\Box\neg\varphi$. But as an instance of (113.2), we know $\neg\Box\Box\neg\varphi \equiv \Diamond\neg\Box\neg\varphi$. So, by biconditional syllogism (64.6.b), it follows that $\neg\Box\Box\neg\varphi$. So by CP, $\Diamond\Diamond\varphi \to \neg\Box\Box\neg\varphi$. ($\leftarrow$) Assume $\neg\Box\Box\neg\varphi$. Complete the proof by reversing the reasoning in the left-to-right direction. ⋈

(113.12) As an instance of the K axiom (32.1), we know:

$$(\vartheta)\quad \Box(\neg\psi \to \varphi) \to (\Box\neg\psi \to \Box\varphi)$$

Independently, as instances of (63.5.k), we know there is a modally strict proof of the following:

(a) $(\neg\psi \to \varphi) \equiv (\neg\neg\psi \lor \varphi)$

(b) $(\Box\neg\psi \to \Box\varphi) \equiv (\neg\Box\neg\psi \lor \Box\varphi)$

By the modally strict equivalence $\neg\neg\psi \equiv \psi$ and Rule of Substitution, (a) implies the following by a modally strict proof:

(c) $(\neg\psi \to \varphi) \equiv (\psi \lor \varphi)$

So from the (c) and the Rule of Substitution, we can infer the following from (ϑ):

(ξ) $\Box(\psi \lor \varphi) \to (\Box\neg\psi \to \Box\varphi)$

Now by definition of $\Diamond$, (b) implies:

(d) $(\Box\neg\psi \to \Box\varphi) \equiv (\Diamond\psi \lor \Box\varphi)$

So from (ξ) and the left-to-right direction of (d), it follows by hypothetical syllogism that:

(ζ) $\Box(\psi \lor \varphi) \to (\Diamond\psi \lor \Box\varphi)$

But we know that there is a modally strict proof of the fact that disjuncts of a disjunction commute (63.3.e). So, by applying appropriate equivalences and the Rule of Substitution to the antecedent and consequence of (ζ), we have:

$$\Box(\varphi \lor \psi) \to (\Box\varphi \lor \Diamond\psi) \qquad \bowtie$$

(114) As an instance of the T schema (32.2), we know $\Box\neg\varphi \to \neg\varphi$. So, by the derived rule of contraposition (60), it follows that $\neg\neg\varphi \to \neg\Box\neg\varphi$. But as an instance of (58.2), we know $\varphi \to \neg\neg\varphi$. So by hypothetical syllogism (56.1, it follows that $\varphi \to \neg\Box\neg\varphi$. Hence, by the definition of $\Diamond$ in (7.4.e), it follows that $\varphi \to \Diamond\varphi$. $\bowtie$

(115.1) Assume $\Diamond\Box\varphi$. By (113.10), it follows by biconditional syllogism that $\neg\Box\Diamond\neg\varphi$. But note that the following is an instance of the 5 schema: $\Diamond\neg\varphi \to \Box\Diamond\neg\varphi$. So by MT (59.1), it follows that $\neg\Diamond\neg\varphi$, i.e., $\Box\varphi$. Hence, by CP, $\Diamond\Box\varphi \to \Box\varphi$. $\bowtie$

(115.2) ($\to$) Assume $\Box\varphi$. Then by the T$\Diamond$ schema (114), it follows that $\Diamond\Box\varphi$. So by CP, $\Box\varphi \to \Diamond\Box\varphi$. ($\leftarrow$) By (115.1) $\bowtie$

(115.3) ($\to$) $\Diamond\varphi \to \Box\Diamond\varphi$ is just the 5 schema. ($\leftarrow$) Assume $\Box\Diamond\varphi$. Then by the T schema (32.2), it follows that $\Diamond\varphi$. So by CP, $\Box\Diamond\varphi \to \Diamond\varphi$. $\bowtie$

(115.4) By (114), we know $\varphi \to \Diamond\varphi$. And as an instance of the 5 schema (32.3), we know: $\Diamond\varphi \to \Box\Diamond\varphi$. So by hypothetical syllogism (56.1, it follows that $\varphi \to \Box\Diamond\varphi$. $\bowtie$

(115.5) As an instance of the B schema (115.4), we have $\neg\varphi \to \Box\Diamond\neg\varphi$. It follows from this by contraposition that $\neg\Box\Diamond\neg\varphi \to \neg\neg\varphi$. Since the equivalence of $\neg\neg\varphi$ and φ is a $\Box$-theorem, it follows by the Rule of Substitution that:

(ϑ) $\neg\Box\Diamond\neg\varphi \to \varphi$

Now, independently, by (113.10), it is a $\Box$-theorem that $\Diamond\Box\varphi \equiv \neg\Box\Diamond\neg\varphi$, and by the commutativity of $\equiv$, that $\neg\Box\Diamond\neg\varphi \equiv \Diamond\Box\varphi$. So by the Rule of Substitution, we may transform (ϑ) into $\Diamond\Box\varphi \to \varphi$. ⋈

(115.6) $\Box\varphi \to \Box\Diamond\Box\varphi$ is an instance of the B axiom (115.4). Independently, since (115.1), i.e., $\Diamond\Box\varphi \to \Box\varphi$, is a $\Box$-theorem, it follows by RM that $\Box\Diamond\Box\varphi \to \Box\Box\varphi$. So by hypothetical syllogism, $\Box\varphi \to \Box\Box\varphi$. ⋈

(115.7) (Exercise)

(115.8) As an instance of (115.6) we have $\Box\neg\varphi \to \Box\Box\neg\varphi$. By a rule of contraposition, this implies $\neg\Box\Box\neg\varphi \to \neg\Box\neg\varphi$. By the definition of $\Diamond$, this becomes $\neg\Box\Box\neg\varphi \to \Diamond\varphi$. But it is a $\Box$-theorem (113.11) that $\Diamond\Diamond\varphi \equiv \neg\Box\Box\neg\varphi$, which by commutativity of $\equiv$ is $\neg\Box\Box\neg\varphi \equiv \Diamond\Diamond\varphi$. So by the Rule of Substitution, it follows that $\Diamond\Diamond\varphi \to \Diamond\varphi$. ⋈

(115.9) (Exercise)

(115.10) As an instance of (113.12), we know:

$$\Box(\varphi \vee \Box\psi) \to (\Box\varphi \vee \Diamond\Box\psi)$$

Since (115.2) establishes a modally strict equivalence between $\Diamond\Box\psi$ and $\Box\psi$, the Rule of Substitution allows us to infer the following from the above:

(ϑ) $\Box(\varphi \vee \Box\psi) \to (\Box\varphi \vee \Box\psi)$

Now, independently, as an instance of (113.7), we know:

$$(\Box\varphi \vee \Box\Box\psi) \to \Box(\varphi \vee \Box\psi)$$

Since (115.7) establishes a modally strict equivalence between $\Box\Box\psi$ and $\Box\psi$, the Rule of Substitution allows us to infer the following from the above:

(ξ) $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \Box\psi)$

Hence, from (ϑ) and (ξ), it follows by definition of $\equiv$ that:

$$\Box(\varphi \vee \Box\psi) \equiv \Box(\varphi \vee \Box\psi) \qquad ⋈$$

(115.11) (Exercise)

(115.12) As an instance of (115.10), we know:

$$\Box(\neg\varphi \vee \Box\neg\psi) \equiv (\Box\neg\varphi \vee \Box\neg\psi)$$

By (63.5.d), we can negate both sides, to conclude:

$$(\vartheta)\quad \neg\Box(\neg\varphi \vee \Box\neg\psi) \equiv \neg(\Box\neg\varphi \vee \Box\neg\psi)$$

Since $\neg\Box\chi \equiv \Diamond\neg\chi$ (113.2) and $\Box\neg\chi \equiv \neg\Diamond\chi$ (113.4) are modally strict equivalences, we can use the Rule of Substitution multiple times to infer the following from (ϑ):

$$\Diamond\neg(\neg\varphi \vee \neg\Diamond\psi) \equiv \neg(\neg\Diamond\varphi \vee \neg\Diamond\psi)$$

By using the De Morgan law (63.6.a) and applying the the Rule of Substitution to both sides, it follows that:

$$\Diamond(\varphi \,\&\, \Diamond\psi) \equiv (\Diamond\varphi \,\&\, \Diamond\psi) \qquad \bowtie$$

(115.13) (Exercise)

(116.1) Assume $\Gamma \vdash_{\Box} \Diamond\varphi \rightarrow \psi$, i.e., that there is a modally-strict derivation of $\Diamond\varphi \rightarrow \psi$ from Γ. So by the RM, it follows that $\Box\Gamma \vdash \Box\Diamond\varphi \rightarrow \Box\psi$. By (115.4), the instances of the B schema are theorems, so by (47.3) we have $\Box\Gamma \vdash \varphi \rightarrow \Box\Diamond\varphi$. Hence, by (55.1), it follows that $\Box\Gamma \vdash \varphi \rightarrow \Box\psi$. $\bowtie$

(116.2) Assume $\Gamma \vdash_{\Box} \varphi \rightarrow \Box\psi$. Then by RM$\Diamond$ (106.2), it follows that $\Box\Gamma \vdash \Diamond\varphi \rightarrow \Diamond\Box\psi$. But the schema B$\Diamond$ (115.5) is a theorem, and so by (47.3) we know $\Box\Gamma \vdash \Diamond\Box\psi \rightarrow \psi$. Hence, by (55.1) it follows that $\Box\Gamma \vdash \Diamond\varphi \rightarrow \psi$. $\bowtie$

(117.1) As an instance of the $\Box$-theorem (79.1), we have $\forall\alpha\varphi \rightarrow \varphi$. So by RM it follows that $\Box\forall\alpha\varphi \rightarrow \Box\varphi$. By GEN, it follows that $\forall\alpha(\Box\forall\alpha\varphi \rightarrow \Box\varphi)$. But since α isn't free in $\Box\forall\alpha\varphi$, it follows by an appropriate instance of (79.2) that $\Box\forall\alpha\varphi \rightarrow \forall\alpha\Box\varphi$. $\bowtie$

(117.2) As an instance of the Barcan Formula (32.4) we have $\forall\alpha\Box\neg\varphi \rightarrow \Box\forall\alpha\neg\varphi$. By a rule of contraposition, it follows that $\neg\Box\forall\alpha\neg\varphi \rightarrow \neg\forall\alpha\Box\neg\varphi$. Given the instance $\neg\Box\forall\alpha\neg\varphi \equiv \Diamond\neg\forall\alpha\neg\varphi$ of the $\Box$-theorem (113.2), the Rule of Substitution yields $\Diamond\neg\forall\alpha\neg\varphi \rightarrow \neg\forall\alpha\Box\neg\varphi$. And given the instance $\neg\forall\alpha\Box\neg\varphi \equiv \exists\alpha\neg\Box\neg\varphi$ of the $\Box$-theorem (86.2), we may apply the Rule of Substitution again to obtain: $\Diamond\neg\forall\alpha\neg\varphi \rightarrow \exists\alpha\neg\Box\neg\varphi$. We can transform the antecedent using the definition of $\exists$ and the consequent using the definition of $\Diamond$ to obtain: $\Diamond\exists\alpha\varphi \rightarrow \exists\alpha\Diamond\varphi$. $\bowtie$

(117.3) As an instance of (117.1), we have $\Box\forall\alpha\neg\varphi \rightarrow \forall\alpha\Box\neg\varphi$. By a rule of contraposition, it follows that $\neg\forall\alpha\Box\neg\varphi \rightarrow \neg\Box\forall\alpha\neg\varphi$. From the instance $\neg\forall\alpha\Box\neg\varphi \equiv \exists\alpha\neg\Box\neg\varphi$ of the $\Box$-theorem (86.2), we may apply the Rule of Substitution to obtain $\exists\alpha\neg\Box\neg\varphi \rightarrow \neg\Box\forall\alpha\neg\varphi$. And given the instance $\neg\Box\forall\alpha\neg\varphi \equiv \Diamond\neg\forall\alpha\neg\varphi$ of the $\Box$-theorem (113.2), we may again apply the Rule of Substitution to obtain:

$\exists\alpha\neg\Box\neg\varphi \to \Diamond\neg\forall\alpha\neg\varphi$. We can transform the antecedent using the definition $\Diamond$ and the consequent using the definition of $\exists$ to obtain: $\exists\alpha\Diamond\varphi \to \Diamond\exists\alpha\varphi$. ⋈

(117.4) Assume $\exists\alpha\Box\varphi$, for conditional proof. Now let τ be an arbitrary such α, so that we have $\Box\varphi^{\tau}_{\alpha}$ (i.e., τ is an arbitrary constant that is substitutable for, and has the same type as the variable α in φ). Independently, note that from the $\Box$-free premise φ^{τ}_{α} it follows by $\exists$I (84.2) that $\exists\alpha\varphi$. Hence, by RN, the premise $\Box\varphi^{\tau}_{\alpha}$ implies the conclusion $\Box\exists\alpha\varphi$. Hence by Rule $\exists$E, (85), the premise $\exists\alpha\Box\varphi$ implies the conclusion $\Box\exists\alpha\varphi$. So by conditional proof, $\exists\alpha\Box\varphi \to \Box\exists\alpha\varphi$. ⋈

(117.5) As an instance of the $\Box$-theorem (79.1), we know $\forall\alpha\varphi \to \varphi$. By RM$\Diamond$ (106.2), then, it follows that $\Diamond\forall\alpha\varphi \to \Diamond\varphi$. So by GEN, it follows that $\forall\alpha(\Diamond\forall\alpha\varphi \to \Diamond\varphi)$. But since α isn't free in $\Diamond\forall\alpha\varphi$, it follows by an appropriate instance of (79.2) that $\Diamond\forall\alpha\varphi \to \forall\alpha\Diamond\varphi$. ⋈

(117.6) From (86.5), by RM$\Diamond$. ⋈

(117.7) Assume:

$$\Box\forall\alpha(\varphi \to \psi) \,\&\, \Box\forall\alpha(\psi \to \chi)$$

to show $\Box\forall\alpha(\varphi \to \chi)$ by conditional proof. Then since a conjunction of necessities implies a necessary conjunction (107.3), it follows that:

$$(\vartheta)\quad \Box(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi))$$

Note, independently, that the following is (83.9):

$$(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \forall\alpha(\varphi \to \chi)$$

Since this is a modally strict theorem, its necessitation follows by RN:

$$(\zeta)\quad \Box[(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \forall\alpha(\varphi \to \chi)]$$

But as an instance of the K axiom (32.1), we know:

$$\begin{aligned}(\xi)\quad &\Box[(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \forall\alpha(\varphi \to \chi)] \to\\ &\quad \Box(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \Box\forall\alpha(\varphi \to \chi)\end{aligned}$$

From (ξ) and (ζ) it follows by MP that:

$$\Box(\forall\alpha(\varphi \to \psi) \,\&\, \forall\alpha(\psi \to \chi)) \to \Box\forall\alpha(\varphi \to \chi)$$

And from this last result and (ϑ), it follows by MP that $\Box\forall\alpha(\varphi \to \chi)$. ⋈

(117.8) By reasoning analogous to (117.7) but starting with (83.10) and using (107.6) instead of the K axiom. ⋈

(118.1) ($\to$) From the $\Box$-theorem (75), it follows *a fortiori* that $\alpha = \beta \to \Box\alpha = \beta$. Since this is a $\Box$-theorem, it follows by (116.2) that $\Diamond\alpha = \beta \to \alpha = \beta$. ($\leftarrow$) This is an instance of the T$\Diamond$ schema (114). ⋈

(118.2) ($\rightarrow$) From (118.1), it follows that $\Diamond\alpha = \beta \rightarrow \alpha = \beta$, by the definition of $\equiv$ and &E. The contraposition of this result is $\neg\alpha = \beta \rightarrow \neg\Diamond\alpha = \beta$. But as an instance of (113.4), we know $\Box\neg\alpha = \beta \equiv \neg\Diamond\alpha = \beta$. So by commuting this equivalence and using it with the Rule of Substitution, it follows that $\neg\alpha = \beta \rightarrow \Box\neg\alpha = \beta$. By applying infix notation, this is equivalent to $\alpha \neq \beta \rightarrow \Box\alpha \neq \beta$. ($\leftarrow$) This is an instance of the T schema. ⋈

(118.3) ($\rightarrow$) From the previous $\Box$-theorem (118.2), it strictly follows that $\alpha \neq \beta \rightarrow \Box\alpha \neq \beta$, by definition of $\equiv$ and &E. So we may apply (116.2) to conclude $\Diamond\alpha \neq \beta \rightarrow \alpha \neq \beta$. ($\leftarrow$) This is an instance of the T$\Diamond$ schema (114). ⋈

(119.1) Before we begin the proof proper, we note the following facts. As an instance of theorem (72.2), we have:

$$\forall x\Box(x = x)$$

Furthermore, the following is an instance of (29.1):

$$\forall x\Box(x = x) \rightarrow (\exists y(y = \imath x\varphi) \rightarrow \Box\imath x\varphi = \imath x\varphi)$$

Hence by MP we know the following fact:

$$(\vartheta)\quad \exists y(y = \imath x\varphi) \rightarrow \Box\imath x\varphi = \imath x\varphi$$

Now to prove our theorem, assume $\exists y(y = \imath x\varphi)$, for conditional proof. Then by MP and (ϑ), we have $\Box\imath x\varphi = \imath x\varphi$. From this last fact and our assumption that $\exists y(y = \imath x\varphi)$, it follows by Rule $\exists$I (84.1) that $\exists y\Box(y = \imath x\varphi)$. ⋈

(119.2) Assume $\exists y(y = \imath x\varphi)$. then by (119.1), it follows that $\exists\Box y(y = \imath x\varphi)$. But then by the Buridan schema (117.4), it follows that $\Box\exists y(y = \imath x\varphi)$. ⋈

(120.1)★ Suppose y doesn't occur free in φ. Now assume $\exists!x\varphi$, for conditional proof. From this and (96)★, it follows by a biconditional syllogism that $\exists y(y = \imath x\varphi)$. So by (119.1), it follows that $\exists y\Box(y = \imath x\varphi)$. ⋈

(120.2)★ (Exercise)

(121.1) ($\rightarrow$) This direction is simply our rigidity of encoding axiom (37). ($\leftarrow$) Assume $\Box xF$. Then by the T schema (32.2), xF. ⋈

(121.2) ($\rightarrow$) Since our axiom (37), i.e., $xF \rightarrow \Box xF$, is a $\Box$-theorem, it follows by the rule (116.2) that $\Diamond xF \rightarrow xF$. ($\leftarrow$) Assume xF. Then, by T$\Diamond$ (114), it follows that $\Diamond xF$. ⋈

(121.3) By biconditional syllogism from (121.2) and (121.1). ⋈

(121.4) ($\rightarrow$) This direction is immediate from an appropriate instance of (107.6), which guarantees $\Box(\varphi \equiv \psi) \rightarrow (\Box\varphi \equiv \Box\psi)$. ($\leftarrow$) For this direction of the proof, we first establish a few facts about the propositional logic of necessity. By appealing to the right-to-left direction of (63.5.i), one can easily establish (exercise) both that:

$$(\varphi \,\&\, \psi) \rightarrow (\varphi \equiv \psi)$$

$$(\neg\varphi \,\&\, \neg\psi) \rightarrow (\varphi \equiv \psi)$$

Since these are both $\Box$-theorems, we may apply RM to both to conclude:

(ϑ) $\Box(\varphi \,\&\, \psi) \rightarrow \Box(\varphi \equiv \psi)$

(ξ) $\Box(\neg\varphi \,\&\, \neg\psi) \rightarrow \Box(\varphi \equiv \psi)$

Now with facts (ϑ) and (ξ) in hand, assume $\Box xF \equiv \Box yG$, for conditional proof. Then by an appropriate instance of (63.5.i), it follows that $(\Box xF \,\&\, \Box yG) \vee (\neg\Box xF \,\&\, \neg\Box yG)$. We now reason by cases, showing both disjuncts lead to the conclusion $\Box(xF \equiv yG)$.

- If $\Box xF \,\&\, \Box yG$, then from this and an appropriate instance of (107.3), which asserts that a necessary conjunction is equivalent to conjunction of necessities, it follows that $\Box(xF \,\&\, yG)$, by a biconditional syllogism. But then it follows from (ϑ) that $\Box(xF \equiv yG)$.

- If $\neg\Box xF \,\&\, \neg\Box yG$, then by &E we have both (a) $\neg\Box xF$ and (b) $\neg\Box yG$. From appropriate instantiations of (121.3) and biconditional syllogism, (a) and (b) imply, respectively, (c) $\Diamond\neg xF$ and (d) $\Diamond\neg yG$. By appropriate instances of (113.4), (c) and (d) become, respectively, $\Box\neg xF$ and $\Box\neg yG$, which gives us $\Box\neg xF \,\&\, \Box\neg yG$ by &I. But we know by (107.3) that a necessary conjunction is equivalent to conjunction of necessities, So it follows that $\Box(\neg xF \,\&\, \neg xG)$, by biconditional syllogism. But then it follows by (ξ) that $\Box(xF \equiv yG)$.

So, by the rule of reasoning by cases (64.4.a), it follows that $\Box(xF \equiv yG)$. ⋈

(121.5) Theorem (121.2) is that $\Diamond xF \equiv xF$. So by a classical tautology (63.5.d), it follows that $\neg\Diamond xF \equiv \neg xF$, which by the commutativity of $\equiv$ implies $\neg xF \equiv \neg\Diamond xF$. Independently, as an instance of (113.5), we know that $\neg\Diamond xF \equiv \Box\neg xF$. So by the transitivity of $\equiv$, it follows that $\neg xF \equiv \Box\neg xF$. ⋈

(121.6) Theorem (121.1) is that $xF \equiv \Box xF$. So by a classical tautology (63.5.d), it follows that $\neg xF \equiv \neg\Box xF$. Independently, as an instance of (113.2), we know that $\neg\Box xF \equiv \Diamond\neg xF$. So by the transitivity of $\equiv$, it follows that $\neg xF \equiv \Diamond\neg xF$, which commutes to $\Diamond\neg xF \equiv \neg xF$. ⋈

(121.7) (Exercise)

(121.8) Given that axiom (38) is $\mathcal{A}xF \rightarrow xF$, we need only show $xF \rightarrow \mathcal{A}xF$, for then our theorem follows by &I and the definition of $\equiv$. But axiom (37) is $xF \rightarrow \Box xF$ and $\Box xF \rightarrow \mathcal{A}xF$ is an instance of theorem (91.1). So by hypothetical syllogism, it follows that $xF \rightarrow \mathcal{A}xF$. ⋈

(122.1) By hypothesis, $y_1, \ldots, y_n$ are substitutable, respectively, for $x_1, \ldots, x_n$ in φ^* and ψ^*, and don't occur free in φ and ψ^*. Now assume:

(ϑ) $\varphi^* \equiv \psi^*$

Since the conditions of the Re-replacement Lemma (81.1) are met by hypothesis, we may reason using β-Conversion (36.2) as follows:

$$\begin{array}{rcll} [\lambda y_1 \ldots y_n\, \varphi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n & \equiv & [\varphi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]^{x_1,\ldots,x_n}_{y_1,\ldots,y_n} & \text{by } \beta\text{-Conversion} \\ & \equiv & \varphi^* & \text{Re-replacement} \\ & \equiv & \psi^* & \text{by } (\vartheta) \\ & \equiv & [\psi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]^{x_1,\ldots,x_n}_{y_1,\ldots,y_n} & \text{Re-replacement} \\ & \equiv & [\lambda y_1 \ldots y_n\, \psi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n & \text{by } \beta\text{-Conversion} \end{array}$$

We've therefore proved from our assumption that:

$$[\lambda y_1 \ldots y_n\, \varphi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n \equiv [\lambda y_1 \ldots y_n\, \psi^{*}{}^{y_1,\ldots,y_n}_{x_1,\ldots,x_n}]x_1 \ldots x_n$$

⋈

(122.2)

(123) This follows from the facts that (1) every instance of the Strengthened β-Conversion is an alphabetic variant of the stated version of β-Conversion (36.2) and (2) alphabetic variants are interderivable (111). ⋈

(124.1) The axiom schema of β-Conversion is:

$$[\lambda y_1 \ldots y_n\, \varphi^*]x_1 \ldots x_n \equiv \varphi^{*}{}^{x_1,\ldots,x_n}_{y_1,\ldots,y_n}$$

But we have taken the closures of this schema as axioms. So the following is an axiom:

$$\Box\forall x_1 \ldots \forall x_n([\lambda y_1 \ldots y_n\, \varphi^*]x_1 \ldots x_n \equiv \varphi^{*}{}^{x_1,\ldots,x_n}_{y_1,\ldots,y_n})$$

Hence, by $\exists$I, it follows that:

$$\exists F^n \Box\forall x_1 \ldots \forall x_n(F^n x_1 \ldots x_n \equiv \varphi^*)$$ ⋈

(124.2) (Exercise)

(125) Assume:

(ϑ) $\forall x(xF \equiv xG)$

as a premise for conditional proof, to show $F = G$. By definition (16.1), we have to show $\Box\forall x(xF \equiv xG)$. By the Barcan formula, it suffices to show $\forall x\Box(xF \equiv xG)$. Now from our initial assumption, it follows by Rule $\forall$E that:

(ζ) $xF \equiv xG$

Now by (121.1), we know both:

(ξ) $xF \equiv \Box xF$

(ω) $xG \equiv \Box xG$

So starting with $\Box xF$, we can appeal to (ξ) (right-to-left), then (ζ) (left-to-right), and then (ω) (left-to-right), to show $\Box xF \equiv \Box xG$. From this, we may infer $\Box(xF \equiv xG)$, by (121.4) (set y in (121.4) to x). Since x isn't free in our initial premise (ϑ), we may apply GEN to conclude $\forall x\Box(xF \equiv xG)$. ⋈

(126.1) As a 0-place instance of η-Conversion (36.3), we have $[\lambda\, p] = p$, where p is a 0-place relation variable. By GEN, it follows that:

(ϑ) $\forall p([\lambda\, p] = p)$

Since λ binds no variables in $[\lambda\, p]$, every propositional formula φ^* in the language is a 0-place relation term that is substitutable for p in the matrix formula of (ϑ), i.e., every φ^* is substitutable for p in $[\lambda\, p] = p$. Moreover, φ^* is not a description. So we may apply our Rule $\forall$E to conclude $[\lambda\, \varphi^*] = \varphi^*$. ⋈

(126.2) By (126.1), we know $[\lambda\, \varphi^*] = \varphi^*$. Moreover, as an instance of the tautology $\psi \equiv \psi$, we know $[\lambda\, \varphi^*] \equiv [\lambda\, \varphi^*]$. (Recall that $[\lambda\, \varphi^*]$ is a 0-place relation term, and hence a formula.) So by Rule SubId (74.2) it follows that $[\lambda\, \varphi^*] \equiv \varphi^*$. ⋈

(130) Assume $\Diamond\neg\forall x(Fx \equiv Gx)$, which by (113.2), implies $\neg\Box\forall x(Fx \equiv Gx)$. For reductio, assume $F = G$. Then by an appropriate instance of our axiom for the substitution of identicals (25), it follows that $\Box\forall x(Fx \equiv Fx) \rightarrow \Box\forall x(Fx \equiv Gx)$. But $Fx \equiv Fx$ is a tautology, which by GEN yields $\forall x(Fx \equiv Fx)$ and by RN yields $\Box\forall F(Fx \equiv Fx)$. Hence, $\Box\forall x(Fx \equiv Gx)$. Contradiction. ⋈

(132.1) By β-conversion (36.2), we know:

(ϑ) $[\lambda y_1 \ldots y_n\, \neg Fy_1 \ldots y_n]x_1 \ldots x_n \equiv \neg Fx_1 \ldots x_n$

Now definition (131.1) is that $\overline{F^n} = [\lambda y_1 \ldots y_n \neg Fy_1 \ldots y_n]$, which by symmetry of relation identity (67.5) yields $[\lambda y_1 \ldots y_n \neg Fy_1 \ldots y_n] = \overline{F^n}$. Hence, by substituting the latter for the former in (ϑ), it follows by substitution of identicals that $\overline{F^n}x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$. ⋈

(132.2) (Exercise)

(132.3) By β-Conversion for 0-place terms (126.2), we know $[\lambda\, \neg p] \equiv \neg p$. But definition (131.2) is $\overline{p} = [\lambda\, \neg p]$, which by symmetry of proposition identity (67.8) yields $[\lambda\, \neg p] = \overline{p}$. So by substitution of identicals it follows that $\overline{p} \equiv \neg p$. ⋈

(132.4) (Exercise)

(132.5) Suppose, for reductio, that $F^n = \overline{F^n}$, which by symmetry of relation identity (67.5) implies $\overline{F^n} = F^n$. Since theorem (132.1) is that $\overline{F^n}x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$, it follows by substitution of identicals that $F^n x_1 \ldots x_n \equiv \neg F^n x_1 \ldots x_n$, which is a contradiction (63.1.b). ⋈

(132.6) Assume for reductio that $p = \overline{p}$, which by symmetry of proposition identity (67.8) implies $\overline{p} = p$. Then, by substitution of identicals into theorem (132.3), i.e., $\overline{p} \equiv \neg p$, it follows that $p \equiv \neg p$, which is a contradiction (63.1.b). ⋈

(132.7) By definition of $\overline{p}$ (131.2), we have $\overline{p} = [\lambda \neg p]$. Moreover, by η-conversion for 0-place terms (126.1), we have $[\lambda\, \neg p] = \neg p$. So it follows by the transitivity of identity for propositions (67.9) that $\overline{p} = \neg p$. ⋈

(132.8) (Exercise)

(134.1) ($\rightarrow$) Assume *NonContingent*(F). Then by the 1-place case of definition (133.3), we know *Necessary*(F)$\vee$*Impossible*(F), and so by definitions (133.1) and (133.2):

(ϑ) $\Box\forall x Fx \vee \Box\forall x \neg Fx$

Now we know by special cases of (132.2) and (132.1), respectively, that the following are $\Box$-theorems: $\neg\overline{F}x \equiv Fx$ and $\overline{F}x \equiv \neg Fx$. Applying the commutativity of $\equiv$ to each, we therefore have the following $\Box$-theorems:

(a) $Fx \equiv \neg\overline{F}x$

(b) $\neg Fx \equiv \overline{F}x$

Hence, by the Rule of Substitution, (ϑ) and (a) yield $\Box\forall x \neg\overline{F}x \vee \Box\forall x \neg Fx$, and from this latter and (b), by the Rule of Substitution, we have:

$$\Box\forall x \neg\overline{F}x \vee \Box\forall x \overline{F}x$$

So by the commutativity of $\vee$, it follows that:

$$\Box\forall x \overline{F}x \vee \Box\forall x \neg\overline{F}x$$

i.e., *Necessary*($\overline{F}$)$\vee$*Impossible*($\overline{F}$), by (133.1) and (133.2). Hence *NonContingent*($\overline{F}$), by (133.3). ($\leftarrow$) Reverse the reasoning. ⋈

(134.2) ($\rightarrow$) Assume *Contingent*(F). By definition (133.4), this is $\neg$(*Necessary*(F)$\vee$ *Impossible*(F)), and so by definitions (133.1) and (133.2):

$$\neg(\Box\forall x Fx \vee \Box\forall x \neg Fx)$$

By De Morgan's Law (63.6.d), it follows that:

$$\neg\Box\forall x Fx \;\&\; \neg\Box\forall x \neg Fx$$

Using (113.2) on both conjuncts, it follows that:

$$\Diamond\neg\forall xFx \ \&\ \Diamond\neg\forall x\neg Fx$$

Using a quantifier-negation $\Box$-theorem (86.2) with the Rule of Substitution on the left conjunct, and the definition of $\exists$ (7.4.d) on the right conjunct, it follows that:

$$\Diamond\exists x\neg Fx \ \&\ \Diamond\exists xFx$$

Finally, by the commutativity of & (63.3.b), it follows that:

$$\Diamond\exists xFx \ \&\ \Diamond\exists x\neg Fx$$

($\leftarrow$) Reverse the reasoning. ⋈

(134.3) ($\rightarrow$) Assume *Contingent*(F). Then by definition (133.4), we know $\neg(\mathit{Necessary}(F) \vee \mathit{Impossible}(F))$, and so by definitions (133.1) and (133.2):

(ϑ) $\neg(\Box\forall xFx \vee \Box\forall x\neg Fx)$

Now we know by special cases of (132.2) and (132.1), respectively, that the following are $\Box$-theorems: $\neg\overline{F}x \equiv Fx$ and $\overline{F}x \equiv \neg Fx$. Applying the commutativity of $\equiv$ to both, we therefore have, respectively:

(a) $Fx \equiv \neg\overline{F}x$

(b) $\neg Fx \equiv \overline{F}x$

Hence, by the Rule of Substitution, (ϑ) and (a) imply $\neg(\Box\forall x\neg\overline{F}x \vee \Box\forall x\neg Fx)$, and the latter and (b) imply, by the Rule of Substitution:

(ξ) $\neg(\Box\forall x\neg\overline{F}x \vee \Box\forall x\overline{F}x)$

Now the following equivalence, based on the commutativity of $\vee$, is a $\Box$-theorem:

$$(\Box\forall x\neg\overline{F}x \vee \Box\forall x\overline{F}x) \vee (\Box\forall x\overline{F}x \vee \Box\forall x\neg\overline{F}x)$$

From this $\Box$-theorem, we can use the Rule of Substitution to transform (ξ) into:

$$\neg(\Box\forall x\overline{F}x \vee \Box\forall x\neg\overline{F}x)$$

i.e., $\neg(\mathit{Necessary}(\overline{F}) \vee \mathit{Impossible}(\overline{F}))$, by (133.1) and (133.2). Hence, *Contingent*($\overline{F}$), by (133.3). ($\leftarrow$) Reverse the reasoning. ⋈

(135.1) By definition (133.1), we have to show $\Box\forall x_1 Lx_1$. Note that as an instance of the $\Box$-theorem (53), we have $E!y \rightarrow E!y$. But the following is an instance of Strengthened β-conversion (123):

$$[\lambda x\, E!x \rightarrow E!x]y \equiv E!y \rightarrow E!y$$

So by biconditional syllogism, we have $[\lambda x\, E!x \rightarrow E!x]y$, i.e., Ly. So, by GEN, $\forall yLy$ is a $\Box$-theorem. So by RN, $\Box\forall yLy$. This conclusion suffices, since it and $\Box\forall x_1 Lx_1$ are alphabetic variants, which by (111) are interderivable. ⋈

(135.2) By the commutativity of $\equiv$, it follows from the 1-place case of theorem (132.2) that $Fx_1 \equiv \neg\overline{F}x_1$. By two applications of GEN, it then follows that $\forall F\forall x_1(Fx_1 \equiv \neg\overline{F}x_1)$. So by $\forall$E, we may instantiate this to L to obtain:

(a) $\forall x_1(Lx_1 \equiv \neg\overline{L}x_1)$

Since theorem (135.1), by definition (133.1), implies $\Box\forall x_1 Lx_1$, it follows that, by an appropriate instance of the T-schema (32.2), that:

(b) $\forall x_1 Lx_1$

Hence using an appropriate instance of (83.3), it follows from (a) and (b) that:

$$\forall x_1\neg\overline{L}x_1$$

Since this is a $\Box$-theorem, it follows by RN that $\Box\forall x_1\neg\overline{L}x_1$, i.e., by definition (133.2), that *Impossible*$(\overline{L})$. ⋈

(135.3) (Exercise)

(135.4) This follows either by theorems (135.3) and (134.1), or by theorem (135.2) and definition (133.3). ⋈

(135.5) (Exercise) [Hint: The simplest way to prove this is to use (135.3), (135.4), and facts about the distinctness of properties that are negations of one another. Another way to prove this is to use (130).]

(136) [Note: The following proof by Uri Nodelman is much simpler than my original.] ($\rightarrow$) We reason may as follows, invoking the Rule of Substitution on each of the three interior reasoning steps on the grounds that the cited principle in these steps are modally strict:

$$\begin{array}{llll}
\Diamond\exists x(Fx \,\&\, \Diamond\neg Fx) & \rightarrow & \exists x\Diamond(Fx \,\&\, \Diamond\neg Fx) & \text{by BF}\Diamond\text{ (117.2)}\\
 & \rightarrow & \exists x(\Diamond Fx \,\&\, \Diamond\neg Fx) & \text{by (115.12)}\\
 & \rightarrow & \exists x(\Diamond\neg Fx \,\&\, \Diamond Fx) & \text{by \&-commutativity (63.3.b)}\\
 & \rightarrow & \exists x\Diamond(\neg Fx \,\&\, \Diamond Fx) & \text{by (115.12)}\\
 & \rightarrow & \Diamond\exists x(\neg Fx \,\&\, \Diamond Fx) & \text{by CBF}\Diamond\text{ (117.3)}
\end{array}$$

($\leftarrow$) By analogous reasoning. ⋈

(137.1) Axiom (32.5) is:

$$\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \,\&\, \Diamond\neg\exists x(E!x \,\&\, \Diamond\neg E!x)$$

So, by &E, we know:

$$\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x)$$

By applying GEN to (136), we also know:

$$\forall F[\Diamond\exists x(Fx \,\&\, \Diamond\neg Fx) \equiv \Diamond\exists x(\neg Fx \,\&\, \Diamond Fx)]$$

and by instantiating $E!$ for $\forall F$, it follows that:

$$\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \equiv \Diamond\exists x(\neg E!x \,\&\, \Diamond E!x)$$

Hence, by biconditional syllogism, it follows that $\Diamond\exists x(\neg E!x \,\&\, \Diamond E!x)$. ⋈

(137.2) In light of (134.2), it suffices to establish:

(ϑ) $\Diamond\exists x E!x \,\&\, \Diamond\exists x\neg E!x$.

Note independently that from (117.6) and $\Diamond(\exists\alpha\varphi \,\&\, \exists\alpha\psi) \rightarrow (\Diamond\exists\alpha\varphi \,\&\, \Diamond\exists\alpha\psi)$, which is an instance of (113.8), it follows by hypothetical syllogism that:

(ξ) $\Diamond\exists\alpha(\varphi \,\&\, \psi) \rightarrow (\Diamond\exists\alpha\varphi \,\&\, \Diamond\exists\alpha\psi)$

But clearly the consequent of (ξ) implies $\Diamond\exists\alpha\varphi$. So it follows by hypothetical syllogism that:

(ζ) $\Diamond\exists\alpha(\varphi \,\&\, \psi) \rightarrow \Diamond\exists\alpha\varphi$

As instances of (ζ), we have both of the following:

(a) $\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \rightarrow \Diamond\exists x E!x$

(b) $\Diamond\exists x(\neg E!x \,\&\, \Diamond E!x) \rightarrow \Diamond\exists x\neg E!x$

But now the left conjunct of (ϑ) follows from (a) and the left conjunct of axiom (32.5) by MP, and the right conjunct of (ϑ) follows from (b) and (137.1) by MP. ⋈

(137.3) From (137.2) and an appropriate instance of (134.3) by biconditonal syllogism. ⋈

(137.4) From (137.2), (137.3), and an appropriate instance of the 1-place case of (132.5). ⋈

(138.1) Assume *NonContingent*(F). By definition (133.3), we have *Necessary*$(F) \vee$ *Impossible*(F). By Double Negation Introduction, it follows that $\neg\neg$(*Necessary*(F) $\vee$*Impossible*(F)). Hence, by the definition of contingent properties (133.4) it follows that $\neg$*Contingent*(F). Now assume, for reductio, that $\exists G$(*Contingent*$(G) \,\&$ $G = F)$. Let P be an arbitrary such property, so that we have *Contingent*$(P) \,\&$ $P = F$. Applying the second conjunct and the substitution of identicals, the first conjunct implies *Contingent*(F), which is a contradiction. Since we've proved a contradiction from $Contingent(P)\ \&\ P = F$, it follows by $\exists$E that the contradiction follows our reductio assumption. Hence, $\neg\exists G(Contingent(G)\ \&\ G = F)$. ⋈

(138.2) (Exercise) [Hint: Use reasoning similar to that of (138.1).]

(138.3) (Exercise)

(138.4) By four applications of $\exists$I to (138.3). ⋈

(139.1) ($\rightarrow$) Assume $NonContingent(p)$. Then by the 0-place case of definition (133.3), we know $Necessary(p) \vee Impossible(p)$, and so by definitions (133.1) and (133.2):

(ϑ) $\Box p \vee \Box\neg p$

We also know (132.4) and (132.3), i.e., that the following are $\Box$-theorems: $\neg\overline{p} \equiv p$ and $\overline{p} \equiv \neg p$. Applying the commutativity of $\equiv$ to both, we therefore have, as $\Box$-theorems:

(a) $p \equiv \neg\overline{p}$

(b) $\neg p \equiv \overline{p}$

From (ϑ) and (a), it follows by the Rule of Substitution that $\Box\neg\overline{p} \vee \Box\neg p$, and from this and (b), the Rule of Substitution implies: $\Box\neg\overline{p} \vee \Box\overline{p}$. By the commutativity of $\vee$, this implies: $\Box\overline{p} \vee \Box\neg\overline{p}$, i.e., $Necessary(\overline{p}) \vee Impossible(\overline{p})$, by (133.1) and (133.2). Hence, $NonContingent(\overline{p})$, by (133.3). ($\leftarrow$) Reverse the reasoning. ⋈

(139.2) Assume $Contingent(p)$. Then by the 0-place case of definition (133.4), we know $\neg(Necessary(p) \vee Impossible(p))$, and so by definitions (133.1) and (133.2):

$$\neg(\Box p \vee \Box\neg p)$$

By De Morgan's Law (63.6.d), it follows that:

$$\neg\Box p\ \&\ \neg\Box\neg p$$

Using (113.2) on the left conjunct and applying the definition of $\Diamond$ to the right conjunct, we obtain:

$$\Diamond\neg p\ \&\ \Diamond p$$

Finally, by the commutativity of & (63.3.b), it follows that:

$$\Diamond p\ \&\ \Diamond\neg p$$

($\leftarrow$) Reverse the reasoning. ⋈

(139.3) ($\rightarrow$) Assume $Contingent(p)$. Then by the 0-place case of definition (133.4), we know $\neg(Necessary(p) \vee Impossible(p))$, and so by definitions (133.1) and (133.2):

(ϑ) $\neg(\Box p \lor \Box\neg p)$

We also know (132.4) and (132.3), i.e., that the following are $\Box$-theorems: $\neg\overline{p} \equiv p$ and $\overline{p} \equiv \neg p$. Applying the commutativity of $\equiv$ to each, we therefore have as $\Box$-theorems:

(a) $p \equiv \neg\overline{p}$

(b) $\neg p \equiv \overline{p}$

From (ϑ) and (a), it follows by the Rule of Substitution that $\neg(\Box\neg\overline{p} \lor \Box\neg p)$, and from this result and (b), it follows by the same rule that:

$$\neg(\Box\neg\overline{p} \lor \Box\overline{p})$$

So we may use an appropriate instance of the commutativity of $\lor$ (which is a $\Box$-theorem) and the Rule of Substitution to transform the last formula into:

$$\neg(\Box\overline{p} \lor \Box\neg\overline{p})$$

i.e., $\neg(\mathit{Necessary}(\overline{p}) \lor \mathit{Impossible}(\overline{p}))$, by (133.1) and (133.2). Hence, $\mathit{Contingent}(\overline{p})$, by (133.3). ($\leftarrow$) Reverse the reasoning. ⋈

(140.1) Since $E!x \to E!x$ is an instance of tautology (53), it follows by GEN that $\forall x(E!x \to E!x)$. Since this is a $\Box$-theorem, it follows by RN that $\Box\forall x(E!x \to E!x)$. By definition of p_0, then, it follows that $\Box p_0$. Hence by the 0-place case of definition (133.1), it follows that $\mathit{Necessary}(p_0)$. ⋈

(140.2) By the reasoning in (140.1), we can establish the following as a theorem:

(a) $\Box p_0$

Note that by the commutativity of $\equiv$, it follows from theorem (132.4) that $p \equiv \neg\overline{p}$, and hence by GEN that $\forall p(p \equiv \neg\overline{p})$. Instantiating this last claim to p_0, we obtain $p_0 \equiv \neg\overline{p_0}$. So from this $\Box$-theorem, we may apply the Rule of Substitution to (a) to obtain $\Box\neg\overline{p_0}$. Hence, by the 0-place case of definition (133.2), it follows that $\mathit{Impossible}(\overline{p_0})$. ⋈

(140.3) (Exercise)

(140.4) (Exercise)

(140.5) (Exercise)

(141.1) (Exercise)

(141.2) (Exercise)

(141.3) (Exercise)

(141.4) (Exercise)

(142.1) Assume *NonContingent*(p). By the 0-place case of definition (133.3), we have *Necessary*$(p) \vee$ *Impossible*(p). By Double Negation Introduction, it follows that $\neg\neg(\mathit{Necessary}(p) \vee \mathit{Impossible}(p))$. Hence, by the definition of contingent properties (133.4) it follows that $\neg\mathit{Contingent}(p)$. Now assume, for reductio, that $\exists q(\mathit{Contingent}(q) \,\&\, q = p)$. Assume further that q_1 is an arbitrary such property, so that $\mathit{Contingent}(q_1) \,\&\, q_1 = p$. Applying the second conjunct and the substitution of identicals, the first conjunct implies *Contingent*(p), which is a contradiction. Since we've proved a contradiction from $\mathit{Contingent}(q_1) \,\&$ $q_1 = p$, we may discharge our second assumption, by $\exists$E, to conclude that a contradiction follows our reductio assumption. Hence, $\neg\exists q(\mathit{Contingent}(q) \,\&$ $q = p)$. ⋈

(142.2) (Exercise)

(142.3) (Exercise)

(142.4) By four applications of $\exists$I to (142.3). ⋈

(143.1) By definition (18), we have to show $\neg(O! = A!)$. For reductio, assume $O! = A!$. By definitions (11.1) and (11.2) it follows that $[\lambda x\, \Diamond E!x] = [\lambda x\, \neg\Diamond E!x]$. Now the following is an instance of Strengthened β-Conversion (123):

$$[\lambda x\, \Diamond E!x]x \equiv \Diamond E!x$$

So by Rule SubId (74.2), it follows that:

(ϑ) $[\lambda x\, \neg\Diamond E!x]x \equiv \Diamond E!x$

But the following is also an instance of Strengthened β-Conversion:

(ξ) $[\lambda x\, \neg\Diamond E!x]x \equiv \neg\Diamond E!x$

So from (ϑ) and (ξ) it follows by biconditional syllogism (64.6.f) that $\Diamond E!x \equiv \neg\Diamond E!x$, which by (63.1.b) is a contradiction. ⋈

(143.2) The following are instances of Strengthened β-Conversion (123):

(a) $[\lambda x\, \Diamond E!x]x \equiv \Diamond E!x$

(b) $[\lambda x\, \neg\Diamond E!x]x \equiv \neg\Diamond E!x$

From (b), it follows by (63.5.d) that:

(c) $\neg[\lambda x\, \neg\Diamond E!x]x \equiv \neg\neg\Diamond E!x$

Note independently that $\Diamond E!x \equiv \neg\neg\Diamond E!x$ an instance of a tautology (63.4.b), which by the commutativity of $\equiv$ (63.8) becomes $\neg\neg\Diamond E!x \equiv \Diamond E!x$. It follows from this and (c) by biconditional syllogism that:

(d) $\neg[\lambda x\, \neg\Diamond E!x]x \equiv \Diamond E!x$

By the commutativity of the biconditional, (d) implies:

(e) $\Diamond E!x \equiv \neg[\lambda x\, \neg\Diamond E!x]x$

So from (a) and (e) it follows by the transitivity of the biconditional that:

(f) $[\lambda x\, \Diamond E!x]x \equiv \neg[\lambda x\, \neg\Diamond E!x]x$

But (f) is, by definitions (11.1), (11.2), just is $O!x \equiv \neg A!x$. ⋈

(143.3) (Exercise)

(143.4) By (134.2), it suffices to establish $\Diamond\exists x O!x \,\&\, \Diamond\exists x \neg O!x$. By &I, it suffices to prove both conjuncts.

To prove the first conjunct, note that by theorem (137.2), *Contingent*($E!$), which by (134.2), is equivalent to: $\Diamond\exists x E!x \,\&\, \Diamond\exists x \neg E!x$. By detaching the first conjunct and applying BF$\Diamond$ (117.2), we obtain $\exists x \Diamond E!x$. This latter, given the Rule of Substitution and the instance of Strengthened β-Conversion (123) that $[\lambda x\, \Diamond E!x]x \equiv \Diamond E!x$ (a $\Box$-theorem), yields $\exists x([\lambda x\, \Diamond E!x]x)$. But by definition of $O!$ (11.1), this is just $\exists x O!x$. So by T$\Diamond$ (114), we have $\Diamond\exists x O!x$.

To prove the second conjunct, pick any instance of the comprehension axiom for abstract objects. It has the form: $\exists x(A!x \,\&\, \ldots)$. By (86.5), it follows that $\exists x A!x$. Now we know by the previous theorem (143.2) that $O!x \equiv \neg A!x$. It is easy to prove from this that $A!x \equiv \neg O!x$ (exercise). This is a $\Box$-theorem and so by the Rule of Substitution, we may infer $\exists x \neg O!x$ from $\exists x A!x$. But by T$\Diamond$, it then follows that $\Diamond\exists x \neg O!x$. ⋈

(143.5) (Exercise)

(143.6) By (18), we have to show $\neg(\overline{O!} = \overline{A!})$. Suppose, for reductio, that $\overline{O!} = \overline{A!}$. By applying the definition of relation negation (131.1) to both terms, it follows that $[\lambda y\, \neg O!y] = [\lambda y\, \neg A!y]$. But as an instance of β-Conversion, we have: $[\lambda y\, \neg O!y]x \equiv \neg O!x$. So by Rule SubId, it follows that $[\lambda y\, \neg A!y]x \equiv \neg O!x$. Since the β-Conversion instance $[\lambda y \neg A!y]x \equiv \neg A!x$ is a $\Box$-theorem, it follows by the Rule of Substitution that $\neg A!x \equiv \neg O!x$ (Note that the Rule of Substitution saves a step here; without it, we have to commute the sides of the last instance of β-Conversion and then apply transivitity of $\equiv$.) But by (143.2), we know $O!x \equiv \neg A!x$, and so by transitivity of the biconditional, it follows that $O!x \equiv \neg O!x$, which by (63.1.b) is a contradiction. Hence $\neg(\overline{O!} = \overline{A!})$, i.e., $\overline{O!} \neq \overline{A!}$. ⋈

(143.7) From (143.2) by (63.5.d), (58) and the Rule of Substitution. ⋈

(143.8) From (143.4) and (134.3). ⋈

(143.9) From (143.5) and (134.3). ⋈

(144.1) Assume $O!x$, for conditional proof. Then by definition of $O!$ (11.1), we know $[\lambda x\, \Diamond E!x]x$. So by Strengthened β-Conversion (123), it follows that $\Diamond E!x$. This implies $\Box\Diamond E!x$, by the 5 schema (32.3). Since the Strengthened β-Conversion (123) instance $[\lambda x\, \Diamond E!x]x \equiv \Diamond E!x$ is a $\Box$-theorem, it follows by the Rule of Substitution that $\Box[\lambda x\, \Diamond E!x]x$, which by the definition of $O!$ yields $\Box O!x$. ⋈

(144.2) Assume $A!x$, for conditional proof. Then by definition of $A!$ (11.2), we know $[\lambda x\, \neg\Diamond E!x]x$. So by Strengthened β-Conversion (123), it follows that $\neg\Diamond E!x$. This implies $\Box\neg E!x$, by (113.4). By the 4 schema (115.6) it follows that $\Box\Box\neg E!x$. Now using our modally-strict equivalence (113.4), we apply the Rule of Substitution to obtain $\Box\neg\Diamond E!x$. By an appropriate instance of Strengthened β-Conversion (123), we can again appeal to the Rule of Substitution to infer $\Box[\lambda x\, \neg\Diamond E!x]x$, which by the definition of $A!$ yields $\Box A!x$. ⋈

(144.3) We may apply Derived Rule (116.2) to the $\Box$-theorem (144.1) to conclude $\Diamond O!x \rightarrow O!x$. ⋈

(144.4) We may apply Derived Rule (116.2) to the $\Box$-theorem (144.2) to conclude $\Diamond A!x \rightarrow A!x$. ⋈

(144.5) ($\rightarrow$) By hypothetical syllogism from (144.3) and (144.1). ($\leftarrow$) By the T and T$\Diamond$ schemata. ⋈

(144.6) ($\rightarrow$) By hypothetical syllogism from (144.4) and (144.2). ($\leftarrow$) By the T and T$\Diamond$ schemata. ⋈

(144.7) ($\rightarrow$) Assume $O!x$. By (144.1), it follows that $\Box O!x$. By (91.1), it follows that $\mathcal{A}O!x$. ($\leftarrow$) Assume $\mathcal{A}O!x$. By definition of $O!$ (11.1), this is just $\mathcal{A}[\lambda x\, \Diamond E!x]x$. Now Strengthened β-Conversion (123) yields the $\Box$-theorem:

(ϑ) $[\lambda x\, \Diamond E!x]x \equiv \Diamond E!x$

So by the Rule of Substitution, it follows that $\mathcal{A}\Diamond E!x$. But then by the right-to-left direction of (91.6), it follows that $\Diamond E!x$. Hence by (ϑ), it follows that $[\lambda x\, \Diamond E!x]x$, i.e., $O!x$, by definition of $O!$. ⋈

(144.8) ($\rightarrow$) Assume $A!x$. By (144.2), it follows that $\Box A!x$. By (91.1), it follows that $\mathcal{A}A!x$. ($\leftarrow$) Assume $\mathcal{A}A!x$. By definition of $A!$ (11.2), this is just $\mathcal{A}[\lambda x\, \neg\Diamond E!x]x$. Now Strengthened β-Conversion (123) yields the $\Box$-theorem:

(ϑ) $[\lambda x\, \neg\Diamond E!x]x \equiv \neg\Diamond E!x$

So by the Rule of Substitution, it follows that $\mathcal{A}\neg\Diamond E!x$. Note independently that it is an instance of a $\Box$-theorem (113.4) that $\Box\neg E!x \equiv \neg\Diamond E!x$, which commutes to $\neg\Diamond E!x \equiv \Box\neg E!x$. So by the Rule of Substitution, it follows that $\mathcal{A}\Box\neg E!x$. By the right-to-left direction of axiom (33.2), it follows that $\Box\neg E!x$, which by our theorem (113.4) implies $\neg\Diamond E!x$. Hence by (ϑ), it follows that $[\lambda x\, \neg\Diamond E!x]x$, i.e., $A!x$, by definition of $A!$. ⋈

(146.1) ($\rightarrow$) Assume *WeaklyContingent*(F). Then by definition (145) and &E, we know both *Contingent*(F) and $\forall x(\Diamond Fx \rightarrow \Box Fx)$. From the former, it follows that *Contingent*($\overline{F}$), by (134.3). So, by the definition of *WeaklyContingent*($\overline{F}$), it remains to establish $\forall x(\Diamond \overline{F}x \rightarrow \Box \overline{F}x)$. Assume $\Diamond \overline{F}x$, to establish $\Box \overline{F}x$ by conditional proof. Since it is a $\Box$-theorem (132.1) that $\overline{F}x \equiv \neg Fx$, it follows by the Rule of Substitution that $\Diamond\neg Fx$, i.e., $\neg\Box Fx$. Now for reductio, assume $\neg\Box\overline{F}x$, i.e., by definition of relation negation, $\neg\Box[\lambda y\, \neg Fy]x$. Since $[\lambda y\, \neg Fy]x \equiv \neg Fx$ is a $\Box$-theorem (it is an instance of β-Conversion), it follows by the Rule of Substitution that $\neg\Box\neg Fx$, i.e., $\Diamond Fx$, by definition of $\Diamond$. Since we already know $\forall x(\Diamond Fx \rightarrow \Box Fx)$, it follows that $\Box Fx$. Contradiction. So, by reductio, $\Box\overline{F}x$, and by conditional proof, $\Diamond\overline{F}x \rightarrow \Box\overline{F}x$. Since we've now discharged the assumption in which the variable x is free, we may invoke GEN to conclude $\forall x(\Diamond\overline{F}x \rightarrow \Box\overline{F}x)$. ($\leftarrow$) Exercise. ⋈

(146.2) (Exercise)

(147.1) By (143.4), we know *Contingent*($O!$). By the left-to-right direction of (144.5), we know $\forall x(\Diamond O!x \rightarrow \Box O!x)$. Hence, by &I and definition (145), *WeaklyContingent*($O!$). ⋈

(147.2) By (143.5), we know *Contingent*($A!$). By the left-to-right direction of (144.6), we know: $\forall x(\Diamond A!x \rightarrow \Box A!x)$. Hence by &I and definition (145), *Weakly-Contingent*($A!$). ⋈

(147.3) (Exercise) [Hint: Show that $E!$ fails the second conjunct in the definition of contingently necessary, i.e., fails to be such that $\forall x(\Diamond E!x \rightarrow \Box E!x)$. Use either (32.5) or (137).]

(147.4) (Exercise) [Hint: Show that L fails the first conjunct in the definition of contingently necessary. Appeal to previous theorems.] ⋈

(147.5) (Exercise)

(147.6) (Exercise)

(147.7) (Exercise)

(148.1) ($\rightarrow$) Assume $x =_E y$, to show $\Box x =_E y$. Then, by theorem (69.1) and &E, we know the following three comma-separated claims:

(ϑ) $O!x,\ O!y,\ \Box\forall F(Fx \equiv Fy)$

By (144.1), the first two claims of (ϑ) imply $\Box O!x$ and $\Box O!y$, respectively. From the third claim in (ϑ), it follows that $\Box\Box\forall F(Fx \equiv Fy)$, by the 4 schema (115.6). Assembling what we have established using &I, we have:

$$\Box O!x \,\&\, \Box O!y \,\&\, \Box\Box\forall F(Fx \equiv Fy)$$

By a basic theorem of K (107.3), a conjunction of necessary truths is equivalent to a necessary conjunction. Hence, it follows that:[265]

(ξ) $\Box(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$

But if we commute □-theorem (69.1), we have the □-theorem:

$$(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)) \equiv x =_E y$$

So by the Rule of Substitution, we can infer from (ξ) that $\Box x =_E y$. Hence, by conditional proof, we've established: $x =_E y \rightarrow \Box x =_E y$. (←) This direction is an instance of the T schema. ⋈

(148.2) (→) From (148.1), it follows *a fortiori* that $x =_E y \rightarrow \Box x =_E y$. Since this is a □-theorem, it follows by (116.2) that $\Diamond x =_E y \rightarrow x =_E y$. (←) This direction is an instance of the T◇ schema. ⋈

(150) We reason as follows:

$$\begin{array}{llll} x \neq_E y & \equiv & \neq_E xy & \text{By (149.2)} \\ & \equiv & \overline{=_E} xy & \text{By (149.1)} \\ & \equiv & [\lambda y_1 y_2\, \neg(=_E y_1 y_2)]xy & \text{By (131.1)} \\ & \equiv & \neg(=_E xy) & \text{By } \beta\text{-Conversion} \\ & \equiv & \neg(x =_E y) & \text{By (13)} \quad ⋈ \end{array}$$

(151.1) We know by (113.4) that $\Box\neg x =_E y \equiv \neg\Diamond x =_E y$. Independently, from (148.2) it follows by biconditional law (63.5.d) that $\neg\Diamond x =_E y \equiv \neg x =_E y$. So by biconditional syllogism, it follows from our first two results that:

(a) $\Box\neg x =_E y \equiv \neg x =_E y$

Given that $\neg x =_E y \equiv x \neq_E y$ is a □-theorem derivable by commuting theorem (150), a single application of the Rule of Substitution to (a) yields $\Box x \neq_E y \equiv x \neq_E y$. By the commutativity of ≡ we are done. ⋈

(151.2) (→) It follows *a fortiori* from (151.1) that $x \neq_E y \rightarrow \Box x \neq_E y$. Since this is a □-theorem, it follows by (116.2), $\Diamond x \neq_E y \rightarrow x \neq_E y$. (←) This is an instance of the T◇ schema. ⋈

(152.1) (→) Assume $x =_E y$. Then by (148.1), it follows that $\Box x =_E y$. So by theorem (91.1), it follows that $\mathcal{A}x =_E y$. (←) Assume $\mathcal{A}x =_E y$. By (69.1), it is a □-theorem that:

[265] Strictly speaking, we have to first assemble $\Box O!x \,\&\, \Box O!y$ from what we know, apply (107.3) to obtain $\Box(O!x \& O!y)$, then conjoin this with $\Box\Box\forall F(Fx \equiv Fy)$ to obtain $\Box(O!x \& O!y) \,\&\, \Box\Box\forall F(Fx \equiv Fy)$, and finally apply (107.3) a second time to obtain $\Box(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$.

$$x =_E y \equiv (O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$$

So it follows by the Rule of Substitution that:

$$\mathcal{A}(O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy))$$

But by theorem (91.2), an actualized conjunction of truths is equivalent to a conjunction of actualized truths. So it follows that:[266]

$$\mathcal{A}O!x \,\&\, \mathcal{A}O!y \,\&\, \mathcal{A}\Box\forall F(Fx \equiv Fy)$$

By the equivalence (144.7), the first two conjuncts imply, respectively, $O!x$ and $O!y$. By biconditional syllogism, the third conjunct and axiom (33.2) imply $\Box\forall F(Fx \equiv Fy)$. So, by using by &I to assemble what we've established, it follows that:

$$O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$$

So by (69.1), it follows that $x =_E y$. ⋈

(152.2) By theorem (150), we know $x \neq_E y \equiv \neg(x =_E y)$. And by the tautology $(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)$, we may infer from (152.1) that $\neg(x =_E y) \equiv \neg\mathcal{A}x =_E y$. So it follows by biconditional syllogism that $x \neq_E y \equiv \neg\mathcal{A}x =_E y$. But by commuting an instance of axiom (31.1), we have $\neg\mathcal{A}x =_E y \equiv \mathcal{A}\neg x =_E y$. So by another biconditional syllogism, it follows that:

$$(\vartheta)\ \ x \neq_E y \equiv \mathcal{A}\neg x =_E y$$

Now commute (150) and we obtain the $\Box$-theorem that $\neg(x =_E y) \equiv x \neq_E y$. So it follows from (ϑ) by the Rule of Substitution that $x \neq_E y \equiv \mathcal{A}x \neq_E y$. ⋈

(152.3) Let α, β be variables of the same type. $(\rightarrow)$ Assume $\alpha = \beta$. Then, by (75), $\Box\alpha = \beta$. So by theorem (91.1), it follows that $\mathcal{A}\alpha = \beta$. $(\leftarrow)$ We reason by cases, where α, β are either both (A) objects, (B) properties, (C) propositions, or (D) n-place relations $(n \geq 2)$.

Case A. $(\rightarrow)$ Assume $\mathcal{A}x = y$. Then by definition of = (15), this is just:

$$\mathcal{A}(x =_E y \vee (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)))$$

But by theorem (91.10), i.e., that $\mathcal{A}(\varphi \vee \psi) \equiv (\mathcal{A}\varphi \vee \mathcal{A}\psi)$, it follows that:

$$(\vartheta)\ \ \mathcal{A}x =_E y \vee \mathcal{A}(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$$

We now reason by cases, using the disjuncts of (ϑ) as our two cases:

[266] Again, strictly speaking, we have to treat the previously displayed formula as:

$$\mathcal{A}((O!x \,\&\, O!y) \,\&\, \Box\forall F(Fx \equiv Fy))$$

Then we have to distribute the $\mathcal{A}$ to the two conjuncts, and then repeat the process to turn $\mathcal{A}(O!x \,\&\, O!y)$ into $\mathcal{A}O!x \,\&\, \mathcal{A}O!y$.

- Assume $\mathcal{A}x =_E y$. Then by (152.1), it follows that $x =_E y$. So by theorem (69.2), it follows that $x = y$.
- Assume $\mathcal{A}(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$. Since a conjunction of actualized truths is equivalent to an actualized conjunction (91.2), it follows that:

  $$\mathcal{A}A!x \,\&\, \mathcal{A}A!y \,\&\, \mathcal{A}\Box\forall F(xF \equiv yF)$$

  From the first two conjuncts, it follows, respectively, that $A!x$ and $A!y$, by the equivalence (144.8). From the third conjunct, it follows, by the axiom (33.2), that $\Box\forall F(xF \equiv yF)$. So, using &I to assemble what we know, we have $A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)$. By $\vee$I, this implies $x =_E y \;\vee\; (A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF))$. By definition of $=$ (15), this is just $x = y$.

So reasoning by cases from (ϑ), we have established $x = y$.

Case B. Assume $\mathcal{A}F = G$. Then, by definition, this becomes $\mathcal{A}\Box\forall x(xF \equiv xG)$. But by axiom (33.2), this latter is equivalent to $\Box\forall x(xF \equiv xG)$ which, by definition, is $F = G$.

Case C. Assume $\mathcal{A}(p = q)$. By definition, this is just $\mathcal{A}([\lambda y\, p] = [\lambda y\, q])$. By *Case B*, this implies $[\lambda y\, p] = [\lambda y\, q]$ which, by definition, is $p = q$.

Case D. Exercise. $\bowtie$

(152.4) By the tautology $(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)$, (152.3) implies that $\neg\alpha = \beta \equiv \neg\mathcal{A}\alpha = \beta$. Now if commute an appropriate instance of axiom (31.1), we know $\neg\mathcal{A}\alpha = \beta \equiv \mathcal{A}\neg\alpha = \beta$. So by biconditional syllogism, $\neg\alpha = \beta \equiv \mathcal{A}\neg\alpha = \beta$. By definition of $\neq$ (18), it follows that $\alpha \neq \beta \equiv \mathcal{A}\alpha \neq \beta$. $\bowtie$

(153) By (152.3), we know $x = y \equiv \mathcal{A}x = y$, which commutes to $\mathcal{A}x = y \equiv x = y$. By GEN, it follows that $\forall x(\mathcal{A}x = y \equiv x = y)$. By the Rule of Alphabetic Variants, it follows that

(ϑ) $\forall z(\mathcal{A}z = y \equiv z = y)$

Now, independently, we may apply GEN to axiom (34), to obtain:

> $\forall x(x = \imath x\varphi \equiv \forall z(\mathcal{A}\varphi^z_x \equiv z = x))$, provided z is substitutable for x in φ and doesn't occur free in φ

Since y is substitutable for x in the matrix of this universal claim, we may instantiate to the variable y and obtain:

> $y = \imath x\varphi \equiv \forall z(\mathcal{A}\varphi^z_x \equiv z = y)$, provided z is substitutable for x in φ and doesn't occur free in φ

Now let φ be the formula $x = y$. Then z is substitutable for x in φ and doesn't occur free in φ. So as an instance of our last result, we know:

$$y = \imath x(x = y) \equiv \forall z(\mathcal{A}z = y \equiv z = y)$$

But from this last fact and (ϑ), it follows that $y = \iota x(x = y)$, by biconditional syllogism. ⋈

(155.1) The axiom of η-Conversion is a particular formula of our language for every $n \geq 0$:

$$[\lambda x_1 \dots x_n\, F^n x_1 \dots x_n] = F^n \tag{36.3}$$

The occurrences of F^n are free, even when we eliminate the identity symbol and expand the above into primitive notation. Since we've taken the universal generalizations of our axiom schemata as axioms, the following is therefore an axiom for η-Conversion, for every $n \geq 0$:

(ϑ) $\forall F^n([\lambda x_1 \dots x_n\, F^n x_1 \dots x_n] = F^n)$

Without loss of generality, pick a variable, say G, that is not free in n-place relation term Π^n, so that following is an instance of axiom (29.2):

(ξ) $\exists G^n(G^n = \Pi^n)$

By hypothesis, $x_1, \dots, x_n$ aren't free in Π^n, and so there are no occurrences of $x_1, \dots, x_n$ that could be captured by $\lambda x_1 \dots x_n$ if we substitute Π^n for F^n in (36.3). So Π^n is substitutable for F^n. Hence, it follows from (ϑ) and (ξ) by Rule $\forall$E that $[\lambda x_1 \dots x_n\, \Pi^n x_1 \dots x_n] = \Pi^n$. ⋈

(155.2) By α-Conversion (36.1), we may equate alphabetically-variant λ-expressions. So the following is an instance of α-Conversion, where the ν_i are any qualifying object variables (i.e., any distinct variables not free in Π^n):

(ϑ) $[\lambda x_1 \dots x_n\, \Pi^n x_1 \dots x_n] = [\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n]$

By the symmetry of identity (71.2), it follows from (ϑ) that:

(ζ) $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n] = [\lambda x_1 \dots x_n\, \Pi^n x_1 \dots x_n]$

Hence, from (ζ) and (155.1), it follows by the transitivity of identity (71.3) that $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n] = \Pi^n$.[267]
⋈

(155.3) Suppose ρ' is an immediate η-variant of ρ. Then, by definition (154.4), ρ' results from ρ either (i) by replacing one n-place relation term Π^n in ρ by an η-expansion $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n]$ or (ii) by replacing one elementary λ-expression $[\lambda \nu_1 \dots \nu_n\, \Pi^n \nu_1 \dots \nu_n]$ in ρ by its η-contraction Π^n. Note that the following is an instance of (155.2):

[267] Alternatively: From (155.1) by the interderivability of alphabetically-variant formulas (111). We know that (155.2) is an an alphabetic variant of (155.1) in virtue of the fact that they differ only by terms that are alphabetically-variant.

(a) $[\lambda \nu_1 \ldots \nu_n\, \Pi^n \nu_1 \ldots \nu_n] = \Pi^n$

By symmetry of identity, we also know:

(b) $\Pi^n = [\lambda \nu_1 \ldots \nu_n\, \Pi^n \nu_1 \ldots \nu_n]$

We also know that since ρ is not a description, Rule ReflId (74.1) has the following instance:

(c) $\rho = \rho$

Then in case (i), we use may (b) to infer $\rho = \rho'$ from (c) by Rule SubId (74.2), and in case (ii), we may use (a) to infer $\rho = \rho'$ from (c) by Rule SubId. ⋈

(156) Whenever ρ' is an η-variant of ρ, then by our definition of η-variants (154.5), there is a finite sequence of λ-expressions such that each member of the sequence is an immediate η-variant of the preceding member of the sequence. So our theorem follows by a finite number of applications of both (155.3) and the transitivity of identity (71.3). ⋈

(157) ($\rightarrow$) Assume $[\lambda\, p] = [\lambda\, q]$. By 0-place η-conversion (36.3), we know both $[\lambda\, p] = p$ and $[\lambda\, q] = q$. So by two applications of Rule SubId, substituting p for $[\lambda\, p]$ and q for $[\lambda\, q]$ into our assumption, we obtain $p = q$. ($\leftarrow$) (Exercise) ⋈

(159.1) Since $[\lambda y\, p]$ is not a description, we know by Rule ReflId that $[\lambda y\, p] = [\lambda y\, p]$. But then, we may also apply $\exists$I to conclude: $\exists F(F = [\lambda y\, p])$. And by GEN, we have $\forall p \exists F(F = [\lambda y\, p])$. ⋈

(159.2) As an instance of β-conversion (36.2), we know $[\lambda y\, p]x \equiv p$. So by GEN, it follows that $\forall x([\lambda y\, p]x \equiv p)$. Since no $\star$-theorems have been used, it follows by RN that $\Box\forall x([\lambda y\, p]x \equiv p)$. Now assume $F = [\lambda y\, p]$. Then by substitution of identicals into what we've already established, we know $\Box\forall x(Fx \equiv p)$. So by conditional proof, $F = [\lambda y\, p] \rightarrow \Box\forall x(Fx \equiv p)$. ⋈

(160.1) Assume $\Diamond\exists p(F = [\lambda y\, p])$, to show $\exists p(F = [\lambda y\, p])$. By the Barcan Formula, it follows that $\exists p \Diamond(F = [\lambda y\, p])$. Assume q_1 is an arbitrary such p, so that we know $\Diamond(F = [\lambda y\, q_1])$. By the definition of property identity (16.1), it follows that $\Diamond\Box\forall x(xF \equiv x[\lambda y\, q_1])$. By 5$\Diamond$ (115.1), it follows that $\Box\forall x(xF \equiv x[\lambda y\, q_1])$. Again, by the definition of property identity, we have $F = [\lambda y\, q_1]$. So by $\exists$I (84.2), $\exists p(F = [\lambda y\, p])$. Hence, $\exists p(F = [\lambda y\, p])$, by $\exists$E (85). ⋈

(160.2) By contraposition on the previous theorem (160.1), we know $\neg\exists p(F = [\lambda y\, p]) \rightarrow \neg\Diamond\exists p(F = [\lambda y\, p])$. By applying the quantifier negation equivalence (86.4) and the Rule of Substitution to the antecedent, we have $\forall p \neg(F = [\lambda y p]) \rightarrow \neg\Diamond\exists p(F = [\lambda y p])$. By applying the Rule of Substitution to the consequent twice, first using (113.4) and then using the quantifier negation equivalence (86.4), we obtain: $\forall p \neg(F = [\lambda y\, p]) \rightarrow \Box\forall p \neg(F = [\lambda y\, p])$. But by the definition of $\neq$ (18), this becomes: $\forall p(F \neq [\lambda y\, p]) \rightarrow \Box\forall p(F \neq [\lambda y\, p])$. ⋈

(160.3) From (160.1) by (116.1), given that the proof of (160.1) is modally-strict. ⋈

(160.4) From (160.2) by (116.2), given that the proof of (160.2) is modally-strict. ⋈

(161.1) Assume $\diamond\forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$, for conditional proof. By the Buridan$\diamond$ formula (117.5), this implies:

(ϑ) $\forall F \diamond(xF \rightarrow \exists p(F = [\lambda y\, p]))$

Now we want to show $\forall F(xF \rightarrow \exists p(F = [\lambda y\, p]))$, so let Q be an arbitrarily chosen property and assume xQ, for conditional proof, to show $\exists p(Q = [\lambda y\, p])$. It follows that $\Box xQ$, by axiom (37). Now if we instantiate (ϑ) to Q, it follows that $\diamond(xQ \rightarrow \exists p(Q = [\lambda y\, p]))$. This together with $\Box xQ$ yields $\diamond\exists p(Q = [\lambda y\, p])$, by (113.9). But by our previous theorem (160.1), it follows that $\exists p(Q = [\lambda y\, p])$. By discharging our second assumption, we've established $xQ \rightarrow \exists p(Q = [\lambda y\, p])$. Since Q was arbitrary, it follows that $\forall F(xF \rightarrow \exists p(F = [\lambda y\, p])$. ⋈

(161.2) From (161.1), by (116.1), given that (161.1) is a $\Box$-theorem. ⋈

(162.1) By excluded middle (63.2), either $\diamond E!x \vee \neg\diamond E!x$. But by the commutativity of $\equiv$, the next two claims follow from instances of β-Conversion (36.2):

$$\diamond E!x \equiv [\lambda y\, \diamond E!y]x$$

$$\neg\diamond E!x \equiv [\lambda y\, \neg\diamond E!y]x$$

So by a disjunctive syllogism (64.4.e), it follows that $[\lambda y\, \diamond E!y]x \vee [\lambda y\, \neg\diamond E!y]x$. By our rule of Substitution of Alphabetically-Variant Relation Terms (68), we may infer $[\lambda x\, \diamond E!x]x \vee [\lambda x\, \neg\diamond E!x]x$, which by definitions (11.1) and (11.2), becomes $O!x \vee A!x$. ⋈

(162.2) Assume, for reductio, that $\exists x(O!x \,\&\, A!x)$. Assume further that b is an arbitrary such x, so that we have $O!b \,\&\, A!b$ and, by &E, both $O!b$ and $A!b$. From the former, it follows that $\neg A!b$, by (143.2). Contradiction. ⋈

(163.1) Assume $O!x$ (to show $x =_E x$). So by (63.3.a), it follows that $O!x \,\&\, O!x$. Independently, by (63.4.a), we know $Fx \equiv Fx$, and so by GEN, $\forall F(Fx \equiv Fx)$, and by RN, $\Box\forall F(Fx \equiv Fx)$. So we have established: $O!x \,\&\, O!x \,\&\, \Box\forall F(Fx \equiv Fx)$. So by theorem (69.1), it follows that $x =_E x$. ⋈

(163.2) Assume $x =_E y$, for conditional proof. Then, by (69.1), it follows that $O!x \,\&\, O!y \,\&\, \Box\forall F(Fx \equiv Fy)$. By &E, we know: $O!x$, $O!y$, and $\Box\forall F(Fx \equiv Fy)$. By the commutativity of the biconditional (63.3.h), we know it is a $\Box$-theorem that $(Fx \equiv Fy) \equiv (Fy \equiv Fx)$. So by the Rule of Substitution, it follows that $\Box\forall F(Fy \equiv Fx)$. Using &I to conjoin what we have established, we obtain: $O!y \,\&\, O!x \,\&\, \Box\forall F(Fy \equiv Fx)$, which by (69.1), yields $y =_E x$. ⋈

(163.3) Assume $x =_E y$, and $y =_E z$. Then, by (69.1) and &E, we know all of the following, comma-separated claims:

$$O!x,\ O!y,\ \Box\forall F(Fx \equiv Fy)$$

$$O!y,\ O!z,\ \Box\forall F(Fy \equiv Fz)$$

We leave it as an exercise to show that from $\Box\forall F(Fx \equiv Fy)$ and $\Box\forall F(Fy \equiv Fz)$, it follows that $\Box\forall F(Fx \equiv Fz)$. So using &I to conjoin some of what we know, we have $O!x \,\&\, O!z \,\&\, \Box\forall F(Fx \equiv Fz)$, which by (69.1), yields $x =_E z$. ⋈

(164) Suppose $O!x$, $O!y$, and $\forall F(Fx \equiv Fy)$ (to show $x =_E y$). Instantiate *being identical$_E$ to x*, i.e., $[\lambda z\, z =_E x]$, into the hypothesis $\forall F(Fx \equiv Fy)$ to obtain $[\lambda z\, z =_E x]x \equiv [\lambda z\, z =_E x]y$. But, by (163.1), we know $x =_E x$, and so by Strengthened β-Conversion (123), we know $[\lambda z\, z =_E x]x$. So by biconditional syllogism, it follows that $[\lambda z\, z =_E x]y$. By Strengthened β-Conversion (123), this implies $y =_E x$, and by (163.2), $x =_E y$.[268] ⋈

(165) Assume $O!x$, $O!z$. ($\rightarrow$) Assume $x \neq z$. Then $x \neq_E z$, by (69.2). For reductio, assume $[\lambda y\, y =_E x] = [\lambda y\, y =_E z]$. Since $O!x$, we know by the reflexivity of $=_E$ (163.1) that $x =_E x$. So by β-Conversion (36.2), it follows that $[\lambda y\, y =_E x]x$. But then by substitution of identicals, $[\lambda y\, y =_E z]x$. By β-Conversion, $x =_E z$. But by (150), this contradicts our assumption. ($\leftarrow$) It suffices to prove the contrapositive. So assume $[\lambda y\, y =_E x] = [\lambda y\, y =_E z]$. Then by the reasoning just given (inside the reductio), $x =_E z$. ⋈

(166) Suppose $A!x$, $A!y$, and $\forall F(xF \equiv yF)$, for conditional proof. Let P be an arbitrary property. So $xP \equiv yP$. From (121.1), we know $xP \equiv \Box xP$, which commutes to $\Box xP \equiv xP$. So by transitivity of $\equiv$, $\Box xP \equiv yP$. But we also know from (121.1) that $yP \equiv \Box yP$. So, by transitivity of $\equiv$, it follows that $\Box xP \equiv \Box yP$. So by (121.4) (set both F and G in (121.4) to P), it follows that $\Box(xP \equiv yP)$. Since P was arbitrary, we have $\forall F\Box(xF \equiv yF)$. Thus, by the Barcan formula, $\Box\forall F(xF \equiv yF)$. So we may conjoin what we have supposed and what we know to obtain: $A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)$, which by (15), is just $x = y$. ⋈

(167) Assume $O!x$. Then by (144.1), it follows that $\Box O!x$. Since the closures of the instances of (39) are axioms, we also know: $\Box(O!x \rightarrow \neg\exists F xF)$. So by the K axiom (32.1), it follows that $\Box\neg\exists F xF$. ⋈

(168) By applying &E to axiom (32.5), we have $\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x)$. By the Barcan Formula, it follows that $\exists x\Diamond(E!x \,\&\, \Diamond\neg E!x)$. Assume a is an arbitrary such object, so that we know $\Diamond(E!a \& \Diamond\neg E!a)$. Then by applying (113.8) (i.e., the conjuncts of a possibly true conjunction are themselves possibly true) and &E,

[268] Thanks go to Johannes Korbmacher for suggesting a simplification of an earlier version of this proof.

we know $\diamond E!a$. By λ-abstraction, it follows that $[\lambda x \diamond E!x]a$, and by definition of $O!$ (11.1), this just becomes $O!a$. Hence by $\exists$I, $\exists x O!x$ becomes a theorem once we discharge our assumption by $\exists$E. ⋈

(169) By contraposing (39) and eliminating the double negation, we have: $\exists F xF \rightarrow \neg O!x$. But by appealing to (143.2), it is easy to establish that $\neg O!x \rightarrow A!x$ (exercise). So by hypothetical syllogism, $\exists F xF \rightarrow A!x$ ⋈

(170) Let φ be any formula with no free xs. Then by (40), $\exists x(A!x \& \forall F(xF \equiv \varphi))$. Assume a is an arbitrary such object, so that we have:

(ϑ) $A!a \,\&\, \forall F(aF \equiv \varphi)$

If we can then show:

(ξ) $\forall y[(A!y \,\&\, \forall F(yF \equiv \varphi)) \rightarrow y = a]$

Then our theorem follows by conjoining (ϑ) and (ξ), existentially generalizing on a from the resulting conjunction, to obtain, by definition (87.1):

$$\exists! x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Finally, this conclusion would then be established once we discharge our assumption about a using $\exists$E. We therefore need to show (ξ) on the assumption that (ϑ). Pick an arbitrary object, say b, and assume $A!b \,\&\, \forall F(bF \equiv \varphi)$, for conditional proof, to show $b = a$. By applying &E to our assumptions about a and b, we know the following:

$$\forall F(aF \equiv \varphi)$$

$$\forall F(bF \equiv \varphi)$$

It is now straightforward to establish $\forall F(aF \equiv bF)$ by reasoning with respect to an arbitrarily chosen property, say P, appealing to the commutativity and transitivity of the biconditional, and universally generalizing (i.e., applying $\forall$I) at the end. Since we also know $A!a$ and $A!b$ by &E, we may now appeal to (166) to infer $b = a$. So by conditional proof, $(A!b \,\&\, \forall F(bF \equiv \varphi)) \rightarrow b = a$. Since b was arbitrary, it follows that (ξ), completing the derivation of (ξ) from (ϑ). ⋈

(171.1) – **(171.7)** These are all instances of theorem (170). ⋈

(172.1) We want to establish:

$$\mathcal{A}\exists!\alpha\varphi \equiv \exists!\alpha\mathcal{A}\varphi$$

By definition of unique existence (87.2), for some variable β that is substitutable for α in φ and that doesn't occur free in φ, we have to show:

(ϑ) $\mathcal{A}\exists\alpha\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$

Our proof strategy is to start with the left condition of (ϑ) and establish a string of equivalences that ends with the right condition of (ϑ). Our first equivalence is an instance of theorem (91.11), which allows us to commute the actuality operator with the existential quantifier in the left condition of (ϑ):

(a) $\mathcal{A}\exists\alpha\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv \exists\alpha\mathcal{A}\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$

Note independently that the actuality operator also commutes with the universal quantifer, by axiom (31.3). The necessitations of the instances of this axiom are axioms, so we know $\Box(\mathcal{A}\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv \forall\beta\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha))$. Hence, by a Rule of Necessary Equivalence (108.2), we know the following about the right condition of (a):

(b) $\exists\alpha\mathcal{A}\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv \exists\alpha\forall\beta\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$

Again, note independently that by applying RN to an instance of the $\Box$-theorem (91.5), we have as a theorem: $\Box(\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv (\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \mathcal{A}\beta = \alpha))$. Hence, by the Rule of Necessary Equivalence (108.2) again, we know the following about the right condition of (b):

(c) $\exists\alpha\forall\beta\mathcal{A}(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \mathcal{A}\beta = \alpha)$

Again, note independently that by commuting $\Box$-theorem (152.3) and applying RN to the result, we have as a theorem: $\Box(\mathcal{A}\alpha = \beta \equiv \alpha = \beta)$. Hence, again by the same Rule of Necessary Equivalence (108.2), we know the following about the right condition of (c):

(d) $\exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \mathcal{A}\beta = \alpha) \equiv \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$

Consequently, by the chain of biconditionals (a) – (d) and the transitivity of $\equiv$, it follows that the left condition of (a) is equivalent to the right condition of (d), i.e.,

$$\mathcal{A}\exists\alpha\forall\beta(\varphi^{\beta}_{\alpha} \equiv \beta = \alpha) \equiv \exists\alpha\forall\beta(\mathcal{A}\varphi^{\beta}_{\alpha} \equiv \beta = \alpha)$$

But this is just (ϑ). ⋈

(172.2) Use (104.1) and (172.1). ⋈

(172.3) Use (104.2) and (172.1). ⋈

(173) Let φ be any formula in which x, y don't occur free. Then by Strengthened Comprehension (170), it is a theorem that:

$$\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

So by the rule of actualization RA (88), it follows that:

$$\mathcal{A}\exists!x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Since, by hypothesis, y isn't free in φ, it follows from this and an instance of theorem (172.2), by biconditional syllogism, that:

$$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))) \qquad \bowtie$$

(175)★ Suppose x doesn't occur free in φ and G is substitutable for F in φ. Without loss of generality, let y be a variable that doesn't occur free in φ. Then we know by (173) that $\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$. Assume a is an arbitrary such object, so that we have: $a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$. Again, without loss of generality, let z be a variable that is substitutable for x and that doesn't occur free in the matrix of our canonical description. Then by the Hintikka schema (94)★, it follows that:

$$A!a \,\&\, \forall F(aF \equiv \varphi) \,\&\, \forall z((A!z \,\&\, \forall F(zF \equiv \varphi)) \rightarrow z = a)$$

So by &E, $\forall F(aF \equiv \varphi)$. Then since G is, by hypothesis, substitutable for F in φ, it is substitutable for F in $\forall F(aF \equiv \varphi)$. So by Rule $\forall$E, we may instantiate $\forall F$ to the variable G and conclude:

(ϑ) $aG \equiv \varphi^G_F$

But by assumption, $a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$. It follows from this and (ϑ) that $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F$, by Rule SubId. This last conclusion can be considered established once we discharge our assumption about a by citing $\exists$E. $\bowtie$

(177) Consider any formula φ where x doesn't occur free and G is substitutable for F. We want to show (without appealing to any ★-theorems):

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \mathcal{A}\varphi^G_F$$

In what follows, we take ψ to be: $A!x \,\&\, \forall F(xF \equiv \varphi)$. Hence, we want to show:

$$\imath x\psi G \equiv \mathcal{A}\varphi^G_F$$

Since $\imath x\psi$ is a canonical description, it follows by (173) that $\exists y(y = \imath x\psi)$, where y is some variable not free in ψ. So by (104.3), it follows that $\mathcal{A}\psi^{\imath x\psi}_x$, i.e.,

(a) $\mathcal{A}(A!\imath x\psi \,\&\, \forall F(\imath x\psi F \equiv \varphi))$

Now it is a $\Box$-theorem (91.2) that actuality distributes over a conjunction, so it follows from (a) that:

(b) $\mathcal{A}A!\imath x\psi \,\&\, \mathcal{A}\forall F(\imath x\psi F \equiv \varphi)$

By &E, we detach the second conjunct of (b), $\mathcal{A}\forall F(\imath x\psi F \equiv \varphi)$, and since the necessary axiom (31.3) guarantees that the actuality operator commutes with the universal quantifier, it follows that $\forall F\mathcal{A}(\imath x\psi F \equiv \varphi)$. Since G is substitutable for F in this last result (given that, by hypothesis, it is substitutable for F in φ), we may instantiate this last result to G, to obtain $\mathcal{A}(\imath x\psi G \equiv \varphi^G_F)$. Now we've also established as a $\Box$-theorem (91.5) that the actuality operator distributes over the biconditional. So it follows that:

(c) $\mathcal{A}\imath x\psi G \equiv \mathcal{A}\varphi^G_F$

But note also that by theorem (121.8), it follows that $\mathcal{A}\imath x\psi G \equiv \imath x\psi G$, which commutes to:

(d) $\imath x\psi G \equiv \mathcal{A}\imath x\psi G$

Hence from (d) and (c) it follows by biconditional syllogism that $\imath x\psi G \equiv \mathcal{A}\varphi^G_F$, which is what we had to show. ⋈

(178.1) Assume $\Box\varphi^G_F$. It follows that $\mathcal{A}\varphi^G_F$, by theorem (91.1). But then by the right-to-left direction of the theorem (177), we may conclude $\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$. ⋈

(178.2) Assume, for conditional proof:

(a) $\Box\varphi^G_F$

Then it follows from (a) by a 'paradox' of strict implication (107.1) that:

(b) $\Box(\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \rightarrow \varphi^G_F)$

Put this result aside for the moment. By our previous theorem (178.1), it also follows from (a) that:

$$\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$$

From this it follows by the rigidity of encoding that:

$$\Box\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G$$

This implies, by the same 'paradox' of strict implication, that:

(c) $\Box(\varphi^G_F \rightarrow \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G)$

By &I we may conjoin (b) and (c), so that by (107.4), we've derived:

$$\Box(\imath x(A!x \,\&\, \forall F(xF \equiv \varphi))G \equiv \varphi^G_F)$$ ⋈

(182) Suppose κ is provably canonical and, in particular, that κ is provably canonical with respect to φ. So by definition (180.2), there is some individual variable not free in φ, say x (without loss of generality), for which there is a modally strict proof of:

(ϑ) $\kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$

Since a term identity implies the logical propriety of the terms (73.1), we may instantiate κ into an appropriate instance of the modally-strict Hintikka schema (101) and infer from (ϑ) that:

$$\mathcal{A}(A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi)) \,\&\, \forall z(\mathcal{A}(A!z \,\&\, \forall F(zF \equiv \varphi)) \to z = \kappa)$$

By (91.2), $\mathcal{A}$ distibutes over a conjunction and so the first conjunct implies:

$$\mathcal{A}A!\kappa \,\&\, \mathcal{A}\forall F(\kappa F \equiv \varphi)$$

The first conjunct of this last result, $\mathcal{A}A!\kappa$, implies $A!\kappa$, by the modally-strict theorem (144.8). ⋈

(183)★ Suppose κ is provably canonical with respect to φ. Then by (180.2), we can suppose that ν is x without loss of generality and conclude that there is a modally strict proof of:

(ϑ) $\kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$

Now, independently, as an instance of (97.2)★, we know:

$$z = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)) \to (A!z \,\&\, \forall F(zF \equiv \varphi))$$

By now familiar reasons, (ϑ) implies that κ is logically proper, and so we can instantiate it into the previous instance of (97.2)★ to obtain:

(ξ) $\kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)) \to (A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi))$

Hence, from (ϑ) and (ξ), it follows that:

$$A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi)$$

So $\forall F(\kappa F \equiv \varphi)$, by &E. ⋈

(186) Assume κ is strictly canonical with respect to φ. Then by (184.2), we know:

(a) κ is provably canonical with respect to φ

(b) φ is a rigid condition.

By (a) and metadefinition (180.2), we can suppose ν is x without loss of generality and conclude that:

(c) $\vdash_{\Box} \kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$

By (b) and metadefinition (184.1), we know:

(d) $\vdash_{\Box} \forall H(\varphi^H_F \to \Box\varphi^H_F)$

Moreover, by now familiar reasons, κ is logically proper. So if we set φ in the modally strict Hintikka schema (101) to $A!x \,\&\, \forall F(xF \equiv \varphi)$, then we may instantiate the resulting instance to κ and independently conclude:

(e) $\vdash_\Box \kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)) \equiv$
$\mathcal{A}(A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi)) \,\&\, \forall z(\mathcal{A}(A!z \,\&\, \forall F(zF \equiv \varphi)) \rightarrow z = \kappa)$

From (e) and (c), we may infer the right condition of (e) and by dropping its second conjunct, we have established:

$$\vdash_\Box \mathcal{A}(A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi))$$

It follows from this by (91.2):

$$\vdash_\Box \mathcal{A}A!\kappa \,\&\, \mathcal{A}\forall F(\kappa F \equiv \varphi)$$

By axiom (31.3), the right conjunct implies:

(ξ) $\vdash_\Box \forall F \mathcal{A}(\kappa F \equiv \varphi)$

With these facts, we may justify of Rule ESCO as follows. Note that as an instance of the T schema (32.2) we know that $\vdash_\Box \Box\varphi^H_F \rightarrow \varphi^H_F$. By GEN, it follows that $\vdash_\Box \forall H(\Box\varphi^H_F \rightarrow \varphi^H_F)$. We may combine this with (d), by the right-to-left direction of (83.2), to conclude:

(ϑ) $\vdash_\Box \forall H(\varphi^H_F \equiv \Box\varphi^H_F)$

Now we have to show $\vdash_\Box \forall F(\kappa F \equiv \varphi)$. By $\forall$I, it suffices to show, for an arbitrary property P:

$$\vdash_\Box \kappa P \equiv \varphi^P_F$$

Before we give the argument, note that we can infer the following facts about P from (ξ) and (ϑ), respectively:

(ξ)′ $\vdash_\Box \mathcal{A}(\kappa P \equiv \varphi^P_F)$

(ϑ)′ $\vdash_\Box \varphi^P_F \equiv \Box\varphi^P_F$

From (ξ)′, it follows by (91.5) that:

(ζ) $\vdash_\Box \mathcal{A}\kappa P \equiv \mathcal{A}\varphi^P_F$

We can now establish that there is a modally strict proof of $\kappa P \equiv \varphi^P_F$, reasoning only by appeal to modally strict theorems:

(1)	$\vdash_\Box \kappa P \equiv \mathcal{A}\kappa P$	by (121.8) and (63.3.h)
(2)	$\vdash_\Box \kappa P \equiv \mathcal{A}\varphi^P_F$	from (1) by (ζ) and Rule $\equiv$E
(3)	$\vdash_\Box \kappa P \equiv \mathcal{A}\Box\varphi^P_F$	from (2) by (ϑ)′ and the Rule of Substitution
(4)	$\vdash_\Box \kappa P \equiv \Box\varphi^P_F$	from (3) by axiom (33.2), (63.3.h), and Rule $\equiv$E
(5)	$\vdash_\Box \kappa P \equiv \varphi^P_F$	from (4) by (ϑ)′ and Rule $\equiv$E

Hence, by ∀I, $\vdash_\Box \forall F(\kappa F \equiv \varphi)$. ⋈

(188) Suppose ψ is strictly equivalent to a canonical matrix constructed from a formula φ in which x isn't free. Then, by definition (187), there is a modally strict proof of:

$$\psi \equiv (A!x \,\&\, \forall F(xF \equiv \varphi))$$

It follows by the Rule of Actualization that:

$$\mathcal{A}(\psi \equiv (A!x \,\&\, \forall F(xF \equiv \varphi)))$$

Hence by (102), it follows that:

$$(\vartheta)\quad \forall x[(x = \imath x\psi) \equiv (x = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))]$$

Now independently, by (173), we know:

$$\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$$

Suppose a is such an individual, so that we know:

$$a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

Instantiating a into (ϑ), we have:

$$(a = \imath x\psi) \equiv (a = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$$

But from the last two displayed results, it follows that:

$$a = \imath x\psi$$

Hence, $\exists y(y = \imath x\psi)$. ⋈

(189) (→) Assume κ is provably canonical with respect to φ. Then by (180.2) we can suppose ν is x without loss of generality and conclude that:

$$(\zeta)\quad \vdash_\Box \kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

We have to show $\vdash_\Box \kappa = \imath x\psi$, for some ψ that is strictly equivalent to a canonical matrix constructed from φ. But this is easy: let ψ be $A!x \,\&\, \forall F(xF \equiv \varphi)$. From the fact that $\vdash_\Box \chi \equiv \chi$, it follows that $\vdash_\Box \psi \equiv (A!x \,\&\, \forall F(xF \equiv \varphi))$, and so by (187), ψ is strictly equivalent to a canonical matrix constructed from φ. It then remains to show $\vdash_\Box \kappa = \imath x\psi$. But this follows immediately from (ζ) by choice of ψ. ⋈

(←) Without loss of generality, suppose that x is not free in φ and assume that $\vdash_\Box \kappa = \imath x\psi$, where ψ is strictly equivalent to a canonical matrix constructed from φ. Then by (187), we know:

$$(\xi)\quad \vdash_\Box \psi \equiv (A!x \,\&\, \forall F(xF \equiv \varphi))$$

If we apply the Rule of Actualization (RA) to (ξ), we obtain:

(ζ) $\vdash_\Box \mathcal{A}(\psi = (A!x \,\&\, \forall F(xF \equiv \varphi)))$

Now by (73.1), the term identity in our initial assumption implies the logical propriety of κ. So from (ζ) and the modally strict theorem governing the identity for equivalent descriptions (102), we may conclude:

$$\vdash_\Box (\kappa = \imath x\psi) \equiv (\kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi)))$$

But from this and our initial assumption, it follows that:

$$\vdash_\Box \kappa = \imath x(A!x \,\&\, \forall F(xF \equiv \varphi))$$

So by (180.2), κ is provably canonical with respect to φ. ⋈

(190) Assume both:

(a) ψ is strictly equivalent to a canonical matrix constructed from some rigid condition, say φ, in which ν doesn't occur free.

(b) $\vdash_\Box \kappa = \imath x\psi$.

We have to show $\vdash_\Box \psi^\kappa_\nu$. By Rule PCI (189), both (a) and (b) imply:

(c) κ is provably canonical with respect to φ.

So by the modally strict theorem (182), we may conclude:

(ϑ) $\vdash_\Box A!\kappa$

Furthermore, from (c) and the fact that φ is a rigid condition, which we know by (a), it follows by metadefinition (184.2) that:

(d) κ is strictly canonical with respect to φ.

Hence, from (d) it follows by Rule ESCO (186) that:

(ξ) $\vdash_\Box \forall F(\kappa F \equiv \varphi)$

Consequently, from (ϑ) and (ξ), it follows by (64.1), that:

(ζ) $\vdash_\Box A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi)$

Now (a) just means, by definition (187), that:

(e) $\vdash_\Box \psi \equiv (A!\nu \,\&\, \forall F(\nu F \equiv \varphi))$

By GEN, (e) implies:

$$\vdash_\Box \forall\nu(\psi \equiv (A!\nu \,\&\, \forall F(\nu F \equiv \varphi)))$$

If we instantiate this last result to κ, we obtain:

$$\vdash_\Box \psi^\kappa_\nu \equiv (A!\kappa \,\&\, \forall F(\kappa F \equiv \varphi))$$

From this result and (ζ), it follows that $\vdash_\Box \psi^\kappa_\nu$. ⋈

(192.1) By (191.1), $Null(x)$ is defined as $A!x \,\&\, \neg\exists FxF$. Now if we let φ be $A!x$, ψ be $\neg\exists FxF$, and χ be $\forall F(xF \equiv F \neq F)$, then in light of the tautology $((\varphi \,\&\, \psi) \equiv \varphi \,\&\, \chi) \equiv (\varphi \to (\psi \equiv \chi))$ (exercise), we can more easily establish the equivalence of $Null(x)$ and $A!x \,\&\, \forall F(xF \equiv F \neq F)$ by showing:

$$A!x \to (\neg\exists FxF \equiv \forall F(xF \equiv F \neq F))$$

So assume $A!x$. We show both directions of the consequent.

($\to$) Assume $\neg\exists FxF$. By GEN, it suffices to show $xF \equiv F \neq F$:

> ($\to$) By our initial assumption, we know $\forall F\neg xF$, and so by $\forall$E, $\neg xF$. Hence, $xF \to F \neq F$, by (58.3).
>
> ($\leftarrow$) Assume $F \neq F$, i.e., $\neg F = F$. Now, for reductio, assume $\neg xF$. By (67.1) or Rule ReflId, we know $F = F$. Contradiction. Hence xF.

($\leftarrow$) Assume $\forall F(xF \equiv F \neq F)$. For reductio, assume $\exists FxF$. Suppose P is such a property, so that we know xP. Then by our assumption, it follows that $P \neq P$, contradicting $P = P$, which we know by Rule ReflId. ⋈

(192.2) (Exercise)

(192.3) By (192.1), $Null(x)$ is strictly equivalent to a canonical matrix constructed from $F \neq F$, as this was defined in (187). Since x isn't free in $F \neq F$, we may apply theorem (188) and conclude $\exists y(y = \imath x Null(x))$. ⋈

(192.4) (Exercise)

(195.1) By applying GEN to (118.2), we know $\forall F(F \neq F \to \Box F \neq F)$. By the Variant version of the Rule of Alphabetic Variants (111), $\forall H(H \neq H \to \Box H \neq H)$. ⋈

(195.2) (Exercise)

(195.3) We've noted both in the text (192) and in the proof of (192.3) that (192.1) implies that $Null(x)$ is strictly equivalent to a canonical matrix constructed from $F \neq F$, as this was defined in (187). Moreover, from (195.1) it follows, by (184.1), that $F \neq F$ is a rigid condition. Hence, $Null(x)$ is strictly equivalent to a canonical matrix constructed from a *rigid* condition in which x isn't free. From this last conclusion and the fact that definition (193.1) yields, by convention (202.2), a modally strict proof of $\boldsymbol{a}_\varnothing = \imath x Null(x)$, it follows by Rule SECM (190) that there is a modally strict proof of $Null(\boldsymbol{a}_\varnothing)$. ⋈

(195.4) (Exercise)

(197.1) – **(197.4)** (Proofs given in the text.)

(198.1) (Note: Readers who have gotten this far should now be well-equipped to follow the more obvious shortcuts in reasoning we take in this proof and the ones that follow.) Consider the following instance of comprehension, in which there is a free occurrence of the 2-place relation variable R:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(A!y \,\&\, F = [\lambda z\, Rzy] \,\&\, \neg yF)))$$

Assume a is an arbitrary such object, so that we have:

$$(\vartheta)\quad A!a \,\&\, \forall F(aF \equiv \exists y(A!y \,\&\, F = [\lambda z Rzy] \,\&\, \neg yF))$$

Now consider the property $[\lambda z\, Rza]$ and ask the question whether a encodes this property. Assume $\neg a[\lambda z\, Rza]$. Then, from the second conjunct of (ϑ) it follows that:

$$\neg\exists y(A!y \,\&\, [\lambda z\, Rza] = [\lambda z\, Rzy] \,\&\, \neg y[\lambda z\, Rza]))),$$

i.e., by quantifier negation:

$$(\xi)\quad \forall y(A!y \,\&\, [\lambda z\, Rza] = [\lambda z\, Rzy] \to y[\lambda z\, Rza]))),$$

i.e., for any abstract object y, if $[\lambda z\, Rza] = [\lambda z\, Rzy]$, then $y[\lambda z\, Rza]$. Instantiate this universal claim to a. We know $A!a$ by the left conjunct of (ϑ) and and we know $[\lambda z\, Rza] = [\lambda z\, Rza]$ by Rule ReflId. So if we instantiate $\forall y$ in (ξ) to a, it follows that $a[\lambda z Rza]$, contrary to assumption. So we've established by reductio that $a[\lambda z\, Rza]$. Then by the second conjunct of (ϑ), there is an abstract object, say b, such that both $[\lambda z\, Rza] = [\lambda z\, Rzb]$ and $\neg b[\lambda z\, Rza]$. But since $a[\lambda z\, Rza]$ and $\neg b[\lambda z\, Rza]$, it follows by the contrapositive of Rule SubId that $a \neq b$.[269] So, by $\exists$I and two applications of $\exists$E, there are abstract objects x and y such that $x \neq y$, yet such that $[\lambda z\, Rzx] = [\lambda z\, Rzy]$. By GEN, this theorem holds for every R. ⋈

(198.2) By reasoning analogous to that used in the proof of (198.1). ⋈

(198.3) Consider the following instance of comprehension in which there is a free occurrence of the 1-place property variable P:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \exists y(A!y \,\&\, F = [\lambda z\, Py] \,\&\, \neg yF)))$$

By reasoning analogous to that used in the proof of (198.1), it is straightforward to establish that there are distinct abstract objects, say k, l, such that $[\lambda z Pk]$ is identical to $[\lambda z Pl]$. But, then by the definition of proposition identity

[269]Since the *derived* Rule SubId is: $\varphi^{\tau}_{\alpha},\ \tau = \tau' \,/\, \varphi'$, its contrapositive rule is: $\varphi^{\tau}_{\alpha},\ \neg\varphi' \,/\, \tau \neq \tau'$. We leave its justification as an exercise.

(16.3), it follows that $[\lambda\, Pk]$ is identical to $[\lambda\, Pl]$. So, there are distinct abstract objects, x, y such that $[\lambda\, Px] = [\lambda\, Py]$, and this holds for any property P. ⋈

(199) Let R_1 be the relation $[\lambda xy\, \forall F(Fx \equiv Fy)]$. So by (198.1), there are abstract objects, say a, b, such that $a \neq b$ and $[\lambda z\, R_1 za] = [\lambda z\, R_1 zb]$. But, from the definition of R_1, Strengthened β-Conversion (123), and the easily-established fact that $\forall F(Fa \equiv Fa)]$, it is easily provable that $R_1 aa$. Hence, by β-Conversion (123), it follows that $[\lambda z\, R_1 za]a$. But, by Rule SubId, it then follows that $[\lambda z\, R_1 zb]a$. Hence, by β-Conversion (123), $R_1 ab$, which by the definition of R_1 and β-Conversion (123), yields $\forall F(Fa \equiv Fb)$. Hence, $\exists x, y(A!x \,\&\, A!y \,\&\, a \neq b \,\&\, \forall F(Fx \equiv Fy))$. ⋈

(207.1) By definition (206) and the Comprehension Principle for Abstract Objects (40). ⋈

(207.2) By definition (206) and the Strengthened Comprehension for Abstract Objects (170). ⋈

(208) It follows from (207.2) that $\mathcal{A}\exists!x\mathit{TruthValueOf}(x, p)$, by the Rule of Actualization. Hence by (172.2), $\exists y(y = \imath x\mathit{TruthValueOf}(x, p))$.[270] ⋈

(212.1)⋆ By definition (209) and (97.2)⋆. ⋈

(212.2)⋆ From (212.1)⋆, it follows by definition (206) that:

$$A!p^\circ \,\&\, \forall F(p^\circ F \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))$$

By &E, we're done. (Alternatively, this theorem follows from theorem (183) and the metatheoretic fact noted in (210) that p° is provably canonical.) ⋈

(212.3)⋆ By instantiating (212.2)⋆ to the property $[\lambda y\, r]$, it follows that:

$$(\vartheta)\;\; p^\circ[\lambda y\, r] \equiv \exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q])$$

Now we have to show $p^\circ\Sigma r \equiv (r \equiv p)$. ($\rightarrow$) Assume that $p^\circ\Sigma r$, for conditional proof. By (211), it follows that $p^\circ[\lambda y\, r]$. From this and (ϑ), we may conclude that:

$$(\xi)\;\; \exists q((q \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q])$$

Let q_1 be such a proposition, so that we know:

$$(\zeta)\;\; (q_1 \equiv p) \,\&\, [\lambda y\, r] = [\lambda y\, q_1]$$

[270] Alternatively, we know by (173) that $\exists y(y = \imath x(A!x \,\&\, \forall F(xF \equiv \exists q((q \equiv p) \,\&\, F = [\lambda y\, q]))))$. So, by definition of *TruthValueOf* (206) and the fact that a definiendum and definiens can be exchanged in any context (203.5), it follows that $\exists y(y = \imath x\mathit{TruthValueOf}(x, p))$.