From a book of Richard L. Epstein?

Appendix 1
Criteria of Formalization

The traditional view of formalizing is aptly described by Richmond H. Thomason and Robert C. Stalnaker "A semantic theory of adverbs":

> Formalization is the procedure of translating statements of a natural language into formulas of an artificial language for the purpose of evaluating arguments using the statements, exposing ambiguities in them, or revealing their "true logical form". The procedure is informal, since the rules for carrying it out are never made fully explicit. One must use his intuitive understanding of the content and structure of the given statements. An adequate formalization must yield a formula that has the same truth-conditions as the given statement, but beyond this, the standards of adequacy are unclear. All one can say is that the "relevant structure" of the given statement should be reflected in the formalized equivalent. Although this standard is unstatisfactory as an abstract account of what formalization is, in concrete cases the procedures for formalizing and evaluating formalizations are often unproblematic. (footnote p. 196)

The criterion to require that the truth-conditions of the formalization must be the same as that of the informal proposition is far too strong, for the truth-conditions of the formalization will always be restricted compared to those of the original, taking account of only the semantic values considered significant in the models of the logic. Rather, the formalization and the informal proposition should both be true or both false in any given way that takes account of the other sentences we are formalizing and the semantic values we are paying attention to. On the other hand, we can set out standards that are significantly clearer than just relying on the intuitive understanding of logicians.

I present here the criteria and agreements for formalization for predicate logic(s) without modifiers, internal connectives, tenses, or indices for time or location which were developed with many examples in *Predicate Logic*.[1] I have not attempted to gather the criteria and agreements discussed in this text.

Criteria of formalization

Analysis is distinct from formalization: logical validity is not concerned with the meaning of categorematic words.

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[1] These are somewhat simplified to avoid the need for extended explanations here.

A very different analysis of formalization as translation from ordinary language is given by Herbert R. Otto in *The Linguistic Basis of Logic Translation*, which I became aware of only after completing *Predicate Logic*.

Each criterion takes precedence over succeeding ones.

1.a. The formalization respects the assumptions that govern our choice of primitive syntactic categories and definition of truth in a model. The constraints we work under when we adopt predicate logic must be observed.

1.b. The formalization respects the assumptions that govern our choice of a particular logic.

2.a. The original proposition would both be true or both false relative to any given universe.

2.b. The formalization and the original proposition have the same (additional) semantic value(s) (if any) in any model of the logic we choose.

2.c. If one proposition follows informally from another or a collection of other propositions, then its formalization is a formal semantic consequence of the formalizations of the other(s) (relative to the logic we adopt).

3. The formalization contains the same categorematic words as the original, allowing for changes of grammar to satisfy Criterion 1 (e.g., “dogs” changed to “is a dog”).

3.a. The formalization of a proposition may contain additional words to make clear the tense of a verb.

3.b. The use of previously agreed-upon words or phrases to replace grammatical constructions not recognized by predicate logic may appear in a formalization.

3.c. If we give an analysis of a particular predicate by formalizing it with a formal symbol whose meaning we will stipulate either by semantic agreements or an axiomatization, we may choose in advance to recognize certain other predicates or grammatical variations as being formalizable by the same symbol.

4. The grammar of the original should be preserved by the formalization. That is, the structure of the original proposition with respect to the grammatical categories we have assumed (names, predicates, connectives, quantifiers, variables, phrase markers) is respected by the formalization.

4.a. A regular translation of certain words into syncategorematic terms should be observed. For example, “not” as $\neg$, “all” as $\forall$, “every” as $\forall$, “there is” as $\exists$, “it” as a variable.

Parity of form Formalizations should be regular in the sense that each proceeds in analogy with agreed upon formalizations of other propositions.

4.b. A proposition containing no words governed by Criterion 4a should be atomic.

4.c. The order of the categorematic words and parts of the original should be respected.

Agreements

Propositional Connectives

'Or' shall be read as inclusive unless context dictates otherwise.

The following formalizations are good:

B only if A	$B \rightarrow A$
A if and only if B	$(A \rightarrow B) \wedge (B \rightarrow A)$
neither A nor B	$\neg A \wedge \neg B$
B provided that A	$A \rightarrow B$
given A, B	$A \rightarrow B$
B if A	$A \rightarrow B$
if A, B	$A \rightarrow B$
either A or B	$A \vee B$
A unless B	$\neg B \rightarrow A$
A or else B	$(A \vee B) \wedge \neg(A \wedge B)$

Semi-colons, commas, and other punctuation are often used in English in place of connectives and should be formalized accordingly.

Nouns into predicates

If we wish to formalize a proposition containing a common noun (phrase) used as a subject or object, we must convert the noun (phrase) into a predicate and supply some form of quantification.

Universal quantification over just those objects covered by a common noun is formalized by taking the converted predicate as (possibly one conjunct of) the antecedent of a conditional that is universally quantified. Existential quantification is formalized by taking the predicate as a conjunct.

Relativizing quantifiers

Letting "γ", "δ" stand for common or collective nouns, and "$\gamma(x)$", "$\delta(x)$" stand for the conversions of those into predicates, we have:

All γ are δ
is formalized as $\forall x\, (\gamma(x) \rightarrow \delta(x))$

Some γ are δ
is formalized as $\exists x\, (\gamma(x) \wedge \delta(x))$

All γ are P to all δ
is formalized as $\forall x\, \forall y\, (\gamma(x) \wedge \delta(y) \rightarrow P(x, y))$

To formalize *two or more universally quantified common nouns* in a proposition that contains no existentially quantified common nouns, proceed by converting each common noun into a predicate, conjoin the predicates in the order in which the nouns appear in the original (associating to the left), and use that conjunction as antecedent to the formalization of the predicate that formalizes the relation among the nouns, taking the universal closure of the whole in the order of the variables in the conjunction.

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When we have mixtures of quantifiers, we relativize each common noun sequentially, proceeding from the left in the order in which the nouns appear in the proposition.

Adjectives and Predicates

We have two choices for formalizing an adjective or adjectival phrase in a proposition:

i. We can convert the adjective (phrase) into a (possibly complex) predicate. If the adjective modifies a name, then that name serves as subject to the predicate, and the predicate is taken as a further conjunct. If the adjective modifies a common noun, then the predicate is treated as a conjunct using the same variable which the formalization of the noun uses.

ii. If (i) fails in the sense that it does not yield a proposition that is equivalent to the original for our logical purposes, then unless we can somehow manage to incorporate into our formalizations a way to respect the role of the proposition in deductions using formalized assumptions, the proposition is not formalizable.

What we take as atomic will depend on what consequences we recognize as valid. As a rule, we want to split up predicates into parts whenever possible.

Adverbs

Deductions depending on adverbs can be formalized only through the use of formalized assumptions governing the adverbs.

Adverbial clauses for location create the same problems for formalizing as do tensed propositions and can be formalized only by adding formalized assumptions.

Tenses

Tenses of verbs that affect deductions can be taken into account only through expanding the reading of the verb (Criterion 3.a) or in the metalogic, unless we countenance moments of time as things.

In most cases we can assimilate uses of the simple present to a timeless verb. But the simple present used for what is typical or normal can require expanding the reading of the tense to make that sense explicit.

The simple present and sometimes the future can indicate a general law for which universal quantification is appropriate in converting common nouns to predicates; other tenses seem to require existential quantification.

Collections
Reasoning about collections, qualities, and properties can be taken account only by treating collections, qualities, and properties as things (in second-order predicate logic) The words “possesses”, “belongs”, “is an element of”, in the sense of an object belonging to a collection, are excluded from realizations by the *Self-Reference Exclusion Principle.*

Mass Terms
Reasoning involving mass terms is generally outside the scope of predicate logic, though we may accept as atomic predicates words involving mass terms, such as “is made of water”, “is made of green cheese”, and we may accept names of specific parts of masses, such as “the snow on R. L. Epstein’s front yard” as long as doing so does not conflict with the roles these play in deductions.

Indexicals
A proposition containing indexical words may be formalized using additional categorematic words to replace the indexicals, though by doing so we obtain a proposition that is not semantically equivalent to the original

Reference
Reference must be understood as timeless in predicate logic.

An inductive approach to formalizing
Formalize parts of propositions in accord with our established agreements whenever possible, then formalize the way those parts are put together in the original.

We take a negation to apply to an entire proposition (or wff) only if it expressly occurs that way in the proposition we are formalizing (for example, “It’s not the case that cats bark”, “Not: cats bark”). In all other cases, we give wider scope to the quantification we use to convert a predicate into a noun than to the negation.

An inductive approach to formalizing will in general lead to minimal scope for quantifiers.

Prefixes
There are a number of prefixes in English that we can formalize as negations: “un-”, “dis-”, “ir–”, “im-”, “in-”, “non”.

'– one', *'– body'*

'Everyone', 'somebody', and other quantifications involving '-one' or '-body' shall be understood as relative quantification over persons.

Possessives

Let u,v stand for terms (names or variables) and γ stand for a common noun.

1. u is (a) γ of v

Take this as atomic, relative to:

(a) $\forall x\, \forall y\, (x \text{ is a } \gamma \text{ of } y \;\rightarrow\; x \text{ is a } \gamma)$

2. u is v's γ

Formalize this as "u is a γ of v" relative to (a), and possibly relative to a formal uniqueness assumption, depending on γ.

3. u has (a) γ

i. Take this as atomic if γ is not a kind of *thing*, for example "cold".

ii. If it makes sense, depending on γ, formalize this as

$$\exists y\, (y \text{ is a } \gamma \;\wedge\; x \text{ has } y)$$

For example, γ could be "dog" or "pen".

iii. Otherwise, assimilate 'has' to the general possessive case, relative to (a):

$$\exists x\, (x \text{ is a } \gamma \text{ of } t)$$

Read "my" as the possessive form of "I", so that if we replace "I" by a name c, replace "my" by "c's". Similar remarks apply to formalizations of the other possessive pronouns.

Passive/Active

Given a sentence containing an active verb with direct object, we can form a passive equivalent by using the direct object as subject and taking the subject of the active verb as indirect object of the passive form.

For sentences which use the passive form of a verb in which there is no indirect object, an indirect object can invariably be supplied, namely, "by something".

The rewriting from passive into active may, however, affect the content of the proposition.

The rewriting from passive into active or vice-versa must be done after quantification is supplied for the conversion of nouns into predicates.

Equality predicate

The equality predicate $\equiv$ is syncategorematic.

Existence and $\exists$

Assertions about existence can only be made using the quantifier $\exists$; the phrase "— exists" will not be accepted as a predicate.

Various Quantifiers

"No γ", where γ is a common noun, shall be formalized as $\neg\, \exists x\, \gamma(x)$.

"Nothing" shall be formalized as $\neg\, \exists x$, though some uses of it as subject cannot be formalized.

"Only γ" shall be understood as "Some γ and not that which is not γ".

Comparatives, Superlatives

The use of a comparative with a common noun will be understood as comparison to other kinds of things.

A superlative makes comparison to all objects other than the object claimed to be the superlative.

Descriptive names

We have two choices in formalizing propositions in which a descriptive name appears, so long as the choice is adopted uniformly.

1. We may treat the descriptive name as an atomic name and add formalized assumptions that guarantee that deductions based on the internal structure of the name are respected.
2. *Russell's Method of Eliminating Descriptive Names in Atomic Propositions*
   Given an atomic proposition B(*a*) where *a* is a descriptive phrase purporting to refer, we select a common noun phrase α using the same categorematic words as appear in *a* such that *a* refers to a unique object iff there is one and only one α. Then we rewrite B(*a*) as:

   There is exactly one thing that is an α *and* B(*it*)

   We then convert α into a predicate A(x) (where x does not appear in B(a)) to obtain a formalization of B(a) as: $\exists!\, x\, (A(x) \wedge B(x))$.

For Russell's method there is a distinction between a primary and secondary occurrence of a descriptive name (the existential assumption governs the entire proposition or only the smallest part in which the name appears). Unless context demands otherwise, occurrences of descriptive names shall be taken to be secondary.

Equivalent propositions

We have an informal notion of what it means for two propositions to be *equivalent.* This requires at least that for any (description of) the way the world could be, they are both true or both false.

That criterion alone allows that the following are equivalent:

(a) Ralph is a dog $\wedge$ $\neg$ (Ralph is a dog)

$9 < 7 \wedge \neg\,(9 < 7)$

They are both false in every model, yet they are about very different things. In other predicate logics that take account of additional semantic values such as subject matter or referential content these can come out as inequivalent if we require that equivalent propositions have all the same semantic values.

Semantically equivalent propositions
Two semi-formal propositions are *semantically equivalent* ifff they have the same semantic values in any model.

Two ordinary language propositions are semantically equivalent iff they can be formalized as equivalent semi-formal propositions.

This definition uses "can be" rather than "are" since the criteria of formalization are probably incomplete: there may be more than one good formalization of an ordinary language proposition.

For classical predicate logic the only semantic value of a proposition as a whole is its truth-value. Hence, we have:

> Two semi-formal propositions are semantically equivalent in classical predicate logic iff they have the same truth-value in any model.

The notion of semantic equivalence does not take account of the form of a proposition. For example, the following are equivalent:

(b) Ralph is a dog $\vee$ $\neg$ (Ralph is a dog)

Ralph is a dog $\rightarrow$ Ralph is a dog

We can use a stronger notion of equivalence and require that two semi-formal propositions are equivalent iff they have the same semantic values and the same form. This would rule out the equivalence of the pair (b), but not the equivalence of the pair (a).

For ordinary language propositions, however, we do have a stronger criterion than semantic equivalence. It depends on our criteria of formalization.

Equivalent propositions   Two ordinary language propositions are equivalent iff they can be formalized as the same proposition.

Appendix 2
The Unique Readability of Wffs and Terms

In many places in the text we need that the wffs and terms of a language are defined correctly, in the sense that each cna be parsed correctly in exactly one way. The proofs for all those are variations on the proofs of the unique readability of the wffs and the terms of classical predicate logic.

Theorem 1 Unique readability of wffs for the language of propositional logic There is one and only one way to parse each wff for the language of propositional logic, $L(\neg, \rightarrow, \wedge, \vee, p_0, p_1, \ldots)$.

Proof To each primitive symbol α of the formal language assign an integer $\lambda(\alpha)$ according to the following chart:

$\neg$	$\rightarrow$	$\wedge$	$\vee$	p_i	(	)	—	,
0	0	0	0	0	– 1	1	0	0

To the concatenation of symbols $\alpha_1 \alpha_2 \cdots \alpha_n$ assign the number:

$$\lambda(\alpha_1 \alpha_2 \cdots \alpha_n) = \lambda(\alpha_1) + \lambda(\alpha_2) + \cdots + \lambda(\alpha_n)$$

First I'll show that for any wff A, $\lambda(A) = 0$, using induction on the number of occurrences of $\neg, \rightarrow, \wedge, \vee$ in A.

If there are no occurrences, then A is atomic, that is, for some $i \geq 0$, A is (p_i). Then $\lambda(\text{‘(’}) = -1$, $\lambda(p_i) = 0$, and $\lambda(\text{‘)’}) = 1$. Adding, we have $\lambda(A) = 0$.

Suppose the lemma is true for every wff that has fewer occurrences of these symbols than A does. Then there are 4 cases, which we can't yet assume are distinct: A arises as $(\neg B)$, $(B \rightarrow C)$, $(B \wedge C)$, or $(B \vee C)$. By induction $\lambda(B) = \lambda(C) = 0$, so in each case by adding, we have $\lambda(A) = 0$.

Now I'll show that reading from the left, if α is an initial segment of a wff (reading from the left) other than the entire wff itself, then $\lambda(\alpha) < 0$; and if α is a final segment other than the entire wff itself, then $\lambda(\alpha) > 0$. So no proper initial or final segment of a wff is a wff. To establish this I will again use induction on the number of occurrences of connectives in the wff. I'll let you establish the base case for atomic wffs, where there are no (that is, zero) connectives.

Now suppose the lemma is true for any wff that contains $\leq n$ occurrences of the connectives. If A contains $n + 1$ occurrences, then it must have (at least) one of the forms given in the definition of wffs. If A has the form $(B \wedge C)$, then an initial segment of A must have one of the following forms:

i. (
ii. (β where β is an initial segment of B
iii. (B∧
iv. (B∧ γ where γ is an initial segment of C

For (ii), $\lambda($ ‘(‘ $) = -1$ and by induction $\lambda(\beta) < 0$, so $\lambda($ ‘(β’ $) < 0$. I’ll leave (i), (iii), and (iv) to you. The other cases (for ¬, →, and ∨) follow similarly, and I’ll leave those and the proof for final segments to you.

Now to establish the theorem we proceed through a number of cases by way of contradiction. Suppose we have a wff that could be read as both (A∧ B) and (C→ D). Then A∧ B) must be the same as C→ D). In that case either A is an initial part of C or C is an initial part of A. But then $\lambda(A) < 0$ or $\lambda(C) < 0$, which is a contradiction, as we proved above that $\lambda(A) = \lambda(C) = 0$. Hence, A is C. But then we have that ∧ B) is the same as → D), which is a contradiction.

Suppose (¬A) could be parsed as (C→ D). Then (¬A and (C→ D must be the same. So D would be a final segment of A other than A itself, but then $\lambda(D) > 0$, which is a contradiction. The other cases are similar, and I’ll leave them to you. ■

Theorem 2 Unique readability of wffs of the language of predicate logic
There is one and only one way to parse each wff of the language
$L(\neg, \rightarrow, \wedge, \vee, \forall, \exists, x_0, x_1, \ldots, P_0^1, P_0^2, P_0^3, \ldots, P_1^1, P_1^2, P_1^3, \ldots, c_0, c_1, \ldots)$.

Proof To each symbol α of the formal language assign an integer $\lambda(\alpha)$ according to the following chart:

¬	→	∧	∨	—	,
0	0	0	0	0	0

(	)	x_i	c_i	P_i^n	∀	∃
– 1	1	1	1	$-n$	– 1	– 1

To the concatenation of symbols $\alpha_1\,\alpha_2 \cdots \alpha_n$ assign the number:

$$\lambda(\alpha_1\,\alpha_2 \cdots \alpha_n) = \lambda(\alpha_1) + \lambda(\alpha_2) + \cdots + \lambda(\alpha_n)$$

We proceed by induction on the number of occurrences of ∀, ∃, ¬, →, ∧, ∨ in A to show that for any wff A, $\lambda(A) = 0$.

If there are no occurrences, then A is atomic. That is, for some $i \geq 0$, $n \geq 1$, A is $(P_i^n(—, \ldots, —)(u_1, \ldots, u_n))$, where there are n blanks and $n - 1$ commas. I’ll let you calculate that $\lambda(A) = 0$.

The inductive stage of the proof, and then the proof that initial segments and final segments of wffs have value ≠ 0, and then that there is only one way to parse each wff, is done almost exactly as for the propositional language in Theorem 1. ■

We define inductively what it means for *a symbol to appear in a wff* :

The symbols that appear in the atomic wff ($P_i^n(u_1, \ldots, u_n)$) are: P_i^n and $u_1, \ldots, u_n$.

The symbols that appear in $(\neg A)$ are '$\neg$' and those that appear in A.

The symbols that appear in $(A \to B)$, $(A \wedge B)$, $(A \vee B)$ are, respectively, $\to$, $\wedge$, $\vee$, and all those that appear in either A or in B.

The symbols that appear in $(\forall x\ A)$ are x, $\forall$, and those that appear in A; and similarly for $(\exists x\ A)$, reading '$\exists$' for '$\forall$'.

We also say that a symbol *occurs in a wff* or *a wff contains a symbol.*

Theorem 3 Unique readability of terms of the language of classical predicate logic with functions

Proof Let γ be the following assignment of numbers to the symbols:

x_i	c_i	f_i^n	(	)	for each $i \geq 0$ and $n \geq 0$
1	1	$2 - n$	2	-2	

To a concatenation of these symbols $\alpha_1\ \alpha_2 \cdots \alpha_n$ assign $\gamma(\alpha_1\ \alpha_2 \cdots \alpha_n) = \gamma(\alpha_1) + \gamma(\alpha_2) + \cdots + \gamma(\alpha_n)$. The proof then follows as for the previous two theorems by showing:

a. For every term u, $\gamma(u) = 1$.

b. If β is an initial segment of a term other than the entire term, then $\gamma(\beta) < 1$.

c. If β is a final segment of a term other than the entire term, then $\gamma(\beta) > 1$.

d. No proper initial or final segment of a term is a term.

e. There is one and only one way to parse each term. ■

By Theorem 3 we have that each term has a unique number assigned as its depth. I'll let you define what it means to say that a *symbol appears in a term*. We often say instead that the *term contains a symbol*.

Appendix 3
Completeness Proofs

- The letter L stands for the formal language of the system under discussion.
- When no proof is appended to a lemma or theorem I've left the proof to you.
- I write "Axiom" for "axiom schema".

0 Classical Propositional Logic

The language is $L(\neg, \rightarrow, \wedge, \vee, p_0, p_1, \ldots)$.

Lemma 1 *Soundness* If $\Gamma \vdash A$, then $\Gamma \vDash A$.

Lemma 2 a. $\vdash A \rightarrow A$

b. $\{A, \neg A\} \vdash B$

Proof

(a) 1. $\vdash A \rightarrow ((A \rightarrow A) \rightarrow A)$ — by Axiom 2

2. $\vdash A \rightarrow (A \rightarrow A)$ — by Axiom 2

3. $\vdash (A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$ — by Axiom 4

4. $\vdash (A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$ — *modus ponens* using (1) and (3)

5. $\vdash A \rightarrow A$ — *modus ponens* using (2) and (4)

(b) 1. $\vdash \neg A \rightarrow (A \rightarrow B)$ — by Axiom 1

2. $\neg A$ — premise

3. $(A \rightarrow B)$ — by *modus ponens* using (1) and (2)

4. A — premise

5. B — by *modus ponens* on (3) and (4) ■

Lemma 3 *a.* Γ is consistent iff there is some B such that $\Gamma \nvdash B$.

b. $\Gamma \cup \{A\}$ is consistent iff $\Gamma \nvdash \neg A$.

c. $\Gamma \cup \{\neg A\}$ is consistent iff $\Gamma \nvdash A$.

d. If Γ is consistent, then $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is consistent.

e. If Γ is consistent and complete, then Γ is a theory.

f. Γ is consistent iff every finite subset of Γ is consistent.

Proof (a) From left to right is immediate. So suppose $\Gamma \cup \{A\}$ is not consistent. Then for some A, $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. So by Lemma 2.b, for every B, $\Gamma \vdash B$.

(b) From left to right is immediate by definitions. So suppose $\Gamma \cup \{A\}$ is inconsistent. Then by (a), $\Gamma \cup \{A\} \vdash \neg A$. Hence by Axiom 1, $\Gamma \vdash A$.

(c) The proof is as for (b).

(d) Suppose both $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ are inconsistent. Then by (b) and (c), $\Gamma \vdash A$ and $\Gamma \vdash \neg A$, and hence Γ is inconsistent.

(e) Suppose that Γ is complete and consistent, and $\Gamma \vdash A$. If A is not in Γ, then $\neg A$ is in Γ, and hence $\Gamma \vdash \neg A$, so that Γ would be inconsistent. So A is in Γ.

(f) If Γ is inconsistent. Then for some A, $\Gamma \vdash A$ and $\Gamma \vdash \neg A$. Let $B_1, \ldots, B_n$ be a proof of A from Γ, and let $C_1, \ldots, C_m$ be a proof of $\neg A$ from Γ. Then $\{B_1, \ldots, B_n, C_1, \ldots, C_m\}$ is a finite subset of Γ that is not consistent. On the other hand, if some finite subset $\Delta \subseteq \Gamma$ is inconsistent, then for some A, $\Delta \vdash A$ and $\Gamma \vdash \neg A$. But the same proofs are proofs from Γ, too. So Γ is inconsistent. ■

The proofs of parts (e) and (f) show that those hold for any axiom system. The proof of Lemma 3 uses only Axiom 1.

Lemma 4 The syntactic deduction theorem

a. $\Gamma, \mathrm{A} \vdash \mathrm{B}$ iff $\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B}$

b. $\Gamma \cup \{\mathrm{A}_1, \ldots, \mathrm{A}_n\} \vdash \mathrm{B}$ iff $\Gamma \vdash \mathrm{A}_1 \rightarrow (\mathrm{A}_2 \rightarrow (\cdots \rightarrow (\mathrm{A}_n \rightarrow \mathrm{B}) \cdots)$

Proof (a) From right to left is immediate, since *modus ponens* is our rule.

To show that if $\Gamma, \mathrm{A} \vdash \mathrm{B}$, then $\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B}$, suppose that $\mathrm{B}_1, \ldots, \mathrm{B}_n$ is a proof of B from $\Gamma \cup \{\mathrm{A}\}$. I'll show by induction that for each i, $1 \le i \le n$, $\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B}_i$.

Either $\mathrm{B}_1 \in \Gamma$ or B_1 is an axiom, or B_1 is A. In the first two cases the result follows by using Axiom 2. If B is A, it follows by Lemma 2.

Now suppose for all $k < i$, $\vdash \mathrm{A} \rightarrow \mathrm{B}_k$. If B_i is an axiom, or $\mathrm{B}_i \in \Gamma$, or B_i is A, we have $\vdash \mathrm{A} \rightarrow \mathrm{B}_i$ as before. The only other case is when B_i is a consequence by *modus ponens* of B_m and $\mathrm{B}_j = \mathrm{B}_m \rightarrow \mathrm{B}_i$ for some $m, j < i$. Then by induction $\Gamma \vdash \mathrm{A} \rightarrow (\mathrm{B}_m \rightarrow \mathrm{B}_i)$ and $\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B}_m$, so by Axiom 4, $\Gamma \vdash \mathrm{A} \rightarrow \mathrm{B}_i$. ■

Lemma 5 Γ is a complete and consistent theory iff there is a model v such that $\Gamma = \{\mathrm{A} : v(\mathrm{A}) = \mathsf{T}\}$.

Proof I'll let you show that the set of wffs true in a model is a complete and consistent theory.

Suppose now that Γ is a complete and consistent theory. Define a function v: Wffs $\rightarrow$ $\{\mathsf{T}, \mathsf{F}\}$ by setting $v(\mathrm{A}) = \mathsf{T}$ iff $\mathrm{A} \in \Gamma$. To show that v is a model we need only show that it evaluates the connectives correctly.

If $v(\neg\mathrm{A}) = \mathsf{T}$, then $\neg\mathrm{A} \in \Gamma$, so $\mathrm{A} \notin \Gamma$ by consistency, so $v(\mathrm{A}) = \mathsf{F}$.

If $v(\mathrm{A}) = \mathsf{F}$, then $\mathrm{A} \notin \Gamma$, so $\neg\mathrm{A} \in \Gamma$ by completeness, so $v(\neg\mathrm{A}) = \mathsf{T}$.

Suppose $v(\mathrm{A} \rightarrow \mathrm{B}) = \mathsf{T}$. Then $\mathrm{A} \rightarrow \mathrm{B} \in \Gamma$. If $v(\mathrm{A}) = \mathsf{T}$, we have $\mathrm{A} \in \Gamma$, so by *modus ponens*, $\mathrm{B} \in \Gamma$, as Γ is a theory. Hence $v(\mathrm{B}) = \mathsf{T}$. Conversely, suppose $v(\mathrm{A}) = \mathsf{F}$ or $v(\mathrm{B}) = \mathsf{T}$. If the former then $\mathrm{A} \notin \Gamma$, so $\neg\mathrm{A} \in \Gamma$, and by Axiom 1, $\mathrm{A} \rightarrow \mathrm{B} \in \Gamma$; so $v(\mathrm{A} \rightarrow \mathrm{B}) = \mathsf{T}$. If the latter then $\mathrm{B} \in \Gamma$, and as Γ is a theory, by Axiom 2, $\mathrm{A} \rightarrow \mathrm{B} \in \Gamma$; so $v(\mathrm{A} \rightarrow \mathrm{B}) = \mathsf{T}$.

For conjunction,

$$\begin{aligned} v(\mathrm{A} \wedge \mathrm{B}) = \mathsf{T} \quad &\text{iff} \quad v(\mathrm{A}) = \mathsf{T} \text{ and } v(\mathrm{B}) = \mathsf{T} \\ &\text{iff} \quad \mathrm{A} \in \Gamma \text{ and } \mathrm{B} \in \Gamma \\ &\text{iff} \quad (\mathrm{A} \wedge \mathrm{B}) \in \Gamma \text{ using Axioms 5, 6 and 7} \end{aligned}$$

For disjunction, we have that if $\mathrm{A} \in \Gamma$ or $\mathrm{B} \in \Gamma$, then $(\mathrm{A} \vee \mathrm{B}) \in \Gamma$ by Axioms 8 and 9. So if $v(\mathrm{A}) = \mathsf{T}$ or $v(\mathrm{B}) = \mathsf{T}$, then $v(\mathrm{A} \vee \mathrm{B}) = \mathsf{T}$. In the other direction, if $v(\mathrm{A} \vee \mathrm{B}) = \mathsf{T}$, then $(\mathrm{A} \vee \mathrm{B}) \in \Gamma$. If both $\mathrm{A} \notin \Gamma$ and $\mathrm{B} \notin \Gamma$, then by Theorem 6.f, for any C, $\Gamma \cup \{\mathrm{A}\} \vdash \mathrm{C}$ and $\Gamma \cup \{\mathrm{B}\} \vdash \mathrm{C}$. So by the syntactic deduction theorem, for any C, $\Gamma \vdash \mathrm{A} \rightarrow \mathrm{C}$ and $\Gamma \vdash \mathrm{B} \rightarrow \mathrm{C}$. Hence by Axiom 10, $\Gamma \vdash (\mathrm{A} \vee \mathrm{B}) \rightarrow \mathrm{C}$. As $(\mathrm{A} \vee \mathrm{B}) \in \Gamma$, this means that for every C, $\Gamma \vdash \mathrm{C}$, which is a contradiction on the consistency of Γ. Hence, $\mathrm{A} \in \Gamma$ or $\mathrm{B} \in \Gamma$, so that $v(\mathrm{A}) = \mathsf{T}$ or $v(\mathrm{B}) = \mathsf{T}$. ■

Lemma 6 If $\nvdash$D, then there is some complete and consistent theory Γ such that $D \notin \Gamma$.

Proof Let $A_0, A_1, \ldots$ be a numbering of the wffs of the formal language. Define:

$$\Gamma_0 = \{\neg D\}$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if this is consistent} \\ \Gamma_n & \text{otherwise} \end{cases}$$

$$\Gamma = \bigcup_n \Gamma_n$$

We have that Γ_0 is consistent by Lemma 3. So by construction, each Γ_n is consistent. Hence Γ is consistent, for if not some finite $\Delta \subseteq \Gamma$ is inconsistent by Lemma 3, and Δ being finite, $\Delta \subseteq \Gamma_n$ for some n, so Γ_n would be inconsistent.

Γ is complete, because if $\Gamma \nvdash A$, then by Lemma 3, $\Gamma \cup \{\neg A\}$ is consistent, and hence by construction, $\neg A \in \Gamma$, and so $\Gamma \vdash \neg A$.

Finally, Γ is a theory, since if $\Gamma \vdash A$, then $\Gamma \cup \{A\}$ is consistent, and hence by construction, $A \in \Gamma$. ■

Theorem 7

a. *Strong completeness* $\Gamma \vdash A$ iff $\Gamma \vDash A$.

b. *Compactness* Γ has a model iff every finite subset of Γ has a model.

Proof a. By Lemma 1 we need only show that if $\Gamma \vDash A$ then $\Gamma \vdash A$. If $\Gamma \vDash A$, suppose that $\Gamma \nvdash A$. Then by Lemma 3, $\Gamma \cup \{\neg A\}$ is consistent, and so by Lemmas 5 and 6 it has a model. So $\Gamma \nvDash A$, a contradiction. So $\Gamma \vdash A$.

b. Γ has a model iff Γ is consistent (Lemmas 5 and 6). Γ is consistent iff every finite subset of Γ is consistent (Lemma 3). Every finite subset of Γ is consistent iff every finite subset of Γ has a model (Lemmas 5 and 6). ■

The proof of Lemma 6 depends on infinitistic assumptions. A finitistic proof of completeness, though not of strong completeness for this system can be found in Carnielli's and my *Computability*.

If we formulate classical propositional logic with only $\neg$ and $\rightarrow$ as primitive and $\wedge$, $\vee$ as defined connectives, then the proof above shows that axiom schema 1–4 with modus ponens is strongly complete.

I   Classical Predicate Logic

The language is L($\neg, \rightarrow, \wedge, \vee, \forall, \exists$; $P_0, P_1, \ldots, c_0, c_1, \ldots$), with no superfluous quantification. We define a *propositional form* of a wff in this language to be a wff B of the language of propositional logic such that there is a way to assign wffs, possibly open, to the propositional variables in B that results in A, where the same variable is assigned the same wff throughout B.

By "PC" I mean classical propositional logic, as given in Chapter 4 and the previous section.

Lemma 8   The syntactic deduction theorem

$\Gamma, A \vdash B$ iff $\Gamma \vdash A \rightarrow B$.

$\Gamma \cup \{A_1, \ldots, A_n\} \vdash B$ iff $\Gamma \vdash A_1 \rightarrow (A_2 \rightarrow (\cdots \rightarrow (A_n \rightarrow B) \cdots))$.

Proof   Since A, B, and all wffs in Γ are closed, the proof is as for PC. ■

Lemma 9   If $\vdash A \rightarrow B$ and $\vdash B \rightarrow C$, then $\vdash A \rightarrow C$.

Proof   By Lemma 8, if $\vdash A \rightarrow B$ and $\vdash B \rightarrow C$, then $A \vdash B$ and $B \vdash C$. Combining those proofs, as you can show, we have a proof of C from A. So by Lemma 8, $\vdash A \rightarrow C$. ■

Lemma 10   a. $\vdash \forall \ldots (A \rightarrow B) \rightarrow (\forall \ldots A \rightarrow \forall \ldots B)$

b. $\forall \ldots (A \rightarrow B),\ \forall \ldots A \} \vdash \forall \ldots B$ generalized *modus ponens*

Proof   a. We proceed by induction on the number of variables in the universal closure for $(A \rightarrow B)$. If $n = 0$, then this is just $\vdash (A \rightarrow B) \rightarrow (A \rightarrow B)$, which we have via the PC axioms as in Lemma 2. If $n = 1$, this is Axiom 1. Suppose the lemma true for n. Let x be the last variable (in alphabetical order) that appears free in $(A \rightarrow B)$. We have three cases, depending on whether x appears free in A, B, or both. I will do one case and leave the others to you. An instance of Axiom 1 is:

$$\forall \ldots (\forall x\, (A \rightarrow B) \rightarrow (\forall x\, A \rightarrow B)).$$

Since the universal closure of this formula has only n variables, we have (a) by induction. Part (b) then follows from (a). ■

Lemma 11   PC *in predicate logic*

If A has the form of a PC-tautology, then $\vdash \forall \ldots A$.

Proof   Suppose that one form of A is the PC-tautology B. By the completeness theorem for PC, there is a proof $B_1, \ldots, B_n = B$ in PC. In that proof we can assume there is no propositional variable appearing in any of $B_1, \ldots, B_n$ that does not also appear in B. Let $B_i{}^*$ be B_i with propositional variables replaced by predicate wffs just as they are replaced in B to obtain A. I'll show by induction on i that $\vdash \forall \ldots B_i{}^*$. The lemma will then follow for $i = n$.

For $i = 1$, B_1 is a PC-axiom, and so we are done by the PC-axioms. Suppose now that for all $j < i$, $\vdash \forall \ldots B_j^*$. If B_i is a PC-axiom we are done. If not, then for some $j, k < i$, B_j is $B_k \rightarrow B_i$. Note that $(B_k \rightarrow B_i)^*$ is $B_k^* \rightarrow B_i^*$. So by induction we have $\vdash \forall \ldots B_k^*$ and $\vdash \forall \ldots (B_k^* \rightarrow B_i^*)$. Hence by generalized *modus ponens*, $\vdash \forall \ldots B_i^*$. ■

When invoking Lemma 10 or Lemma 11 I'll say "by PC". In particular we have the transitivity of $\rightarrow$:

$$\forall \ldots (A \rightarrow B),\ \forall \ldots (B \rightarrow C) \vdash \forall \ldots (A \rightarrow C)$$

Lemma 12 If $y_1 \ldots y_n$ are the variables free in A in alphabetic order, then:

$$\vdash \forall \ldots A \rightarrow \forall y_k \forall y_1 \ldots \forall y_{k-1} \forall y_{k+1} \ldots \forall y_n A .$$

Proof We induct on n. For $n = 0$ or 1, this is by PC. For $n = 2$ this is an instance of Axiom 2. Suppose now the lemma is true for fewer than n variables. If $k \neq n$, we are done by induction (replacing A by $\forall y_n$ A). If $k = n$, then by Axiom 1 and Lemma 2:

$$\vdash \forall y_1 \ldots \forall y_{n-2} (\forall y_{n-1} \forall y_n A) \rightarrow \forall y_1 \ldots \forall y_{n-2} (\forall y_n \forall y_{n-1} A)$$

By induction we have:

$$\vdash \forall y_1 \ldots \forall y_{n-2} \forall y_n (\forall y_{n-1} A) \rightarrow \forall y_n \forall y_1 \ldots \forall y_{n-2} (\forall y_{n-1} A)$$

Hence, by Lemma 9, we are done. ■

Lemma 13 Let $B(x)$ be a formula with one free variable x.

a. If $\vdash B(c|x)$, then $\vdash \forall x\, B(x)$.

b. If $\Gamma \vdash B(c|x)$ and c does not appear in any wff in Γ, then $\Gamma \vdash \forall x\, B(x)$.

Proof a. We proceed by induction on the length of a proof of $B(c|x)$. If the length of a proof is 1, then $B(c|x)$ is an axiom. I will show that (a) holds for certain instances of Axiom 1; the other instances of Axiom 1 and the other schema follow similarly.

So suppose $\vdash (\forall \ldots)_1 (C(c|x) \rightarrow D) \rightarrow (\forall \ldots C(c|x) \rightarrow \forall \ldots D)$. Another instance of Axiom 1 is $\vdash (\forall \ldots)_2 (C(x) \rightarrow D) \rightarrow (\forall \ldots C(x) \rightarrow \forall \ldots D)$, where the only difference between $(\forall \ldots)_2$ and $(\forall \ldots)_1$ is that $\forall x$ appears in the former and not in the latter. So by Lemma 12 we are done.

Suppose now that (a) is true for theorems with proofs of length m, $1 \leq m < n$, and the shortest proof of $B(c|x)$ has length n. Then for some closed A, $\vdash$A and $\vdash A \rightarrow B(c|x)$, both of which have proofs shorter than length n. By induction, $\vdash \forall x\, (A \rightarrow B(x))$, so by Axiom 1 and PC, $\vdash A \rightarrow \forall x\, B(x)$. Hence, $\vdash \forall x\, B(x)$.

b. Suppose $\Gamma \vdash B(c|x)$. Then for some closed $D_1, \ldots, D_n \in \Gamma$, $\{D_1, \ldots, D_n\} \vdash B(c|x)$. Hence by the syntactic deduction theorem:

$$\vdash D_1 \rightarrow (D_2 \rightarrow \cdots \rightarrow (D_n \rightarrow B(c|x)) \ldots)$$

Since c does not appear in any of $D_1, \ldots, D_n$, we have by (a):

$\vdash \forall x\, (D_1 \to (D_2 \to \cdots \to (D_n \to B(x)) \ldots))$

So by Axiom 1 (repeated if necessary):

$\vdash D_1 \to (D_2 \to \cdots \to (D_n \to \forall x\, B(x)) \ldots)$

By repeated use of *modus ponens*, we have $\{D_1, \ldots, D_n\} \vdash \forall x\, B(x)$.
So $\Gamma \vdash \forall x\, B(x)$. ■

NOTE: Only Axioms 1, 2, and the propositional axioms are invoked in the proofs of Lemmas 8–13.

Lemma 14 When x is free in A,

a. $\vdash \forall \ldots (A(c|x) \to \exists x\, A(x))$ $\exists$-*introduction*

b. $\vdash \forall \ldots (\forall x\, A(x) \to \exists x\, A(x))$ $\forall$ *implies* $\exists$

Proof a. By Axiom 3, $\vdash \forall \ldots (\forall x \neg A(x) \to \neg A(c|x))$. So by PC, $\vdash \forall \ldots (A(c|x) \to \neg \forall x \neg A(x))$. So by axiom 4.b and PC, $\vdash \forall \ldots (A(c|x) \to \exists x\, A(x))$.

b. We have by Axiom 3, $\vdash \forall \ldots (\forall x\, A(x) \to A(c|x))$ for the first name symbol in the language. So by (a) and the transitivity of $\to$, we have (b). ■

Theorem 15 Soundness If $\Gamma \vdash A$, then $\Gamma \vDash A$.

Theorem 16 Let Γ be a consistent set of closed wffs of L. Let $L(w_0, w_1, \ldots)$ be L with the additional name symbols $w_0, w_1, \ldots$. Then there is a collection of closed wffs Σ in $L(w_0, w_1, \ldots)$ such that:

a. $\Gamma \subseteq \Sigma$.

b. Σ is a complete and consistent theory.

c. If $\exists x B \in \Sigma$ and x is free in B, then for some m, $B(w_m|x) \in \Sigma$.

d. For every wff $B(x)$ in $L(w_0, w_1, \ldots)$ with one free variable, if for each i, $B(w_i|x) \in \Sigma$, then $\forall x\, B(x) \in \Sigma$.

Proof Let $A_0, A_1, \ldots$ be a numbering of the closed wffs of the expanded language $L(w_0, w_1, \ldots)$. Let $\vdash$ refer to derivations in this language. Define Σ by stages:

$\Sigma_0 = \Gamma$

Σ_{n+1} is defined by cases:

i. If $\Sigma_n \vdash \neg A_n$, then $\Sigma_{n+1} = \Sigma_n \cup \{\neg A_n\}$.

If $\Sigma_n \nvdash \neg A_n$, then:

ii. If A_n is not $\exists x\, B$, then $\Sigma_{n+1} = \Sigma_n \cup \{A_n\}$.

iii. If $\Sigma_n \nvdash \neg A_n$ and A_n is $\exists x\, B$, and w_m is the least w_i that does not appear in Σ_n, then $\Sigma_{n+1} = \Sigma_n \cup \{\exists x\, B,\ B(w_m|x),\ \exists x\, (x \equiv w_m)\}$.

$\Sigma = \bigcup_n \Sigma_n$

Part (a) follows by construction. For (b), I'll first show by induction that for each n, Σ_n is consistent. We have $n = 0$ by hypothesis. If Σ_n is consistent, then if Σ_{n+1} is defined by (i) it's immediate. If it is defined by (ii) it follows by induction and Lemma 3. So suppose it is defined by (iii). Then $\Delta = \Sigma_n \cup \{\exists x\, \mathrm{B}(x)\}$ is consistent by Lemma 3. Suppose now that Σ_{n+1} is not consistent. Then by Lemma 3, $\Delta \vdash \neg\mathrm{B}(\mathrm{w}_m|\,x)$. So by Lemma 13, $\Delta \vdash \forall x \neg \mathrm{B}(x)$. Hence, by Axiom 4 and PC, $\Delta \vdash \neg\exists x\, \mathrm{B}(x)$. But then Δ is not consistent, which is a contradiction. So Σ_{n+1} is consistent.

It then follows that Σ is consistent, for if it were not, then some finite subset of it would be inconsistent, and hence some Σ_n would be inconsistent. For every A, by construction either $\mathrm{A} \in \Sigma$ or $\neg\mathrm{A} \in \Sigma$. So Σ is complete, and hence by Lemma 3, Σ is a theory, and (b) is proved.

For (c), suppose $\exists x\, \mathrm{B}(x) \in \Sigma$ and x is free in B. Then for some n and m, $\Sigma_{n+1} = \Sigma_n \cup \{\exists x\, \mathrm{B}, \mathrm{B}(\mathrm{w}_m\,|\,x)\} \subseteq \Sigma$.

For (d), I'll show the contrapositive. Suppose $\forall x\, \mathrm{B}(x) \notin \Sigma$. Then by (b), $\neg\forall x\, \mathrm{B}(x) \in \Sigma$. Hence by Axiom 4 and PC, $\exists x\, \neg\mathrm{B}(x) \in \Sigma$. So by (c), for some m, $\neg\mathrm{B}(\mathrm{w}_m|\,x) \in \Sigma$. So by the consistency of Σ, $\mathrm{B}(\mathrm{w}_m|\,x) \notin \Sigma$. ■

Theorem 17 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in $\mathrm{L}(\mathrm{w}_0, \mathrm{w}_1, \ldots)$ be as in Theorem 16. We define a model M of $\mathrm{L}(\mathrm{w}_0, \mathrm{w}_1, \ldots)$. We take the formal language to be the semi-formal language.

$\mathsf{U} = \{\mathrm{w}_i, \mathrm{c}_i : i \geq 0\}$

For every σ, $\sigma(\mathrm{c}_i) = \mathrm{c}_i$ and $\sigma(\mathrm{w}_i) = \mathrm{w}_i$.

$\mathsf{v}_\sigma \models \mathrm{P}_i^n(v_1, \ldots, v_n)$ iff $\mathrm{P}_i^n(\sigma(v_1), \ldots, \sigma(v_n)) \in \Sigma$

I'll let you show by induction on the length of wffs that for $\mathrm{d} \in \mathsf{U}$, and any σ, and any wff B:

(‡) If $\sigma(x) = \mathrm{d}$, then $\mathsf{v}_\sigma \models \mathrm{B}(x)$ iff $\mathsf{v}_\sigma \models \mathrm{B}(\mathrm{d}\,|\,x)$

Now we'll show by induction on the length of wffs that for every closed wff A in $\mathrm{L}(\mathrm{w}_0, \mathrm{w}_1, \ldots)$, $\mathsf{M} \models \mathrm{A}$ iff $\mathrm{A} \in \Sigma$. It is immediate if A is atomic. Suppose it is true for wffs shorter than A. I'll do the cases when A is $\exists x\, \mathrm{B}$ or $\forall x\, \mathrm{B}$ and leave the others to you.

Suppose A is $\exists x\, \mathrm{B}$ and $\exists x\, \mathrm{B}(x) \in \Sigma$. Then x is the only variable free in $\mathrm{B}(x)$. Since Σ satisfies the conditions in Theorem 16, for some m, $\mathrm{B}(\mathrm{w}_m|x) \in \Sigma$. Hence by induction, for any σ, $\mathsf{v}_\sigma \models \mathrm{B}(\mathrm{w}_m|x)$, and so by (‡), for any σ, if $\sigma(x) = \mathrm{w}_m$, then $\mathsf{v}_\sigma \models \mathrm{B}(x)$. So $\mathsf{M} \models \exists x\, \mathrm{B}(x)$.

Suppose A is $\exists x\, \mathrm{B}$ and $\mathsf{M} \models \exists x\, \mathrm{B}(x)$. Then for some σ, $\mathsf{v}_\sigma \models \mathrm{B}(x)$. Take one where $\sigma(x) = \mathrm{d}$. Then by (‡), $\mathsf{v}_\sigma \models \mathrm{B}(\mathrm{d}\,|\,x)$. Since $\mathrm{B}(\mathrm{d}\,|\,x)$ is closed, $\mathsf{M} \models \mathrm{B}(\mathrm{d}\,|\,x)$, so $\mathrm{B}(\mathrm{d}\,|\,x) \in \Sigma$. By Lemma 14.a, $\vdash \mathrm{B}(\mathrm{d}\,|\,x) \rightarrow \exists x\, \mathrm{B}(x)$, so $\exists x\, \mathrm{B}(x) \in \Sigma$.

Suppose A is $\forall x$ B and $\mathsf{M} \vDash \forall x\, B(x)$. Then for every $i \geq 0$, if $\sigma(x) = w_i$, then $\mathsf{v}_\sigma \vDash B(x)$. So by (‡), for every i, for every σ, $\mathsf{v}_\sigma \vDash B(w_i|x)$. So by induction, for every i, $B(w_i|x) \in \Sigma$. Since Σ satisfies the condition (d) of Theorem 16, $\forall x\, B(x) \in \Sigma$.

Suppose A is $\forall x$ B and $\forall x\, B(x) \in \Sigma$. Since Σ is a theory, by Axiom 3 (universal instantiation), for all $d \in \mathsf{U}$, $B(d|x) \in \Sigma$. Hence by induction, for every σ, $\mathsf{v}_\sigma \vDash B(d|x)$. So by (‡), for every σ, $\mathsf{v}_\sigma \vDash \forall x\, B(x)$. Hence, $\mathsf{M} \vDash \forall x\, B(x)$.

To complete the proof of the theorem, define a structure N for L by taking M and deleting the interpretations of the w_i's (the universes are the same). Then for any closed wff A in L, $\mathsf{M} \vDash A$ iff $\mathsf{N} \vDash A$ by the partial interpretation theorem.[1] So $\mathsf{N} \vDash \Gamma$. And N is countable. ■

Theorem 18

a. For any collection of closed wffs Σ in L, Σ is a complete and consistent theory iff there is a model M such that $\mathsf{M} \vDash A$ iff $A \in \Sigma$.

b. For any model M of L, there is a countable model M^* such that $\mathsf{M}^* \vDash A$ iff $\mathsf{M} \vDash A$.

c. *Strong completeness* $\Gamma \vdash A$ iff $\Gamma \vDash A$.

d. *Compactness* Γ has a model iff every finite subset of Γ has a model.

Proof a. If $\Sigma = \{A\colon \mathsf{M} \vDash A\}$, then Σ is a complete and consistent theory. The converse follows by Theorem 17.

b. If M is a model, then Th(M) is complete and consistent, so by Theorem 17, there is a countable model M^* such that Th(M) = Th(M^*).

Parts (c) and (d) follow as for Theorem 7. ■

The proof of Theorem 16 is where infinitistic assumptions enter into our proof of strong completeness. In *Models and Ultraproducts* J. Bell and A. B. Slomson show that the strong completeness theorem for classical predicate logic is equivalent to the axiom of choice for countable sets.

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[1] Two models M and N *agree on* L(Γ) when the models have the same universe and the same interpretations of the names and the predicates that are in L(Γ). That is, the following all hold:

i. $\mathsf{U}_\mathsf{M} = \mathsf{U}_\mathsf{N}$.

ii. For every name c appearing in Γ, for every σ on M and τ on N, $\sigma(c) = \tau(c)$.

iii. For every n-ary predicate P appearing in Γ, whenever $\sigma(y_i) = \tau(y_i)$ for each $i \leq n$, $\sigma \vDash P(y_1, \ldots, y_n)$ iff $\tau \vDash P(y_1, \ldots, y_n)$.

The partial interpretation theorem If M and N agree on L(A), then $\mathsf{M} \vDash A$ iff $\mathsf{N} \vDash A$.

II Classical Predicate Logic with Equality

Lemma 19

a. $\vdash \forall x \forall y\, (x \equiv y \rightarrow y \equiv x)$

b. $\vdash \forall x\, \forall y\, \forall z\, (x \equiv y \rightarrow (y \equiv z \rightarrow x \equiv z))$

Proof a.

1.	$\vdash \forall x\, (x \equiv x)$	identity Axiom
2.	$\vdash \forall x\, \forall y\, (x \equiv y \rightarrow (x \equiv x \rightarrow y \equiv x))$	extensionality Axiom
3.	$\vdash \forall x\, \forall y\, (x \equiv x \rightarrow (x \equiv y \rightarrow y \equiv x))$	(2) and PC
4.	$\vdash \forall x\, \forall y\, (x \equiv y \rightarrow y \equiv x)$	(1), (3), and Lemma 5

b.

1.	$\vdash \forall z\, \forall y\, \forall x\, (y \equiv x \rightarrow (y \equiv z \rightarrow x \equiv z))$	extensionality Axiom
2.	$\vdash \forall x\, \forall y\, (x \equiv y \rightarrow y \equiv x)$	part (a)
3.	$\vdash \forall x\, \forall y\, \forall z\, (x \equiv y \rightarrow (y \equiv z \rightarrow x \equiv z))$	(1), (2), PC, Lemma 5 ■

Note: The only axioms used in proving Lemma 19 are the propositional axioms, Axioms 1 and 2, and the equality axioms.

In proving the strong completeness of this axiomatization, the lemmas and theorems through Theorem 17, along with their proofs, are the same as for classical predicate logic without equality.

Theorem 20 Every consistent set of closed wffs of L has a countable model in which "≡" is interpreted as the identity of the universe.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ be constructed as in the proof of Theorem 16. By Theorem 17, there is a countable model M of Σ, and hence of Γ. In that model, write '≃' for the interpretation of '≡', that is, $c \simeq d$ iff $(c \equiv d) \in \Sigma$. That need not be the identity on the universe of M.

However, $\simeq$ is an equivalence relation by virtue of Axiom 5 (identity), Lemma 19, and Axiom 3. Due to Axiom 6 (extensionality), $\simeq$ is a congruence relation, too. Denote by [d] the equivalence class of d for $\simeq$. We define a model $\mathsf{M}_{/\simeq}$ in which '≡' is interpreted as the identity on the universe such that $\mathsf{M}_{/\simeq}$ validates exactly the same wffs as M.

The semi-formal language of $\mathsf{M}_{/\simeq}$ is the same as for M. The universe of $\mathsf{M}_{/\simeq}$ is {[d] : d is in the universe of M}. For each assignment of references σ of M define an assignment of references $\sigma_{/\simeq}(v)$ by setting for every term v, $\sigma_{/\simeq}(v) = [\sigma(v)]$. Since the collection for M is complete, the collection of these assignments is complete. For each assignment of references define the valuation:

$$\mathsf{v}_{\sigma/\simeq} \vDash P_i^n(v_1, \ldots, v_n) \quad \text{iff} \quad \mathsf{v}_\sigma \vDash P_i^n(v_1, \ldots, v_n)$$
$$\text{iff} \quad \mathrm{P}_i^n(\sigma(v_1), \ldots, \sigma(v_n)) \in \Sigma$$

I'll let you show that for every formal wff A and every σ for M, $\mathsf{v}_{\sigma/\simeq} \vDash \mathrm{A}$ iff $\mathsf{v}_\sigma \vDash \mathrm{A}$. Hence, for every A, $\mathsf{M} \vDash \mathrm{A}$ iff $\mathsf{M}_{/\simeq} \vDash \mathrm{A}$. In $\mathsf{M}_{/\simeq}$ we have:

$$\begin{aligned} \mathrm{v}_{\sigma/\simeq} \vDash v \equiv u \quad & \text{iff} \;\; (\sigma(v) \equiv \sigma(u)) \in \Sigma \\ & \text{iff} \;\; \sigma(v) \simeq \sigma(u) \\ & \text{iff} \;\; [\sigma(v)] = [\sigma(u)] \end{aligned} \qquad \blacksquare$$

Theorem 18 then follows for this axiom system.

III Classical Predicate Logic with Functions

The proof of strong completeness is just as for classical predicate logic with equality, noting only that in the proof of Theorem 20, by the extensionality axiom for functions, we can define:

$$\sigma/\!\simeq(f(u_1, \ldots, u_n)) = [\sigma(f(v_1, \ldots, v_n))]$$

IV Classical Predicate Logic with Non-Referring Names

This logic is presented in Chapter 36.

From Section I on classical predicate logic, Lemmas 1–13 hold as the schema needed for those are the same here. The proof of Lemma 19 proceeds as before using the new version of the extensionality axiom. I'll let you prove that the axiomatization is sound.

Theorem 21 Let Γ be a consistent set of closed wffs of L. Then there is a collection of closed wffs Σ in $L(w_0, w_1, \ldots)$ such that:

a. $\Gamma \subseteq \Sigma$.

b. Σ is a complete and consistent theory.

c. If $\exists x\, B \in \Sigma$, then for some m, $B(w_m|x) \in \Sigma$ and $\exists x\,(x \equiv w_m) \in \Sigma$.

d. If $\neg\forall x\, B \in \Sigma$, then for some m, $\neg B(w_m|x) \in \Sigma$.

e. For every wff $B(x)$ in $L(w_0, w_1, \ldots)$ with one free variable, if for each i, $B(w_i|x) \in \Sigma$, then $\forall x\, B(x) \in \Sigma$.

Proof Let $A_0, A_1, \ldots$ be a numbering of all the closed wffs of the expanded language $L(w_0, w_1, \ldots)$. Let '$\vdash$' refer to derivations in this language.

Define Σ by stages:

$\Sigma_0 = \Gamma$

Σ_{n+1} is defined by cases:

i. If $\Sigma_n \vdash \neg A_n$, then $\Sigma_{n+1} = \Sigma_n \cup \{\neg A_n\}$.

If $\Sigma_n \nvdash \neg A_n$, then:

ii. If A_n is not $\exists x\, B$ or $\neg\forall x\, B$, then $\Sigma_{n+1} = \Sigma_n \cup \{A_n\}$.

iii. If $\Sigma_n \nvdash \neg A_n$ and A_n is $\exists x\, B$, and w_m is the least w_i that does not appear in Σ_n, then $\Sigma_{n+1} = \Sigma_n \cup \{\exists x\, B, B(w_m|x), \exists x\,(x \equiv w_m)\}$.

iv. if $\Sigma_n \nvdash \neg A_n$ and A_n is $\neg\forall x\, B$, and w_m is the least w_i that does not appear in Σ_n, then $\Sigma_{n+1} = \Sigma_n \cup \{\neg\forall x\, B, \neg B(w_m|x)\}$.

$\Sigma = \bigcup_n \Sigma_n$

I'll show that Σ satisfies (a)–(e). Part (a) follows by construction.

For part (b) we first show by induction that for each n, Σ_n is consistent. For $n = 0$ it's true by hypothesis. If it's true for n and Σ_{n+1} is defined by (i) it's immediate. If Σ_{n+1} is defined by (ii) it follows by induction and Lemma 3.

Suppose Σ_{n+1} is defined by (iii). Then $\Delta = \Sigma_n \cup \{\exists x\, B(x)\}$ is consistent by induction and Lemma 3. Suppose Σ_{n+1} is not consistent. Then we have that $\Delta \vdash \neg(\exists x\,(x \equiv w_m) \wedge B(w_m|x))$. Hence, by Lemma 13, there is a y not appearing in B such that $\Delta \vdash \forall y\, \neg(\exists x\,(x \equiv y) \wedge B(y|x))$. But then by Axiom 14, $\Delta \vdash \neg\exists x\, B(x)$, which is a contradiction.

Suppose Σ_{n+1} is defined by (iv). Then by Lemma 3 and PC, $\Delta = \Sigma_n \cup \{\neg\forall x\, B(x)\}$ is consistent. Suppose that Σ_{n+1} is not consistent. Then $\Delta \vdash B(w_m|x)$. So by Lemma 13, for some y, $\Delta \vdash \forall y\, B(y)$. Hence by Axiom 3, $\Delta \vdash \forall x\, B(x)$, which contradicts the consistency of Δ.

It then follows that Σ is consistent, for if it were not then some finite subset of it would be inconsistent, and hence some Σ_n would be inconsistent. By construction, for every A, either $A \in \Sigma$ or $\neg A \in \Sigma$, and so Σ is complete. By Lemma 3, Σ is also a theory.

Parts (c) and (d) follows by construction, since if $A \in \Sigma$, then for some n, A is $A_n$and the appropriate formulas are put into Σ at stage $n + 1$.

For part (e) I'll show the contrapositive. Suppose $\forall x\, B(x) \notin \Sigma$. Then by (b), $\neg\forall x\, B(x) \in \Sigma$. Hence by (d), for some m, $\neg\, B(w_m|x) \in \Sigma$. So $B(w_m|x) \notin \Sigma$. ■

Theorem 22 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in $L(w_0, w_1, \ldots)$ be as in Theorem 21. We define a model M of $L(w_0, w_1, \ldots)$.

$\mathsf{U} = \{c_i\text{: for some } x, \exists x\, (x \equiv c_i) \in \Sigma\} \cup \{w_i\text{: for some } x, \exists x\, (x \equiv w_i) \in \Sigma\}$

Assignments of references:

For every σ and every x, $\sigma(x)\downarrow$, and the collection of such σ is complete.

For every atomic name d:

$\sigma(d)\downarrow$ iff $d \in \mathsf{U}$. If $\sigma(d)\downarrow$, then $\sigma(d) = d$.

Evaluation of the equality predicate:

$\mathsf{v}_\sigma \vDash v \equiv u$ iff $\sigma(v)\downarrow = c$ and $\sigma(u)\downarrow = d$, and $(c \equiv d) \in \Sigma$
or both are undefined and $(v \equiv u) \in \Sigma$

The interpretation of "$\equiv$" satisfies restrictions (ii)–(vi) on the evaluation of the equality predicate: conditions (ii) and (iii) follow by definition; condition (iv) follows because the axiom of identity and universal instantiation are in Σ; conditions (v) and (vi) follow by Lemma 19. Note that if $(v \equiv u) \in \Sigma$ and $\sigma(v)\downarrow$, then for some x, $\exists x\, (x \equiv v) \in \Sigma$, so by the axioms of extensionality of atomic predications and Σ being a theory, $\exists x\, (x \equiv u) \in \Sigma$, so that $\sigma(u) \in \Sigma$.

Valuations of atomic wffs other than the equality predicate:

Given $A(v_1, \ldots, v_n)$ and σ, let $y_1, \ldots, y_n$ be a list of all the variables appearing in A, and let $\sigma(y_i) = d_i$. Let $A(v_1, \ldots, v_n)[d_i|y_i]$ denote A with each y_i replaced by d_i. Then:

$\mathsf{v}_\sigma \vDash A(v_1, \ldots, v_n)$ iff $A(v_1, \ldots, v_n)[d_i|y_i] \in \Sigma$.

I'll let you show that the valuations of atomic wffs satisfy the extensionality condition. This completes the definition of M.

I'll let you show by induction on the length of wffs that for any d in the universe and for any assignment of references σ, for any wff B:

(‡) If $\sigma(x) = \mathrm{d}$, then $v_\sigma \vDash \mathrm{B}(x)$ iff $v_\sigma \vDash \mathrm{B}(\mathrm{d}|x)$.

Now we show by induction on the length of A that for every closed wff A in $\mathrm{L}(\mathrm{w}_0, \mathrm{w}_1, \ldots)$, $\mathsf{M} \vDash \mathrm{A}$ iff $\mathrm{A} \in \Sigma$. It is true for wffs of length 1 by definition. Suppose that the lemma is true for all wffs of length *n*, and let A be a wff of length $n + 1$. I'll leave to you the cases when A is $\neg\mathrm{B}$, $\mathrm{B} \to \mathrm{C}$, $\mathrm{B} \wedge \mathrm{C}$, and $\mathrm{B} \wedge \mathrm{C}$.

Suppose that A is $\exists x\, \mathrm{B}$ and $\mathsf{M} \vDash \exists x\, \mathrm{B}(x)$. Then for some σ, $v_\sigma \vDash \mathrm{B}(x)$. Take one such σ, where $\sigma(x) = \mathrm{d}$. By (‡), $v_\sigma \vDash \mathrm{B}(\mathrm{d}|x)$. Since d is in the universe, for some *y*, $\exists y\, (y \equiv \mathrm{d}) \in \Sigma$. Since $\mathrm{B}(\mathrm{d}|x)$ is closed, $\mathsf{M} \vDash \mathrm{B}(\mathrm{d}|x)$, so by induction $\mathrm{B}(\mathrm{d}|x) \in \Sigma$. Hence, using Axiom 9 (existential generalization for referring terms), $\exists x\, \mathrm{B}(x) \in \Sigma$.

Suppose that A is $\exists x\, \mathrm{B}$ and $\exists x\, \mathrm{B}(x) \in \Sigma$. Since Σ satsifies condition (c) of Theorem 1, for some *m*, $\mathrm{B}(\mathrm{w}_m|x) \in \Sigma$ and $\mathrm{w}_m \in \mathsf{U}$. Hence by induction, $\mathsf{M} \vDash \mathrm{B}(\mathrm{w}_m| x)$, and so for any σ, if $\sigma(x) = \mathrm{w}_m$, then $v_\sigma \vDash \mathrm{B}(x)$. So $\mathsf{M} \vDash \exists x\, \mathrm{B}(x)$.

Suppose that A is $\forall x\, \mathrm{B}$ and $\mathsf{M} \vDash \forall x\, \mathrm{B}$. Then for each *i*, $\mathsf{M} \vDash \mathrm{B}(\mathrm{w}_i|x)$. Hence by induction, for each *i*, $\mathrm{B}(\mathrm{w}_i|x) \in \Sigma$. Since Σ satisfies condition (e), $\forall x\, \mathrm{B} \in \Sigma$.

Suppose that A is $\forall x\, \mathrm{B}$ and $\forall x\, \mathrm{B} \in \Sigma$. Then by universal instantiation, for every d, $\mathrm{B}(\mathrm{d}|x) \in \Sigma$. Hence by induction, $\mathsf{M} \vDash \mathrm{B}(\mathrm{d}|x)$. Hence, for all σ, for all d, $v_\sigma \vDash \mathrm{B}(\mathrm{d}|x)$ as $\mathrm{B}(\mathrm{d}|x)$ is closed. So by (‡), for each σ, $v_\sigma \vDash \mathrm{B}(x)$, so $\mathsf{M} \vDash \mathrm{B}(x)$.

I'll let you show using Lemma 20 that the interpretation of "≡", which I'll denote by "≃", is an equivalence relation on the universe. Using the extensionality axioms you can show that it is also a congruence relation. Define a model $\mathsf{M}_{/\simeq}$ by:

$\mathsf{U} = \{ [\mathrm{d}] : \mathrm{d} \in \Sigma \}$

$\sigma_{/\simeq}(v) \equiv_{\mathrm{Def}} [\sigma(v)]$ if $\sigma(v)\!\downarrow$, and undefined otherwise

For every atomic wff $\mathrm{A}(u_1, \ldots, u_n)$,
$v_{\sigma/\simeq} \vDash \mathrm{A}(u_1, \ldots, u_n)$ iff $v_\sigma \vDash \mathrm{A}(u_1, \ldots, u_n)$

I'll let you show that $\mathsf{M}_{/\simeq} \vDash \mathrm{A}$ iff $\mathsf{M} \vDash \mathrm{A}$. It then follows that N is a model of Σ in which "≡" satisfies the restrictions on the equality predicate, including that it is the identity on the universe.

Now define a model N' for L by taking N and deleting the interpretations of the w_i's (the universes are the same). For any closed wff A in L, $\mathsf{N} \vDash \mathrm{A}$ iff $\mathsf{N}' \vDash \mathrm{A}$ by the partial interpretation theorem, which you can show holds here. So $\mathsf{N} \vDash \Gamma$. ∎

The strong completeness theorem then follows in the usual way.

V Classical Predicate Logic with Partial Functions

The language L is that of classical predicate logic with equality and functions. Theorem 21 is proved as before. Lemma 19 is proved as before using the new extensionality axiom. I'll let you to prove the axiomatization is sound.

Theorem 23 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in $L(w_0, w_1, \ldots)$ be as in Theorem 21. We define a model M of $L(w_0, w_1, \ldots)$.

$\mathsf{U} = \{c_i\text{: for some } x, \exists x\,(x \equiv c_i) \in \Sigma\} \cup \{w_i\text{: for some } x, \exists x\,(x \equiv w_i) \in \Sigma\}$

Assignments of references:

For every σ and every x, $\sigma(x)\downarrow$, and the collection of such σ is complete.

For every atomic name c:

$\sigma(c)\downarrow$ iff $c \in \mathsf{U}$. If $\sigma(c)\downarrow$, then $\sigma(c) = c$.

For terms $f(v_1, \ldots, v_n)$ of depth 1:

If some v, $\sigma(v_i)\not\downarrow$, then $\sigma(f(v_1, \ldots, v_n))\not\downarrow$.

If for all i, $\sigma(v_i)\downarrow$, let $\sigma(v_i) = d_i$. If there is a c such that $(f(d_1, \ldots, d_n) \equiv c) \in \Sigma$ and some x, $(\exists x\,(x \equiv c)) \in \Sigma$, let d be the least such (where each c_i precedes each w_i) and set $\sigma(f(v_1, \ldots, v_n))\downarrow = d$. Otherwise, $\sigma(f(v_1, \ldots, v_n))\not\downarrow$.

For terms $f(v_1, \ldots, tv_n)$ of depth > 1 :

If some i, $\sigma(v_i)\not\downarrow$, then $\sigma(f(v_1, \ldots, v_n))\not\downarrow$.

If for all i, $\sigma(v_i)\downarrow$, let $z_1, \ldots, z_n$ be the first variables not appearing in $f(v_1, \ldots, v_n)$. Let τ be an assignment of references that differs from σ only in that $\tau(z_i) = \sigma(v_i)$. Then $\sigma(f(v_1, \ldots, v_n)) \approx \tau(f(z_1, \ldots, z_n))$.

The definition of assignments for terms of depth 1 gives a unique value for each term, as we choose the least such name symbol d. I'll let you show that this definition satisfies the extensionality condition for atomic values of functions.

Evaluation of the equality predicate:

Atomic terms

$\mathsf{v}_\sigma \vDash v \equiv u$ iff $\sigma(v)\downarrow = c$ and $\sigma(u)\downarrow = d$, and $(c \equiv d) \in \Sigma$
or both are undefined and $(v \equiv u) \in \Sigma$

Complex terms:

a. $f(v_1, \ldots, v_n) \equiv x$.
For each variable y_i appearing in $f v_1, \ldots, v_n)$, there is a d_i such that $\sigma(y_i) = d_i$, and there is some d, such that $\sigma(x) = d$. Denote by

$f(v_1, \ldots, v_n)[d_i|y_i]$ the result of substituting d_i for y_i everywhere in $f(v_1, \ldots, v_n)$. Then

$$\mathsf{v}_\sigma \vDash f(v_1, \ldots, v_n) \equiv x \text{ iff } (f(v_1, \ldots, v_n)[d_i|y_i] \equiv d) \in \Sigma.$$

b. $x \equiv f(v_1, \ldots, v_n)$
$\mathsf{v}_\sigma \vDash x \equiv f(v_1, \ldots, v_n)$ iff $\mathsf{v}_\sigma \vDash f(v_1, \ldots, v_n) \equiv x$.

c. $f(v_1, \ldots, v_n) \equiv d$.
Proceed as in (i), deleting the clause that has $\sigma(x) = d$.

d. $d \equiv f(v_1, \ldots, v_n)$
$\mathsf{v}_\sigma \vDash d \equiv f(v_1, \ldots, v_n)$ iff $\mathsf{v}_\sigma \vDash f(v_1, \ldots, v_n) \equiv d$.

e. $f(v_1, \ldots, v_n) \equiv g(u_1, \ldots, u_m)$.
Define y_i and d_i as in (i), and similarly for $g(u_1, \ldots, u_m)$ define z_i and e_i. Then $\mathsf{v}_\sigma \vDash f(v_1, \ldots, v_n) \equiv g(u_1, \ldots, u_n)$ iff $(f(v_1, \ldots, v_n)[d_i|y_i] \equiv g(u_1, \ldots, u_n)[e_i|z_i]) \in \Sigma$.

Using the equality axiom schema (5) and (6*) you can show that this interpretation of "≡" satisfies restrictions (ii), (iii*), (iv), (v), and (vi) on equality.

The proof then follows as for Theorem 22, using Axiom 7.*a to show that the interpretation of "≡" in U is a congruence relation. ■

VI Classical Predicate Logic with Non-Referring Names as Nil

Because the propositional axioms and Axioms 1 and 2 are as for classical predicate logic, Lemmas 8–13 hold. I'll let you prove the axiomatization is sound.

Theorem 24 Let Γ be a consistent set of closed wffs of L. Then there is a collection of closed wffs Σ in L($w_0, w_1, \ldots$) such that:

a. $\Gamma \subseteq \Sigma$.

b. Σ is a complete and consistent theory.

c. If $\exists x\, B \in \Sigma$, then for some m, $B(w_m|x) \in \Sigma$ and $\exists x\, (x \equiv w_m) \in \Sigma$.

d. For every wff $B(x)$ in L($w_0, w_1, \ldots$) with one free variable,
if for each i, $B(w_i|x) \in \Sigma$, then $\forall x\, B(x) \in \Sigma$.

Proof The proof is the same as for Theorem 16 for classical predicate logic except for the proof that if $\Sigma_{n+1} = \Sigma_n \cup \{\exists x\, B,\ B(w_m|x), \exists x\, (x \equiv w_m)\}$ then Σ_{n+1} is consistent if Σ_n is. We have that $\Delta = \Sigma_n \cup \{\exists x\, B(x)\}$ is consistent by induction and Lemma 3. Suppose Σ_{n+1} is not consistent. Then $\Delta \vdash \neg (\exists x\, (x \equiv w_m) \wedge B(w_m|x))$. Hence, by Lemma 13, there is a y not appearing in B such that $\Delta \vdash \forall y \neg\, (\exists x\, (x \equiv y) \wedge\ B(y|x))$. But then by Axiom 14, $\Delta \vdash \neg\, \exists x\, B(x)$, which is a contradiction. ■

Theorem 25 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in L($w_0, w_1, \ldots$) be as in Theorem 24. We define a model M of L($w_0, w_1, \ldots$).

$\mathsf{U} = \{c_i$: for some x, $\exists x\, (x \equiv c_i) \in \Sigma \} \cup \{w_i$: $i \geq 0\}$

By construction, for each i, $\exists x\, (x \equiv w_i) \in \Sigma$.

Assignments of references:

For every σ and every x, $\sigma(x)\downarrow$, and the collection of such σ is complete.

For every atomic name c:

$\sigma(c)\downarrow$ iff $c \in \mathsf{U}$. If $\sigma(c)\downarrow$, then $\sigma(c) = c$.

For terms $f(v_1, \ldots, v_n)$ of depth 1:

If some t, $\sigma(v_i)\not\downarrow$, then $\sigma(f(v_1, \ldots, v_n))\not\downarrow$.

If for all i, $\sigma(v_i)\downarrow$, let $\sigma(v_i) = d_i$. If there is a c such that
$(f(d_1, \ldots, d_n) \equiv c) \in \Sigma$ and some x, $(\exists x\, (x \equiv c)) \in \Sigma$,
let d be the least such (where each c_i precedes each w_i) and
set $\sigma(f(v_1, \ldots, v_n))\downarrow = d$. Otherwise, $\sigma(f(v_1, \ldots, v_n))\not\downarrow$.

For terms $f(v_1, \ldots, v_n)$ of depth > 1:

If some i, $\sigma(v_i)\not\downarrow$, then $\sigma(f(v_1, \ldots, v_n))\not\downarrow$.
If for all i, $\sigma(v_i)\downarrow$, let $z_1, \ldots, z_n$ be the first variables

not appearing in $f(v_1, \ldots, v_n)$. Let τ be an assignment of references that differs from σ only in that $\tau(z_i) = \sigma(t_i)$. Then $\sigma(f(v_1, \ldots, v_n)) \approx \tau(f(z_1, \ldots, z_n))$.

Evaluation of the equality predicate:

$\mathsf{v}_\sigma \vDash v \equiv u$ iff $\sigma(v)\downarrow = \mathrm{c}$ and $\sigma(u)\downarrow = \mathrm{d}$, and $(\mathrm{c} \equiv \mathrm{d}) \in \Sigma$

Valuations on atomic wffs:

If for some i, $\sigma(v_i)\not\downarrow$, then $\mathsf{v}_\sigma \nvDash \mathrm{A}(v_1, \ldots, v_n)$.
Otherwise, $\mathsf{v}_\sigma \vDash \mathrm{A}(v_1, \ldots, v_n)$ iff $\mathrm{A}(\sigma(v_1), \ldots, \sigma(v_n)) \in \Sigma$.

I'll let you show:

(‡) If $\sigma(x) = \mathrm{d}$, then $\mathsf{v}_\sigma \vDash \mathrm{B}(x)$ iff $\mathsf{v}_\sigma \vDash \mathrm{B}(\mathrm{d}|x)$.

Now we'll show by induction on the length of wffs that for every closed wff A in $\mathrm{L}(\mathrm{w}_0, \mathrm{w}_1, \ldots)$, $\mathsf{M} \vDash \mathrm{A}$ iff $\mathrm{A} \in \Sigma$. It is immediate if A is atomic. Suppose it is true for wffs shorter than A. I'll do the cases when A is $\exists x$ B or $\forall x$ B and leave the others to you.

Suppose A is $\exists x$ B and $\exists x\, \mathrm{B}(x) \in \Sigma$. Then x must be the only variable free in $\mathrm{B}(x)$. Since Σ satisfies the conditions in Theorem 24, for some m, $\mathrm{B}(\mathrm{w}_m|x) \in \Sigma$ and $\exists x\, (x \equiv \mathrm{w}_m) \in \Sigma$. So $\mathrm{w}_m \in \mathsf{U}$, and by induction, for any σ, $\mathsf{v}_\sigma \vDash \mathrm{B}(\mathrm{w}_m|x)$. So for any σ, if $\sigma(x) = \mathrm{w}_m$, then $\mathsf{v}_\sigma \vDash \mathrm{B}(x)$. So $\mathsf{M} \vDash \exists x\, \mathrm{B}(x)$.

Suppose that A is $\exists x$ B and $\mathsf{M} \vDash \exists x\, \mathrm{B}(x)$. Then for some σ, $\mathsf{v}_\sigma \vDash \mathrm{B}(x)$. Take one such σ, where $\sigma(x) = \mathrm{d}$. By (‡), $\mathsf{v}_\sigma \vDash \mathrm{B}(\mathrm{d}|x)$. Since d is in the universe, for some y, $\exists y\, (y \equiv \mathrm{d}) \in \Sigma$. Since $\mathrm{B}(\mathrm{d}|x)$ is closed, $\mathsf{M} \vDash \mathrm{B}(\mathrm{d}|x)$, so $\mathrm{B}(\mathrm{d}|x) \in \Sigma$. Hence, using Axiom 9, $\exists x\, \mathrm{B}(x) \in \Sigma$.

Suppose that A is $\forall x$ B and $\mathsf{M} \vDash \forall x$ B. Then for each i, $\mathsf{M} \vDash \mathrm{B}(\mathrm{w}_i|x)$. So by induction, $\mathrm{B}(\mathrm{w}_i|x) \in \Sigma$. Since Σ satisfies condition (d) of Theorem 24, $\forall x\, \mathrm{B} \in \Sigma$.

Suppose now that A is $\forall x$ B and $\forall x\, \mathrm{B}(x) \in \Sigma$. By Axiom 3*, for all $\mathrm{d} \in \mathsf{U}$, $\mathrm{B}(\mathrm{d}|x) \in \Sigma$. So by induction, for every σ, for every $\mathrm{d} \in \mathsf{U}$, $\mathsf{v}_\sigma \vDash \mathrm{B}(\mathrm{d}|x)$ and so by (‡), $\mathsf{v}_\sigma \vDash \forall x\, \mathrm{B}(x)$. Hence, $\mathsf{M} \vDash \forall x\, \mathrm{B}(x)$.

The proof then follows as for Theorem 23, using Axiom 6.*a to show that the interpretation of "≡" in U is a congruence relation. ■

The strong completeness theorem (Theorem 18) then follows in the usual way.

VII Classical Predicate Logic with Descriptive Names and Descriptive Functions

Because the propositional axioms and Axioms 1 and 2 are the same as for classical predicate logic, Lemmas 8–13 hold. I'll let you show that the axiomatization is sound.

The proof of the following is the same as for Theorem 24.

Theorem 26 Let Γ be a consistent set of closed wffs of L. Then there is a collection of closed wffs Σ in $L(w_0, w_1, \ldots)$ such that:

a. $\Gamma \subseteq \Sigma$.

b. Σ is a complete and consistent theory.

c. If $\exists x\, B \in \Sigma$, then for some m, $B(w_m|x) \in \Sigma$ and $\exists x\,(x \equiv w_m) \in \Sigma$.

d. For every wff $B(x)$ in $L(w_0, w_1, \ldots)$ with one free variable,
if for each i, $B(w_i\,|x) \in \Sigma$, then $\forall x\, B(x) \in \Sigma$.

Theorem 27 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in $L(w_0, w_1, \ldots)$ be as in Theorem 26. For each i, $\exists x\,(x \equiv w_i) \in \Sigma$.

We define a model M^* of $L(w_0, w_1, \ldots)$. For the model M of classical predicate logic with equality for stage 0 we take:

$$\mathsf{U} = \{c_i : i \geq 0\} \cup \{w_i : i \geq 0\}$$

We also take a complete collection of assignments of references where for every name symbol d, $\sigma_0(d) = d$. For valuations of atomic wffs we set:

$$v_{\sigma_0} \vDash A(t_1, \ldots, t_n) \text{ iff } A(\sigma_0(t_1), \ldots, \sigma_0(t_n)) \in \Sigma$$

This determines all stages $n \geq 0$ and hence M^*.

By Axiom 5.b*, for every i, $\exists x\,(x \equiv c_i) \in \Sigma$. Hence by Axiom 4.a and 5.a, for every i, $(c_i \equiv c_i) \in \Sigma$ and $(w_i \equiv w_i) \in \Sigma$.

I'll let you show that for every n:

(‡) $v_{\sigma_n} \vDash B(x)$ iff if $\sigma_n(x) = d$, then $v_{\sigma_n} \vDash B(d|\,x)$

We now must prove that for every closed atomic wff A, $\mathsf{M}^* \vDash A$ iff $A \in \Sigma$. By virtue of the stability of references and truth-values, it suffices to show that if A is first defined at stage n, then:

(a) $\vDash_n A$ iff $A \in \Sigma$

To show this we will need to show simultaneously:

(b) v is a closed referring term at stage n iff $\exists z\,(z \equiv v) \in \Sigma$.

I'll write "$\vDash_n A$" to mean that for every σ_n, $v_{\sigma_n} \vDash A$.

For $n = 0$ we have (b) as noted above, and the proof of (a) is as for classical predicate logic with non-referring names as nil (to take account of the different axioms used here).

Suppose (a) and (b) are true for all stages $\leq n$.

To show (b) the only new cases are when t is a closed term "$\underline{\text{the}}\ x\ A(x)$" where A is a wff of stage n and not of any earlier stage. If $\exists z\ (z \equiv \underline{\text{the}}\ x\ A(x)) \in \Sigma$, then by Axiom 11, $\exists x\ A(x) \wedge \forall x\ \forall y\ (A(x) \wedge A(y) \rightarrow (x \equiv y)) \in \Sigma$. This is a wff of stage n, so by induction $\vDash_n \exists x\ A(x) \wedge \forall x\ \forall y\ (A(x) \wedge A(y) \rightarrow (x \equiv y))$. Hence, there is a unique d in the universe that satisfies $A(x)$. Hence, "$\underline{\text{the}}\ x\ A(x)$" has reference d. On the other hand, if "$\underline{\text{the}}\ x\ A(x)$" is a referring term at stage n, then $\vDash_n \exists x\ A(x)$ and $\vDash_n \forall x\ \forall y\ (A(x) \wedge A(y) \rightarrow (x \equiv y))$, and so by induction and PC, $\exists x\ A(x) \wedge \forall x\ \forall y\ (A(x) \wedge A(y) \rightarrow (x \equiv y)) \in \Sigma$. Hence by Axiom 11, $\exists z\ (z \equiv \underline{\text{the}}\ x\ A(x)) \in \Sigma$.

We show (a) by induction on the length of A. First note that for every B of stage $\leq n$, if "$\underline{\text{the}}\ x\ B(x)$" is a closed term then:

(†) $\sigma_{n+1}(\underline{\text{the}}\ x\ B(x)) = \text{d}$

iff d is the unique object satisfying $B(x)$ at stage n

iff (by (‡)) $\vDash_n B(\text{d}|x)$ and $\vDash_n \forall x\ (B(x) \rightarrow (x \equiv \text{d}))$

iff (by induction on stages) $B(\text{d}|x) \in \Sigma$ and $\forall x\ (B(x) \rightarrow (x \equiv \text{d})) \in \Sigma$ and (by (b)) $\exists z\ (z \equiv \underline{\text{the}}\ x\ A(x)) \in \Sigma$

iff (by Axioms 12 and 6) $(\text{d} \equiv \underline{\text{the}}\ x\ B(x)) \in \Sigma$

Suppose now that $A(v_1, \ldots, v_m)$ is atomic. Since A is closed, each v_i is closed. If $\vDash_{n+1} A(v_1, \ldots, v_m)$ then every v_i is referring and $\vDash_0 A(u_1|v_1, \ldots, u_m|v_m)$, where if v_i is a name, then u_i is v_i, and if t_i is "$\underline{\text{the}}\ x\ B(x)$", by (‡), u_i is the reference of v_i. So for all i, by (†) $(v_i \equiv u_i) \in \Sigma$, and by induction $A(u_1|v_1, \ldots, u_m|v_m) \in \Sigma$. So by Axiom 6* (extensionality), $A(v_1, \ldots, v_m) \in \Sigma$. On the other hand, if $A(v_1, \ldots, v_m) \in \Sigma$, then by Axiom 10 (falsity is the default), for every i, for some x, $\exists x\ (x \equiv v_i) \in \Sigma$. So via Axioms 11 and 12 and induction, if v_i is a "$\underline{\text{the}}$"-term, there is a reference for it, say d_i. Then by (†), $(\text{d}_i \equiv v_i) \in \Sigma$. Hence, by extensionality, $A(u_1|v_1, \ldots, u_m|tv_m) \in \Sigma$ where $u_1, \ldots, u_m$ are as before. By induction, $\vDash_0 A(u_1|t_1, \ldots, u_m|t_m)$, and so $\vDash_0 A(v_1, \ldots, v_m)$. So (a) is true for atomic wffs at stage n.

The proof now proceeds just as for Theorem 25, including showing that the equality predicate is evaluated as the identity on the universe. ■

The strong completeness theorem (Theorem 18) then follows in the usual way.

VII Classical Predicate Logic with Non-Referring Atomic Names and Descriptive Functions

The proof of the following is the same as for Theorem 21, using Axiom 3† instead of Axiom 3.

Theorem 28 Let Γ be a consistent set of closed wffs of L. Then there is a collection of closed wffs Σ in L($w_0, w_1, \ldots$) such that:

a. $\Gamma \subseteq \Sigma$.

b. Σ is a complete and consistent theory.

c. If $\exists x\, B \in \Sigma$, then for some m, $B(w_m|x) \in \Sigma$ and $\exists x\, (x \equiv w_m) \in \Sigma$.

d. If $\neg\forall x\, B \in \Sigma$, then for some m, $\neg\, B(w_m|x) \in \Sigma$.

e. For every wff $B(x)$ in L($w_0, w_1, \ldots$) with one free variable, if for each i, $B(w_i|x) \in \Sigma$, then $\forall x\, B(x) \in \Sigma$.

Theorem 29 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in L($w_0, w_1, \ldots$) be as in Theorem 28. We define a model M of L($w_0, w_1, \ldots$).

$$\mathsf{U} = \{c_i : \text{for some } x,\ \exists x\, (x \equiv c_i) \in \Sigma\} \cup \{w_i : i \geq 0\}$$

By construction, for each i, $\exists x\, (x \equiv w_i) \in \Sigma$. By Axiom 5*.a and b we have:

(*) For every i, $(w_i \equiv w_i) \in \Sigma$ and $(c_i \equiv c_i) \in \Sigma$.

Stage 0

Assignments of references:

For every σ and every x, $\sigma(x)\downarrow$, and the collection of such σ is complete.

For every atomic name c:

$\sigma(c)\downarrow$ iff $c \in \mathsf{U}$. If $\sigma(c)\downarrow$, then $\sigma(c) = c$.

Evaluation of the equality predicate:

$v_\sigma \vDash v \equiv u$ iff $\sigma(v)\downarrow = c$ and $\sigma(u)\downarrow = d$, and $(c \equiv d) \in \Sigma$
or both are undefined and $(v \equiv u) \in \Sigma$

Valuations on atomic wffs:

$v_\sigma \vDash A(v_1, \ldots, v_n)$ iff $A(\sigma(v_1), \ldots, \sigma(v_n)) \in \Sigma$

This determines all stages $n \geq 0$ and hence M*.

I'll let you show that for every n:

(‡) $v_{\sigma_n} \vDash B(x)$ iff if $\sigma_n(x) = d$, then $v_{\sigma_n} \vDash B(d|\,x)$

We now must prove that for every closed atomic wff A, M* $\vDash$ A iff $A \in \Sigma$. By virtue of the stability of references and truth-values, it suffices to show that if A is first defined at stage n, then:

(a) $\vDash_n$A iff A$\in\Sigma$

To show that we will need to show simultaneously:

(b) At stage n, for every closed t, if t is referring or pseudo-referring, then $(\mathrm{t}\equiv\mathrm{t})\in\Sigma$.

I'll write "$\vDash_n$A" to mean that for every σ_n, $\mathsf{v}_{\sigma_n}\vDash$A.

For $n = 0$ we have (b) as noted above since, $\sigma(\mathrm{t}) = \mathrm{t}$. The proof that (a) holds at stage 0 is the same as for classical predicate logic with non-referring names except that universal instantiation for non-nil terms is used instead of universal instantiation. That is possible by virtue of (*).

Suppose (a) and (b) are true for all stages $\leq n$.

To show (b) at stage $n + 1$, the only new cases are when t is "$\underline{\text{the}}\ x\ \mathrm{A}(x)$" where A is a wff of stage n and not of any earlier stage.

Suppose that $(\underline{\text{the}}\ x\ \mathrm{A}(x)\equiv\underline{\text{the}}\ x\ \mathrm{A}(x))\in\Sigma$. Then by Axiom 13 and PC, $(\mathsf{R}x\,A(x)\vee\ \mathsf{P}x\,A(x))\in\Sigma$. This wff is of stage n. Hence by induction, $\mathsf{M}\vDash\mathsf{R}x\,A(x)\vee\ \mathsf{P}x\,A(x)$. Hence by Lemma 2 of the chapter, "$\underline{\text{the}}\ x\ \mathrm{A}(x)$" is referring or pseudo-referring.

Now suppose that "$\underline{\text{the}}\ x\ \mathrm{A}(x)$" is referring or pseudo-referring. Then by Lemma 2 of the chapter, $\mathsf{M}\vDash\mathsf{R}x\,A(x)\vee\ \mathsf{P}x\,A(x)$ and by induction $(\mathsf{R}x\,A(x)\vee\ \mathsf{P}x\,A(x))\in\Sigma$. Hence by Axiom 11 and PC, $\exists z\,(z\equiv\underline{\text{the}}\ x\ \mathrm{A}(x))\vee\neg\,\forall z\,\neg\,(z\equiv\underline{\text{the}}\ x\ \mathrm{A}(x))\in\Sigma$. Hence by Axiom 5* and PC, $(\underline{\text{the}}\ x\ A(x)\equiv\underline{\text{the}}\ x\ A(x))\in\Sigma$. So (b) holds at stage $n + 1$.

We can use (b) to show something more:

(c) If $\sigma_{n+1}(\underline{\text{the}}\ x\ \mathrm{B}(x)) = \mathrm{d}$ or d is the pseudo-reference of "$\underline{\text{the}}\ x\ \mathrm{B}(x)$", then $(\mathrm{d}\equiv\underline{\text{the}}\ x\ \mathrm{B}(x))\in\Sigma$.

If $\sigma_{n+1}(\underline{\text{the}}\ x\ \mathrm{B}(x)) = \mathrm{d}$ then d is the unique object satisfying B(x). So by (‡), $\vDash_n\mathrm{B}(\mathrm{d}|x)$. If d is the pseudo-reference of "$\underline{\text{the}}\ x\ \mathrm{B}(x)$)", then $\vDash_n\mathrm{B}(\mathrm{d}|x)$. In both cases $\vDash_n\forall x\,\forall y\,(B(x)\wedge\ B(y)\rightarrow\ \ (x\equiv y))$. So by induction, $\mathrm{B}(\mathrm{d}|x)\in\Sigma$ and $\forall x\,\forall y\,(B(x)\wedge\ B(y)\rightarrow\ \ (x\equiv y))\in\Sigma$. By Axiom 12, $\mathrm{B}(\underline{\text{the}}\ x\ \mathrm{B}(x))\in\Sigma$, and by (b) $(\underline{\text{the}}\ x\ \mathrm{B}(x)\equiv\underline{\text{the}}\ x\ \mathrm{B}(x))\in\Sigma$. So by Axiom 3†, $(\mathrm{d}\equiv\underline{\text{the}}\ x\ \mathrm{B}(x))\in\Sigma$. Hence, (c) holds at stage $n + 1$.

We now show (a) at stage $n + 1$ by induction on the length of A.

Suppose that $\mathrm{A}(v_1, \ldots, v_m)$ is atomic. Since A is closed, each v_i is closed. If $\vDash_{n+1}\mathrm{A}(v_1, \ldots, v_m)$, then every v_i is referring or pseudo-referring then $\mathsf{v}_{\tau_0}\vDash_{n+1}A(u_1|v_1, \ldots, u_m|v_m)$ where if v_i is an atomic name then u_i is v_i, if v_i is a pseudo-referring term, then u_i is its pseudo-reference, and if t_i is referring, $\tau_0(u_i) =$ the reference of v_i. Hence by (‡), $\vDash_0\mathrm{A}(u_1{}'|v_1, \ldots, u_m{}'|v_m)$ where for referring terms t_i, $v_i{}'$ is the reference of v_i, and in the other cases $u_i{}'$ is u_i. Hence by (c) and Axiom 6* (extensionality), $\mathrm{A}(v_1, \ldots, v_m)\in\Sigma$.

Now suppose that $\mathrm{A}(v_1, \ldots, v_m)\in\Sigma$. By Axiom 10 (falsity is the default), for every i, $(v_i \equiv v_i) \in \Sigma$. Hence by Axiom 13, for every i, either i is an atomic name or a referring or pseudo-referring "the"-term. Hence by (c) and Axiom 6*, $A(u_1', \ldots, u_m') \in \Sigma$, where if t_i is an atomic name then u_i' is v_i, and if v_i is a "the"-term, then u_i' is its reference or pseudo-reference. Since $A(u_1', \ldots, u_m')$ is a wff of stage 0, we have by induction $\vDash_0 A(u_1', \ldots, u_m')$. By (‡), we then have $\vDash_0 A(u_1, \ldots, u_m)$, where the u_i are as in the definition of satisfaction for $A(v_1, \ldots, v_m)$. Hence, $\vDash_{n+1} A(v_1, \ldots, v_m)$. Hence, (a) is true at stage $n + 1$ for atomic wffs.

Suppose now that the lemma is true for all wffs of length k, and let A be a wff of length $k + 1$. I'll leave to you the cases when A is $\neg$B, B$\rightarrow$ C, B$\wedge$ C, and B$\wedge$ C.

Suppose that A is $\exists x$ B and $\vDash_{n+1} \exists x\, B(x)$. Then for some σ_{n+1}, $\mathsf{v}_{\sigma_{n+1}} \vDash B(x)$. Take one such, where $\sigma_{n+1}(x) = d$. By (‡), $\mathsf{v}_{\sigma_{n+1}} \vDash B(d\,|\,x)$. Since d is in the universe, for some y, $\exists y\,(y \equiv d) \in \Sigma$. Since $B(d\,|\,x)$ is closed, $\vDash_{n+1} B(d\,|\,x)$, so $B(d\,|\,x) \in \Sigma$. Hence, using Axiom 9 (existential generalization for referring terms), $\exists x\, B(x) \in \Sigma$.

Suppose that A is $\exists x$ B and $\exists x\, B(x) \in \Sigma$. Since Σ satisfies condition (c) of Theorem 28, for some m, $B(w_m|x) \in \Sigma$. Hence by induction, $\vDash_{n+1} B(w_m|x)$. We also have that $w_m \in \mathsf{U}$. So by (‡), for any σ_{n+1}, if $\sigma_{n+1}(x) = w_m$, then $\mathsf{v}_{\sigma_{n+1}} \vDash B(x)$. So $\vDash_{n+1} \exists x\, B(x)$.

Suppose that A is $\forall x$ B and $\vDash_{n+1} \forall x$ B. Then by (‡), for each i, $\vDash_{n+1} B(w_i|x)$. Hence by induction, for each i, $B(w_i|x) \in \Sigma$. Since Σ satisfies condition (e) of Theorem 28, $\forall x\, B \in \Sigma$.

Finally, suppose that A is $\forall x$ B and $\forall x\, B \in \Sigma$. Then by universal instantiation for non-nil terms and (*), for every d, $B(d|x) \in \Sigma$. Hence by induction, $\vDash_{n+1} B(d|x)$. Hence, for all σ_{n+1}, for all d, $\mathsf{v}_{\sigma_{n+1}} \vDash B(d|x)$ as $B(d|x)$ is closed. So by (‡), for each σ_{n+1}, $\mathsf{v}_{\sigma_{n+1}} \vDash B(x)$, so $\vDash_{n+1} B(x)$.

The model M* satisfies the restrictions on the equality predicate except that it need not be the identity on the universe. To create a model N that validates the same wffs as M* and in which the equality predicate is evaluated as the identity on the universe we can proceed as in the proof of Theorem 22. Though we do not have the identity axiom $\forall x\,(x \equiv x)$ to prove Lemma 19, that formula is true in M* and hence is in Σ. ∎

The strong completeness theorem (Theorem 18) then follows in the usual way.

VIII PC with Termporal Connectives Assuming Time is Linear (TL-PC)

Theorem 30 If $\nvdash$D, then there is a complete and consistent collection of wffs Γ such that D $\notin \Gamma$.

Proof The proof is the same as for Lemma 6. ■

Theorem 31 Γ is complete and consistent iff there is some model $\langle \mathsf{v}, \mathsf{t}, \mathsf{S} \rangle$ such that A $\in \Gamma$ iff v(A) = T.

Proof The direction from right to left I'll leave to you.

Let v be the valuation such that for every variable p,

v(p) = T iff p $\in \Gamma$

Let R = { p : p is propositional variable}. Define:

p $\mathsf{=}$ q iff (p $\approx$ q) $\in \Gamma$

I'll let you show using axiom schema (a)–(c) that $\mathsf{=}$ is an equivalence relation. Denote by p the equivalence class of p.

Let S = { p : p is propositional variable}. Define on S:

$\mathsf{p} < \mathsf{q}$ iff $(p \prec q) \in \Gamma$

Using axiom schema (d)–(g) I'll let you show that $<$ is a linear order on S that, by schema (h) and (i), does not depend on the choice of representative for the equivalence class.

Define the time assignment:

$\mathsf{t}(p) = \{\mathsf{p}\}$

This completes the definition of the model. The valuation v is extended to all wffs of *L* by the standard evaluations of the connectives for this logic.

We need to show that for every A, v(A) = T iff A $\in \Gamma$. This can be proved by induction. If A has length 1, this is immediate from the definition of v. The following inductive cases can be proved as for classical propositional logic (Lemma 5):

$\mathsf{v}(\neg\, \mathrm{A}) = \mathsf{T}$ iff $\neg\, \mathrm{A} \in \Gamma$

$\mathsf{v}(\mathrm{A} \rightarrow \mathrm{B}) = \mathsf{T}$ iff $(\mathrm{A} \rightarrow \mathrm{B}) \in \Gamma$

$\mathsf{v}(\mathrm{A} \wedge \mathrm{B}) = \mathsf{T}$ iff $(\mathrm{A} \wedge \mathrm{B}) \in \Gamma$

$\mathsf{v}(\mathrm{A} \vee \mathrm{B}) = \mathsf{T}$ iff $(\mathrm{A} \vee \mathrm{B}) \in \Gamma$

It then remains to show the inductive step:

$\mathsf{v}(\mathrm{A} \leq \mathrm{B}) = \mathsf{T}$ iff $(\mathrm{A} \leq \mathrm{B}) \in \Gamma$

I'll first show that $(A \preceq B) \in \Gamma$ iff for every p that appears in A and q that appears in B, $(p \preceq q) \in \Gamma$. The proof is by induction, first on the length of B and then on the length of A. It is immediate if B is of length 1. Suppose now that it is true for all wffs shorter than B. Then if B is of the form $\neg C$, we have the proof by schema (j) and induction. If B is of the form $(C \wedge D)$, we have the proof by schema (l) and induction. If B is of the form $(C \leq D)$, we have the proof by schema (n) and induction. The proof inducting on the length of *A* is similar, using schema (k), (m), and (o).

So by (*) we now have:

$$\begin{aligned} \mathsf{t}(A) \leq \mathsf{t}(B) \quad & \text{iff for all p in A, q in B, } \mathsf{p} \leq \mathsf{q} \\ & \text{iff for all p in A, q in B, } (p \preceq q) \in \Gamma \\ & \text{iff } (A \preceq B) \in \Gamma \end{aligned}$$

Finally,

$$\begin{aligned} \mathsf{v}(A \leq B) = \mathsf{T} \quad & \text{iff } \mathsf{v}(A) = \mathsf{v}(B) = \mathsf{T} \text{ and } \mathsf{t}(A) \leq \mathsf{t}(B) \\ & \text{iff } A, B \in \Gamma \text{ and } (A \preceq B) \in \Gamma \\ & \text{iff } (A \wedge B) \in \Gamma \text{ and } (A \preceq B) \in \Gamma \qquad \text{by axiom schema (3)–(5)} \\ & \text{iff } (A \leq B) \in \Gamma \qquad \text{by axiom schema (n)} \end{aligned}$$

We then have that for every A, $\mathsf{v}(A) = \mathsf{T}$ iff $A \in \Gamma$ by the usual proof for classical propositional logic. ■

I'll let you show that the axiomatization is sound: If $\Gamma \vdash A$, then $\Gamma \vDash A$. Then strong completeness follows in the usual way from what we have shown.

Theorem 32 For every collection of wffs Γ and wff A in L, $\Gamma \vdash A$ iff $\Gamma \vDash A$.

IX CPL with Temporal Connectives

In the proof for CPL (Section II), replace "PC" with "TL-PC" everywhere and everything in that section follows as before except for Theorem 17.

Theorem 33 Every consistent collection of closed wffs in L has a countable model.

Proof Let Γ be a consistent collection of closed wffs of L. Let $\Sigma \supseteq \Gamma$ in $L(w_0, w_1, \ldots)$ be as in Theorem 16 for this logic. We define a model M of $L(w_0, w_1, \ldots)$, taking the formal language to be the semi-formal language, too. Let P, Q stand for any predicate symbols.

$\mathsf{U} = \{w_i, c_i : i \geq 0\}$

For every σ, $\sigma(c_i) = c_i$ and $\sigma(w_i) = w_i$.

$\mathsf{v}_\sigma \vDash P(v_1, \ldots, v_n)$ iff $P(\sigma(v_1), \ldots, \sigma(v_n)) \in \Sigma$

Let $\mathsf{R} = \{\langle \sigma, A \rangle : A$ is an atomic wff and σ is an assignment of references$\}$. Define

$$\langle \sigma, P(v_1, \ldots, v_n) \rangle \equiv \langle \tau, Q(u_1, \ldots, u_m) \rangle$$
$$\text{iff } (P(\sigma(v_1), \ldots, \sigma(v_n) \approx Q(\tau(u_1), \ldots, \tau(u_m))) \in \Gamma$$

By Axioms (a)–(c) of LT-PC, $\equiv$ is an equivalence relation. Denote by $\mathbf{\langle} \sigma, A \mathbf{\rangle}$ the equivalence class of $\langle \sigma, A \rangle$.

Let $\mathsf{S} = \{\mathbf{\langle} \sigma, A \mathbf{\rangle} : A$ is an atomic wff and σ is an assignment of references$\}$. Define on S :

$$\mathbf{\langle} \sigma, P(v_1, \ldots, v_n) \mathbf{\rangle} \prec \mathbf{\langle} \tau, Q(u_1, \ldots, u_m) \mathbf{\rangle} \text{ iff}$$
$$(P(\sigma(v_1), \ldots, \sigma(v_n) \approx Q(\tau(u_1), \ldots, \tau(u_m))) \in \Gamma$$

By Axioms (d)–(g) of TL-PC, $\prec$ is a linear order on S that, by Axioms (h) and (i), does not depend on the choice of representative for the equivalence class.

Define the time assignment:

$$\mathsf{t}_\sigma(A) = \mathbf{\langle} \sigma, A \mathbf{\rangle}$$

This completes the definition of the model. The time assignments are extended to all wffs by the standard conditions (p. 256) and the valuation v is then extended to all wffs of *L* by the standard evaluations (p. 257).

The rest of the proof follows as for Theorem 17 except for the proof by induction on the length of wffs that for every closed wff A in $L(w_0, w_1, \ldots)$, $\mathsf{M} \vDash A$ iff $A \in \Sigma$, to which we must add the case when A is $B \preceq C$. The proof for that follows as in the proof of Theorem 31 for TL-PC. ■

Appendix 4
Events as a Foundation for Predicate Logic

It is difficult if not impossible to specify exactly what we mean by "an individual thing". We start with an intuition, develop our logic, and then, at best, we can say that an individual thing is whatever we can reason about in predicate logic.[1]

Some people believe that events are things: the burning of a flame in a fireplace or Juney barking. They say that by recognizing that events are things we can formalize a great deal more in predicate logic. Consider, for example:

Juney is barking loudly

This would be formalized as:

$\exists x$ (x is a barking $\wedge$ x is by Juney $\wedge$ x is loud)

Formalizing in this manner there is no need for an analysis of adverbs.[2] Indeed, there is no need for an analysis of verbs or recognizing the possibility that processes are not things. Every verb is replaced by a gerund acting as a noun (or in some languages, like Latin, with an infinitive), so the only verb left is the copula of being: "to be" declined in all tenses. Terence Parsons writes:[3]

> The basic assumption is that a sentence such as
>
> Caesar died
>
> says something like the following:
>
> For some event e,
> e is a dying, *and*
> the object of e is Caesar, *and*
> e culminates before now.

If we formalize along these lines then propositions we previously viewed as atomic in predicate logic will be parsed as compound, requiring quantification with a variable that ranges over events. A two-sorted logic will be essential to distinguish the use of such variables from variables used to refer to other kinds of things that would normally comprise the universe of a realization. Thus, though "Juney barks" would seem to be atomic, we should understand it as having a hidden variable and

[1] See the discussion in *Predicate Logic* and Chapter 45 here.
[2] Event-talk accounts for adverbs by converting them into adjectives. But without a theory of relative adjectives the formalizations will be wrong.
[3] *Events in the Semantics of English*, p. 6.

quantifier: "$\exists e$ (e is a barking of Juney)". Advocates of this view have to argue that this is how we should understand such propositions, for there is no good evidence that we actually do understand them this way.[4] They are giving a prescriptive theory of how to reason.

By excising verbs from their reasoning, the advocates of this view are implicitly, if not explicitly adopting a much stronger version of *Things, the world, and propositions*.[5]

Only things, the world, and propositions The world is made up of individual things only; propositions are about individual things only.

The reason they give for adopting such a view is that it allows us to formalize a great deal more in predicate logic than we could before.[6] That is, they argue for their theory in terms of it having good consequences, which, as I discussed in Chapter 3, is not an adequate reason for adopting a theory. Most certainly, it is not reason enough to adopt such a metaphysical view.

There are many problems with adopting events as basic to our metaphysics.[7] One, however, is so great that there seems to be no way to proceed in using events as things in predicate logic: we have no way to distinguish events. We have no way to pick out one event rather than another when we wish to give a reference to a variable, and this is essential for the semantics of any predicate logic.[8] When we say that "*x*" is to refer to the stabbing of Caesar, is that the same event as the

[4] When I asked Donald Davidson why we should take events as basic and formalize in this way in predicate logic he replied that it's because we talk that way. Yes, we do talk about events sometimes, but not in the way that such a theory requires.

Roberto Casati and Achille Varzi explain that view more carefully and review contemporary debates about the nature of events as things in "Events". In Davidson's comments and Casati and Varzi's article there is no acknowledgement that the talk they are considering is in English. At best, then, they could be said to be discussing the implicit metaphysics of English-language speakers.

[5] Henry Laycock in "Theories of matter" gives a good survey of works in which this stronger assumption is made. The most explicit formulation I know is in "The fundamental ideas of pansomatism" by Tadeusz Kotarbin´ski:

> Since every object is a thing, and since, therefore, only things exist . . . p. 489
> The whole of reality consists entirely of bodies. p. 495

[6] From Parsons, *Events in the Semantics of English*, p. 146:

> I don't cite these results as evidence for the theory, or even as philosophically desirable consequences. The evidence for the theory lies in its ability to explain a wide range of data better than other existing theories. The existence and nature of events and states are by-products, in the same way that the symmetry of space and time are by-products of investigations in physics.

This also misconstrues the work of physicists; see my "On models and theories".

[7] A general critique can be found in E. J. Borowski, "Adverbials in action sentences".

[8] See *Predicate Logic* and Chapter 45 here.

stabbing of Caesar with a knife? Is it the same event as the stabbing of Caesar with a knife by Brutus? When did the event start: with Brutus conceiving of the action? with Brutus lifting his hand? with Brutus pushing the knife into Caesar? If the last, how far into Caesar did the knife go in that event?

The one and only way anyone has been able to individuate events is by invoking propositions that are meant to describe them. Thus, each of the above is a different event because each of the following is a different proposition: “Caesar was stabbed”, “Caesar was stabbed with a knife”, “Caesar was stabbed with a knife by Brutus”, “Caesar was stabbed with a knife by Brutus only when Brutus lifted his hand with the knife”, ...[9] No other way has been presented that is clear enough to use as the basis of naming in our models. Thus, to understand how to reason with events, we already need to know how to parse and hence how to reason with the propositions that were supposed to be explicated by rewriting them by appeal to hidden variables ranging over events.[10]

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[9] From Parsons, *Events in the Semantics of English*, pp. 145–146:

> Most events and states are concrete entities, not abstract ones. First, they are located in space. Since Brutus stabbed Caesar in the marketplace, the theory tells us that there was a stabbing, by Brutus, of Caesar, and the stabbing was in the marketplace.

[10] Nicholas Unwin in “The individuation of events” presents a survey of this problem. Arthur Prior in *Past*, *Present and Future*, p. 18 paraphrases talk of events in favor of talk of propositions.

Donald Davidson in “Causal relations” says events are needed to clarify and to give the truth conditions of causal claims, since we apparently talk of events in our ordinary speech. He says that we need events as things, because otherwise we wouldn’t be able to give the logical form of causal claims, meaning a predicate-logic form. But after surveying all the possibilities for criteria of individuating events, Davidson in “The individuation of events” comes to the conclusion that the best criterion we can muster is that events are different if and only if they differ in their causes and/or effects. That is, we need events to explain cause and effect, but we first need to understand causes and effects to be able to distinguish events. However, if we cast talk of events as talk of propositions we can analyze cause and effect quite well, as I show in *Five Ways of Saying* “*Therefore*” and describe briefly in Chapter 13X here.

Davidson’s dilemma is part of a general problem of specifying entities that correspond to propositions. Gottlob Frege in “Negation” takes propositions to be expressions of thoughts and says:

> How, indeed, could a thought be dissolved? How could the interconnexion of its parts be split up? The world of thoughts has a model in the world of sentences, expressions, words, signs. To the structure of the thought there corresponds the compounding of words into a sentence; and here the order is in general not indifferent. To the dissolution or destruction of the thought there must accordingly correspond a tearing apart of the words, such as happens, e.g., if a sentence written on paper is cut up with scissors, so that on each scrap of paper there stands the expression for a part of a thought. p. 123

To distinguish one thought from another we have to point to the sentences which are said to express them. It is precisely because of this problem that I take sentences as used in such a way that we can agree that they are either true or false as primitive in our studies of reasoning, for it is these that we have clear standards for individuating. See “Language, thought, and meaning” for a richer analysis of thoughts that does not identify them with sentences.

Some people disagree. Robin Le Poidevin in "Relatonism and temporal topology" says:

> Not only are events, on the face of it, less mysterious entities than instants, they are clearly things with which we can causally interact. p. 152

But instants are no more mysterious than points in geometry: we arrive at them through a process of abstraction. In any application of a theory of time in which instants play a role, we take as instants intervals of time so short that their duration is negligible relative to the rest of what we are paying attention to.[11] Events, on the other hand, are not obviously entities at all. As Benson Mates pointed out to me, "The cat is on the mat" is supposed to be made true by or describe an event. But what is included in that event? The cat is touching the mat? The cat is upon the mat? The earth, because up and down can only be determined relative to that? Where do we stop? Probably only at the entire universe. But simply, the event is that the cat is on the mat. We use "that" to restate the claim.

One further problem arises that I have not seen discussed elsewhere. Consider:

(*) Juney is a dog.

If events are used in the foundations of predicate logic, then this should be converted into event talk as something like:

There is an event and it is a dogging and it is by Juney.

But that is to convert "is a dog" into a process, and then to convert that back to a thing. I've never seen anything like this. Rather, what is assumed is that (*) is already in primitive form. But then we have a huge assumption that verbs of "activity" and verbs of classification are quite distinct metaphysically, so that every use of a non-copula verb is an implicit quantification while every use of a copula verb isn't. As I discuss in Chapter 116, there seems to be no justification for this metaphysical distinction other than following the implicit metaphysics of some particular language, in this case English, which doesn't seem to be based on anything like an understanding of the nature of the world, just accreted habits of speech.

It might seem that I, too, am committed to events and actions as things. In Example 16 of Chapter 21 I say "She (Juney) is doing something". But that kind of speech is an artefact of English, which seems to reify much that on reflection neither you nor I would consider to be a thing, e.g., a huff, a snit, a hurry. When I say "Juney is doing something" I can be understood as meaning nothing more than "There is some verb such that "Juney is (verb)ing" ", or "There is some predicate *P* such that *P*(Juney)".

[11] See the discussion of points and lines in geometry in my "On mathematics"

Appendix 5
Collections and Qualities

A fundamental agreement concerning formalizations into predicate logic says that when we encounter a collective common noun in a proposition, such as “dogs”, we convert it into a predicate, such as “— is a dog”.[1] Actually, the reverse conversion is often more straightforward:

Predicate or open wff	*Collective noun* (*phrase*)
a. — is a dog	dogs
b. — is a father	fathers
c. (—is human) (x) $\wedge$ (—is male) (x) $\wedge$ $\exists y$ (— is the child of —) (y, x)	human males who have a child
d. (—is a sheep) (x) $\wedge$ (— is in Alan’s corral) (x)	the flock of sheep in Alan’s corral
e. (— $\equiv$ Ralph) (x) $\vee$ (— is a canary) (x) $\vee$ (—is the Eiffel tower) (x)	Ralph, canaries, and the Eiffel Tower
f. (— is a king of a country in North America) (x)	kings of countries in North America
g. (— is a natural number) (x) $\wedge$ $1 < x \wedge x < 2$	natural numbers between 1 and 2

We convert a predicate into a collective noun (phrase). Or, if you like, we correlate predicates with collections of things. Thus, “— is a dog” is correlated to the collection of all dogs; “— is Ralph $\vee$ — is a canary $\vee$ — is the Eiffel Tower” is correlated to the collection consisting of Ralph, all canaries, and the Eiffel Tower.
If the correlation is between linguistic predicates and collections, rather than between linguistic predicates and other parts of speech, we should say that (b) and (c) are correlated to the same collection, namely all fathers, and (f) and (g) are correlated to the same collection, the collection with nothing in it. The platonist does not believe that every collection can be represented by a predicate. But in predicate logic we use linguistic representations of propositions, and hence we are concerned with specific collections only as they appear via specific noun (phrases).

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[1] See the criterion of formalization “Nouns into predicates” in Appendix 1.

In English we often correlate an adjective to a word for a property or quality. For example:

Predicate	*Quality or property word or phrase*
wise	wisdom
intelligent	intelligence
blue	blueness, blue
beautiful	beauty

This suggests or leads to viewing predicates generally as denoting or corresponding to properties. For example:

Predicate	*Property word or phrase*
— is a dog	doggieness, the property of being a dog
— is a domestic canine	the property of being a domestic canine
— is a father	fatherhood
— is running	the property of running, runningness
— is taller than —	the property of being taller than

To take the words or phrases on the right-hand side of these lists as denoting things is to accept that there are abstract objects, for they are neither parts of a language nor things of time and space. More, to take adjectives as correlated to quality words is to take them to be absolute, not relative: if there is a quality of wisdom it is only because "— is wise" is a predicate.

Those more linguistically oriented believe that talk of properties can and should be reduced to talk about objects via predicates. But there is no straightforward way to convert "property talk" or "quality talk" into "object talk". Consider:

(1) Fatherhood is good

This is not equivalent to:

$$\forall x\ (\ (\text{—is a father})\ (x) \rightarrow\ (\text{—is good})\ (x)\ )$$

It may be that fatherhood is good, though some fathers are bad. Rewrite (1) as:

Fatherhood is a good thing

It then becomes clearer that "fatherhood" is being used to designate a thing: the word is being used as a name. It may be possible to reduce (1) or the more problematical "Blue is a color" to talk about objects satisfying "— is a father" or "— is blue", but the conversion is hardly obvious.

Whether we take "fatherhood", "blue", and other property or quality words as names depends on what we believe exists. For platonists these denote things, and the naming is by directing our intellects to those abstract objects: "blue" does not denote something blue, it is the color, beyond our world of sensation and becoming. But if we take "fatherhood" as a name of something, it must be primitive, and logic alone will yield no connection between "fatherhood" and "— is a father".

The same problem arises for talk about collections. Consider:

(2) The flock of sheep in Alan's corral is more homogeneous than the collection consisting of Ralph, all canaries, and the Eiffel Tower.

There is no obvious way to reduce this to talk about objects satisfying predicates such as "— is a sheep" and "— is a canary". The predicate "— is homogeneous" applies to collections, not to objects in the collections. In (2) we are using "the flock of sheep in Alan's corral" to denote a single thing; we are using it as a name. In that case it is atomic, so our logic recognizes no connection between that name and the predicate "— is a sheep".

Perhaps we can recognize such connections by adding meaning axioms:

$\forall x$ ((— belongs to the flock of sheep in Alan's corral) (x)
$\quad\rightarrow$ (—is a sheep) (x))

$\forall x$ ((— possesses intelligence) (x) $\leftrightarrow$ (— is intelligent) (x))

But this creates a problem. Consider:

(3) The collection of all things that do not belong to themselves belongs to itself.

(4) The property of not being possessed by itself is possessed by itself.

Possesses, *belongs to* are recastings of *applies to* and *satisfies* in terms of properties and collections; (3) and (4) are recastings of the liar paradox. Seen in this light, we must exclude these words from realizations according to the self-reference exclusion principle.

With some artifice we might be able to transform a particular proposition involving collections or qualities and what belongs to or is possessed by them into a semi-formal proposition. For example, we might formalize:

The flock of sheep in Alan's corral has a leader.

$\exists y\, \forall x$ ((—is a sheep) (y) $\wedge$ (— is in Alan's corral) (y) $\wedge$
$\quad$(— is a sheep) (x) $\wedge$ (— is in Alan's corral) (x) $\rightarrow$ y leads x) (y, x)

But in general *we cannot formalize talk of properties or collections in predicate logic along with talk of the objects that possess or belong to them*. We cannot formalize, for example:

Every pack of dogs has a leader.
Marilyn Monroe had all the qualities of a great actress.
Ralph has some quality that distinguishes him from every other thing.

This is a significant restriction on the scope of predicate logic. Not only in colloquial language, but in mathematics and science, reasoning about collections and the things in the collections is important.

Extensions of predicate logic have been devised for formalizing such reasoning. What is called *second-order predicate logic* is discussed generally in Chapter X of *Predicate Logic*, and formal systems of classical second-order logic are presented in Chapter XIV of *Classical Mathematical Logic*. In both those places I discuss the inadequacy of formal theories of collections, what is called *set theory*, to formalize such reasoning.

Appendix 6
A Mathematical Abstraction of the Semantics

The nature of formal semantics

Semantics is the study of meaning. We have given semantics for the system of classical predicate logic with predicate modifiers and internal conjunctions in Chapter 30. Those take as given the truth or falsity of each atomic predication and then use an inductive definition of truth in a model. The relation of truth-values of some atomic predications to others, as codified in the semantic conditions on the model, are also part of those semantics. We arrived at those conditions by reflection on what we mean in asserting propositions of certain kinds, reflecting on what follows from certain kinds of assertions involving various parts of language. This is the notion of meaning we have incorporated in our models.

We did not give an analysis of the meaning of each part of speech, such as predicates or restrictors, independently of that. We can't, if we wish our semantics to be generally acceptable to people who hold very different views of the nature of language, truth, and the world. A platonist, a pragmatist, an idealist all can adopt our formal system, each giving a different explanation of the meaning of those parts of language. They need only agree that we will reason about things, that a name is used to pick out one thing, and that an atomic predicate is true or false of each thing or sequence of things under discussion.

Some would say we have not given semantics for a formal system until we have supplemented what we have given with a fuller analysis or explanation of what it means for an atomic predicate to apply to an object or objects. That is part of a larger project, a metaphysics beyond that of assuming the world is made up at least in part of things. It can be a descriptive project for which the logician looks to the linguist and anthropologist: this is what people take as the meaning of such assertions. It can be a descriptive project for which the logician looks to the metaphysician: this is the underlying reality of the world that makes such assertions true or false. Or it can be a prescriptive project: this is what people should believe or mean when they make such assertions.

It has become common in modern formal logic to take one particular approach to giving a fuller analysis of meaning in formal semantics. Some variant of set-theory is used as giving the meaning of atomic predications. In classical predicate logic a predicate is identified with its extension, those things of ordered *n*-tuples of things of which it is true. That extension is considered to be a set. A predicate is true of a thing or an *n*-tuple iff that thing or *n*-tuple is in the set that is the extension of the predicate.

This can be viewed as giving a fuller account of the meaning of the parts of speech and propositions of the formal system only if we accept that sets have an independent existence as abstract objects. Such semantics then would either be prescriptive for how we should understand our assertions, or descriptive of an underlying reality of the nature of the world, language, and truth that give meaning to our assertions, a distinctly platonic reality. Those semantics are certainly not descriptive of how we actually do ascribe meaning to the various parts of speech.

The underlying reality of those sets is explicated in a formal set-theory, a theory that is itself given in classical predicate logic. A circle of meaning dependence is present in such analyses, though it is not clearly vicious. Different set-theories are proposed, some inconsistent with others, that are supposed to give the meaning of atomic predications, leading to a proliferation of possible platonic interpretations of the world.

There is a very different view of the role of set-theory in formal semantics. Set-theory is an abstraction of our notion of a predicate applying to an object or objects.[1] We devise it in the same way we devise other mathematics, through a process of abstraction.[2] It is useful to take it as a supplement to the "informal" semantics we have given so that we can apply the methods of mathematics to derive structural analyses of our formal system. This is especially useful for applications of classical logic to analyses of mathematical reasoning.

A mathematical abstraction of the semantics

I will present here a mathematical abstraction of the semantics of classical predicate logic with predicate modifiers and internal conjunctions of Chapter 30, based on an informal set-theory. I will leave to you to decide what role this has in giving meaning to the formal language.

The universe

We view the universe of the realization as a set U. What this means depends on what mathematical theory of sets you wish to adopt, whether constructive or nonconstructive, allowing for only finite sets or infinite ones too, with strong cardinality axioms or not.

Assignments of references

An assignment of references σ is a function:

$$\sigma : \{u : u \text{ is a term of the semi-formal language}\} \to \mathsf{U}$$

The collection of such functions is a set, and together they satisfy the condition:

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1 See Chapter XIV, "Second-order classical predicate logic" in *Classical Mathematical Logic.*

2 See "On mathematics".

If a is a name of the semi-formal language and σ, τ are assignments of references, then $\sigma(a) = \tau(a)$

We assume that in whatever set theory we are employing the set of assignments of references is complete (p. 28).

I'll first present the mathematical abstraction for the system without conjunctions of terms. Then I'll show how to modify that abstraction to allow for those. I'll give only the conditions on the extensions of atomic predicates, for the extension to all wffs is given in the usual way for classical predicate logic.

Simple predicates

To each simple n-ary predicate P of the semi-formal language we associate a set P, called the *extension of P* such that $\mathsf{P} \subseteq \mathsf{U}^n$. This set is meant to be all those things in the universe that satisfy P. That is, for every assignment of references σ,

$$\mathsf{v}_\sigma \models (P)(y_1, \ldots, y_n) \text{ where } \sigma(y_i) = \mathsf{b}_i \text{ iff } <\mathsf{b}_1, \ldots, \mathsf{b}_n> \in \mathsf{P}$$

I use the convention that predicates are notated in italic, A, B, C, . . . and their extensions in script, A, B, C, D . . . .

The predicate conjoiner

For the predicate conjoiner $+$ there is a function $\boldsymbol{+}$ such that:

$$\boldsymbol{+} : \{ (A, B) : A \text{ and } B \text{ are unary atomic predicates } \} \rightarrow \text{ subsets of } \mathsf{U}$$

$$\boldsymbol{+}((A, B)) \subseteq \mathsf{A} \cap \mathsf{B}$$

$$\boldsymbol{+}((A, B + C)) = \boldsymbol{+}((A + B, C))$$

The set $\boldsymbol{+}((A, B))$ is the extension of $A + B$ and is meant to be all those things in the universe that satisfy $A + B$. That is,

$$\mathsf{v}_\sigma \models (A + B)(x) \text{ where } \sigma(x) = \mathsf{b} \text{ iff } \mathsf{b} \in \boldsymbol{+}((A, B))$$

Since this function is on unordered pairs, we have $\boldsymbol{+}((A, B)) = \boldsymbol{+}((B, A))$.

Non-variable restrictors

To each simple non-variable restrictor R we associate a function R such that:

$$\mathsf{R} : \bigcup_n \{ <A, \mathsf{A}> : A \text{ is an } n\text{-ary atomic predicate and } \mathsf{A} \text{ is its extension} \}$$
$$\rightarrow \text{ subsets of } \bigcup_n \mathsf{U}^n$$

$$\mathsf{R}(<A, \mathsf{A}>) \subseteq \mathsf{A}$$

The set $\mathsf{R}(<A, \mathsf{A}>)$ is the extension of A/R and is meant to be all those things in the universe that satisfy A/R. That is,

$$\mathsf{v}_\sigma \models (A/R)(y_1, \ldots, y_n) \text{ where } \sigma(y_i) = \mathsf{b}_i \text{ iff}$$
$$<\mathsf{b}_1, \ldots, \mathsf{b}_n> \in \mathsf{R}(<A, \mathsf{A}>)$$

Variable restrictors

For each k-ary variable restrictor R we associate a function R such that:

$$\mathsf{R} : \bigcup_n \{ <A, \mathsf{A}> : A \text{ is an } n\text{-ary atomic predicate and } \mathsf{A} \text{ is its extension} \} \times \mathsf{U}^k$$
$$\rightarrow \text{ subsets of } \bigcup_n \mathsf{U}^n$$

$$\mathsf{R}(< <A, \mathsf{A}>, <\mathsf{b}_1, \ldots, \mathsf{b}_k> >) \subseteq \mathsf{A}$$

The set $\mathsf{R}(<A, \mathsf{A}>, <\mathsf{b}_1, \ldots, \mathsf{b}_k>))$ is the extension of $A/R(y_1, \ldots, y_k)$ when $\sigma(y_i) = \mathsf{b}_i$ and is meant to be all those things in the universe that satisfy $A/R(y_1, \ldots, y_k)$ for that assignment. That is,

$$\mathsf{v}_\sigma \vDash (A/R(y_1, \ldots, y_k))\,(z_1, \ldots, z_n) \text{ where } \sigma(z_i) = \mathsf{c}_i \text{ iff}$$
$$<\mathsf{c}_1, \ldots, \mathsf{c}_n> \in \mathsf{L}(< <A, \mathsf{A}>, <\mathsf{b}_1, \ldots, \mathsf{b}_n> >)$$

Negators

For each negator N we associate a function N such that:

$$\mathsf{N} : \bigcup_n \{ <A, \mathsf{A}> : A \text{ is an } n\text{-ary atomic predicate and } \mathsf{A} \text{ is its extension} \}$$
$$\rightarrow \text{ subsets of } \bigcup_n \mathsf{U}^n$$

$$\mathsf{N}(<A, \mathsf{A}>) \cap \mathsf{A} = \varnothing$$

The set $\mathsf{N}(<A, \mathsf{A}>)$ is the extension of A/N and is meant to be all those things in the universe that satisfy A/N. That is,

$$\mathsf{v}_\sigma \vDash (A/N)(y_1, \ldots, y_n) \text{ where } \sigma(y_i) = \mathsf{b}_i \text{ iff}$$
$$<\mathsf{b}_1, \ldots, \mathsf{b}_n> \in \mathsf{N}(<A, \mathsf{A}>)$$

Modifiers of modifiers

For pairs of modifiers R, N the function F associated with the restrictor R/N satisfies in addition to the conditions above:

$$\mathsf{F}(<A, \mathsf{A}>) \cap \mathsf{R}(<A, \mathsf{A}>) = \varnothing$$

For pairs of negators N, N' the function F associated with the restrictor N/N' satisfies:

$$\mathsf{F}(<A, \mathsf{A}>) \cap \mathsf{N}(<A, \mathsf{A}>) = \varnothing$$

For any restrictor R the function F associated with the restrictor $\sim R$ satisfies:

$$\mathsf{K}(<A, \mathsf{A}>) = \mathsf{A} \cap \text{ the complement of } \mathsf{R}(<A, \mathsf{A}>)$$

The restrictor conjoiner

For the restrictor conjoiner & there is a function **&** such that:

$$\mathbf{\&}\,(R, R') = \text{the function associated with the restrictor } R + R'$$

$$\mathbf{\&}\,((R_1, R_2 + R_3)) = +\,((R_1 + R_2, R_3))$$

Since this function is on unordered pairs, we have $\mathbf{\&}\,((R, R')) = +\,((R', R))$.

This concludes the presentation of the mathematical abstraction of the semantics. I'll let you show that relative to the set-theory you adopt all the semantic assumptions of the original theory are satisfied.

The abstraction of the extensions of atomic predicates can be given in an inductive definition as in the aside in Chapter 14, which may be more compatible with an abstraction given within a constructivist set-theory.

Conjunctions of terms

Allowing for conjunctions of terms complicates the extensions of all unary atomic predicates. If A is a unary atomic predicate, we assign to it $<\mathsf{A}_1, \mathsf{A}_2, \ldots, \mathsf{A}_n, \ldots>$ where for each n, $\mathsf{A}_n \subseteq \mathsf{U}^n$. We write A for A_1. We say that A is the *extension* of A, and A_n is the *n-ary extension of A*. For an atomic wff $A(d)$ where the terms appearing in d are, in order reading from the left with repetitions, $u_1, \ldots, u_n$,

$$\sigma \vDash A(d) \text{ iff } <\sigma(u_1), \ldots, \sigma(u_n)> \in \mathsf{A}_n$$

The extensions satisfy:

If $<\mathsf{a}_1, \ldots, \mathsf{a}_n> \in \mathsf{A}_n$, then any permutation of that sequence is also an element of A_n.

If $<\mathsf{a}_1, \ldots, \mathsf{a}_n> \in \mathsf{A}_n$, then for every i, $\mathsf{a}_i \in \mathsf{A}$.

The conditions for the functions associated with restrictors, negators, +, and & must be modified, too, and I will leave those to you.

Appendix 7
Function-Names in Classical Predicate Logic with Descriptive Functions

Syntax

The base language L is that of classical predicate logic with equality and function-names. For the full language we gain need an induction on stages of languages, but now within each of those stages we now need an inductive definition of "term" to allow for all function-name terms.

Stage 0
The terms and wffs of stage 0 are those of L. Call this language L_0.

Stage n+1
Terms

Depth 0

a. Every term of stage *n* is a term of stage *n* + 1 of depth 0.
It is *closed* iff it is closed at stage *n*.

b. If A is a wff of stage *n* and not of stage *m* for any *m* < *n*, and *x* is free in A, then "<u>the</u> *x* A" is a term of stage *n* + 1. Every occurrence of *x* in "<u>the</u> *x* A(*x*)" is *bound.* All free occurrences of other variables in A are *free* in "<u>the</u> *x* A". The term "<u>the</u> *x* A" is *closed* if there is no variable free in it; otherwise it is *open*. The *scope* of the operator "<u>the</u> *x*" is A(*x*).

Depth *k* + 1
If $u_1, \ldots, u_m$ are terms of stage *n* + 1 of depth ≤ *k*, and at least one of them has depth *k*, and f is an *m*-ary function symbol, then $f(u_1, \ldots, u_m)$ is a term of depth *k* + 1. It is *closed* iff each t_i is closed.

A concatenation of symbols is a term of stage *n* + 1 iff it is a term of stage *n* + 1 of depth *k* for some *k*.

Wffs Given this definition of "term" the definition of wffs at stage *n* is the same as for classical predicate logic (p. 18).

Semantics

When we considered names for partial functions in Chapter 39 we treated all non-referring terms as nil. To extend the semantics to non-referring function-name terms we will treat those as nil, too.

Stage 0

We start with a model M of classical predicate logic with partial functions for L_0 in which every atomic name refers. To summarize that, we have a realization, semi-formal language, and universe. We also have a complete collection of assignments of references $\sigma_0, \tau_0, \gamma_0, \ldots$ satisfying:

For all x, $\sigma(x)\downarrow$.

For every atomic name c, $\sigma(c)\downarrow$

For each closed term u, for all σ and τ, $\sigma(u) \approx \tau(u)$.

Non-referring is the default application for compound terms.

The collection of such assignments satisfies the extensionality condition for function terms (p. 164).

The collection of such assignments is complete.

Valuations of the atomic wffs are given based on these assignments satisfying the extensionality condition for atomic predications as well:

$\mathsf{v}_\sigma \vDash u = v$ iff $\sigma(t)\downarrow = \sigma(u)\downarrow$.

From this and the extensionality condition for function terms we have:

iii–functions. If for every variable x appearing in u or v, $\sigma(x) = \tau(x)$, then $\mathsf{v}_\sigma \vDash u \equiv v$ iff $\mathsf{v}_\tau \vDash u \equiv v$.

We also have:

iv–referring. For all u, $\sigma(u)\downarrow$ iff $\mathsf{v}_\sigma \vDash u = u$.

Falsity is taken as the default truth-value for atomic predications:

If $A(u_1, \ldots, u_n)$ is an atomic wff and for some i, $\sigma(u_i)\not\downarrow$, then $\mathsf{v}_\sigma \nvDash A(u_1, \ldots, u_n)$.

The extension of the valuations on atomic wffs to all wffs is by the usual definition of classical predicate logic (p. 29).

Stage $n + 1$

Assignments of references

We extend each assignment of references σ_n of stage n to all terms of stage $n + 1$, denoting the new assignment as σ_{n+1}.

Terms of depth 0:
If u is a term of stage 0, then $\sigma_{n+1}(u) \approx \sigma_n(u)$.

If u is a term of the form "$\underline{\text{the}}\ x\ A(x)$" where $A(x)$ is a wff of stage n:

$\sigma_{n+1}(\underline{\text{the}}\ x\ A(x)) = a$ if for every τ_n such that $\tau_n \sim_x \sigma_n$,
$\mathsf{v}_{\tau_n} \vDash A$ iff $\tau_n(x) = a$.

In this case we say that "<u>the</u> $x\, A(x)$" is *referring* with respect to (the assignments made by) σ_{n+1}, and that its reference is a.

σ_{n+1} (<u>the</u> $A(x)) \not\downarrow$ otherwise.

In this case we say that "<u>the</u> $x\, A(x)$" is *nil* with respect to σ_{n+1}, or that σ_{n+1} assigns no value to "<u>the</u> $x\, A(x)$".

Terms of depth $k + 1$:
Given $f(u_1, \ldots, u_m)$ of depth $k + 1$ and an assignment of references σ_{n+1}, if for some i, $\sigma_{n+1}(u_i)\not\downarrow$, then $\sigma_{n+1}(f(u_1, \ldots, u_m))\not\downarrow$. In that case we say that $f(u_1, \ldots, u_m)$ is *nil* with respect to σ_{n+1}.

Otherwise, let $z_1, \ldots, z_n$ be the first variables not appearing in $f(u_1, \ldots, u_m)$ and τ_0 an assignment of references that differs from σ_{n+1} only in that $\tau_0(z_i) = \sigma_{n+1}(u_i)$. Then $\sigma_{n+1}(f(u_1, \ldots, u_n)) = \tau_0(f(z_1, \ldots, z_n))$.

For this to be an adequate definition we need that for every term u of stage $n + 1$ of depth $\leq k$ and every assignment of references σ_{n+1}, $\sigma_{n+1}(u)$ is given, subject to the collections of such assignments satisfying the extensionality condition and non-referring is the default application, which I'll let you prove.

Valuations We extend all valuations v_{σ_n} of stage n to all atomic wffs $A(u_1, \ldots, u_m)$ of stage $n + 1$, including the equality predicate:

If for some i, u_i is nil, then $v_{\sigma_{n+1}} \not\models A(u_1, \ldots, u_m)$.

Otherwise, proceeding from $i = 1$ to m, define v_i by:

If u_i is a variable or atomic name, let v_i be u_i.

If u_i is a "<u>the</u>"-term and $\sigma_{n+1}(u_i)\downarrow$, let v_i be the least variable that does not appear in $A(u_1, \ldots, u_m)$ and is not v_j for any $j < i$.

Let τ_0 be an assignment of references such that for each i, $\tau_0(v_i) = \sigma_{n+1}(u_i)$ and τ_0 agrees with σ_{n+1} on all other variables. Then:

$$v_{\sigma_{n+1}} \models A(u_1, \ldots, u_m) \text{ iff } v_{\tau_0} \models A(v_1|u_1, \ldots, v_m|u_m).$$

These valuations are extended to all wffs of stage $n + 1$ by the usual definition for classical predicate logic.

The proof that references and truth-values are stable is as before (p. 188). The model M^* is then defined via:

$\sigma^*(u) \approx \sigma_n(u)$ for the least n such that u is defined at stage n.

$v_{\sigma^*}(A) = v_{\sigma_n}(A)$ for the least n such that A is defined at stage n.

For every closed wff A, $v(A) = \mathsf{T}$ iff for all σ^*, $v_{\sigma^*}(A) = \mathsf{T}$.

Axiomatization

We add to the axiomatization of classical predicate logic with descriptive names and descriptive functions the axioms for partial functions.

I. Propositional axioms

The axiom schema of classical propositional logic in L($\neg, \rightarrow, \wedge, \vee$), where A, B, C are replaced by wffs of L and the universal closure is taken.

II. Axioms governing $\forall$

1. a. $\forall \ldots (\forall x\, (A \rightarrow B) \rightarrow (\forall x\, A \rightarrow \forall x\, B))$ — distribution of $\forall$
if x is free in both A and B

b. $\forall \ldots (\forall x\, (A \rightarrow B) \rightarrow (\forall x\, A \rightarrow B))$
if x is free in A and not free in B

c. $\forall \ldots (\forall x\, (A \rightarrow B) \rightarrow (A \rightarrow \forall x\, B))$
if x is free in B and not free in A

2. $\forall \ldots (\forall x\, \forall y\, A \rightarrow \forall y\, \forall x\, A)$ — commutativity of $\forall$

3*. a. $\forall \ldots (\forall x\, A(x) \wedge \exists x\, (x \equiv u)) \rightarrow A(u \mid x))$ — universal instantiation for referring terms
when u is free for x in A

III. Axioms governing the relation between $\forall$ and $\exists$

4. a. $\forall \ldots (\exists x\, A \rightarrow \neg \forall x \neg A)$

b. $\forall \ldots (\neg \forall x \neg A \rightarrow \exists x\, A)$

Axioms for equality

5. $\forall x\, (x \equiv x)$ — identity (for referring terms)

b.* $\exists x\, (x \equiv c)$ for every atomic name c — atomic names are referring

6.* $\bigwedge_i (u_i \equiv v_i) \rightarrow (A(u_1, \ldots, u_n) \rightarrow A(v_1, \ldots, v_n))$ — extensionality of atomic predications
for n-ary atomic wffs A

Axioms for partial functions

7.* a. $\forall \ldots (\bigwedge_i (u_i \equiv v_i) \rightarrow (f(u_1, \ldots, u_n) \equiv f(v_1, \ldots, v_n))$
for n-ary function symbols f — extensionality of functions

7.* b. $\forall \ldots (\neg \exists x\, (x \equiv u) \rightarrow \neg \exists y\, (y \equiv f(u_1, \ldots, u, \ldots, u_n)))$
for n-ary function symbols f — non-referring is the default application

Axioms for non-referring terms

8. $\forall \ldots (\forall y\, (\neg (\exists x\, (x \equiv y) \wedge A(y \mid x)) \rightarrow \neg \exists x\, A(x))$
where y is free for x in A

9. $\forall \ldots ((A(u \mid x) \wedge \exists x\, (x \equiv u)) \rightarrow \exists x\, A(x))$ — existential generalization for referring terms

10. $\forall \ldots (\neg \exists x\, (x \equiv u) \rightarrow \neg A(u_1, \ldots, u, \ldots, u_n))$ — nil terms use falsity as default
    for n-ary atomic wffs A and terms $t_1, \ldots, t, \ldots, t_n$

Axioms for "<u>the</u>"-terms

11. $\forall \ldots\ [\exists x\, A(x) \wedge \forall x\, \forall y\, (A(x) \wedge A(y|x) \rightarrow (x \equiv y))]$
    $\leftrightarrow \exists z\, (z \equiv \underline{\text{the}}\ x\, A(x))$
    where y is the least variable not appearing in A — referring descriptions create referring terms

12. $\forall \ldots\ \exists z\, (z \equiv \underline{\text{the}}\ x\, A(x)) \rightarrow A(\underline{\text{the}}\ x\, A(x))$ — referring descriptive terms satisfy their own description

Rule *modus ponens* for closed wffs

The proof that this axiomatization is strongly complete is a modification of the proof in Appendix 3 that classical predicate logic with descriptive functions is strongly complete, taking into account the more complex terms as in the proof in Appendix 3 that classical predicate logic with non-referring terms treated as nil is strongly complete.

Appendix 8
Modifying a Partially-Filled Relation

In the language of classical predicate logic presented in Chapter 4 there is only one way to formalize "Helga and Leopold are twins", namely:

(1) (— and — are twins) (Helga, Leopoldo)

But we could fill the blanks differently:

(2) (— and Leopoldo are twins) (Helga)

(3) (Helga and — are twins) (Leopoldo)

Each of these is true if and only if the informal "Helga and Leopoldo are twins" is true. There seems to be no point in allowing for (2) and (3). But with modifiers there are distinctions we can make that do depend on having all three of these available as well-formed formulas.

The best example I can give of this is with a ternary predicate. Consider:

(4) Alex and Betty and Clara are a ménage a trois.

We could say that these people are in a passionate ménage a trois with:

(4) ((— , —, — are a ménage a trois) / passionate) (Alex, Betty, Clara)

This is to say that each of them is passionate. But how do we formalize:

(5) Within the relation of being in a ménage a trois both Alex and Betty are passionate and Clara is not passionate.

We can do this with:

(6) ((Alex and Betty and — are a ménage a trois) / passionate) (Clara) $\wedge$
$\neg$ ((— and — and Clara are a ménage a trois) / passionate) (Alex, Betty)

By allowing for modifiers to be applied to predicates in which some but not all blanks are filled, we allow for formalizations that would otherwise be outside the scope of our formal systems. We can *modify some but not all terms within the relation*. I can see no other way to formalize a proposition like (5), yet it is straightforward in (6).

So we will allow for all of (1), (2), and (3) as well-formed formulas. And these are equivalent. I'll let you write out the seven different wff we get by replacing none or some but not all of the blanks in "(— and — and — are a ménage a trois)" with the names "Roger", "Clara," "Betty" in that order and using the remaining names in the usual way. Those wffs, too, when there is no modifier, are equivalent. To state this in a general way for all relations we need some notation.

Partially filled simple predicates Let P be an *n*-ary atomic predicate symbol with $n \geq 2$. Let $\vec{u}$ be a sequence $u_1, \ldots, u_n$ of terms. Let $\vec{v}$ be a subsequence $u_{i_1}, \ldots, u_{i_j}$ of $\vec{u}$, where $j < k$. Let $\vec{u} - \vec{v}$ be the sequence $\vec{u}$ with u_{i_r} deleted from the i_rth place in $\vec{u}$ for each *r*. Let $P\,|\vec{v}\,|$ be P with, for each *r*, the i_rth blank reading from the left filled with u_{i_r}.

A predicate symbol is realized by a *simple atomic predicate*: one that contains no restrictor symbol or name. An *atomic predicate* is a simple atomic predicate in which not all blanks are filled and which may or may not be modified. Thus, "(Helga and — are twins)" and "((— are twins —) / fraternal)" are atomic predicates but not simple atomic. An *atomic proposition* is, as before, an atomic predicate with all the blanks filled with terms.

We said that (1), (2), and (3) are equivalent. Generally, all the following are equivalent, for each is true or each is false according to whether the informal *P*(*a*, *b*) is true:

P(—, —) (*a*, *b*)
P(—, *b*) (*a*)
P(*a*, —) (*b*)

And the same equivalence holds for ternary, and quaternary, and *n*-ary relations generally. With our new notation we can state that.

Equivalence of partially filled simple predicates If *P* is a simple atomic predicate:

$P(\vec{u})$ is semantically equivalent to $P|\vec{v}|\ (\vec{u} - \vec{v})$

That is, a sequence of objects $\alpha_1, \ldots \alpha_m$ assigned to the terms in $\vec{u}$ satisfy $P(\vec{u})$ iff those same objects assigned to the same terms in $\vec{v}$ and $u - \vec{v}$ satisfy $P|\vec{v}|\ (\vec{u} - \vec{v})$.

Now consider:

(7) ((Alex, Betty, — are a ménage a trois) / passionate) (Clara) $\wedge$
((— , —, Clara are a ménage a trois) / passionate) (Alex, Betty)

This asserts that within the relation of being in a ménage a trois with Clara, Alex and Betty are passionate, and also within the relation of being in a ménage a trois with Alex and Betty, Clara is passionate. But this is just to say that all three are passionate within that relation of being in a ménage a trois. And that's all there is to saying that the ménage a trois is passionate for those three. That is, (7) is true iff the following is true:

((Alex and — and — are a ménage a trois) / passionate) (Betty, Clara) $\wedge$
((— and Betty — are a ménage a trois) / passionate) (Alex, Clara) $\wedge$
((— , —, Clara are a ménage a trois) / passionate) (Alex, Betty)

For binary predicates it's simple to state the semantic principle of such equivalences:

$$[\ (P(a, —)/L)\ (b) \wedge\ (P(—, b)/L)\ (a)\ ] \leftrightarrow (P(—, —)/L)\ (a, b)$$

The equivalences comparable to this for ternary relations are more complicated. Roughly, if a restrictor applies to each term within a predicate, then it applies to the predicate when applied to all the terms, and vice-versa.

Equivalence of modified predicates cov ering all terms Given P a simple atomic predicate and $\vec{u}$ with subsequences $\vec{v}_1, \ldots, \vec{v}_k$ of $\vec{u}$ such that each term in $\vec{u}$ appears in at least one of those subsequences, then:

$(P/L)(\vec{u})$ is semantically equivalent to

$$(P|\vec{v}_1|\ /L)\ (\vec{u} - \vec{v}_1) \wedge\ \ldots \wedge\ (P|\vec{v}_k|\ /L)\ (\vec{u} - \vec{v}_k)$$

Appendix 9
The Tapestry of Time

There is a view of time that does not take it to flow, to be a mass, in which time is pointillistic and static but not linearly ordered. It provides a contrast to the views that are explored in the main text, and also shows how a conception of time is linked to issues of free will and theology.[1] At the end of Chapter 49 I discuss how it may be possible to base a tense logic on such a view without invoking modal notions.

Let us suppose: Some of what we do we choose freely to do.

Consider, then, the following picture:

At every moment at which a free choice is made, the world branches. So there is a split at the moment at which I chose to sit down to type: in one branch I sit down to type, in another I do not. But that branching is not just a single splitting: on one branch I sit down to type, on the other I don't. Rather, there is a different path according to any of the choices I might have made: go for a walk, play with the dog, go inspect the sheep corral, stick my head under a faucet to ease my allergies, ...

Similarly, the world branches depending on whether this electron moves to this or that energy level in this atom, if such movements are random. Particular random happenings and particular free choices in any combination lead each to a different branch.

At each "moment" there is a multiplicity of branchings, beyond our ability to comprehend except one branch in comparison to another specific branch. These branchings continue on. What we call "a branch" is just a particular path through all such moments at which a free choice and/or random happening occurs. We do not have any reason to believe that any branch ever stops; nor do we have any reason to think that any branch does not stop. Nor do we have any reason to believe that each point on each branch has a unique path to it: there might have been many ways to arrive at the same point. If we count memory as part of our path, though, then a conscious free choice creates a branching that cannot be reached by any other branch. But a random act of an electron could lead to a point that could be reached by other random physical happenings. This is a way in which the internal world could differ from the external world.

Laws of nature, if there be any, give the substratum of all these branchings. When I choose to sit down and type, I do not have any choice in how the chemical reactions in my blood continue. Given this particular disintegration of the radon atom, the Geiger counter will make a sound.

[1] I do not claim that the ideas here are new, but I have not found a concise exposition of them.

Each branch is real, as real as any other. These are not “alternate worlds”, “alternate possibilities” compared to the world I am in. They are all real, equally real. At a branching, the “I” up to that point continues in a multiplicity of branches, and in each one it is reasonable to say it is the same “I”, for they all come from the same branching. Thus, the “I” of this branch is the same “I” as the one in which I chose to go for a walk instead of typing because they can be traced back to that moment at which the “I”s branched apart due to a free choice. If we had the ability to see all these branches, we could say “That is one way I might have been had I done that instead of this”.

The world, then, is a tapestry of branchings so multitudinous as to be beyond comprehension in their details: only the general form is conceivable to us. The tapestry is not flowing; there is no movement in the tapestry, only threads that make up the whole. What we call “time” is a branch: the tapestry is oriented, and that orientation is what we call the arrow from the past to the future. What we call “now” is the consciousness we have of being right here on this branch. Every point on every branch is as real as any other: the unreality of the past and of the future, for me, is that they are not at the point that I call “now” on this branch where I am.

We, each of us, choose which branch we follow, though equally, another person, exactly the same “I” up to a particular branching point, chooses a different branch, and then a different branch, and a different branch again forever. There are some branchings in which the “I” of when I was seventeen chose to be mean to her, and one branch, followed through all its multiple branchings, in which “I” from seventeen on lived a blameless life, good to the point of being saintly.

In this conception, free will is fully compatible with the assumption that there is an omniscient God who knows the future as well as the past. God would be the only intelligence that could comprehend all branches at once. He can see the point on the branch of Jesus’ life at which Judas betrayed him and see a branching where Judas did not betray Jesus: in all the branchings that followed the one choice, Judas is damned; in all the branchings that followed the other choice, Judas is saved. Or perhaps not: in some of those branchings he may have repented and been saved; in some of those branchings he could have betrayed Jesus also. Only God knows.

Now all we need is some empirical evidence to support this view.

Appendix 10
"Water is H_2O"

Philosophers have been discussing for a long time the status of the sentence:

(1) Water is H_2O.

They say that it is an identity, but differ on whether it is an essential or a contingent one. It is neither.

If taken at face-value, it is too vague to be a proposition. A mass term, "water", is said to pick out the same part of the world as the word "H_2O". But "H_2O" is a descriptive term for identifying molecules: two hydrogen atoms joined with one oxygen atom. It doesn't pick out anything at all in the world, though it distinguishes some molecules from others, as in "That's an H_2O molecule". If "H_2O" is meant to pick out some part of the world, it must mean some molecule or molecules of that kind. The purported identity then is:

(2) Any quantity of water is a collection of H_2O molecules.

On the face of it, however, (2) is a category mistake: a mass is identified with a collection of things. Like saying a horse is a number or an idea jumps, it's nonsense. To hold that it is not nonsense but true we have to accept that a mass is a collection of things. This is part of the general program of trying to reduce ontological categories such as masses and processes and events to the linguistic and logical paradigm of Western thought: the world is made up of things, and propositions are about things. Actually, a much stronger assumption is usually taken as the basis of that view: the world is made up of *only* things.[1]

Suppose, though, that (2) is true: any quantity of water is a collection of H_2O molecules. Then we have to reject almost every way we have of learning what "water" means. We point to a river and say "That's water", yet it isn't a collection of H_2O molecules: it's a mixture, with sand, and dirt, and minerals. We never encounter collections of just H_2O molecules. No one, on this account, has ever seen or experienced or drunk water, except perhaps for a few chemists in a laboratory.

If (1) is an identity, then the reverse of it is, too. That is, from (1) we have:

H_2O is water.

For this to be a proposition we have to understand it as:

(3) Any collection of H_2O molecules is a quantity of water.

---

[1] Harry C. Bunt in *Mass Terms and Model Theoretic Semantics* gives a good survey and history of this reductionism and stronger assumption.

It is (3) that Quine assumes in his defense of the principle that each mass has least parts. In particular, he asserts that a least part of any mass of water is an H_2O molecule.[2] A single H_2O molecule is water.

To accept (3) as true we have to reject almost every claim we believe is true about water: "Water flows", "Water freezes", "Water gives rise to a sensation of wetness", "Water evaporates". None of those are true of a single molecule, not even of four molecules of water. To rescue those claims we have to modify both (2) and (3) to give a lower limit for the number of molecules of H_2O that can count as a quantity of water. Perhaps such a number could be established: to do so would be to substitute a precise notion for an imprecise one, avoiding thereby a sorites paradox but, as with all such solutions, leading to anomalous classifications. In any case, we would need a notion of water that is independent of any chemical theory before we could establish such a minimal bound. We would be giving an analysis of water in chemical terms, but water as a mass would be primary. We would be attempting to reduce the notion of a mass, in this case water, to a collection of things, in this case molecules of H_2O, as part of the larger reductive program.

Still, everyday people talk of water as if it were the stuff that is basic in the liquids we encounter. We talk of dirty water, alkaline water, acidic water, salt water. But compare: a brown dog, a white dog, a tiny dog. Just as each of those is a dog, so, too, alkaline water is water. Yet we also talk of pure water, as if we could eliminate what's not really water in a muddy river or a cup of salt water, while we don't talk of a pure dog. This is the first step in science: we take some part of our experience and choose to ignore all but one aspect, to say that dissolved minerals and suspended bits of earth are not what we're interested in, are not important to our discussion of this liquid. Until chemists isolated and named H_2O molecules, to assume that there was a pure substance which was the substratum, the key ingredient in all the liquids we call "water" was a hope, a conjecture, but not something one had good reason to believe.[3]

Chemists have now isolated that key ingredient. They will tell us, as one told a colleague of mine who asked him if water is H_2O:

> To answer the question simply, water is H_2O. In general, a material is defined by its chemical formula and an atom is defined by the number of protons it contains.

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[2] Willard Van Orman Quine says in *Word and Object:*

> In general a mass term in predicative position may be viewed as a general term which is true of each portion of the stuff in question, excluding only the parts too small to count. Thus "water" and "sugar", in the role of general terms, are true of each part of the world's water or sugar, down to single molecules but not to atoms; and "furniture", in the role of general term, is true of each part of the world's furniture down to single chairs but not to legs and spindles. pp. 97–98

[3] For a reduction of masses to collections of things generally we would need a notion of pure honey, and pure butter, and pure mud. Platonists have no problem with this; they even say there is a pure dog.

But this can’t be right, for it takes “Water is H_2O” to be a definition. No one, then, could have discovered that water is H_2O.

Rather, “Water is H_2O” is an abstraction from experience, as are all scientific laws. Like “Force = mass x acceleration” it is not true. Neither is it false. A scientific law tells us that if we abstract from our experience in this way, ignoring this, paying attention to that, then the claim is true.[4] Or rather, it is true enough.

I told my coleague to go to the chemistry professor, take a glass and fill it with water from a tap in front of him, and ask him if that’s water. If he says “yes”, he’s wrong by his definition. If he says “no” then ask him what good are his theories and analysis of water as H_2O. He’ll start backtracking and meandering until he ends up at the view that “Water is H_2O” is an abstraction that is not true or false until we say what we are paying attention to in our experience.

“Water is H_2O” is not an identity in the sense that philosophers think, not a discovery that what we call water is, for any particular quantity, a collection of H_2O molecules. Yet it is meaningful. It shows us how to use chemical and physical theories. We can use those theories to analyze water and how water is in the world so long as what those theories take into account is what we want to take into account and so long as what those theories ignore is what we are willing to ignore. For example, a chemical theory will not say “water freezes”, because “water” is not a term of the theory. What it will say is that given a large enough quantity of H_2O molecules, and only H_2O molecules, they will freeze at 0 degrees centigrade.[5] We then use that information to find our way in the world. Actually, though, we knew long before there were any theories of molecules that water—certainly not “pure” water—freezes at some temperature that we designate as 0 degrees centigrade. It was for the chemical theory to explain that. And it does, if we agree to use the theory correctly in applying it to experience: we only use it to talk about water if we have a large enough quantity of H_2O molecules with few enough “impurities”.

In doing philosophy of science it’s good to talk to scientists to see if your theory is compatible with what they do. But it need not be compatible with what they say. Many chemists have forgotten that theirs is a science of the natural world. They retreat to chemical formulas in place of thought about the world just as many mathematicians retreat to platonism when first asked if their theorems are true. My colleague’s chemist friend is a naive platonist as much as most mathematicians, thinking that abstractions are true, for there is no other way for an abstraction to be true, rather than a guide to experience.

We know how to use the word “water” quite well. The chemists’ analyses have not changed how we use it, nor should they. Water is a mass, not a collection of things.

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[4] This is a view of science that I develop in “On models and theories”.

[5] “Cluster physics” is the study of the minimal number of molecules of a particular kind that is necessary for a collection of those molecules to have the macroscopic properties we associate with the substance.

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