Automation of the Principia Metaphysica in HOL: Part I

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1 Introduction

This work is related to [1], which is significantly extends ...

2 Preliminaries

typedecl i
— the type possible worlds; the formalism explicitly encodes Kripke style semantics
type-synonym $io = (i \Rightarrow bool)$
— formulas are essentially of this type
— predicates on worlds
typedecl e
— the raw type of entities/objects (abstract or ordinary)
datatype $'a\ opt = Error\ 'a \mid Term\ 'a \mid Form\ 'a \mid PropForm\ 'a$

consts $cw :: i$
— the distinguished actual world
consts $dE{::}e\ dIO{::}io\ dEIO{::}e{\Rightarrow}io\ dEEIO{::}e{\Rightarrow}e{\Rightarrow}io\ dEEEIO{::}e{\Rightarrow}e{\Rightarrow}e{\Rightarrow}io\ dA{::}'a$
— some fixed dummy symbols; we anyway assume that the domains are non-empty
— needed as dummy object in some cases below

Meta-logical predicates.

abbreviation $isWff :: io\ opt{\Rightarrow}bool$ **where** $isWff\ \varphi \equiv case\ \varphi\ of\ Error\ \psi \Rightarrow False \mid Term\ \psi \Rightarrow False$ $|\text{-} \Rightarrow True$
abbreviation $isForm :: io\ opt{\Rightarrow}bool$ **where** $isForm\ \varphi \equiv case\ \varphi\ of\ Form\ \psi \Rightarrow True \mid \text{-} \Rightarrow False$
abbreviation $isPropForm :: io\ opt{\Rightarrow}bool$ **where** $isPropForm\ \varphi \equiv case\ \varphi\ of\ PropForm\ \psi \Rightarrow True \mid$ $\text{-} \Rightarrow False$
abbreviation $isTerm :: io\ opt{\Rightarrow}bool$ **where** $isTerm\ \varphi \equiv case\ \varphi\ of\ Term\ \psi \Rightarrow True \mid \text{-} \Rightarrow False$
abbreviation $isError :: io\ opt{\Rightarrow}bool$ **where** $isError\ \varphi \equiv case\ \varphi\ of\ Error\ \psi \Rightarrow True \mid \text{-} \Rightarrow False$
abbreviation $valid :: io\ opt{\Rightarrow}bool$ **where** $[\varphi] \equiv case\ \varphi\ of$
  $PropForm\ \psi \Rightarrow \forall w.(\psi\ w)$
  $\mid Form\ \psi \Rightarrow \forall w.(\psi\ w)$
  $\mid \text{-} \Rightarrow False$
abbreviation $satisfiable :: io\ opt{\Rightarrow}bool$ **where** $[\varphi]^{sat} \equiv case\ \varphi\ of$
  $PropForm\ \psi \Rightarrow \exists w.(\psi\ w)$
  $\mid Form\ \psi \Rightarrow \exists w.(\psi\ w)$
  $\mid \text{-} \Rightarrow False$
abbreviation $countersatisfiable :: io\ opt{\Rightarrow}bool$ **where** $[\varphi]^{csat} \equiv\ case\ \varphi\ of$

PropForm $\psi \Rightarrow \exists\, w.\neg(\psi\ w)$
| *Form* $\psi \Rightarrow \exists\, w.\neg(\psi\ w)$
| - $\Rightarrow$ *False*

abbreviation *invalid* :: *io opt*$\Rightarrow$*bool* **where** $[\varphi]^{inv} \equiv$ *case* φ *of*
PropForm $\psi \Rightarrow \forall\, w.\neg(\psi\ w)$
| *Form* $\psi \Rightarrow \forall\, w.\neg(\psi\ w)$
| - $\Rightarrow$ *False*

3 Encoding of the language

abbreviation $\mathcal{A}$::*io opt* $\Rightarrow$ *io opt* **where** $\mathcal{A}\ \varphi \equiv$ *case* φ *of*
Form $\psi \Rightarrow$ *Form* $(\lambda w.\ \psi\ cw)$
| *PropForm* $\psi \Rightarrow$ *PropForm* $(\lambda w.\ \psi\ cw)$
| - $\Rightarrow$ *Error dIO*

actuality operator; φ is evaluated wrt the current world; Error is passed on

abbreviation *Enc*::*e opt*$\Rightarrow$(*e*$\Rightarrow$*io*) *opt*$\Rightarrow$*io opt* **where** $<x \circ P> \equiv$ *case* (x,P) *of*
(*Term* y,*Term* Q) $\Rightarrow$ *Form* $(\lambda w.(Q\ y)\ w)$
| (-,-) $\Rightarrow$ *Error dIO*

$\kappa_1\Pi^1$ will be written here as $<\kappa_1\circ\Pi^1>$; $\kappa_1\Pi^1$ is a Form; Error is passed on

abbreviation *Exe1*::(*e*$\Rightarrow$*io*) *opt*$\Rightarrow$*e opt*$\Rightarrow$*io opt* **where** $<P\bullet x> \equiv$ *case* (P,x) *of*
(*Term* Q,*Term* y) $\Rightarrow$ *PropForm* $(\lambda w.(Q\ y)\ w)$
| - $\Rightarrow$ *Error dIO*

$\Pi^1\kappa_1$ will be written here as $<\Pi^2\bullet\kappa_1>$; $\Pi^1\kappa_1$ is a PropForm; Error is passed on

abbreviation *Exe2*::(*e*$\Rightarrow$*e*$\Rightarrow$*io*) *opt*$\Rightarrow$*e opt*$\Rightarrow$*e opt*$\Rightarrow$*io opt* **where** $<P\bullet x1,x2> \equiv$ *case* $(P,x1,x2)$ *of*
(*Term* Q,*Term* $y1$,*Term* $y2$) $\Rightarrow$ *PropForm* $(\lambda w.(Q\ y1\ y2)\ w)$
| - $\Rightarrow$ *Error dIO*

$\Pi^2\kappa_1\kappa_2$ will be written here as $<\Pi^2\bullet\kappa_1,\kappa_2>$; $\Pi^2\kappa_1\kappa_2$ is a PropForm; Error is passed on

abbreviation *Exe3*::(*e*$\Rightarrow$*e*$\Rightarrow$*e*$\Rightarrow$*io*) *opt*$\Rightarrow$*e opt*$\Rightarrow$*e opt*$\Rightarrow$*e opt*$\Rightarrow$*io opt* **where** $<P\bullet x1,x2,x3> \equiv$ *case* $(P,x1,x2,x3)$ *of*
(*Term* Q,*Term* $y1$,*Term* $y2$,*Term* $y3$) $\Rightarrow$ *PropForm* $(\lambda w.(Q\ y1\ y2\ y3)\ w)$
| - $\Rightarrow$ *Error dIO*

$\Pi^3{}_1\kappa_2\kappa_3$ will be written here as $<\Pi^2\bullet\kappa_1,\kappa_2,\kappa_3>$; $\Pi^3{}_1\kappa_2\kappa_3$ is a PropForm; Error is passed on; we could, of course, introduce further operators: Exe4, Exe5, etc.

abbreviation *z-not*::*io opt*$\Rightarrow$*io opt* **where** $\neg^z\ \varphi \equiv$ *case* φ *of*
Form $\psi \Rightarrow$ *Form* $(\lambda w.\ \neg\ \psi\ w)$
| *PropForm* $\psi \Rightarrow$ *PropForm* $(\lambda w.\ \neg\ \psi\ w)$
| - $\Rightarrow$ *Error dIO*

negation operator; $\neg^z\ \varphi$ inherits its type from φ if φ is Form or PropForm; Error is passed on

abbreviation *z-implies*::*io opt*$\Rightarrow$*io opt*$\Rightarrow$*io opt* **where** $\varphi \rightarrow^z \psi \equiv$ *case* (φ,ψ) *of*
(*PropForm* α,*PropForm* β) $\Rightarrow$ *PropForm* $(\lambda w.\ \alpha\ w \longrightarrow \beta\ w)$
| (*Form* α,*Form* β) $\Rightarrow$ *Form* $(\lambda w.\ \alpha\ w \longrightarrow \beta\ w)$
| - $\Rightarrow$ *Error dIO*

implication operator; $\varphi \rightarrow^z \psi$ returns returns a PropForm if both are PropForms, Form if both are Forms, otherwise it returns Error

abbreviation *z-forall*::$('a \Rightarrow io\ opt) \Rightarrow io\ opt$ **where** $\forall\ \Phi \equiv case\ (\Phi\ dA)\ of$
$\quad PropForm\ \varphi \Rightarrow PropForm\ (\lambda w.\ \forall x.\ case\ (\Phi\ x)\ of\ PropForm\ \psi \Rightarrow \psi\ w)$
$\mid Form\ \varphi \Rightarrow Form\ (\lambda w.\ \forall x.\ case\ (\Phi\ x)\ of\ Form\ \psi \Rightarrow \psi\ w)$
$\mid \text{-} \Rightarrow Error\ dIO$

universal quantification; $\forall\,(\lambda x.\varphi)$ inherits its kind (Form or PropForm) from φ; Error is passed on $\forall\,(\lambda x.\varphi)$ is mapped to $(\lambda w.\forall\,x.\varphi xw)$ as expected

abbreviation *z-box*::$io\ opt \Rightarrow io\ opt$ **where** $\Box\ \varphi \equiv case\ \varphi\ of$
$\quad Form\ \psi \Rightarrow Form\ (\lambda w.\ \forall v.\ \psi\ v)$
$\mid PropForm\ \psi \Rightarrow PropForm\ (\lambda w.\ \forall v.\ \psi\ v)$
$\mid \text{-} \Rightarrow Error\ dIO$

box operator; $\Box\ \varphi$ inherits its type (Form or PropForm) from φ; Error is passed on. Note that the $\Box$-operator is defined here without an accessibility relation; this is ok since we assume logic S5.

abbreviation *lam0*::$io\ opt \Rightarrow io\ opt$ **where** $\lambda^0\ \varphi \equiv case\ \varphi\ of$
$\quad PropForm\ \psi \Rightarrow PropForm\ \psi$
$\mid \text{-} \Rightarrow Error\ dIO$

0-arity lambda abstraction; $\lambda^0\ \varphi$ returns PropForm φ if φ is a PropForm, otherwise Error

abbreviation *lam1*::$(e \Rightarrow io\ opt) \Rightarrow (e \Rightarrow io)\ opt$ **where** $\lambda^1\ \Phi \equiv case\ (\Phi\ dE)\ of$
$\quad PropForm\ \varphi \Rightarrow Term\ (\lambda x.\ case\ (\Phi\ x)\ of\ PropForm\ \varphi \Rightarrow \varphi)$
$\mid \text{-} \Rightarrow Error\ (\lambda x.\ dIO)$

1-arity lambda abstraction; $\lambda^1(\lambda x.\varphi)$ returns Term $(\lambda x.\varphi)$ if φ is a PropForm, otherwise Error

abbreviation *lam2*::$(e \Rightarrow e \Rightarrow io\ opt) \Rightarrow (e \Rightarrow e \Rightarrow io)\ opt$ **where** $\lambda^2\ \Phi \equiv case\ (\Phi\ dE\ dE)\ of$
$\quad PropForm\ \varphi \Rightarrow Term\ (\lambda x\ y.\ case\ (\Phi\ x\ y)\ of\ PropForm\ \varphi \Rightarrow \varphi)$
$\mid \text{-} \Rightarrow Error\ (\lambda x\ y.\ dIO)$

2-arity lambda abstraction; $\lambda^2(\lambda xy.\varphi)$ returns Term $(\lambda xy.\varphi)$ if φ is a PropForm, otherwise Error

abbreviation *lam3*::$(e \Rightarrow e \Rightarrow e \Rightarrow io\ opt) \Rightarrow (e \Rightarrow e \Rightarrow e \Rightarrow io)\ opt$ **where** $\lambda^3\ \Phi \equiv case\ (\Phi\ dE\ dE\ dE)\ of$
$\quad PropForm\ \varphi \Rightarrow Term\ (\lambda x\ y\ z.\ case\ (\Phi\ x\ y\ z)\ of\ PropForm\ \varphi \Rightarrow \varphi)$
$\mid \text{-} \Rightarrow Error\ (\lambda x\ y\ z.\ dIO)$

3-arity lambda abstraction; $\lambda^2(\lambda xyz.\varphi)$ returns Term $(\lambda xyz.\varphi)$ if φ is a PropForm, otherwise Error; we could, of course, introduce further operators: λ^4, λ^5, etc.

abbreviation *that*::$(e \Rightarrow io\ opt) \Rightarrow e\ opt$ **where** $\varepsilon\ \Phi \equiv case\ (\Phi\ dE)\ of$
$\quad PropForm\ \varphi \Rightarrow Term\ (THE\ x.\ case\ (\Phi\ x)\ of\ PropForm\ \psi \Rightarrow \psi\ cw)$
$\mid \text{-} \Rightarrow Error\ dE$

that operator; that $(\lambda x.\varphi)$ returns Term $(THE\ x.\ \varphi\ x\ cw)$, that is the inbuilt THE operator is used and evaluation is wrt to the current world cw; moreover, application of that is allowed if $(\Phi\ sRE)$ is a PropForm, otherwise Error is passed on for some someRawEntity

4 Further logical connectives

abbreviation *z-and*::*io opt*⇒*io opt*⇒*io opt* **where** $\varphi \wedge^z \psi \equiv \neg^z(\varphi \rightarrow^z \neg^z\psi)$
abbreviation *z-or*::*io opt*⇒*io opt*⇒*io opt* **where** $\varphi \vee^z \psi \equiv (\neg^z\varphi \rightarrow^z \psi)$
abbreviation *z-equiv*::*io opt*⇒*io opt*⇒*io opt* **where** $\varphi \equiv^z \psi \equiv (\varphi \rightarrow^z \psi) \wedge^z (\psi \rightarrow^z \varphi)$
abbreviation *z-exists*::(*'a*⇒*io opt*)⇒*io opt* **where** $\exists\ \Phi \equiv$ *case* $(\Phi\ dA)$ *of*
  PropForm $\varphi \Rightarrow$ *PropForm* $(\lambda w.\ \exists x.$ *case* $(\Phi\ x)$ *of PropForm* $\psi \Rightarrow \psi\ w)$
  | *Form* $\varphi \Rightarrow$ *Form* $(\lambda w.\ \exists x.$ *case* $(\Phi\ x)$ *of Form* $\psi \Rightarrow \psi\ w)$
  | - ⇒ *Error dIO*
abbreviation *z-dia*::*io opt*⇒*io opt* **where** $\Diamond\ \varphi \equiv \neg^z\ (\Box\ (\neg^z\ \varphi))$

5 Some shortcuts for the constructors

abbreviation *mkPropForm* :: *io*⇒*io opt* **where** ,*p*, ≡ *PropForm p*
abbreviation *mkForm* :: *io*⇒*io opt* **where** ;*p*; ≡ *Form p*
abbreviation *mkTerm* :: *'a*⇒*'a opt* **where** .*t*. ≡ *Term t*
abbreviation *mkError* :: *'a*⇒*'a opt* **where** ∗*t*∗ ≡ *Term t*

6 Some Basic Tests

6.1 Meta-Logic

lemma $(\forall\,\varphi.\ [,\varphi,]) \longleftrightarrow [\forall\,(\lambda\varphi.\ ,\varphi,)]$ **apply** *simp* **by** *auto*
lemma $(\forall\,\varphi.\ [;\varphi;]) \longleftrightarrow [\forall\,(\lambda\varphi.\ ;\varphi;)]$ **apply** *simp* **by** *auto*

6.2 Verifying Modal Logic Principles

Necessitation holds

lemma *necessitation-PropForm*: $[,\varphi,] \longrightarrow [\Box\ ,\varphi,]$ **apply** *simp* **done**
lemma *necessitation-Form*: $[;\varphi;] \longrightarrow [\Box\ ;\varphi;]$ **apply** *simp* **done**

Modal Collapse does not hold

lemma *modalCollapse-PropForm*: $[,\varphi, \rightarrow^z \Box\ ,\varphi,]$ **apply** *simp* **nitpick oops**
lemma *modalCollapse-Form*: $[;\varphi; \rightarrow^z \Box\ ;\varphi;]$ **apply** *simp* **nitpick oops**

6.3 S5 Principles

lemma *axiom-T-PF*: $[(\Box\ ,\varphi,) \rightarrow^z\ ,\varphi,]$ **apply** *simp* **done**
lemma *axiom-T-F*: $[(\Box\ ;\varphi;) \rightarrow^z\ ;\varphi;]$ **apply** *simp* **done**

lemma *axiom-B-PF*: $[,\varphi, \rightarrow^z (\Box\ (\Diamond\ ,\varphi,))]$ **apply** *simp* **done**
lemma *axiom-B-F*: $[;\varphi; \rightarrow^z (\Box\ (\Diamond\ ;\varphi;))]$ **apply** *simp* **done**

lemma *axiom-D-PF*: $[\Box\ ,\varphi, \rightarrow^z \Box\ (\Box\ ,\varphi,)]$ **apply** *simp* **done**
lemma *axiom-D-F*: $[\Box\ ;\varphi; \rightarrow^z \Box\ (\Box\ ;\varphi;)]$ **apply** *simp* **done**

lemma *axiom-4-PF*: $[\Box\ ,\varphi, \rightarrow^z \Diamond\ ,\varphi,]$ **apply** *simp* **by** *auto*
lemma *axiom-4-F*: $[\Box\ ;\varphi; \rightarrow^z \Diamond\ ;\varphi;]$ **apply** *simp* **by** *auto*

lemma *axiom-5-PF*: $[\Diamond\ ,\varphi, \rightarrow^z \Box\ (\Diamond\ ,\varphi,)]$ **apply** *simp* **done**
lemma *axiom-5-F*: $[\Diamond\ ;\varphi; \rightarrow^z \Box\ (\Diamond\ ;\varphi;)]$ **apply** *simp* **done**

lemma *test-A-PF*: $[\Box\ (\Diamond\ ,\varphi,) \rightarrow^z \Diamond\ ,\varphi,]$ **apply** *simp* **done**
lemma *test-A-F*: $[\Box\ (\Diamond\ ;\varphi;) \rightarrow^z \Diamond\ ;\varphi;]$ **apply** *simp* **done**

lemma *test-B-PF*: $[\Diamond\ (\Box\ ,\varphi,) \rightarrow^z \Diamond\ ,\varphi,]$ **apply** *simp* **by** *auto*
lemma *test-B-F*: $[\Diamond\ (\Box\ ;\varphi;) \rightarrow^z \Diamond\ ;\varphi;]$ **apply** *simp* **by** *auto*

lemma *test-C-PF*: $[\Box\ (\Diamond\ ,\varphi,) \rightarrow^z \Box\ ,\varphi,]$ **apply** *simp* **nitpick oops**
lemma *test-C-F*: $[\Box\ (\Diamond\ ;\varphi;) \rightarrow^z \Box\ ;\varphi;]$ **apply** *simp* **nitpick oops**

lemma *test-D-PF*: $[\Diamond\ (\Box\ ,\varphi,) \rightarrow^z \Box\ ,\varphi,]$ **apply** *simp* **done**
lemma *test-D-F*: $[\Diamond\ (\Box\ ;\varphi;) \rightarrow^z \Box\ ;\varphi;]$ **apply** *simp* **done**

6.4 Validity, Satisfiabilty, Countersatisfiability and Invalidity

lemma $[,\varphi,] \longleftrightarrow \neg\ [,\varphi,]^{csat}$ **apply** *simp* **done**
lemma $[,\varphi,]^{sat} \longleftrightarrow \neg\ [,\varphi,]^{inv}$ **apply** *simp* **done**
lemma $[;\varphi;] \longleftrightarrow \neg\ [;\varphi;]^{csat}$ **apply** *simp* **done**
lemma $[;\varphi;]^{sat} \longleftrightarrow \neg\ [;\varphi;]^{inv}$ **apply** *simp* **done**

For Terms and Error objects these correspondences do not apply

lemma $[.\varphi.] \longleftrightarrow \neg\ [.\varphi.]^{csat}$ **nitpick oops**
lemma $[.\varphi.]^{sat} \longleftrightarrow \neg\ [.\varphi.]^{inv}$ **nitpick oops**
lemma $[{*}\varphi{*}] \longleftrightarrow \neg\ [{*}\varphi{*}]^{csat}$ **nitpick oops**
lemma $[{*}\varphi{*}]^{sat} \longleftrightarrow \neg\ [{*}\varphi{*}]^{inv}$ **nitpick oops**

6.5 Example signature; entities and relations

consts *a-0* :: *e* **abbreviation** *a* **where** $a \equiv .a\text{-}0.$
consts *b-0* :: *e* **abbreviation** *b* **where** $b \equiv .b\text{-}0.$
consts *c-0* :: *e* **abbreviation** *c* **where** $c \equiv .c\text{-}0.$

consts *R-0* :: *io* **abbreviation** *R0* **where** $R0 \equiv .R\text{-}0.$
consts *R-1* :: $e \Rightarrow io$ **abbreviation** *R1* **where** $R1 \equiv .R\text{-}1.$
consts *R-2* :: $e \Rightarrow e \Rightarrow io$ **abbreviation** *R2* **where** $R2 \equiv .R\text{-}2.$
consts *R-3* :: $e \Rightarrow e \Rightarrow e \Rightarrow io$ **abbreviation** *R3* **where** $R3 \equiv .R\text{-}3.$

Testing term and formula constructions

lemma $[<R1 \cdot a>]$ **nitpick oops**
lemma *isPropForm* $<R1 \cdot a>$ **apply** *simp* **done**
lemma $<R1 \cdot a> = X$ **apply** *simp* **oops**

lemma $[<a \circ R1>]$ **nitpick oops**
lemma *isPropForm* $<a \circ R1>$ **apply** *simp* **oops**
lemma *isForm* $<a \circ R1>$ **apply** *simp* **done**
lemma $<a \circ R1> = X$ **apply** *simp* **oops**

lemma $[<\lambda^1(\lambda x.\ <R1 \cdot .x.> \rightarrow^z <R1 \cdot .x.>) \cdot a>]$ **apply** *simp* **done**
lemma $<\lambda^1(\lambda x.\ <R1 \cdot .x.> \rightarrow^z <R1 \cdot .x.>) \cdot a> = X$ **apply** *simp* **oops**

lemma $\neg$ *isWff* $(<R1 \cdot .x.> \rightarrow^z <.x. \circ R1>)$ **apply** *simp* **done**
lemma $\lambda^1(\lambda x.\ <R1 \cdot .x.> \rightarrow^z <.x. \circ R1>) = X$ **apply** *simp* **oops**

lemma $[<\lambda^1(\lambda x.\ <R1{\cdot}.x.> \to^z <.x.\circ R1>){\cdot}a>]$ **apply** *simp* **oops**
lemma $<\lambda^1(\lambda x.\ <R1{\cdot}.x.> \to^z <.x.\circ R1>){\cdot}a> = X$ **apply** *simp* **oops**

lemma $[\forall(\lambda x.\ <R1{\cdot}.x.> \to^z <R1{\cdot}.x.>)]$ **apply** *simp* **done**
lemma $[\forall(\lambda R.\ \forall(\lambda x.\ <.R.{\cdot}.x.> \to^z <.R.{\cdot}.x.>))]$ **apply** *simp* **done**
lemma $\forall(\lambda x.\ <R1{\cdot}.x.> \to^z <R1{\cdot}.x.>) = X$ **apply** *simp* **oops**

lemma $[\forall(\lambda x.\ <.x.\circ R1> \to^z <.x.\circ R1>)]$ **apply** *simp* **done**
lemma $\forall(\lambda x.\ <.x.\circ R1> \to^z <.x.\circ R1>) = X$ **apply** *simp* **oops**

lemma $[\forall(\lambda x.\ <R1{\cdot}.x.> \to^z <.x.\circ R1>)]$ **apply** *simp* **oops**
lemma $\forall(\lambda x.\ <R1{\cdot}.x.> \to^z <.x.\circ R1>) = X$ **apply** *simp* **oops**
lemma $[\forall(\lambda R.\ <.R.{\cdot}.x.> \to^z <.x.\circ.R.>)]$ **apply** *simp* **oops**
lemma $\forall(\lambda R.\ <.R.{\cdot}.x.> \to^z <.x.\circ.R.>) = X$ **apply** *simp* **oops**

7 Are the priorities set correctly?

lemma $,\varphi,\ \wedge^z\ ,\psi,\ \to^z\ ,\chi,\ \equiv (,\varphi,\ \wedge^z\ ,\psi,) \to^z\ ,\chi,$ **apply** *simp* **done**
lemma $,\varphi,\ \wedge^z\ ,\psi,\ \to^z\ ,\chi,\ \equiv\ ,\varphi,\ \wedge^z\ (,\psi,\ \to^z\ ,\chi,)$ **apply** *simp* **nitpick oops**

lemma $(,\varphi,\ \wedge^z\ ,\psi,\ \equiv^z\ ,\varphi,\ \wedge^z\ ,\psi,) \equiv ((,\varphi,\ \wedge^z\ ,\psi,) \equiv^z (,\varphi,\ \wedge^z\ ,\psi,))$ **apply** *simp* **done**
lemma $(,\varphi,\ \wedge^z\ ,\psi,\ \equiv^z\ ,\varphi,\ \wedge^z\ ,\psi,) \equiv (,\varphi,\ \wedge^z\ (,\psi,\ \equiv^z\ ,\varphi,)\ \wedge^z\ ,\psi,)$ **apply** *simp* **nitpick oops**

8 E!, O!, A! and =E

consts $E{::}(e{\Rightarrow}io)$

Distinguished 1-place relation constant: E! (read: being concrete or concreteness)

abbreviation *z-ordinary*$::(e{\Rightarrow}io)$ *opt* **where** $O^! \equiv \lambda^1(\lambda x.\ \Diamond <.E.{\cdot}.x.>)$

Being ordinary is being possibly concrete.

abbreviation *z-abstract*$::(e{\Rightarrow}io)$ *opt* **where** $A^! \equiv \lambda^1(\lambda x.\ \neg^z\ (\Diamond <.E.{\cdot}.x.>))$

Being abstract is not possibly being concrete.

abbreviation *z-identity*$::(e{\Rightarrow}e{\Rightarrow}io)$ *opt* **where** $=_e{}^z \equiv$
$\lambda^2(\lambda x\ y.\ ((<O^!{\cdot}.x.> \wedge^z <O^!{\cdot}.y.>) \wedge^z \Box\ (\forall(\lambda F.\ <.F.{\cdot}.x.> \equiv^z <.F.{\cdot}.y.>))))$

abbreviation *z-identityE*$::(e\ opt{\Rightarrow}e\ opt{\Rightarrow}io\ opt)$ **where** $x =_E y \equiv (Exe2 =_e{}^z\ x\ y)$

9 Further test examples

lemma $[\forall(\lambda x.\ \exists(\lambda R.\ (<.x.\circ.R.> \to^z <.x.\circ R1>)))]$ **apply** *simp* **by** *auto*
lemma $[\forall(\lambda x.\ \forall(\lambda R.\ (<.x.\circ.R.> \to^z <.x.\circ R1>)))]$ **apply** *simp* **nitpick oops**

lemma $[a =_E a]$ **apply** *simp* **nitpick oops**

lemma $[<O^!{\cdot}a> \to^z a =_E a]$ **apply** *simp* **done**

lemma $[(\forall(\lambda F.\ <.F.{\cdot}.x.> \equiv^z <.F.{\cdot}.x.>))]$ **apply** *simp* **done**

lemma $[{<}O^!{\cdot}a{>} \to^z {<}\lambda^1(\lambda x.\ .x.\ =_E\ a){\cdot}a{>}]$ **apply** *simp* **done**

lemma $[(\exists\,(\lambda F.\ {<}a{\circ}.F.{>}))]$ **apply** *simp* **by** *auto*

lemma *isWff* $,(\lambda w.\ \mathit{True}),$ **apply** *simp* **done**

lemma $[(\exists\,(\lambda\varphi.\ ,\varphi,))]$ **apply** *simp* **by** *auto*
lemma $[(\exists\,(\lambda\varphi.\ ;\varphi;))]$ **apply** *simp* **by** *auto*

10 Axioms

10.1 Axioms for Negations and Conditionals

lemma *a21-1-PF*: $[,\varphi,\ \to^z\ (,\varphi,\ \to^z\ ,\varphi,)]$ **apply** *simp* **done**
lemma *a21-1-F*: $[;\varphi;\ \to^z\ (;\varphi;\ \to^z\ ;\varphi;)]$ **apply** *simp* **done**
lemma *a21-2-PF*: $[(,\varphi,\ \to^z\ (,\psi,\ \to^z\ ,\chi,))\ \to^z\ ((,\varphi,\ \to^z\ ,\psi,)\ \to^z\ (,\varphi,\ \to^z\ ,\chi,))]$ **apply** *simp* **done**
lemma *a21-2-F*: $[(;\varphi;\ \to^z\ (;\psi;\ \to^z\ ;\chi;))\ \to^z\ ((;\varphi;\ \to^z\ ;\psi;)\ \to^z\ (;\varphi;\ \to^z\ ;\chi;))]$ **apply** *simp* **done**
lemma *a21-3-PF*: $[(\neg^z\ ,\varphi,\ \to^z\ \neg^z\ ,\psi,)\ \to^z\ (\neg^z\ ,\varphi,\ \to^z\ ,\psi,)\ \to^z\ ,\varphi,]$ **apply** *simp* **done**
lemma *a21-3-F*: $[(\neg^z\ ;\varphi;\ \to^z\ \neg^z\ ;\psi;)\ \to^z\ (\neg^z\ ;\varphi;\ \to^z\ ;\psi;)\ \to^z\ ;\varphi;]$ **apply** *simp* **done**

10.2 Axioms of Identity

todo

10.3 Axioms of Quantification

todo

10.4 Axioms of Actuality

lemma *a31-1-PF*: $[\mathcal{A}\ (\neg^z\ ,\varphi,)\ \equiv^z\ (\neg^z\ (\mathcal{A}\ ,\varphi,))]$ **apply** *simp* **done**
lemma *a31-1-F*: $[\mathcal{A}\ (\neg^z\ ;\varphi;)\ \equiv^z\ (\neg^z\ (\mathcal{A}\ ;\varphi;))]$ **apply** *simp* **done**
lemma *a31-2-PF*: $[\mathcal{A}\ (,\varphi,\ \to^z\ ,\psi,)\ \equiv^z\ (\mathcal{A}\ ,\varphi,\ \to^z\ \mathcal{A}\ ,\psi,)]$ **apply** *simp* **done**
lemma *a31-2-F*: $[\mathcal{A}\ (;\varphi;\ \to^z\ ;\psi;)\ \equiv^z\ (\mathcal{A}\ ;\varphi;\ \to^z\ \mathcal{A}\ ;\psi;)]$ **apply** *simp* **done**
lemma *a31-3-PF*: $[(\mathcal{A}\ (\forall\,(\lambda x.\ ,\varphi,))\ \equiv^z\ \forall\,(\lambda x.\ \mathcal{A}\ ,\varphi,))]$ **apply** *simp* **done**
lemma *a31-3-F*: $[(\mathcal{A}\ (\forall\,(\lambda x.\ ;\varphi;))\ \equiv^z\ \forall\,(\lambda x.\ \mathcal{A}\ ;\varphi;))]$ **apply** *simp* **done**
lemma *a31-4-PF*: $[\mathcal{A}\ ,\varphi,\ \equiv^z\ \mathcal{A}\ (\mathcal{A}\ ,\varphi,)]$ **apply** *simp* **done**
lemma *a31-4-F*: $[\mathcal{A}\ ;\varphi;\ \equiv^z\ \mathcal{A}\ (\mathcal{A}\ ;\varphi;)]$ **apply** *simp* **done**

10.5 Axioms of Necessity

lemma *a32-1-PF*: $[\Box\ (,\varphi,\ \to^z\ ,\varphi,)\ \to^z\ (\Box\ ,\varphi,\ \to^z\ \Box\ ,\varphi,)]$ **apply** *simp* **done**
lemma *a32-1-F*: $[\Box\ (;\varphi;\ \to^z\ ;\varphi;)\ \to^z\ (\Box\ ;\varphi;\ \to^z\ \Box\ ;\varphi;)]$ **apply** *simp* **done**
lemma *a32-2-PF*: $[\Box\ ,\varphi,\ \to^z\ ,\varphi,]$ **apply** *simp* **done**
lemma *a32-2-F*: $[\Box\ ;\varphi;\ \to^z\ ;\varphi;]$ **apply** *simp* **done**
lemma *a32-3-PF*: $[\Box\ (\Diamond\ ,\varphi,)\ \to^z\ (\Diamond\ ,\varphi,)]$ **apply** *simp* **done**
lemma *a32-3-F*: $[\Box\ (\Diamond\ ;\varphi;)\ \to^z\ (\Diamond\ ;\varphi;)]$ **apply** *simp* **done**
lemma *a32-4-PF*: $[(\forall\,(\lambda x.\ \Box\ ,\varphi,))\ \to^z\ \Box\ (\forall\,(\lambda x.\ ,\varphi,))]$ **apply** *simp* **done**
lemma *a32-4-F*: $[(\forall\,(\lambda x.\ \Box\ ;\varphi;))\ \to^z\ \Box\ (\forall\,(\lambda x.\ ;\varphi;))]$ **apply** *simp* **done**

The following needs to be an axiom; it does not follow for free: it is possible that there are contingently concrete individuals and it is possible that there are not:

axiomatization where
a32-5-PF: $[\Diamond\ (\exists(\lambda x.\ <.E.\cdot.x.>\ \wedge^z\ (\Diamond\ (\neg^z\ <.E.\cdot.x.>))))\ \wedge^z\ \Diamond\ (\neg^z\ (\exists(\lambda x.\ <.E.\cdot.x.>\ \wedge^z\ (\Diamond\ (\neg^z\ <.E.\cdot.x.>)))))]$

10.6 Axioms of Necessity and Actuality

lemma *a33-1-PF*: $[\mathcal{A}\ ,\varphi,\ \to^z\ \Box\ (\mathcal{A}\ ,\varphi,)]$ **apply** *simp* **done**
lemma *a33-1-F*: $[\mathcal{A}\ ;\varphi;\ \to^z\ \Box\ (\mathcal{A}\ ;\varphi;)]$ **apply** *simp* **done**
lemma *a33-2-PF*: $[\Box\ ,\varphi,\ \equiv^z\ (\mathcal{A}\ (\Box\ ,\varphi,))]$ **apply** *simp* **done**
lemma *a33-2-F*: $[\Box\ ;\varphi;\ \equiv^z\ (\mathcal{A}\ (\Box\ ;\varphi;))]$ **apply** *simp* **done**

References

[1] C. Benzmüller and L. Paulson. Quantified multimodal logics in simple type theory. *Logica Universalis (Special Issue on Multimodal Logics)*, 7(1):7–20, 2013.