Chapter 8

Axioms

Now that we have a precisely-specified philosophical language that allows to express claims using primitive and defined notions, we next assert the fundamental axioms of our theory in terms of these notions. We may group these axioms as follows:

- Classical axioms governing the negation and the conditional operators.
- Classical axioms for identity formulas.
- Classical axioms of quantification theory, modified only to accommodate rigid definite descriptions.
- Classical axioms for the actuality operator, including an axiom schema whose instances are not assumed to be necessary truths.
- Classical axioms (including the Barcan Formula) for the necessity operator, supplemented by an axiom about the possibility of contingently nonconcrete objects.
- Classical axioms for the interaction of the necessity operator and actuality operator.
- Classical axioms governing complex terms, modified only to accommodate for rigid definite descriptions.
- Axioms for encoding formulas.

The statement of some axiom groups require preliminary definitions.

(**20**) **Metadefinition:** Closures and Necessitation-Free Closures. We define:

- φ is a *universal closure* or *generalization* of ψ if and only if for some variables $\alpha_1, \ldots, \alpha_n$ ($n \geq 0$), φ is $\forall\alpha_1 \ldots \forall\alpha_n\psi$.

- φ is an *actualization* of ψ iff φ is the result of prefacing ψ by a string of zero or more occurences of $\mathcal{A}$.

- φ is a *necessitation* of ψ iff φ is the result of prefacing ψ by a string of zero or more occurrences of $\Box$.

Given these preliminary definitions, we say:

- φ is a *closure* of ψ if and only if for some variables $\alpha_1, \dots, \alpha_n$ ($n \geq 0$), φ is the result of prefacing ψ by any string of zero or more occurrences of universal quantifiers, $\mathcal{A}$ operators, and $\Box$ operators.

- φ is a *necessitation-free closure* (written "$\Box$-free closure") of ψ if and only if for some variables $\alpha_1, \dots, \alpha_n$ ($n \geq 0$), φ is the result of prefacing ψ by any string of zero or more occurrences of universal quantifiers and $\mathcal{A}$ operators.[84]

Since we're counting the empty string as a string, the definitions yield that every formula is a closure of itself. In what follows, when we say that we are taking the ($\Box$-free) closures of a *schema* as axioms, we mean that the ($\Box$-free) closures of every instance of the schema is an axiom.

8.1 Axioms for Negations and Conditionals

(**21**) **Axioms**: Negations and Conditionals. To ensure that negation and conditionalization behave classically, we take the closures of the following schemata as axioms:

(.1) $\varphi \to (\psi \to \varphi)$

(.2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

(.3) $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

8.2 Axioms of Identity

(22) **Remark:** The Symbol '=' in Both Object Language and Metalanguage. In this section, we begin with two metalinguistic definitions, (23) and (24), in which we use the symbol '=' in the metalanguage to represent a primitive metalinguistic notion of identity. Metalinguistic identity helps us to define certain metalinguistic notions concerning our syntax, such as the formula (term) that

[84]Note that to say φ is a $\Box$-free closure of ψ is not to imply that φ is free of $\Box$s. If ψ is $\Box\chi$, then if φ is $\forall\alpha\Box\chi$, φ is a $\Box$-free closure of ψ and it is not free of $\Box$s. However, if ψ is $\Box$-free, then so are its $\Box$-free closures.

results when a term τ is substituted for the occurrences of a variable α in some given formula (term). We state various axioms using such notions, such as the axiom of for the substitution of identicals (25). Of course, in these axioms, the formulas involving the symbol = are defined in items (15) and (16) of the previous chapter. But we need metalinguistic identity to formulate the notions needed to state these axioms.

Consequently, in what follows, we continue to use the = symbol for both object-theoretic and metatheoretic identity. In general, it should always be clear when = is being used to assert something in the object language and when it is being used to assert something in the metalanguage. Of course, if our philosophical project is correct, the theory of identity developed in the object language provides an analysis of the pretheoretical and unanalyzed metalinguistic notion of identity.

(23) **Metadefinitions:** Substitutions. One or more of the definitions in this item and the next are required to state the axioms of identity in item (25), the axioms of quantification in item (29), and the axioms for complex relation terms in item (36).

- Where τ is any term and α any variable, we use the notation φ^τ_α and ρ^τ_α, respectively, to stand for the result of substituting the term τ for every free occurrence of the variable α in formula φ and in term ρ.

This notion may be defined more precisely by recursion, based on the syntactic complexity of ρ and φ as follows, where the parentheses serve only to eliminate ambiguity and we suppress the obvious superscript indicating arity on the metalinguistic relation variable Π:

- If ρ is a constant or variable other than α, $\rho^\tau_\alpha = \rho$.
  If ρ is α, $\rho^\tau_\alpha = \tau$
- If φ is $\Pi\kappa_1 \dots \kappa_n$, then $\varphi^\tau_\alpha = \Pi^\tau_\alpha {\kappa_1}^\tau_\alpha \dots {\kappa_n}^\tau_\alpha$.
  If φ is $\kappa_1\Pi$, then $\varphi^\tau_\alpha = {\kappa_1}^\tau_\alpha \Pi^\tau_\alpha$.
- If φ is $\neg\psi$, $\Box\psi$ or $\mathcal{A}\psi$, then $\varphi^\tau_\alpha = \neg(\psi^\tau_\alpha)$ or $\Box(\psi^\tau_\alpha)$, or $\mathcal{A}(\psi^\tau_\alpha)$, respectively.
  If φ is $\psi \to \chi$, then $\varphi^\tau_\alpha = \psi^\tau_\alpha \to \chi^\tau_\alpha$.
- If φ is $\forall\beta\psi$, then $\varphi^\tau_\alpha = \begin{cases} \forall\beta\psi, \text{ if } \alpha = \beta \\ \forall\beta(\psi^\tau_\alpha), \text{ if } \alpha \neq \beta \end{cases}$
- If ρ is $\imath\nu\psi$, then $\rho^\tau_\alpha = \begin{cases} \imath\nu\psi, \text{ if } \alpha = \nu \\ \imath\nu(\psi^\tau_\alpha), \text{ if } \alpha \neq \nu \end{cases}$
- If ρ is $[\lambda\nu_1 \dots \nu_n \psi^*]$, then $\rho^\tau_\alpha = \begin{cases} [\lambda\nu_1 \dots \nu_n\, \psi^*], \text{ if } \alpha \text{ is one of } \nu_1, \dots, \nu_n \\ [\lambda\nu_1 \dots \nu_n\, {\psi^*}^\tau_\alpha], \text{ if } \alpha \text{ is none of } \nu_1, \dots, \nu_n \end{cases}$

We shall also want to define multiple simultaneous substitutions of terms for variables in φ and ρ, but since such a recursive definition would be extremely difficult to read, we simply rest with the following definition: where $\tau_1, \ldots, \tau_m$ are any terms and $\alpha_1, \ldots, \alpha_m$ are any distinct variables, we let $\varphi^{\tau_1,\ldots,\tau_m}_{\alpha_1,\ldots,\alpha_m}$ stand for the result of simultaneously substituting the term τ_i for each free occurrence of the corresponding variable α_i in φ, for each i such that $1 \leq i \leq m$. In other words, $\varphi^{\tau_1,\ldots,\tau_m}_{\alpha_1,\ldots,\alpha_m}$ is the result of making all of the following substitutions simultaneously: (a) substituting τ_1 for every free occurrence of α_1 in φ, (b) substituting τ_2 for every free occurrence of α_2 in φ, etc. Similarly, where $\tau_1, \ldots, \tau_m$ are any terms and $\alpha_1, \ldots, \alpha_m$ are any distinct variables, we let $\rho^{\tau_1,\ldots,\tau_m}_{\alpha_1,\ldots,\alpha_m}$ stand for the result of simultaneously substituting the term τ_i for each free occurrence of the corresponding variable α_i in ρ, for each i such that $1 \leq i \leq m$.

(24) **Metadefinitions:** Substitutable at an Occurrence and Substitutable For. We say:

- Term τ is *substitutable at an occurrence* of α in formula φ or term ρ just in case that occurrence of α does not appear within the scope of any operator binding a variable that has a free occurrence in τ.

In other words, τ is substitutable at an occurrence of α in φ or ρ just in case every occurrence of any variable β free in τ remains an occurrence that is free when τ is substituted for that occurrence of α in φ or ρ. Then we say:

- τ is *substitutable for* α in φ or ρ just in case τ is substitutable at every free occurrence of α in φ or ρ.

In other words, τ is substitutable for α in φ or ρ just in case every occurrence of any variable β free in τ remains an occurrence that is free when τ is substituted for every free occurrence of α in φ^{τ}_{α} or ρ^{τ}_{α}.

The following are consequences of this definition:

- Every term τ is trivially substitutable for α in φ if there are no free occurrences of α in φ.
- α is substitutable for α in φ or ρ.
- If τ contains no free variables, then τ is substitutable for any variable α in any formula φ or complex term ρ.
- If none of the free variables in τ occur bound in φ or ρ, then τ is substitutable for any α in φ or ρ.

(25) **Axioms:** The Substitution of Identicals. The identity symbol '=' is not a primitive expression of our object language. Instead, identity was defined in items (15) and (16) for both individuals and n-place relations ($n \geq 0$). The

classical law of the reflexivity of identity, i.e., $\alpha = \alpha$, where α is any variable, will be derived in a subsequent chapter — see item (71.1) in Chapter 9. By contrast, we take the classical law of the substitution of identicals as an axiom. Therefore the closures of the following schema are axioms of our system:

> $\alpha = \beta \rightarrow (\varphi \rightarrow \varphi')$, whenever β is substitutable for α in φ, and φ' is the result of replacing zero or more free occurrences of α in φ with occurrences of β.

This is an unrestricted principle of substitution of identicals: (a) if x and y are identical individuals, then anything true of x is true of y, and (b) if F^n and G^n are identical n-place relations, then anything true of F^n is true of G^n.

(**26**) **Remark:** Observation about the Substitution of Identicals. Note that Substitution of Identicals will guarantee that if abstract objects x and y necessarily encode the same properties, then they necessarily exemplify the same properties. For the former implies (by definition) that $x = y$ and so Substitution of Identicals would let us infer $\Box\forall F(Fx \equiv Fy)$ from the provable fact that $\Box\forall F(Fx \equiv Fx)$. Similarly, this axiom will guarantee that if properties F and G are necessarily encoded by the same objects, then they are necessarily exemplified by the same objects. For the former implies (by definition) that $F = G$ and so Substitution of Identicals would let us infer $\Box\forall x(Fx \equiv Gx)$ from the provable fact that $\Box\forall x(Fx \equiv Fx)$.

Indeed, given such facts, one might wonder whether we could (a) replace the Substitution of Identicals with the following axioms:

- $(A!x \,\&\, A!y \,\&\, \Box\forall F(xF \equiv yF)) \rightarrow \Box\forall F(Fx \equiv Fy)$
- $\Box\forall x(xF \equiv xG) \rightarrow \Box\forall x(Fx \equiv Gx)$

and then (b) prove the Substitution of Identicals.

It would seem that these two axioms are *not* sufficient to prove the substitution of identicals. Here is an instance of the substitution of identicals that would apparently remain unprovable:

$$x = y \rightarrow (\exists F(xF \,\&\, \neg Fx) \rightarrow \exists F(yF \,\&\, \neg Fy))$$

The reasoning that suggests this is unprovable goes as follows.

> Suppose we assume both $x = y$ and $\exists F(xF \,\&\, \neg Fx)$, to prove $\exists F(yF \,\&\, \neg Fy)$. From $x = y$, we know that x and y are either (a) both ordinary and necessarily exemplify the same properties or (b) both abstract and necessarily encode the same properties, in which case, we know by the first of the alternative axioms above that they exemplify the same properties. Now from $\exists F(xF \,\&\, \neg Fx)$, then it follows by axiom (39) stated below, which asserts that ordinary objects don't encode any properties, that x is abstract.

Hence y is abstract, given $x = y$, and so by definition, x and y encode the same properties. So x and y not only exemplify the same properties but also encode the same properties. But it isn't immediately clear how either fact will yield $\exists F(yF \,\&\, \neg Fy)$.

However, if we could abstract out from $\exists F(xF \,\&\, \neg Fx)$ a property that x exemplifies by using the principle of β-Conversion described in (36.2) below, then the fact that x and y exemplify the same properties would help us to prove $\exists F(yF \,\&\, \neg Fy)$. But this we cannot do; we may not infer $[\lambda z \exists F(zF \& \neg Fz)]x$ from $\exists F(xF \& \neg Fx)$ because the λ-expression isn't well-formed. Hence $\exists F(xF \,\&\, \neg Fx)$ doesn't yield a property that x exemplifies. So we can't infer $[\lambda z \exists F(zF \,\&\, \neg Fz)]y$ (from the fact that x and y exemplify the same properties) and then use β-Conversion in the reverse direction to obtain $\exists F(yF \,\&\, \neg Fy)$.

If, notwithstanding this reasoning, someone were to find a proof of substitution of identicals from the proposed alternative axioms, we would have an interesting question to consider: is it more elegant to formulate the system by *deriving* Substitution of Identicals from the proposed alternative axioms?

8.3 Axioms of Quantification

We begin with some preliminary definitions.

(27) **Metadefinitions:** Terms of the Same Type. We say that τ and τ' are terms *of the same type* iff τ and τ' are both individual terms or are, for some $n \geq 0$, both n-place relation terms.

(28) **Remark:** Logically Proper Terms. The sentences $\exists x(x = a)$ and $\exists F(F = P)$ *explicitly* assert, respectively, that there is such a thing as a and there is such a thing as P and, as we shall soon see, these will both be axioms. By contrast, $\exists x(x = \imath y\varphi)$ may be false if there is no unique individual such that φ; semantically, in such a case, $\imath y\varphi$ fails to denote. We shall say that axioms and theorems of the form $\exists\beta(\beta = \tau)$ (for any variable β of the same type as τ) assert that term τ is *logically proper*, and we will appeal to this notion of logical propriety instead of saying semantically that τ has a denotation. Using this terminology, we may say, also informally, that the axioms for quantification listed below guarantee: (a) that every term τ other than a description is logically proper,[85] (b) that quantification with respect to logically proper terms is classical, and

[85]Recall the discussion in a previous chapter, where we saw that complex n-place relation terms containing non-denoting descriptions nevertheless have denotations. The denotations of such terms are assigned on the basis of the truth conditions of the matrix formula in the relation term; since those truth conditions are always well-defined, the relation term will get assigned a denotation, even if its matrix formula has a non-denoting term.

(c) that descriptions appearing in true exemplification and encoding formulas are logically proper.

(29) **Axioms:** Quantification. Let α and β be variables of the same type, and let τ be a term of the same type as α and β. Then we assert that the closures of the following are axioms:

(.1) $\forall\alpha\varphi \rightarrow (\exists\beta(\beta=\tau) \rightarrow \varphi^{\tau}_{\alpha})$, provided τ is substitutable for α in φ.

(.2) $\exists\beta(\beta=\tau)$, provided τ is not a description and β doesn't occur free in τ.[86]

(.3) $\forall\alpha(\varphi \rightarrow \psi) \rightarrow (\forall\alpha\varphi \rightarrow \forall\alpha\psi)$

(.4) $\varphi \rightarrow \forall\alpha\varphi$, provided α doesn't occur free in φ

(.5) $\psi^{\imath x\varphi}_{\mu} \rightarrow \exists\nu(\nu = \imath x\varphi)$, provided (a) ψ is either an exemplification formula $\Pi^n\kappa_1 \ldots \kappa_n$ $(n \geq 1)$ or an encoding formula $\kappa_1\Pi^1$, (b) μ is an individual variable that occurs in ψ and only as one or more of the κ_i $(1 \leq i \leq n)$, and (c) ν is any individual variable that doesn't occur free in φ.[87]

The above axioms employ metatheoretical definitions (24) – (27) and the theoretical definitions (15) – (16).

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[86]We need the proviso that β doesn't occur free in τ to rule out instances like $\exists F(F = [\lambda y\, \neg Fy])$, in which β (= F) is free in τ (= $[\lambda y\, \neg Fy]$) and which asserts that some F is identical to its negation. This would yield a contradiction given β-Conversion (36.2), for suppose P is such an F, so that we know $P = [\lambda y\, \neg Py]$. By β-Conversion (36.2), it is axiomatic that $[\lambda y\, \neg Py]x \equiv \neg Px$. But then substituting P for $[\lambda y\, \neg Py]$ in this latter formula, we would end up with $Px \equiv \neg Px$, which will provably be a contradiction.

Interestingly, however, we need not similarly restrict (29.1) to those cases where β isn't free in τ. If in fact $\exists\beta(\beta=\tau)$ when β is free in τ, then we should be able to substitute τ everywhere for α in $\forall\alpha\varphi$. For example, F is free in $[\lambda y\, Fy]$, and it is true that $\exists F(F = [\lambda y\, Fy])$. Indeed, as we shall see, every property is a witness to this claim, by η-Conversion (36.3). So we should be able to instantiate $[\lambda y\, Fy]$ into universal claims of the form $\forall G\varphi$. Moreover, the first example in this footnote isn't a problem for (29.1), since the following instances of (29.1) remain valid:

$$\forall G\varphi \rightarrow (\exists F(F = [\lambda y\, \neg Fy]) \rightarrow \varphi^{[\lambda y\, \neg Fy]}_{G})$$

In this case, the antecedent of the consequent is false in every interpretation (as we just saw), and so the consequent,

$$\exists F(F = [\lambda y\, \neg Fy]) \rightarrow \varphi^{[\lambda y\, \neg Fy]}_{G}),$$

is true in every interpretation. So the instance of (29.1) is true in every interpretation.

[87]The restrictions (a) – (c) can be explained as follows. (a) and (b) are required because the truth of 'atomic' (i.e., exemplification or encoding) formulas imply that $\imath x\varphi$ has a denotation when the description appears as one or more of the principal individual terms; a molecular formula having the form of a tautology, e.g., $F\imath x\varphi \rightarrow F\imath x\varphi$, is true even when the $\imath x\varphi$ fails to denote. (c) We don't want ν to have a free occurrence in φ for then the quantifier $\exists\nu$ in the right side of the conditional would capture it.

8.4 Axioms of Actuality

(**30**) ⋆**Axioms:** Necessitation-Averse Axioms of Actuality. We now introduce axioms whose necessitations are not valid. We shall call these *necessitation-averse* axioms because the Rule of Necessitation won't be applicable to them. We've used the label '⋆**Axiom**' to signpost this fact and, henceforth, we reference the item number for this axiom as (30)⋆. We take only the *necessitation-free* ($\Box$-free) closures of the following axiom schema to be axioms of the system, where φ is any formula:

$$\mathcal{A}\varphi \equiv \varphi$$

The presence of these axioms explains why we do not assume the Rule of Necessitation as a primitive rule, for otherwise it would allow us to derive the necessitations of this axiom.

(**31**) **Axioms:** Actuality. By constrast we take the all of the closures of the following axiom schemata to be axioms of the system:

(.1) $\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi$

(.2) $\mathcal{A}(\varphi \to \psi) \equiv (\mathcal{A}\varphi \to \mathcal{A}\psi)$

(.3) $\mathcal{A}\forall\alpha\varphi \equiv \forall\alpha\mathcal{A}\varphi$

(.4) $\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi$

We leave the axioms governing the interaction between the actuality operator and the necessity operator for Section 8.6.

8.5 Axioms of Necessity

(**32**) **Axioms:** Necessity. We take the closures of the following principles as axioms:

(.1) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ K schema

(.2) $\Box\varphi \to \varphi$ T schema

(.3) $\Diamond\varphi \to \Box\Diamond\varphi$ 5 schema

(.4) $\forall\alpha\Box\varphi \to \Box\forall\alpha\varphi$ Barcan Formula (= BF)

Furthermore, let us read the formula $\exists x(E!x \,\&\, \Diamond\neg E!x)$ as asserting that there are contingently concrete individuals, i.e., individuals that exemplify being concrete but possibly don't. Then the following axiom asserts that it is possible that there are contingently concrete individuals and it is possible that there are not:

(.5) $\Diamond\exists x(E!x \,\&\, \Diamond\neg E!x) \,\&\, \Diamond\neg\exists x(E!x \,\&\, \Diamond\neg E!x)$

(32.1) – (32.4) are well known and axiom (32.5) was discussed in Chapter 6.[88]

As previously noted (e.g., in Chapter 6), many (quantified S5) systematizations of the necessity operator *derive* BF using a primitive Rule of Necessitation (RN). But as also previously mentioned, we are taking BF as an axiom and deriving Rule RN. We do the latter in item (52) (Chapter 9), where RN is derived in a form that implies that if there is a proof of φ in which φ doesn't depend on the necessitation-averse axiom of actuality $\mathcal{A}\varphi \equiv \varphi$ (30)★, then there is a proof of $\Box\varphi$. We can't formulate such a rule as a *primitive* rule because it requires the notions of *proof* and *dependence*, which haven't yet been defined. Note also that the Converse Barcan Formula will be derived as item (117.1).

8.6 Axioms of Necessity and Actuality

(**33**) **Axioms:** Necessity and Actuality. We take the closures of the following principles as axioms:

(.1) $\mathcal{A}\varphi \to \Box\mathcal{A}\varphi$

(.2) $\Box\varphi \equiv \mathcal{A}\Box\varphi$

8.7 Axioms for Descriptions

(**34**) **Axioms:** Descriptions. We take the closures of the following axiom schema as axioms:

> $x = \imath x\varphi \;\equiv\; \forall z(\mathcal{A}\varphi^z_x \equiv z = x)$, provided z is substitutable for x in φ and doesn't occur free in φ

We may read this as: x is the individual that is (in fact) such that φ just in case all and only those individuals that are actually such that φ are identical to x. The notation $\mathcal{A}\varphi^z_x$ used in the axiom is a harmless ambiguity; it should strictly be formulated as $(\mathcal{A}\varphi)^\tau_\nu$. By the third bullet point in definition (23), we know that $(\mathcal{A}\varphi)^\tau_\nu = \mathcal{A}(\varphi^\tau_\nu)$. Note also that one other axiom for descriptions has already been stated, namely, (29.5).

[88]One might wonder why we didn't take (.5) to be the simpler claim: $\Diamond\exists xE!x \,\&\, \Diamond\neg\exists xE!x$. The reason has to do with the second conjunct of the simpler claim — it rules out *necessarily concrete* objects like Spinoza's God. If Spinoza is correct that God just is Nature and that God is a necessary being, then it would follow that God (g) is necessarily concrete, i.e., that $\Box E!g$. If so, it wouldn't be correct to assert that it is possible that there are no concrete objects; at least, we shouldn't assert this *a priori*. Our axiom in the text doesn't contradict Spinoza's thesis, since it allows for the possible existence and possible nonexistence of *contingently* concrete objects.

8.8 Axioms for Complex Relation Terms

> Note: We have been using the prime symbol $'$ to avoid overspecificity. When we attach a prime symbol to a metavariable, the resulting metavariable represents an expression that, in most cases, is distinct from the expression represented by the metavariable without the prime. For example, when we defined terms *of the same type* in (27), we used τ and τ' to represent any two *terms* in the language. In the axiom for the Substitution of Identicals (25), we use φ' to indicate the result of replacing zero or more free occurrences of the variable α in φ with occurrences of the variable β. (So in the case where zero occurrences are replaced, φ' is φ.) Sometimes we shall place primes on expressions in the object language; for example, in a later chapter, c is introduced as a object-language restricted variable ranging over classes, and we let $c', c'', \ldots$ be *distinct* restricted variables for classes (and so on for other restricted variables). In the next item, however, we shall be use φ' to stand for an alphabetical-variant of the formula φ, and use ρ' to stand for an alphabetical-variant of the term ρ. Later we shall use ρ' to denote an η-variant of the relation term ρ. The context should always make it clear how the prime symbol $'$ is being used.

(**35**) **Metadefinitions:** Alphabetic Variants. To state one of the axioms governing complex relation terms in item (36.1), we require a definition of *alphabetically-variant* relation terms. In general, *alphabetically-variant* formulas and terms are complex expressions that intuitively have the same meaning because they exhibit an insignificant syntactic difference in their bound variables. In the simplest cases:

- alphabetically-variant formulas $\forall x F x$ and $\forall y F y$ have the same truth conditions
- alphabetically-variant descriptions $\imath x F x$ and $\imath y F y$ either both denote the unique individual that in fact exemplifies F if there is one, or both denote nothing if there isn't
- alphabetically-variant relation terms $[\lambda x\, \neg F x]$ and $[\lambda y\, \neg F y]$ denote the same relation

However, we shall also need to define the notion of *alphabetic variant* for formulas and terms of arbitrary complexity, so that the following pairs of expressions count as alphabetic variants:[89]

[89] Traditionally, two formulas or complex terms are defined to be *alphabetic variants* just in case some sequence of uniform permutations of the bound variables (in which no variable is captured

- $\forall F(Fx \equiv Fy)$ / $\forall G(Gx \equiv Gy)$
- $\imath x\forall yMyx$ / $\imath y\forall zMzy$
- $[\lambda z\, Rz\imath yQy]$ / $[\lambda x\, Rx\imath zQz]$
- $[\lambda\, Pa \rightarrow \forall F\, Fa]$ / $[\lambda\, Pa \rightarrow \forall G\, Ga]$
- $[\lambda x\, \neg Fx]a \rightarrow \forall yMy$ / $[\lambda z\, \neg Fz]a \rightarrow \forall xMx$

We shall eventually prove that alphabetically variant formulas are equivalent, and that alphabetically variant definite descriptions have the same denotation if they have one. But in the case of n-place relation terms, we have to stipulate, as axioms, that alphabetically variant n-place relation terms denote the same relation.

The last three examples displayed above can be reformulated as instances of an axiomatic *equation schema* asserted below. The axiom schema α-Conversion introduced in (36.1) will assert that alphabetically-variant relation terms can be put into an equation:

$$[\lambda z\, Rz\imath yQy] = [\lambda x\, Rx\imath zQz]$$

$$[\lambda\, Pa \rightarrow \forall F\, Fa] = [\lambda\, Pa \rightarrow \forall G\, Ga]$$

$$[\lambda x\, \neg Fx]a \rightarrow \forall yMy = [\lambda z\, \neg Fz]a \rightarrow \forall xMx$$

Note that the third example is an equation between two 0-place relation terms. The above examples shed light on the opening sentence of this numbered item, where we said that the notion of *alphabetic variant* is needed for the statement of α-Conversion.

To precisely define the general notion of *alphabetic variant*, i.e., for formulas and terms of arbitrary complexity, we first define *linked* and *independent* occurrences of a variable.

(.1) Let α_1 and α_2 be occurrences of the variable α in the formula φ or in term ρ. Then we say that α_1 is *linked to* α_2 in φ or ρ (or say that α_1 and α_2 are *linked* in φ or ρ) just in case:

(a) either both α_1 and α_2 are free, or

(b) both α_1 and α_2 are bound by the same occurrence of a variable-binding operator.

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during a permutation) transforms one expression into the other. So, in the first example below, the permutation sequence $F \rightarrow G$ transforms the first formula into the second; in the second example, the permutation sequence $y \rightarrow z, x \rightarrow y$ transforms the first description into the second; in the third pair, the permutation sequence $z \rightarrow x, y \rightarrow z$ transforms the first λ-expression into the second. And so on. However, we shall *not* follow the traditional definition but rather develop a definition that identifies alphabetic variants by way of symmetries among bound variable occurrences.

Otherwise, we say that α_1 and α_2 are *independent* in φ or ρ.

For example:

- In the formula $\forall F(Fa \equiv Fb)$, each occurrence of the variable F is linked to every other occurrence.
- In the formula $\forall F Fa \equiv \forall F Fb$, the first two occurrences of F are linked to each other and the last two occurrences of F are linked to each other, while each of the first two occurrences is independent of each of the last two occurrences and vice versa.
- In the formula $\forall x Fx \rightarrow Fx$, the first two occurrences of x are linked, and both are independent of the third, given that this formula is shorthand for $(\forall x Fx) \rightarrow Fx$.
- In the term $\imath x(Fx \rightarrow Gy)$, the two occurrences of x are linked.
- In the term $[\lambda x\, \forall y Gy \rightarrow (Gy \;\&\; Gx)]$, the two occurrences of x are linked (both are bound by the λ), the first two occurrences of y are linked, and both of those occurrences of y are independent of the third occurrence of y. Also, all three occurrences of G are free and hence linked.

Thus, we have:

> **Metatheorem** ⟨8.1⟩
> *Linked* is an equivalence condition on variable occurrences in a formula (or term).

A proof is given in the Appendix to this chapter. Thus, the occurrences of each variable in a formula (or term) can be partitioned into linkage groups; in any linkage group for the variable α, each occurrence of α in the group is linked to every other occurrence, while occurrences of α in different linkage groups are independent of one another. In the first example above, there is one linkage group for F, and in the fourth example, there is one linkage group for x. In the second example, there are two linkage groups for F and in the third example, there are two linkage groups for x. In the fifth example, there is one linkage group for x, two linkage groups for y, and one linkage group for G.

Now let us introduce some notation ('BV-notation') that makes explicit the occurrences of *bound* variables in formulas and terms:

(.2) When $\alpha_1,\ldots,\alpha_n$ is the list of all the variable occurrences bound in formula φ or complex term ρ, in order of occurrence and including repetitions of the same variable, the BV-notation for φ is $\varphi[\alpha_1,\ldots,\alpha_n]$, and the BV-notation for ρ is $\rho[\alpha_1,\ldots,\alpha_n]$, respectively, .

Some examples should help:

- When $\varphi = \forall F(Fx \equiv Fy)$, then φ in BV-notation is $\varphi[F,F,F]$
- When $\rho = \imath x\forall yMyx$, then ρ in BV-notation is $\rho[x,y,y,x]$
- When $\rho = [\lambda z\, Rz\imath yQy]$, then ρ in BV-notation is $\rho[z,z,y,y]$
- When $\rho = [\lambda\, Pa \rightarrow \forall F\, Fa]$, then ρ in BV-notation is $\rho[F,F]$
- When $\rho = [\lambda x\, \neg Fx]a \rightarrow \forall yMy$, then ρ in BV-notation is $\rho[x,x,y,y]$

Next we introduce notation for replacing bound variables:

(.3) We write $\varphi[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$ to refer to the result of replacing α_i by β_i in $\varphi[\alpha_1,\ldots,\alpha_n]$, for $1 \leq i \leq n$. Analogously, we write $\rho[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$ to refer to the result of replacing α_i by β_i in term $\rho[\alpha_1,\ldots,\alpha_n]$.

Finally, we may define:

(.4) φ' is an *alphabetic variant* of φ just in case, for some n:

- $\varphi' = \varphi[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$, and
- for $1 \leq i,j \leq n$, α_i and α_j are linked in $\varphi[\alpha_1,\ldots,\alpha_n]$ if and only if β_i and β_j are linked in $\varphi'[\beta_1,\ldots,\beta_n]$.

(.5) ρ' is an *alphabetic variant* of ρ just in case, for some n:

- $\rho' = \rho[\beta_1/\alpha_1,\ldots,\beta_n/\alpha_n]$, and
- for $1 \leq i,j \leq n$, α_i and α_j are linked in $\rho[\alpha_1,\ldots,\alpha_n]$ if and only if β_i and β_j are linked in $\rho'[\beta_1,\ldots,\beta_n]$.

Though we saw numerous examples of alphabetically-variant λ-expressions above, it may still be useful to offer the following additional examples:

- $\varphi = \forall F\, Fa \equiv \forall G\, Gb$
$\varphi' = \forall F\, Fa \equiv \forall F\, Fb$
- $\varphi = \forall xRxx \rightarrow \exists ySyz$
$\varphi' = \forall yRyy \rightarrow \exists xSxz$
- $\rho = \imath z(Fz \rightarrow Gy)$
$\rho' = \imath x(Fx \rightarrow Gy)$
- $\rho = [\lambda z\, \forall xGx \rightarrow (Gy \,\&\, Gz)]$
$\rho' = [\lambda x\, \forall zGz \rightarrow (Gy \,\&\, Gx)]$

Note that our definitions require that if β is to replace α to produce an alphabetic variant, then β must not occur free within the scope of a variable-binding operator that binds an occurrence of α in φ. For example, in the formula $\forall xRxy$ $(= \varphi)$, y occurs free within the scope of $\forall x$. We don't obtain an alphabetic variant of φ by substituting occurrences of y for the occurrences of x in the linkage group of bound occurrences of x. The formula that results from such a replacement, $\forall yRyy$ $(= \varphi')$, is very different in meaning from the original. In this example, the single occurrence of y in φ gets captured when occurrences of y replace the bound occurrences of x. Thus, φ in BV-notation is $\varphi[x,x]$ and φ' in BV-notation is $\varphi'[y,y,y]$, and so the first condition in the definition of *alphabetic variant* fails because the number of bound variables, counting multiple occurrences, is different for φ and φ'.

Note that though all free occurrences of variable α in φ (or ρ) are linked, they cannot be changed in any alphabetic variant φ' (or ρ') since the free occurrences are not listed in BV-notation.

Note also that our definitions imply that:[90]

> If ρ is a term occurring in φ and ρ' is an alphabetic variant of ρ, then if φ' is the result of replacing one or more occurrences of ρ in φ by ρ', then φ' is an alphabetic variant of φ.
>
> If φ is a formula occurring in ρ and φ' is an alphabetic variant of φ, then if ρ' is the result of replacing one or more occurrences of φ in ρ by φ', then ρ' is an alphabetic variant of ρ.

Here are some example pairs of the preceding facts (the first two pairs are examples of the first fact and the second two pairs are examples of the second fact):

- Where $\varphi_1 = \forall F(F\imath xPx \equiv Fb)$, $\rho_1 = \imath xPx$, and $\rho_1{}' = \imath yPy$, then $\forall F(F\imath yPy \equiv Fb)$ is an alphabetic variant of φ_1
- Where $\varphi_2 = \forall x([\lambda y\, \neg Fy]x \equiv \neg Fx)$, $\rho_2 = [\lambda y\, \neg Fy]$, and $\rho_2{}' = [\lambda z\, \neg Fz]$, then $\forall x([\lambda z\, \neg Fz]x \equiv \neg Fx)$ is an alphabetic variant of φ_2
- Where $\rho_3 = [\lambda y\, \forall xGx \rightarrow Gy]$, $\varphi_3 = \forall xGx$, and $\varphi_3{}' = \forall zGz$, then $[\lambda y\, \forall zGz \rightarrow Gy]$ is an alphabetic variant of ρ_3
- Where $\rho_4 = \imath y\forall F(Fy \equiv Fa)$, $\varphi_4 = \forall F(Fy \equiv Fa)$, and $\varphi_4{}' = \forall G(Gy \equiv Ga)$, then $\imath y\forall G(Gy \equiv Ga)$ is an alphabetic variant of ρ_4

[90] The first of the following two facts holds because all the bound variable occurrences in $\rho[\alpha_1,\ldots,\alpha_n]$ will appear in $\varphi[\gamma_1,\ldots,\gamma_m]$ and the replacements for the γ_is needed to replace ρ with $\rho'[\beta_1,\ldots,\beta_n]$ can not break any linkage groups. The only variable occurrences in ρ that could become linked to non-ρ variable occurrences in φ are the free variables of ρ and those cannot be changed in ρ' as noted above. The same reasoning explains why the second of the following two facts holds.

In the first case, $\imath y Py$ is an alphabetic variant of $\imath x Px$, and replacing the latter by the former in φ_1 yields $\varphi_1{}'$. In this case, φ_1 in BV-notation is $\varphi[F,F,x,x,F]$ and $\varphi_1{}'$ in BV-notation is $\varphi_1{}'[F,F,y,y,F]$. So (a) $\varphi_1{}' = \varphi_1[F/F,F/F,y/x,y/x,F/F]$, and (b) any two variables in the list of bound variables for φ_1 are linked in φ_1 if and only the corresponding variables in the list of bound variables for $\varphi_1{}'$ are linked in $\varphi_1{}'$. We leave the explanation of the remaining cases as exercises for the reader.

It follows from our definitions that:

Metatheorem $\langle 8.2 \rangle$
Alphabetic variance is an equivalence condition on the complex formulas of our language, i.e., alphabetic variance is reflexive, symmetric, and transitive.

A proof can be found in the Appendix to this chapter.

Another important metatheorem can be approached by way of an example. Let φ be a formula of the form $\neg\psi$. Then any alphabetic variant of φ we pick will be a formula of the form $\neg(\psi')$, for some alphabetic variant ψ' of ψ. For example, if φ is $\neg\forall xFx$, so that ψ is $\forall xFx$. Then pick any alphabetic variant of φ, say, $\neg\forall yFy$. In this case, the formula $\forall yFy$ is the witness ψ' such that φ' is $\neg(\psi')$. This generalizes to formulas of arbitrary complexity, so that we have:

Metatheorem $\langle 8.3 \rangle$: Alphabetic Variants of Complex Formulas and Terms.

(a) If φ is a formula of the form $\neg\psi$ (or $\mathcal{A}\psi$, or $\Box\psi$), then each alphabetic variant φ' is a formula of the form $\neg(\psi')$ (or $\mathcal{A}(\psi')$ or $\Box(\psi')$, respectively), for some alphabetic variant ψ' of ψ.

(b) If φ is a formula of the form $\psi \to \chi$, then each alphabetic variant φ' is a formula of the form $\varphi' \to \psi'$, for some alphabetic variants φ' and ψ', respectively, of φ and ψ.

(c) If φ is a formula of the form $\forall\alpha\psi$, then each alphabetic variant φ' is a formula of the form $\forall\beta(\psi'^{\beta}_{\alpha})$, for some alphabetic variant ψ' of ψ and some variable β substitutable for α in ψ' and not free in ψ'.[91]

(d) If τ is a term of the form $\imath\nu\varphi$, then each alphabetic variant τ' is a term of the form $\imath\mu(\varphi'^{\mu}_{\nu})$, for some alphabetic variant φ' of φ and individual variable μ substitutable for ν in φ' and not free in φ'.

(e) If τ is a term of the form $[\lambda\nu_1 \ldots \nu_n\, \varphi^*]$, then each alphabetic variant τ' is a term of the form $[\lambda\mu_1 \ldots \mu_n\, (\varphi^{*\prime\,\mu_1,\ldots,\mu_n}_{\;\;\nu_1,\ldots,\nu_n})]$, for some alphabetic variant φ' of φ and individual variables μ_i $(1 \leq i \leq n)$ substitutable, respectively, for the ν_i in $\varphi^{*\prime}$ and not free in $\varphi^{*\prime}$).[92]

[91] Note that this allows for the case where β is α or the case where ψ' is ψ.

[92] Note that this allows for the case where $\mu_1,\ldots,\mu_n$ are, respectively, $\nu_1,\ldots,\nu_n$.

Again, a proof can be found in the Appendix to this chapter.

Finally, given the semantics defined in previous chapters, it follows that alphabetically-variant formulas logically imply one another:

Metatheorem $\langle 8.4 \rangle$
$\varphi \mathrel{=\!\mid\!\models} \varphi'$, where φ' is an alphabetic variant of φ

The proof of this Metatheorem is left as an exercise.

(**36**) **Axioms:** Complex n-Place Relation Terms. As previously mentioned, the definition of alphabetically-variant terms and formulas is needed to state one of the axioms for complex n-place relation terms. In what follows, the classical axioms of Church's functional λ-calculus are formulated so as to apply both to the λ-expressions in our relational λ-calculus and propositional formulas (which, by (3.5.b), are 0-place relation terms). In these axioms, remember that the identity symbol = is the defined notion of identity in our object language.

(.1) α-Conversion. Where φ^* is any propositional formula and $\nu_1, \ldots, \nu_n$ are any distinct object variables, the closures of the following are axioms ($n \geq 0$):

(.a) $[\lambda \nu_1 \ldots \nu_n\, \varphi^*] = [\lambda \nu_1 \ldots \nu_n\, \varphi^*]'$,
where $[\lambda \nu_1 \ldots \nu_n\, \varphi^*]'$ is any alphabetic variant of $[\lambda \nu_1 \ldots \nu_n\, \varphi^*]$

(.b) $\varphi^* = \varphi^{*\prime}$,
where $\varphi^{*\prime}$ is any alphabetic variant of φ^*

(.2) β-Conversion. Where φ^* is any propositional formula, the closures of the following are axioms ($n \geq 1$):

$[\lambda y_1 \ldots y_n\, \varphi^*]x_1 \ldots x_n \equiv \varphi^{*x_1,\ldots,x_n}_{\;y_1,\ldots,y_n}$, provided the x_i are substitutable, respectively, for the y_i in φ^* ($1 \leq i \leq n$).

(.3) η-Conversion: Where $n \geq 0$, the closures of the following are axioms:

$[\lambda x_1 \ldots x_n\, F^n x_1 \ldots x_n] = F^n$

(.4) $\imath$-Conversion:[93] Where $\chi^{*\prime}$ is the result of substituting $\imath x\psi$ for zero or more occurrences of $\imath x\varphi$ anywhere the latter occurs in χ^*:

(.a) $\mathcal{A}(\varphi \equiv \psi) \rightarrow ([\lambda x_1 \ldots x_n\, \chi^*] = [\lambda x_1 \ldots x_n\, \chi^{*\prime}])$ $\quad (n \geq 0)$

(.b) $\mathcal{A}(\varphi \equiv \psi) \rightarrow (\chi^* \rightarrow \chi^{*\prime})$

[93] We may call this *iota*-Conversion even though we're using the LaTeX character $\imath$ instead of the Greek letter ι to represent the definite description operator. The Greek letter ι is standardly used for definite descriptions, but since we are in a modal context, we use the LaTeX character $\imath$ to signal this fact that our descriptions are rigid. With this warning, it does no harm to call the following axiom *iota*-Conversion.

α-Conversion guarantees that alphabetically-variant relation terms denote the same relation. β-Conversion is well-known: the left-to-right direction is sometimes referred to as λ-Conversion, while the right-to-left direction is sometimes referred to as λ-Abstraction. Note that we don't assert the 0-place case (i.e., $[\lambda\, \varphi^*] \equiv \varphi^*$) of β-Conversion as an axiom because it is derivable as a theorem; see item (126.2). Moreover, a stronger form of β-Conversion, which applies to λ-expressions having any variables (not just $y_1,\ldots,y_n$) bound by the leftmost λ and to any argument variables (not just $x_1,\ldots,x_n$), is derived in item (123).

Note also that the above statement of η-Conversion is not a schema. However, in item (155.2), we derive the schema $[\lambda \nu_1 \ldots \nu_n \Pi^n \nu_1 \ldots \nu_n] = \Pi^n$ for $n \geq 0$, where $[\lambda \nu_1 \ldots \nu_n\, \Pi^n \nu_1 \ldots \nu_n]$ is any elementary λ-expression, Π^n is any n-place relation term, and $\nu_1,\ldots,\nu_n$ are any distinct object variables that don't occur free in Π^n. Furthermore, in item (126.1), we derive the case where $n = 0$, that is, $[\lambda\, \varphi^*] = \varphi^*$.

Finally, note that (.4.a) intuitively tells us that if $\mathcal{A}(\varphi \equiv \psi)$, then the denotation of a λ-expression ρ whose matrix χ^* contains an occurrence of $\imath x\varphi$ is the same as that of a λ-expression just like ρ but whose matrix $\chi^{*\prime}$ is the result of replacing zero or more occurrences of $\imath x\varphi$ by $\imath x\varphi$. This holds because if $\mathcal{A}(\varphi \equiv \psi)$, the $\imath x\varphi$ and $\imath x\psi$ have exactly the same denotation conditions and so substituting the latter for the former in χ^* makes no difference to the denotation conditions of the λ-expression ρ. Similarly, (.4.b) intuitively tells us that if $\mathcal{A}(\varphi \equiv \psi)$, then if a propositional formula χ^* containing $\imath x\varphi$ is true, then so is the propositional formula that results by replacing zero or more occurrences of $\imath x\varphi$ by $\imath x\psi$. This holds because if the descriptions have the same denotation conditions, then they make the same contribution to both the truth conditions and denotation conditions of χ^* and $\chi^{*\prime}$. (Question: Can we extend (.4.b) to *any* formula and not limit it to propositional formulas?)

8.9 Axioms of Encoding

(**37**) **Axiom:** Rigidity of Encoding. If an object x encodes a property F, it does so necessarily. That is, the closures of the following are axioms of the system:

$$xF \rightarrow \Box xF$$

In other words, encoded properties are rigidly encoded. From this axiom, we will be able to prove that the properties an object possibly encodes are necessarily encoded.

(**38**) **Axiom:** Actuality and Encoding. If it is actually the case that an object x encodes property F, then x encodes F. That is, the closures of the following are axioms of the system:

$$\mathcal{A}xF \to xF$$

In the next chapter, we'll learn that although $\mathcal{A}xF \to xF$ would have been derivable from the necessitation-averse axiom of actuality (30)$\star$, we would not have been able to apply the Rule of Necessitation to obtain $\Box(\mathcal{A}xF \to xF)$. Since the latter is valid, we are taking all the closures of $\mathcal{A}xF \to xF$ to be axioms.

(**39**) **Axiom:** Ordinary Objects Fail to Encode Properties. The closures of the following are axioms:

$$O!x \to \neg\exists F\, xF$$

We prove in the next chapter that if x is ordinary, then *necessarily* x fails to encode any properties, i.e., that $O!x \to \Box\neg\exists F\, xF$.

(**40**) **Axioms:** Comprehension Principle for Abstract Objects ('Object Comprehension'). The closures of the following schema are axioms:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \varphi)), \text{ provided } x \text{ doesn't occur free in } \varphi$$

When x doesn't occur free in φ, we may think of φ as presenting a condition on properties F, whether or not F is free in φ (the condition φ being a vacuous one when F doesn't occur free). So this axiom guarantees that for every condition φ on properties F expressible in the language, there exists an abstract object x that encodes just the properties F such that φ.

(**41**) **Remark:** The Restriction on Comprehension. In the formulation of the Comprehension Principle for Abstract Objects in (40), the formula φ used in comprehension may not contain free occurrences of x. This is a traditional constraint on comprehension schemata. Without such a restriction, a contradiction would be immediately derivable by using the formula '$\neg xF$' as φ, so as to produce the instance:

$$\exists x(A!x \,\&\, \forall F(xF \equiv \neg xF))$$

Any such object, say a, would be such that $\forall F(aF \equiv \neg aF)$, and a contradiction of the form $\varphi \equiv \neg\varphi$ would follow once we instantiate the quantifier $\forall F$ to any property term. Instances such as the above are therefore ruled out by the restriction.