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IDENTITY AND EXISTENCE IN INTUITIONISTIC LOGIC

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Standard formulations of intuitionistic logic, whether by logicians or by category theorists, generally do not take into account partially defined elements. (For a recent reference see Makkai and Reyes [18] , esp. pp. 144-163.) Perhaps there is a simple psychological reason: we dislike talking of those things not already *proved* to exist. Certainly we should not *assume* that things exist without making this assumption explicit. In classical logic the problem is not important, because it is always possible to split the definition (or theorem) into cases according as the object in question does or does not exist. In intuitionistic logic this way is not open to us, and the circumstance complicates many constructions, the theory of descriptions, for example. Many people I find do not agree with me, but I should like to advocate in a mild way in this paper what I consider a simple extension of the usual formulation of logic allowing reference to partial elements. The discussion will be entirely formal here, but for the model theory of the system the reader should consult Fourman and Scott [10] for interpretations over a complete Heyting algebra (and this includes the so-called Kripke models) and Fourman [8] (the paper was written in 1975) for the interpretation in an arbitrary topos.

Technically the idea is to permit a wider interpretation of *free* variables. All bound variables retain their usual existential import (when we say something exists it does exist), but free variables behave in a more "schematic" way. Thus there will be no restrictions on the use of *modus ponens* or on the rule of *substitution* involving free variables and their occurrences. The laws of quantifiers require some modification, however, to make the existential assumptions explicit. The modification is very straightforward, and I shall argue that what has to be done is simply what is done naturally in making a *relativization* of quantifiers from a larger domain to a subdomain. Again in intuitionstic logic we have to take care over relativization, because we cannot say that either the subdomain is empty or not - thus a given element may be only "partially" in the subdomain.

FOOTNOTE The first draft of this paper was written during a visit on leave at the ETH, Zürich in March, 1975, and it has been revised since the Durham Symposium. The style of formalization was developed in seminars at Oxford starting in 1972/73 . Thanks for contributions and remarks are owed to D. van Dalen, M.P. Fourman, R. Grayson, J.M.E. Hyland, C. Smorynski, and R. Statman.

In Section 1, I discuss the idea of allowing existence as a predicate and the laws of quantifiers. Section 2 treats the theory of identity and the connections with existence. Questions of strictness and extensionality of relations and functions are discussed in Section 3 along with some examples of first-order theories. As further examples of the use of the system, the familiar theories of apartness and ordering in intuitionistic logic are presented in Section 4. Section 5 goes briefly into relativization, and Section 6 details the principles of descriptions. Finally, Section 7 reviews the axioms for higher-order intuitionistic logic from this general viewpoint.

The idea of schematic free variables is not new for classical logic, and the literature on "free" logic (or logic without existence assumptions) is extensive. (For some earlier references see Scott [21].) All I have done in this essay is to make what seems to me to be the obvious carryover to intuitionistic logic, because I think it is necessary and convenient. For those who do not like this formulation, some comfort can be taken from the fact that in topos theory both kinds of systems are completely equivalent, and the domains of partial elements can be *defined* at higher types (this is analogous to passing from a sheaf to its "flabbyfication", which is a subsheaf of the power sheaf). However, in first-order logic something is lost in not allowing partial elements, as I shall try to argue along the way.

1. EXISTENCE AND THE LAWS OF QUANTIFIERS

It has often been suggested that identity is a trivial relation, since to say " $a = b$ " is trivially true in case a and b are the same and otherwise trivially false. If "a" and "b" are "constant" names, this criticism may be reasonable; but when the expressions depend on *parameters*, it is obviously useful to express *properties* by equations. If an example is needed, take the equation:

$$x^2 = x + 1 \quad .$$

Whether this is true or false depends on x, and such equations (generally) define a whole class of solutions. We can, of course, in this case investigate by well-known methods exactly which x make the equation true; but with only the most superficial knowledge of the laws of algebra, we can easily assert a *conditional* like:

$$x^2 = x + 1 \rightarrow x^6 = 8x + 5 \quad .$$

Indeed, all the values of x^n can be simplified under the assumption that $x^2 = x + 1$. Passing to the many examples we are familiar with in several variables, we see that conditional equations may often be verified even when a complete analysis of the solution set corresponding to the hypothesis is lacking. The assumption is used as if it were true even though by itself it has no determinate truth value owing to the occurrence of parameters.

If we are willing to employ complex equations in this way, why should we not feel free to use complex expressions (terms) without demanding that they always denote? Just as we have to make certain equations conditional on the truth of other equations in order that they be valid, we may also have to make some statements conditional on the existence of certain complex terms. In algebra (say, in ring theory), the implication:

$$\forall x . \phi(x) \rightarrow \phi(0)$$

is unconditionally valid because the constant 0 is taken as always denoting in all rings. However, the statement:

$$\forall x . \phi(x) \rightarrow \phi(1/a)$$

cannot be valid in general because not every element a has an inverse. We can circumvent the difficulty by rephrasing the statement:

$$\forall x . \phi(x) \rightarrow \forall y \; [\, a \cdot y = 1 \rightarrow \phi(y) \,] \quad ,$$

but though correct this seems clumsy. Why not say more directly:

$$\forall x . \phi(x) \wedge E(1/a) \rightarrow \phi(1/a) \quad ,$$

where " $E(1/a)$ " is to be read as " $1/a$ exists" ? Even if we agree that

$$E(1/a) \leftrightarrow \exists y . \, a \cdot y = 1$$

(which avoids the notation $1/a$ on the right-hand side of the equivalence), we still want to use $1/a$ in the conclusion. The desire to keep to fractional notation will become even more urgent when more complex rational functions (say, $3x + 4 \,/\, 2x^2 + x + 1$) are to be manipulated.

Is the existence predicate E an illusion? Was the equality predicate an illusion? No. We shall find in the next section, with a full statement of the laws of equality, that E can always be defined in terms of quantification:

$$E\tau \leftrightarrow \exists y . \, y = \tau \quad ,$$

where "y" is a variable not free in the term τ (and where the equations *may* be further simplified as in the case of $1/a$). However, both in conception and in the models of (intuitionistic) logic we have in mind, the existence predicate is more basic than equality and prior to it.

For the time being we consider only a one-sorted first-order logic (and postpone the theory of equality to the next section). Higher-order logic (last section) will be regarded as a *theory* with its own special axioms in a many-sorted first-order (or quantifier) logic, and the passage from one sort to many sorts is essentially trivial once the quantifier laws are clear. Mainly the problem is a notational one of giving variables, predicates, functions and compound terms *types*. We shall do this precisely in the last section. As far as intuitionistic propositional calculus is concerned, no revision is necessary for predicate calculus, and we assume this as known. (A recent reference is Dummett [5] .) In predicate calculus, then, we use all the usual propositional laws (as applied to arbitrary formulae of the first-order language) together with a completely unrestricted rule of modus ponens:

$$(\mathrm{MP}) \qquad \frac{\phi \ , \ \phi \to \psi}{\psi}$$

without regard as to which free variables are shared between ϕ and ψ .

1.1 THE RULE OF SUBSTITUTION. If $\phi(x)$ is any formula with (possibly) the variable x free, and if τ is any individual term of the language, then the rule is the passage

$$(\mathrm{S}) \qquad \frac{\phi(x)}{\phi(\tau)}$$

where is making the substitution of τ for x , bound variables of $\phi(x)$ must be rewritten to avoid capturing the free variables of τ .

In other words, when we state a logical law (or axiom of a theory) with some free variables, then we intend that these variables have the broadest universal force and are freely replaceable by any (well-formed) term (of the appropriate type when types are important). On the other hand, when we state axioms with *quantified* variables, we intend that the universally quantified ones can only be replaced by terms whose values exist.

1.2 THE RULES OF THE UNIVERSAL QUANTIFIER. There is only one axiom (schema) and one rule. With the same understanding about substitution as in 1.1 , the axiom reads:

$$(\forall) \qquad \forall x \, . \, \phi(x) \wedge E\tau \to \phi(\tau) \quad .$$

This eliminates an initial quantifier. To adjoin a quantifier, we set down the rule:

$$(\forall^{+}) \qquad \frac{\phi \wedge Ex \to \psi(x)}{\phi \to \forall x \, . \, \psi(x)}$$

where x is a variable *not* free in ϕ .

Thus, the existence predicate and the quantifier are inextricably linked. Further, these rules already implicitly determine the meaning of the existence predicate.

1.3 METATHEOREM If the language had another existence predicate E^* also satisfying the axiom and rule of 1.2 , then we could show for all terms τ :

$$E\tau \leftrightarrow E^*\tau \ .$$

Proof. A direct consequence of $(\forall^+)$ is:

(1) $\forall x . Ex$,

because we can take ϕ as true and $\psi(x)$ as Ex . Using this in $(\forall)$ for the other predicate E^* , we derive:

(2) $E^*\tau \rightarrow E\tau$.

The converse implication to (2) has an analogous proof. □

In classical logic the existential quantifier $\exists$ can be defined from $\forall$ by negation; however, in intuitionistic logic we must give the dual axiom and rule as independent principles (in first-order logic, that is).

1.4 THE RULES OF THE EXISTENTIAL QUANTIFIER. Dually to 1.2 we have:

$(\exists)$ $\quad \phi(\tau) \wedge E\tau \rightarrow \exists x . \phi(x)$

and

$$(\exists^+) \quad \frac{\phi(x) \wedge Ex \rightarrow \psi}{\exists x . \phi(x) \rightarrow \psi}$$

where x is a variable *not* free in ϕ .

We shall not go into the proofs since they are so obvious; but, as in 1.3 , if we had another quantifier $\forall^*$ satisfying the rules of 1.2 , it could be proved equivalent to $\forall$. A similar result holds for $\exists$. Note also the following equivalences:

$$\exists x [Ex \wedge \phi(x)] \leftrightarrow \exists x . \phi(x)$$

and

$$\forall x [Ex \rightarrow \phi(x)] \leftrightarrow \forall x . \phi(x) \ ,$$

which show that the existence predicate is superfluous when bound in these ways within the scope of a quantifier.

2. EQUALITY AND EQUIVALENCE

Consider an equation like " $\tau = \sigma$ " . What should it mean? Our point of view is purely extensional, so the meaning should depend just on the "values" of the terms

τ and σ and not on how they are defined (or written) syntactically. There would seem to be naturally two senses possible: (i) *both* τ and σ exist and are equal; and (ii) *in so far as* one of τ and σ exists, then so does the other and they are equal. We shall take the first as the meaning of the simple equation " $\tau = \sigma$ ", because we think it is the one more often intended. The second is important, however, and will be written " $\tau \equiv \sigma$ " and called <u>equivalence</u>. The two notions are interdefinable on the basis of the axioms to be presented below. Within the scope of quantifiers *and* where there are no compound terms involved, there is no distinction between $=$ and $\equiv$. In the next section some algebraic axiom systems are presented where the distinction is essentially used. In this section we are concerned only with the logical laws.

For the first sense " $\tau = \sigma$ " implies existence, so – in free variable form – the relation $=$ is no longer reflexive. But this is the only modification we need to make in the usual axioms.

2.1 AXIOMS FOR SIMPLE EQUALITY. The three axioms are:

(refl) $\quad x = x \leftrightarrow Ex$

(symm) $\quad x = y \rightarrow y = x$

(trans) $\quad x = y \wedge y = z \rightarrow x = z$

Discussion. In effect E is definable in terms of $=$ (without the aid of the quantifiers). Still, we could not explain $=$ without mentioning existence. Thus it seems best just to regard (refl) as an axiom. Of course in quantified form we have:

(1) $\quad \forall x. x = x$,

but this is weaker than (refl) . Note that from the axioms, we can prove at once:

(2) $\quad x = y \rightarrow Ex \wedge Ey$,

which was as intended. Since (symm) and (trans) are conditionals, we see by (2) – if it has been taken as an axiom – that the second two axioms could have been stated equivalently in their universally quantified form. This will often happen in other systems, and predicates satisfying something like (2) are called <u>strict</u>. $\square$

The axioms of 2.1 are sufficient only when there are no other primitives in the system. This is just the theory of equality *by itself*. If other primitives are involved, we must adjoin a principle of replacement of equals by equals. But there is a difficulty in stating this, because an equation always implies existence. A step toward a more relaxed replaceability is to consider:

$$\forall z\, [\, x = z \leftrightarrow y = z \,] \quad .$$

Note that this does not imply that x and y exist, but it does imply that they are replaceable in equality contexts. The formula can be simplified. Specializing z first to x and then to y, we derive from the above:

$$[\ Ex \rightarrow x = y\ ] \wedge [\ Ey \rightarrow x = y\ ] \quad .$$

Since $x = y$ already implies Ex and Ey, this formula just expresses the statement that *if* one exists, *then* so does the other and they are equal; that is, this is the relation of equivalence mentioned at the start of this section. We can, by the way, shorten the writing to:

$$[\ Ex \vee Ey \rightarrow x = y\ ] \quad .$$

Assume this and work backwards. If $x = z$ (and Ez, but this is redundant), then Ex follows. Thus $x = y$; so $y = z$. And we argue similarly if $y = z$. So the unquantified expression of equivalence is *equivalent* to the quantified version. This shows we are onto the right idea.

It is clear that non-existent things are equivalent; and of two equivalent things, their existence (or non-existence) are equivalent. We could say that they are the same whether or not they exist. Hence, for *general* extensional predicates, equivalence is the right assumption for making an exchange.

2.2 THE AXIOM OF EQUIVALENCE. For arbitrary formulae $\phi(x)$ we assume:

$$(\mathrm{eq}) \qquad [\ Ex \vee Ey \rightarrow x = y\ ] \wedge \phi(x) \rightarrow \phi(y) \quad .$$

Discussion. Note that in the axioms of 2.1 and 2.2 we have formulated them with free variables. By the rule of substitution, we can replace the x, y, and z by any terms whatsoever. Thus the axioms are really schematic. Note too that (eq) is quite strong. If we had assumed the strictness of $=$ (that is (2) in 2.1), we could have derived both (symm) and (trans). $\square$

2.3 METATHEOREM. Using the following definition of equivalence:

$$(\equiv) \qquad x \equiv y \leftrightarrow [\ Ex \vee Ey \rightarrow x = y\ ] \quad ,$$

the theory of equality can be axiomatized by:

$$(\mathrm{in}) \qquad \forall z\ [\ x \equiv z \leftrightarrow y \equiv z\ ] \rightarrow x \equiv y$$

$$(\mathrm{eq}) \qquad x \equiv y \wedge \phi(x) \rightarrow \phi(y)$$

where equality can be redefined by:

$$(=) \qquad x = y \leftrightarrow Ex \wedge Ey \wedge x \equiv y \quad .$$

Furthermore, existence can be defined by quantification:

$$(\mathrm{E}) \qquad Ex \leftrightarrow \exists y . x \equiv y \quad .$$

Proof. Under the definition $(\equiv)$, what is called (eq) above is just the (eq) of 2.2. Also $(=)$ is easily provable by the axioms of 2.1 from $(\equiv)$. Principle (in) is another form of *indiscernability*. To prove it, assume the hypothesis, and assume $Ex \vee Ey$. In either case we can substitute for z in view of $(\forall)$. Suppose Ex. Then $x \equiv x$ follows at once (by definition), so $x \equiv y$ follows.

Having shown that (in), (eq), and $(=)$ are provable in the theory of equality, we now assume (in) and (eq) and use $(=)$ as the definition. Note that from (in), we can at once derive $x \equiv x$. By (eq), $x \equiv y \rightarrow y \equiv x$ comes out in the usual way. Similarly $x \equiv y \wedge y \equiv z \rightarrow x \equiv z$. By $(=)$ as a definition, then, it is easy to prove the original (refl), (symm), (trans). It remains to prove $(\equiv)$ to show that the systems are completely equivalent.

Assume $x \equiv y$ and $Ex \vee Ey$. By (eq) $Ex \rightarrow Ey$ and $Ey \rightarrow Ex$. Thus $Ex \wedge Ey$ follows, and so $x = y$. Assume next that $[Ex \vee Ey \rightarrow x = y]$. We wish to use (in) to derive $x \equiv y$. So assume Ez and $x \equiv z$. Then Ex, so $x = y$ and $x \equiv y$. Thus $y \equiv z$. Similarly $Ez \wedge y \equiv z \rightarrow x \equiv z$, and we are done.

That existence is definable by (E) follows easily from the laws of $\exists$ together with $x \equiv x$ and (eq). □

In some ways the system based on $\equiv$ is formally simpler. But both systems are very close. In the model theory (either the usual Kripke models or the Ω-sets presented in Fourman-Scott [9]) it is more natural to interpret Ex and $x = y$ *before* going on to the quantifiers. Except possibly for (eq) all the axioms of logic are pretty clear directly from the model-theoretic definitions. For (eq), it is *assumed* for atomic formulae at any rate, and then the induction on up to compound formulae is automatic by the logical rules after that.

Though the theory of equality is a very early and very simple chapter of logic, it is perhaps good to point out here that in *intuitionistic* logic even the theory of simple equality (no other predicates) is as undecidable as full predicate calculus (cf. Smorynski [22], p.117). This is stark contrast to classical logic where the theory of equality is relatively trivial. Some other shocks of a similar kind, which show that intuitionistic logic is more difficult than classical, are mentioned in the next two sections.

3. RELATIONS AND FUNCTIONS

The equivalence rule (eq), which is a principle of extensionality, has many consequences for relations (more generally, n-ary predicates) and functions. For example, two special cases are:

$$x \equiv x' \wedge y \equiv y' \wedge xRy \rightarrow x'Ry'$$

and

$$x \equiv x' \rightarrow f(x) \equiv f(x') \quad .$$

The first statement would be valid (but weaker) if we replaced $\equiv$ by $=$; however, for the second statement, with $=$ in the conclusion we obtain something stronger:

$$x = x' \rightarrow f(x) = f(x') \quad .$$

This means in particular (putting x for x'):

$$Ex \rightarrow Ef(x) \quad ,$$

and in view of (eq) , the previous implication is reprovable. Having this last statement hold for free variable x is the same as having

$$\text{(tot)} \qquad \forall x\ \exists y\, ,\, y = f(x) \quad ,$$

which means that the function is <u>total</u> (or totally defined on the whole domain for the variable x). Not all functions are total, and in Algebra we can have

$$x \equiv y \rightarrow 1/x \equiv 1/y \quad ,$$

even though $1/x$ is not total.

The converse implication to that defining total functions is:

$$\text{(str)} \qquad Ef(x) \rightarrow Ex \quad .$$

Clearly this can be extended in the obvious way to functions of many variables; we call such functions <u>strict</u>, since we do not allow the use of the value of the function unless the argument exists. Now *constant* functions are not strict if the constant is given without even looking at the argument. Perhaps this is unnatural; in any case, most ways of defining functions give strict functions. In the case of predicates the rule reads:

$$\text{(str)} \qquad Px_0 \ldots x_{n-1} \rightarrow Ex_0 \wedge \ldots \wedge Ex_{n-1} \quad .$$

An example of a non-strict predicate is $\neg\, x = y$.

3.1 METATHEOREM. If we assume that all primitive predicates and functions are strict, then the extensionality principle (eq) reduces to the special case

$$\text{(eq')} \qquad x = y \wedge \phi(x) \rightarrow \phi(y) \quad ;$$

and in fact this schema follows from the atomic cases:

$$\text{(i)} \qquad x_0 = y_0 \wedge \ldots \wedge x_{n-1} = y_{n-1} \wedge Px_0 \ldots x_{n-1} \rightarrow Py_0 \ldots y_{n-1}$$

$$\text{(ii)} \qquad x_0 = y_0 \wedge \ldots \wedge x_{n-1} = y_{n-1} \wedge z = f(x_0, \ldots, x_{n-1}) \rightarrow z = f(y_0, \ldots, y_{n-1}) \quad ,$$

where for total functions (ii) can be replaced by:

(ii') $x_0 = y_0 \wedge \ldots \wedge x_{n-1} = y_{n-1} \rightarrow f(x_0,\ldots,x_{n-1}) = f(y_0,\ldots,y_{n-1})$.

Further, all these statements are equivalent to their universal closures.

Proof. Obviously (eq') implies (i) and (ii) . But by strictness of P and f , we can replace $=$ by $\equiv$ in (i) and (ii) . This gives (eq) for all stomic cases. The compound cases can be built up inductively in the standard way. Thus, from

$$x \equiv y \rightarrow [\ \phi(x,z) \leftrightarrow \phi(y,z)\ ]$$

we can deduce:

$$x \equiv y \rightarrow [\ \neg\phi(x,z) \leftrightarrow \neg\phi(y,z)\ ] \quad ,$$

and then:

$$x \equiv y \rightarrow [\ \exists z.\neg\phi(x,z) \leftrightarrow \exists z\ \neg\phi(y,z)\ ] \quad .$$

And so on.

As for (ii') , it not only expresses the fact that f is total, but (ii') at once implies (ii) . □

In formalizing pure logic (as distinguished from a theory with its special axioms), some may not want to assume that all atomic predicates are strict. The reason is that not all *formulae* define strict predicates, and it is not unreasonable to require *logical* validities to be preserved under substitution of formulae for predicate symbols. (The predicate letters are like free predicate variables.) However, in model theory, there are technical advantages to restricting attention to the strict predicates and functions.

We conclude this section with three examples of theories where questions of strictness and totalness are relevant to a convenient formulation.

3.2 THE THEORY OF HEMILATTICES. An (upper) semilattice is a structure with a zero element $\perp$ and a join operation $x \sqcup y$ satisfying the usual zero, idempotent, sommutative and associative laws. Usually we assume that $\sqcup$ is a total function. But examples come up where a restriction to special elements seems necessary. Consider, say, *consistent* theories: the join of two theories may not be consistent, so the join operation is not always well defined. We could, of course, allow the inconsistent theory as an element of the lattice of theories; but this may not always be desirable, since we may not want to permit it as a value of other functions. Thus we have to agree that $\sqcup$ is a partial function. The functional notation can be avoided in this case entirely, by regarding a hemilattice as a partially ordered set in which every finite subset with *an* upper bound has a *least* upper bound. As we need only the empty and two-element subsets to pass to all finite subsets, it is easy to give the required axioms in quantified form. This solution to the axiomatic problem does not seem very elegant. The following is the solution with a partial join operation:

(1)	$E\bot$	(Zero exists)
(2)	$E(x \sqcup y) \rightarrow Ex \wedge Ey$	($\sqcup$ is strict)
(3)	$x \sqcup \bot \equiv x$	(The zero law)
(4)	$x \sqcup x \equiv x$	(The idempotent law)
(5)	$x \sqcup y \equiv y \sqcup x$	(The commutative law)
(6)	$x \sqcup (y \sqcup z) \equiv (x \sqcup y) \sqcup z$	(The associative law)

The partial ordering is defined by:

(7) $\quad x \sqsubseteq y \leftrightarrow x \sqcup y = y$.

Note that by using $=$ in (7) we make $\sqsubseteq$ strict. We then prove in succession:

(8)	$Ex \rightarrow \bot \sqsubseteq x$	(From (3))
(9)	$Ex \rightarrow x \sqsubseteq x$	(From (4))
(10)	$x \sqsubseteq y \wedge y \sqsubseteq x \rightarrow x = y$	(From (7) and (5))
(11)	$x \sqsubseteq y \wedge y \sqsubseteq z \rightarrow x \sqsubseteq z$	(From (7) and (6))
(12)	$E(x \sqcup y) \rightarrow x \sqsubseteq x \sqcup y \wedge y \sqsubseteq x \sqcup y$	(From (4) , (5) , and (6))
(13)	$x \sqsubseteq y \wedge y \sqsubseteq z \leftrightarrow x \sqcup y \sqsubseteq z$	(From (4) , (5) , (6) , (11) , and (12))
(14)	$E(x \sqcup y) \leftrightarrow \exists z\, [x \sqsubseteq z \wedge y \sqsubseteq z]$	(From (12) and (13))

Though this is simple minded, the algebraic properties (1) - (6) seem marginally simpler than the partial-ordering properties that require quantifiers to state the existence and uniqueness of the appropriate joins. The algebraic formulation also suggests the proper notion of <u>morphism</u> for such structures: namely, a strict, total function f where in addition:

$$E(x \sqcup y) \rightarrow f(x \sqcup y) = f(x) \sqcup f(y) \quad .$$

It is tempting to write:

$$f(x \sqcup y) \equiv f(x) \sqcup f(y) \quad ,$$

but this is probably not right because $f(x) \sqcup f(y)$ might exist (owing to some trivial feature of f) when we would not want $x \sqcup y$ to exist.

3.3 THE THEORY OF CATEGORIES. The example is similar to the hemilattices: a category is an associative (though not commutative) semigroupoid with appropriate left and right identities. The condition for the composition to exist is the coincidence of identities. The equational form of the axioms carries out a suggestion of Peter Freyd and seems simpler than the statement in MacLane [17], p.9.

(1) $Ex \leftrightarrow E\mathrm{dom}(x)$

(2) $Ex \leftrightarrow E\mathrm{cod}(x)$

(3) $E(x \circ y) \leftrightarrow \mathrm{dom}(x) = \mathrm{cod}(y)$

(4) $x \circ (y \circ z) \equiv (x \circ y) \circ z$

(5) $x \circ \mathrm{dom}(x) \equiv x$

(6) $\mathrm{cod}(x) \circ x \equiv x$

All the functions are strict, and dom and cod are total. We leave the reader the proofs of:

(7) $\mathrm{dom}(x) \equiv \mathrm{cod}(\mathrm{dom}(x))$

(8) $\mathrm{cod}(x) \equiv \mathrm{dom}(\mathrm{cod}(x))$

(9) $E(x \circ y) \rightarrow \mathrm{dom}(x \circ y) = \mathrm{dom}(y)$

(10) $E(x \circ y) \rightarrow \mathrm{cod}(x \circ y) = \mathrm{cod}(x)$

(11) $E(x \circ y) \wedge E(y \circ z) \rightarrow E(x \circ (y \circ z))$

As a hint for (9) and (10), note that (4) can be used after a substitution of identities. Of course heavy use is made of (3).

Functors are nothing more than morphisms of categories. They are strict total functions with these three properties:

(i) $f(\mathrm{dom}(x)) \equiv \mathrm{dom}(f(x))$

(ii) $f(\mathrm{cod}(x)) \equiv \mathrm{cod}(f(x))$

(iii) $E(x \circ y) \rightarrow f(x \circ y) \equiv f(x) \circ f(y)$

Functors satisfying (iii) *without* the existence assumption have been studied, but they are very special because they are *one-one* on identities (or objects of the category).

3.4 THE THEORY OF LOCAL RINGS. We need not repeat the familiar axioms for a commutative ring with unity. There are the usual operations and constants $+$, $-$, $\cdot$, 0, 1 which are *all* strict and total. In a local ring, certain elements may have inverses (such elements are called units), but not all elements need be units. The notion of inverse can be taken as primitive and implicitly defined by:

(inv) $y = x^{-1} \leftrightarrow x \cdot y = 1$.

The characteristic property of local rings can then be stated as:

(local) $Ex \rightarrow Ex^{-1} \vee E(1-x)^{-1}$.

As a brief exercise, we prove that in a local ring the following additive property of units holds:

$$E(x+y)^{-1} \rightarrow Ex^{-1} \vee Ey^{-1} \quad .$$

This, by the way, in turn implies (local) ; because if Ex , then $1 = x + (1-x)$ and 1 is trivially its own inverse. In the other direction, assume $E(x+y)^{-1}$. Let $z = y \cdot (x+y)^{-1}$. A standard calculation shows $1 - z = x \cdot (x+y)^{-1}$. If z^{-1} exists then y^{-1} must exist, because $y^{-1} = z^{-1} \cdot (x+y)^{-1}$. Similarly, if $(1-z)^{-1}$ exists, then $x^{-1} = (1-z)^{-1} \cdot (x+y)^{-1}$ exists. Thus from (local) we have shown $Ex^{-1} \vee Ey^{-1}$.

In defining local ring morphism, the situation is formally different from the examples 3.2 and 3.3 : this time the partial function x^{-1} must be exactly preserved:

$$f(x^{-1}) \equiv f(x)^{-1} \quad ;$$

that is, an element is invertible *if and only if* its image is. (Of course, f must preserve the other ring operations.)

3.5 PROBLEMS WITH CHOICE FUNCTIONS. The fact that the axiom of choice in higher-order intuitionistic logic implies the law of the excluded middle (and, hence, classical logic) will be discussed in the last section of this paper. In first-order logic, the use of a *few* choice functions would not seem to be harmful. In classical logic, Skolem functions are just choice functions, and they allow us to simplify the quantifier complexity of axioms without producing additional theorems in the original notation. (For a traditional proof see Church [3] .) In intuitionistic logic, however, the situation is quite different. Minc and Osswald found examples first, but Smorynski [24] contains probably the simplest result. (That paper can also be consulted for references to the other work.)

Consider the theory of equality in 2.1 . We add just one more axiom, a principle of diversity:

$$\text{(div)} \qquad \forall x \; \exists y . \neg\, x = y \quad .$$

The point is that in most models the choice of y is far from unique. Suppose we strengthen this theory by expanding the primitives to include a strict, total function f where:

$$\text{(div')} \qquad \forall x . \neg\, x = f(x) \quad .$$

We have made only one kind of choice, and it does not seem especially remarkable. The point, however, is that f is an *extensional* function. Thus, an easy argument from (div') proves:

$$\forall x \; \exists y \; \forall x' \; \exists y' \; [\neg\, x = y \wedge \neg\, x' = y' \wedge [x = x' \rightarrow y = y']] \quad .$$

Smorynski shows that this conclusion is *not* derivable from (div) alone. A general result in Smorynski [24] axiomatizes *all* the consequences of (div') not involving the function symbol.

A related example of Smorynski [24] shows that the existence of a non-trivial automorphism is *not* equivalent, as it is in classical logic, to $\exists x,y. \neg x = y$. The failure of excluded middle (particularly for $=$), makes intuitionistic logic into a quite different story.

4. APARTNESS AND ORDERING

As is well known, not only does the law of the excluded middle fail in intuitionistic logic but also the law of double negation; therefore, there is a considerable lack of symmetry between theories of equations, say, and theories of inequalities. Notions are no longer interdefinable with the aid of a simple negation; an independent "positive" theory of inequalities (and ordering relations) is required. Whereas equality is a logical notion, the positive inequality (called the apartness relation) can not be so considered. Some domains have no apartness, others have more than one (we are more explicit below on how this is possible). Nevertheless the concept is a very natural one - particularly for ordered sets. Other types of examples will also be provided.

4.1 THE THEORY OF APARTNESS. The four axioms are:

(str) $x \neq y \rightarrow Ex \wedge Ey$

(irr) $\neg x \neq x$

(symm) $y \neq x \rightarrow x \neq y$

(trans) $x \neq z \wedge Ey \rightarrow x \neq y \vee y \neq z$.

Discussion. How are these axioms obtained? If we refer back to 2.1 , the axioms for $=$, we note the obvious parallelism. That $=$ is strict was a consequence of the axioms in 2.1 ; here we have to assume $\neq$ is strict. Were we to be able to think classically, we could consider the remaining axioms of 4.1 just the *contrapositive* versions of those in 2.1 - except that in (irr) we have simplified the statement and made $x \neq x$ uniformly false, and in (trans) (or should it have been called "contratransitive" ?) we have to assume that the y exists because $\neq$ is strict. In reading these axioms " $\neq$ " must be taken as a symbol *in itself*, and " $x \neq y$ " should not be read as short for " $\neg x = y$ " ; the relation $\neq$ of apartness is something new, but it has properties dual to $=$ which would hold classically if it were defined by a negation.

How reasonable from a constructive viewpoint is this last axiom of apartness ? In the case of the real numbers given by decimal expansions, we can think informally of $x \neq z$ as meaning that we know the decimal place where they differ. Thus, given y , we have only to approximate it to that degree of accuracy, so that in a finite number of steps we will know whether $x \neq y$ or $y \neq z$. The "positive" character of these discriminations is quite clear.

The principle of contraposition (for putting *on* negations) holds intuitionistically, so the contrapositives of the axioms of 4.1 give us an equivalence relation in the predicate $\neg\, x \neq y$. Moreover, we see from 4.1 (trans) that we can easily prove:

$$\neg\, x \neq y \wedge Ey \wedge x \neq z \;\rightarrow\; y \neq z \quad .$$

This means that the negation of apartness acts like equality as far as formulae defined in terms of $\neq$ itself are concerned. In an attempt to make the notion of apartness more like a logical notion, we could be led to assume a further axiom.

4.2 DEFINITION. An apartness relation is said to be <u>tight</u> if in addition to the axioms of 4.1 it also satisfies:

$$\text{(tight)} \qquad x = y \leftrightarrow Ex \wedge Ey \wedge \neg\, x \neq y \quad .$$

Discussion. This puts apartness as close to equality as possible; in effect, $=$ is definable in terms of $\neq$ (but definitely not conversely, as we shall see). But the definition has consequences; the equality relation becomes "stable" in the well known, double-negation sense:

$$\text{(stable)} \qquad \forall x,y \;[\, x = y \leftrightarrow \neg\neg\, x = y \,] \quad .$$

Not all domains have stable equality. Indeed, let p be any proposition and let 0 be a "point". We shall consider subsets of the one-element set $\{0\}$. In fact, only two subsets will be needed; namely, $\{0\}$ and $\{\, x \mid x = 0 \wedge p \,\}$. (Let us call this $\{\, 0 \mid p \,\}$ for short.) With the usual rules for equality of sets (and more on this in §7), it is obvious that:

$$\{0\} = \{0 \mid p\} \;\leftrightarrow\; p \quad .$$

Therefore, stability of equality on $\{\, \{0\} \,,\, \{0 \mid p\} \,\}$ would imply the law of double negation for p . We would not assume this in general unless we were allowing the logic to be classical.

The question thus presents itself as to whether there are other equality consequences of 4.1 and 4.2 beyond (stable) . It came as something of a surprise to the author that there are. Van Dalen and Statman [4] were able to axiomatize the equality fragment of the theory of apartness, and the reader is directed to Smorynski [23] and [24] for a detailed discussion of the model theory needed for such problems. With the aid of some notation, we state the result in 4.4 .

4.3 DEFINITION. The following recursive definition gives a sequence of stronger and stronger <u>diversity</u> relations all defined in terms of equality:

(i) $x\,D_0\,y \leftrightarrow Ex \wedge Ey \wedge \neg\, x = y$

(ii) $x\,D_{n+1}\,y \leftrightarrow Ex \wedge Ey \wedge \forall z\,[\,x\,D_n\,z \vee y\,D_n\,z\,]$.

4.4 METATHEOREM. The equality fragment of the theory of (tight) apartness is axiomatized by the usual axioms (2.1) together with an infinite list of higher-degree stability conditions:

(stable_n) $\neg\, x\,D_n\,y \wedge Ex \wedge Ey \rightarrow x = y$.

No finite subset of these axioms is sufficient.

Van Dalen and Statman also show that in the theory of apartness, no equality formula can be used to define $\neq$. This is curious because in many models for intuitionistic analysis the relation D_1 in the real numbers is already equivalent to $\neq$. It would be interesting to have "natural" domains where $\neq$ was D_{n+1} but not D_n for $n > 0$.

Of course axioms (stable_0) and (stable) are the same. To prove (stable_n) from the axioms of 4.1 and 4.2 , one need only remark that there is an easy inductive proof of:

$x \neq y \rightarrow x\,D_n\,y$,

which for $n > 0$ makes heavy use of (trans) for $\neq$. We have to refer to the papers mentioned for the sufficiency of these axioms.

To give examples of domains on which it is *impossible* to have an apartness relation, we have to go beyond finite sets such as $\{\{0\}, \{0 \mid p\}\}$ where stability "almost" holds because $\neg\neg[\,\neg\neg p \rightarrow p\,]$ is intuitionistically valid. The obvious candidate is $\mathcal{P}\{0\}$, the full power set (or set of *all* subsets) of $\{0\}$ - assuming that such an all-inclusive set is conceivable. What $\mathcal{P}\{0\}$ amounts to is the set of all propositions, since each element x of $\mathcal{P}\{0\}$ can be identified as the set $\{0 \mid p\}$ where p is the proposition $0 \in x$. Equality on $\mathcal{P}\{0\}$ transfers over to $\leftrightarrow$ on propositions:

$\{0 \mid p\} = \{0 \mid q\} \leftrightarrow [\,p \leftrightarrow q\,]$.

Having an apartness on $\mathcal{P}\{0\}$ would imply a *quantified* statement:

$\forall p\, [\neg\neg p \rightarrow p\,]$.

As remarked in Fourman and Scott [9] , it is consistent to assume the *negation* of this.

Let Ω be the set of all propositions (or better: the propositional values). We have just pointed out that Ω has no apartness because equality on Ω is not stable. The obvious way to get a domain with stable equality (other than domains with *decidable* equality) is to restrict Ω to the stable propositions. By the law of triple negation, this is the same as the set of negative propositions:

$$\neg\Omega = \{\neg p \mid p \in \Omega\} \quad .$$

It is a simple exercise to verify that $\neg\Omega$ with $\leftrightarrow$ as equality satisfies (stable_0) . It does not, in general, satisfy (stable_1) , however. On $\neg\Omega$ what (stable_1) comes down to is:

$$\forall p,q\ [\neg\forall r\ [\neg[\neg p \leftrightarrow \neg r] \vee \neg[\neg q \leftrightarrow \neg r]] \to [\neg p \leftrightarrow \neg q]] \quad .$$

By making p *true* and q *false* , we derive:

$$\neg\neg\forall r\ [\neg r \vee \neg\neg r] \quad .$$

But this need not hold in some models either; in fact, it is consistent to assume $\neg\forall r\ [\neg r \vee \neg\neg r]$. So even $\neg\Omega$ can be a domain without any apartness relation.

Having examples of domains without an apartness relation, we should ask next whether when there is one it is *unique*. (This only makes sense assuming (tight) , of course.) Alas, the answer is no. Perhaps the reader would have guessed this from the undefinability result mentioned above, but the following easy argument due to M.P. Fourman may be instructive. Let $\neq$ be a tight apartness relation. Let p be any fixed "dense" proposition; that is, $\neg\neg p$ is to hold but, in order to be nontrivial, p itself is not assumed. (Oh, $p \leftrightarrow [q \vee \neg q]$ for some suitable q is what we have in mind.) Define a new relation by:

$$x \# y \leftrightarrow p \wedge x \neq y \quad .$$

The verification of the axioms in 4.1 is immediate. We need only prove (tight) . So assume Ex , Ey , and $\neg x \neq y$. This last yields $x \neq y \to \neg p$. But $\neg\neg p$, therefore $\neg x \neq y$; thus $x = y$ by (tight) for $\neq$. Now to say that the two relations $\neq$ and $\#$ are the same is to say $\forall x,y [x \neq y \to p]$, or, equivalently $\exists x,y\ [x \neq y] \to p$. But we can easily consider a domain where $\exists x,y\ [x \neq y]$ holds (for example, on the integers or on the reals), this last implication is not correct (unless the law of the excluded middle holds). Thus we have shown that the uniqueness hardly ever can be expected. Whether there is some interesting structure to the totality of all (tight) apartness relations on a set has not really been investigated.

Turning now from simple (symmetric) inequalities, we give some thought to ordering relations.

4.5 THE THEORY OF A TOTAL ORDERING. The axioms are:

(str) $x < y \rightarrow Ex \wedge Ey$

(asymm) $\neg [x < y \wedge y < x]$

(cover) $x < z \wedge Ey \rightarrow x < y \vee y < z$

(tight) $x = y \leftrightarrow Ex \wedge Ey \wedge \neg x < y \wedge \neg y < x$.

And we can regard apartness introduced by definition:

(apart) $x \neq y \leftrightarrow x < y \vee y < x$.

Discussion. One might have expected transitivity of $<$ as an axiom, but it follows easily. If $x < y$ and $y < z$, then by (cover) we find $x < z$ or $z < y$. The second alternative is ruled out by (asymm) .

The axiom (cover) is a bit odd to someone who only thinks in classical logic. Formally it is parallel to the (trans) property of $\neq$, and indeed it is obvious that using (apart) as the definition, all of 4.1 and 4.2 follow. It is also clear that (cover) is connected with being a linear ordering - but it is not quite that. If we could assume (after supposing $x < z$ and Ey) that $y < z \vee y = z \vee z < y$ (that is, an instance of trichotomy for $<$), then by transitivity of $<$, (cover) would follow. But the trichotomy law does not often enough hold in intuitionistic logic, say in the theory of the real numbers. It is fair to say that there are good constructive reasons (just as with $\neq$) why a disjunction about $<$ can be established under a certain hypothesis, whereas the disjunction of trichotomy is too demanding.

Another way to view the axioms of 4.5 formally is to define the inclusive order by:

$x \leq y \leftrightarrow Ex \wedge Ey \wedge \neg y < x$.

The reflexive, transitive and antisymmetric laws for $\leq$ then follow from 4.5 . And the axioms of 4.5 are just what we would expect if we could redefine $<$ from $\leq$ by negation. But we cannot. The exclusive notion $<$ is the "positive" one, even if $\leq$ is the more familiar classically. There is even a technical problem here: it is not known how to axiomatize the $\leq$-fragment of the theory of 4.5 . We have given *some* consequences of defining $\leq$ in terms of $<$, but the problem is to axiomatize *all* the consequences. As this was not altogether obvious for $=$, it is not surprising it is harder for $\leq$.

As regards $\neq$ we are much better off. Smorynski [24] contains a very nice argument by Kripke models which shows that the $\neq$ fragment of 4.5 is exactly the theory of (tight) apartness. A question remains, however, in higher-order logic: If we have a domain with apartness, can we show there *exists* an ordering relation compatible in the sense of satisfying the axioms of 4.5 . Classically this requires the Axiom of Choice (or a part of that axiom) to provide a (well) ordering. Intuitionistically, the choice principles fail. Sometimes a form of Zorn's Lemma holds, but there are difficulties in using it effectively. Smorynski's argument dealt with sufficiently many models for first-order consequences (and he used a non-constructive metalanguage), but we are asking whether something can be done for *all* models. Probably the answer is no, but it would be good to see why.

5. RELATIVIZATION OF QUANTIFIERS

Ordinary intuitionistic logic, without existence worries, would result by simply dropping all the E-formulae from the axioms and rules of Section 1. A formula with free variables is provable in the ordinary system if and only if its universal generalization (i.e. the result of binding up all the free variables by prefixed universal quantifiers) is provable in our present system - provided we assume that all functions are total. (That is, in ordinary theories the values of all terms are usually assumed to exist, and we would have to make such assumptions explicit *axioms*. If no function letters occur in the formula, so that all terms are just single variables, then no additional axioms are needed.) In this way the ordinary logic has a formal reduction to (is a kind of sublogic of) the slightly more general system of this paper. What we tried to argue in Section 3 was that this generality is *interesting* because there are good mathematical reasons for *not* assuming all functions to be total.

Now in this section we wish to go a step further and argue that the generality is even *necessary*, because there is simply no way of avoiding the passage from a structure to a *sub*structure. Formally this passage can be expressed by the relativization of the quantifiers to the predicate defining the substructure. The problems come in when we realize that the *closure* of a substructure under certain operations may - intuitionistically - be an undecided statement. Thus, there may be no natural way to define the desired functions as total functions on this substructure. If we wish to reason about the substructure as given, the more general logic is seen as being entirely appropriate. In a certain way the ordinary first-order intuitionistic logic has persisted only owing to a lack of imagination about the possible variety of structures. Of course, we admit that partial functions can be replaced by predicates, but this is very unmathematical (and certainly unalgebraic).

5.1 THE CAUCHY REALS. As a first example consider the definition of the real numbers in terms of Cauchy sequences. We proceed in the style of Bishop [1] (Chapter 2); but we do not discuss here whether this construction gives us *all* real numbers, since the question is beside the point in the present context. We are assuming as known the rationals, $\mathbb{Q}$, with their usual structure. Let $\mathbb{Q}^\infty$

be the space of (simple) infinite sequences, $\langle x_n \rangle_{n=1}^{\infty}$, of rationals; that is, we assume enough (intuitionistic) set theory to be able to do a completion by sequences. We regard $\mathbb{Q}^{\infty}$ as a perfectly nice set, where two sequences are equal if and only if they are termwise equal. (We could also define an apartness relation on $\mathbb{Q}^{\infty}$, but it is not needed.) In the well-worn manner we are going to single out a subset of $\mathbb{Q}^{\infty}$, call it S, and the Cauchy reals, $\mathbb{R}^c$, will be a quotient of S by an equivalence relation. (Bishop makes a longish story of avoiding equivalence relations, but for the point of the example it really does not matter whether you take the quotient or just work modulo equivalence.)

In fact S is easily defined in terms of the equivalence relation: for $x = \langle x_n \rangle_{n=1}^{\infty}$ and $y = \langle y_n \rangle_{n=1}^{\infty}$ define

$$x \approx y \quad \text{iff} \quad |x_n - y_m| \leq 1/n + 1/m \quad \text{for } n, m = 1,2,3, \ldots .$$

This is not an equivalence relation on all of $\mathbb{Q}^{\infty}$ but just on the subset defined by

$$x \in S \quad \text{iff} \quad x \approx x .$$

Thus a Cauchy real number (generator) is given by a sequence with modulus of convergence $1/n$. Without much trouble we see that the relation is symmetric and transitive. This last follows because if $x \approx y$ and $y \approx z$, then by the triangle inequality we have:

$$|x_n - z_k| \leq |x_n - y_m| + |y_m - z_k| \leq 1/n + 2/m + 1/k .$$

Since m is arbitrary (and we work with rationals), we obtain $x \approx z$ as desired. But to prove that x *is* a generator, we have to prove $x \in S$; the real number only *exists* when it is given by a *convergent* sequence. *Existence* for reals means $x \in S$.

Next suppose we want to define addition. Of course, if $x, y \in S$, then $\langle x_n + y_n \rangle_{n=1}^{\infty}$ converges – but with the wrong modulus. What we want to do is define:

$$x + y \;=\; \langle x_{2n} + y_{2n} \rangle_{n=1}^{\infty} .$$

Note the "$=$" here: the operation is clearly well defined on $\mathbb{Q}^{\infty}$. To prove it is well defined (total) on $\mathbb{R}^c = S/\approx$, we have to show something more, namely:

$$x \approx x' \wedge y \approx y' \rightarrow x + y \approx x' + y' .$$

This is not all difficult, but it takes a short proof involving more triangle inequalities. Before we prove this we do not really know that $+$ is a function on these reals.

In this example, because an equivalence relation is involved, it is perhaps not quite so clear how quantifiers are relativized. The point is, of course, that starting out with "ordinary" logic on $\mathbb{Q}^{\infty}$, to get the theory of $\mathbb{R}^c$ we need to replace

$=$ by $\approx$. This introduces partial elements, because $x \approx x$ does not hold throughout $\mathbb{Q}^\infty$. Even if we relativize to $S \subseteq \mathbb{Q}^\infty$ it does not at once obviate the question, since we can define operations under which S may not be closed. In the case of $+$, it works out; but for $^{-1}$, as in 3.4 , it really may be only a partial function.

It is more elementary to use elements in such cases instead of classes (as Bishop does), but the language of classes (which we formalize in Section 7) makes the act of relativization particularly simple. Let $P\mathbb{Q}^\infty$ be the powerset of $\mathbb{Q}^\infty$, a domain on which we can use ordinary logic. We define $\mathbb{R}^c \subseteq P\mathbb{Q}^\infty$ as the class of equivalence classes:

$$\mathbb{R} = \{ X \subseteq \mathbb{Q}^\infty \mid \exists x \in S \; \forall y \; [y \in X \leftrightarrow x \approx y] \} \quad .$$

The operation $+$ can be lifted to classes:

$$X + Y = \{ z \in \mathbb{Q}^\infty \mid X \in \mathbb{R} \wedge Y \in \mathbb{R} \wedge \exists x \in X \; \exists y \in Y \;\; z \approx x + y \} \quad .$$

Doing it this way, $+$ becomes strict relative to $\mathbb{R}$:

$$X + Y \in \mathbb{R} \rightarrow X \in \mathbb{R} \wedge Y \in \mathbb{R} \quad .$$

Thus, if we define - by relativization - a new $\equiv_{\mathbb{R}^c}$ by:

$$X \equiv_{\mathbb{R}^c} Y \leftrightarrow [X \in \mathbb{R}^c \vee Y \in \mathbb{R}^c \rightarrow X = Y] \quad ,$$

it follows trivially from the way we defined $\mathbb{R}^c$ and $+$ that

$$X \equiv_{\mathbb{R}^c} X' \wedge Y \equiv_{\mathbb{R}^c} Y' \rightarrow X + Y \equiv_{\mathbb{R}^c} X' + Y' \quad .$$

That is, axiom (eq) of 2.3 holds for formulae only involving $+$; here we regard all quantifiers such as " $\exists X$ " as <u>relativized</u> to $\mathbb{R}$ (that is, replaced by " $\exists X \in \mathbb{R}^c$ "). From this point of view " $X \in \mathbb{R}^c$ " is the new existence predicate. Thus, logic in the sense of Sections 1-3 is validated without any special reference to the meaning of the construction. As a *theorem*, proved from the specific definitions, we still have to show:

$$\forall X \in \mathbb{R}^c \; \forall Y \in \mathbb{R}^c . \; X + Y \in \mathbb{R}^c \quad ,$$

but obviously this is equivalent to the well-definedness of $+$ that we remarked on earlier.

5.2 THE GROUP OF INVERTIBLE ELEMENTS. Consider any commutative monoid M with its multiplication $x \cdot y$ and its unit 1 . The axioms are well known and we can regard them as given without worry of partial elements. Now, among the elements of M , some have inverses and some do not - in general this property is not decidable. Formally, we can define

$$G = \{ x \in M \mid \exists y \in M . \; x \cdot y = 1 \} \quad .$$

Obviously G is nonempty since $1 \in G$, but there is little more to say on that score generally; however, G has pleasant closure properties:

$$x \in G \wedge y \in G \leftrightarrow x \cdot y \in G \quad .$$

Note the biconditional, which we now make use of. Define a new $\equiv_G$ by:

$$x \equiv_G y \leftrightarrow [x \in G \vee y \in G \to x = y] \quad .$$

We can also define:

$$x =_G y \leftrightarrow x = y \in G \quad .$$

Clearly by the converse of the biconditional:

$$x \equiv_G x' \wedge y \equiv_G y' \to x \cdot y \equiv_G x' \cdot y' \quad ;$$

by the other direction, multiplication is total. Therefore, it is easy to verify that the structure with M as partial elements, G as total elements, and with $\equiv_G$ and $=_G$ becomes a *group*.

Classically, all we have done is to collapse the complement of G to the (unique) undefined element. Intuitionistically, the situation is not so black and white; but there are no technical difficulties in disregarding the exterior of G *via* the definition of $\equiv_G$ given above.

5.3 A METATHEOREM. A formula ϕ of ordinary intuitionistic logic (that is, without the E-predicate and without $\equiv$, but with $=$) is provable in the ordinary system if and only if the universal generalization of ϕ is provable in the system of this paper from the universal generalization of all formulae $E\tau$, where τ is any atomic term (that is, the formal axioms to the effect that all functions are total). We remarked this before. What we wish is a converse that reduces our system to the ordinary one.

We do not attempt a general converse, but we content ourselves with the case where all predicates and functions are *strict*. Thus we assume (str) of Section 5 as additional axioms. Now we can read these axioms just as well in the ordinary system as in our own: the assumption of strictness is an assumption about a distinguished predicate E. Having fixed on such a predicate, we translate every formula ϕ into a formula ϕ^E of the ordinary system which has the E made explicit. Thus:

$(P\tau_0\tau_1 \ldots \tau_{n-1})^E$ is $P\tau_0\tau_1 \ldots \tau_{n-1}$;

$(\tau_0 = \tau_1)^E$ is $E\tau_0 \wedge E\tau_1 \wedge \tau_0 = \tau_1$;

$(\tau_0 \equiv \tau_1)^E$ is $E\tau_0 \vee E\tau_1 \to \tau_0 = \tau_1$;

$(\phi \wedge \psi)^E$ is $\phi^E \wedge \psi^E$;

$(\phi \vee \psi)^E$ is $\phi^E \vee \psi^E$;

etc.

$(\forall x . \phi)^E$ is $\forall x [Ex \to \phi^E]$; and

$(\exists x.\phi)^E$ is $\exists x\,[\,Ex \wedge \phi^E\,]$.

Then the desired metatheorem states that ϕ *is provable from* (str) *in our present system if and only if* ϕ^E *is provable from* (str) *in the ordinary system.* Just as in our examples, ϕ^E may be read as a relativization of the property ϕ to the predicate E .

In one direction the proof is easy: if ϕ is provable in our system, then ϕ^E is provable in the ordinary system, as can be seen by looking at our logical axioms and rules. Of course, (str) is needed for such axioms as (eq) . In the other direction the only proof we know is *model theoretic.* There might be a good reason to look for a more constructive argument, because the completeness theorem for first-order intuitionistic logic is not intuitionistically provable (for an extended discussion of this point, see Dummett [6]).

In Fourman [7] details were given (along lines standard for such completeness proofs) to show that if ϕ is not provable from (str) in our system, then it fails in an Ω-set where the predicates and functions are strict. (For the theory of Ω-sets see Fourman-Scott [10] .) But if an Ω-set is used with $\equiv$ in place of $=$ (and the predicate E is momentarily forgotten), it is a model for ordinary intuitionistic logic (as is usual for the topological interpretation). But then, if E is brought back, ϕ^E will clearly get just the truth value in Ω that ϕ got originally. Therefore, ϕ^E fails in a model for the ordinary system.

In outline, and by means of examples, we have thus shown that the two systems are very closely related and that the one we advocate results naturally by a simple relativization of quantifiers - but once this relativization is done, partial elements are unavoidable. Our argument, then, is that it is simpler to have used them from the start.

6. DESCRIPTIONS.

Not all functions can be introduced by explicit formulae for their values; as with inverses or roots, values may only be singled out through certain properties. This indirect method is called definition by description, and the symbol we choose to employ is an (inverted ?) capital I similar to the quantifier symbols: $Ix.\phi(x)$. We read this as "the (unique) x such that $\phi(x)$ " .

6.1 THE AXIOM FOR DESCRIPTIONS. For any formula $\phi(x)$ where y is not free:

(I) $\forall y\,[\,y = Ix.\phi(x) \leftrightarrow \forall x\,[\,\phi(x) \leftrightarrow x = y\,]\,]$.

Informally this axiom (schema) can be construed as saying that something equals a described value if and only if it is the one and only thing satisfying the stated property. And what if there is no such thing? What does $Ix.\phi(x)$ denote then? Answer: the non-existing or undefined object. This sounds mildly paradoxical, but there is no formal reason to avoid expressions for un- or partially defined objects. For example, $Ix.\neg x = x$ can never exist; while $Ix.x = x$ exists just in case the domain has but a single element. More generally we can prove:

6.2 THEOREM. (i) For any formula $\phi(x)$ where y is not free:

$$EIx.\phi(x) \leftrightarrow \exists y\ \forall x\ [\ \phi(x) \leftrightarrow x = y\ ] \quad ;$$

$$\text{(ii)} \quad EIx.\phi(x) \rightarrow \phi(Ix.\phi(x)) \quad .$$

Proof. The proofs are immediate from (I) by the laws of equality and quantifiers, if we note that:

$$EIx.\phi(x) \leftrightarrow \exists y .\ y = Ix.\phi(x) \quad . \qquad \square$$

Of course we must remember that, once the formal language has been expanded to include I, there are many more terms in the language, and these must be allowed in all axioms and rules. In a similar way to the above we can also prove:

6.3 THEOREM. For formulae $\phi(x)$ and $\psi(x)$, in neither of which y is free, we have:

$$\text{(i)} \quad Ix.\phi(x) = Ix.\psi(x) \leftrightarrow \exists y\ [\ \forall x [\ \phi(x) \leftrightarrow x = y\ ] \wedge \forall x [\ \psi(x) \leftrightarrow x = y\ ]\ ] \quad ;$$

$$\text{(ii)} \quad Ix.\phi(x) \equiv Ix.\psi(x) \leftrightarrow \forall y\ [\ \forall x [\ \phi(x) \leftrightarrow x = y\ ] \leftrightarrow \forall x [\ \psi(x) \leftrightarrow x = y\ ]\ ] \quad ;$$

$$\text{(iii)} \quad \forall x [\ \phi(x) \leftrightarrow \psi(x)\ ] \rightarrow Ix.\phi(x) \equiv Ix.\psi(x) \quad ;$$

$$\text{(iv)} \quad y \equiv Ix.x = y \quad ;$$

$$\text{(v)} \quad Ix.\phi(x) \equiv Ix.\neg x = x \leftrightarrow \neg \exists y \forall x [\ \phi(x) \leftrightarrow x = y\ ] \quad ;$$

$$\text{(vi)} \quad Ix.\phi(x) \equiv Ix [\ Ex \wedge \phi(x)\ ] \quad .$$

From such results one can be led to conjecture that at the price of somewhat indirect references all descriptions can be eliminated. This is not quite true; an exact and reasonably comprehensive result is given next.

6.4 METATHEOREM. Under the assumption (str) that all primitive predicates and functions are strict, any formula ϕ is equivalent to a formula ϕ^* without descriptions in such a way that ϕ is provable in the extended system if and only if ϕ^* is provable in the system without descriptions.

Proof in outline. We need to catalogue the forms of the atomic formulae first. Aside from equations $\tau = \sigma$ we can have predications $P\tau_0\tau_1 \ldots \tau_{n-1}$. Now owing to (str) we can rewrite these as they occur by using the equivalences:

(1) $\tau = \sigma \leftrightarrow \exists y [y = \tau \wedge y = \sigma]$

(2) $P\tau_0\tau_1 \ldots \tau_{n-1} \leftrightarrow \exists y_0, y_1, \ldots, y_{n-1} [Py_0y_1 \ldots y_{n-1} \wedge$
$$y_0 = \tau_0 \wedge y_1 = \tau_1 \wedge \ldots \wedge y_{n-1} = \tau_{n-1}] \quad ,$$

where the variables $y, y_0, \ldots, y_{n-1}$ are chosen not to clash with any free variables of $\tau, \sigma, \tau_0, \ldots, \tau_{n-1}$. Compound terms are thus displaced to the right-hand sides of equations (and out of the grasp of predicates). There are just two kinds of compound terms (i.e., terms other than variables), namely, $f\tau_0\tau_1 \ldots \tau_{n-1}$ and $Ix.\phi(x)$, where f is an n-ary operation symbol. Now (I) in itself is a rewrite rule, or perhaps we should formulate it as:

(3) $y = Ix.\phi(x) \leftrightarrow Ey \wedge \forall x [\phi(x) \leftrightarrow x = y]$.

Its application eliminates a description. Similarly,

(4) $y = f\tau_0\tau_1 \ldots \tau_{n-1} \leftrightarrow \exists y_0, y_1, \ldots, y_{n-1} [y = fy_0y_1 \ldots y_{n-1} \wedge$
$$y_0 = \tau_0 \wedge y_1 = \tau_1 \wedge \ldots \wedge y_{n-1} = \tau_{n-1}] \quad ;$$

again, because we assume (str). This puts terms outside the grasp of operations. Layer by layer we transform ϕ until atomic formulae no longer involve compound formulae. The resulting transform ϕ^* is such that $\phi \leftrightarrow \phi^*$ is provable from (str) in the theory with descriptions. Hence, if ϕ^* is provable in the theory without I, then ϕ must be provable in the system *with* I.

For the converse argument about provability, we again resort to a (non-constructive) model-theoretic argument. If ϕ^* is not provable, it fails in an Ω-set A. But because all the assumed structure is strict, we can extend A to a *sheaf* $\hat{A}$ in which ϕ^* obtains the *same* value. (See Fourman-Scott [10] §5 for details.) But now in a sheaf we can interpret descriptions so that axiom (I) holds. Therefore in $\hat{A}$, both ϕ and ϕ^* will have the same truth value; therefore, ϕ fails in $\hat{A}$, and so cannot be provable. □

It might be interesting to have a constructive proof of this result; whatever the situation it seems to depend heavily on (str). The model-theoretic approach does make one useful point obvious, however. If the Ω-set A is taken as a model for a *theory*, all of whose axioms are given as sentences without free variables, then the sheaf $\hat{A}$ will satisfy the *same* theory (because any sentence in strict predicates and functions has the same truth value in A as in $\hat{A}$). This means that any such theory has a *conservative* extension to a theory with descriptions (conservative in the sense of no new theorems in the old notation). No doubt this stronger result has a proof-theoretic – and constructive – proof, but the question does not seem to have been investigated by logicians with regard to intuitionistic logic.

In this paper only glancing references to models have been made; model theory is the purpose of Fourman-Scott [10] . And it is only through seeing the natures of certain models that the necessity for various formulations becomes clear. It is hoped, however, that sufficient informal explanation has been given to make this paper readable. A particular case in point is the special case of descriptions called restriction:

6.5 DEFINITION. If x is not free in τ and in ϕ we set:

(rest) $\quad \tau \upharpoonleft \phi \equiv \mathrm{I}x[\, x = \tau \wedge \phi \,]$.

The idea is that $\tau \upharpoonleft \phi$ exists (and is equal to τ) only in so far as ϕ is true. It seems on the face of it silly to make τ exist *less* of the time; but such control is very useful in higher-order logic, say, when $\tau \upharpoonleft \phi$ can be an element of a class in which we *do not* want to put the whole of ϕ . Some general formal properties of restrictions are catalogued next.

6.6 THEOREM. (i) $x \equiv x \upharpoonleft \mathrm{E}x$

(ii) $x \upharpoonleft \phi \equiv x \upharpoonleft [\, \mathrm{E}x \wedge \phi \,]$

(iii) $(x \upharpoonleft \phi) \upharpoonleft \psi \equiv x \upharpoonleft [\, \phi \wedge \psi \,]$

(iv) $\phi \rightarrow x \equiv x \upharpoonleft \phi$

(v) $[\, \phi \leftrightarrow \psi \,] \rightarrow x \upharpoonleft \phi \equiv x \upharpoonleft \psi$

(vi) $x \equiv y \rightarrow x \upharpoonleft \phi \equiv y \upharpoonleft \phi$

(vii) $\mathrm{E}(x \upharpoonleft \phi) \leftrightarrow \mathrm{E}x \wedge \phi$

(viii) $y = x \upharpoonleft \phi \leftrightarrow y = x \wedge \phi$

(ix) $y \equiv x \upharpoonleft \phi \leftrightarrow [\, \phi \rightarrow x \equiv y \,] \wedge [\, \mathrm{E}y \rightarrow \phi \,]$

(x) $x \upharpoonleft \phi \equiv y \upharpoonleft \phi \leftrightarrow [\, \phi \rightarrow x \equiv y \,]$

(xi) $(\mathrm{I}x.\phi(x)) \upharpoonleft \psi \equiv \mathrm{I}x[\, \phi(x) \wedge \psi \,]$

(xii) $\tau(x) \upharpoonleft \psi \equiv \tau(x \upharpoonleft \psi) \upharpoonleft \psi$

(xiii) $\phi(x) \wedge \psi \leftrightarrow \phi(x \upharpoonleft \psi) \wedge \psi$.

7. HIGHER-ORDER LOGIC

Up to now we have dealt only with a one-sorted first-order logic; that is, the variables range over just one domain at one level. The theory of topological spaces - even particular spaces like the real numbers - requires a stronger logic, however, since many properties to be given full force have to be stated with quantifiers on arbitrary subsets or on arbitrary functions (sequences). From the philosophical standpoint of intuitionism, there are serious questions as to how much of the theory of "species" is constructive; often much less is required than is commonly used in an uncritical way in classical mathematics. We shall not discuss the issue of constructivism here but will formulate the *strongest* system with an eye to the model theory for higher-order logic as described in Fourman-Scott [10]. Our attitude for the moment is that there are enough difficulties in understanding how this "naive" higher-order intuitionistic logic works, because it already introduces structures (or properties of structures) quite different from those familiar to the classical mathematician. A full assessment of constructive content will have to wait for another study. (There already is a big literature in logic; see, e.g., Troelstra [27,28].) In any case the theory of topoi shows that there is a great variety of models of the full higher-order theory quite unsuspected by logicians and for any number of mathematical reasons, interesting in their own right.

The language for higher-order logic can be given many different formulations. Church [2] provides a primitive notation for *functions* of all types. (The *expressions* are the same, note, whether the system is classical or intuitionistic.) In Fourman [8], the formation of n-ary *relations* was the main primitive, a well-known approach. In Johnstone [16] (there called the Mitchell-Benabou language in §5.4) the notation combined functions, products and power types in a mixture convenient to the categorical background; the description there, however, is fairly informal and the use of partial elements was not adopted (see pp. 155 f.). A middle course seems to be to use power sets and products as primitives and to introduce very quickly a full range of defined types. As we employ descriptions, functional terms are also easily defined. Essentially the same system was used by Grayson [13].

7.1 DEFINITION OF SORT. (i) A stock of given sorts (for the moment not further specified) is allowed; these are called the <u>ground sorts</u>.

(ii) If $A_0, A_1, \ldots, A_{n-1}$ is any (finite) sequence of sorts (including $n=0$), the expression $(A_0 \times A_1 \times \ldots \times A_{n-1})$ is called the <u>product sort</u>.

(iii) If A is a sort, the expression $P(A)$ is called the <u>power sort</u>.

Discussion. The only primitive sorts are those expressions given by (i) - (iii); and these are *very* primitive in the sense that they just cut a fairly brisk "cofinal" path through the mass of conceivable higher types. Note, too, that all sorts are *constant*; types depending on variables can be defined (though we have no variables ranging *over* sorts), but we do not get involved in the notational problems on infinite products (and coproducts).

In (ii) the empty product ( ) is allowed, as well as the one-termed product

(A) . Often we may wish to abbreviate $(A_0 \times A_1 \times \ \dots \ \times A_{n-1})$ as $\mathop{\times}\limits_{i<n} A_i$. In writing $P(A_0 \times A_1 \times \ \dots \ \times A_{n-1})$ the parentheses may be left out; but we must take care to distinguish $P(A)$ and $P((A))$, because A and (A) are different sorts. (As sorts are *symbols*, equality between sorts means notational identity and not identity under a semantical interpretation.) □

We next define term and formula, but the definitions are separated because each involves several clauses; note, however, that the definitions depend one on the other.

7.2 DEFINITION OF TERM. (i) Associated with each sort is an (infinite) stock of *variables*. We do not try to make a precise syntax for these variables; and, if x is a variable, we write $\# x$ for its uniquely determined sort. All variables are terms.

(ii) Associated with each sort a stock of *constants*, not further specified, is allowed. Every constant c is a term, and $\# c$ denotes its sort.

(iii) If $A_0, A_1, \dots, A_{n-1}$ are sorts and $\tau_0, \tau_1, \dots, \tau_{n-1}$ are terms, where $\#\tau_i = A_i$ for $i < n$, then the *tuple* $\langle\tau_0, \tau_1, \dots, \tau_{n-1}\rangle$ is a term and $\# \langle\tau_0, \tau_1, \dots, \tau_{n-1}\rangle = (A_0 \times A_1 \times \dots \times A_{n-1})$.

(iv) If $A_0, A_1, \dots, A_{n-1}$ are sorts and τ is a term with $\# \tau = (A_0 \times A_1 \times \dots \times A_{n-1})$, then the *projection* $\pi_i \tau$ is a term, provided $i < n$, and $\# \pi_i \tau = A_i$.

(v) If A is a sort and x a variable with $\# x = A$, and if ϕ is any formula, then the *description* $Ix.\phi$ is a term and $\# Ix.\phi = A$.

(vi) If A is a sort, then it is also a term with $\# A = P(A)$.

Discussion. Every term has a unique sort - the intention being that a sort denotes a domain and a term of that sort an element of that domain. Note that in (iv) the term $\pi_i \tau$ is not well formed unless τ is a product sort and i is less than the length of the product. As we said before, we do not include function symbols and other compound terms, because these will be defined by descriptions. Clause (vi) is not strictly necessary; but, since we have symbols for sorts already, there is no reason not to let them be used as terms. As we will specify in the axioms, A will denote the *universal set* of sort A .

7.3 DEFINITION OF FORMULA. (i) If σ and τ are terms, then

$$E\tau \ , \ \sigma = \tau \ , \ \sigma \in \tau$$

are formulae, provided in the second case that $\# \sigma = \# \tau$, and in the third that $\# \tau = P(\# \sigma)$. These are the *atomic formulae*.

(ii) As *compound formulae* we take all the usual propositional combinations and quantifications as in first-order logic.

Discussion. Formulae are all of the same "sort": the propositional type; but we did not introduce it especially since it can be defined as $\mathcal{P}(())$. As variables carry their sorts along with them, we do not have to incorporate any sort indicator into the quantifier symbols themselves. It is only in the case of atomic formulae that we need to take care to check sorts to secure well-formedness. In an *equation* both sides have, obviously, to have the same sort; and in a membership relationship one side is the sort of an element, while the other side is the sort of a set. As sorts are uniquely determined, we could have decreed that ill-formed formulae are *false*, but it seems pointless to write them at all.

7.4 THE AXIOMS AND RULES OF HIGHER-ORDER LOGIC. (i) All logical rules and axioms of Sections 1 , 2 and 6 carry over with the proviso that in substitutions (the passage from $\phi(x)$ to $\phi(\tau)$), the sort of the variable and the term must be the *same* ($\# x = \# \tau$).

(ii) The axioms (and axiom schema) particular to the higher-order theory are as follows:

(memb) $[x \in y \to Ex \wedge Ey]$;

(prod) $[E\langle x_0, x_1, \ldots, x_{n-1}\rangle \leftrightarrow Ex_0 \wedge Ex_1 \wedge \ldots \wedge Ex_{n-1}] \wedge \bigwedge\!\!\!\bigwedge_{i<n} [E\pi_i z \leftrightarrow Ez]$;

(sort) EA ;

(comp) $EIy.\, \forall x\, [x \in y \leftrightarrow \phi(x)]$;

(proj) $\forall x_0, \ldots, x_{n-1}.\, \langle x_0, \ldots, x_{n-1}\rangle = Iz.\, \bigwedge\!\!\!\bigwedge_{i<n} \pi_i z = x_i$;

(univ) $\forall x.\, x \in A$;

where the sorts must be chosen to make all formulae well formed, and in (comp) the variable y is not free in $\phi(x)$.

Discussion. The first three axioms make all the primitives *strict*, in the now familiar sense; and in the case of (prod) the tupling and projection operations are *total*. We also want A to exist. The next three axioms specify the content of the primitives; sets can be formed from their members, tuples are characterized by their projections, and A functions as the universal set of sort $\mathcal{P}(A)$.

If it is now desired to avoid the use of descriptions, the existence and uniqueness parts of the axioms can be divided as follows:

$\forall y_0, y_1\ [\forall x\ [x \in y_0 \leftrightarrow x \in y_1] \to y_0 = y_1]$;

$\exists y\ \forall x\ [x \in y \leftrightarrow \phi(x)]$;

$\forall x_0, \ldots, x_{n-1}\ \bigwedge\!\!\!\bigwedge_{i<n} \pi_i \langle x_0, \ldots, x_{n-1}\rangle = x_i$;

$$\forall z \,.\, z = \langle \pi_0 z, \ldots, \pi_{n-1} z \rangle \quad .$$

In the above the sorts are chosen so that $\# x = A$, $\# y = \# y_0 = \# y_1 = P(A)$, $\# x_i = A_i$, $\# z = \bigtimes_{i<n} A_i$. □

We now introduce auxiliary notation and the immediate consequence of the axioms.

7.5 DEFINITION OF TYPE. (i) The *set abstraction* notation $\{ x \mid \phi(x) \}$ is short for $\mathrm{I}y \, \forall x \, [x \in y \leftrightarrow \phi(x)]$. It is a term of sort $P(A)$ if $\# x = A$. Such terms are called types.

(ii) If $\# \tau_i = P(A_i)$, for $i<n$, then we write:

$$\tau_0 \times \tau_1 \times \ldots \times \tau_{n-1} = \{ z \in \bigtimes_{i<n} A_i \mid \bigwedge_{i<n} \pi_i z \in \tau_i \} \quad .$$

(iii) If $\# \tau = P(A)$, then we write:

$$P(\tau) = \{ y \in P(A) \mid \forall x \, [x \in y \leftrightarrow x \in \tau] \} \quad .$$

Discussion. The import of this definition is that general types are just subsets of our sorts; moreover, the types can be subjected to the same operations as the sorts. Strictly speaking, sorts are not (notationally) types, but this distinction is obviated by the following easy result. □

7.6 LEMMA. For any sort A , where $\# x = A$,

$$A = \{ x \mid x = x \} \quad .$$

Proof. By (sort) , A "exists" at the right level, and clearly $\forall x \, [x \in A \leftrightarrow x = x]$ by the second part of (sort) . We can now apply the uniqueness part of (comp) and 6.2 . □

In fact, we can regard types as "given" sorts and relativize all the results of higher-order logic to the compound types of 7.5 (ii) and (iii) constructed from them. First we need the definitions for relativization of variables.

7.7 DEFINITION OF RELATIVIZATION.

(i) $\forall x \in X \,.\, \phi(x)$ for $\forall x \, [x \in X \rightarrow \phi(x)]$

(ii) $\exists x \in X \,.\, \phi(x)$ for $\exists x \, [x \in X \wedge \phi(x)]$

(iii) $\mathrm{I}x \in X \,.\, \phi(x)$ for $\mathrm{I}x \, [x \in X \wedge \phi(x)]$

(iv) $\{ x \in X \mid \phi(x) \}$ for $\{ x \mid x \in X \wedge \phi(x) \}$.

Discussion. In the above we should understand $\# x = A$ and $\# X = P(A)$. Actually it would make good sense to write such expressions where X is *any* term of sort $P(A)$. However, to obtain the relativized forms of all the axioms, we need to know X "exists". Note that there is no definition of "existence within X", because we can just write " $x \in X$ " for that. If desired, one could write " $x =_X y$ " for " $[x \in X \wedge y \in X \wedge x = y]$ ", in a way similar to what was done in Section 5 . □

7.8 THEOREM. Assuming that sorts are chosen to make the following well formed, we have:

(i) $\{x \in \tau \mid \phi(x)\} \in P(\tau)$

(ii) $\sigma \in \{x \in \tau \mid \phi(x)\} \leftrightarrow \sigma \in \tau \wedge \phi(\sigma)$

(iii) $\{x \in \tau \mid \phi(x)\} = \{x \in \tau \mid \psi(x)\} \leftrightarrow \forall x \in \tau [\phi(x) \leftrightarrow \psi(x)]$

(iv) $\bigwedge_{i<n} \sigma_i \in \tau_i \rightarrow \langle\sigma_0, \ldots, \sigma_{n-1}\rangle \in \tau_0 \times \ldots \times \tau_{n-1} \wedge \bigwedge_{i<n} \pi_i \langle\sigma_0, \ldots, \sigma_{n-1}\rangle = \sigma_i$

(v) $z \in \tau_0 \times \ldots \times \tau_{n-1} \rightarrow \bigwedge_{i<n} \pi_i z \in \tau_i \wedge z = \langle\pi_0 z, \ldots, \pi_{n-1} z\rangle$

(vi) $\langle x_0, \ldots, x_{n-1}\rangle = \langle y_0, \ldots, y_{n-1}\rangle \leftrightarrow \bigwedge_{i<n} x_i = y_i$.

7.9 METATHEOREM. All the axioms for higher-order logic hold for types in place of sorts.

The proofs of (i) - (vi) in 7.8 come directly out of the definitions and the properties of descriptions. Then 7.9 is almost a corollary. For the most part it is a matter of relativizing quantifiers, checking uniqueness, and making sure that things turn out at the right type. Note that 7.6 for types just means $X = \{x \mid x \in X\}$.

7.10 A DISCUSSION OF RELATIONS. In view of our postulation of products and powers, n-*ary relations* can be accommodated in the usual way. Indeed $P(X_0 \times \ldots \times X_{n-1})$ is the type of (mixed) n-ary relations as aubsets of a cartesian product. We write " $rx_0 \ldots x_{n-1}$ " for " $\langle x_0, \ldots, x_{n-1}\rangle \in r$ " . Note at once this makes a strict predicate; thus, the relations of type $P(X_0 \times \ldots \times X_{n-1})$ are strict relations, and higher-order logic allows us to quantify over such relations. If we wish to look at predicates with all variables from the same domain, we simply restrict to $P(X^n)$, where X^n is the n-fold product $X \times \ldots \times X$ (n-times).

As for definitions of relations, we can easily introduce as needed a notation such as

$$\{\langle x_0, \ldots, x_{n-1}\rangle \in X^n \mid \phi(x_0, \ldots, x_{n-1})\}$$

as short for:

$$\{z \mid \exists x_0 \in X \ldots \exists x_{n-1} \in X [\phi(x_0, \ldots, x_{n-1}) \wedge z = \langle x_0, \ldots, x_{n-1}\rangle]\} .$$

One standard use of relation theory is the formation of the *quotient* of a set

X under an equivalence relation $R \in P(X^2)$. (We had an extended example in 5.1.) We may write:

$$X/R = \{ z \in P(X) \mid \exists x \in X \ [Rxx \wedge \forall y \ [y \in z \leftrightarrow Rxy] \} \quad .$$

If R is reflexive on X , we can drop the first clause; but it is convenient to use this construction when R is just symmetric and transitive.

7.11 A DISCUSSION OF FUNCTIONS. In the context of the present theory, it is easiest to reduce functions to relations. Thus

$$f : X \to Y$$

should be regarded as short for

$$f \in P(X \times Y) \wedge \forall x \in X \ (\ Iy.fxy \ ) \in Y \quad .$$

And " f(x) " is taken for " $Iy \in Y.fxy$ " . Such a move is standard, and we need not go into details, except to note that the f(x)-notation makes f *strict* and the $(f:X \to Y)$-notation makes f *total.* Thus

$$Y^X = \{ f \in P(X \times Y) \mid f:X \to Y \}$$

is the space of all strict, total functions from X to Y . We could also consider n-ary functions in Y^{X^n} in the usual way.

7.12 TRUTH VALUES AND PARTIAL ELEMENTS. There is a subject called "second-order intuitionistic propositional calculus", meaning that you can quantify propositions. For example, a dyed-in-the-wool intuitionist may want to assert: $\neg \forall p [p \vee \neg p]$. We do not formally introduce this notation and the required axioms as primitive, because the theory is already incorporated into the theory of sets. (There was some discussion after 4.4 .) Indeed consider the sort $\mathbb{1} = (\ )$, the empty product. (Note that $\mathbb{1} = A^0$, too.) There is only one element in $\mathbb{1}$, namely $0 = \langle \ \rangle$. (In construing axiom (proj) , the empty list of quantifiers is simply not written, the empty conjunction becomes *true*, and $0 = Iz.\mathrm{true}$ means 0 exists as the only element of its sort.) Consider $\Omega = P(\mathbb{1})$. There are two obvious elements:

$$\{ 0 \} = \{ x \in \mathbb{1} \mid \mathrm{true} \} \quad , \text{ and}$$

$$\emptyset = \{ x \in \mathbb{1} \mid \mathrm{false} \} \quad .$$

As there is only one possible element of $\mathbb{1}$, we shorten the writing of $\{ x \in \mathbb{1} \mid \phi(x) \}$ to a simple $\{ 0 \mid \phi(0) \}$. If we use the singleton notation $\{ a \} = \{ x \mid x = a \}$, this could also be written as $\{ 0 \restriction \phi(0) \}$, where restriction was defined in 6.5 . Now because

$$\{ 0 \mid \phi \} = \{ 0 \mid \psi \} \leftrightarrow (\phi \leftrightarrow \psi) \text{ , and}$$

$$0 \in \{ 0 \mid \phi \} \leftrightarrow \phi \quad ,$$

we have an isomorphism between sets in $P(\mathbf{1})$ and propositions. (We could carry this over to an isomorphism involving propositional connectives, if we defined such obvious set-theoretical operations as union, intersection, complement, etc.) Propositional quantification can then be defined by:

$$\forall p.\phi(p) \leftrightarrow \forall x \in \Omega . \phi(0 \in x) \;,$$

and similarly for $\exists$.

By the way, though propositional quantification is trivial in classical logic (there are, classically, only two truth values), in intuitionistic logic the theory is *undecidable* (see Gabbay [11]).

Another interesting feature of propositional quantification is the ability to define all connectives in terms of $\forall$ and $\rightarrow$. (This was discovered by the author in 1956/57 but never published; it has since been remarked several times, see especially Prawitz [20] .) We have these definitions

(1) $p \wedge q$ for $\forall r\; [[p \rightarrow [q \rightarrow r]] \rightarrow r]$;

(2) $p \leftrightarrow q$ for $[p \rightarrow q] \wedge [q \rightarrow p]$;

(3) $p \vee q$ for $\forall r\; [[[p \rightarrow r] \wedge [q \rightarrow r]] \rightarrow r]$;

(4) $\exists x . \phi(x)$ for $\forall r\; [\forall x\; [\phi(x) \rightarrow r] \rightarrow r]$.

The trick here is that even though $p \leftrightarrow \neg\neg p$ fails intuitionistically, $p \leftrightarrow \forall r\; [[p \rightarrow r] \rightarrow r]$ holds. Negation, by the way, is defined as:

(5) $\neg p$ for $\forall r\; [p \rightarrow r]$.

A consequence of this reduction (noted by Fourman as a generalization of Henkin [15]) is that higher-order intuitionistic logic can be axiomatized with $\rightarrow$, $\forall$, $=$, $\in$, $\{\cdot \mid \cdot\}$, $\langle \cdots \rangle$, and π_i as the only primitives; in fact, we have essentially done this - except that the existential quantifier in the comprehension axiom should be replaced by the class abstract. The reason for this choice of primitives is that all axioms (and rules of inference) can be easily stated in *primitive notation*. At some expense of readability we could even eliminate $=$ by the definition:

(6) $x = y$ for $x \in A \wedge y \in A \wedge \forall z\; [x \in z \leftrightarrow y \in z]$.

Or the other way round, we could define:

(7) $p \leftrightarrow q$ for $\{0 \mid p\} = \{0 \mid q\}$,

and then use the higher-order idea of Tarski [26] (see also Henkin [15]) to define:

(8) $p \wedge q$ for $\forall x\; [p \leftrightarrow [\{0 \mid p\} \in x \leftrightarrow \{0 \mid q\} \in x]]$,

where x has sort $P(P(\mathbf{1}))$. Then we could get:

(9) $\quad p \to q \quad$ for $\quad [\, p \leftrightarrow p \wedge q \,]$.

Definitions (7) - (9) thus eliminate $\to$, but at very considerable cost in complexity and not much gain in understanding. (Conjunction could also be defined in terms of pairing:

(10) $\quad p \wedge q \quad$ for $\quad \langle \{ 0 \mid p \}, \{ 0 \mid q \} \rangle = \langle \{ 0 \}, \{ 0 \} \rangle$,

but this is not particularly neat either.)

The construction of the space of propositions, Ω , as $P(\mathbb{1})$ is a special case of the construction of the *space of partial elements*. As remarked several times, $x \upharpoonright p$ does not exist as much as x , since $E(x \upharpoonright p) \leftrightarrow Ex \wedge p$. Also when we quantify as in $\forall x . \phi(x)$ we mean $\forall x [\, Ex \to \phi(x) \,]$, that is, quantification over existing - not partial - elements. Thus to quantify over partial elements we need something like propositional quantification. To be specific, let X be any type. When we say $x \in X$, this implies x exists (is total). To get at the partial elements, define:

$$\tilde{X} = \{ z \in P(X) \mid \forall x,y \, [\, x \in z \wedge y \in z \to x = y \,] \} .$$

We could also write:

$$\tilde{X} = \{ z \in P(X) \mid z = \{ Ix . x \in z \} \} .$$

This device (due to Lawvere and Tierney, see [12]) works because sets $\{ \cdot \mid \cdot \}$ are always total even if the potential elements are partial. Writing $\{ x \mid p \} \in P(X)$ does *not* imply x exists. For instance $\emptyset_X = \{ x \in X \mid \text{false} \} \in \tilde{X}$; it *exists* but it corresponds to the totally undefined element of X . The extreme case of this construction is $\tilde{\emptyset} = \{ \emptyset \}$. The next is $\Omega = \tilde{\mathbb{1}}$ (which is isomorphic to $\tilde{\tilde{\emptyset}}$).

It should also be noted that the $\tilde{X}$-construction also gives a nice way to compare $=$ and $\equiv$. For x, y variables of sort A , we have:

$$x \equiv y \leftrightarrow \{ x \} = \{ y \} .$$

Of course $\{ \tau \} \in \tilde{A}$ holds if τ is any term of sort A . Thus the theory of $\equiv$ on A is just the theory of $=$ on $\tilde{A}$: this move completely internalizes talk of partial elements.

7.13 DISJOINT SUMS. Within a fixed sort A , we can form *unions* of subsets in the usual way:

$X \cup Y \quad$ for $\quad \{ t \mid t \in X \vee t \in Y \}$,

where $\# t = A$ and $\# X = \# Y = P(A)$. But if the sort of X is different from the sort of Y , there is no coherent meaning to give to $X \cup Y$. Sorts could just as well be taken to be pairwise disjoint; the only *primitive* way we have of joining them is by the (cartesian) product. In case A and B are inhabited (non-empty), then a disjoint union can be found as a subset of $A \times B$ - but not in a canonical way. If one might be empty, then $A \times B$ might also; thus the union is *not* always a subset. The canonical solution is quite well known (and was also employed by Mikkelsen [19] in simplifying the definition of an elementary topos); write:

$$X + Y \quad \text{for} \quad \{ z \in P(X) \times P(Y) \mid \exists x \in X.z = \langle\{x\},\emptyset_Y\rangle \vee \exists y \in Y.z = \langle\emptyset_X,\{y\}\rangle \} .$$

As there is no danger that $\langle\{x\},\emptyset\rangle = \langle\emptyset,\{y\}\rangle$, this is a *disjoint* sum. Of course, it is not difficult to axiomatize disjoint sums (coproducts), but here is a case where there is a definite gain in simplicity in reducing a notion to other primitives.

7.14 NATURAL NUMBERS. As an application of 7.13, we could use disjoint sums to define $\mathbf{2} = \mathbf{1} + \mathbf{1}$, $\mathbf{3} = \mathbf{1} + \mathbf{2}$, etc. In this way finite sets of any size are guaranteed to exist. We do not obtain *infinite* sets (sorts) for free, however. We cannot "define" the *natural numbers* by the equation $\mathbb{N} = \mathbf{1} + \mathbb{N}$, even if in a certain sense this is true (up to isomorphism). It is necessary to take $\mathbb{N}$ as a given sort and have primitives 0 and S for *zero* and *successor* and to assume the usual postulates:

$$
\begin{array}{ll}
\text{(Peano)} & E0 \wedge \\
& [x = y \leftrightarrow Sx = Sy] \wedge \\
& \neg 0 = Sx \wedge \\
& [0 \in z \wedge \forall x [x \in z \to Sx \in z] \to z = \mathbb{N}]
\end{array}
$$

where $\# x = \# y = \mathbb{N}$ and $\# z = P(\mathbb{N})$. The formal development of the properties of $\mathbb{N}$ from these axioms (even in intuitionistic logic) is standard (see [6]). Similarly once we have $\mathbb{N}$, there is no problem in obtaining the *ring* $\mathbb{Z}$ of integers and the *field* $\mathbb{Q}$ of rational numbers. The reals $\mathbb{R}$ can also be constructed now in higher-order logic by the use of Dedekind cuts (see Fourman and Scott [10]).

7.15 A NOTE ON THE AXIOM OF CHOICE. Stated most simply with functional notation, choice is the following principle:

$$\text{(choice)} \qquad \forall x \, \exists y.\phi(x,y) \to \exists f \, \forall x.\phi(x,f(x)) ,$$

where $\# x = X$, $\# y = Y$, and $\# f = Y^X$ (that is, f is *total*). For fixed X, Y we should call this (choice $- X,Y$) or (AC$-X,Y$). But there are problems. Some intuitionists would accept (choice $- \mathbb{N},\mathbb{N}$); but for general types they cannot: the general axiom implies the law of the excluded middle. (The argument is due to Diaconescu [5].)

Write as usual:

$$\{x_0,\ldots,x_{n-1}\} \quad \text{for} \quad \{x_0\} \cup \ldots \cup \{x_{n-1}\} .$$

For simplicity, rename the elements of $\mathbf{2}$ so that $\mathbf{2} = \{0,1\}$. Consider the following subtype of $\mathbf{2}$:

$$X = \{\{0, 1 \upharpoonleft p\}, \{0 \upharpoonleft p, 1\}\} \quad .$$

Clearly we have:

$$\forall x \in X \; \exists y \in \mathbf{2} . y \in x \quad .$$

Now suppose there is a function $f \in \mathbf{2}^X$ where $\forall x \in X . f(x) \in x$ is true. If we look at the first element $a = \{0, 1 \upharpoonleft p\}$ of X, we must have $f(a) = 0$ or $f(a) = 1 \upharpoonleft p$. The second case implies p . Now consider $b = \{0 \upharpoonleft p, 1\}$. Then $f(b) = 0 \upharpoonleft p$ or $f(b) = 1$. The first case also implies p . The only case so far where we do not at once get information about p is the case where $f(a) = 0$ *and* $f(b) = 1$. Now $[p \to a = b]$ and $[a = b \to f(a) = f(b)]$, so $\neg p$ must follow in view of $\neg 0 = 1$. This shows the existence of f implies $p \vee \neg p$, and p was arbitrary.

There are models (complete for propositional calculus, so not validating excluded middle) in which (choice-$\mathbb{N}, Y$) always holds. This is not, however, strong enough for some intuitionistic purposes; a better principle is *dependent choices*:

$$\text{(dep. choice} - Y) \qquad \forall x \in Y \; \exists y \in Y . \phi(x,y) \to \forall z \in Y \; \exists f \in Y^{\mathbb{N}} [\forall n \in \mathbb{N} . \phi(f(n), f(Sn)) \wedge f(0) = z] \quad .$$

This also holds in many models. But even (choice - $\mathbb{N}, \mathbb{N}$) fails in other models which have independent interest (see Fourman and Hyland [9]). Perhaps some more extensive investigations about connections between different forms would be fruitful.

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