Automating Free Logic in HOL, with an Experimental Application in Category Theory

Christoph Benzmüller and Dana S. Scott

Abstract A shallow semantical embedding of free logic in classical higher-order logic is presented, which enables the off-the-shelf application of higher-order interactive and automated theorem provers for the formalisation and verification of free logic theories. Subsequently, this approach is exemplarily employed in a selected domain of mathematics: starting from a generalization of the standard axioms for a monoid we present a stepwise development of various, mutually equivalent foundational axiom systems for category theory. As a side-effect of this work some (minor) issue in a prominent category theory textbook has been revealed.

The purpose of this article is not to claim any novel results in category theory, but to demonstrate an elegant way to "implement" and utilize interactive and automated reasoning in free logic, and to present respective experiments.

Keywords Free Logic · Classical Higher-Order Logic · Category Theory · Interactive and Automated Theorem Proving

1 Introduction

Partiality and undefinedness are prominent challenges in various areas of mathematics and computer science. Unfortunately, however, modern proof assistant systems and automated theorem provers based on traditional classical or intuitionistic logics provide rather inadequate support for these challenge concepts. Free logic [22,28,23,26] offers a theoretically appealing solution, but is has been considered as rather unsuited towards practical utilisation.

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In the first part of this article (§2 and §3) we show how free logic can be elegantly "implemented" in any theorem proving system for classical higher-order logic (HOL) [8]. The proposed solution employs a semantic embedding of free in HOL. We present, as an example, one implementation of this idea in the mathematical proof assistant Isabelle/HOL [25]. Various state-of-the-art first-order and higher-order automated theorem provers and model finders are integrated (modulo suitable logic translations) with Isabelle via the Sledgehammer tool [13], so that our solution can be utilized, via Isabelle as foreground system, with a whole range of other background reasoners. As a result we obtain an elegant and powerful implementation of an interactive and automated theorem proving (and model finding) system for free logic.

To demonstrate the practical relevance of our new system, we present in §4 a stepwise development of axioms systems for category theory by generalizing the standard axioms for a monoid to a partial composition operation. Our purpose is not to make or claim any contribution to category theory but rather to show how formalizations involving the kind of logic required (free logic) can be implemented and validated within modern proof assistants such as Isabelle/HOL. We also address the relation of our axiom systems to alternative proposals from the literature, including an axiom set proposed by Freyd and Scedrov in their textbook "Categories, Allegories" [20] for which we reveal a technical flaw: either all operations, e.g. morphism composition, are total or their axiom system is inconsistent. The repair for this problem is quite straightforward, however. The solution essentially corresponds to a set of axioms proposed by Scott [29] in the 1970s.

Our exploration has been significantly supported by series of experiments in which automated reasoning tools have been called from within the proof assistant Isabelle/HOL via the Sledgehammer tool [13]. Moreover, we have obtained very useful feedback at various stages from the model finder Nitpick [14] saving us from making several mistakes.

At the conceptual level this paper exemplifies a new style of explorative mathematics which rests on a significant amount of human-machine interaction with integrated interactive-automated theorem proving technology. The experiments we have conducted are such that the required reasoning is often too tedious and time-consuming for humans to be carried out repeatedly with highest level of precision. It is here where cycles of formalization and experimentation efforts in Isabelle/HOL provided significant support. Moreover, the technical inconsistency issue for axiom system VII was discovered by automated theorem provers, which further emphasizes the added value of automated theorem proving in this area.

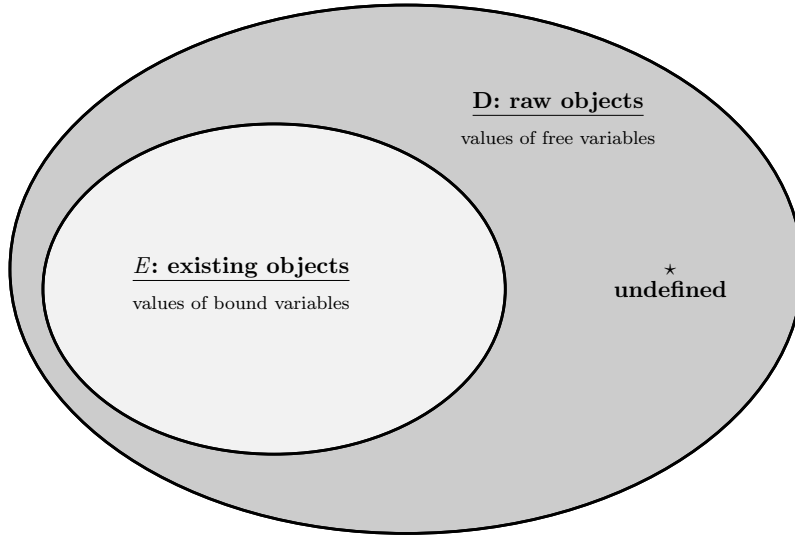The content of article is based on the contributions reported in two previous papers [9,10].

Fig. 1 Illustration of the Semantical Domains of Free Logic

2 Preliminaries

2.1 Free Logic

Free logic (respectively inclusive logic) [22, 28, 23, 26] refers to a class of logic formalisms that are free of basic existence assumptions regarding the denotation of terms. Remember that terms in e.g. traditional classical and intuitionistic predicate logics always denote an (existing) object in a given (non-empty) domain D, and that D is also exactly the set the quantifiers range over. In free logic these basic assumption are abolished. Terms do still denote objects in a (non-empty) domain D, but a (possibly empty) set $E \subseteq D$ is chosen to characterize the subdomain of "*existing*" resp. "*defined*" objects in D. Quantification is now restricted to set E of existing/defined objects only.

It is obvious how this can be used to model undefideness and partiality: problematic terms, e.g. division by zero or improper definite descriptions, still denote, but they refer to undefined objects, that is, objects d in $D \setminus E$ lying outside of the scope of quantification. Moreover, a function f is *total* if and only if for all x we have $Ex \longrightarrow E(fx)$.[1] For *partial* functions f we may have some x such that Ex but not $E(fx)$. A function f is *strict* if and only if for all x we have $E(fx) \longrightarrow Ex$.

The particular notion of free logic as exploited in the remainder of this article has been proposed by Scott [28]. A graphical illustration of this notion of free logic is presented in Fig. 1. It employs a distinguished undefined object $\star$.

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[1] The predication Ex represents that x is a member of E.

We formally introduce the syntax and semantics of free logic as assumed in the remainder of this article. We refer to this logic with *FFOL*.

Definition 1 (Syntax of *FFOL*) We start with a denumerable set V of variable symbols, a denumerable set F of n-ary function symbols ($n \geq 0$), and a denumerable set P of n-ary predicate symbols ($n \geq 0$).

The *terms and formulas of FFOL* are formally defined as the smallest sets such that:

1. each variable $x \in V$ is a term of *FFOL*,
2. given any n-ary ($n \geq 0$) function symbol $f \in F$ and terms $t_1, \ldots, t_n$ of *FFOL*, then $f(t_1, \ldots, t_n)$ is a term of *FFOL*,
3. given terms t_1 and t_2 of *FFOL*, then $t_1 = t_2$ is an (atomic) formula of *FFOL*,
4. given any n-ary ($n \geq 0$) predicate symbol $p \in P$ and terms $t_1, \ldots, t_n$ of *FFOL*, then $p(t_1, \ldots, t_n)$ is an (atomic) formula of *FFOL*,
5. given formulas r and s of *FFOL*, then $\neg r$, $r \rightarrow s$ and $\forall x. r$ are (compound) formulas of *FFOL*, and
6. given a formula r of *FFOL*, then $\imath x. r$ is a term of *FFOL* (definite description).

Further terms and formulas of *FFOL*, including various defined notions of equality, can be introduced as abbreviations.

A *variable assignment* g maps variables $x \in V$ to elements in D. $g[d/x]$ denotes the assignment that is identical to g, except for variable x, which is now mapped to d.

Regarding the semantics different options have been proposed in the literature. For example, instead of a possible empty set of existing objects E, we could postulate non-emptiness of E. In fact, our approach below can be easily adapted for different variants of free and inclusive logic. Here we closely follow the notion of free logic as proposed by Scott [28].

Definition 2 (Model of *FFOL*) A *model (structure) for FFOL* consists of a quadruple $M = \langle D, E, I, \star \rangle$, where D is a non-empty raw domain of objects, $E \subseteq D$ a possible empty set of existing/defined objects, and I an interpretation function mapping 0-ary function symbols (constants) to defined objects $d \in E$, 0-ary predicate symbols (propositions) to *True* or *False*, n-ary function symbols (for $n \geq 1$) to n-ary functions $D \times \ldots \times D \longrightarrow D$ and n-ary predicate symbols (for $n \geq 1$) to n-ary relations $D \times \ldots \times D$. *True* or *False* denote truth and felshood respectively. Finally, $\star \in D \setminus E$ is a designated (non-existing/undefined) object.

Definition 3 (Evaluation Function for *FFOL*) The *value* $\|s\|^{M,g}$ of a term or formula $s \in FFOL$ in a model $M = \langle D, E, I, \star \rangle$ under assignment g defined in the following way:

Terms
  1. $\|x\|^{M,g} = g(x)$ for variable symbols $x \in V$

2. $\|c\|^{M,g} = I(c)$, where $c \in F$ is an 0-ary function symbol
3. $\|f(t_1, \ldots, t_n)\|^{M,g} = I(f)(\|t_1\|^{M,g}, \ldots, \|t_n\|^{M,g})$, where $f \in F$ is an n-ary $(n \geq 1)$ function symbol
4. $\|\imath x_{\blacksquare}r\|^{M,g} = d \in E$, such that $\|r\|^{M,g[d/x]} = \textit{True}$ and $\|r\|^{M,g[d'/x]} = \textit{False}$ for all $d' \neq d \in E$ (i.e. d is the unique existing object for which r holds); if there is no such $d \in E$, then $\|\imath x_{\blacksquare}r\|^{M,g} = \star$

Formulas

5. $\|q\|^{M,g} = I(q)$, where $q \in P$ is an 0-ary predicate symbol
6. $\|t_1 = t_2\|^{M,g} = \textit{True}$ if and only if $\|t_1\|^{M,g} = \|t_2\|^{M,g}$ (this basic notion of primitive equality on D implies that equations between "undefined" terms such as $1/0 = 1/0$ are evaluated to *True*; later, in §4, we will define and utilize further notions of equality, including *Kleene equality* and *existing equality*, which behave differently).
7. $\|p(t_1, \ldots, t_n)\|^{M,g} = \textit{True}$ if and only if $(\|t_1\|^{M,g}, \ldots, \|t_n\|^{M,g}) \in I(p)$ for n-ary $(n \geq 1)$ predicate symbols $p \in P$
8. $\|\neg r\|^{M,g} = \textit{True}$ if and only if $\|r\|^{M,g} = \textit{False}$
9. $\|r \rightarrow s\|^{M,g} = \textit{True}$ if and only if $\|r\|^{M,g} = \textit{False}$ or $\|s\|^{M,g} = \textit{True}$
10. $\|\forall x_{\blacksquare}r\|^{M,g} = \textit{True}$ if and only if for all $d \in E$ we have $\|r\|^{M,g[d/x]} = \textit{True}$

Definition 4 (Validity) A formula s_o is *true* in model M under assignment g if and only if $\|s_o\|^{M,g} = T$; this is also denoted as $M, g \models^{FFOL} s_o$. A formula s_o is called *valid* in M, which is denoted as $M \models^{FFOL} s_o$, if and only if $M, g \models^{FFOL} s_o$ for all assignments g. Finally, a formula s_o is called *valid*, which we denote by $\models^{FFOL} s_o$, if and only if s_o is valid for all M.

2.2 Classical Higher-Order Logic

Simple type theory, also referred to as classical higher-order logic (HOL) [2], is an expressive logic formalism which is based on the simply typed λ-calculus [3]. HOL has its origin in the work by Church [17].

For a detailed discussion of the syntax, semantics and automation of HOL we refer to the literature (see e.g. [2,8,6] and the references therein). Below we introduce a variant of HOL with primitive equality and description.

Definition 5 (Types) The set $\boldsymbol{T}$ of simple types freely generated from a set of basic types $\{\boldsymbol{o}, \boldsymbol{i}\}$ using the function type constructor $\rightarrow$. $\boldsymbol{o}$ is the type of Booleans and $\boldsymbol{i}$ is the type of individuals. We may avoid parentheses if the structure of a complex type is clear in context.

Definition 6 (Syntax of HOL) The language of higher-order logic HOL with primitive equality and description is defined by the following grammar:[2]

$$s, t \quad ::= \quad p_\alpha \mid X_\alpha \mid (\lambda X_\alpha.s_\beta)_{\alpha \to \beta} \mid (s_{\alpha \to \beta}\, t_\alpha)_\beta \mid s_\alpha = t_\alpha \mid \neg_{o \to o}\, s_o \mid$$
$$((\vee_{o \to o \to o}\, s_o)\, t_o) \mid \forall_{(\alpha \to o) \to o}(\lambda X_\alpha.s_o) \mid \imath(\lambda X_\alpha.s_o)$$

where $\alpha, \beta \in T$. p_α denotes typed constants and X_α typed variables (distinct from p_α). Complex typed terms are constructed via abstraction and application. The type of each term is given as a subscript. Terms s_o of type o are called formulas. The logical connectives of choice are $\neg_{o \to o}$, $\vee_{o \to o \to o}$, $=_{\alpha \to \alpha \to o}$, $\forall_{(\alpha \to o) \to o}$ and $\imath_{(\alpha \to o) \to \alpha}$ (where $\alpha \in T$). Type subscripts may be dropped if irrelevant or obvious. Similarly, parentheses may be avoided. Binder notation $\forall X_\alpha.s_o$ and $\imath X_\alpha.s_o$ is used as shorthand for $\forall_{(\alpha \to o) \to o}(\lambda X_\alpha.s_o)$ and $\imath(\lambda X_\alpha.s_o)$, and infix notation $s \vee t$ is employed instead of $((\vee s)\, t)$. From the above connectives, other logical connectives, such as $\top$, $\bot$, $\wedge$, $\to$, $\equiv$ and $\exists$, can be defined in the usual way. For example, $\to$ can be defined as $\lambda X_o.\lambda Y_o.\neg X \vee Y$ and $\wedge$ as $\lambda X_o.\lambda Y_o.\neg(\neg X \vee \neg Y)$.

We assume familiarity with λ-conversion (e.g. *α-renaming* and *$\beta\eta$-reduction*) and *$\beta\eta$-normal forms* [3].

A *variable assignment* g maps variables X_α to elements in D_α. $g[d/W]$ denotes the assignment that is identical to g, except for variable W, which is now mapped to d.

Definition 7 (Frame for HOL) A *frame D* is a collection $\{D_\alpha\}_{\alpha \in T}$ of nonempty sets D_α, such that $D_o = \{True, False\}$ (where, as before, *True* and *False* denote truth and falsehood). The $D_{\alpha \to \beta}$ are collections of functions mapping D_α into D_β.

Definition 8 (Model for HOL) A *model* for HOL is a tuple $M = \langle D, I \rangle$, where D is a frame, and I is a family of typed interpretation functions mapping constant symbols p_α to appropriate elements of D_α, called the *denotation of p_α* (the logical connectives $\neg$, $\vee$, and $\forall$ are always given the standard denotations, see below). Moreover, we assume that the domains $D_{\alpha \to \alpha \to o}$ contain the respective identity relations.

Definition 9 (Evaluation Function for HOL) The *value* $\|s_\alpha\|^{M,g}$ of a HOL term s_α on a model $M = \langle D, I \rangle$ under assignment g is an element $d \in D_\alpha$ defined in the following way:

1. $\|p_\alpha\|^{M,g} = I(p_\alpha)$
2. $\|X_\alpha\|^{M,g} = g(X_\alpha)$
3. $\|(s_{\alpha \to \beta}\, t_\alpha)_\beta\|^{M,g} = \|s_{\alpha \to \beta}\|^{M,g}(\|t_\alpha\|^{M,g})$

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[2] It is well known that we could work with a much smaller set of logical connectives, see e.g. §1.4 of Andrews's overview article [2]. The choice here closely reflects the set of primitive connectives as chosen in higher-order automated theorem provers such as LEO-II [12], Leo-III [11] and Satallax [16].

4. $\|(\lambda X_\alpha \cdot s_\beta)_{\alpha \to \beta}\|^{M,g}$ = the function f from D_α to D_β such that $f(d) = \|s_\beta\|^{M,g[d/X_\alpha]}$ for all $d \in D_\alpha$

5. $\|s_\alpha = t_\alpha\|^{M,g}$ = *True* if and only if $\|s_\alpha\|^{M,g} = \|t_\alpha\|^{M,g}$

6. $\|(\neg_{o \to o} s_o)_o\|^{M,g}$ = *True* if and only if $\|s_o\|^{M,g}$ = *False*

7. $\|((\vee_{o \to o \to o} s_o) t_o)_o\|^{M,g}$ = *True* if and only if $\|s_o\|^{M,g}$ = *True* or $\|t_o\|^{M,g}$ = *True*

8. $\|(\forall_{(\alpha \to o) \to o}(\lambda X_\alpha \cdot s_o))_o\|^{M,g}$ = *True* if and only if for all $d \in D_\alpha$ we have $\|s_o\|^{M,g[d/X_\alpha]}$ = *False*

9. $\|(\iota_{(\alpha \to o) \to \alpha}(\lambda X_\alpha \cdot s_o))_o\|^{M,g} = d$ if there exists a unique $d \in D_\alpha$ such that $\|s_o\|^{M,g[d/X_\alpha]}$ = *True*, otherwise $\|(\iota_{(\alpha \to o) \to \alpha}(\lambda X_\alpha \cdot s_o))_o\|^{M,g} = e$ for an arbitrary element $e \in D_\alpha$

Definition 10 (Standard and Henkin models) A model $M = \langle D, I \rangle$ is called a *standard model* if and only if for all $\alpha, \beta \in T$ we have $D_{\alpha \to \beta} = \{f \mid f : D_\alpha \longrightarrow D_\beta\}$. In a *Henkin model* function spaces are not necessarily full. Instead it is only required that $D_{\alpha \to \beta} \subseteq \{f \mid f : D_\alpha \longrightarrow D_\beta\}$ (for all $\alpha, \beta \in T$) and that the valuation function $\| \cdot \|^{M,g}$ from above is total (i.e., every term denotes). Any standard model is obviously also a Henkin model.

We consider Henkin models in the remainder. For more details on Henkin semantics, its proof theory and examples of sound and complete calculi we refer to the literature (e.g. [6,7]).

Definition 11 (Validity) A formula s_o is *true* in model M under assignment g if and only if $\|s_o\|^{M,g}$ = *True*; this is also denoted as $M, g \models^{\mathrm{HOL}} s_o$. A formula s_o is called *valid* in M, which is denoted as $M \models^{\mathrm{HOL}} s_o$, if and only if $M, g \models^{\mathrm{HOL}} s_o$ for all assignments g. Finally, a formula s_o is called *valid*, which we denote by $\models^{\mathrm{HOL}} s_o$, if and only if s_o is valid for all M.

3 Shallow Semantical Embedding of *FFOL* in HOL

We now present a shallow embedding of *FFOL* in HOL by identifying the language constructs of *FFOL* with corresponding terms of HOL. In this embedding the raw domain D of a *FFOL* model is identified with the domain of individuals D_i in a corresponding HOL model. The subdomain E of existing objects is characterized in the embedding by a HOL predicate E of type $i \to o$. Hence, we assume in the remainder that a respective uninterpreted constant symbol $E_{i \to o}$ is given in the signature of HOL. Moreover, we assume that an uninterpreted constant symbol $\star$ of type i be in the signature of HOL. Finally, we assume that $\|E \star_i\|^{M,g} = F$ for all M, g, i.e. that the element denoted by $\star_i$ is not an element of the domain of existing objects denoted by $E_{i \to o}$.

Definition 12 (Embedding of *FFOL* in HOL) Given a formula $s \in FFOL$. We map s to a corresponding term $\hat{s}$ of HOL. This mapping is defined as follows:

$$
\begin{array}{lll}
\hat{x} & := & \boldsymbol{X_i} \qquad\qquad\qquad\qquad \text{for all } x \in V \\
f(t^1,\stackrel{\wedge}{\ldots},t^n) & := & (\hat{f}\ \hat{t^1}\ldots\hat{t^n}) \qquad\qquad \text{for all } n\text{-ary } f \in F\ (n \geq 0) \\
&&\qquad\qquad\qquad\qquad\qquad \text{where } \hat{f} = \boldsymbol{f} \text{ is an uninterpreted} \\
&&\qquad\qquad\qquad\qquad\qquad \text{constant symbol of type } \underbrace{i \to \ldots \to i \to}_{n \geq o} i \\
s \stackrel{\wedge}{=} t & := & \hat{s} = \hat{t} \\
p(t^1,\stackrel{\wedge}{\ldots},t^n) & := & (\hat{p}\ \hat{t^1}\ldots\hat{t^n}) \qquad\qquad \text{for all } n\text{-ary } p \in P\ (n \geq 0) \\
&&\qquad\qquad\qquad\qquad\qquad \text{where } \hat{p} = \boldsymbol{p} \text{ is an uninterpreted} \\
&&\qquad\qquad\qquad\qquad\qquad \text{constant symbol of type } \underbrace{i \to \ldots \to i \to}_{n \geq o} o \\
\hat{\neg} s & := & \neg \hat{s} \\
s \stackrel{\wedge}{\to} r & := & \hat{s} \to \hat{r} \\
\forall \hat{x}. r & := & \forall \boldsymbol{X_i}. \boldsymbol{E X_i} \to \hat{r} \\
\imath \hat{x}. r & := & \textbf{IfThenElse} \\
&&(\exists \boldsymbol{X_i}. \boldsymbol{E X} \wedge \hat{r} \wedge (\forall \boldsymbol{Y_i}. (\boldsymbol{E Y} \wedge ((\lambda \boldsymbol{X_i}.\ \hat{r})\boldsymbol{Y})) \to \boldsymbol{Y = X})) \\
&&(\imath \boldsymbol{X_i}. \hat{r}) \\
&&\star
\end{array}
$$

where **IfThenElse** is an abbreviation for the term

$$
\lambda \boldsymbol{S_o}. \lambda \boldsymbol{X_i}. \lambda \boldsymbol{Y_i}. \imath \boldsymbol{Z_i}. (\boldsymbol{S_o} \wedge \boldsymbol{Z = X}) \vee (\neg \boldsymbol{S_o} \wedge \boldsymbol{Z = Y})
$$

The above mapping induces mappings from the sets $\hat{V}$, $\hat{F}$ and $\hat{P}$ of *FFOL* to corresponding variables (of type i), uninterpreted function symbols and uninterpreted predicate symbols in HOL.

To prove soundness and completeness[3] for the embedding, a mapping from *FFOL* models into Henkin models is employed. This mapping utilizes a mapping of *FFOL* variable assignments g into corresponding HOL variable assignments $\boldsymbol{g}$ (remember that *FFOL* domains D are identified with HOL domains $\boldsymbol{D_i}$, i.e. $\hat{D} = \boldsymbol{D_i}$): Let g be a variable assignment for *FFOL*. Then $\hat{g} : \hat{V} \mapsto \hat{D}$ for HOL is defined such that $\hat{g}(\boldsymbol{X_i}) = \hat{g}(\hat{x}) = g(x)$ for all $\boldsymbol{X_i} \in \hat{V}$. Finally, $\hat{g}$ is extended to an assignment for arbitrary variables by choosing $\hat{g}(\boldsymbol{Y_\alpha}) = \boldsymbol{d} \in \boldsymbol{D_\alpha}$ arbitrary whenever $\boldsymbol{Y_\alpha} \notin \hat{V}$.

Definition 13 (Henkin model $\hat{M}$ for *FFOL* model M) Given a *FFOL* model $M = \langle D, E, I, \star \rangle$. The Henkin model $\hat{M} = \langle \{\boldsymbol{D_\alpha}\}_{\alpha \in T}, \boldsymbol{I} \rangle$ for M is defined as follows:

- $\boldsymbol{D_i} = D$
- $\boldsymbol{D_o} = \{True, False\}$
- $\boldsymbol{D_{\alpha \to \beta}}$ are chosen as (not necessarily full) collections of functions from $\boldsymbol{D_\alpha}$ to $\boldsymbol{D_\beta}$. Remember, however, that the choice of $\boldsymbol{D_{\alpha \to \beta}}$ must always ensure that the evaluation function $\| \cdot \|^{\hat{M}, \hat{g}}$ below remains total, i.e. that all terms denote. In particular, it is required that $\boldsymbol{D_{i \to o}}$ contains the element $\boldsymbol{I E_{i \to o}}$ as characterized below.
- The interpretation function $\boldsymbol{I}$ of $\hat{M}$ is chosen as follows:

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[3] Similar soundness and completeness proofs for shallow semantical embeddings have been presented in [4] and [5].

- $I\star_i = \star \in D_i$
- For all $d \in D_i$ we have: $(IE_{i\to o})(d) = T$ if and only if $d \in E$. Note that this implies $(IE_{i\to o})(\star_i) = F$.
- For all $f = \hat{f} \in \hat{F}$ we have: $(If)(d^1, \ldots, d^n) = (If)(d^1, \ldots, d^n)$ for all $d^i \in D_i$ $(i = 1, \ldots, n$ and $n \geq 0)$.
- For all $p = \hat{p} \in \hat{P}$ we have: $(Ip)(d^1, \ldots, d^n) = T$ if and only if $(d^1, \ldots, d^n) \in (Ip)$ for all $d^i \in D_i$ $(i = 1, \ldots, n$ and $n \geq 0)$.
- For all other constants c_α, choose $Ic_\alpha \in D_\alpha$ arbitrary.[4]

It is not hard to verify that $\hat{M}$ is a Henkin model.[5]

Lemma 1 *Let $\hat{M}$ be a Henkin model for FFOL model M. For all terms and formulas $s \in FFOL$ and variable assignments g we have $\|s\|^{M,g} = \|\hat{s}\|^{\hat{M},\hat{g}}$.*

Proof The proof is by induction on the structure of s.

For $s = x \in V$ the claim follows from the definition of $\hat{g}$. For $s = c \in F$, where c is 0-ary, we get the claim by the choice of I in $\hat{M}$, and for $s = f(t^1, \ldots, t^n)$, where $f \in F$ is n-ary $(n \geq 1)$, we additionally need to apply the induction hypothesis. The arguments for $s = q \in P$, where q is 0-ary, and for $s = p(t^1, \ldots, t^n)$, where $p \in P$ is n-ary $(n \geq 1)$, are similar. The most complicated case is when $s = \imath x_\bullet r$. We here consider two cases. We either have $d \in E$ with $\|r\|^{M,g[d/x]} = True$ and $\|r\|^{M,g[d'/x]} = False$ for all $d' \neq d \in E$ (i.e. d is the unique existing object for which r holds) or there is no such $d \in E$. In the former case we have $\|\imath x_\bullet r\|^{M,g} = d \in E$. By the definition of the embedding $\hat{\cdot}$, definition of $\hat{M}$ and $\hat{g}$, λ-conversion, induction hypothesis and a series of tedious evaluation steps in HOL we get $\|\imath x \hat{}_\bullet r\|^{\hat{M},\hat{g}} = \|\imath x_\bullet \hat{r}\|^{\hat{M},\hat{g}} = d \in E$ (for the very same d as above). In the second case we have $\|\imath x_\bullet r\|^{M,g} = \star$. Again we apply the definition of the embedding $\hat{\cdot}$, the definition of $\hat{M}$ and $\hat{g}$, λ-conversion, induction hypothesis and a series of tedious evaluation steps in HOL to verify that $\|\imath x \hat{}_\bullet r\|^{\hat{M},\hat{g}} = \star = \star$. The remaining cases are similar (actually simpler) and left to the reader. $\square$

Theorem 1 (Soundness and Completeness of the Embedding)
For all formulas $s \in FFOL$ we have $\models^{FFOL} s$ if and only if $\models^{HOL} \hat{s}$.

Proof (Soundness, $\leftarrow$) The proof is by contraposition. Assume $\not\models^{FFOL} s$, i.e, there is a FFOL model M and an assignment g such that $\|s\|^{M,g} = False$. By Lemma 1 we have $\|\hat{s}\|^{\hat{M},\hat{g}} = False$. Hence, we get $\not\models^{HOL} \hat{s}$.

(Completeness, $\to$) Analogous to above by contraposition and Lemma 1.

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[4] In fact, it may be safely assumed that there are no other constant symbols given in a HOL signature, except for the symbols in $\hat{F}$ and $\hat{P}$, the symbols $E_{i\to o}$ and $\star_i$ and the logical connectives.

[5] The fixings introduced in $\hat{M}$ are not in conflict with any of the requirements regarding frames and interpretations. The existence of a valuation function V for an HOL interpretation crucially depends on how sparse the function spaces have been chosen in frame $\{D_\alpha\}_{\alpha \in T}$. Andrews [1] discusses criteria that are sufficient to ensure the existence of a valuation function; in $\hat{M}$ these requirements are met.

The above results enable the employment of any theorem prover that supports HOL with definite description to reason with *FFOL*, including TPTP THF [30] compliant systems such as Satallax, Nitpick, LEO-II and Leo-III. Alternatively, this theory can be encoded in interactive proof assistants, such as Isabelle/HOL. ¡

Properties of *FFOL* could now be experimentally explored with automated reasoning tools for HOL. We have conducted such experiments in an earlier paper [9]. These experiments confirm the illustrating examples discussed in Scott's paper [28].

4 Exploring Axioms Systems for Category Theory

In an experimental theory exploration study, in which we utilized the free logic reasoning farmework from above, we have shown how Scott's [29] axiom system for category theory can be derived from a notion of partial monoids. These axioms systems are presented in Table 1.

The stepwise evolution has been described in detail in[10]. Below we summarize these experiments. However, first we describe some basic modeling decisions for the technical encoding in Isabelle/HOL. The sources of our experiments, as discussed in [10], are available at `http://christoph-benzmueller.de/papers/2017-arXiv.zip`. Note that in these sources we added neither the definite description operator nor the designated undefined object $\star$, since both were not required in this particular application.

Figure 3 displays (parts of) the embedding of *FFOL* in HOL, encoded in Isabelle/HOL, that we have employed in our experiments, as illustrated in Fig. **??**. A related Isabelle/HOL encoding of the above theory with description has been presented and discussed in [9].[6]

4.1 Modeling of basic concepts

Morphisms in the category are modeled as objects in D (respectively, $\boldsymbol{D_i}$). We introduce three partial functions, *dom* (domain), *cod* (codomain), and $\cdot$ (morphism composition). Partiality of composition is handled exactly as expected: we generally may have non-existing compositions $x \cdot y$ (i.e. $\neg(E(x \cdot y))$) for some existing morphisms x and y (i.e. Ex and Ey).

For composition $\cdot$ we assume set-theoretical composition here (i.e., functional composition from right to left). This means that

$$(cod\ x) \cdot (x \cdot (dom\ x)) \cong x$$

and that

$$(x \cdot y)a \cong x(ya) \quad \text{when} \quad dom\ x \simeq cod\ y$$

---

[6] In the remainder of this article, and inline with our text so far, we present the formulas of *FFOL* in non-boldface font. These formulas have been encoded in Isabelle/HOL using the abbreviations as introduced in Fig. 3. In the actual source encoding, however, the usage of boldface and non-boldface is (for technical reasons) reversed.

Axioms Set I

S_i $E(x \cdot y) \longrightarrow (Ex \wedge Ey)$

E_i $E(x \cdot y) \longleftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$

A_i $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

C_i $\forall y. \exists i. Ii \wedge i \cdot y \cong y$

D_i $\forall x. \exists j. Ij \wedge x \cdot j \cong x$

Axioms Set II

S_{ii} $E(x \cdot y) \longrightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \longrightarrow Ex) \wedge (E(cod\ y) \longrightarrow Ey)$

E_{ii} $E(x \cdot y) \longleftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$

A_{ii} $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

C_{ii} $Ey \longrightarrow (I(cod\ y) \wedge (cod\ y) \cdot y \cong y)$

D_{ii} $Ex \longrightarrow (I(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Axioms Set III

S_{iii} $E(x \cdot y) \longrightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \longrightarrow Ex) \wedge (E(cod\ y) \longrightarrow Ey)$

E_{iii} $E(x \cdot y) \longleftarrow (dom\ x \cong cod\ y \wedge E(cod\ y)))$

A_{iii} $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

C_{iii} $Ey \longrightarrow (I(cod\ y) \wedge (cod\ y) \cdot y \cong y)$

D_{iii} $Ex \longrightarrow (I(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Axioms Set IV

S_{iv} $E(x \cdot y) \longrightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \longrightarrow Ex) \wedge (E(cod\ y) \longrightarrow Ey)$

E_{iv} $E(x \cdot y) \longleftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y)))$

A_{iv} $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

C_{iv} $(cod\ y) \cdot y \cong y$

D_{iv} $x \cdot (dom\ x) \cong x$

Axioms Set V (Scott 79, [29])

$S1$ $E(dom\ x) \longrightarrow Ex$

$S2$ $E(cod\ y) \longrightarrow Ey$

$S3$ $E(x \cdot y) \longleftrightarrow dom\ x \simeq cod\ y$

$S4$ $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

$S5$ $(cod\ y) \cdot y \cong y$

$S6$ $x \cdot (dom\ x) \cong x$

Table 1 Stepwise evolution of Scott's [29] axiom system for category theory from partial monoids. The axiom names are motivated as follows: S stands for strictness, E for existence, A for associativity, C for codomain, D for Domain. The free variables x, y, z range over the raw domain D. The quantifiers in Axioms Sets I and II are free logic quantifiers, that is, they range over the domain E of existing objects.

```
  98
  99  typedecl i -- {* Type for individuals *}
 100  consts fExistence:: "i⇒bool" ("E") --{* Existence/definedness predicate in free logic *}
 101
 102  abbreviation fNot ("¬") --{* Free negation *}
 103    where "¬φ ≡ ¬φ"
 104  abbreviation fImplies (infixr "→" 13) --{* Free implication *}
 105    where "φ → ψ ≡ φ ⟶ ψ"
 106  abbreviation fIdentity (infixr "=" 13) --{* Free identity *}
 107    where "l = r ≡ l = r"
 108  abbreviation fForall ("∀") --{* Free universal quantification guarded by existence
 109                                 predicate @{text "E"}*}
 110    where "∀Φ ≡ ∀x. E x ⟶ Φ x"
 111  abbreviation fForallBinder (binder "∀" [8] 9) --{* Binder notation *}
 112    where "∀x. φ x ≡ ∀φ"
 113
 114  text {* Further free logic connectives can now be defined as usual. *}
 115
 116  abbreviation fOr (infixr "∨" 11)
 117    where "φ ∨ ψ ≡ (¬φ) → ψ"
 118  abbreviation fAnd (infixr "∧" 12)
 119    where "φ ∧ ψ ≡ ¬(¬φ ∨ ¬ψ)"
 120  abbreviation fImplied (infixr "←" 13)
 121    where "φ ← ψ ≡ ψ → φ"
 122  abbreviation fEquiv (infixr "↔" 15)
 123    where "φ ↔ ψ ≡ (φ → ψ) ∧ (ψ → φ)"
 124  abbreviation fExists ("∃")
 125    where "∃Φ ≡ ¬(∀(λy. ¬(Φ y)))"
 126  abbreviation fExistsBinder (binder "∃" [8]9)
 127    where "∃x. φ x ≡ ∃φ"
 128
```
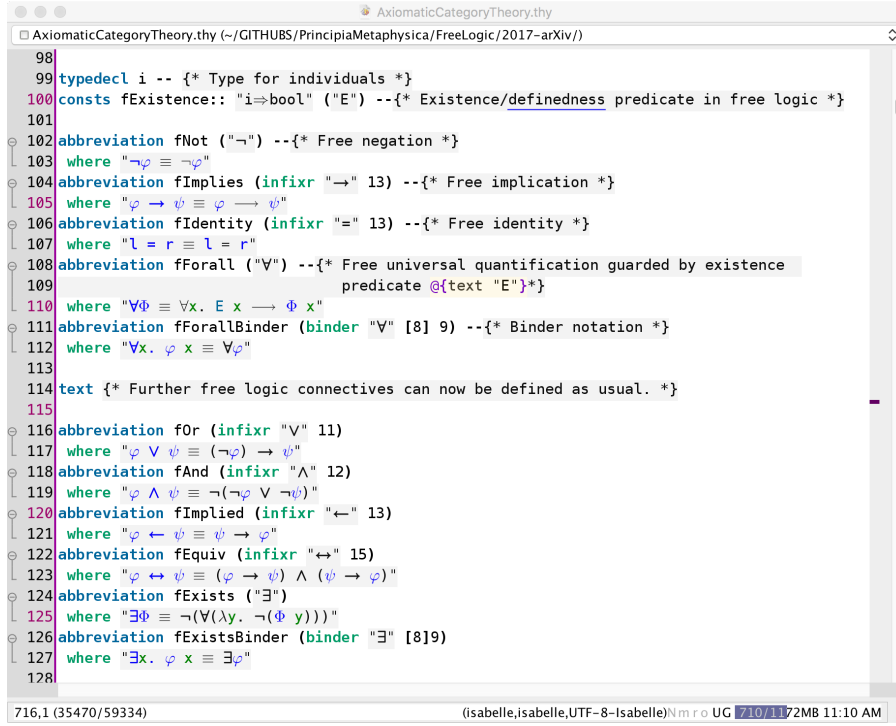
Fig. 2 Isabelle/HOL encoding of *FFOL* (without description)

The equality symbol $\cong$ denotes Kleene equality and it is defined as follows (where $=$ is identity on all objects, existing or non-existing, of type i):

$$x \cong y := (Ex \vee Ey) \longrightarrow x = y$$

Existing identity $\simeq$ is defined as:

$$x \simeq y := Ex \wedge Ey \wedge x = y$$

$\cong$ is an equivalence relation. $\simeq$, in contrast, is only symmetric and transitive, and lacks reflexivity. These observations are quickly confirmed by Sledgehammer in Isabelle.

Next, we define the identity morphism predicate I as follows:

$$Ii := (\forall x. \, E(i \cdot x) \longrightarrow i \cdot x \cong x) \wedge (\forall x. \, E(x \cdot i) \longrightarrow x \cdot i \cong x)$$

This definition was suggested by an exercise in the textbook by Freyd and Scedrov [20] on p. 4. In earlier experiments we used a longer definition which can be proved equivalent on the basis of the other axioms. For monoids, where composition is total, Ii means i is a two-sided identity — and such are unique. For categories the property is much weaker.
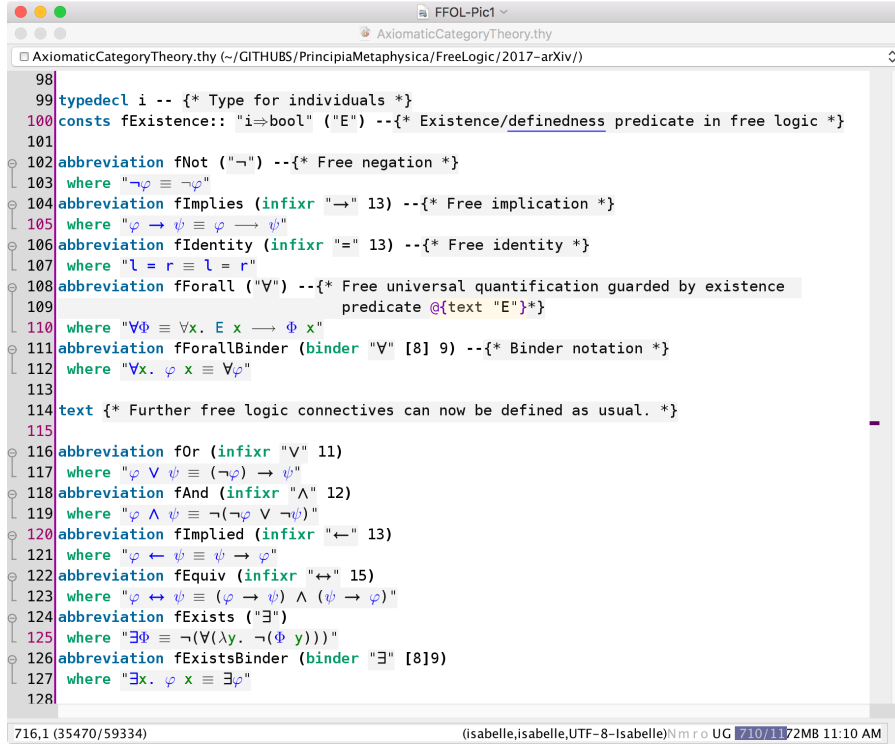
```
 98
 99 typedecl i -- {* Type for individuals *}
100 consts fExistence:: "i⇒bool" ("E") --{* Existence/definedness predicate in free logic *}
101
102 abbreviation fNot ("¬") --{* Free negation *}
103  where "¬φ ≡ ¬φ"
104 abbreviation fImplies (infixr "→" 13) --{* Free implication *}
105  where "φ → ψ ≡ φ ⟶ ψ"
106 abbreviation fIdentity (infixr "=" 13) --{* Free identity *}
107  where "l = r ≡ l = r"
108 abbreviation fForall ("∀") --{* Free universal quantification guarded by existence
109                                  predicate @{text "E"}*}
110  where "∀Φ ≡ ∀x. E x ⟶ Φ x"
111 abbreviation fForallBinder (binder "∀" [8] 9) --{* Binder notation *}
112  where "∀x. φ x ≡ ∀φ"
113
114 text {* Further free logic connectives can now be defined as usual. *}
115
116 abbreviation fOr (infixr "∨" 11)
117  where "φ ∨ ψ ≡ (¬φ) → ψ"
118 abbreviation fAnd (infixr "∧" 12)
119  where "φ ∧ ψ ≡ ¬(¬φ ∨ ¬ψ)"
120 abbreviation fImplied (infixr "←" 13)
121  where "φ ← ψ ≡ ψ → φ"
122 abbreviation fEquiv (infixr "↔" 15)
123  where "φ ↔ ψ ≡ (φ → ψ) ∧ (ψ → φ)"
124 abbreviation fExists ("∃")
125  where "∃Φ ≡ ¬(∀(λy. ¬(Φ y)))"
126 abbreviation fExistsBinder (binder "∃" [8]9)
127  where "∃x. φ x ≡ ∃φ"
128
```

Fig. 3 Encoding of Axioms Set V in Isabelle/HOL using the semantically embedded logic *FFOL*

4.2 Consistency

The model finder Nitpick confirms consistency for all of the axioms sets from Table 1. For example, when asked to consider at least one defined and one undefined object, then Nitpick generates for all cases the following model (called M_1 in the remainder): $D = \{i_i, i_2\}$ and $E = \{i_1\}$; $i_1 \cdot i_1$ is i_1, and i_2 in all other cases; *cod* and *dom* are identity on D. Without constraining the request, Nitpick generates an even simpler model (called M_0 in the remainder): $D = \{i_i\}$ and $E = \emptyset$; $i_1 \cdot i_1$ is i_1; *cod* and *dom* are identity on D. It is trivial to check that these models indeed confirm the consistency of all axioms sets from Table 1.

4.3 Axioms Sets I and II

Axioms Set I is our most basic set of axioms for category theory generalizing the axioms for a monoid to a partial composition operation. Remember that a monoid is an algebraic structure $(S, \circ)$, where $\circ$ is a binary operator on set

S, satisfying the following properties:

Closure:	$\forall a, b \in S\text{.}\ a \circ b \in S$
Associativity:	$\forall a, b, c \in S\text{.}\ a \circ (b \circ c) = (a \circ b) \circ c$
Identity:	$\exists id_S \in S\text{.} \forall a \in S\text{.}\ id_S \circ a = a = a \circ id_S$

That is, a monoid is a semigroup with a two-sided identity element.

Axioms Set I generalizes the notion of a monoid by introducing a partial, strict binary composition operation $\cdot$. The existence of left and right identity elements is addressed in the last two axioms. The notions of dom (Domain) and cod (Codomain) abstract from their common meaning in the context of sets. In category theory we work with just a single type of objects (the type $\boldsymbol{i}$ in our setting) and therefore identity morphisms are employed to suitably characterize their meanings.

We can prove that the i in axiom C_i and the j in axiom D_i are unique. The proofs and the dependencies can be found automatically by Sledgehammer.

$$\forall y\text{.} \exists i\text{.} Ii \wedge i \cdot y \cong y \wedge (\forall j\text{.} d(Ij \wedge j \cdot y \cong y) \longrightarrow i \cong j) \quad \text{(by } A_i, C_i, S_i)$$

$$\forall x\text{.} \exists j\text{.} Ij \wedge x \cdot j \cong x \wedge (\forall i\text{.} (Ii \wedge x \cdot i \cong x) \longrightarrow j \cong i) \quad \text{(by } A_i, D_i, S_i)$$

However, the i and j need not be equal. Using existential variables C and D, this can be encoded in our formalization as follows:

$$\exists C\text{.} \exists D\text{.} (\forall y\text{.} I(Cy) \wedge (Cy) \cdot y \cong y) \wedge (\forall x\text{.} I(Dx) \wedge x \cdot (Dx) \cong x) \wedge D \neq C$$

The model finder Nitpick confirms that this formula is satisfiable: e.g. choose domain $D = \{i_1, i_2\}$ and $E = \{i_2\}$; $i_2 \cdot i_2$ returns i_2, and i_1 in all other cases; variable D is identity on domain D, but C maps both i_1 and i_2 to i_2.

Axioms Set II is developed from Axioms Set I by Skolemization of the existentially quantified variables i and j in axioms C_i and D_i. We can argue semantically that every model of Axioms Set I has such functions. Hence, we get a conservative extension of Axioms Set I. This could be done for any theory with an "$\forall x\text{.} \exists i\text{.}$"-axiom. The strictness axiom S is extended, so that strictness is now also postulated for the new Skolem functions dom and cod. Note that the values of Skolem functions outside E can just be given by the identity function.

The left-to-right direction of existence axiom E_{ii} is implied.

$$E(x \cdot y) \longrightarrow (Ex \wedge Ey \wedge (\exists z\text{.} z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y)) \quad \text{(by } A_{ii}, C_{ii}, S_{ii})$$

Axioms C_{ii} and D_{ii}, together with S_{ii}, show that dom and cod are total functions, as intended:

$$Ex \longrightarrow E(dom\ x) \quad \text{(by } D_{ii}, S_{ii})$$

$$Ex \longrightarrow E(cod\ x) \quad \text{(by } C_{ii}, S_{ii})$$

The proofs are found by Sledgehammer and verified in Isabelle/HOL. Using Sledgehammer we have also shown that Axioms Set II implies Axioms Set I. Vice versa, Axioms Set I also implies Axioms Set II. This can easily be shown by semantical means on the meta-level.

4.4 Remark on the Experiments

All proofs above and all proofs in the rest of this paper (unless stated otherwise) have been obtained fully automatically in very reasonable time (typically just a few seconds) with the Sledgehammer tool in Isabelle/HOL. This tool interfaces to prominent first-order automated theorem provers such as CVC4 [18], Z3 [24], E [27] and Spass [15]. Remotely, also provers such as Vampire [21], or the higher-order provers Satallax [16] and LEO-II [12] can be reached. For example, to prove axiom E_{iii} from Axioms Set II, we have called Sledgehammer on all axioms of Axioms Set II. The provers then, via Sledgehammer, suggested to call trusted/verified tools in Isabelle/HOL with the exactly required dependencies they detected. With the provided dependency information the trusted tools in Isabelle/HOL were then able to reconstruct the external proofs on their own. This way we obtain a verification of our claims in Isabelle/HOL, in which all the proofs have nevertheless been contributed by automated theorem provers. However, there are still a few calls of the SMT-tactic contained in our document, which we found are not entirely trivial to replace. In this sense, our Isabelle/HOL verification is modulo the correctness of the SMT solvers CVC4 [18] and Z3 [24].

4.5 Axioms Set III, IV and V

In Axioms Set III the existence axiom E_{ii} from Axioms Set II is simplified by taking advantage of the two new Skolem functions dom and cod.

The left-to-right direction of existence axiom E_{iii} is implied.

$$E(x \cdot y) \longrightarrow (dom\ x \cong cod\ y \land E(cod\ y)) \quad (\text{by } A_{iii}, C_{iii}, D_{iii}, S_{iii})$$

Axioms Set IV simplifies the axioms C_{iii} and D_{iii}. However, as it turned out, these simplifications also require the existence axiom E_{iii} to be strengthened into an equivalence.

Axioms Set V has been proposed by Scott [29] in the 1970s. This set of axioms is equivalent to the axioms set presented by Freyd and Scedrov in their textbook "Categories, Allegories" [20], when encoded in free logic, corrected/adapted and further simplified. Their axioms set is technically flawed when encoded in our given context. This issue has been detected by automated theorem provers with the same technical infrastructure as employed so far. See §5 for more details.

Axioms Sets II, III, IV and V are equivalent; this has been automatically confirmed by the automated theorem provers and verified in Isabelle/HOL.

5 Assessment of the Axiom System by Freyd and Scedrov

In this section we study the axioms set of Freyd and Scedrov from their textbook "Categories, Allegories" [20]. In §5.1 we show, that their axioms set,

Axioms Set FS-I: Freyd and Scedrov in original notation (with issues)

$A1$ $E(x \circ y) \longleftarrow (x\square \cong \square y)$
$A2a$ $((\square x)\square) \cong \square x$
$A2b$ $\square(x\square) \cong \square x$
$A3a$ $(\square x) \circ x \cong x$
$A3b$ $x \circ (x\square) \cong x$
$A4a$ $\square(x \circ y) \cong \square(x \circ (\square y))$
$A4b$ $(x \circ y)\square \cong ((x\square) \circ y)\square$
$A5$ $x \circ (y \circ z) \cong (x \circ y) \circ z$

Axioms Set FS-II: Freyd and Scedrov in our notation (with issues)

$A1$ $E(x \cdot y) \longleftrightarrow dom\ x \cong cod\ y$
$A2a$ $cod\ (dom\ x) \cong dom\ x$
$A2b$ $dom\ (cod\ y) \cong cod\ y$
$A3a$ $x \cdot (dom\ x) \cong x$
$A3b$ $(cod\ y) \cdot y \cong y$
$A4a$ $dom\ (x \cdot y) \cong dom\ ((dom\ x) \cdot y)$
$A4b$ $cod\ (x \cdot y) \cong cod\ (x \cdot (cod\ y))$
$A5$ $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Axioms Set VI: Freyd and Scedrov in our notation and corrected

$A1'$ $E(x \cdot y) \longleftrightarrow dom\ x \simeq cod\ y$
$A2a$ $cod\ (dom\ x) \cong dom\ x$
$A2b$ $dom\ (cod\ y) \cong cod\ y$
$A3a$ $x \cdot (dom\ x) \cong x$
$A3b$ $(cod\ y) \cdot y \cong y$
$A4a$ $dom\ (x \cdot y) \cong dom\ ((dom\ x) \cdot y)$
$A4b$ $cod\ (x \cdot y) \cong cod\ (x \cdot (cod\ y))$
$A5$ $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Table 2 The axioms set of Freyd and Scedrov in their and our notation, together with a proposed correction.

replicated in Table 2 as Axioms Set FS-I, becomes inconsistent in our free logic setting if we assume non-existing objects in D, respectively, if we assume that the operations are non-total.

Note, however, that the free variables in this first study range over the existing and non-existing objects in D. One may argue, that this is not the intention of Freyd and Scedrov. Therefore, we add a second study in §5.2,

in which we restrict the variables to range only over existing objects in E. However, also in this case the axiom system of Freyd and Scedrov remains unsatisfactory. Now it turns out incomplete, since strictness conditions/axioms are required which are not mentioned in the textbook.

Freyd and Scedrov employ a different notation for $dom\ x$ and $cod\ x$. They denote these operations by $\Box x$ and $x\Box$. Moreover, they employ diagrammatic composition $(f \circ g)x \cong g(fx)$ (functional composition from left to right) instead of the set-theoretic definition $(f \cdot g)x \cong f(gx)$ (functional composition from right to left) used so far. We leave it to the reader to verify that their Axioms Set FS-I corresponds to Axioms Set FS-II modulo an appropriate conversion of notation.[7]

5.1 Constricted Inconsistency in Free Logic Setting

A main difference in the system by Freyd and Scedrov to our Axioms Set V from Table 1 concerns axiom $S3$ respectively $A1$. Namely, instead of the non-reflexive existing identity $\simeq$, they use Kleene equality $\cong$, cf. definition 1.11 on page 3 of their textbook [20].[8] The difference seems minor, but in our free logic setting it has the effect to cause the mentioned constricted inconsistency issue.[9]

The (constricted) inconsistency of Axioms Set FS-I, respectively Axioms Set FS-II, from Table 2 has been detected first by the model finder Nitpick. When we asked Nitpick to generate a model with at least one non-existing object, it claimed that there is no such model. However, a model can still be constructed if we do not make any assumptions about non-existing objects. In fact, the model presented by Nitpick for this case consists of a single, existing morphism.

However, one can see directly that Axiom $A1$ is problematic as written: If x and y are undefined, then (presumably) $dom\ x$ and $cod\ y$ are undefined as well, and by the definition of Kleene equality, $dom\ x \cong cod\ y$. $A1$ stipulates that $x \cdot y$ should be defined in this case, which appears unintended.

As we will demonstrate now, the consequences of this version of the axiom are even stronger. It implies that *all* objects are defined, that is, composition (as well as dom and cod) become total operations. The theory described by these axioms "collapses" to the theory of monoids: If all objects are defined, then one can conclude from $A1$ that $dom\ x \cong dom\ y$ (resp. $dom\ x \cong cod\ y$)

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[7] A recipe for this translation is as follows: (i) replace all $x \circ y$ by $y \cdot x$, (ii) rename the variables to get them again in alphabetical order, (iii) replace $\varphi\Box$ by $cod\ \varphi$ and $\Box\varphi$ by $dom\ \varphi$, and finally (iv) replace $cod\ y \cong dom\ x$ (resp. $cod\ y \simeq dom\ x$) by $dom\ x \cong cod\ y$ (resp. $dom\ x \simeq cod\ y$).

[8] Def. 1.11 in Freyd Scedrov: "The ordinary equality sign $=$ [i.e., our $\cong$] will be used in the symmetric sense, to wit: if either side is defined then so is the other and they are equal. ..."

[9] This could perhaps be an oversight, or it could indicate that Freyd and Scedrov actually mean the Axioms Set discussed in §5.2 below.

and $cod\ x \cong cod\ y$), and according to 1.14 of [20], the category reduces to a monoid provided that it is not empty.

In fact, the automated theorem provers, via Sledgehammer, quickly prove falsity from Axioms Sets FS-II and FS-I when assuming a non-existing object of type i:

$$(\exists x\blacksquare \neg Ex) \longrightarrow False$$

The provers identify the axioms $A1$, $A2a$ and $A3a$ to cause the problem under this assumption. A corresponding human-intuitive proof argument is as follows:

Let $a \in D$ be an undefined object, that is, assume Ea. By instantiating axiom $A3a$ with a we have $a \cdot (dom\ a) \cong a$. From this and definition of $\cong$ we know that $a\cdot(dom\ a)$ is not defined. This is easy to see, since if $a\cdot(dom\ a)$ were defined, we also had that a is defined, which is not the case by assumption. Hence, $\neg E(a \cdot (dom\ a)$. Next, we instantiate $A1$ with a and $dom\ a$ to obtain $E(a \cdot (dom\ a)) \longleftrightarrow dom\ a \cong cod\ (dom\ a)$. Moreover, by instantiating $A2a$ with a we obtain $cod\ (dom\ a) \cong dom\ a$, which we use (modulo symmetry and transitivity of $\cong$) to rewrite the former result into $E(a\cdot(dom\ a)) \longleftrightarrow dom\ a \cong dom\ a$. By reflexivity of $\cong$ we thus get $E(a \cdot (dom\ a))$, i.e. that $a \cdot (dom\ a)$ is defined, which contradicts $\neg E(a \cdot (dom\ a)$. $\square$

As a corollary from the above constricted inconsistency result we get that all morphisms (objects in D) must be defined: $\forall x\blacksquare Ex$.

Obviously Axioms Sets FS-I and FS-II are also redundant, and we have previously reported on respective redundancies [9].[10] For the corrected Axioms Set VI we still get redundancies. The different options to reduce this system are reported in Table 3.

Attempts to remove axioms A1', A3a, A3b, and A5 from Axiom Set VI failed. Nitpick shows that they are independent.

However, when assuming strictness of dom and cod, the axioms A2a, A2b, A4a and A4b are all implied. Hence, under this assumptions, the reasoning tools quickly identify (A1' A3a A3b A5) as a minimal axiom set, which then exactly matches the Axioms Set V of Scott from Table 1.[11]

5.2 Missing Strictness Axioms in Alternative Setting

We study the axiom system by Freyd and Scedrov once again. However, this time we restrict the free variables in their system to range over existing objects only. In the context of algebraic theories, it could be argued that this is the

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[10] The discussion in our releated conference paper [9] was before the discovery of the above constricted inconsistency issue, which tells us that the system (in our setting) can even be reduced to axioms A1, A2a, and A3a (when we assume undefined objects).

[11] This minimal set of axioms has also been mentioned by Freyd in a note [19] and attributed to Martin Knopman. However, the proof sketch presented there seems to fail when the adapted version of A1 (with $\simeq$) is employed.

Freyd and Scedrov in our notation (corrected and reduced I)

$A1'$ $E(x \cdot y) \longleftrightarrow dom\ x \simeq cod\ y$
$A3a$ $x \cdot (dom\ x) \cong x$
$A3b$ $(cod\ y) \cdot y \cong y$
$A4a$ $dom\ (x \cdot y) \cong dom\ ((dom\ x) \cdot y)$
$A4b$ $cod\ (x \cdot y) \cong cod\ (x \cdot (cod\ y))$
$A5$ $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Freyd and Scedrov in our notation (corrected and reduced II)

$A1'$ $E(x \cdot y) \longleftrightarrow dom\ x \simeq cod\ y$
$A2a$ $cod\ (dom\ x) \cong dom\ x$
$A2b$ $dom\ (cod\ y) \cong cod\ y$
$A3a$ $x \cdot (dom\ x) \cong x$
$A3b$ $(cod\ y) \cdot y \cong y$
$A5$ $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Freyd and Scedrov in our notation (corrected and reduced III)

S_v^1 $E(dom\ x) \longrightarrow Ex$
S_v^2 $E(cod\ y) \longrightarrow Ey$
$A1'$ $E(x \cdot y) \longleftrightarrow dom\ x \simeq cod\ y$
$A3a$ $x \cdot (dom\ x) \cong x$
$A3b$ $(cod\ y) \cdot y \cong y$
$A5$ $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Table 3 Reduced variants of Axioms Set VI.

preferred reading of free variables. By employing the free logic universal quantifier $\forall$, which realizes such a restriction, we thus modify Axioms Set FS-II into Axioms-Set FS-III as displayed in Table 4.

For Axioms Set FS-III the consistency checks with Nitpick succeeds, even if we assume undefined objects. However, this axioms set is obviously weaker than Axioms Set V from Table 1. In fact, as has been shown by Nitpick, none of the axioms of this set are implied. The situation changes when we explicitly postulate strictness of dom, cod and $\cdot$. Doing so we obtain Axioms Set FS-IV from Table 4, which, as Nitpick confirms, is consistent even if we assume undefined objects. And the automated theorem provers via Sledgehammer confirm that Axioms Set FS-IV is equivalent to Axioms Set V, as intended.

Axioms Set FS-III: Freyd and Scedrov in our notation (with issues)

$B1$ $\forall x. \forall y. E(x \cdot y) \longleftrightarrow dom\ x \cong cod\ y$
$B2a$ $\forall x. cod\ (dom\ x) \cong dom\ x$
$B2b$ $\forall y. dom\ (cod\ y) \cong cod\ y$
$B3a$ $\forall x. x \cdot (dom\ x) \cong x$
$B3b$ $\forall y. (cod\ y) \cdot y \cong y$
$B4a$ $\forall x. \forall y. dom\ (x \cdot y) \cong dom\ ((dom\ x) \cdot y)$
$B4b$ $\forall x. \forall y. cod\ (x \cdot y) \cong cod\ (x \cdot (cod\ y))$
$B5$ $\forall x. \forall y. \forall z. x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Axioms Set FS-IV: Freyd and Scedrov in our notation (without issues)

$B0a$ $E(x \cdot y) \longrightarrow (Ex \wedge Ey)$
$B0b$ $E(dom\ x) \longrightarrow Ex$
$B0b$ $E(cod\ x) \longrightarrow Ex$
$B1$ $\forall x. \forall y. E(x \cdot y) \longleftrightarrow dom\ x \cong cod\ y$
$B2a$ $\forall x. cod\ (dom\ x) \cong dom\ x$
$B2b$ $\forall y. dom\ (cod\ y) \cong cod\ y$
$B3a$ $\forall x. x \cdot (dom\ x) \cong x$
$B3b$ $\forall y. (cod\ y) \cdot y \cong y$
$B4a$ $\forall x. \forall y. dom\ (x \cdot y) \cong dom\ ((dom\ x) \cdot y)$
$B4b$ $\forall x. \forall y. cod\ (x \cdot y) \cong cod\ (x \cdot (cod\ y))$
$B5$ $\forall x. \forall y. \forall z. x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Table 4 The axioms set of Freyd and Scedrov in our notation and with variable restriction to existing objects only.

Unfortunately, however, respective strictness conditions are not mentioned in the textbook by Freyd and Scedrov.

6 Summary and Further Work

We have developed a new reasoning framework for free logic, and we have experimentally applied it for some first experiments in category theory. We have demonstrated how modern proof assistants and theorem provers for classical higher-order logic may well support the reasoning in free logic. More concretely, we have applied our new free logic reasoning framework for the systematic exploration of axiom systems for category theory. Without tools, support of such experiments would be extremely tedious and error prone. In the course of our experiments, automated theorem provers have revealed some (minor) issue in the textbook of Freyd and Scedrov [20], which we were able to

correct. The correction essentially corresponds to the axiom system by Scott proposed earlier [28]. All our findings were achieved directly by or in close interaction with automated reasoning tools.

Comparisons with other theorem provers for free logic are not possible at this stage, since we are not aware of any other existing systems.

Further work includes the continuation of our formalization studies in category theory and the application of free logic to various other mathematical domains, including, for example, projective geometry.

Moreover, abstract level proof tactics could be provided in proof assistants to support — as an alternative to always unfolding the mapping from *FFOL* to HOL — intuitive interaction (and even automation) in *FFOL* on top the semantical embedding.

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