ABSTRACT AND CONCRETE CATEGORIES
THE JOY OF CATS

Dedicated to Bernhard Banaschewski

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3.51 REMARK

(1) Clearly each category is a quasicategory.

(2) **CAT** is a proper quasicategory in the sense that it is not a category. [Notice that hom(**Set**, **Set**) is not a set.]

(3) Virtually every categorical concept has a natural analogue or interpretation for quasicategories. The names for such quasicategorical concepts will be the same as those of their categorical analogues. Thus we have, for example, the notions of functor between quasicategories, equivalence of quasicategories, discrete and thin quasicategories, etc. Because the main object of our study is categories, most notions will only be specifically formulated for categories. However, we will freely make use of implied quasicategorical analogues, especially when it allows clearer or more convenient expression. For example, at this point it is clear that an isomorphism between categories (3.24) is precisely the same as an isomorphism in **CAT** (3.8). Not every categorical concept has a reasonable quasicategorical interpretation. An outstanding example of this is the fact that quasicategories in general lack hom-functors into **Set**.

(4) Dealing with quasicategories and forming **CAT** gives us the possibility of applying category theory to itself. There are advantages to doing this (some of which are indicated above) as well as certain dangers. One danger is the tendency to want to form something like the "quasicategory of all quasicategories". However, to do so causes a Russell-like paradox that cannot be salvaged within our foundational system, as outlined in §2. Because our main interest is in categories, as opposed to quasicategories, we will never need to consider such an entity as the "quasicategory of all quasicategories".

OBJECT-FREE DEFINITION OF CATEGORIES

Because of the bijection between the class of objects and the class of identity morphisms in any category (given by $A \mapsto id_A$) and the fact that identities in a category can be characterized by their behavior with respect to composition, it is possible to obtain an "object-free" definition of category. This definition, given below, is formally simpler than the original one and is "essentially" equivalent to it (3.55). The reason for choosing the definition given in 3.1 is that it is more closely associated with standard examples of categories.

3.52 DEFINITION

(1) A **partial binary algebra** is a pair $(X, *)$ consisting of a class X and a partial binary operation $*$ on X; i.e., a binary operation defined on a subclass of $X \times X$. [The value of $*(x, y)$ is denoted by $x * y$.]

(2) If $(X, *)$ is a partial binary algebra, then an element u of X is called a **unit** of $(X, *)$ provided that $x * u = x$ whenever $x * u$ is defined, and $u * y = y$ whenever $u * y$ is defined.

3.53 DEFINITION

An **object-free category** is a partial binary algebra $\mathbf{C} = (M, \circ)$, where the members of M are called **morphisms**, that satisfies the following conditions:

(1) *Matching Condition*: For morphisms f, g, and h, the following conditions are equivalent:

  (a) $g \circ f$ and $h \circ g$ are defined,

  (b) $h \circ (g \circ f)$ is defined, and

  (c) $(h \circ g) \circ f$ is defined.

(2) *Associativity Condition*: If morphisms f, g, and h satisfy the matching conditions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) *Unit Existence Condition*: For every morphism f there exist units u_C and u_D of $(M, \circ)$ such that $u_C \circ f$ and $f \circ u_D$ are defined.

(4) *Smallness Condition*: For any pair of units (u_1, u_2) of $(M, \circ)$ the class $\hom(u_1, u_2) = \{f \in M \mid f \circ u_1 \text{ and } u_2 \circ f \text{ are defined}\}$ is a set.

3.54 PROPOSITION

If $\mathbf{A}$ *is a category, then*

(1) $(Mor(\mathbf{A}), \circ)$ *is an object-free category, and*

(2) an $\mathbf{A}$*-morphism is an* $\mathbf{A}$*-identity if and only if it is a unit of* $(Mor(\mathbf{A}), \circ)$.

Proof: $(Mor(\mathbf{A}), \circ)$ is clearly a partial binary algebra, where $f \circ g$ is defined if and only if the domain of f is the codomain of g. Thus each $\mathbf{A}$-identity is a unit. If $A \xrightarrow{u} B$ is a unit in $(Mor(\mathbf{A}), \circ)$, then $u = u \circ id_A = id_A$, where the first equality holds since id_A is a $\mathbf{A}$-identity and the second one holds since u is a unit. Thus (2) is established. From this, (1) is immediate. □

3.55 REMARK

We now have two versions of the concept of category, the “standard” one (3.1), which is more intuitive and is more easily associated with familiar examples, and the “object-free” one (3.53), which is more succinctly stated and so, in many cases, more convenient to use. Next we will see that these two concepts are equivalent. Proposition 3.54 shows that with every category we can associate an object-free category. Even though this correspondence is neither injective nor surjective, it provides an *equivalence* between the “standard” and the “object-free” definitions of category. This claim can be made precise as follows:

(1) One can define functors between object-free categories to be functions between their classes of morphisms that preserve both units (= identities) and composition.

(2) Parallel to the definition of the quasicategory **CAT** of all categories one can define the quasicategory $\mathbf{CAT}_{of}$ of all object-free categories and functors between them.

(3) The correspondence from Proposition 3.54 is the object part of a functor between the quasicategories **CAT** and $\mathbf{CAT}_{of}$ that can be shown to be an equivalence in the sense of Definition 3.33.

In this sense the two concepts of category are essentially the same; i.e., essentially the same "category theory" will result if one proceeds from either of the two formulations.

EXERCISES

3A. Graphs of Categories

A **graph** is a quadruple (V, E, d, c) consisting of a set V (of vertices), a set E (of (directed) edges), and functions $d, c\colon E \to V$ (giving the domain and codomain of an edge). A **large graph** is the same concept except that V and E are allowed to be classes. The **graph** $G(\mathbf{A})$ **of a category A** is the obvious large graph with $V = Ob(\mathbf{A})$ and $E = Mor(\mathbf{A})$.

(a) Verify that a thin category is determined up to isomorphism by its graph.

(b) Find two non-isomorphic categories with the same graph.

(c) Determine which of the following graphs are of the form $G(\mathbf{A})$ for some category $\mathbf{A}$ (where vertices and identity edges are indicated by dots, and non-identity edges are indicated by arrows).

(1) $\bullet \rightleftarrows \bullet \longrightarrow \bullet$ (2) $\bullet \leftleftarrows \bullet \longrightarrow \bullet$

(3)
$$\begin{tikzcd}
\bullet \arrow[rr, shift left] \arrow[dr] & & \bullet \arrow[ll, shift left] \arrow[dl] \\
& \bullet &
\end{tikzcd}$$

(4)
$$\begin{tikzcd}
\bullet \arrow[r, shift right=2] & \bullet \arrow[l, shift right=2] \arrow[l]
\end{tikzcd}$$

(d) Show that for each of the following graphs G there exists up to isomorphism precisely one category $\mathbf{A}$ with $G(\mathbf{A}) = G$.

(1)
$$\begin{tikzcd}
\bullet \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[loop right]
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[loop above right] \arrow[loop below right]
\end{tikzcd}$$

(e) The **free category** generated by a graph (V, E, d, c) is the category $\mathbf{A}$ with $Ob(\mathbf{A}) = V$, $Mor(\mathbf{A})$ = all paths (= all finite sequences in E in which the domain of each edge is the codomain of the preceding one), composition is the obvious composition of paths, and identity morphisms are the empty paths. Verify that $\mathbf{A}$ is indeed a category.