that for every M'' with $n_1 + n_2$ elements satisfying $M' \subset M'' \subset M$ there is an isomorphism $\phi_2: M'' \to N$ such that ϕ_2 is an extension of ϕ_1^{-1} and $\phi_2(M'') = N'' \supset N'$.

In order to obtain the definition of $N \leqslant (N', n_1, n_2, n_3)\, M$ we additionally require that for every extension N''' of N'' which has $n_1 + n_2 + n_3$ elements there is an isomorphism ϕ_3 of N''' onto a submodel M''' of M such that ϕ_3 is an extension of ϕ_2^{-1}. Obvious continuation of these requirements leads to a full definition of the relation $N \leqslant (N', n_1, \cdots, n_k)\, M$.

If $N \leqslant (N', n_1, \cdots, n_k)\, M$ for arbitrary $N' \subset N$ with n_1 elements, then the author writes $N \leqslant (n_1, \cdots, n_k)\, M$; if $n_1 = n_2 = \cdots = n_k = n$ then we replace in the above the sequence $(n_1, \cdots, n_k)$ by ${}^kn$.

Now let A be the set of first-order sentences in prenex normal form with k alternations of quantifiers and beginning with a general quantifier. The author proves that a class K consists of all models of an axiom in A if and only if the following holds: There is an integer $n > 0$ such that any N belongs to K provided that for every submodel N' of N with at most n elements there is in K a model M satisfying $N \leqslant (N', {}^kn)\, M$.

Let A' be the set of axioms dual to A. A second result established by the author says that K consists of all models of an axiom in A' if and only if there is an integer $n > 0$ such that any M belongs to K' provided that there is N in K such that $N \leqslant ({}^kn)\, M$.

The paper contains other results of similar character but their formulation requires additional symbols. ANDRZEJ MOSTOWSKI

DANA SCOTT. *Existence and description in formal logic.* ***Bertrand Russell, Philosopher of the century, Essays in his honour,*** edited by Ralph Schoenman, Little, Brown and Company, Boston and Toronto, and George Allen & Unwin Ltd, London, 1967, pp. 181–200.

To facilitate comparisons we use Russell's notation for (definite) descriptions throughout this review, rather than either Frege's or Scott's; and we also sometimes otherwise modify Fregean notation and terminology in the direction of notation and terminology which is now more familiar.

In 49*10*, which represents the final form of Frege's logic, the description $(\iota x)\Phi$ is made to mean the same as $\hat{x}\Phi$ ("the class of x's such that Φ") in all cases except one; the one exception is that if $\hat{x}\Phi$ denotes a unit class, then $(\iota x)\Phi$ denotes the single member of the unit class. The parallel in English would be that e.g. "the son of Queen Elizabeth I" denotes the null class, "the author of ***Principia mathematica***" denotes the pair-class consisting of A. N. Whitehead and Bertrand Russell, and "the author of ***Waverley***" denotes the man Sir Walter Scott. The point of Frege's convention is that, although denying neither the possibility nor the naturalness of denotationless names, he held that they are an unnecessary complication in a formalized language and should be avoided. The convention is a natural one but is ill adapted to type theory. As an adaptation to a context in which each variable, such as x, ranges over a restricted domain A (which may be different for different variables) it has often been suggested to select a designated element in each domain; and if $(\iota x)\Phi$ is an *improper description*, i.e., a description which would otherwise be denotationless, it is then taken instead to be a name of the designated element in the range of x. When descriptions are understood in such a sense as this, let us call them Frege descriptions.

Instead of either Frege descriptions or Russell's contextually defined descriptions, Scott prefers to take the denotation of an improper description $(\iota x)\Phi$ to be something outside the range of the bound variable x. This requires selecting for every domain A a "null entity" $*_A$ such that $*_A \notin A$. Then an improper description $(\iota x)\Phi$ denotes $*_A$, where A is the range of x.

To support this, Scott formulates a first-order logic which is in fact a "free description theory" in the sense of Lambert and van Fraassen, providing for it both an axiomatization and a semantics.

Scott's semantics makes use of a domain A, which may be empty, and a non-empty superdomain A_* of A whose members are the members of A, the null entity $*_A$, and possibly others. Bound variables range over A, free variables range over A_*, and improper descriptions denote $*_A$. Remaining details of the semantics are routine. As the language is actually formulated, there is, besides the equality sign, only a single (binary) predicate R, for which a binary relation

R over A_* must therefore be provided as denotation. The generalization to any number of predicates is, however, obvious and we shall assume this.

An oddity which Scott does not mention is that $y = (\imath x) \,.\, x = y$ is not (universally) valid, as may be seen by taking for y a value which is neither $*_A$ nor a member of A. But if α is a description, $\alpha = (\imath x) \,.\, x = \alpha$ is valid, as the value of a description is always either $*_A$ or a member of A.

In the light of this it appears that Scott's description theory is not quite the same as that of Lambert in XXXII 252(3), but it reduces to Lambert's (semantically) by restricting the domain A_* to have no other members than $*_A$ and the members of A. It may be thought a defect that Lambert does not mention the appropriate semantics, but with this limitation he has anticipated Scott's idea about the meaning to be attributed to improper descriptions. Scott mentions a number of other authors as having had ideas similar to his own but overlooks this much closer anticipation by Lambert.

It is evident that Scott's theory and Lambert's, whether taken purely syntactically or taken with the above described semantics, are each a fragment of an ordinary (i.e., non-free) two-sorted logic, but a two-sorted logic so formulated, contrary to usual custom, that one sort contains the other, and the latter sort may be empty. The reviewer supposes that such a fragmentary two-sorted logic, once admitted, deserves filling out to the full two-sorted logic. For if the free variables may take extra values, not allowed to the bound variables, it becomes appropriate to use a different alphabet for the free variables. Then why not quantification with respect to this second sort of variables?

Scott then turns to a proposed revised treatment of Quine's theory of virtual classes. As they appear in Quine's XXXVII 768, virtual classes are logical constructions in the sense of Russell, i.e., notations for them are contextually defined. But Scott, wishing to make a semantical analysis, provides an axiomatization in which the operation of class abstraction is primitive, and in which therefore the abstraction terms are to be regarded as actually denoting virtual classes. The semantics uses a domain A and a binary relation E over A. Free variables range over all subdomains of A, including A itself, while bound variables range over only those special subdomains of A which are determined from some member a of A as $\hat{x} \,.\, xEa$. It is allowed that A may be empty.

Thus there arises a situation in which there is a distinction between "proper" and "improper" individuals, analogous to that in Scott's free description theory as described above.

Now Quine makes use of the abstraction operator to give (what is in terms of it) an explicit definition of the description $(\imath x)\Phi$. This makes $(\imath x)\Phi$ a Frege description in the sense that, when the description is improper, its denotation is still something in the range of the bound variable x. In fact Quine's $(\imath x)\Phi$, when improper, denotes the empty class—which Quine regards as being a "real" class, i.e., a set.

To accord with his own program, Scott amends the definition of Quine in such a way that $(\imath x)\Phi$ comes to denote an "improper" individual when the description is improper. This necessitates a change in the definition of the notation $\phi(\xi)$ ("the value of the function ϕ for the argument ξ"), which Scott also provides.

In a letter quoted by Scott, Quine protests: "The redefinition of description and of function value that you propose sacrifices an advantage that I had gone out of my way for: the freedom to substitute descriptions for bound variables without regard to special existence premisses. . . . Without this freedom the book would be appreciably more laboured. Perhaps you could devise alternative conventions, on your basis, that would work smoothly too; but then, I'd want to see some trial runs for comparison." Scott defends his program against such objection, but agrees that Quine is justified in asking for trial runs for comparison.

In the reviewer's judgment Scott's conventions about descriptions are a minor change in standard conventions which is likely to be useful in some contexts, but whose importance he has on the following grounds greatly overestimated: 1. It is indeed important for some purposes to formulate first- (and higher-) order logic in such a way that the domain of individuals is not assumed to be non-empty; but this is better done by allowing only closed formulas to be asserted than by introducing a second domain to bear the burden of non-emptiness. 2. Pending the trial runs, one must suspend judgment in regard to the merits of the modification of

Quine's theory, but the presumption is in favor of Quine; Scott remarks (p. 181) that "it is quite natural to employ descriptions *before* they have been proved to be proper," and it seems that this natural and useful procedure will generally be facilitated if substitution of $(\iota x)\Phi$ for a bound variable of the same type as x does not depend on a proof that the description is proper. 3. Scott claims (pp. 181–182) that there is a difficulty in the theory of recursive definition which is avoided by using his conventions about descriptions; but the claim is made only in his introductory statement and is not substantiated in the body of the paper; it seems to be almost certainly erroneous.

This last claim requires more detailed examination, as it may have played a key role in convincing Scott of the merits of his approach. He writes at the top of page 182: "In axiomatic set theory in the discussion of recursive definitions, it is very tempting to give an explicit definition of the required function by means of a description and then prove a theorem of the form: *for all a in a set well-ordered by* $<$, *if* $f(x)$ *exists for all* $x < a$, *then* $f(a)$ *exists.* It will then follow by transfinite induction that $f(a)$ exists for all a in the well-ordered set. Only a logician would have objections to this use of 'exists'. It is the purpose of this paper to lay these objections to rest by presenting a formal theory of descriptions that corresponds quite faithfully to such natural modes of reasoning."

In discussing this we may suppose as an example that the set well-ordered by $<$ is the set N of finite ordinals, so that the transfinite recursion reduces to an ordinary recursion. And we assume that Scott's reference is to the well-known difficulty in the theory of definition by recursion which troubled Landau (419½*1*). He does not give details of the "explicit definition by means of a description," and the reviewer can only report that he has tried without success to find another interpretation that would bear out Scott's claim.

We understand functions as classes of ordered pairs in the way that is usual in axiomatic set theory. We assume the notation $\phi(\xi)$, and the notation $\arg\phi$ for "the class of x's such that $\phi(x)$ exists." For Quine, $\phi(\xi)$ is $(\iota y) . \langle y, \xi\rangle \epsilon \phi$ and $\arg\phi$ is $\hat{x}(\exists z)(y) . y = z \equiv \langle y, x\rangle \epsilon \phi$ (where suitable restrictions are to be understood to avoid conflict of variables, in case ϕ has free variables). In Scott's modification of Quine's theory, $\phi(\xi)$ is $(\iota y)(\exists x) . x = \xi . \langle y, x\rangle \epsilon \phi$ and $\arg\phi$ is $\hat{x}(\exists y) . y = \phi(x)$. In what follows we seek to be neutral among these definitions and still other definitions which may be appropriate in other systems of set theory. We define $\phi \operatorname{fn} \nu$ ("ϕ is a function on ν") as $a \epsilon \nu \supset_a a \epsilon \arg\phi . x \epsilon \phi \supset_x (\exists y)(\exists a) . a \epsilon \nu . x = \langle y, a\rangle$.

Suppose a recursive definition $f(0) = \kappa$, $f(x + 1) = \psi(x, f(x))$, where κ is a given finite ordinal and ψ is a given (previously known) function on $N \times N$ whose values belong to N. To show the existence of a function f on N that is determined or "defined" by this recursion, we may begin with an explicit definition of F by the description:

$$(1) \qquad (\iota f) . f \operatorname{fn} N . f(0) = \kappa . x \epsilon N \supset_x f(x + 1) = \psi(x, f(x)).$$

Then as Scott suggests we may seek to prove by induction that $F(a)$ exists for all a in N (that $a \epsilon N \supset_a a \epsilon \arg F$). But the obvious method of doing this is fallacious unless we first show that the description (1) is proper. For if the description were improper, not even $F(0) = \kappa$ would follow; and this objection remains the same whether we use a Russell description, a Frege description, or a Scott description. One usual method of meeting the objection avoids the circularity by first defining F_a as:

$$(2) \qquad (\iota f) . f \operatorname{fn} \hat{x}[x \leqq a] . f(0) = \kappa . x < a \supset_x f(x + 1) = \psi(x, f(x)).$$

Then the existence of F_a for all $a \epsilon N$, in the sense that the description (2) is proper, is proved by mathematical induction with respect to a. Then F is defined as $\hat{u}(\exists a) . a \epsilon N . u \epsilon F_a$, the union of the partial functions F_a. This process is not hindered by the use of Scott descriptions, but neither is it facilitated.

The last two sections of Scott's paper (pp. 193–199) are the most valuable part of it, but because non-controversial they will receive brief treatment in this review. A first-order language is set up which has "a variable binding operator O of the same syntactical category as the operators of description and abstraction." As before, bound variables range over a domain A while free variables range over a non-empty superdomain A_* of A. The values of the terms

$\mathrm{O}x\Phi$ are not necessarily confined to A but may lie also in the superdomain A_*. The operator O obeys the law of extensionality

$$\Phi \equiv_x \Psi \supset \mathrm{O}x\Phi = \mathrm{O}x\Psi$$

and the law of alphabetic change of bound variable, but its meaning is not further determined. An axiomatization of this language is given, and then a completeness proof is provided (briefly). There follow as corollaries, completeness proofs for Scott's free description theory and for his version of Quine's virtual-class theory.

A *theory* T is defined as a set of sentences (of the language just described) which contains all universally valid sentences and is closed under *modus ponens*—where it is understood that a *sentence* has no free variables. A necessary and sufficient model-theoretic condition is found that O shall be eliminable in T. The result is, as Scott remarks, reminiscent of Beth's definability theorem.

The eliminability which results by this condition may mean that, in order to eliminate $\mathrm{O}x\Phi$, we must examine and make use of the internal structure of Φ. If we ask for *uniform* eliminability in the sense that the elimination of $\mathrm{O}x\Phi$ makes no use of the internal structure of Φ, a new problem arises, to find a model-theoretic condition. Scott leaves this as an interesting open problem.

It should be noticed that, while Scott's model-theoretic result is stated as a result about (a certain fragment of) two-sorted logic, it includes a parallel result about one-sorted logic. And the like is true of the open problem. ALONZO CHURCH

ABRAHAM ROBINSON. *Obstructions to arithmetical extension and the theorem of Łoś and Suszko.* ***Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings,*** series A vol. 62 (1959), pp. 489–495; also ***Indagationes mathematicae,*** vol. 21 (1959), pp. 489–495.

MICHAEL O. RABIN. *Diophantine equations and non-standard models of arithmetic.* ***Logic, methodology and philosophy of science, Proceedings of the 1960 International Congress,*** edited by Ernest Nagel, Patrick Suppes, and Alfred Tarski, Stanford University Press, Stanford, Calif., 1962, pp. 151–158. [Cf. XXXVIII 159(1).]

É. GOLOD. Review of Ax and Kochen's *Diophantine problems over local fields I* (XXXVI 683). Russian. ***Référativnyj žurnal, Matématika,*** no. 10A (1966), pp. 31–32.

V. P. KLASSEN. *Inclusion problem for a certain class of groups.* English translation of XXXVII 206. ***Algebra and logic,*** vol. 9 (for 1970, pub. 1971), pp. 183–186.

A. I. MAL'TSEV. *On the history of algebra in the USSR during her first twenty-five years.* English translation of XXXVII 218(8). Ibid., vol. 10 (for 1971, pub. 1972), pp. 68–75.

PH. DWINGER. *On the completeness of the quotient algebras of a complete Boolean algebra. II.* ***Koninklijke Nederlandse Akademie van Wetenschappen, Proceedings,*** series A vol. 62 (1959), pp. 26–35; also ***Indagationes mathematicae,*** vol. 21 (1959), pp. 26–35.

FAWZI M. YAQUB. *A theorem on the existence of the generalized free α-products of Boolean algebras.* ***Nieuw archief voor wiskunde,*** ser. 3 vol. 16 (1968), pp. 179–183.

GIULIO ANDREOLI. *Automorfismi in un algebra di Boole determinati da funzioni algebriche e trascedenti invertibili e gruppo dell'ipercubo.* ***La ricerca,*** vol. 6 no. 2 (1955), pp. 3–9, and no. 3, pp. 3–7.

GIULIO ANDREOLI. *Strutture booleane e topologia combinatorica.* Ibid., ser. 2 vol. 12 (May–Aug. 1961), pp. 1–7.

GIULIO ANDREOLI. *Algebricità delle funzioni booleane.* Ibid., ser. 2 vol. 13 (Jan.–Apr. 1962), pp. 1–6.

GIULIO ANDREOLI and GIULIO COLONNESE. *Estensioni del concetto di circuiti ad interruttori ed algebra relativa.* Ibid., ser. 2 vol. 16 (Jan.–Apr. 1965), pp. 3–25.

GIULIO ANDREOLI. *Una proprietà caratteristica per la soluzione dei sistemi di equazioni booleane e loro discussione.* Ibid., ser. 2 vol. 13 (May–Aug. 1962), pp. 1–9.