Combining Logics in Simple Type Theory

Christoph Benzmüller

Articulate Software, Angwin, CA, USA

CLIMA XI, Lisbon, August 16, 2010

Research funded by
DFG grant BE 2501/6-1

synonyms in this talk

Church's Simple Type Theory

Classical Higher Order Logic (HOL)

HOL (Alonzo Church, 1940)

- simple types $\alpha, \beta ::= \iota | o | \alpha \to \beta$ (opt. further base types)
- HOL defined by

$$
\begin{aligned}
s, t \quad ::= \quad & p_\alpha \mid X_\alpha \\
& \mid (\lambda X_\alpha . s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid (\forall X_\alpha . t_o)_o
\end{aligned}
$$

- HOL is well understood
  - Origin (Church, J.Symb.Log., 1940)
  - Henkin semantics (Henkin, J.Symb.Log., 1950)
    (Andrews, J.Symb.Log., 1971, 1972)
  - Extens./Intens. (BenzmüllerEtAl., J.Symb.Log., 2004)
    (Muskens, J.Symb.Log., 2007)

- simple types $\alpha, \beta ::= \iota | o | \alpha \to \beta$ (opt. further base types)
- HOL defined by

$$
\begin{aligned}
s, t \quad ::= \quad & p_\alpha \mid X_\alpha \\
& \mid (\lambda X_\alpha . s_\beta)_{\alpha \to \beta} \mid (s_{\alpha \to \beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o \to o}\, s_o)_o \mid (s_o \vee_{o \to o \to o} t_o)_o \mid (\Pi_{(\alpha \to o) \to o} (\lambda X_\alpha . t_o))_o
\end{aligned}
$$

- HOL is well understood
  - Origin (Church, J.Symb.Log., 1940)
  - Henkin semantics (Henkin, J.Symb.Log., 1950)
    (Andrews, J.Symb.Log., 1971, 1972)
  - Extens./Intens. (BenzmüllerEtAl., J.Symb.Log., 2004)
    (Muskens, J.Symb.Log., 2007)

HOL (Alonzo Church, 1940)

- simple types $\alpha, \beta ::= \iota | o | \alpha \to \beta$ (opt. further base types)
- HOL defined by

$$
\begin{aligned}
s,t \quad ::= \quad & p_\alpha \mid X_\alpha \\
& \mid (\lambda X_\alpha . s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \\
& \mid (\neg_{o\to o}\, s_o)_o \mid (s_o \vee_{o\to o\to o} t_o)_o \mid (\Pi_{(\alpha\to o)\to o}\,(\lambda X_\alpha . t_o))_o
\end{aligned}
$$

- HOL is well understood
  - Origin (Church, J.Symb.Log., 1940)
  - Henkin semantics (Henkin, J.Symb.Log., 1950)
    (Andrews, J.Symb.Log., 1971, 1972)
  - Extens./Intens. (BenzmüllerEtAl., J.Symb.Log., 2004)
    (Muskens, J.Symb.Log., 2007)

Opinions about HOL:

- HOL is expressive

but ...

- HOL can not be effectively automated
- HOL is a classical logic and not easily compatible with
  - modal logics
  - intuitionistic logic
  - ...
- HOL can not fruitfully serve as a basis for combining logics

Opinions about HOL: Position of this talk

- HOL is expressive and we exploit this here

but ...

- HOL can ~~not~~ be effectively automated (at least partly)
- HOL is a classical logic and ~~not~~ easily compatible with
  - (normal) modal logics
  - intuitionistic logic
  - ...
- HOL can ~~not~~ fruitfully serve as a basis for combining logics
  (interesting application area: multi-agent systems)

... I will give theoretical and practical evidence

Quantified Multimodal Logics (QML) as HOL Fragments

(jww Larry Paulson)

Quantified Multimodal Logics (QML)

- QML defined by

$$\begin{aligned} s, t \quad ::= \quad & P \mid (k\, X^1 \ldots X^n) \\ & \mid \neg\, s \mid s \vee t \\ & \mid \Box_r\, s \\ & \mid \forall^i X_{\blacksquare} s \mid \forall^p P_{\blacksquare} s \end{aligned}$$

- Kripke style semantics
  - notion of (QS5) models: (Fitting, J.Symb.Log., 2005)

    QS5π

(BenzmüllerPaulson, Techn.Report, 2009)

Quantified Multimodal Logics (QML)

- QML defined by

$$\begin{aligned} s, t \quad ::= \quad & P \mid (k\, X^1 \ldots X^n) \\ & \mid \neg\, s \mid s \vee t \\ & \mid \Box_r\, s \\ & \mid \forall^i X_\blacksquare s \mid \forall^p P_\blacksquare s \end{aligned}$$

- Kripke style semantics
  - notion of (QS5) models: (Fitting, J.Symb.Log., 2005)

  $\mathbf{QS5}\pi \longrightarrow \mathbf{QK}\pi$ (correspondence to Henkin models)

  (BenzmüllerPaulson, Techn.Report, 2009)

(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \to o$

QML operators as abbreviations for specific HOL terms

$$\neg = \lambda\phi_{\iota\to o}.\lambda W_\iota.\neg(\phi\, W)$$

$$\vee = \lambda\phi_{\iota\to o}.\lambda\psi_{\iota\to o}.\lambda W_\iota.\phi\, W \vee \psi\, W$$

$$\Box = \lambda R_{\iota\to\iota\to o}.\lambda\phi_{\iota\to o}.\lambda W_\iota.\forall V_\iota.\neg(R\, W\, V) \vee \phi\, V$$

$$(\forall^i) \quad \Pi^\mu = \lambda\tau.\lambda W.\forall X.(\tau\, X)\, W$$

$$(\forall^p) \quad \Pi^{\iota\to o} = \lambda\tau.\lambda W.\forall P.(\tau\, P)\, W$$

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

QML operators as abbreviations for specific HOL terms

$$\neg = \lambda \phi_{\iota\to o} . \lambda W_\iota . \neg(\phi\, W)$$

$$\vee = \lambda \phi_{\iota\to o} . \lambda \psi_{\iota\to o} . \lambda W_\iota . \phi\, W \vee \psi\, W$$

$$\Box = \lambda R_{\iota\to\iota\to o} . \lambda \phi_{\iota\to o} . \lambda W_\iota . \forall V_\iota . \neg(R\, W\, V) \vee \phi\, V$$

$$(\forall^i) \quad \Pi^\mu = \lambda \tau_{\mu\to(\iota\to o)} . \lambda W_\iota . \forall X_\mu . (\tau\, X)\, W$$

$$(\forall^p) \quad \Pi^{\iota\to o} = \lambda \tau_{(\iota\to o)\to(\iota\to o)} . \lambda W_\iota . \forall P_{\iota\to o} . (\tau\, P)\, W$$

(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

QML operators as abbreviations for specific HOL terms

$$\neg = \lambda \phi_{\iota\rightarrow o}. \lambda W_{\iota}. \neg(\phi\, W)$$

$$\vee\ \phi = \lambda \psi_{\iota\rightarrow o}. \lambda W_{\iota}. \phi\, W \vee \psi\, W$$

$$\Box = \lambda R_{\iota\rightarrow\iota\rightarrow o}. \lambda \phi_{\iota\rightarrow o}. \lambda W_{\iota}. \forall V_{\iota}. \neg(R\, W\, V) \vee \phi\, V$$

$$(\forall^i) \qquad \mathbf{\Pi}^{\mu} = \lambda \tau_{\mu\rightarrow(\iota\rightarrow o)}. \lambda W_{\iota}. \forall X_{\mu}. (\tau\, X)\, W$$

$$(\forall^p) \qquad \mathbf{\Pi}^{\iota\rightarrow o} = \lambda \tau_{(\iota\rightarrow o)\rightarrow(\iota\rightarrow o)}. \lambda W_{\iota}. \forall P_{\iota\rightarrow o}. (\tau\, P)\, W$$

(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \to o$

QML operators as abbreviations for specific HOL terms

$$\begin{aligned}
\neg &= \lambda \phi_{\iota\to o}.\, \lambda W_\iota.\, \neg(\phi\, W) \\
\vee\ \phi\ \psi &= \lambda W_\iota.\, \phi\, W \vee \psi\, W \\
\Box &= \lambda R_{\iota\to\iota\to o}.\, \lambda \phi_{\iota\to o}.\, \lambda W_\iota.\, \forall V_\iota.\, \neg(R\ W\ V) \vee \phi\ V \\
(\forall^i) \qquad \Pi^\mu &= \lambda \tau_{\mu\to(\iota\to o)}.\, \lambda W_\iota.\, \forall X_\mu.\, (\tau\, X)\, W \\
(\forall^p) \qquad \Pi^{\iota\to o} &= \lambda \tau_{(\iota\to o)\to(\iota\to o)}.\, \lambda W_\iota.\, \forall P_{\iota\to o}.\, (\tau\, P)\, W
\end{aligned}$$

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

QML operators as abbreviations for specific HOL terms

$$\neg = \lambda \phi_{\iota\rightarrow o} \lambda W_{\iota} \neg(\phi\, W)$$

$$(\vee\, \phi\, \psi)\, W = \phi\, W \vee \psi\, W$$

$$\Box = \lambda R_{\iota\rightarrow\iota\rightarrow o} \lambda \phi_{\iota\rightarrow o} \lambda W_{\iota} \forall V_{\iota} \neg(R\, W\, V) \vee \phi\, V$$

$$(\forall^i) \qquad \Pi^{\mu} = \lambda \tau_{\mu\rightarrow(\iota\rightarrow o)} \lambda W_{\iota} \forall X_{\mu} (\tau\, X)\, W$$

$$(\forall^p) \qquad \Pi^{\iota\rightarrow o} = \lambda \tau_{(\iota\rightarrow o)\rightarrow(\iota\rightarrow o)} \lambda W_{\iota} \forall P_{\iota\rightarrow o} (\tau\, P)\, W$$

(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

QML operators as abbreviations for specific HOL terms

$$
\begin{array}{lrl}
 & \neg & = \lambda \phi_{\iota\to o} \lambda W_{\iota} \neg(\phi\, W) \\
 & \vee & = \lambda \phi_{\iota\to o} \lambda \psi_{\iota\to o} \lambda W_{\iota} \phi\, W \vee \psi\, W \\
 & \Box & = \lambda R_{\iota\to\iota\to o} \lambda \phi_{\iota\to o} \lambda W_{\iota} \forall V_{\iota} \neg(R\, W\, V) \vee \phi\, V \\
(\forall^{i}) & \Pi^{\mu} & = \lambda \tau_{\mu\to(\iota\to o)} \lambda W_{\iota} \forall X_{\mu} (\tau\, X)\, W \\
(\forall^{p}) & \Pi^{\iota\to o} & = \lambda \tau_{(\iota\to o)\to(\iota\to o)} \lambda W_{\iota} \forall P_{\iota\to o} (\tau\, P)\, W
\end{array}
$$

(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

QML operators as abbreviations for specific HOL terms

$$\begin{aligned}
\neg &= \lambda \phi_{\iota\rightarrow o} . \lambda W_{\iota} . \neg(\phi\, W) \\
\vee &= \lambda \phi_{\iota\rightarrow o} . \lambda \psi_{\iota\rightarrow o} . \lambda W_{\iota} . \phi\, W \vee \psi\, W \\
\Box_R &= \lambda \phi_{\iota\rightarrow o} . \lambda W_{\iota} . \forall V_{\iota} . \neg(R\, W\, V) \vee \phi\, V \\
(\forall^i) \quad \Pi^{\mu} &= \lambda \tau_{\mu\rightarrow(\iota\rightarrow o)} . \lambda W_{\iota} . \forall X_{\mu} . (\tau\, X)\, W \\
(\forall^p) \quad \Pi^{\iota\rightarrow o} &= \lambda \tau_{(\iota\rightarrow o)\rightarrow(\iota\rightarrow o)} . \lambda W_{\iota} . \forall P_{\iota\rightarrow o} . (\tau\, P)\, W
\end{aligned}$$

(Normal) QML as Fragment of HOL

— related, but significantly extending (Ohlbach, 1988/93) —

Straightforward encoding

- base type ι: non-empty set of possible worlds
- base type μ: non-empty set of individuals

QML formulas $\longrightarrow$ HOL terms of type $\iota \rightarrow o$

QML operators as abbreviations for specific HOL terms

$$\neg = \lambda \phi_{\iota\rightarrow o} . \lambda W_\iota . \neg(\phi\, W)$$

$$\vee = \lambda \phi_{\iota\rightarrow o} . \lambda \psi_{\iota\rightarrow o} . \lambda W_\iota . \phi\, W \vee \psi\, W$$

$$\Box = \lambda R_{\iota\rightarrow\iota\rightarrow o} . \lambda \phi_{\iota\rightarrow o} . \lambda W_\iota . \forall V_\iota . \neg(R\, W\, V) \vee \phi\, V$$

$$(\forall^i) \quad \Pi^\mu = \lambda \tau_{\mu\rightarrow(\iota\rightarrow o)} . \lambda W_\iota . \forall X_\mu . (\tau\, X)\, W$$

$$(\forall^p) \quad \Pi^{\iota\rightarrow o} = \lambda \tau_{(\iota\rightarrow o)\rightarrow(\iota\rightarrow o)} . \lambda W_\iota . \forall P_{\iota\rightarrow o} . (\tau\, P)\, W$$

Encoding of validity

$$\text{valid} \quad = \quad \lambda\phi_{\iota\to o}\blacksquare\forall W_{\iota}\blacksquare\phi\, W$$

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \, \exists^p P_{\iota\to o} \, P$$

then automatically rewrite abbreviations

$$\Box_r \xrightarrow{\text{rewrite}} \ldots$$

$$\exists^p \xrightarrow{\text{rewrite}} \ldots$$

$$\text{valid} \xrightarrow{\text{rewrite}} \ldots$$

$$\xrightarrow{\beta\eta\downarrow} \forall W_\iota \, \forall Y_\iota \, \neg r \, W \, Y \vee (\neg \forall P_{\iota\to o} \, \neg (P \, Y))$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ...
here the provers need to generate witness term $P = \lambda Y_\iota \, \top$)

Example: In all r-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \exists^p P_{\iota\to o} \centerdot P$$

then automatically rewrite abbreviations

$$\begin{array}{rcl} \Box_r & \xrightarrow{\text{rewrite}} & \ldots \\ \exists^p & \xrightarrow{\text{rewrite}} & \ldots \\ \text{valid} & \xrightarrow{\text{rewrite}} & \ldots \\ & \xrightarrow{\beta\eta\downarrow} & \forall W_\iota \centerdot \forall Y_\iota \centerdot \neg r\ W\ Y \vee (\neg \forall P_{\iota\to o} \centerdot \neg (P\ Y)) \end{array}$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ...

here the provers need to generate witness term $P = \lambda Y_\iota \centerdot \top$)

Example: In all r-accessible worlds exists truth

Formulate problem in HOL using original QML syntax

$$\text{valid } \Box_r \exists^p P_{\iota\to o}. P$$

then automatically rewrite abbreviations

$$\begin{array}{rcl} \Box_r & \xrightarrow{\text{rewrite}} & \ldots \\ \exists^p & \xrightarrow{\text{rewrite}} & \ldots \\ \text{valid} & \xrightarrow{\text{rewrite}} & \ldots \\ & \xrightarrow{\beta\eta\downarrow} & \forall W_\iota. \forall Y_\iota. \neg r\ W\ Y \vee (\neg \forall P_{\iota\to o}. \neg (P\ Y)) \end{array}$$

and prove automatically (LEO-II, IsabelleP, TPS, Satallax, ...
here the provers need to generate witness term $P = \lambda Y_\iota. \top$)

Soundness and Completeness Theorem:

$$\models^{QML}_{\mathbf{QK}\pi} s \quad \text{if and only if} \quad \models^{HOL}_{Henkin} \text{valid } s_{\iota\to o}$$

(BenzmüllerPaulson, Techn.Report, 2009)

Soundness and Completeness Theorem for Propositional Multimodal Logic

(BenzmüllerPaulson, Log.J.IGPL, 2010)

Further interesting Fragments of HOL

- Intuitionistic Logic
  (exploiting Gödel's translation to S4)
  (BenzmüllerPaulson, Log.J.IGPL, 2010)
- Access Control Logics
  (exploiting a translation by Garg and Abadi)
  (Benzmüller, IFIP SEC, 2009)
- Region Connection Calculus — later in this talk
- ...

Reasoning *about* Combinations of Logics

Correspondences between properties of accessibility relations like

$$\text{symmetric} = \lambda R.\forall S, T. R\ S\ T \Rightarrow R\ T\ S$$
$$\text{serial} = \lambda R.\forall S.\exists T. R\ S\ T$$

and corresponding axioms

$$\forall R.\text{symmetric}\ R \overset{0.0s}{\Leftarrow}$$
$$\overset{0.0s}{\Rightarrow} \text{valid}\ \forall^p \phi. \phi \supset \Box_R \Diamond_R \phi \qquad (B)$$

$$\forall R.\text{serial}\ R \overset{0.0s}{\Leftarrow}$$
$$\overset{0.0s}{\Rightarrow} \text{valid}\ \forall^p \phi. \Box_R \phi \supset \Diamond_R \phi \qquad (D)$$

Such proofs — including axioms D, M, 4, B, 5 — can be automated with LEO-II in no-time!

Reasoning *about* Combinations of Logics: Correspondence

Correspondences between properties of accessibility relations like

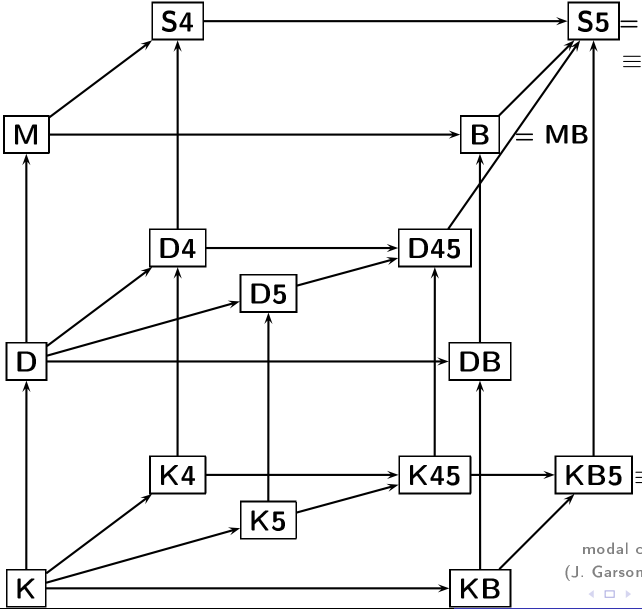
$$\text{symmetric} = \lambda R.\forall S, T.R\,S\,T \Rightarrow R\,T\,S$$
$$\text{serial} = \lambda R.\forall S.\exists T.R\,S\,T$$

and corresponding axioms

$$\forall R.\text{symmetric}\,R \overset{0.0s}{\Leftarrow}$$
$$\overset{0.0s}{\Rightarrow} \quad \text{valid}\ \forall^p \phi.\phi \supset \Box_R \Diamond_R \phi \qquad (B)$$

$$\forall R.\text{serial}\,R \overset{0.0s}{\Leftarrow}$$
$$\overset{0.0s}{\Rightarrow} \quad \text{valid}\ \forall^p \phi.\Box_R \phi \supset \Diamond_R \phi \qquad (D)$$

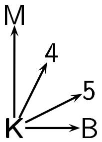
Such proofs — including axioms D, M, 4, B, 5 — can be automated with LEO-II in no-time!



S5 ≡ M5 ≡ MB5 ≡ M4B5 ≡ M45 ≡ M4B ≡ D4B ≡ D4B5 ≡ DB5

B ≡ MB

KB5 ≡ K4B5 ≡ K4B



modal cube reproduced from [?]
(J. Garson, Modal Logic, SEP 2009)

$$\forall R.$$

$$\left.\begin{array}{ll} & \text{valid } \forall^p \phi.\, \Box_R\, \phi \supset \phi \\ \wedge & \text{valid } \forall^p \phi.\, \Diamond_R\, \phi \supset \Box_R \Diamond_R\, \phi \end{array}\right\} M5$$

$$\Leftrightarrow$$

$$\left.\begin{array}{ll} & \text{valid } \forall^p \phi.\, \Box_R\, \phi \supset \Diamond_R\, \phi \\ \wedge & \text{valid } \forall^p \phi.\, \Box_R\, \phi \supset \Box_R \Box_R\, \phi \\ \wedge & \text{valid } \forall^p \phi.\, \phi \supset \Box_R \Diamond_R\, \phi \end{array}\right\} D4B$$

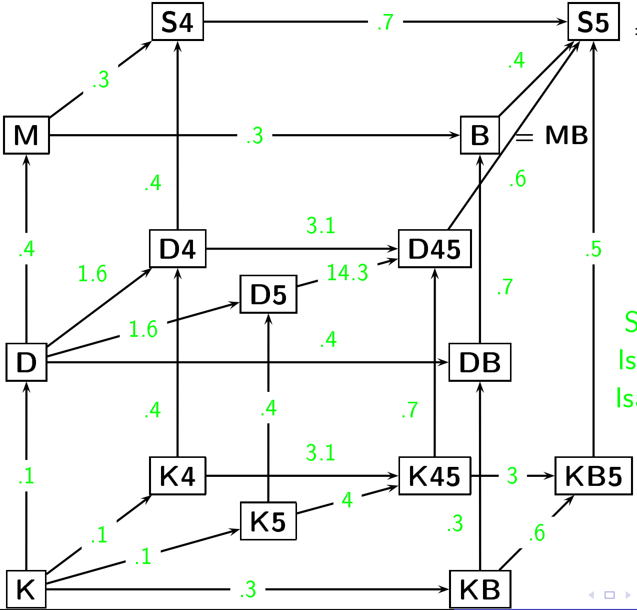
$$
\begin{array}{l}
\forall R_{\blacksquare} \\
\quad \left.\begin{array}{ll}
 & \text{valid } \forall^p \phi_{\blacksquare} \Box_R \phi \supset \phi \\
\wedge & \text{valid } \forall^p \phi_{\blacksquare} \Diamond_R \phi \supset \Box_R \Diamond_R \phi
\end{array}\right\} M5 \\
\Leftrightarrow \\
\quad \left.\begin{array}{ll}
 & \text{serial } R \\
\wedge & \text{valid } \forall^p \phi_{\blacksquare} \Box_R \phi \supset \Box_R \Box_R \phi \\
\wedge & \text{symmetric } R
\end{array}\right\} D4B
\end{array}
$$

$$
\forall R. \;
\left.
\begin{array}{ll}
 & \text{reflexive } R \\
\wedge & \text{euclidean } R
\end{array}
\right\} M5
\quad \Leftrightarrow \quad
\left.
\begin{array}{ll}
 & \text{serial } R \\
\wedge & \text{transitive } R \\
\wedge & \text{symmetric } R
\end{array}
\right\} D4B
$$

$$\forall R.\; \underbrace{\text{reflexive } R \;\wedge\; \text{euclidean } R}_{M5} \;\overset{0.1s}{\Leftrightarrow}\; \underbrace{\text{serial } R \;\wedge\; \text{transitive } R \;\wedge\; \text{symmetric } R}_{D4B}$$

Proof with LEO-II in 0.1s

Reasoning *about* Combinations of Logics: Cube Verification



$= \mathbf{M5} \overset{.1}{\equiv} \mathbf{MB5} \overset{.2}{\equiv} \mathbf{M4B5}$
$\overset{.2}{\equiv} \mathbf{M45} \overset{.1}{\equiv} \mathbf{M4B} \overset{.1}{\equiv} \mathbf{D4B}$
$\overset{.2}{\equiv} \mathbf{D4B5} \overset{.1}{\equiv} \mathbf{DB5}$

<u>fastest results by:</u>

LEO-II (prover)
Satallax (prover)
TPS (prover)
Satallax (mod.find.)
IsabelleN (mod.find.)
IsabelleM (mod.find.)

$\overset{.2}{\equiv} \mathbf{K4B5} \overset{.1}{\equiv} \mathbf{K4B}$

$\sum <$ 40sec.

Reasoning *about* Combinations of Logics: Segerberg

(Segerberg, 1973) discusses a 2-dimensional logic providing two S5 modalities $\Box_a$ and $\Box_b$. He adds further axioms stating that these modalities are commutative and orthogonal. It actually turns out that orthogonality is already implied in this context.

$$\begin{array}{ll} & \text{reflexive } a, \text{transitive } a, \text{euclid. } a, \\ & \text{reflexive } b, \text{transitive } b, \text{euclid. } b, \\ & \text{valid } \forall\phi.\ \Box_a \Box_b \phi \Leftrightarrow \Box_b \Box_a \phi \\ \vdash^{HOL} & \\ & \text{valid } \forall\phi,\psi.\ \Box_a (\Box_a \phi \vee \Box_b \psi) \supset (\Box_a \phi \vee \Box_a \psi) \\ & \wedge \\ & \text{valid } \forall\phi,\psi.\ \Box_b (\Box_a \phi \vee \Box_b \psi) \supset (\Box_b \phi \vee \Box_b \psi) \end{array}$$

(Segerberg, 1973) discusses a 2-dimensional logic providing two S5 modalities $\Box_a$ and $\Box_b$. He adds further axioms stating that these modalities are commutative and orthogonal. It actually turns out that orthogonality is already implied in this context.

$$\vdash^{HOL} \begin{array}{l} \text{reflexive } a, \text{transitive } a, \text{euclid. } a, \\ \text{reflexive } b, \text{transitive } b, \text{euclid. } b, \\ \text{valid } \forall \phi.\ \Box_a \Box_b \phi \Leftrightarrow \Box_b \Box_a \phi \\ \qquad\qquad\qquad\qquad\qquad \text{proof by LEO-II in 0.2s} \\ \text{valid } \forall \phi, \psi.\ \Box_a (\Box_a \phi \vee \Box_b \psi) \supset (\Box_a \phi \vee \Box_a \psi) \\ \wedge \\ \text{valid } \forall \phi, \psi.\ \Box_b (\Box_a \phi \vee \Box_b \psi) \supset (\Box_b \phi \vee \Box_b \psi) \end{array}$$

Reasoning *within* Combined Logics

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapted from (Baldoni, PhD, 1998))

- epistemic modalities:

  $\Box_a$, $\Box_b$, $\Box_c$: three wise men

  $\Box_{fool}$: common knowledge

- predicate constant:

  ws: 'has white spot'

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapted from (Baldoni, PhD, 1998))

- common knowledge:
  at least one of the wise men has a white spot

$$\text{valid } \Box_{\text{fool}} (ws\ a) \vee (ws\ b) \vee (ws\ c)$$

  if X one has a white spot then Y can see this

$$(\text{valid } \Box_{\text{fool}} (ws\ X) \Rightarrow \Box_Y (ws\ X))$$

  if X has not a white spot then Y can see this

$$\text{valid } \Box_{\text{fool}} \neg (ws\ X) \Rightarrow \Box_Y \neg (ws\ X))$$

$$X \neq Y \in \{a, b, c\}$$

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapted from (Baldoni, PhD, 1998))

- if X knows ϕ then Y knows this

$$\text{valid}\,\forall^p\phi.\,(\Box_X\,\phi \Rightarrow \Box_Y\,\Box_X\,\phi)$$

- if X does not know ϕ then Y knows this

$$\text{valid}\,\forall^p\phi.\,(\neg\,\Box_X\,\phi \Rightarrow \Box_Y\,\neg\,\Box_X\,\phi)$$

$$X \neq Y \in \{a, b, c\}$$

- axioms for common knowledge

$$\text{valid}\,\forall^p\phi.\,\Box_{\text{fool}}\,\phi \Rightarrow \phi \qquad (M)$$

$$\text{valid}\,\forall^p\phi.\,\Box_{\text{fool}}\,\phi \Rightarrow \Box_{\text{fool}}\,\Box_{\text{fool}}\,\phi \qquad (4)$$

$$\forall R.\,\text{valid}\,\forall^p\phi.\,\Box_{\text{fool}}\,\phi \Rightarrow \Box_R\,\phi$$

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapted from (Baldoni, PhD, 1998))

- a, b do not know that they have a white spot

$$\text{valid}\, \neg\, \Box_a\, (ws\ a) \qquad \text{valid}\, \neg\, \Box_b\, (ws\ b)$$

- prove that c does know he has a white spot:

$$\ldots \vdash^{HOL} \text{valid}\, \Box_c\, (ws\ c)$$

Wise Men Puzzle

Once upon a time, a king wanted to find the wisest out of his three wisest men. He arranged them in a circle and told them that he would put a white or a black spot on their foreheads and that one of the three spots would certainly be white. The three wise men could see and hear each other but, of course, they could not see their faces reflected anywhere. The king, then, asked to each of them to find out the color of his own spot. After a while, the wisest correctly answered that his spot was white.

(adapted from (Baldoni, PhD, 1998))

- a, b do not know that they have a white spot

$$\text{valid}\,\neg\,\Box_a\,(ws\ a) \qquad \text{valid}\,\neg\,\Box_b\,(ws\ b)$$

- prove that c does know he has a white spot:

$$\ldots \vdash^{HOL} \text{valid}\,\Box_c\,(ws\ c)$$

LEO-II can prove this result in 0.4s

Region Connection Calculus (RCC) (RandellCuiCohn, 1992)
as fragment of HOL:

disconnected :	*dc*	$= \lambda X_\tau . \lambda Y_\tau . \neg(c\ X\ Y)$
part of :	*p*	$= \lambda X_\tau . \lambda Y_\tau . \forall Z . ((c\ Z\ X) \Rightarrow (c\ Z\ Y))$
identical with :	*eq*	$= \lambda X_\tau . \lambda Y_\tau . ((p\ X\ Y) \wedge (p\ Y\ X))$
overlaps :	*o*	$= \lambda X_\tau . \lambda Y_\tau . \exists Z . ((p\ Z\ X) \wedge (p\ Z\ Y))$
partially o :	*po*	$= \lambda X_\tau . \lambda Y_\tau . ((o\ X\ Y) \wedge \neg(p\ X\ Y) \wedge \neg(p\ Y\ X))$
ext. connected :	*ec*	$= \lambda X_\tau . \lambda Y_\tau . ((c\ X\ Y) \wedge \neg(o\ X\ Y))$
proper part :	*pp*	$= \lambda X_\tau . \lambda Y_\tau . ((p\ X\ Y) \wedge \neg(p\ Y\ X))$
tangential pp :	*tpp*	$= \lambda X_\tau . \lambda Y_\tau . ((pp\ X\ Y) \wedge \exists Z . ((ec\ Z\ X) \wedge (ec\ Z\ Y)))$
nontang. pp :	*ntpp*	$= \lambda X_\tau . \lambda Y_\tau . ((pp\ X\ Y) \wedge \neg\exists Z . ((ec\ Z\ X) \wedge (ec\ Z\ Y)))$

Reasoning *within* Combined Logics: Epistemic & Spatial

A trivial problem for RCC:

Catalunya is a border region of Spain	(*tpp* catalunya spain),
Spain and France share a border	(*ec* spain france),
Paris is a region inside France	(*ntpp* paris france)
	$\vdash^{\text{HOL}}$
Catalunya and Paris are disconnected	(*dc* catalunya paris)
	$\wedge$
Spain and Paris are disconnected	(*dc* spain paris)

Reasoning *within* Combined Logics: Epistemic & Spatial

A trivial problem for RCC:

Catalunya is a border region of Spain	(tpp catalunya spain),
Spain and France share a border	(ec spain france),
Paris is a region inside France	($ntpp$ paris france)
	$\vdash^{\mathsf{HOL}}_{2.3s}$
Catalunya and Paris are disconnected	(dc catalunya paris)
	$\wedge$
Spain and Paris are disconnected	(dc spain paris)

Reasoning *within* Combined Logics: Epistemic & Spatial

$$\begin{array}{ll}
& \text{valid } \forall\phi\text{.}\ \Box_{\text{fool}}\,\phi \supset \Box_{\text{bob}}\,\phi, \\
& \text{valid } \Box_{\text{fool}}\,(\lambda W\text{.}(\mathit{ec}\ \text{spain france})), \\
& \text{valid } \Box_{\text{bob}}\,(\lambda W\text{.}(\mathit{tpp}\ \text{catalunya spain})), \\
& \text{valid } \Box_{\text{bob}}\,(\lambda W\text{.}(\mathit{ntpp}\ \text{paris france})) \\
\vdash^{HOL} & \text{valid } \Box_{\text{bob}}\,(\lambda W\text{.}((\mathit{dc}\ \text{catalunya paris}) \wedge (\mathit{dc}\ \text{spain paris})))
\end{array}$$

Reasoning *within* Combined Logics: Epistemic & Spatial

$$\begin{array}{ll}
& \text{valid } \forall\phi\text{.}\ \Box_{\text{fool}}\ \phi \supset \Box_{\text{bob}}\ \phi, \\
& \text{valid } \Box_{\text{fool}}\ (\lambda W\text{.}(\mathit{ec}\ \text{spain france})), \\
& \text{valid } \Box_{\text{bob}}\ (\lambda W\text{.}(\mathit{tpp}\ \text{catalunya spain})), \\
& \text{valid } \Box_{\text{bob}}\ (\lambda W\text{.}(\mathit{ntpp}\ \text{paris france})) \\
\vdash^{\mathit{HOL}}_{20.4s} & \text{valid } \Box_{\text{bob}}\ (\lambda W\text{.}((\mathit{dc}\ \text{catalunya paris}) \wedge (\mathit{dc}\ \text{spain paris})))
\end{array}$$

Reasoning *within* Combined Logics: Epistemic & Spatial

$$\text{valid } \forall \phi_{\blacksquare}\ \Box_{\text{fool}}\, \phi \supset \Box_{\text{bob}}\, \phi,$$

$$\text{valid } \Box_{\text{fool}}\, (\lambda W_{\blacksquare} (ec \text{ spain france})),$$

$$\text{valid } \Box_{\text{bob}}\, (\lambda W_{\blacksquare} (tpp \text{ catalunya spain})),$$

$$\text{valid } \Box_{\text{bob}}\, (\lambda W_{\blacksquare} (ntpp \text{ paris france}))$$

$$\vdash^{HOL}_{20.4s} \quad \text{valid } \Box_{\text{bob}}\, (\lambda W_{\blacksquare} ((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))$$

$$\not\vdash^{HOL} \quad \text{valid } \Box_{\text{fool}}\, (\lambda W_{\blacksquare} ((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))$$

Reasoning *within* Combined Logics: Epistemic & Spatial

$$\text{valid } \forall \phi. \ \Box_{\text{fool}}\ \phi \ \supset \ \Box_{\text{bob}}\ \phi,$$
$$\text{valid } \Box_{\text{fool}}\ (\lambda W. (ec \text{ spain france})),$$
$$\text{valid } \Box_{\text{bob}}\ (\lambda W. (tpp \text{ catalunya spain})),$$
$$\text{valid } \Box_{\text{bob}}\ (\lambda W. (ntpp \text{ paris france}))$$
$$\vdash^{HOL}_{20.4s} \quad \text{valid } \Box_{\text{bob}}\ (\lambda W. ((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))$$
$$\nvdash^{HOL}_{39.7s} \quad \text{valid } \Box_{\text{fool}}\ (\lambda W. ((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))$$

Reasoning *within* Combined Logics: Epistemic & Spatial

$$\begin{array}{ll}
& \text{valid } \forall \phi.\ \Box_{\text{fool}}\, \phi \supset \Box_{\text{bob}}\, \phi, \\
& \text{valid } \Box_{\text{fool}}\, (\lambda W.(ec \text{ spain france})), \\
& \text{valid } \Box_{\text{bob}}\, (\lambda W.(tpp \text{ catalunya spain})), \\
& \text{valid } \Box_{\text{bob}}\, (\lambda W.(ntpp \text{ paris france})) \\
\vdash^{HOL}_{20.4s} & \text{valid } \Box_{\text{bob}}\, (\lambda W.((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris}))) \\
\not\vdash^{HOL}_{39.7s} & \text{valid } \Box_{\text{fool}}\, (\lambda W.((dc \text{ catalunya paris}) \wedge (dc \text{ spain paris})))
\end{array}$$

Key idea is "Lifting" of RCC propositions to modal predicates:

$$\underbrace{(tpp \text{ catalunya spain})}_{\text{type } o} \longrightarrow \underbrace{(\lambda W.(tpp \text{ catalunya spain}))}_{\text{type } \iota \to o}$$

Conclusion

- HOL seems well suited as framework for combining logics
- automation of object-/meta-level reasoning — scalability?
- embeddings can possibly be fully verified in Isabelle/HOL?
  (consistency of QML embedding: 3.8s – IsabelleN)
- current work: application to ontology reasoning (SUMO)

You can use this framework right away! Try it!

- new TPTP infrastructure for automated HOL reasoning
  (SutcliffeBenzmüller, J.Formalized Reasoning, 2010)
  - standardized input / output language (THF)
  - problem library: 3000 problems
  - yearly CASC competitions
- provers and examples are online; demo: `http://tptp.org`
  Wise Men Puzzle:

  `http://www.cs.miami.edu/~tptp/cgi-bin/SeeTPTP?Category=Problems&Domain=PUZ&File=PUZ087^1.p`

Current Work

Application to Ontology Reasoning

- possible worlds semantics for SUMO ontology
- mapping of modal operators in SUMO to appropriate modal logic operators
- logic combinations
- automation with LEO-II (and other THF0 reasoners)

$\longrightarrow$ see my presentation ARCOE-10 (tomorrow)

SUMO ontology and Sigma ontology engineering tool

$\longrightarrow$ two more presentations at IKBET-10 (tomorrow) and ARCOE-10 (today)

LEO-II

(EPRSC grant EP/D070511/1 at Cambridge University)

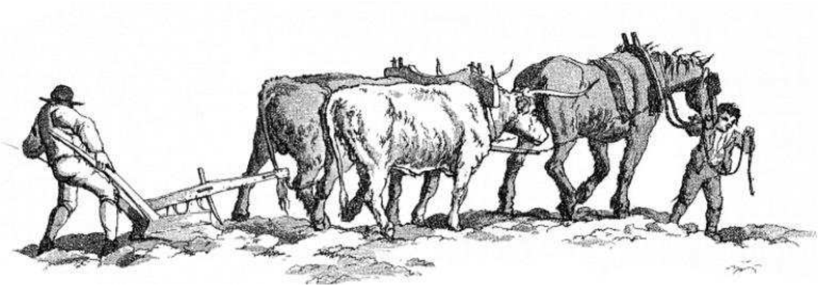
Thanks to Larry Paulson

LEO-II

An Effective Higher-Order Theorem Prover

UNIVERSITY OF CAMBRIDGE

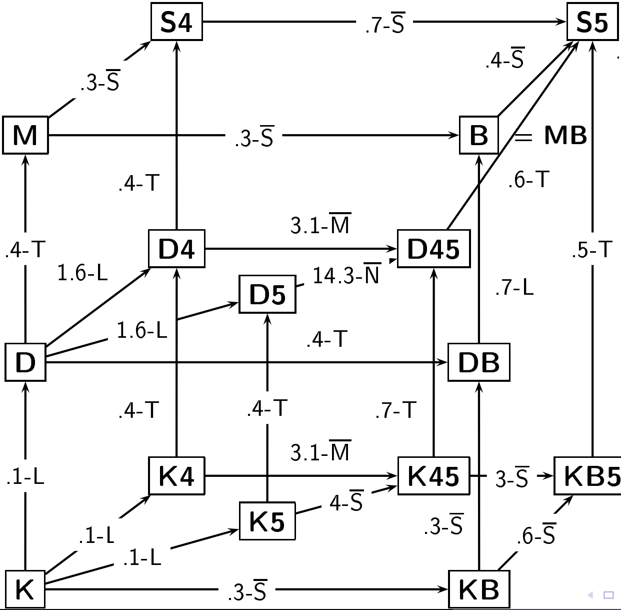
UNIVERSITÄT DES SAARLANDES



LEO-II employs FO-ATPs: E, Spass, Vampire

http://www.ags.uni-sb.de/~leo

Reasoning *about* Combinations of Logics: Cube Verification



$= \text{M5} \overset{.1\text{-L}}{\equiv} \text{MB5} \overset{.2\text{-L}}{\equiv} \text{M4B5} \overset{.2\text{-L}}{\equiv} \text{M45} \overset{.1\text{-L}}{\equiv} \text{M4B} \overset{.1\text{-L}}{\equiv} \text{D4B} \overset{.2\text{-L}}{\equiv} \text{D4B5} \overset{.1\text{-L}}{\equiv} \text{DB5}$

$= \text{MB}$

best proving time by

n-L: LEO-II

n-S: Satallax

n-T: TPS

n-$\overline{S}$: Satallax

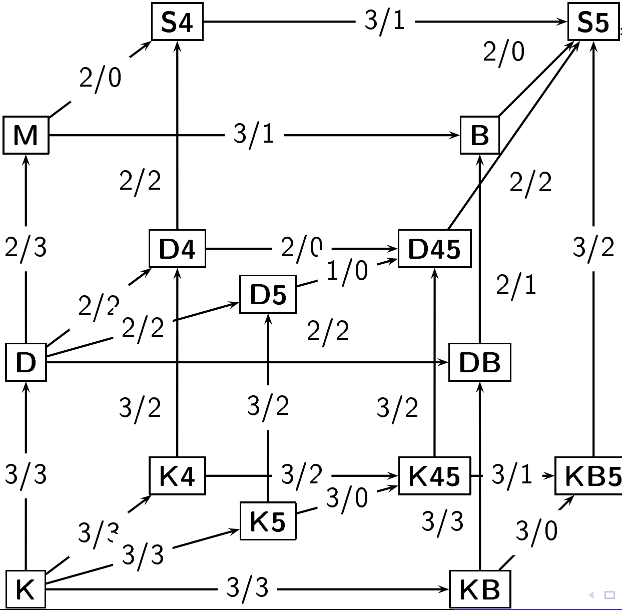
n-$\overline{N}$: IsabelleN

n-$\overline{M}$: IsabelleM

$\overset{.2\text{-L}}{\equiv} \text{K4B5} \overset{.1\text{-L}}{\equiv} \text{K4B}$

$\sum < 40\text{sec.}$

Reasoning *about* Combinations of Logics: Cube Verification



$= \mathbf{M5} \overset{4}{\equiv} \mathbf{MB5} \overset{4}{\equiv} \mathbf{M4B5} \overset{4}{\equiv} \mathbf{M45} \overset{4}{\equiv} \mathbf{M4B} \overset{3}{\equiv} \mathbf{D4B} \overset{2}{\equiv} \mathbf{D4B5} \overset{3}{\equiv} \mathbf{DB5}$

$\overset{4}{\equiv} \mathbf{K4B5} \overset{4}{\equiv} \mathbf{K4B}$