LEO-II - A Cooperative Automatic Theorem Prover for Higher-Order Logic

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[1]Thanks to EPSRC grant EP/D070511/1 at Cambridge University

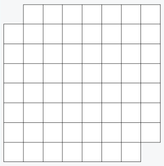
Overview

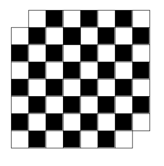
- Motivation and Project Hypothesis
- LEO-II's Architecture
- Reasoning within and about Multimodal Logics
- Conclusion

Motivation and Project Hypothesis

Representation (and the right System Architecture) Matters!

A general lesson in AI . . .

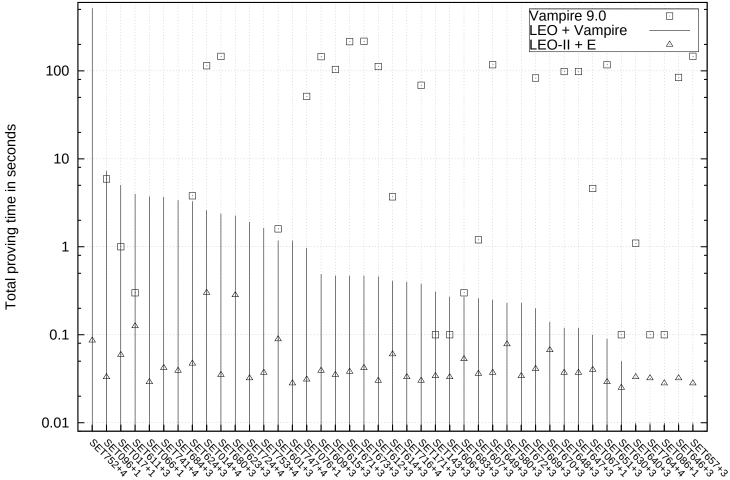




. . . and a specific research hypothesis here

FOL	HOL
+	+
FO-ATP	LEO-II + FO-ATP

Reasoning about Sets, Relations, and Functions



LEO-II's Architecture

A Cooperative Prover

LEO-II

An Effective Higher-Order Theorem Prover

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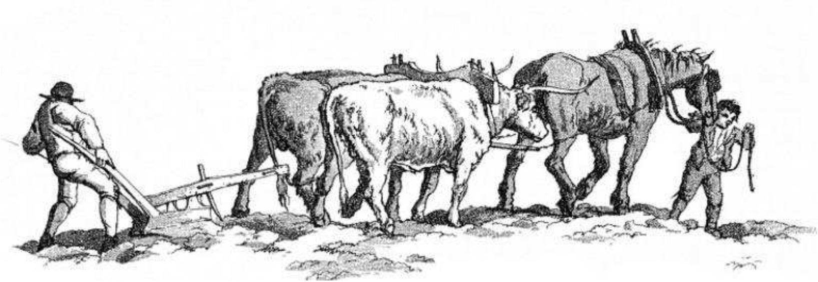
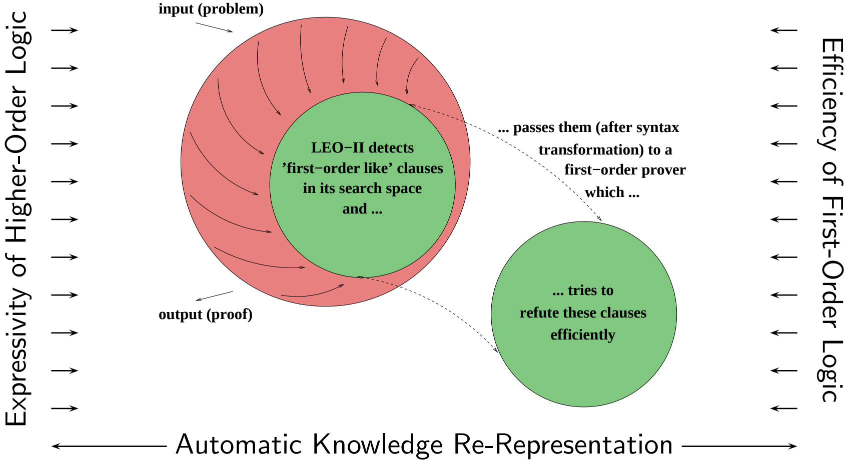


LEO-II employs FO-ATPs: E, Spass, Vampire

Architecture of LEO-II



An Illustrating Example

$$(p\ (\lambda X_{\iota\to\iota}.((q\ X) \Rightarrow (R\ X))))$$

$$\neg(p\ (\lambda Y_{\iota\to\iota}.(\neg(q\ Y) \vee (r\ Y))))$$

An Illustrating Example

$$(\mathsf{p}\ (\lambda X_{\iota\to\iota}.((\mathsf{q}\ X) \Rightarrow (R\ X))))$$

$$\neg(\mathsf{p}\ (\lambda Y_{\iota\to\iota}.(\neg(\mathsf{q}\ Y) \vee (\mathsf{r}\ Y))))$$

- resolution:

$$(\mathsf{p}\ (\lambda X_{\iota\to\iota}.((\mathsf{q}\ X) \Rightarrow (R\ X)))) \neq (\mathsf{p}\ (\lambda Y_{\iota\to\iota}.(\neg(\mathsf{q}\ Y) \vee (\mathsf{r}\ Y))))$$

An Illustrating Example

$$(p\ (\lambda X_{\iota\to\iota}.((q\ X) \Rightarrow (R\ X))))$$
$$\neg(p\ (\lambda Y_{\iota\to\iota}.(\neg(q\ Y) \vee (r\ Y))))$$

- resolution:

$$(p\ (\lambda X_{\iota\to\iota}.((q\ X) \Rightarrow (R\ X)))) \neq (p\ (\lambda Y_{\iota\to\iota}.(\neg(q\ Y) \vee (r\ Y))))$$

- decomposition:

$$(\lambda X_{\iota\to\iota}.((q\ X) \Rightarrow (R\ X))) \neq (\lambda Y_{\iota\to\iota}.(\neg(q\ Y) \vee (r\ Y)))$$

An Illustrating Example

$$(p\ (\lambda X_{\iota\rightarrow\iota}.((q\ X) \Rightarrow (R\ X))))$$

$$\neg(p\ (\lambda Y_{\iota\rightarrow\iota}.(\neg(q\ Y) \vee (r\ Y))))$$

- resolution:

$$(p\ (\lambda X_{\iota\rightarrow\iota}.((q\ X) \Rightarrow (R\ X)))) \neq (p\ (\lambda Y_{\iota\rightarrow\iota}.(\neg(q\ Y) \vee (r\ Y))))$$

- decomposition:

$$(\lambda X_{\iota\rightarrow\iota}.((q\ X) \Rightarrow (R\ X))) \neq (\lambda Y_{\iota\rightarrow\iota}.(\neg(q\ Y) \vee (r\ Y)))$$

- functional and Boolean extensionality:

$$\neg\forall Z_{\iota\rightarrow\iota}.(((q\ Z) \Rightarrow (R\ Z)) \Leftrightarrow (\neg(q\ Z) \vee (r\ Z)))$$

An Illustrating Example

$$(\mathsf{p}\ (\lambda X_{\iota\to\iota}.((\mathsf{q}\ X) \Rightarrow (R\ X))))$$

$$\neg(\mathsf{p}\ (\lambda Y_{\iota\to\iota}.(\neg(\mathsf{q}\ Y) \vee (\mathsf{r}\ Y))))$$

- clause normalisation

$$\neg(\mathsf{q}\ \mathsf{s}_{\iota\to\iota}) \vee (R\ \mathsf{s}_{\iota\to\iota})$$

$$(\mathsf{q}\ \mathsf{s}_{\iota\to\iota}) \qquad \neg(\mathsf{r}\ \mathsf{s}_{\iota\to\iota})$$

An Illustrating Example

$$(p\ (\lambda X_{\iota\to\iota}.((q\ X) \Rightarrow (R\ X))))$$

$$\neg(p\ (\lambda Y_{\iota\to\iota}.(\neg(q\ Y) \vee (r\ Y))))$$

- clause normalisation

$$\neg(q\ s_{\iota\to\iota}) \vee (R\ s_{\iota\to\iota})$$

$$(q\ s_{\iota\to\iota}) \qquad \neg(r\ s_{\iota\to\iota})$$

- mapping to first-order

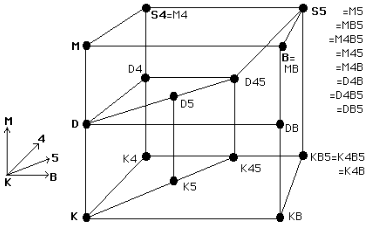
$$\neg @_{((\iota\to\iota)\to o)\_(\iota\to\iota)}(q, s) \vee @_{((\iota\to\iota)\to o)\_(\iota\to\iota)}(R, s)$$

$$@_{((\iota\to\iota)\to o)\_(\iota\to\iota)}(q, s) \qquad \neg @_{((\iota\to\iota)\to o)\_(\iota\to\iota)}(r, s)$$

Reasoning within and about Multimodal Logics

Logic Systems Interrelationships

Modal Logics Challenge

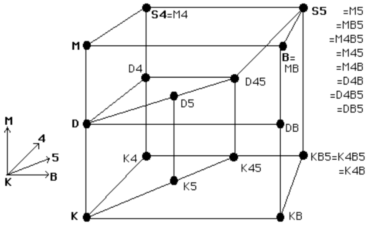


John Halleck (U Utah):
`http://www.cc.utah.edu/~nahaj/`

$100 Modal Logic Challenge:
`www.tptp.org`

Logic Systems Interrelationships

Modal Logics Challenge



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Example

$$
\begin{aligned}
S4 &= K \\
&+ \; M: \quad \Box_R A \Rightarrow A \\
&+ \; 4: \quad \Box_R A \Rightarrow \Box_R \Box_R A
\end{aligned}
$$

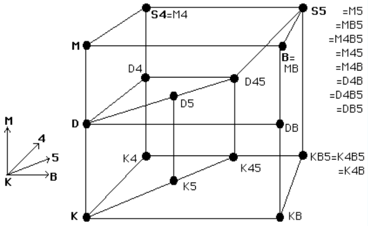
Theorems:

$$S4 \not\subseteq K \tag{1}$$

$$(M \wedge 4) \Leftrightarrow (\mathit{refl}.(R) \wedge \mathit{trans}.(R)) \tag{2}$$

Logic Systems Interrelationships

Modal Logics Challenge



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Example

$$\begin{array}{ccll} S4 & = & K & \\ & + & M: & \Box_R A \Rightarrow A \\ & + & 4: & \Box_R A \Rightarrow \Box_R \Box_R A \end{array}$$

Theorems:

$$\begin{array}{cccr} S4 & \not\subseteq & K & (1) \\ (M \wedge 4) & \Leftrightarrow & (\mathit{refl}.(R) \wedge \mathit{trans}.(R)) & (2) \end{array}$$

Experiments

	FO-ATPs **[SutcliffeEtal-07]**	LEO-II + E
(1)	16min + 2710s	17.3s
(2)	???	2.4s

(Normal) Multimodal Logic in HOL – on one slide –

Simple, Straightforward Encoding of Multimodal Logic

- base type ι: set of possible worlds
  terms of type $\iota \to o$: multimodal logic formulas
- multimodal logic operators:

$$
\begin{aligned}
\neg_{(\iota\to o)\to(\iota\to o)} &:= \lambda A_{\iota\to o}.(\lambda x_\iota.\neg A(x)) \\
\vee_{(\iota\to o)\to(\iota\to o)\to(\iota\to o)} &:= \lambda A_{\iota\to o}, B_{\iota\to o}.(\lambda x_\iota.A(x) \vee B(x)) \\
\Box_{R\,(\iota\to\iota\to o)\to(\iota\to o)\to(\iota\to o)} &:= \lambda R_{\iota\to\iota\to o}, A_{\iota\to o}. \\
&\qquad (\lambda x_\iota.\forall y_\iota.R(x,y) \Rightarrow A(y))
\end{aligned}
$$

Encoding of Validity

$$\mathtt{valid} := \lambda A_{\iota\to o}.(\forall w_\iota.A(w))$$

Even simpler: Reasoning within Multimodal Logics

Problem	LEO-II + E
$\mathtt{valid}(\Box_r \top)$	0.025s
$\mathtt{valid}(\Box_r a \Rightarrow \Box_r a)$	0.026s
$\mathtt{valid}(\Box_r a \Rightarrow \Box_s a)$	–
$\mathtt{valid}(\Box_s (\Box_r a \Rightarrow \Box_r a))$	0.026s
$\mathtt{valid}(\Box_r (a \wedge b) \Leftrightarrow (\Box_r a \wedge \Box_r b))$	0.044s
$\mathtt{valid}(\Diamond_r (a \Rightarrow b) \Rightarrow \Box_r a \Rightarrow \Diamond_r b)$	0.030s
$\mathtt{valid}(\neg \Diamond_r a \Rightarrow \Box_r (a \Rightarrow b))$	0.029s
$\mathtt{valid}(\Box_r b \Rightarrow \Box_r (a \Rightarrow b))$	0.026s
$\mathtt{valid}((\Diamond_r a \Rightarrow \Box_r b) \Rightarrow \Box_r (a \Rightarrow b))$	0.027s
$\mathtt{valid}((\Diamond_r a \Rightarrow \Box_r b) \Rightarrow (\Box_r a \Rightarrow \Box_r b))$	0.029s
$\mathtt{valid}((\Diamond_r a \Rightarrow \Box_r b) \Rightarrow (\Diamond_r a \Rightarrow \Diamond_r b))$	0.030s

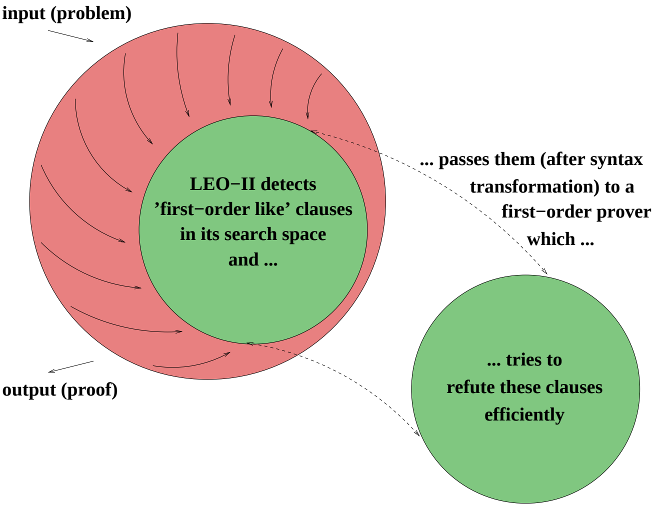
Example (2)

In modal logic **K**, the axioms M and 4 are equivalent to reflexivity and transitivity of the accessibility relation R

$$\forall R. (\forall A. \mathtt{valid}(\Box_R A \Rightarrow A) \wedge \mathtt{valid}(\Box_R A \Rightarrow \Box_R \Box_R A)) \\ \Leftrightarrow (\mathtt{reflexive}(R) \wedge \mathtt{transitive}(R))$$

- 70 clauses are passed to E
- E generates **21769** clauses before finding the empty clause
- total proving time 2.4sec (more than 2sec used by E)
- proof cannot be found in LEO-II alone

Architecture of LEO-II



More about Multimodal Logic in Higher-Order Logic

- C. Benzmüller, L. Paulson: **Exploring Properties of Normal Multimodal Logics in Simple Type Theory with LEO-II.** Festschrift in Honour of Peter B. Andrews 70th Birthday. To appear soon.

  - Interesting examples and limitations
  - $+$ First-Order Multimodal Logic
  - $+$ Higher-Order Multimodal Logic

Conclusion

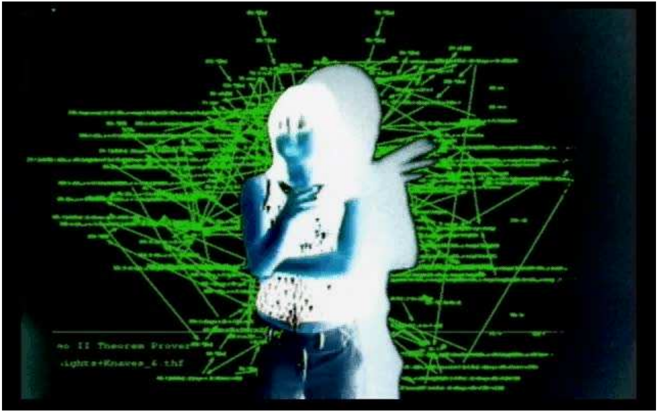
What makes LEO-II strong? The combination of

- expressive higher-order representations
- reduction to first-order representations
- cooperation with first-order ATPs
- higher-order termsharing and termindexing techniques

Try LEO-II (running under Ocaml 3.10)

- Website: `http://www.ags.uni-sb.de/~leo`
  - download version, very easy to install
  - online demo
- Systems on TPTP:
  `http://www.cs.miami.edu/~tptp/cgi-bin/SystemOnTPTP`

Latest Application of LEO-II: Dancefloor Animation



Grooving to an animation of LEO-II's dynamically growing termgraph (while LEO-II is proving Cantor's theorem)

Example (1)

$\mathtt{S4} \not\subseteq \mathtt{K}$: Axioms M and 4 are not valid in modal logic K

$$\neg \forall R. \forall A. \forall B. (\mathrm{valid}(\Box_R A \Rightarrow A)) \wedge (\mathrm{valid}(\Box_R B \Rightarrow \Box_R \Box_R B))$$

- LEO-II shows that axiom M is not valid
- R is instantiated with $\neq$ via primitive substitution
- total proving time 17.3s

Architecture of LEO-II