System Demonstration

The ΩMEGA Group
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Overview

I. Introduction

II. Interactive theorem proving with ΩMEGA

III. Proof Planning in ΩMEGA

IV. Exploration

I. INTRODUCTION

- Aims and Philosophy
- Background
- System

Aims

Goal: Mathematical Assistant System for proof development

- human-oriented
- abstract (top-down)
- knowledge-based
- mixed-initiative

Current status: implementation as a joint research platform for a collection of related and integrated research projects

Philosopy

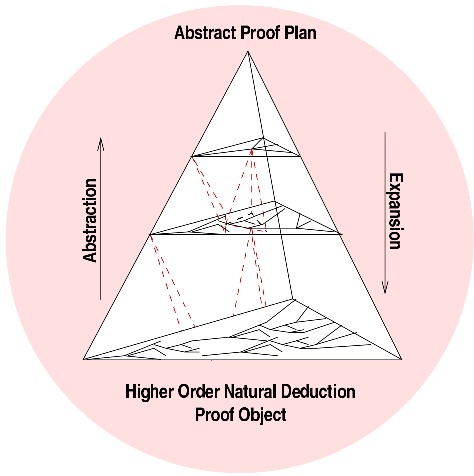
- 'Top-down' approach to theorem proving
  - Proof construction with abstract steps
  - Expansion onto a basic logic level
  - Proof checking in a small ND calculus

$\Rightarrow$ Proving and expansion are equivalent problems

- Do not re-invent the wheel!
  (i.e., use existing technology)

Proof Object

Proof Data Structure of ΩMEGA



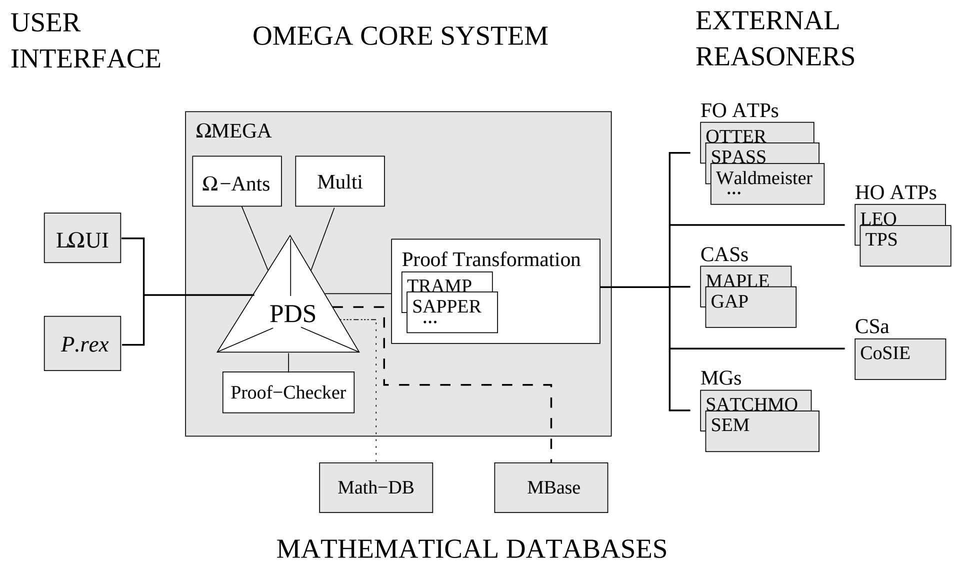
maintains simultaneously
a proof at
different levels of abstraction

Higher-order language
based on a
simply-typed λ-calculus

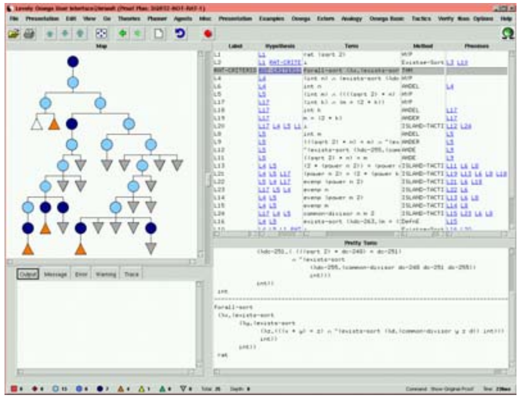
Proof Construction

- Interactive proof construction with
  - 'failing' tactics (correctness not guranteed, unlike LCF)
  - embedded external reasoners (ATP, CAS, . . .)
  - facts from knowledge base
- Usable for both proving and expanding
- Automation via proof planning and agent mechanism

Architecture



$\mathcal{L}\Omega\mathcal{U}\mathcal{I}$: Graphical User Interface



multi modal:
proof tree
linearized proof
term browser

II. Interactive Theorem Proving

- Applying rules and tactics
- Using suggestion mechanism Ω-ANTS
- Proof construction with islands
- Proof expansion and proof explanation

The $\sqrt{2}$-Problem

Theorem: $\sqrt{2}$ is irrational.

Proof: (by contradiction)

Assume $\sqrt{2}$ is rational, that is, there exist natural numbers m, n with no common divisor such that $\sqrt{2} = m/n$. Then $n\sqrt{2} = m$, and thus $2n^2 = m^2$. Hence m^2 is even and, since odd numbers square to odds, m is even; say $m = 2k$. Then $2n^2 = (2k)^2 = 4k^2$, that is, $n^2 = 2k^2$. Thus, n^2 is even too, and so is n. That means that both n and m are even, contradicting the fact that they do not have a common divisor.

Formalization

The Problem:

```
(th~defproblem sqrt2-not-rat
  (in real)
  (conclusion (not (rat (sqrt 2))))
  (help "sqrt 2 is not a rational number."))
```

Formalization

SQRT:

```
(th~defdef sqrt
  (in real)
  (definition
    (lam (x num)
       (that (lam (y num) (= (power y 2) x)))))
  (help "Definition of square root."))
```


Formalization

Rat-Criterion:

```
(th~deftheorem rat-criterion
  (in rational)
  (conclusion
    (forall-sort (lam (x num)
     (exists-sort (lam (y num)
      (exists-sort (lam (z num)
       (and (times x y) z)
            (not (exists-sort (lam (d num)
                    (common-divisor y z d))
                   int))))
       int)) int)) rat))
  (help "x rational implies there exist integers y,z

         which have no common divisor with x=y*z."))
```


Interactive Proof Construction

Successive proof construction by

- applying rules
- applying tactics
- using external systems
- using facts from the database

Problems:

- Which facts are needed from database?
- Which rules/tactics/external systems are applicable?
  How are they applicable?

Suggestion Mechanism Ω-ANTS

Goal:

- Compute possible next proof step
- Consider rules, tactics, external systems, theorems etc.
- Suggest commands + parameters to the user

Realization:

- Realized in concurrent processes
- Computations in the background
- Exhibits anytime behavior

Proof Construction with Islands

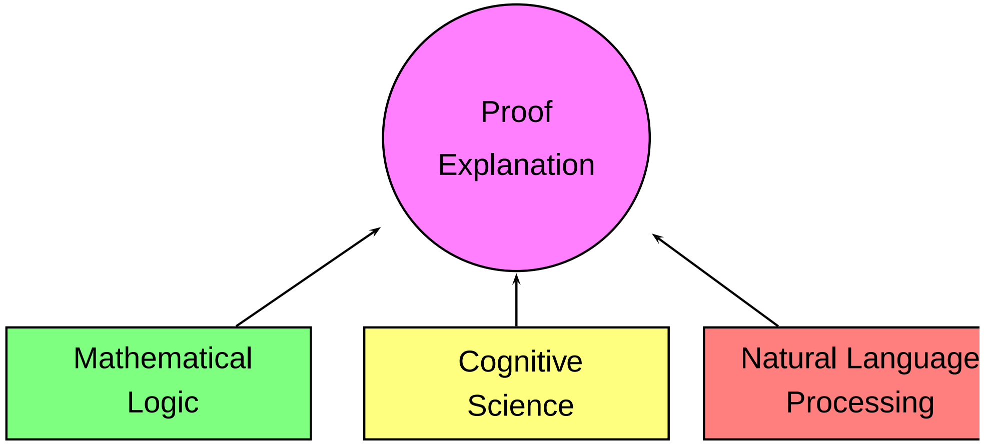
Problem: How do we get the desired proof, if the tactics do not corrspond to the necessary steps?

Insert steps as proof 'islands'

$$\cfrac{\cfrac{2*n^2=m^2}{Even(m^2)}\;Island}{Even(m)}\;Island$$
$$\vdots$$

Valid proof is generated by expansion.

P.rex: An Interactive Proof Explainer



Calculus of Constructions ACT-R Discourse Planning

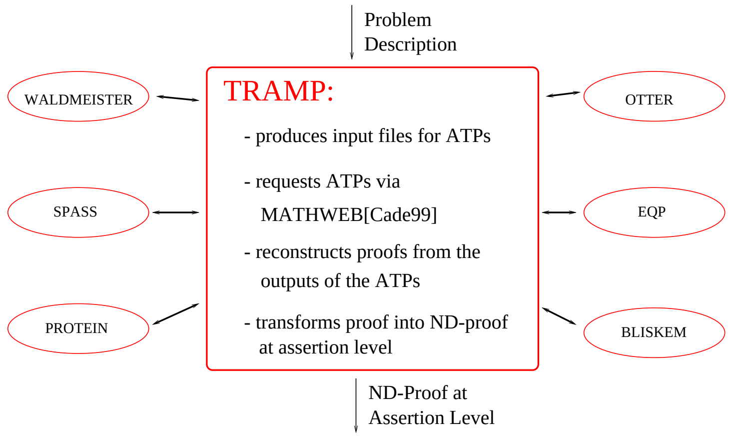
Expansion of Proofs

Recursive and interleaved process:

- Normal 'failing' tactics
  - expansion (hopefully) automatic
- Island Tactic
  - manually construct expansion
  - close gap with external reasoners
- External reasoners
  - compute automatically expansions with special systems/interfaces (e.g. TRAMP, SAPPER, ...)

The TRAMP System

Transformation of ATP output into ND-proofs at assertion level

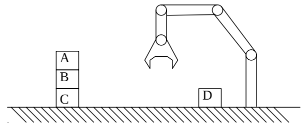


III. Proof Planning

- Automated theorem proving at abstract level
- AI planning paradigm
- Knowledge Acquisition

AI Planning

- Initial State



  $on(A,B)$, $on(B,C)$, $on\_table(C)$, $on\_table(D)$, $free(A)$, ...
- Goal: $on\_table(B)$
- Operators

  PUTDOWN(X)
  prec: $holding(X)$
  effect:
  $\oplus$ $on\_table(X)$, $hand\_empty$
  $\ominus$ $holding(X)$
- Plan $pick(A), putdown(A), pick(B), putdown(B)$

Theorem Proving as Planning

Initial State: proof assumptions
Goal: theorem
Operators: called methods

- Methods representing (abstract) proving steps
- Method = tactic + specification

Proof Plan: sequence of actions, i.e., instantiated methods
Planning Process: precondition achievement planning

Proof Planning in ΩMEGA

Knowledge-based proof planning

- domain-specific methods
- use of domain-specific external systems
- control-rules prune search space in particular domains

with multiple strategies

- strategies define different plan refinements (e.g. supply planner with different sets of methods and control rules)
- flexible combination and interleaving of strategies

Proof Planning the $\sqrt{2}$-Problem

Knowledge acquisition, e.g.

PrimeFacs-Product-m-f		
prec.	L: $n * t = t'$	
appl.-cond.	$n = p_1 * \ldots * p_n$ (CAS)	
effect:	$\oplus$ L_1: prime-divisor(p_1,t')	(Expand-PrimeFacs(L))
	$\vdots$	
	$\oplus$ L_n: prime-divisor(p_n,t')	(Expand-PrimeFacs(L))

Methods include specification of expansion scheme

Case Studies

- Limit Theorems

  (employs COSIE, MAPLE)
- Residue Class Domain

  (employs MAPLE, GAP, WALDMEISTER, SEM, HR)
- Homomorphism Theorems

Only with older version of ΩMEGA:

- Diagonalization proofs
- Completeness of resolution calculi

IV. Exploration

- Motivation
- Exploration with Oants
- Exploration with Multi

Motivation

Do not only re-prove known theorems, but

- Experiment with new conjectures
- Explore properties of structures

$\Rightarrow$ Postulate, prove, and refute conjectures

Set Problems

Examine different arbitrary set equations:

$$A \cup (B \cap C) = (A \cup B) \cap C$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Show validity or invalidity

Exploring with Ω-ANTS

- Consider some tactics and external reasoners
- Automated application of suggestions
- Involve complementary reasoning specialists
  - prove with automated theorem prover
  - refute with model generator
- Set examples: Proofs are simplified with some tactics and concluded by external reasoners

The Residue Class Domain

What kind of algebraic structures are ...

$$(\mathbb{Z}_4, \bar{+}), \qquad (\mathbb{Z}_5, (x \bar{*} y) \bar{+} \bar{1}_5), \quad (\{\bar{0}_6, \bar{2}_6, \bar{4}_6\}, (x \bar{+} x) \bar{+} (y \bar{+} y))$$

...?

Successively check properties such as associativity, unit element, etc.

Which structures are isomorphic ?

$$(\mathbb{Z}_3, \bar{+}) \cong (\{\bar{0}_6, \bar{2}_6, \bar{4}_6\}, \bar{+}) \quad (\mathbb{Z}_2 \otimes \mathbb{Z}_2, \bar{+} \otimes \bar{+}) \ncong (\mathbb{Z}_4, \bar{+})$$

Exploration with Proof Planning

Examine properties step-by-step

- Exploration module employs Multi
  - Several strategies in Multi
  - Supported by CAS, Model Generator
- Proof plan a conjecture or its negation
- Guidance by example computation

Credits

`http://www.ags.uni-sb.de/~omega`

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