PD Dr.-Ing. Christoph Benzmüller

Exercise sheet 1
Semantics of Higher-Order Logics
(2007)

For exercises 1-3, let $\mathcal{D}$ be the standard frame with $\mathcal{D}_o = \{\bot, \top\}$ and $\mathcal{D}_\iota = \{1\}$.

Exercise 1 *Assume* $(\mathcal{E}_\alpha)_{\alpha \in \mathcal{T}}$ *is a standard frame with*
$\mathcal{E}_o = \{\bot, \top\}$
$\mathcal{E}_\iota = \{1\}$
Prove: $\forall \alpha \in \mathcal{T} : \mathcal{E}_\alpha = \mathcal{D}_\alpha$

Exercise 2 *Prove:* $\forall \alpha \in \mathcal{T} : \mathcal{D}_\alpha$ *is finite.*

Exercise 3 *Define inductively an infinite set* $\mathcal{T}^1 \subseteq \mathcal{T}$ *s.t.*
$\forall \alpha \in \mathcal{T}^1 \quad |\mathcal{D}_\alpha| = 1$

Exercise 4 *Prove every functional* Σ*-evaluation is* ξ*-functional.*

Exercise 5 *Let* $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ *be a functional* Σ*-evaluation,* φ *be an assignment into* $\mathcal{J}$*,* $\mathbf{F} \in$ *wff*$_{\alpha \to \beta}(\Sigma)$ *and* $X_\alpha \notin \mathbf{Free}(\mathbf{F})$*. Prove*

$$\mathcal{E}_\varphi(\lambda X_\alpha . \mathbf{F} X) = \mathcal{E}_\varphi(\mathbf{F}).$$

Exercise 6 *Let* $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ *be a* Σ*-model. Prove if either* $\top, \bot \in \Sigma$ *or* $\neg \in \Sigma$*, then* υ *is surjective.*

Exercise 7 *Let* $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ *be a* Σ*-model. Suppose either* $\top, \bot \in \Sigma$ *or* $\neg \in \Sigma$*. Prove* $\mathcal{M}$ *satisfies* $\mathfrak{b}$ *iff* $\mathcal{D}_o$ *has two elements.*

Exercise 8 *Assume that the signature contains only the logical connective* $\supset$ *and the quantifier* Π^o*. Construct a* Σ*-model* $\mathcal{M}$ *such that*

1. $\mathcal{M} \models \forall P_o . P$

Exercise 9 *What are the weakest calculi* $\mathfrak{NK}_*$ *in which the following sentences can be derived? Please give the derivations.*

1. $\forall X_o . \forall Y_o . X \lor Y \Leftrightarrow Y \lor X$
2. $\forall X_o . \forall Y_o . X \lor Y \doteq Y \lor X$
3. $\lambda X_o . \lambda Y_o . X \lor Y \doteq \lambda X_o . \lambda Y_o . Y \lor X$
4. $\lor \doteq \lambda X_o . \lambda Y_o . Y \lor X$