TITLE[1]

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Overview

1. Background: Tutorial DIALOG on Mathematical Proof
2. Step Size - Observations & Modeling
3. Study Environment & Experiments
4. Granularity-Adaptive Proof Presentation
5. Outlook & Discussion

Background: The DIALOG Project

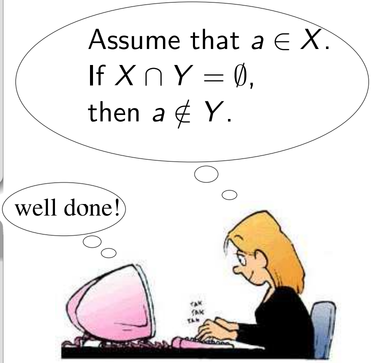
Tutorial Dialog for Mathematics.

Employed Techniques

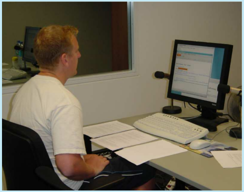
- Dyn. Domain Reasoning for math proofs: math assistant ΩMEGA
- NL processing, dialogue management, teaching model

Research Processes (in spiral model)

- DIALOG system design
- Prototype development
- Empirical studies



Simulation (Wizard-of-Oz)



Let R and S be relations in a set M. It holds that: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. Do the proof interactively with the system!

A pair (x, y) is element of $R \circ S$ iff there is a z in M such that $(x, z) \in R$ and $(z, y) \in S$

Correct!

Therefore a pair (x, y) is element of $(R \circ S)^{-1}$ if there is a z in M, such that $(x, z) \in S$ and $(z, y) \in R$

That's not correct!

Simulation (Wizard-of-Oz) [Ctd.]

Tutor: Let R and S be relations in a set M. It holds that: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. Do the proof interactively with the system!

Student: Execute $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$!

Tutor: That's what you have to prove. Would you like a hint?

Student: yes

Tutor: At first, try to show $(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}$.

Student: How do I do that?

Tutor: You have to show that any element in $(R \circ S)^{-1}$ is also in $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. You could start with: Let $(a, b) \in (R \circ S)^{-1}$. ...Then it holds ...?

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Step Size in Mathematics (Granularity)

Mathematical practice: skip intermediate steps when appropriate:

Proof Exercise: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

- We show $(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}$ and $S^{-1} \circ R^{-1} \subseteq (R \circ S)^{-1}$
- We assume $(y, x) \in (R \circ S)^{-1}$ and show $(y, x) \in S^{-1} \circ R^{-1}$
- Hence, $(x, y) \in R \circ S$
- Hence, $\exists z$ s.t. $(x, z) \in R \wedge (z, y) \in S$
- Hence, $\exists z$ s.t. $(z, x) \in R^{-1} \wedge (z, y) \in S$
- Hence, $\exists z$ s.t. $(z, x) \in R^{-1} \wedge (y, z) \in S^{-1}$
- Hence, $\exists z$ s.t. $(y, z) \in S^{-1} \wedge (z, x) \in R^{-1}$
- Hence, $(y, x) \in S^{-1} \circ R^{-1}$

$\vdots$

Step Size in Mathematics (Granularity)

Mathematical practice: skip intermediate steps when appropriate:

Proof Exercise: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

~~We show $(R \circ S)^{-1} \subseteq S^{-1} \circ R^{-1}$ and $S^{-1} \circ R^{-1} \subseteq (R \circ S)^{-1}$~~

- We assume $(y, x) \in (R \circ S)^{-1}$ and show $(y, x) \in S^{-1} \circ R^{-1}$
- Hence, $(x, y) \in R \circ S$
- Hence, $\exists z$ s.t. $(x, z) \in R \wedge (z, y) \in S$

~~Hence, $\exists z$ s.t. $(z, x) \in R^{-1} \wedge (z, y) \in S$~~

- Hence, $\exists z$ s.t. $(z, x) \in R^{-1} \wedge (y, z) \in S^{-1}$

~~Hence, $\exists z$ s.t. $(y, z) \in S^{-1} \wedge (z, x) \in R^{-1}$~~

- Hence, $(y, x) \in S^{-1} \circ R^{-1}$

$\vdots$

Step Size in the Experiments

Granularity: The question of the appropriate step size/complexity.

Exercise:	z.Z. $(R \circ S)^{-1} = (x, y) \in S^{-1} \circ R^{-1}$
$\vdots$	
student]	$(x, y) \in (R \circ S)^{-1}$
tutor]	Now try to draw conclusions from this!
	correct | appropriate | relevant
student]	$(x, y) \in S^{-1} \circ R^{-1}$
tutor]	This cannot be concluded directly. You need some intermediate steps!
	correct | too coarse-grained | relevant

Step size annotated by tutors as appropriate, too coarse-grained (too big a step) or too detailed (too small a step)

Modeling (Suitable) Granularity

Goal: diagnose student's step size, granularity-adapted proof presentation.

Previous Approach

Reconstruct proofs in ND, relate student step size to ND step size.

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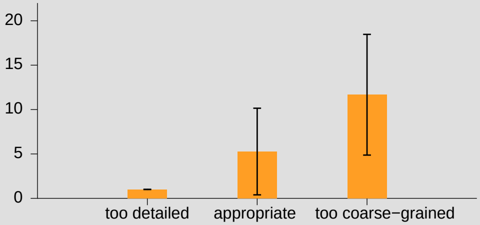

⇒ not promising (cf. [Schiller et al. 2006]).

Modeling (Suitable) Granularity

Goal: diagnose student's step size, granularity-adapted proof presentation.

Recent Approach

- Modeling/representation of proofs: choice of suitable proof calculus/mechanism (assertion level vs. ND or resolution)
- Analysis: granularity-relevant criteria
- Classification: classify (multi-inference) proof steps (as appropriate, too big or too small)

Learn classifier from empirical samples.

Approach: Model Student Proofs as Assertion Level Proofs

Student's Proof

Ex: Show
$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$!

Assertion Level Proof

$$\texttt{Exercise:} \quad \vdash \underbrace{(R \circ S)^{-1}}_{\Gamma} = \underbrace{S^{-1} \circ R^{-1}}_{\Theta}$$

Approach: Model Student Proofs as Assertion Level Proofs

Student's Proof

Ex: Show $(R\circ S)^{-1} = S^{-1}\circ R^{-1}$!

s1: $\boxed{\text{Let } (x,y)\in(R\circ S)^{-1}}$.

Assertion Level Proof

$$\text{Def.}= \; \dfrac{\dfrac{\dfrac{}{\texttt{s1:}\,(x,y)\in(R\circ S)^{-1}\vdash (x,y)\in\Theta}}{\vdash (R\circ S)^{-1}\subseteq S^{-1}\circ R^{-1}}\,\text{Def.}\subseteq \qquad \vdash \Theta\subseteq\Gamma}{\texttt{Exercise:}\quad \vdash \underbrace{(R\circ S)^{-1}}_{\Gamma} = \underbrace{S^{-1}\circ R^{-1}}_{\Theta}}$$

Approach: Model Student Proofs as Assertion Level Proofs

Student's Proof

Ex: Show $(R\circ S)^{-1} = S^{-1}\circ R^{-1}$!

s1: Let $(x,y)\in(R\circ S)^{-1}$.

s2: Hence $(y,x)\in(R\circ S)$.

Assertion Level Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{s2}: (y,x)\in(R\circ S)\vdash (x,y)\in\Theta}}{\texttt{s1}: (x,y)\in(R\circ S)^{-1}\vdash(x,y)\in\Theta}\,\text{Def.}^{-1}}{\vdash (R\circ S)^{-1}\subseteq S^{-1}\circ R^{-1}}\,\text{Def.}\subseteq \qquad \vdash \Theta\subseteq\Gamma}{\texttt{Exercise}: \quad \vdash \underbrace{(R\circ S)^{-1}}_{\Gamma} = \underbrace{S^{-1}\circ R^{-1}}_{\Theta}}\,\text{Def.} =$$

Approach: Model Student Proofs as Assertion Level Proofs

Student's Proof

Ex: Show $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$!

s1: Let $(x,y) \in (R \circ S)^{-1}$.

s2: Hence $(y,x) \in (R \circ S)$.

s3: Hence $(y,z) \in R \wedge (z,x) \in S$.

Assertion Level Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{s3:}\,(y,z)\in R \wedge (z,x)\in S \vdash (x,y)\in\Theta}}{\texttt{s2:}\,(y,x)\in(R\circ S)\vdash (x,y)\in\Theta}\;\textbf{Def.}\circ}{\texttt{s1:}\,(x,y)\in(R\circ S)^{-1}\vdash (x,y)\in\Theta}\;\textbf{Def.}^{-1}}{\vdash (R\circ S)^{-1}\subseteq S^{-1}\circ R^{-1} \qquad \vdash \Theta\subseteq\Gamma}\;\textbf{Def.}\subseteq}{\texttt{Exercise:}\quad \vdash \underbrace{(R\circ S)^{-1}}_{\Gamma} = \underbrace{S^{-1}\circ R^{-1}}_{\Theta}}\;\textbf{Def.} =$$

Approach: Model Student Proofs as Assertion Level Proofs

Student's Proof

Ex: Show $(R\circ S)^{-1} = S^{-1}\circ R^{-1}$!

s1: Let $(x,y)\in(R\circ S)^{-1}$.

s2: Hence $(y,x)\in(R\circ S)$.

s3: Hence $(y,z)\in R \wedge (z,x)\in S$.

s4: Hence $(z,y)\in R^{-1}\wedge(x,z)\in S^{-1}$.

Assertion Level Proof

$$\frac{\texttt{s4:}\ (z,y)\in R^{-1}\wedge(x,z)\in S^{-1}\vdash(x,y)\in\Theta}{(y,z)\in R\wedge(x,z)\in S^{-1}\vdash(x,y)\in\Theta}\ \mathbf{Def}^{-1}$$

$$\frac{(y,z)\in R\wedge(x,z)\in S^{-1}\vdash(x,y)\in\Theta}{\texttt{s3:}\ (y,z)\in R\wedge(z,x)\in S\vdash(x,y)\in\Theta}\ \mathbf{Def.}^{-1}$$

$$\frac{\texttt{s3:}\ (y,z)\in R\wedge(z,x)\in S\vdash(x,y)\in\Theta}{\texttt{s2:}\ (y,x)\in(R\circ S)\vdash(x,y)\in\Theta}\ \mathbf{Def.}\circ$$

$$\frac{\texttt{s2:}\ (y,x)\in(R\circ S)\vdash(x,y)\in\Theta}{\texttt{s1:}\ (x,y)\in(R\circ S)^{-1}\vdash(x,y)\in\Theta}\ \mathbf{Def.}^{-1}$$

$$\frac{\texttt{s1:}\ (x,y)\in(R\circ S)^{-1}\vdash(x,y)\in\Theta}{\vdash(R\circ S)^{-1}\subseteq S^{-1}\circ R^{-1}}\ \mathbf{Def.}\subseteq$$

$$\frac{\vdash(R\circ S)^{-1}\subseteq S^{-1}\circ R^{-1}\qquad \vdash\Theta\subseteq\Gamma}{\texttt{Exercise:}\quad\vdash\underbrace{(R\circ S)^{-1}}_{\Gamma}=\underbrace{S^{-1}\circ R^{-1}}_{\Theta}}\ \mathbf{Def.}=$$

Typically: 1 student step $\cong$ 1 or several assertion level steps (experiment: usually 1-3, seldomly more), cf. [Buckley & Dietrich 2007], [Benzmüller et al. 2007]

Granularity Criteria

Possible criteria for size of a (multi-)inference step ("features")

- How many assertion level inference applications? (total)
- What concepts are used? (concepts)
- How many concepts are not yet known to the student? (unmastered)
- Are the concepts named? (verb)
- etc.

Student step	Infs	Features	Verdict
1. We assume $(y,x) \in (R \circ S)^{-1}$ and show $(y,x) \in S^{-1} \circ R^{-1}$	Def.$=$, Def.$\subseteq$	total:2, concepts:2, relations:0, verb:0,...	?
2. Hence, $(x,y) \in R \circ S$	Def^{-1}	total:1, concepts:1, relations:1, verb:0,...	?
...			

Example Classifier

Sample ruleset classifier

* total $\in \{0,1,2\} \Rightarrow$ "appropriate"
* unmastered $\in \{2,3,4\} \wedge$ relations $\in \{2,3,4\} \Rightarrow$ "step-too-big"
* total $\in \{3,4\} \wedge$ relations $\in \{0,1\} \Rightarrow$ "step-too-big"
* unmastered $\in \{0,1\} \Rightarrow$ "appropriate"
* _ $\Rightarrow$ "appropriate"

Student step	Infs	Features	Verdict
1. We assume $(y,x) \in (R \circ S)^{-1}$ and show $(y,x) \in S^{-1} \circ R^{-1}$	Def.=, Def.$\subseteq$	total:2, concepts:2, relations:0, verb:0,...	appropriate
2. Hence, $(x,y) \in R \circ S$	Def^{-1}	total:1, concepts:1, relations:1, verb:0,...	appropriate

...

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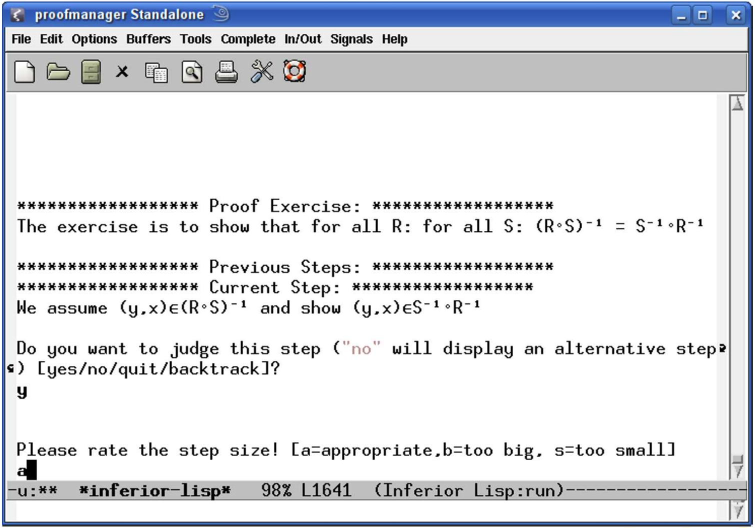
Study Environment - Motivation

- Learn classifiers via annotations from expert tutors using standard machine learning
- WoZ experiments not ideal for focused study on granularity

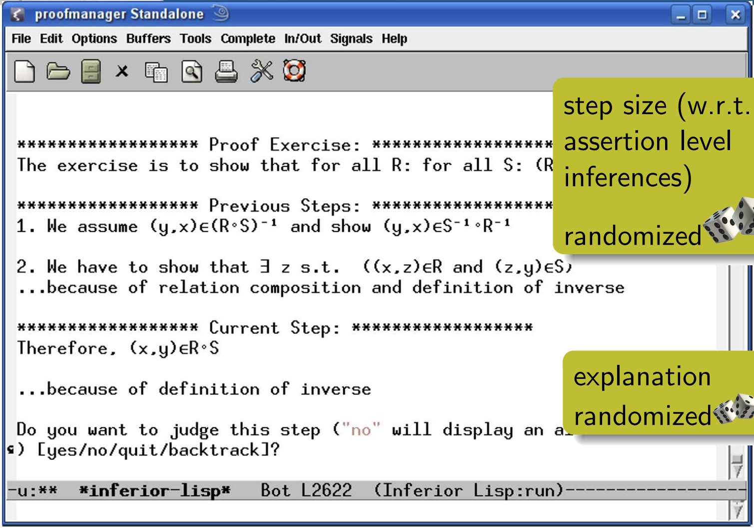
Idea

- Automate student's role using ΩMEGA
- More control over "student"
- More granularity annotations in less time (compared to WoZ)

Study Environment



Study Environment



step size (w.r.t. assertion level inferences) randomized

explanation randomized

Study Environment

proofmanager Standalone

File Edit Options Buffers Tools Complete In/Out Signals Help

Exercise:

Show that for all relations R and S: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

What you should know about the student:

student has basic university knowledge (in particular set membership, equality, subset)

Goal of the exercise:

becoming familiar with the concepts of relation inverse and relation composition.

Hit <enter> when ready

-u:** *inferior-lisp* Bot L1536 (Inferior Lisp:run)

ΩMEGA & student model initialised

Evaluation

Evaluation

- 2 experiments with 2 expert tutors (using different exercises in naive set theory, relations, topology)

	Tutor 1	Tutor 2
Steps annotated:	135	207
Perf. learnt classifier[1] -mean correct - κ	86.7% κ=0.68	68.9% κ=0.47
Interrater reliability[2]	κ=0.37	

[1] best rule-based classifier, evaluated on full dataset using 10-fold cross validation

[2] on common subset of 108 steps

Granularity-Adaptive Proof Presentation

Reproduce the granularity of textbook proofs, e.g. the proof for $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$:

"$\boxed{1}$ *Let* x *be an element of* $A \cap (B \cup C)$, $\boxed{2}$ *then* $x \in A$ *and* $x \in B \cup C$. $\boxed{3}$ *This means that* $x \in A$, *and either* $x \in B$ *or* $x \in C$. $\boxed{4}$ *Hence we either have (i)* $x \in A$ *and* $x \in B$, *or we have (ii)*$x \in A$ *and* $x \in C$. $\boxed{5}$ *Therefore, either* $x \in A \cap B$ *or* $x \in A \cap C$, *so* $\boxed{6}$ $x \in (A \cap B) \cup (A \cap C)$. $\boxed{7}$ *This shows that* $A \cap (B \cup C)$ *is a subset of* $(A \cap B) \cup (A \cap C)$. $\boxed{8}$ *Conversely, let* y *be an element of* $(A \cap B) \cup (A \cap C)$. $\boxed{9}$ *Then, either (iii)* $y \in A \cap B$, *or (iv)* $y \in A \cap C$. $\boxed{10}$ *It follows that* $y \in A$, *and either* $y \in B$ *or* $y \in C$. $\boxed{11}$ *Therefore,* $y \in A$ *and* $y \in B \cup C$ *so that* $y \in A \cap (B \cup C)$. $\boxed{12}$ *Hence* $(A \cap B) \cup (A \cap C)$ *is a subset of* $A \cap (B \cup C)$. $\boxed{13}$ *In view of Definition 1.1.1, we conclude that the sets* $A \cap (B \cup C)$ *and* $(A \cap B) \cup (A \cap C)$ *are equal.*" [from Bartle/Sherbert 1982]

Proof Presentation from Assertion Level Proof

1) We show that $((A \cap B) \cup (A \cap C) \subseteq A \cap B \cup C)$ and $(A \cap B \cup C \subseteq (A \cap B) \cup (A \cap C))$ ...because of definition of equality
2) We assume $x \in A \cap B \cup C$ and show $x \in (A \cap B) \cup (A \cap C)$
3) Therefore, $x \in A \wedge x \in B \cup C$
4) Therefore, $x \in A \wedge (x \in B \vee x \in C)$
5) Therefore, $x \in A \wedge x \in B \vee x \in A \wedge x \in C$
6) Therefore, $x \in A \cap B \vee x \in A \wedge x \in C$
7) Therefore, $x \in A \cap B \vee x \in A \cap C$
8) We are done with the current part of the proof (i.e., to show that $x \in (A \cap B) \cup (A \cap C)$). [It remains to be shown that $(A \cap B) \cup (A \cap C) \subseteq A \cap B \cup C$]
9) We assume $y \in (A \cap B) \cup (A \cap C)$ and show $y \in A \cap B \cup C$
10) Therefore, $y \in A \cap B \vee y \in A \cap C$
11) Therefore, $(y \in A \wedge y \in B) \vee y \in A \cap C)$
12) Therefore, $(y \in A \wedge y \in B) \vee (y \in A \wedge y \in C)$
13) Therefore, $y \in A \wedge (y \in B \vee y \in C)$
14) Therefore, $y \in A \wedge y \in B \cup C$
15) This finishes the proof. Q.E.D.

Ruleset:
_ $\Rightarrow$ "appropriate"

Proof Presentation from Assertion Level Proof

1) We show that $((A \cap B) \cup (A \cap C) \subseteq A \cap B \cup C)$ and $(A \cap B \cup C \subseteq (A \cap B) \cup (A \cap C))$ ...because of definition of equality
2) We assume $x \in A \cap B \cup C$ and show $x \in (A \cap B) \cup (A \cap C)$
3) Therefore, $x \in A \wedge x \in B \cup C$
4) Therefore, $x \in A \wedge (x \in B \vee x \in C)$
5) Therefore, $x \in A \wedge x \in B \vee x \in A \wedge x \in C$
6) ~~Therefore, $x \in A \cap B \vee x \in A \wedge x \in C$~~
7) Therefore, $x \in A \cap B \vee x \in A \cap C$
8) We are done with the current part of the proof $x \in (A \cap B) \cup (A \cap C))$. [It remains to be show $(A \cap B) \cup (A \cap C) \subseteq A \cap B \cup C]$
9) We assume $y \in (A \cap B) \cup (A \cap C)$ and show y
10) Therefore, $y \in A \cap B \vee y \in A \cap C$
11) ~~Therefore, $(y \in A \wedge y \in B) \vee y \in A \cap C)$~~
12) ~~Therefore, $(y \in A \wedge y \in B) \vee (y \in A \wedge y \in C)$~~
13) Therefore, $y \in A \wedge (y \in B \vee y \in C)$
14) Therefore, $y \in A \wedge y \in B \cup C$
15) This finishes the proof. Q.E.D.

Ruleset:

* Hypintro=1 $\wedge$ total$> 1 \Rightarrow$ step-too-big
* $\cup$-Defn $\in 1, 2 \wedge \cap$-Defn $\in 1, 2 \Rightarrow$ step-too-big
* $\cap$-Defn$< 3 \wedge \cup$-Defn$=0$ $\wedge$ masteredconceptsunique=1 $\wedge$ unmasteredconceptsunique=0 $\Rightarrow$ step-too-small
* _ $\Rightarrow$ step-appropriate

Discussion & Outlook

Discussion

- Empirical modeling of granularity
- ...independent of introspection/justification of experts' judgments
- Thus, we imitate the behavior of expert tutors
- Is it desirable/possible to establish a best practice for judging proof step granularity?

Outlook

- Further experiment sessions planned with different experts
- Measure performance of learnt classifiers, agreement between tutors
- What are the most useful granularity criteria for the classification task?

Thank you!

Questions ?

Diversity in Wizard-of-Oz Corpora

Proof exercise: $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

Student X	Student Y
st[0]: $(R \circ S)^{-1} = \{(y,x) \mid (x,y) \in (R \circ S)\}$	st[0]: One needs to show equality between two sets.
tu[0]: This statement is correct.	tu[0]: That's right! How do you proceed?
st[1]: $(R \circ S)^{-1} = \{(y,x) \mid \exists z (z \in M \wedge (x,z) \in R \wedge (z,y) \in S\}$	st[1]: I use the extensionality principle.
tu[1]: This formula is also correct.	tu[1]: That's right.
st[2]: $(R \circ S)^{-1} = \{(y,x) \mid \exists z (z \in M \wedge (z,x) \in R^{-1} \wedge (y,z) \in S^{-1}\}$	st[2]: Let $(s,r) \in (R \circ S)^{-1}$. According to the definition of the inverse relation it then holds that $(r,s) \in (R \circ S)$.
tu[2]: This is correct. You are on a good way.	tu[2]: That's right!