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Exercise sheet 1
Semantics of Higher-Order Logics
(2007)

For exercises 1-3, let $\mathcal{D}$ be the standard frame with $\mathcal{D}_o = \{\bot, \top\}$ and $\mathcal{D}_\iota = \{1\}$.

Exercise 1 *Assume* $(\mathcal{E}_\alpha)_{\alpha \in \mathcal{T}}$ *is a standard frame with*
$\mathcal{E}_o = \{\bot, \top\}$
$\mathcal{E}_\iota = \{1\}$
Prove: $\forall \alpha \in \mathcal{T} : \mathcal{E}_\alpha = \mathcal{D}_\alpha$

Solution: We argue by induction on types. At both base types we know

$$\mathcal{E}_\iota = \{1\} = \mathcal{D}_\iota$$

and

$$\mathcal{E}_o = \{\bot, \top\} = \mathcal{D}_o$$

For the induction step, we assume $\mathcal{E}_\alpha = \mathcal{D}_\alpha$ and $\mathcal{E}_\beta = \mathcal{D}_\beta$.

Since both $\mathcal{D}$ and $\mathcal{E}$ are standard frames,

$$\mathcal{D}_{\alpha\beta} = (\mathcal{D}_\alpha)^{\mathcal{D}_\beta} = (\mathcal{E}_\alpha)^{\mathcal{E}_\beta} = \mathcal{E}_{\alpha\beta}.$$

That was easy!

Exercise 2 *Prove:* $\forall \alpha \in \mathcal{T} : \mathcal{D}_\alpha$ *is finite.*

Solution: We argue by induction on types. Obviously $\mathcal{D}_o = \{\bot, \top\}$ and $\mathcal{D}_\iota = \{1\}$ are both finite. For the induction step, we assume $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ are both finite. Since $\mathcal{D}$ is a standard frame, we know $\mathcal{D}_{\alpha\beta} = (\mathcal{D}_\alpha)^{\mathcal{D}_\beta}$. Hence we calculate

$$|\mathcal{D}_{\alpha\beta}| = |\mathcal{D}_\alpha|^{|\mathcal{D}_\beta|} < \infty$$

Thus all the domains are finite.

Exercise 3 *Define inductively an infinite set* $\mathcal{T}^1 \subseteq \mathcal{T}$ *s.t.*
$\forall \alpha \in \mathcal{T}^1 \quad |\mathcal{D}_\alpha| = 1$

Solution: Let $\mathcal{T}^1$ be the least set of types such that

- $\iota \in \mathcal{T}^1$
- $(\alpha\beta) \in \mathcal{T}^1$ whenever $\alpha \in \mathcal{T}^1$ and $\beta \in \mathcal{T}$

Intuitively, $\mathcal{T}^1$ is the set of types of the form $(\iota\beta^n \cdots \beta^1)$ for $n \geq 0$ and arbitrary types $\beta^1, \ldots, \beta^n$.

We can inductively prove $|\mathcal{D}_\alpha| = 1$ for each $\alpha \in \mathcal{T}^1$.

- Base case: $|\mathcal{D}_\iota| = 1$.
- Induction case: Assume the type is $(\alpha\beta)$ where $\alpha \in \mathcal{T}^1$ and $\beta \in \mathcal{T}$. Assume $|\mathcal{D}_\alpha| = 1$. (We do *not* assume $|\mathcal{D}_\beta| = 1$ – why not?) We calculate:

$$|\mathcal{D}_{\alpha\beta}| = |\mathcal{D}_\alpha|^{|\mathcal{D}_\beta|} = 1^{|\mathcal{D}_\beta|} = 1.$$

Exercise 4 *Prove every functional Σ-evaluation is ξ-functional.*

Solution: To show functionality implies ξ-functionality, let $\mathbf{M}, \mathbf{N} \in \mathit{wff}_\beta(\Sigma)$, an assignment φ and a variable X_α be given. Suppose $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N})$ for every $\mathsf{a} \in \mathcal{D}_\alpha$. We need to show $\mathcal{E}_\varphi(\lambda X.\mathbf{M}) = \mathcal{E}_\varphi(\lambda X.\mathbf{N})$. This follows from functionality since

$$\begin{aligned}\mathcal{E}_\varphi(\lambda X.\mathbf{M})@\mathsf{a} &= \mathcal{E}_{\varphi,[\mathsf{a},X]}((\lambda X.\mathbf{M})X) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{M}) \\ &= \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{N}) = \mathcal{E}_{\varphi,[\mathsf{a},X]}((\lambda X.\mathbf{N})X) = \mathcal{E}_\varphi(\lambda X.\mathbf{N})@\mathsf{a}\end{aligned}$$

for every $\mathsf{a} \in \mathcal{D}_\alpha$.

Exercise 5 *Let $\mathcal{J} := (\mathcal{D}, @, \mathcal{E})$ be a functional Σ-evaluation, φ be an assignment into $\mathcal{J}$, $\mathbf{F} \in \mathit{wff}_{\alpha\to\beta}(\Sigma)$ and $X_\alpha \notin \mathbf{Free}(\mathbf{F})$. Prove*

$$\mathcal{E}_\varphi(\lambda X_\alpha.\mathbf{F}X) = \mathcal{E}_\varphi(\mathbf{F}).$$

Solution: Let $\mathsf{a} \in \mathcal{D}_\alpha$ be given. Since $X_\alpha \notin \mathbf{Free}(\mathbf{F})$, we have $\mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{F}) = \mathcal{E}_\varphi(\mathbf{F})$. Since $\mathcal{E}$ respects β-equality, we can compute

$$\mathcal{E}_\varphi(\lambda X.\mathbf{F}X)@\mathsf{a} = \mathcal{E}_{\varphi,[\mathsf{a}/X]}((\lambda X.\mathbf{F}X)X) = \mathcal{E}_{\varphi,[\mathsf{a}/X]}(\mathbf{F}X) = \mathcal{E}_\varphi(\mathbf{F})@\mathsf{a}.$$

Generalizing over a, we conclude $\mathcal{E}_\varphi(\lambda X.\mathbf{F}X) = \mathcal{E}_\varphi(\mathbf{F})$ by functionality.

Exercise 6 *Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model. Prove if either $\top, \bot \in \Sigma$ or $\neg \in \Sigma$, then υ is surjective.*

Solution: Suppose $\top, \bot \in \Sigma$. By $\mathfrak{L}_\top(\mathcal{E}(\top))$ and $\mathfrak{L}_\bot(\mathcal{E}(\bot))$, we have $\upsilon(\mathcal{E}(\top)) = \mathtt{T}$ and $\upsilon(\mathcal{E}(\bot)) = \mathtt{F}$. Thus υ is surjective.

Suppose $\neg \in \Sigma$. Choose any $\mathsf{a} \in \mathcal{D}_o$. We know $\upsilon(\mathcal{E}(\neg)@\mathsf{a}) \neq \upsilon(\mathsf{a})$ by $\mathfrak{L}_\neg(\mathcal{E}(\neg))$. Thus υ is surjective.

Exercise 7 *Let $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ be a Σ-model. Suppose either $\top, \bot \in \Sigma$ or $\neg \in \Sigma$. Prove $\mathcal{M}$ satisfies $\mathfrak{b}$ iff $\mathcal{D}_o$ has two elements.*

Solution: By the previous exercise, we know υ is surjective. Thus $\mathcal{M}$ satisfies property $\mathfrak{b}$ iff υ is bijective iff $\mathcal{D}_o$ has two elements.

Exercise 8 *Assume that the signature contains only the logical connective $\supset$ and the quantifier Π^o. Construct a Σ-model $\mathcal{M}$ such that*

1. $\mathcal{M} \models \forall P_o.P$

Solution: There is a model such that $\mathcal{M} \models \forall P_o.P$. Let $\mathcal{D}$ be the standard frame with $\mathcal{D}_o = \{\mathtt{T}\}$ and $\mathcal{D}_\iota = \{1\}$. Note that every $\mathcal{D}_\alpha$ has only one element. Let $@$ be the application operator. For every assignment φ and $\mathbf{A} \in \mathit{wff}_\alpha$, let $\mathcal{E}_\varphi(\mathbf{A})$ be the unique member of $\mathcal{D}_\alpha$. It is easy to check $\mathcal{E}$ is an evaluation function. Let $\upsilon : \mathcal{D}_o \to \{\mathtt{T}, \mathtt{F}\}$ be the inclusion function given by $\upsilon(\mathtt{T}) := \mathtt{T}$. It is easy to check $\mathfrak{L}_\supset(()\mathcal{E}(\supset))$ and $\mathfrak{L}_{\Pi^o}(()\mathcal{E}(\Pi^o))$ hold where $\mathcal{E}(\supset)$ is the unique element of $\mathcal{D}_{ooo}$ (the "constantly constant true function") and $\mathcal{E}(\Pi^o)$ is the unique element of $\mathcal{D}_{o(oo)}$ (the "constant true function"). Hence $\mathcal{M} := (\mathcal{D}, @, \mathcal{E}, \upsilon)$ is a Σ-model.

Now to check $\mathcal{M} \models \forall P_o.P$, simply note that $\mathcal{M} \models_{\varphi,[\mathtt{T}/P]} P$ and so $\mathcal{M} \models_{\varphi,[\mathsf{a}/P]} P$ for all $\mathsf{a} \in \mathcal{D}_o$.

Exercise 9 *What are the weakest calculi $\mathfrak{NK}_*$ in which the following sentences can be derived? Please give the derivations.*

1. $\forall X_o.\forall Y_o.X \vee Y \Leftrightarrow Y \vee X$
2. $\forall X_o.\forall Y_o.X \vee Y \doteq Y \vee X$
3. $\lambda X_o.\lambda Y_o.X \vee Y \doteq \lambda X_o.\lambda Y_o.Y \vee X$
4. $\vee \doteq \lambda X_o.\lambda Y_o.Y \vee X$