Extensional Higher-Order Resolution Paramodulation and RUE-Resolution

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Motivation

- Theorem Proving in Higher-Order Logic
- Examples:
  1. $p_{o\to o}\ a_o \wedge p_{o\to o}\ b_o \Rightarrow p_{o\to o}\ (a_o \wedge b_o)$
  2. $(\lambda X_\alpha\text{.}\ joung_{\alpha\to o}\ X \wedge male_{\alpha\to o}\ X) = (\lambda X_\alpha\text{.}\ male\ X \wedge young\ X)$
  3. $\wp(\emptyset_{\gamma\to o}) = \{\emptyset\}$

     Definitions:

     $$\wp := \lambda A_{\gamma\to o}.\lambda B_{\gamma\to o}.\overbrace{\forall X_\gamma.B\ X \Rightarrow A\ X}^{B\ \subseteq\ A}$$
     $$\emptyset_{\gamma\to o} := \lambda X_\gamma\text{.}\ \bot$$
     $$\{\emptyset\}_{(\gamma\to o)\to o} := \lambda Y_{\gamma\to o}.(Y = (\lambda X_\gamma\text{.}\ \bot))$$

     Expanded:

     $$\neg((\lambda B_{\gamma\to o}.(\forall X_\gamma.(B\ X) \Rightarrow \bot)) = (\lambda Y_{\gamma\to o}.(Y = (\lambda X_\gamma\text{.}\ \bot)))$$

- Challenge: Equality and Extensionality / avoid additional axioms

Overview

- Type Theory: Syntax and Semantics
- Leibniz Equality, Extensionality
- Extensional Higher-Order Resolution
- Primitive Equality
  - Extensional Higher-Order Paramodulation
  - Extensional Higher-Order RUE-Resolution
- (Completeness: Adaption of the Abstract Consistency Method)
- Implementation and Examples

Type Theory: HOL & Simply Typed λ-Calc.

- Types: $(i) \quad \{i, o\} \in T \qquad (ii) \quad \alpha, \beta \in T,\ then\ \alpha \rightarrow \beta \in T$
- Terms:
  - (i) $V_\alpha \subseteq \Lambda;\ \ V_\alpha\ (infinite)\ Sets\ of\ Variables\ (\alpha \in T)$
  - (ii) $C_\alpha \subseteq \Lambda;\ \ C_\alpha\ Sets\ of\ Constants\ (\alpha \in T)$
    Required: $\neg_{o\rightarrow o} \in C_{o\rightarrow o},\ \ \vee_{o\rightarrow(o\rightarrow o)} \in C_{o\rightarrow(o\rightarrow o)},$
    $\Pi_{(\alpha\rightarrow o)\rightarrow o} \in C_{(\alpha\rightarrow o)\rightarrow o}$ $(\alpha \in T)$
  - (iii) Application: $\mathbf{A}_{\alpha\rightarrow\beta}, \mathbf{B}_\alpha \in \Lambda,\ then\ (\mathbf{A}\ \mathbf{B})_\beta \in \Lambda$
  - (iii) Abstraction: $X_\alpha \in V_\alpha, \mathbf{A}_\beta \in \Lambda,\ then\ (\lambda X.\mathbf{A})_{\alpha\rightarrow\beta} \in \Lambda$
- λ-Conversion / β-Normalform / $\beta\eta$-(Head-)Normalform:

$$\lambda X_\gamma.\mathbf{A} \leftrightarrow^\alpha \lambda Y_\gamma.\mathbf{A}[Y/X]$$

$$(\lambda X_\gamma.\mathbf{A})\ \mathbf{B}_\gamma \rightarrow^\beta \mathbf{A}[\mathbf{B}/X] \qquad \lambda X.\mathbf{A}\ X \rightarrow^\eta \mathbf{A}\ \ if\ X \notin Free(\mathbf{A})$$

Standard Semantics

- Domains: Choose: D_ι
  Required: $D_o = \{\bot, \top\}, \quad D_{\alpha\to\beta} = Funcs(D_\alpha, D_\beta)$
- Interpretation: Choose: $I_\alpha : C_\alpha \longrightarrow D_\alpha$
  Required: $I(\neg_{o\to o})$ *and* $I(\vee_{o\to(o\to o)})$ as intended
  $I(\Pi_{(\alpha\to o)\to o})$ is a predicate $p \in D_{(\alpha\to o)\to o}$, such that for every $q_{\alpha\to o} \in D_{\alpha\to o} : p\, q_{\alpha\to o} = \top$ iff q holds for all $a \in D_\alpha$ $\Rightarrow \forall X_\alpha.\mathbf{A}_o$ is coded as $\Pi\, (\lambda X_\alpha.\mathbf{A}_o)$
- Variable Assignment: $\varphi_\alpha : V_\alpha \longrightarrow D_\alpha$
- Interpretation of terms: $I_\varphi : \Lambda_\alpha \longrightarrow D_\alpha$
  $I_\varphi(X) = \varphi(X), \quad I_\varphi(c) = I(c), \quad I_\varphi(\mathbf{A}\,\mathbf{B}) = I_\varphi(A)\ I_\varphi(B),$
  $I_\varphi(\lambda X_\alpha.\mathbf{B}_\beta) = f \in D_{\alpha\to\beta}$, such that $fa = I_{\varphi[a/X]}(\mathbf{B})$ for all $a \in D_\alpha$
- Model: $\mathcal{M} = (\mathcal{D} : \{D_\alpha\}, \mathcal{I} : \{I_\alpha\})$, Satisfisfiability and Validity as usual

Henkin Semantics

- like Standard Semantics but: $D_{\alpha \rightarrow \beta} \subseteq Functions(D_\alpha, D_\beta)$
- Required: I_Φ is total (i.e. each term has a Denotation)
- It holds:
  - Each Standard Model is a Henkin Model
  - There are more Henkin Models as Standard Models
  - A formula that is valid in Henkin Sem. is also valid in Standard Sem.
  - There are less formulae valid in Henkin semantics

$\Rightarrow$ Goedel 1931: Standard Semantics allows no complete calculi

$\Rightarrow$ Henkin 1950: Most general notion that allows complete calculi

Properties of Type Theory

- Comprehension Principles are built-in $(\exists F_{\alpha\to\beta}\centerdot \forall X_\alpha \centerdot (F\ X) = \mathbf{A}_\beta)$
- Optional: Axiom of Choice $(\exists F_{(\alpha\to o)\to\alpha}\centerdot \forall M_{\alpha\to o}\centerdot (\exists X_\alpha \centerdot M\ X) \Rightarrow M\ (F\ M))$ and Descriptionoperator ι
- Leibniz Equality denotes intended Equalityrelation (i.e. a functional congruencerelation)

$$\doteq^\alpha \ := \lambda X_\alpha \centerdot \lambda Y_\alpha \centerdot \forall P_{\alpha\to o}\centerdot PX \Rightarrow PY$$

$$i.e.:\quad a_\alpha \doteq^\alpha b_\alpha \qquad expands\ to \qquad \forall P_{\alpha\to o}\centerdot Pa \Rightarrow Pb$$

$\Rightarrow$ Equality is built-in in Type Theory (with Standard or Henkin Semantics)

but . . .

Disadvantages of Leibniz Equality

- Extensionality Axioms needed: E.g. Andrews' Higher Order Resolution (1971), Huet's Constrained Resolution (1972), Jensen & Pietrowski (1972), Wolfram (1993), Kohlhase (1994), TPS-System, HOL-System

  – EXT-Func$^{\doteq}$: $\forall F_{\alpha\to\beta}\centerdot \forall G_{\alpha\to\beta}(\forall X_{\beta}\centerdot F\ X \doteq G\ X) \Rightarrow F \doteq G$

    expanded: $\forall F_{\alpha\to\beta}\centerdot \forall G_{\alpha\to\beta}\centerdot (\forall X_{\beta}\centerdot \forall P_{\beta\to o}\centerdot P\ (F\ X) \Rightarrow P\ (G\ X) \Rightarrow \forall Q_{(\alpha\to\beta)\to o}\centerdot P\ F \Rightarrow P\ G$

    clauses: $\mathcal{C}_1 : [p_{\beta\to o}\ (F\ s_{\beta})]^T \vee [Q\ F]^F \vee [Q\ G]^T, \mathcal{C}_2 : [p_{\beta\to o}\ (G\ s_{\beta})]^T \vee [Q\ F]^F \vee [Q\ G]^T$

  – EXT-Bool$^{\doteq}$: $\forall A_o\centerdot \forall B_o\centerdot (A \Leftrightarrow B) \Leftrightarrow A \doteq^o B$

    expanded: $\forall A_o\centerdot \forall B_o\centerdot (A \Leftrightarrow B) \Leftrightarrow (\forall Q_{o\to o}\centerdot Q\ A \Rightarrow Q\ B$

    clauses: $\mathcal{C}_1 : [A]^F \vee [B]^F \vee [P\ A]^F \vee [P\ B]^T, \mathcal{C}_2 : [A]^T \vee [B]^T \vee [P\ A]^F \vee [P\ B]^T, \mathcal{C}_3 :$ $[A]^F \vee [B]^T \vee [p\ A]^T, \mathcal{C}_4 : [A]^F \vee [B]^T \vee [p\ B]^F, \mathcal{C}_5 : [A]^T \vee [B]^F \vee [p\ A]^T, \mathcal{C}_6 : [A]^T \vee [B]^F \vee [p\ B]^F$

Extensional HO Resolution $\mathcal{ER}$ I

Clause Normalisation $\mathcal{CNF}$

$$\frac{\mathbf{C} \vee [\mathbf{A} \vee \mathbf{B}]^T}{\mathbf{C} \vee [\mathbf{A}]^T \vee [\mathbf{B}]^T}\ \vee^T \qquad \frac{\mathbf{C} \vee [\mathbf{A} \wedge \mathbf{B}]^F}{\mathbf{C} \vee [\mathbf{A}]^F}\ \vee_l^F \qquad \frac{\mathbf{C} \vee [\mathbf{A} \wedge \mathbf{B}]^F}{\mathbf{C} \vee [\mathbf{B}]^F}\ \vee_r^F$$

$$\frac{\mathbf{C} \vee [\neg \mathbf{A}]^T}{\mathbf{C} \vee [\mathbf{A}]^F}\ \neg^T \qquad \frac{\mathbf{C} \vee [\neg \mathbf{A}]^F}{\mathbf{C} \vee [\mathbf{A}]^T}\ \neg^F \qquad \frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^T \quad X_\alpha \text{ new variable}}{\mathbf{C} \vee [\mathbf{A}\ X]^T}\ \Pi^T$$

$$\frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^F \quad \mathsf{s}_\alpha \text{ is a Skolem term for this clause}}{\mathbf{C} \vee [\mathbf{A}\ \mathsf{s}_\alpha]^F}\ \Pi^F$$

Extensional HO Resolution $\mathcal{ER}$ II

Constrained Resolution

$$\frac{[\mathbf{A}]^\alpha \vee \mathbf{C} \quad [\mathbf{B}]^\beta \vee \mathbf{D} \quad \alpha \neq \beta}{\mathbf{C} \vee \mathbf{D} \vee [\mathbf{A} = \mathbf{B}]^F}\ Res \qquad \frac{[\mathbf{A}]^\alpha \vee [\mathbf{B}]^\alpha \vee \mathbf{C} \quad \alpha \in \{T, F\}}{[\mathbf{A}]^\alpha \vee \mathbf{C} \vee [\mathbf{A} = \mathbf{B}]^F}\ Fac$$

$$\frac{[Q_\gamma\ \overline{\mathbf{U}^k}]^\alpha \vee \mathbf{C} \quad \mathbf{P} \in \mathcal{GB}_\gamma^{\{\neg,\vee\}\cup\{\Pi^\beta|\beta\in\mathcal{T}^k\}}}{[Q_\gamma\ \overline{\mathbf{U}^k}]^\alpha \vee \mathbf{C} \vee [Q = \mathbf{P}]^F}\ Prim$$

Note: Resolution or Factorization on Unification Constraints is not allowed

Primitive Substitution: $\quad \exists P_{\alpha\to o}.\, P\ a_\alpha \quad \xrightarrow{CNF} \quad \mathcal{C}_1 : [P\ a]^F$

$$Prim(\mathcal{C}_1, [\lambda X_\alpha.\, \neg(P'X)/P]) : \qquad \mathcal{C}_2 : [P'a]^T$$

Extensional HO Resolution $\mathcal{ER}$ III

Higher-Order Pre-Unification

$$\frac{\mathbf{C} \vee [\mathbf{A}_{\alpha\to\beta}\ \mathbf{C}_\alpha = \mathbf{B}_{\alpha\to\beta}\ \mathbf{D}_\alpha]^F}{\mathbf{C} \vee [\mathbf{A} = \mathbf{B}]^F \vee [\mathbf{C} = \mathbf{D}]^F}\ Dec$$

$$\frac{\mathbf{C} \vee [\mathbf{A} = \mathbf{A}]^F}{\mathbf{C}}\ Triv \qquad \frac{\mathbf{C} \vee [F_\gamma\ \overline{\mathbf{U}^n} = h\ \overline{\mathbf{V}^m}]^F \quad \mathbf{G} \in \mathcal{GB}^h_\gamma}{\mathbf{C} \vee [F = \mathbf{G}]^F \vee [F\ \overline{\mathbf{U}^n} = h\ \overline{\mathbf{V}^m}]^F}\ FlexRigid$$

$$\frac{\mathbf{C} \vee [(\lambda X_\alpha\text{.}\mathbf{M}_\beta) = \mathbf{N}_{\alpha\to\beta}]^F}{\mathbf{C} \vee [(\lambda X_\alpha\text{.}\mathbf{M})\ s = \mathbf{N}\ s]^F}\ Func_1 \qquad \frac{\mathbf{C} \vee [(\lambda X_\alpha\text{.}\mathbf{M}_\beta) = (\lambda Y_\alpha\text{.}\mathbf{N}_\beta)]^F}{\mathbf{C} \vee [(\lambda X_\alpha\text{.}\mathbf{M})\ s = (\lambda X_\alpha\text{.}\mathbf{N})\ s]^F}\ Func_2$$

$$\frac{\mathbf{C} \vee [X = \mathbf{A}]^F \quad X \notin Free(\mathbf{A})}{(\mathbf{C}[\mathbf{A}/X])}\ Subst \qquad \frac{\mathcal{D} \quad \mathcal{C} \in \mathcal{CNF}(\mathcal{D})}{\mathcal{C}}\ Cnf$$

Extensional HO Resolution $\mathcal{ER}$ IV

Extensionality: Recursive calls of te overall refutation process from within the unification process

$$\frac{\mathbf{C} \vee [\mathbf{M}_o = \mathbf{N}_o]^F}{\mathbf{C} \vee [\mathbf{M}_o \Leftrightarrow \mathbf{N}_o]^F}\ Equiv \qquad \frac{\mathbf{C} \vee [\mathbf{M}_\alpha = \mathbf{N}_\alpha]^F}{\mathbf{C} \vee [\forall P_{\alpha \to o}\text{.}\, P\ M \Rightarrow P\ N]^F}\ Leib$$

$$\frac{\mathbf{C} \vee [\mathbf{M}_{\alpha \to \beta} = \mathbf{N}_{\alpha \to \beta}]^F \quad s_\alpha \text{ Skolem term for this clause}}{\mathbf{C} \vee [\mathbf{M}\ s = \mathbf{N}\ s]^F}\ Func$$

- Claim: Rule Leib can be restricted to type ι

$\Rightarrow$ First Henkin complete refutation calculus that does not need additional axiom in the search space (CADE-15)

$\Rightarrow$ Difference to Huet (1972): Eager Unification becomes essential

Example: $(\lambda X_\alpha \blacksquare joung_{\alpha\to o}\ X \land male_{\alpha\to o}\ X) = (\lambda X_\alpha \blacksquare male\ X \land joung\ X)$

$$\forall P_{(\alpha\to o)\to o} \blacksquare P\ (\lambda X_\alpha \blacksquare joung_{\alpha\to o}\ X \land male_{\alpha\to o}\ X) \Rightarrow P\ (\lambda X_\alpha \blacksquare male\ X \land joung\ X)$$

c1: $\boxed{[p\ (\lambda X \blacksquare joung\ X \land male\ X]^T}$ *c2:* $\boxed{[p\ (\lambda X \blacksquare male\ X \land joung\ X]^F}$

c3: $\boxed{[(p\ (\lambda X \blacksquare joung\ X \land male\ X)) = (p\ (\lambda X \blacksquare male\ X \land joung\ X))]^F}$

c4: $\boxed{[(\lambda X \blacksquare joung\ X \land male\ X) = (\lambda X \blacksquare male\ X \land joung\ X)]^F}$

c5: $\boxed{[(joung\ s \land male\ s) = (male\ s \land joung\ s)]^F}$

c6: $\boxed{[(joung\ s \land male\ s) \equiv (male\ s \land joung\ s)]^F}$

c7: $\boxed{[joung\ s]^T \lor [male\ s]^T}$ *c8:* $\boxed{[joung\ s]^T}$ *c9:* $\boxed{[male\ s]^T}$ *c10:* $\boxed{[joung\ s]^F \lor [male\ s]^F}$

Addinq Primitive Equality

- Why: Leibniz Equality introduces flexible heads (Primitive Substitution)

$\Rightarrow$ Improve Equality Treatment by adding a Primitive Equality Treatment

- Reflexivity Definition:
  $\doteq^{\alpha} \; := \lambda X_{\alpha\blacksquare} \lambda Y_{\alpha\blacksquare} \forall Q_{\alpha\to\alpha\to o\blacksquare} (\forall Z_{\alpha\blacksquare} (Q\ Z\ Z)) \Rightarrow (Q\ X\ Y)$

- Modified Leibniz Equality:
  $\dddot{=}^{\alpha} \; := \lambda X_{\alpha\blacksquare} \lambda Y_{\alpha\blacksquare} \forall P_{\alpha\to o\blacksquare} ((a_o \vee \neg\, a_o) \wedge P\ X) \Rightarrow ((b_o \vee \neg\, b_o) \wedge P\ Y)$

$\Rightarrow$ It is not decidable wheter a given input problem contains Defined Equations

$\Rightarrow$ We still have to take care of Defined Equations even if we now add a Primitive Equality Treatment to the calculus

Extensional HO Paramodulation EP I

$$\frac{[\mathbf{A}[\mathbf{T}_\beta]]^\alpha \vee C \quad [\mathbf{L} =^\beta \mathbf{R}]^T \vee D}{[\mathbf{A}[\mathbf{R}]]^\alpha \vee C \vee D \vee [\mathbf{T} =^\beta \mathbf{L}]^F}\ Para$$

$$\frac{[\mathbf{A}]^\alpha \vee C \quad [\mathbf{L} =^\beta \mathbf{R}]^T \vee D}{[P_{\alpha\to o}\ \mathbf{R}]^\alpha \vee C \vee D \vee [\mathbf{A} =^o P_{\beta\to o}\ \mathbf{L}]^F}\ Para'$$

- negative Equation Literals are still interpreted as Unification Constraints
- No Resolution on Unification Constraints

$$\frac{[p\ (f\ (f\ a))]^T \quad [f = g]^T}{[p\ (f\ (g\ a))]^T}\ Para, Uni$$

$$\frac{[p\ (f\ (f\ a))]^T \quad [f = g]^T}{[p\ (g\ (f\ a))]^T}\ Para, Uni$$

$$\cfrac{\cfrac{[p\ (f\ (f\ a))]^T \quad [f = g]^T}{[P\ g]^T \vee [(P\ f) = (p\ (f\ (f\ a)))]^F}\ Para'}{\begin{array}{ll}[p\ (f\ (f\ a))]^T & with\ [\lambda X_\blacksquare (p\ (f\ (f\ a)))/P] \\ [p\ (f\ (g\ a))]^T & with\ [\lambda X_\blacksquare (p\ (f\ (X\ a)))/P] \\ [p\ (g\ (f\ a))]^T & with\ [\lambda X_\blacksquare (p\ (X\ (f\ a)))/P] \\ [p\ (g\ (g\ a))]^T & with\ [\lambda X_\blacksquare (p\ (X\ (X\ a)))/P]\end{array}}\ UNI$$

Extensional HO Paramodulation EP II

- In FO Reflexivity Rule needed $\longrightarrow$ already given here by UNI

$$\frac{[(fX) = (fa)]^F}{\square}\ Ref \qquad\qquad \frac{[(fX) = (fa)]^F}{\square}\ UNI$$

- $\mathcal{C}_1 : [\overbrace{nempty\ (\lambda X_\iota\text{▪}((joung\ X \wedge male\ X) \wedge smart\ X))}^{\{X|joung\ X \wedge male\ X \wedge smart\ X\} \in nempty_{(\iota \to o) \to o}}]^T \qquad \mathcal{C}_2 : [\ \overbrace{joung\ X \wedge male\ X = boy\ X}^{\forall X_\iota\text{▪}(joung\ X \wedge male\ X) = boy\ X}\ ]^T$

  To show: $\mathcal{C}_3 : [\underbrace{nempty\ (\lambda X_\iota\text{▪}boy\ X \wedge smart\ X)}_{\{X|boy\ X \wedge smart\ X\} \in nempty}]^F$

- Employ Term-Rewriting Idea with Paramodulation rule:

$$Para(\mathcal{C}_1, \mathcal{C}_2), UNI : \quad \mathcal{C}_4 : [nempty\ (\lambda X\text{▪}(boy\ X \wedge smart\ X))]^T$$

$$Res(\mathcal{C}_4, \mathcal{C}_3), UNI : \quad \square$$

New Problem in EP: Positive Equation Literals

- Contrast to FO: Contradictory positive Equation Literals

$$[\mathbf{A}_o = \neg\mathbf{A}_o]^T \qquad [\overbrace{(\lambda X \mathbf{.} male\ X) = (\lambda X \mathbf{.} \neg(male\ X))}^{\{X|male\ X\}=\overline{\{X|male\ X\}}}]^T$$

- Additional positive Extensionality Rules needed

$$\frac{\mathbf{C} \vee [\mathbf{M}_o = \mathbf{N}_o]^T}{\mathbf{C} \vee [\mathbf{M}_o \Leftrightarrow \mathbf{N}_o]^T}\ Equiv' \qquad \frac{\mathbf{C} \vee [\mathbf{M}_{\alpha\to\beta} = \mathbf{N}_{\alpha\to\beta}]^T \quad X \text{ new}}{\mathbf{C} \vee [\mathbf{M}\ X = \mathbf{N}\ X]^T}\ Func'$$

- New Rules further strengthen the Difference-Reduction character
- Henkin Completeness without additional axioms
  (proved yet only with additional FlexFlex-Rule in UNI)

Extensional HO Paramodulation EP III

- Slightly modify the problem: No Term-Rewriting possible at all !

$$\mathcal{C}_1' : [nempty(\lambda X_{\iota \blacksquare}((joung\ X \wedge smart\ X) \wedge male\ X))]^T \qquad \mathcal{C}_2 : [(joung\ X \wedge male\ X) = boy\ X]^T$$

To show: $\mathcal{C}_3 : [nempty\ (\lambda X_{\iota \blacksquare} boy\ X \wedge smart\ X)]^F$

$$Para(\mathcal{C}_1, \mathcal{C}_2) : \quad \mathcal{C}_4 : [nempty\ (\lambda X_{\blacksquare}(boy\ X \wedge smart\ X))]^T \vee [(joung\ X \wedge male\ X) = (joung\ X \wedge smart\ X)]^F$$

$$\ldots$$

Instead one has to employ the Difference-Reduction idea

$$\begin{array}{ll} Res(\mathcal{C}_1', \mathcal{C}_3) : & \mathcal{C}_4 : [(nempty(\lambda X_{\iota \blacksquare}((joung\ X \wedge smart\ X) \wedge male\ X))) = \\ & \quad (nempty\ (\lambda X_{\iota \blacksquare} boy\ X \wedge smart\ X))]^F \\ Dec(\mathcal{C}_4), Func, Equiv : & \mathcal{C}_5 : [((joung\ s \wedge smart\ s) \wedge male\ s) \equiv (boy\ s \wedge smart\ s)]^F \\ Equiv'(\mathcal{C}_2) : & \mathcal{C}_6 : [(joung\ X \wedge male\ X) \equiv boy\ X]^T \\ CNF(\mathcal{C}_5, \mathcal{C}_6) : & \ldots \end{array}$$

$\Box$

$\Rightarrow$ Unavoidable Mix of Term-Rewriting & Difference-Reduction

Extensional HO RUE-Resolution ERUE I

- Motivation: Try to find a pure Difference-Reducing calculus
- Extensional HO RUE-Resolution:
  - Remove Paramodulation Rule and instead ...
  - Allow to resolve and factorize also on Unification Constraints

- $\mathcal{C}_1 : [nempty\ (\lambda X_\iota \blacksquare ((joung\ X \wedge male\ X) \wedge smart\ X))]^T \qquad \mathcal{C}_2 : [(joung\ X \wedge male\ X) = boy\ X]^T$

$$\mathcal{C}_3 : [nempty\ (\lambda X_\iota \blacksquare boy\ X \wedge smart\ X)]^F$$

$$\begin{array}{ll} Res(\mathcal{C}_1, \mathcal{C}_3), Dec, Func : & \mathcal{C}_4 : [((joung\ s \wedge male\ s) \wedge smart\ s) = (boy\ s \wedge smart\ s)]^F \\ Dec(\mathcal{C}_4), Triv : & \mathcal{C}_5 : [(joung\ s \wedge male\ s) = boy\ s]^F \\ Res(\mathcal{C}_5, \mathcal{C}_2), UNI : & \ldots\ \Box \end{array}$$

Extensional HO RUE-Resolution ERUE II

- Slightly modified example

- $\mathcal{C}_1 : [nempty\ (\lambda X_\iota\blacksquare((joung\ X \land smart\ X) \land male\ X))]^T \qquad \mathcal{C}_2 : [(joung\ X \land male\ X) = boy\ X]^T$

$$\mathcal{C}_3 : [nempty\ (\lambda X_\iota\blacksquare boy\ X \land smart\ X)]^F$$

$$\begin{array}{ll}
Res(\mathcal{C}_1, \mathcal{C}_3), Dec, Func : & \mathcal{C}_4 : [((joung\ s \land smart\ s) \land male\ s) = (boy\ s \land smart\ s)]^F \\
Equiv(\mathcal{C}_4) : & \mathcal{C}_5 : [((joung\ s \land smart\ s) \land male\ s) \equiv (boy\ s \land smart\ s)]^F \\
Equiv'(\mathcal{C}_2) : & \mathcal{C}_6 : [(joung\ X \land male\ X) \equiv boy\ X]^T \\
CNF(\mathcal{C}_5, \mathcal{C}_6) : & \ldots\ \Box
\end{array}$$

$\Rightarrow$ Pure Difference-Reduction approach probably better to handle in practice

- Henkin Completeness without additional axioms
  (proved yet only with additional FlexFlex-Rule in UNI)

Conclusion

- Henkin complete refutation approaches for classical Type Theory that do not need additional axiom and which treat equality and extensionality in a rather appropriate way:
  - Extensional HO Resolution ER
  - Extensional HO Paramodulation EP (Compl. modulo FlexFlex)
  - Extensional HO RUE-Resolution ERUE (Compl. modulo FlexFlex)
- Implementation LEO
- Experiments very promising for examples about sets
- For Completeness Proofs: Adaption of Smullyan's / Andrews' Unifying Principle to HOL with Henkin Semantics

Further Work

- Clarify the many open questions
- Compare EP EP and ERUE
- Integration of LEO as powerful Deductive Agent in ΩMEGA's Agentmechanism for supporting Interactive Theorem Proving (talk of last week)
- Cooperation with TPS and First-Order ATP's via Proof Planning layer in ΩMEGA
- Reimplementation of LEO in Oz: Exploit Concurrency

Higher-Order Abstract Consistency

Definition 0.1 (Properties for Abstract Consistency Classes). Let Γ_Σ be a class of sets of Σ-sentences.

∇_c If $\mathbf{A}$ is atomic, then $\mathbf{A} \notin \Phi$ or $\neg\mathbf{A} \notin \Phi$.

$\nabla_\neg$ If $\neg\neg\mathbf{A} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$.

∇_β If $\mathbf{A} \in \Phi$ and $\mathbf{B}$ is the β-normal form of $\mathbf{A}$, then $\mathbf{B} * \Phi \in \Gamma_\Sigma$.

$\nabla_\mathfrak{f}$ If $\mathbf{A} \in \Phi$ and $\mathbf{B}$ is the $\beta\eta$-normal form of $\mathbf{A}$, then $\mathbf{B} * \Phi \in \Gamma_\Sigma$.

$\nabla_\vee$ If $\mathbf{A} \vee \mathbf{B} \in \Phi$, then $\Phi * \mathbf{A} \in \Gamma_\Sigma$ or $\Phi * \mathbf{B} \in \Gamma_\Sigma$.

$\nabla_\wedge$ If $\neg(\mathbf{A} \vee \mathbf{B}) \in \Phi$, then $\Phi \cup \{\neg\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$.

$\nabla_\forall$ If $\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \mathbf{F}\mathbf{W} \in \Gamma_\Sigma$ for each $\mathbf{W} \in$ *cwff*$_\alpha(\Sigma)$.

$\nabla_\exists$ If $\neg\Pi^\alpha \mathbf{F} \in \Phi$, then $\Phi * \neg(\mathbf{F}w) \in \Gamma_\Sigma$ for any constant $w \in \Sigma_\alpha$, which does not occur in Φ.

$\nabla_\mathfrak{b}$ If $\neg(\mathbf{A} \doteq^o \mathbf{B}) \in \Phi$, then $\Phi \cup \{\mathbf{A}, \neg\mathbf{B}\} \in \Gamma_\Sigma$ or $\Phi \cup \{\neg\mathbf{A}, \mathbf{B}\} \in \Gamma_\Sigma$.

$\nabla_\mathfrak{q}$ If $\neg(\mathbf{F} \doteq^{\alpha\to\beta} \mathbf{G}) \in \Phi$, then $\Phi * \neg(\mathbf{F}w \doteq^\beta \mathbf{G}w) \in \Gamma_\Sigma$ for any constant $w \in \Sigma_\alpha$, which does not occur in Φ.

$\nabla_\mathfrak{e}$ **(r)** $\neg(\mathbf{A} =^\alpha \mathbf{A}) \notin \Phi$

(s) if $\mathbf{F}[\mathbf{A}]_p \in \Phi$ and $\mathbf{A} = \mathbf{B} \in \Phi$, then $\Phi * \mathbf{F}[\mathbf{B}]_p \in \Gamma_\Sigma$

Higher-Order Abstract Consistency