Working with Automated Reasoning Tools
– Higher-Order Resolution –

Christoph Benzmüller and Geoff Sutcliffe

Department of Computer Science, Saarland University

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Approaches to Higher-Order Resolution: Andrews' $\mathcal{R}$

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We present and discuss Andrews' higher-order resolution calculus [Andrews71]; we call this calculus $\mathcal{R}$

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- we omit explicit rules for α- and β-convertibility and instead treat them implicitly, i.e. we assume that the presented rules operate on input and generate output in β-normal form and we automatically identify terms which differ only with respect to the names of bound variables

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Clause Normalisation

- $\mathcal{R}$ introduces only four rules belonging to clause normalisation: negation elimination, conjunction elimination, existential elimination, and universal elimination

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- negation elimination: $$\frac{\mathbf{C} \vee [\neg \mathbf{A}]^{\mathsf{T}}}{\mathbf{C} \vee [\mathbf{A}]^{\mathsf{F}}}\ \neg^{\mathsf{T}} \qquad \frac{\mathbf{C} \vee [\neg \mathbf{A}]^{\mathsf{F}}}{\mathbf{C} \vee [\mathbf{A}]^{\mathsf{T}}}\ \neg^{\mathsf{F}}$$

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- conjunction/disjunction elimination:

$$\frac{\mathbf{C} \vee [\mathbf{A} \vee \mathbf{B}]^T}{\mathbf{C} \vee [\mathbf{A}]^T \vee [\mathbf{B}]^T}\ \vee^T \qquad \frac{\mathbf{C} \vee [\mathbf{A} \vee \mathbf{B}]^F}{\mathbf{C} \vee [\mathbf{A}]^F}\ \vee^F_l \qquad \frac{\mathbf{C} \vee [\mathbf{B} \vee \mathbf{B}]^F}{\mathbf{C} \vee [\mathbf{B}]^F}\ \vee^F_r$$

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$$\frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^\mathsf{T}}{\mathbf{C} \vee [\mathbf{A}\ \mathsf{X}_\alpha]^\mathsf{T}}\ \Pi^\mathsf{T} \qquad \frac{\mathbf{C} \vee [\Pi^\alpha \mathbf{A}]^\mathsf{F}}{\mathbf{C} \vee [\mathbf{A}\ \mathsf{s}_\alpha]^\mathsf{F}}\ \Pi^\mathsf{F}$$

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X_α is a new free variable and s_α is a new Skolem term

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- additionally Andrews presents rules addressing commutativity and associativity of the $\vee$-operator connecting the clauses literals; we treat this implicit here
- we refer with $\mathsf{Cnf}(\mathbf{A})$ to the set of clauses obtained from formula $\mathbf{A}$ by exhaustive clause normalisation

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Resolution & Factorisation

- Instead of a resolution and a factorisation rule — which work in connection with unification — Andrews presents a simplification and a cut rule. The cut rule is only applicable to clauses with two complementary literals which have identical atoms. Similarly Sim is defined only for clauses with two identical literals. In order to generate identical literal atoms during the refutation process these two rules have to be combined with the substitution rule Sub presented below.

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- Simplification: $$\frac{[\mathbf{A}]^{\mu} \vee [\mathbf{A}]^{\mu} \vee \mathbf{C}}{[\mathbf{A}]^{\mu} \vee \mathbf{C}}\ \text{Sim}$$

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- Cut: $$\frac{[\mathbf{A}]^{\mu} \vee \mathbf{C} \quad [\mathbf{A}]^{\nu} \vee \mathbf{D}}{\mathbf{C} \vee \mathbf{D}}\ \text{Cut}$$

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- As higher-order unification was still an open problem in 1971 calculus $\mathcal{R}$ employs the British museum method instead, i.e. it provides a substitution rule that allows to blindly instantiate free variables by arbitrary terms. As the instantiated terms may contain logical constants, instantiation of variables in proper clauses may lead to pre-clauses, which must be normalised again with the clause normalisation rules.

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- Extensionality axioms

$$\mathbf{EXT}^{\dot=}_{\alpha\to\beta}: \quad \forall F_{\alpha\to\beta}.\forall G_{\alpha\to\beta}.(\forall X_\beta.F\,X \doteq G\,X) \Rightarrow F \doteq G$$

$$\mathbf{EXT}^{\dot=}_{o}: \quad \forall A_o.\forall B_o.(A \Leftrightarrow B) \Rightarrow A \doteq^o B$$

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Extensionality Treatment (contd.)

- The extensionality clauses derived from the extensionality axioms have the following form (note the many free variables, especially at literal head position, that are introduced into the search space – they heavily increase the amount of blind search in any attempt to automate the calculus):

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$$\mathcal{E}_{\mathbf{1}}^{\alpha\rightarrow\beta} : [\mathsf{p}\ (\mathbf{F}\ \mathsf{s})]^{\mathsf{T}} \vee [\mathbf{Q}\ \mathbf{F}]^{\mathsf{F}} \vee [\mathbf{Q}\ \mathbf{G}]^{\mathsf{T}}$$
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$\mathsf{p}_{\beta\to o}$, s_{α} are Skolem terms and A_o, B_o, $\mathsf{P}_{o\to o}$, $\mathsf{Q}_{(\alpha\to\beta)\to o}$ are new free variables.

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- the extensionality treatment in $\mathcal{R}$ simply assumes to add at the beginning of the refutation process the above clauses obtained from the extensionality axioms
- the proof search can be graphically illustrated as follows:

ext. axioms	proof search & blind variable instantiation

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Completeness

- [Andrews71] gives a completeness proof for calculus $\mathcal{R}$ with respect to the semantical notion of V-complexes (corresponds to our weakest model class $\mathfrak{M}_\beta(\Sigma)$)

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- as the extensionality principles are not valid in this rather weak semantical structures, the extensionality axioms are not needed in this completeness proof
- Theorem: (V-completeness of $\mathcal{R}$) The calculus $\mathcal{R}$ is (sound and) complete with respect to the notion of V-complexes.

  Proof: [Andrews71].

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Example Proofs

Exercise: How are the following theorems proved in calculus $\mathcal{R}$?

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- The set of all red balls equals the set of all balls that are red: $\{\mathsf{X}|\mathsf{red}\ \mathsf{X} \wedge \mathsf{ball}\ \mathsf{X}\} = \{\mathsf{X}|\mathsf{ball}\ \mathsf{X} \wedge \mathsf{red}\ \mathsf{X}\}$. This problem can be encoded as

$$(\lambda \mathsf{X}_{\iota}\text{.}\mathsf{red}\ \mathsf{X} \wedge \mathsf{ball}\ \mathsf{X}) = (\lambda \mathsf{X}_{\iota}\text{.}\mathsf{ball}\ \mathsf{X} \wedge \mathsf{red}\ \mathsf{X})$$

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- All unary logical operators $O_{o \to o}$ which map the propositions a and b to $\top$ consequently also map $a \wedge b$ to $\top$:

$$\forall O_{o \to o} . (O\ a_o) \wedge (O\ b_o) \Rightarrow (O\ (a_o \wedge b_o))$$

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Exercise: How are the following theorems proved in calculus $\mathcal{R}$?

- In Henkin semantics the domain $\mathcal{D}_o$ of all Booleans contains exactly the truth values $\bot$ and $\top$. Consequently the domain of all mappings from Booleans to Booleans contains exactly contains in each Henkin model at most four elements. And because of the requirement, that the function domains in Henkin models must be rich enough such that every term has a denotation, it follows that $\mathcal{D}_{o\to o}$ contains exactly the pairwise distinct denotations of the following four terms: $\lambda X_o. X_o$, $\lambda X_o. \neg X_o$, $\lambda X_o. \bot$, and $\lambda X_o. \top$. This theorem can be formulated as follows (where $f_{o\to o}$ is a constant):

$$(f = \lambda X_o. X_o) \vee (f = \lambda X_o. \neg X_o) \vee (f = \lambda X_o. \bot) \vee (f = \lambda X_o. \top)$$

Approaches to Higher-Order Resolution: Huet's $\mathcal{CR}$

Huet's Constrained Resolution $\mathcal{CR}$

We transform Huet's constrained resolution approach [Huet72,Huet73]. The calculus here is the unsorted fragment of the variant of Huet's approach as presented in [Kohlhase94]. In the remainder of this paper we refer to this calculus with $\mathcal{CR}$.

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- Calculus $\mathcal{CR}$ assumes that terms, literals, and clauses are implicitly reduced to β-normal form.

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- Calculus $\mathcal{CR}$ assumes that terms, literals, and clauses are implicitly reduced to β-normal form.
- Furthermore, we assume that α-equality is treated implicitly, i.e. we identify all terms that differ only with respect to the names of bound variables.

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Clause Normalisation

- [Huet72] does not explicitly present clause normalisation rules but assumes that they are given. Here we employ the rules $\neg^T$, $\neg^F$, $\vee^T$, $\vee^F_l$, $\vee^F_r$, Π^T, and Π^F as already defined for calculus $\mathcal{R}$ before.

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- As first-order unification is decidable and unitary it can be employed as a strong filter in first-order resolution [Robinson65].

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- As first-order unification is decidable and unitary it can be employed as a strong filter in first-order resolution [Robinson65].
- Unfortunately higher-order unification is not decidable (cf. [Lucchesi72,Huet73,Goldfarb81]) and thus it can not be applied in the sense of a terminating side computation in higher-order theorem proving.

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- As first-order unification is decidable and unitary it can be employed as a strong filter in first-order resolution [Robinson65].
- Unfortunately higher-order unification is not decidable (cf. [Lucchesi72,Huet73,Goldfarb81]) and thus it can not be applied in the sense of a terminating side computation in higher-order theorem proving.
- Huet therefore suggests in [Huet72,Huet73] to delay the unification process and to explicitly encode unification problems occurring during the refutation search as unification constraints.

Huet's Constrained Resolution $\mathcal{CR}$

Resolution & Factorisation (contd.)

- In his original approach Huet presented a hyper-resolution rule which simultaneously resolves on the resolution literals $\mathbf{A}^1, \ldots \mathbf{A}^{\mathsf{n}}$ $(1 \leq \mathsf{n})$ and $\mathbf{B}^1, \ldots \mathbf{B}^{\mathsf{m}}$ $(1 \leq \mathsf{m})$ of two given clauses and adds the unification constraint $[\neq^? (\mathbf{A}^1, \ldots \mathbf{A}^{\mathsf{n}}, \mathbf{B}^1, \ldots \mathbf{B}^{\mathsf{m}})]$ to the resolvent:

$$\frac{[\mathbf{A}^1]^\mu \vee \ldots \vee [\mathbf{A}^{\mathsf{n}}]^\mu \vee \mathbf{C} \quad [\mathbf{B}^1]^\nu \vee \ldots \vee [\mathbf{B}^{\mathsf{m}}]^\nu \vee \mathbf{D}}{\mathbf{C} \vee \mathbf{D} \vee [\neq^? (\mathbf{A}^1, \ldots \mathbf{A}^{\mathsf{n}}, \mathbf{B}^1, \ldots \mathbf{B}^{\mathsf{m}})]} \text{ Hres}$$

(where $\mu \neq \nu$).

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- In order to ease the comparison with the two other approaches discussed in this paper we instead employ a resolution rule Res and a factorisation rule Fac.

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- Constrained resolution:

$$\frac{[\mathbf{A}]^{\mu} \vee \mathbf{C} \quad [\mathbf{B}]^{\nu} \vee \mathbf{D}}{\mathbf{C} \vee \mathbf{D} \vee [\mathbf{A} \neq^{?} \mathbf{B}]} \text{Res}$$

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- Constrained factorisation:

$$\frac{[\mathbf{A}]^\mu \vee [\mathbf{B}]^\mu \vee \mathbf{C}}{[\mathbf{A}]^\mu \vee \mathbf{C} \vee [\mathbf{A} \neq^? \mathbf{B}]^{\mathsf{F}}} \text{ Fac}$$

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- One can easily prove by induction on $n + m$ that each proof step applying rule Hres can be replaced by a corresponding derivation employing Res and Fac.

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- One can easily prove by induction on $n + m$ that each proof step applying rule Hres can be replaced by a corresponding derivation employing Res and Fac.
- For a formal proof note that the unification constraint $[\neq^? (\mathbf{A}^1, \ldots \mathbf{A}^n, \mathbf{B}^1, \ldots \mathbf{B}^m)]$ is equivalent to $[\mathbf{A}^1 \neq^? \mathbf{A}^2] \vee [\mathbf{A}^2 \neq^? \mathbf{A}^3] \vee \ldots \vee [\mathbf{A}^{n-1} \neq^? \mathbf{A}^n] \vee [\mathbf{A}^n \neq^? \mathbf{B}^1] \vee [\mathbf{B}^1 \neq^? \mathbf{B}^2] \vee [\mathbf{B}^2 \neq^? \mathbf{B}^3] \vee \ldots \vee [\mathbf{B}^{n-1} \neq^? \mathbf{B}^n]$.

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- The higher-order pre-unification rules presented here are discussed in detail in [Benzmüller-PhD-99]. They furthermore closely reflect the rules as presented in [SnyderGallier89].

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- Elimination of trivial pairs:

$$\frac{\mathbf{C} \vee [\mathbf{A} \neq^? \mathbf{A}]}{\mathbf{C}}\ \text{Triv}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting

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- Elimination of trivial pairs:

$$\frac{\mathbf{C} \lor [\mathbf{A} \neq^? \mathbf{A}]}{\mathbf{C}}\ \mathsf{Triv}$$

- Decomposition

$$\frac{\mathsf{C} \lor [\mathsf{h}\overline{\mathbf{U}^{\mathsf{n}}} \neq^? \mathsf{h}\overline{\mathbf{V}^{\mathsf{n}}}]}{\mathsf{C} \lor [\mathbf{U}^1 \neq^? \mathbf{V}^1] \lor \ldots \lor [\mathbf{U}^{\mathsf{n}} \neq^? \mathbf{V}^{\mathsf{n}}]}\ \mathsf{Dec}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Elimination of λ-binders: (weak functional extensionality)

$$\frac{\mathbf{C} \vee [\mathbf{M}_{\alpha\to\beta} \neq^? \mathbf{N}_{\alpha\to\beta}]}{\mathbf{C} \vee [\mathbf{M}\ \mathsf{s}_\alpha \neq^? \mathbf{N}\ \mathsf{s}_\alpha]}\ \text{Func}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

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s_{α} is a new Skolem term.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

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- Imitation of rigid heads:

$$\frac{\mathbf{C} \vee [\mathsf{F}_\gamma\ \overline{\mathbf{U}^\mathsf{n}} \neq^? \mathsf{h}\ \overline{\mathbf{V}^\mathsf{m}}] \quad \mathbf{G} \in \mathcal{AB}^\mathsf{h}_\gamma}{\mathbf{C} \vee [\mathsf{F} \neq^? \mathbf{G}] \vee [\mathsf{F}\ \overline{\mathbf{U}^\mathsf{n}} \neq^? \mathsf{h}\ \overline{\mathbf{V}^\mathsf{m}}]}\ \text{FlexRigid}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

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$\mathcal{AB}^{\mathsf{h}}_\gamma$ is the set of general bindings of type γ for head h.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Huet points to the usefulness of eager unification to filter out clauses with non-unifiable unification constraints or to back-propagate the solutions of easily solvable constraints (e.g., in case of first-order unification problems occurring during the proof search): many of the higher-order unification problems occurring in practice are decidable and have only finitely many solutions.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

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- Hence, even though higher-order unification is generally not decidable it is sensible in practice to apply the unification algorithm with a particular resource, such that only those unification problems which may have further solutions beyond this bound need to be delayed.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- In our presentation of calculus $\mathcal{CR}$ we explicitly address the aspect of eager unification and substitution by rule Subst. This rule back-propagates eagerly computed unifiers to the literal part of a clause.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- In our presentation of calculus $\mathcal{CR}$ we explicitly address the aspect of eager unification and substitution by rule Subst. This rule back-propagates eagerly computed unifiers to the literal part of a clause.
- Eager unification & substitution:

$$\frac{\mathbf{C} \vee [\mathsf{X} \neq^{?} \mathbf{A}] \quad \mathsf{X} \notin \mathsf{free}(\mathbf{A})}{\mathbf{C}_{[\mathbf{A}/\mathsf{X}]}}\ \mathsf{Subst}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- The literal heads of our clauses may consist of set variables and it may be necessary to instantiate them with terms introducing new logical constant at head position in order to find a refutation.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- The literal heads of our clauses may consist of set variables and it may be necessary to instantiate them with terms introducing new logical constant at head position in order to find a refutation.
- Unfortunately not all appropriate instantiations can be computed with the calculus rules presented so far.

Huet’s Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- The literal heads of our clauses may consist of set variables and it may be necessary to instantiate them with terms introducing new logical constant at head position in order to find a refutation.
- Unfortunately not all appropriate instantiations can be computed with the calculus rules presented so far.
- To address this problem Huet’s approach provides the following splitting rules:

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Instantiate
  set variables:

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{T}} \lor \mathbf{C}}{[\mathsf{Q}]^{\mathsf{T}} \lor [\mathsf{R}]^{\mathsf{T}} \lor \mathbf{C} \lor [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{o} \lor \mathsf{R}_{o})]}\ \mathsf{S}_{\lor}^{\mathsf{T}}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{T}} \vee \mathbf{C}}{[\mathsf{Q}]^{\mathsf{T}} \vee [\mathsf{R}]^{\mathsf{T}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{\mathsf{o}} \vee \mathsf{R}_{\mathsf{o}})]}\ \mathsf{S}^{\mathsf{T}}_{\vee}$$

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{F}} \vee \mathbf{C}}{\begin{array}{c}[\mathsf{Q}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{\mathsf{o}} \vee \mathsf{R}_{\mathsf{o}})] \\ [\mathsf{R}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{\mathsf{o}} \vee \mathsf{R}_{\mathsf{o}})]\end{array}}\ \mathsf{S}^{\mathsf{F}}_{\vee}$$

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mu} \vee \mathbf{C}}{[\mathsf{Q}]^{\nu} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} \neg \mathsf{Q}_{\mathsf{o}}]}\ \mathsf{S}^{\mathsf{TF}}_{\neg}$$

(where $\mu \neq \nu$)

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{T}} \vee \mathbf{C}}{[\mathsf{Q}]^{\mathsf{T}} \vee [\mathsf{R}]^{\mathsf{T}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{\mathsf{o}} \vee \mathsf{R}_{\mathsf{o}})]}\ \mathsf{S}_{\vee}^{\mathsf{T}}$$

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{F}} \vee \mathbf{C}}{\begin{array}{c}[\mathsf{Q}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{\mathsf{o}} \vee \mathsf{R}_{\mathsf{o}})] \\ [\mathsf{R}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} (\mathsf{Q}_{\mathsf{o}} \vee \mathsf{R}_{\mathsf{o}})]\end{array}}\ \mathsf{S}_{\vee}^{\mathsf{F}}$$

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(where $\mu \neq \nu$)

$$\frac{[\mathsf{P}\ \mathbf{A}_{\alpha\rightarrow\mathsf{o}}]^{\mathsf{T}} \vee \mathbf{C}}{[\mathsf{M}_{\alpha\rightarrow\mathsf{o}}\ \mathsf{Z}]^{\mathsf{T}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} \Pi^{\alpha}\mathsf{M}]}\ \mathsf{S}_{\Pi}^{\mathsf{T}}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Instantiate set variables:

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mu} \vee \mathbf{C}}{[\mathsf{Q}]^{\nu} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A} \ \neq^{?} \neg \mathsf{Q}_{o}]}\ \mathsf{S}_{\neg}^{\mathsf{TF}}$$

(where $\mu \neq \nu$)

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{T}} \vee \mathbf{C}}{[\mathsf{Q}]^{\mathsf{T}} \vee [\mathsf{R}]^{\mathsf{T}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A} \ \neq^{?} (\mathsf{Q}_{o} \vee \mathsf{R}_{o})]}\ \mathsf{S}_{\vee}^{\mathsf{T}}$$

$$\frac{[\mathsf{P}\ \mathbf{A}]^{\mathsf{F}} \vee \mathbf{C}}{\begin{array}{c}[\mathsf{Q}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A} \ \neq^{?} (\mathsf{Q}_{o} \vee \mathsf{R}_{o})] \\ [\mathsf{R}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A} \ \neq^{?} (\mathsf{Q}_{o} \vee \mathsf{R}_{o})]\end{array}}\ \mathsf{S}_{\vee}^{\mathsf{F}}$$

$$\frac{[\mathsf{P}\ \mathbf{A}_{\alpha\rightarrow o}]^{\mathsf{T}} \vee \mathbf{C}}{[\mathsf{M}_{\alpha\rightarrow o}\ \mathsf{Z}]^{\mathsf{T}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A} \ \neq^{?} \Pi^{\alpha}\mathsf{M}]}\ \mathsf{S}_{\Pi}^{\mathsf{T}}$$

$$\frac{[\mathsf{P}\ \mathbf{A}_{\alpha\rightarrow o}]^{\mathsf{F}} \vee \mathbf{C}}{[\mathsf{M}_{\alpha\rightarrow o}\ \mathsf{s}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A} \ \neq^{?} \Pi^{\alpha}\mathsf{M}]}\ \mathsf{S}_{\Pi}^{\mathsf{F}}$$

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- Instantiate set variables:

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(where $\mu \neq \nu$)

$$\frac{[\mathsf{P}\ \mathbf{A}_{\alpha\to o}]^{\mathsf{T}} \vee \mathbf{C}}{[\mathsf{M}_{\alpha\to o}\ \mathsf{Z}]^{\mathsf{T}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} \Pi^{\alpha}\mathsf{M}]}\ \mathsf{S}_{\Pi}^{\mathsf{T}}$$

$$\frac{[\mathsf{P}\ \mathbf{A}_{\alpha\to o}]^{\mathsf{F}} \vee \mathbf{C}}{[\mathsf{M}_{\alpha\to o}\ \mathsf{s}]^{\mathsf{F}} \vee \mathbf{C} \vee [\mathsf{P}\ \mathbf{A}\ \neq^{?} \Pi^{\alpha}\mathsf{M}]}\ \mathsf{S}_{\Pi}^{\mathsf{F}}$$

- $\mathsf{S}_{\Pi}^{\mathsf{T}}$ and $\mathsf{S}_{\Pi}^{\mathsf{F}}$ are infinitely branching as they are parameterised over type α. $\mathsf{Q_o}, \mathsf{R_o}, \mathsf{M}_{\alpha\to o}, \mathsf{Z}_{\alpha}$ are new variables and s_{α} is a new Skolem constant.

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- A theorem which is not refutable in $\mathcal{CR}$ if the splitting rules are not available is $\exists A_o . A$:

Huet's Constrained Resolution $\mathcal{CR}$

Unification & Splitting (contd.)

- A theorem which is not refutable in $\mathcal{CR}$ if the splitting rules are not available is $\exists A_o.A$:
- After negation this statement normalises to clause $\mathcal{C}_1 : [A]^F$, such that none but the splitting rules are applicable. With the help of rule $S_{\neg}^{TF}$ and eager unification, however, we can derive $\mathcal{C}_2 : [A']^T$ which is then successfully resolvable against $\mathcal{C}_1$.

Huet's Constrained Resolution $\mathcal{CR}$

Extensionality Treatment

- On the one hand η-convertibility is built-in in higher-order unification, such that calculus $\mathcal{CR}$ already supports functional extensionality reasoning to a certain extend.

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- On the one hand η-convertibility is built-in in higher-order unification, such that calculus $\mathcal{CR}$ already supports functional extensionality reasoning to a certain extend.
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Extensionality Treatment

- On the one hand η-convertibility is built-in in higher-order unification, such that calculus $\mathcal{CR}$ already supports functional extensionality reasoning to a certain extend.
- On the other hand $\mathcal{CR}$ nevertheless fails to address full extensionality as it does not realise the required subtle interplay between the functional and Boolean extensionality principles.
- Without employing additional (Boolean and functional!) extensionality axioms $\mathcal{CR}$ is, e.g., not able to prove the rather simple examples presented before.

Huet's Constrained Resolution $\mathcal{CR}$

Proof Search

- Initially the proof problem is negated and normalised. The main proof search then operates on the generated clauses by applying the resolution, factorisation, and splitting rules.

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- Despite the possibility of eager unification $\mathcal{CR}$ generally foresees to delay the higher-order unification process in order to overcome the undecidability problem.

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- Initially the proof problem is negated and normalised. The main proof search then operates on the generated clauses by applying the resolution, factorisation, and splitting rules.
- Despite the possibility of eager unification $\mathcal{CR}$ generally foresees to delay the higher-order unification process in order to overcome the undecidability problem.
- When deriving a potentially empty clause (no normal literals), $\mathcal{CR}$ then tests whether the accumulated unification constraints justifying this particular refutation are solvable.

Huet's Constrained Resolution $\mathcal{CR}$

Proof Search (contd.)

- Like $\mathcal{R}$, the extensionality treatment of $\mathcal{CR}$ requires to add infinitely many extensionality axioms to the search space.

Huet's Constrained Resolution $\mathcal{CR}$

Proof Search (contd.)

- Like $\mathcal{R}$, the extensionality treatment of $\mathcal{CR}$ requires to add infinitely many extensionality axioms to the search space.
- The following figure graphically illustrates the main ideas of the proof search in $\mathcal{CR}$.

Huet's Constrained Resolution $\mathcal{CR}$

Completeness Results

- [Huet72,Huet73] analyses completeness of $\mathcal{CR}$ formally only with respect to Andrews V-complexes, i.e. Huet verifies that the set of non-refutable sentences in $\mathcal{CR}$ is an abstract consistency class for V-complexes.

Huet's Constrained Resolution $\mathcal{CR}$

ATP in FOL and HOL

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- Theorem (V-completeness of $\mathcal{CR}$): The calculus $\mathcal{CR}$ is complete with respect to the notion of V-complexes.

  Proof: [Huet72,Huet73]

Huet's Constrained Resolution $\mathcal{CR}$

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- Theorem (V-completeness of $\mathcal{CR}$): The calculus $\mathcal{CR}$ is complete with respect to the notion of V-complexes.

  Proof: [Huet72,Huet73]
- Theorem (Henkin completeness of $\mathcal{CR}$): The calculus $\mathcal{CR}$ is complete wrt. Henkin semantics provided that the infinitely many extensionality axioms are given.

  Proof: exercise

Example Proofs

Exercise: How are the following theorems proved in calculus $\mathcal{CR}$?

- Leibniz equality and η-equality:

$$\mathsf{f}_{\iota\to\iota} \doteq \lambda \mathsf{X}_\iota\mathbf{.}\,\mathsf{f}\ \mathsf{X}$$

Example Proofs

Exercise: How are the following theorems proved in calculus $\mathcal{CR}$?

- Leibniz equality and η-equality:

$$\mathsf{f}_{\iota\to\iota} \doteq \lambda \mathsf{X}_\iota \centerdot \mathsf{f}\ \mathsf{X}$$

- The set of all red balls equals the set of all balls that are red: $\{\mathsf{X} | \mathsf{red}\ \mathsf{X} \land \mathsf{ball}\ \mathsf{X}\} = \{\mathsf{X} | \mathsf{ball}\ \mathsf{X} \land \mathsf{red}\ \mathsf{X}\}$. This problem can be encoded as

$$(\lambda \mathsf{X}_\iota \centerdot \mathsf{red}\ \mathsf{X} \land \mathsf{ball}\ \mathsf{X}) = (\lambda \mathsf{X}_\iota \centerdot \mathsf{ball}\ \mathsf{X} \land \mathsf{red}\ \mathsf{X})$$

Example Proofs

Exercise: How are the following theorems proved in calculus $\mathcal{CR}$?

- All unary logical operators $O_{o \to o}$ which map the propositions a and b to $\top$ consequently also map $a \wedge b$ to $\top$:

$$\forall O_{o \to o} . (O\ a_o) \wedge (O\ b_o) \Rightarrow (O\ (a_o \wedge b_o))$$

Example Proofs

Exercise: How are the following theorems proved in calculus $\mathcal{CR}$?

- In Henkin semantics the domain $\mathcal{D}_o$ of all Booleans contains exactly the truth values $\bot$ and $\top$. Consequently the domain of all mappings from Booleans to Booleans contains exactly contains in each Henkin model at most four elements. And because of the requirement, that the function domains in Henkin models must be rich enough such that every term has a denotation, it follows that $\mathcal{D}_{o\rightarrow o}$ contains exactly the pairwise distinct denotations of the following four terms: $\lambda X_o \centerdot X_o$, $\lambda X_o \centerdot \neg X_o$, $\lambda X_o \centerdot \bot$, and $\lambda X_o \centerdot \top$. This theorem can be formulated as follows (where $f_{o\rightarrow o}$ is a constant):

$$(f = \lambda X_o \centerdot X_o) \vee (f = \lambda X_o \centerdot \neg X_o) \vee (f = \lambda X_o \centerdot \bot) \vee (f = \lambda X_o \centerdot \top)$$

Approaches to Higher-Order Resolution: Benzmüller's $\mathcal{ER}$

Extensional HO Resolution $\mathcal{ER}$

Clause normalization

$$\frac{C \vee [A \vee B]^T}{C \vee [A]^T \vee [B]^T}\ \vee^T \qquad \frac{C \vee [A \vee B]^F}{C \vee [A]^F}\ \vee^F_l \qquad \frac{C \vee [A \vee B]^F}{C \vee [B]^F}\ \vee^F_r$$

$$\frac{C \vee [\neg A]^T}{C \vee [A]^F}\ \neg^T \qquad \frac{C \vee [\neg A]^F}{C \vee [A]^T}\ \neg^F$$

$$\frac{C \vee [\Pi^\alpha A]^T \quad X_\alpha \text{ new variable}}{C \vee [A\ X]^T}\ \Pi^T$$

$$\frac{C \vee [\Pi^\alpha A]^F \quad sk_\alpha \text{ Skolem term}}{C \vee [A\ sk_\alpha]^F}\ \Pi^F$$

This rules may be combined into a single rule Cnf.

Extensional HO Resolution $\mathcal{ER}$

Resolution and Factorisation

$$\frac{[\mathsf{N}]^{\alpha} \vee \mathsf{C} \quad [\mathsf{M}]^{\beta} \vee \mathsf{D} \quad \alpha \neq \beta}{\mathsf{C} \vee \mathsf{D} \vee [\mathsf{N} \neq^{?} \mathsf{M}]}\ \mathsf{Res}$$

$$\frac{[\mathsf{N}]^{\alpha} \vee [\mathsf{M}]^{\alpha} \vee \mathsf{C} \quad \alpha \in \{\mathsf{T}, \mathsf{F}\}}{[\mathsf{N}]^{\alpha} \vee \mathsf{C} \vee [\mathsf{N} \neq^{?} \mathsf{M}]}\ \mathsf{Fac}$$

$$\frac{[\mathsf{Q}_{\gamma}\overline{\mathsf{U}^{\mathsf{k}}}]^{\alpha} \vee \mathsf{C} \quad \mathsf{P} \in \mathcal{GB}_{\gamma}^{\{\neg,\vee\} \cup \{\Pi^{\beta} | \beta \in \mathcal{T}^{\mathsf{k}}\}}}{[\mathsf{Q}_{\gamma}\overline{\mathsf{U}^{\mathsf{k}}}]^{\alpha} \vee \mathsf{C} \vee [\mathsf{Q} \neq^{?} \mathsf{P}]}\ \mathsf{Prim}^{\mathsf{k}}$$

Extensional HO Resolution $\mathcal{ER}$

(Pre-)unification rules

$$\frac{C \vee [M_{\alpha\rightarrow\beta} \neq^? N_{\alpha\rightarrow\beta}] \quad s_\alpha \text{ Skolem-Term}}{C \vee [M\ s \neq^? N\ s]}\ \text{Func}$$

$$\frac{C \vee [h\overline{U^n} \neq^? h\overline{V^n}]}{C \vee [U^1 \neq^? V^1] \vee \ldots \vee [U^n \neq^? V^n]}\ \text{Dec} \qquad \frac{C \vee [A \neq^? A]}{C}\ \text{Triv}$$

$$\frac{C \vee [F_\gamma\overline{U^n} \neq^? h\overline{V^n}] \quad G \in \mathcal{GB}^h_\gamma}{C \vee [F \neq^? G] \vee [F\overline{U^n} \neq^? h\overline{V^n}]}\ \text{Flex/Rigid}$$

$$\frac{C \vee E \quad E \text{ solved for } C}{Cnf(subst_E(C))}\ \text{Subst}$$

Extensional HO Resolution $\mathcal{ER}$

Extensionality rules

$$\frac{C \vee [M_o \neq^? N_o]}{Cnf(C \vee [M_o \Leftrightarrow N_o]^F)}\ \text{Equiv}$$

$$\frac{C \vee [M_\alpha \neq^? N_\alpha] \quad \alpha \in \{o, \iota\}}{Cnf(C \vee [\forall P_{\alpha\to o}.PM \Rightarrow PN]^F)}\ \text{Leib}$$

Extensional HO Resolution $\mathcal{ER}$

Extensionality Treatment

- Instead of adding infinitely many extensionality axioms to the search space $\mathcal{ER}$ provides two new extensionality rules which closely connect refutation search and eager unification.

Extensional HO Resolution $\mathcal{ER}$

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- Instead of adding infinitely many extensionality axioms to the search space $\mathcal{CR}$ provides two new extensionality rules which closely connect refutation search and eager unification.
- The idea is to allow for recursive calls from higher-order unification to the overall refutation process.

Extensional HO Resolution $\mathcal{ER}$

Extensionality Treatment

- Instead of adding infinitely many extensionality axioms to the search space $\mathcal{CR}$ provides two new extensionality rules which closely connect refutation search and eager unification.
- The idea is to allow for recursive calls from higher-order unification to the overall refutation process.
- This turns the rather weak syntactical higher-order unification approach considered so far into a most general approach for *dynamic* higher-order theory unification.

Extensional HO Resolution $\mathcal{ER}$

Proof Search

- Initially the proof problem is negated and normalised. The main proof search then closely interleaves the refutation process on resolution layer and unification, i.e. the main proof search rules Res, Fac, and Prim and the unification rules are integrated at a common conceptual level. The calls from unification to the overall refutation process with rules *Leib* and *Equiv* introduce new clauses into the search space which can be resolved against already given ones.

Extensional HO Resolution $\mathcal{ER}$

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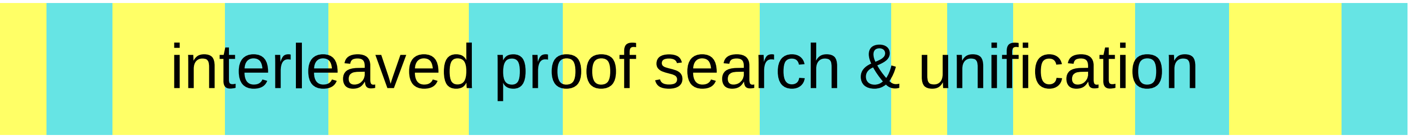
- The following figure graphically illustrates the main ideas of the proof search in $\mathcal{ER}$.



Ex.: Extensional HO Resolution $\mathcal{ER}$

$$\forall P_{o\to o}\centerdot (P\ a_o) \wedge (P\ b_o) \Rightarrow (P\ (a_o \wedge b_o))$$

Negation and clause normalization

$$\mathcal{C}_1 : [p\ a]^T \quad \mathcal{C}_2 : [p\ b]^T \quad \mathcal{C}_3 : [p\ (a \wedge b)]^F$$

Resolution between $\mathcal{C}_1$ and $\mathcal{C}_3$ and between $\mathcal{C}_2$ and $\mathcal{C}_3$

$$\mathcal{C}_4 : [p\ a \neq^? p\ (a \wedge b)] \quad \mathcal{C}_5 : [p\ b \neq^? p\ (a \wedge b)]$$

Decomposition

$$\mathcal{C}_6 : [a \neq^? (a \wedge b)] \quad \mathcal{C}_7 : [b \neq^? (a \wedge b)]$$

Recursive call of proof process with rules Equiv and $\mathcal{C}nf$

$$\mathcal{C}_8 : [a]^F \vee [b]^F \quad \mathcal{C}_9 : [a]^T \vee [b]^T \quad \mathcal{C}_{10} : [a]^T \quad \mathcal{C}_{11} : [b]^T$$

Ex.: Extensional HO Resolution $\mathcal{ER}$

ATP in FOL and HOL

$$\forall B_{\alpha\to o}, C_{\alpha\to o}, D_{\alpha\to o}.B \cup (C \cap D) = (B \cup C) \cap (B \cup D)$$

Negation and definition expansion with

$\cup = \lambda A_{\alpha\to o}, B_{\alpha\to o}, X_{\alpha}.(A\ X) \vee (B\ X) \qquad \cap = \lambda A_{\alpha\to o}, B_{\alpha\to o}, X_{\alpha}.(A\ X) \wedge (B\ X)$

leads to:

$$C_1 : [\lambda X_{\alpha}.(b\ X) \vee ((c\ X) \wedge (d\ X)) \neq^? \lambda X_{\alpha}.((b\ X) \vee (c\ X)) \wedge ((b\ X) \vee (d\ X)))]$$

Goal directed functional and Boolean extensionality treatment:

$$C_2 : [(b\ x) \vee ((c\ x) \wedge (d\ x)) \Leftrightarrow ((b\ x) \vee (c\ x)) \wedge ((b\ x) \vee (d\ x)))]^F$$

Clause normalization results then in a pure propositional, i.e. decidable, set of clauses. Only these clauses are still in the search space of LEO(in total there are 33 clauses generated and LEO finds the proof on a 2,5GHz PC in 820ms).

Similar proof in case of embedded propositions:

$$\forall P_{(\alpha\to o)\to o}, B_{\alpha\to o}, C_{\alpha\to o}, D_{\alpha\to o}.P(B \cup (C \cap D)) \Rightarrow P((B \cup C) \cap (B \cup D))$$

Ex.: Extensional HO Resolution $\mathcal{ER}$

Further small examples which test Henkin completeness:

$$\forall F_{o\to o}.(F \doteq \lambda X_o.X_o) \vee (F \doteq \lambda X_o.\neg X_o) \vee (F \doteq \lambda X_o.\bot) \vee (F \doteq \lambda X_o.\top)$$

$$\forall H_{o\to o}.H\ \bot \doteq H\ (H\ \top \doteq H\ \bot)$$

$$\dots$$

Higher-Order ATPs

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  extensional higher-order resolution (plus primitive equality treatment)   and cooperation with FO ATPs

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- TPS [AndrewsEtAl06]: mating method
- OTTER-λ [Beeson06]: not operating simple type theory but λ-logic, which is a novel combination of λ-calculus and first-order logic; employs first-order resolution