

Topology and measure in logics for region-based theories of space

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ABSTRACT

Space, as we typically represent it in mathematics and physics, is composed of dimensionless, indivisible points. On an alternative, region-based approach to space, extended regions together with the relations of ‘parthood’ and ‘contact’ are taken as primitive; points are represented as mathematical abstractions from regions.

Region-based theories of space have been traditionally modeled in regular closed (or regular open) algebras, in work that goes back to [5] and [21]. Recently, logics for region-based theories of space were developed in [3] and [19]. It was shown that these logics have both a nice topological and relational semantics, and that the minimal logic for contact algebras, \mathbb{L}_{min}^{cont} (defined below), is complete for both.

The present paper explores the question of completeness of \mathbb{L}_{min}^{cont} and its extensions for individual topological spaces of interest: the real line, Cantor space, the rationals, and the infinite binary tree. A second aim is to study a different, algebraic model of logics for region-based theories of space, based on the Lebesgue measure algebra (or algebra of Borel subsets of the real line modulo sets of Lebesgue measure zero). As a model for point-free space, the algebra was first discussed in [2]. The main results of the paper are that \mathbb{L}_{min}^{cont} is weakly complete for any zero-dimensional, dense-in-itself metric space (including, e.g., Cantor space and the rationals); the extension $\mathbb{L}_{min}^{cont} + (Con)$ is weakly complete for the real line and the Lebesgue measure contact algebra. We also prove that the logic $\mathbb{L}_{min}^{cont} + (Univ)$ is weakly complete for the infinite binary tree.

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1. Introduction

For a long time, logicians have wondered whether our ways of representing space are in some sense too idealized. Euclidean space is made up of *points*: dimensionless, indivisible regions. These are the smallest parts of space—the atoms beyond which we can divide no further. But such spatial atoms do not seem to correspond to anything in our ordinary experience of the world. On an alternative, region-based approach to space, extended regions together with some mereological and topological relations are taken as primitive; points are constructed as mathematical abstractions from regions.

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The project of giving a region-based theory of space is perhaps most famously associated with the work of de Laguna and Whitehead in the first half of last century.¹ Typically, modern incarnations of that project take as primitive the mereological relation ‘parthood,’ and the topological relation ‘contact.’² (Here topology is understood not in terms of standard ‘pointy’ structures, but more loosely in terms of the way in which parts of space are, as it were, glued together.) Intuitively, two regions of space are in contact if they overlap or are adjacent to one another. Such region-based theories of space are interpreted in *contact algebras* (B, C) , where $B = (\mathbf{B}, \vee, \wedge, -, 0, 1)$ is a non-degenerate Boolean algebra, and C is a binary relation on B that satisfies the axioms for contact (given below). Regions are understood as the non-zero elements of \mathbf{B} . The partial order \leq in the Boolean algebra interprets the mereological parthood relation, and C interprets the topological contact relation.

Many nice results in this area concern what Vakarelov calls the ‘equivalence’ of point-based and region-based approaches to space. From region-based models of space (i.e., contact algebras), it is shown that we can recover some interesting class of pointy topological spaces, and from pointy topologies we can construct region-based models. Important results of this kind were proved in, e.g., [16], [9], [6], [7], and [19]. One particularly important class of contact algebras arises from topological spaces in the following way. If X is a topological space, the set of regular closed subsets of X forms a complete Boolean algebra, $RC(X)$. The standard contact relation on the algebra is defined by:

$$AC_X B \text{ iff } A \cap B \neq \emptyset$$

Regular closed algebras of this sort are standard models for region-based theories of space. Part of the importance of such contact algebras lies in their role in representation theorems. [19], for example, shows that every contact algebra (B, C) can be embedded in the algebra $RC(X)$ of regular closed subsets of some compact, semi-regular, T_0 topology, X ; the embedding is, moreover, an isomorphism if B is a complete Boolean algebra.³

My starting point in this paper is the family of *formal logics* for region-based theories of space developed in [3] and further discussed in [19]. The language of these logics is a first-order language without quantifiers, containing a countable set of Boolean variables, Boolean constants $\mathbf{0}$ and $\mathbf{1}$, Boolean functions $\sqcup, \sqcap, *$, as well as two binary predicates \leq and C that express, respectively, parthood and contact. Terms are built from Boolean variables and constants using the functions $\sqcup, \sqcap, *$. Atomic sentences in the language are sentences of the form $t_1 \leq t_2$ and $t_1 C t_2$, where t_1 and t_2 are terms; complex formulas are built from atomic ones using the connectives \neg, \vee, \wedge , and the constants \top and \perp . The logics of region-based theories of space developed in [3] have much in common with propositional modal logics. As [3] shows, these logics have both a relational and topological semantics. In the relational semantics, the logics are interpreted in reflexive, symmetric Kripke frames, or pairs $F = (W, R)$, where W is a non-empty set and R is a reflexive and symmetric relation on W . Surprisingly, every consistent axiomatic extension of the minimal logic for contact algebras, \mathbb{L}_{min}^{cont} , is weakly complete for the class of frames determined by the given extension—and indeed, for the subclass of *finite* frames determined by the extension. Thus each such extension is decidable, and there are no ‘Kripke-incomplete’ logics.

In the topological semantics, on the other hand, logics for region-based theories of space are interpreted in the Boolean algebra $RC(X)$ for some space X , together with the standard contact relation, C_X . [3] shows that \mathbb{L}_{min}^{cont} is weakly complete for the class of all topological spaces—and indeed, for the smaller class of all compact, semi-regular, T_0 , κ -normal topological spaces.⁴ This result leaves open the question

¹ See [5] and [21].

² Some recent examples of work in this area include, e.g., Randell et al. [15] and Roeper [16]. Randell et al. [15] takes as primitive only the contact relation; Roeper [16] takes as primitive the relations contact and parthood, together with the predicate ‘limited.’

³ See [19], Lemma 2.3.6. and Theorem 2.3.9.

⁴ It is also shown in that paper that the logic \mathbb{L}_{min}^{cont} is strongly complete for the class of all compact, semi-regular, T_0 topologies. See [3], Theorems 9.1 and 9.2.

of completeness for particular topological spaces of interest: *e.g.*, the real line, \mathbb{R} , Cantor space, \mathcal{C} , the rationals, \mathbb{Q} , and the infinite binary tree. We can compare the situation here with what we know about the topological semantics for propositional modal logics. The celebrated completeness result of [12] shows that the propositional modal logic $S4$ is weakly complete with respect to each separable, dense-in-itself metric space.⁵ This result includes as special cases the real line, Cantor space, and the rationals. Simplified proofs of completeness for those particular topological spaces can be found, respectively, in, *e.g.*, [1], [13], and [20]. It was proved independently by Dov Gabbay and Johan van Benthem that $S4$ is also complete for the infinite binary tree.

Turning therefore to the topological semantics for logics of region-based theories of space, it is natural to ask what the region-based logic of particular topological spaces is. In other words, what set of formulas is valid in $RC(X)$, where X is some particular topological space of interest. The first aim of this paper is to study that question. A second aim is to study a different, algebraic model for region-based theories of space that was first proposed in [2]. Arntzenius was concerned with constructing a model of region-based theories of space that had, in addition to certain topological features, certain nice measure-theoretic features. He noted that Lebesgue measure is not even finitely additive over the standard model of region-based theories of space, $RC(\mathbb{R}^n)$, and indeed, there is *no* non-zero, countably additive measure over $RC(\mathbb{R}^n)$. For this reason, Arntzenius proposed to model space in the Lebesgue measure algebra instead, or algebra of Borel subsets of \mathbb{R}^n modulo (Borel) sets of Lebesgue-measure zero. Lebesgue measure is countably additive over the algebra; moreover, the algebra admits of a natural contact relation, which provides it with topological structure. We study this algebra as a model for the formal logic \mathbb{L}_{min}^{cont} below; this is the first attempt to bring the work of Arntzenius into contact with the logics of region-based theories of space developed in [3]. The main results of this paper are that \mathbb{L}_{min}^{cont} is weakly complete for any zero-dimensional, dense-in-itself metric space (and thus for Cantor space, \mathcal{C} , the rationals, \mathbb{Q} , and the irrationals, \mathbb{P}), and $\mathbb{L}_{min}^{cont} + (Con)$ (defined below) is weakly complete for the real line, \mathbb{R} , and the Lebesgue measure (contact) algebra. We also prove that $\mathbb{L}_{min}^{cont} + (Univ)$ (defined below) is weakly complete for the infinite binary tree.

The paper is organized as follows. In §2, we review the algebraic, relational, and topological semantics for the logic \mathbb{L}_{min}^{cont} , as given in [3] and [19]. We recall, in particular, a representation theorem for contact algebras in terms of spaces of clans (defined below). To this we add a short discussion of truth-preserving maps between topological spaces. In §3, we study the infinite binary tree \mathcal{T}_2 , and show that the logic $\mathbb{L}_{min}^{cont} + (Univ)$ is weakly complete for that topology. In §4, we show that \mathbb{L}_{min}^{cont} is weakly complete for any zero-dimensional, dense-in-itself metric space. In §5, we study the real line together with the Lebesgue measure (contact) algebra, and show that the logic $\mathbb{L}_{min}^{cont} + (Con)$ is weakly complete for both.

2. Preliminaries

Let the language \mathcal{L} consist of a countable set BV of Boolean variables; Boolean constants **0** and **1**; Boolean functions \sqcup , \sqcap , and $*$; propositional connectives \vee , \wedge , \neg ; propositional constants \top and \perp ; and two binary relations \leq and C . Boolean variables will be denoted by lower-case letters $a, b, c \dots$

Terms of the language are defined inductively as follows. Every Boolean variable is a term, and the Boolean constants **0** and **1** are terms; if t_1 and t_2 are terms, then $t_1 \sqcup t_2$, $t_1 \sqcap t_2$, and t_1^* are terms. Atomic formulas are of the form $t_1 \leq t_2$ or $t_1 C t_2$, where t_1 and t_2 are terms; the full set of formulas is given by the formation rules:

$$\neg\varphi \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \top \mid \perp$$

We also adopt the following shorthand for complex formulas in the language:

⁵ This result was improved in [14], where the authors show that $S4$ is weakly complete for any dense-in-itself metric space.

- $\varphi \rightarrow \psi =_{\text{df}} \neg\varphi \vee \psi$;
- $\varphi \leftrightarrow \psi =_{\text{df}} (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$;
- $a = b =_{\text{df}} a \leq b \wedge b \leq a$;
- $a \neq b =_{\text{df}} \neg a = b$;
- $a\overline{C}b =_{\text{df}} \neg aCb$.

For any formula $\varphi(a_1, \dots, a_n)$, where a_1, \dots, a_n are Boolean variables, we denote by $\varphi(t_1|a_1, \dots, t_n|a_n)$ the result of simultaneously substituting the term t_k for a_k in φ , $1 \leq k \leq n$.

Definition 2.1. The logic $\mathbb{L}_{\min}^{\text{cont}}$ is given by the axioms:

- I. Some complete set of axioms for classical propositional logic;
- II. Some complete set of axioms for Boolean algebra in terms of the partial order, \leq . For example:
 - $a \leq a$;
 - $(a \leq b \wedge b \leq c) \rightarrow a \leq c$;
 - $0 \leq a$; $a \leq 1$;
 - $c \leq a \sqcap b \leftrightarrow (c \leq a \wedge c \leq b)$;
 - $a \sqcup b \leq c \leftrightarrow (a \leq c \wedge b \leq c)$;
 - $(a \sqcap (b \sqcup c)) \leq ((a \sqcap b) \sqcup (a \sqcap c))$;
 - $c \sqcap a \leq 0 \leftrightarrow c \leq a^*$;
 - $a^{**} \leq a$.
- III. The following axioms for the contact relation:
 1. $aCb \rightarrow (a \neq 0 \wedge b \neq 0)$;
 2. $(aCb \wedge b \leq c) \rightarrow aCc$;
 3. $aC(b \sqcup c) \rightarrow (aCb \vee aCc)$;
 4. $a \neq 0 \rightarrow aCa$;
 5. $aCb \rightarrow bCa$;

and the rules of inference:

$$\begin{array}{ll} \text{Modus Ponens} & \frac{\varphi, \varphi \rightarrow \psi}{\psi} \\ \text{Substitution} & \frac{\varphi(a_1, \dots, a_n)}{\varphi(t_1|a_1, \dots, t_n|a_n)} \end{array}$$

In what follows, we will be concerned not just with $\mathbb{L}_{\min}^{\text{cont}}$, but with various extensions of that logic obtained by adding finitely many new axioms. A set of formulas \mathbb{L} is an *extension* of $\mathbb{L}_{\min}^{\text{cont}}$ if $\mathbb{L}_{\min}^{\text{cont}} \subseteq \mathbb{L}$, and \mathbb{L} is closed under Modus Ponens and Substitution. We sometimes refer to the formulas of \mathbb{L} as *theorems* of \mathbb{L} . For a formula φ , let $\mathbb{L}_{\min}^{\text{cont}} + \varphi$ be the smallest extension of $\mathbb{L}_{\min}^{\text{cont}}$ containing φ .

Let \mathbb{L} be an extension of $\mathbb{L}_{\min}^{\text{cont}}$. An \mathbb{L} -theory is a set of formulas containing \mathbb{L} and closed under Modus Ponens. If Γ is an \mathbb{L} -theory, we say that Γ is *consistent* if $\perp \notin \Gamma$. We say that Γ is *maximal* if Γ is consistent and for each consistent \mathbb{L} -theory Δ , $\Gamma \subseteq \Delta$ implies $\Gamma = \Delta$.⁶

In subsequent sections of this paper, we will be particularly concerned with extensions of $\mathbb{L}_{\min}^{\text{cont}}$ by the axioms:

⁶ We could have also defined an arbitrary set of formulas Γ in the language \mathcal{L} to be \mathbb{L} -consistent if $(\gamma_1 \wedge \dots \wedge \gamma_n) \rightarrow \perp$ is not a theorem of \mathbb{L} for any $\gamma_1, \dots, \gamma_n \in \Gamma$. A *maximal* \mathbb{L} -consistent set is then an \mathbb{L} -consistent set Γ such that for any \mathbb{L} -consistent set Δ , $\Gamma \subseteq \Delta$ implies $\Gamma = \Delta$. Maximal \mathbb{L} -consistent sets are equivalent to maximal \mathbb{L} -theories as defined above.

- (Con) $(a \neq 0 \wedge a \neq 1) \rightarrow aCa^*$;
(Univ) $(a \neq 0 \wedge b \neq 0) \rightarrow aCb$.

Informally, (Con) states that any non-zero, non-unit region is in contact with its complement, while (Univ) says that every two non-zero regions are in contact.

2.1. Algebraic semantics

The simplest semantics for the language \mathcal{L} is the algebraic one, introduced in [19]. Here we interpret formulas in contact algebras, defined below.⁷

Definition 2.2. A *contact algebra* is a pair (B, C) where B is a non-degenerate Boolean algebra, and C is a binary relation on B satisfying:

1. If aCb , then $a \neq 0$ and $b \neq 0$;
2. If aCb and $b \leq c$, then aCc ;
3. If $aC(b \vee c)$, then aCb or aCc ;
4. If $a \neq 0$, then aCa ;
5. If aCb , then bCa .

We say that C is a *contact relation* on B .⁸

If B is a Boolean algebra, and $a, b \in B$, we say that a *overlaps* b (aOb) if $a \wedge b \neq 0$. The next lemma shows that O is a contact relation—indeed, the smallest contact relation on the algebra B .

Lemma 2.3. Let B be a Boolean algebra. Then O is the smallest contact relation on B .

Proof. The reader can verify that O satisfies the conditions of Definition 2.2. To see that O is contained in every contact relation on B , suppose that C is a contact relation on B . If aOb , then $a \wedge b \neq 0$. By part 4. of Definition 2.2, $(a \wedge b)C(a \wedge b)$. By parts 2. and 5. of Definition 2.2, aCb . \square

An *algebraic model* is a triple $M = (B, C, V)$ where (B, C) is a contact algebra, and V is a function assigning to each Boolean variable an element of B . We say that the model M is *defined over* the contact algebra (B, C) . We extend V to the set of all terms in the language as follows: $V(t \sqcup s) = V(t) \vee V(s)$; $V(t \sqcap s) = V(t) \wedge V(s)$; $V(t^*) = -V(t)$; $V(\mathbf{0}) = 0$; $V(\mathbf{1}) = 1$ (where 0 and 1 on the RHS denote the top and bottom elements of the Boolean algebra B , and $\vee, \wedge, -$ denote respectively the join, meet, and complement in B). Note that we use the same symbols ‘ C ’ and ‘ \leq ’ for the contact and order relations in a contact algebra and the binary predicates in our formal language \mathcal{L} . We trust that this will not lead to confusion.

The relation of truth ($M \models \varphi$) between a model and an \mathcal{L} -formula is given for atomic formulas by:

- $M \models t \leq s$ iff $V(t) \leq V(s)$;
- $M \models tCs$ iff $V(t) C V(s)$,

⁷ In fact, the semantics given here is a restricted version of the algebraic semantics introduced in [19], where the language \mathcal{L} is interpreted more generally in *pre-contact algebras* (algebras (B, C) , where the relation C does not necessarily satisfy conditions 4. and 5. of Definition 2.2). We use contact algebras instead of pre-contact algebras, because we are interested in giving a semantics for \mathbb{L}_{min}^{cont} and its extensions, rather than the more minimal logic \mathbb{L}_{min} , which does not contain axioms 4. and 5. of Definition 2.1.

⁸ We could have instead used the equivalent axiomatization of contact algebras given in [6], and [19]: (C1) if xCy , then $x, y \neq 0$; (C2) $xC(y \vee z)$ if and only if xCy or xCz ; (C3) if xCy , then yCx ; (C4) if $x \wedge y \neq 0$, then xCy . To see that the axiomatizations are equivalent, note that (C2) follows from 2. and 3. above; (C4) follows from 2, 4, and 5; (C1) and (C3) are just 1. and 5., respectively. Conversely, 2. and 3. follow from (C2); 4 follows from (C4).

and the definition of truth is extended to complex formulas in the usual way:

- $M \models \varphi \vee \psi$ iff $M \models \varphi$ or $M \models \psi$;
- $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$;
- $M \models \neg\varphi$ iff $M \not\models \varphi$.

Suppose that \mathcal{A} is a contact algebra, M is a model over \mathcal{A} , and φ is a formula. We say that φ is *true in* M if $M \models \varphi$. We say that φ is *true in* \mathcal{A} if φ is true in every model defined over \mathcal{A} . If φ is not true in \mathcal{A} , we say that φ is *refuted* in \mathcal{A} . If Σ is a class of contact algebras, we say that φ is *true in* Σ if φ is true in every contact algebra $\mathcal{A} \in \Sigma$. For any class Σ of contact algebras, we let

$$\text{Log}(\Sigma) = \{\varphi \mid \varphi \text{ is true in } \Sigma\}$$

With slight abuse of notation, we write $\text{Log}(\mathcal{A})$ instead of $\text{Log}(\{\mathcal{A}\})$ for any contact algebra, \mathcal{A} .

We state without proof the following simple proposition.

Proposition 2.4. *For any class Σ of contact algebras, $\text{Log}(\Sigma)$ is closed under Modus Ponens, Substitution, and contains all theorems of $\mathbb{L}_{\min}^{\text{cont}}$. Therefore $\text{Log}(\Sigma)$ is an extension of $\mathbb{L}_{\min}^{\text{cont}}$. In particular, if \mathcal{A} is a contact algebra, every theorem of $\mathbb{L}_{\min}^{\text{cont}}$ is true in \mathcal{A} .*

Suppose that $\mathcal{A}_1 = (B_1, C_1)$ and $\mathcal{A}_2 = (B_2, C_2)$ are contact algebras. A function $h : B_1 \rightarrow B_2$ is an *embedding* if h is a Boolean embedding that preserves contact. More precisely, h is an embedding if h is injective, and for any $a, b \in B_1$,

1. $h(a \vee b) = h(a) \vee h(b)$;
2. $h(-a) = -h(a)$;
3. aC_1b iff $h(a)C_2h(b)$.

An *isomorphism* is a surjective embedding. The contact algebras \mathcal{A}_1 and \mathcal{A}_2 are *isomorphic* ($\mathcal{A}_1 \cong \mathcal{A}_2$) if there is an isomorphism from \mathcal{A}_1 to \mathcal{A}_2 .

In what follows we will often wish to transfer countermodels from one contact algebra to another. The following lemma tells us that we can do this via embeddings of contact algebras.

Lemma 2.5. *Suppose \mathcal{A}_1 and \mathcal{A}_2 are contact algebras, and $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an embedding. Let V_1 be a valuation over \mathcal{A}_1 , and define the valuation V_2 over \mathcal{A}_2 by putting $V_2(a) = h(V_1(a))$, for every Boolean variable a . Let $M_1 = (\mathcal{A}_1, V_1)$ and $M_2 = (\mathcal{A}_2, V_2)$. Then for any term t ,*

$$V_2(t) = h(V_1(t))$$

and for any formula φ ,

$$M_1 \models \varphi \text{ iff } M_2 \models \varphi$$

Proof. By induction on terms and formulas. \square

Corollary 2.6. *If \mathcal{A}_1 and \mathcal{A}_2 are contact algebras, and $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an embedding, then*

$$\text{Log}(\mathcal{A}_2) \subseteq \text{Log}(\mathcal{A}_1)$$

Proof. Suppose $\varphi \notin \text{Log}(\mathcal{A}_1)$. Then there is a model $M_1 = (\mathcal{A}_1, V_1)$ such that $M_1 \not\models \varphi$. Define V_2 and M_2 as in Lemma 2.5. Then by Lemma 2.5, $M_2 \not\models \varphi$, and $\varphi \notin \text{Log}(\mathcal{A}_2)$. \square

2.2. Relational semantics

In the previous section, we recalled the algebraic semantics for the language \mathcal{L} . Let us now turn to a relational semantics given in [3], in which the language is interpreted in structures consisting of a set W together with a binary reflexive and symmetric relation R on W . Informally, we can think of W as a set of indivisible regions, and the binary relation R as specifying which indivisible regions are in contact with which others. Regions of space are taken to be arbitrary sets of indivisible regions. [3] gives as an example the collection of squares on a chess board, where the binary relation R holds between two squares if and only if they are adjacent (*i.e.*, share a point).⁹

More formally, a *relational frame* (or simply *frame*) is a pair $F = (W, R)$ where W is a non-empty set, and R is a reflexive, symmetric relation on W . A *relational model* is a triple $M = (W, R, V)$ where $F = (W, R)$ is a frame, and V is a function assigning to each Boolean variable a subset of W . We say that the model M is *defined over* the frame F . We call V a *valuation*, and extend it to the set of all Boolean terms as follows: $V(t \sqcup s) = V(t) \cup V(s)$; $V(t \sqcap s) = V(t) \cap V(s)$; $V(t^*) = W \setminus V(t)$; $V(0) = \emptyset$; $V(1) = W$.

Truth in M for atomic formulas is defined as follows:

- $M \models t \leq s$ iff $V(t) \subseteq V(s)$;
- $M \models tC_s$ iff there exists $x \in V(t)$ and there exists $y \in V(s)$ such that xRy .

Thus in the chess board example, two regions A and B (*i.e.*, sets of squares) are in contact if there is a square S_1 in A and a square S_2 in B such that S_1 and S_2 share a point. We extend the definition of truth in M to complex formulas in the usual way.

Note that there is no notion of truth at a point $w \in W$ as in Kripke semantics for modal logics; truth is defined simply for the model M as a whole.

If $F = (W, R)$ is a frame, M is a relational model over F , and φ is a formula, we say that φ is *true in* M if $M \models \varphi$. We say that φ is *true in* F if φ is true in every model defined over F . If φ is not true in F , we say that φ is *refuted* in F . If Σ is a class of frames, φ is *true in* Σ if φ is true in every frame $F \in \Sigma$. If Φ is a set of formulas, and Σ a class of frames, Φ is true in Σ if φ is true in Σ , for every $\varphi \in \Phi$. A class of frames Σ is *determined by* Φ if Σ is the class of all frames in which Φ is true. Finally, Σ is *determined* if it is determined by some set of formulas, Φ .

It is not difficult to see that the relational semantics defined above is a special case of the more general algebraic semantics. Indeed, let $F = (W, R)$ be a frame, and consider the binary relation C_R defined over the field of sets $\mathcal{P}(W)$ by

$$AC_RB \text{ iff there exists } x \in A \text{ and } y \in B \text{ such that } xRy$$

The reader can verify that $(\mathcal{P}(W), C_R)$ is a contact algebra. We denote this contact algebra by $\mathcal{B}(F)$. Any valuation V assigning to each term in the language a subset of W gives rise to both a relational model $M = (W, R, V)$ and an algebraic model $M_{alg} = (\mathcal{B}(F), V)$. Moreover, for any formula φ ,

$$M \models \varphi \text{ iff } M_{alg} \models \varphi$$

We state without proof the following corollary to that fact.

⁹ See [3], p. 34.

Lemma 2.7. *Let F be a frame, and let φ be a formula. Then φ is true in F iff φ is true in $\mathcal{B}(F)$.*

The following result is proved in [3]. It states that every consistent extension \mathbb{L} of \mathbb{L}_{min}^{cont} is weakly complete for the class of frames determined by \mathbb{L} , and that each such extension has the finite model property.¹⁰

Theorem 2.8. *Let \mathbb{L} be a consistent extension of \mathbb{L}_{min}^{cont} , and let $\Sigma_{\mathbb{L}}$ be the class of frames determined by the set of all theorems of \mathbb{L} . Then the following conditions are equivalent for any formula φ :*

1. φ is a theorem of \mathbb{L} ;
2. φ is true in $\Sigma_{\mathbb{L}}$;
3. φ is true in every finite frame in $\Sigma_{\mathbb{L}}$

where a frame $F = (W, R)$ is finite if W is finite.

Proof sketch. The full proof is given in [3], §4. Since this theorem is relied upon heavily in what follows, we give an outline of the proof. The implications from 1. to 2. and 2. to 3. are obvious. For 3. to 1., let φ be a non-theorem of \mathbb{L} . Then $\neg\varphi$ is included in some maximal, \mathbb{L} -consistent set T . Construct a canonical algebraic model $M = (B, C, V)$ of T , where elements of B are equivalence classes of terms, under the equivalence relation $t_1 \equiv t_2$ iff $t_1 = t_2 \in T$. The formula φ is refuted in M . Via a representation theorem, we can embed the algebra (B, C) in an algebra $\mathcal{B}(F)$, where F is a (reflexive, symmetric) Kripke frame. Then φ is refuted in a canonical relational model (F, V^*) . Take a filtration F' of F through the set of subformulas of φ . Then φ is refuted in F' . Moreover, F' belongs to the set of finite frames in $\Sigma_{\mathbb{L}}$. This last fact is proved by Lemma 4.2 in [3], where it is shown that if a formula A is refuted in F' , then a substitution instance of A is refuted in (F, V^*) . But (F, V^*) verifies all (substitution instances of) theorems of \mathbb{L} , so no theorem of \mathbb{L} can be refuted in F' . \square

It follows from Proposition 2.4 that the class of frames determined by \mathbb{L}_{min}^{cont} is simply the class of all frames. Therefore we have the following special case of Theorem 2.8.

Corollary 2.9. *The following are equivalent:*

1. φ is a theorem of \mathbb{L}_{min}^{cont} .
2. φ is true in every frame.
3. φ is true in every finite frame.

As we already mentioned, we will be interested below in the logics obtained by adding to \mathbb{L}_{min}^{cont} the axioms (Con) and $(Univ)$, and in the class of frames determined by those logics.

Definition 2.10. A frame (W, R) is *path-connected* if for every $w, w' \in W$, there exists a sequence (v_1, \dots, v_n) in W such that $v_1 = w$, $v_n = w'$, and $v_i R v_{i+1}$ for each $i < n$.¹¹

The following Lemma is proved in [8].¹²

Lemma 2.11. *The class of frames determined by $\mathbb{L}_{min}^{cont} + (Con)$ is the class of path-connected frames.*

¹⁰ See [3], Theorem 4.2. In fact, Theorem 4.2 in [3] is more general—it proves completeness for all consistent extensions of the logic \mathbb{L}_{min} with respect to the determined class of frames, where \mathbb{L}_{min} does not contain axioms 4. and 5. for contact (see Note 7).

¹¹ The usual definition of path-connectedness for arbitrary Kripke frames (*i.e.*, not necessarily reflexive or symmetric) states instead that $v_i R v_{i+1}$ or $v_{i+1} R v_i$ for each $i < n$. But since for us R is symmetric, we can use the simpler clause given above.

¹² See [8], Theorem 5 (4). See also [19], Lemma 2.8.1 (6).

Corollary 2.12. *The following are equivalent:*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Con)$.
2. φ is true in every path-connected frame.
3. φ is true in every finite, path-connected frame.

Proof. Immediate from Lemma 2.11 and Theorem 2.8. \square

Definition 2.13. A frame (W, R) is *universal* if $R = W \times W$.

Lemma 2.14. *The class of frames determined by $\mathbb{L}_{min}^{cont} + (Univ)$ is the class of universal frames.*

Proof. Suppose $F = (W, R)$ is a universal frame, and $M = (F, V)$ is a relational model defined over F . If $M \models a \neq 0 \wedge b \neq 0$, then $V(a) \neq \emptyset$ and $V(b) \neq \emptyset$. So there exists $x \in V(a)$ and $y \in V(b)$. Since F is universal, xRy . But then $M \models aCb$. Thus $(Univ)$ is true in F . Moreover, by Proposition 2.4, all axioms in \mathbb{L}_{min}^{cont} are true in F , and $Log(F)$ is closed under Modus Ponens and Substitution. Thus every theorem of $\mathbb{L}_{min}^{cont} + (Univ)$ is true in F .

Conversely, suppose that every theorem of $\mathbb{L}_{min}^{cont} + (Univ)$ is true in $F = (W, R)$. Let $x, y \in W$ and define the valuation V over F as follows: $V(a) = \{x\}$, $V(b) = \{y\}$. (We can let $V(c) = \emptyset$ for all other Boolean variables c .) Let $M = (F, V)$. Since $(Univ)$ is true in F , and $M \models a \neq 0 \wedge b \neq 0$, we have $M \models aCb$. But then xRy . Thus F is a universal frame. \square

Proposition 2.15. *The following are equivalent:*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Univ)$.
2. φ is true in every universal frame.
3. φ is true in every finite, universal frame.

Proof. Immediate from Lemma 2.14 and Theorem 2.8. \square

2.3. Topological semantics

We turn our attention now to the topological semantics for region-based theories of space. This semantics was first presented in [3]; we add here a brief discussion of truth-preserving maps in the topological setting that will allow us to transfer completeness from one class of topological spaces to another.

Recall that a topology is a pair (X, τ) where X is a non-empty set, and τ is a collection of subsets of X containing X , \emptyset , and closed under finite intersections and arbitrary unions. We call the elements of τ *open sets*. Where the meaning is clear, we will often leave off mention of τ , denoting the topology simply by X . An *Alexandroff topology* is a topology in which the open sets are closed under arbitrary intersections. Thus if X is an Alexandroff topology, each point $x \in X$ has a smallest open neighborhood: the intersection of all open sets containing x .

A *quasi-ordered set* (or *qoset*) is a pair (X, \leq) , where X is a non-empty set, and \leq is a reflexive, transitive relation on X . It is well-known that Alexandroff topologies correspond to qosets in the following way. Let (X, τ) be an Alexandroff topology. Define the binary relation \leq_τ on X by putting: $x \leq_\tau y$ iff $x \in Cl(\{y\})$. Then \leq_τ is both reflexive and transitive, so (X, \leq_τ) is a qoset. The relation \leq_τ is called the *specialization quasi-order*. Moving in the opposite direction, let (X, \leq) be a qoset. Say that a set $O \subseteq X$ is an *upset* if $x \in O$ and $x \leq y$ imply that $y \in O$. Then the collection of all upsets, which we denote by τ_\leq , contains \emptyset , X , and is closed under arbitrary unions and intersections. Thus (X, τ_\leq) is an Alexandroff topology.

One can show that $\tau_{\leq\tau} = \tau$ and $\leq_{\tau\leq} = \leq$. Therefore qosets are in one-to-one correspondence with Alexandroff topologies. In what follows, we do not distinguish between qosets and Alexandroff topologies, often referring to an Alexandroff topology as a qoset when convenient to view it in this way. We note that every finite topological space is Alexandroff, and therefore finite topologies are in one-to-one correspondence with finite qosets.

Let X be a topological space. For any set $A \subseteq X$, we denote the interior of A by $\text{Int}(A)$; we denote the closure of A by $\text{Cl}(A)$. Recall that $A \subseteq X$ is *regular closed* if $A = \text{Cl}(\text{Int}(A))$, and A is *regular open* if $A = \text{Int}(\text{Cl}(A))$. It is well-known that the regular closed subsets of X form a complete Boolean algebra with operations defined by:

$$\begin{aligned} A \vee B &= A \cup B \\ A \wedge B &= \text{Cl} \text{Int}(A \cap B) \\ -A &= \text{Cl}(X \setminus A) \\ \bigvee \{A_i \mid i \in I\} &= \text{Cl}(\bigcup \{A_i \mid i \in I\}) \\ \bigwedge \{A_i \mid i \in I\} &= \text{Cl} \text{Int}(\bigcap \{A_i \mid i \in I\}) \end{aligned}$$

As above, we denote this algebra by $\text{RC}(X)$.

Let (X, τ) (or simply X) be a topological space. We define the relation C_X on the algebra $\text{RC}(X)$ by:

$$AC_X B \text{ iff } A \cap B \neq \emptyset$$

It is not difficult to see that $(\text{RC}(X), C_X)$ is a contact algebra. With slight abuse of notation, we denote by ‘ $\text{RC}(X)$ ’ both the Boolean algebra of regular closed sets, and the contact algebra. A *topological contact algebra* is a contact algebra of the form $\text{RC}(X)$ for some topological space, X .

A *topological model* is a pair, $M = (X, V)$, where X is a topological space, and V is a function assigning to each Boolean variable an element in $\text{RC}(X)$. We say that the topological model M is *defined over* the space X . We extend V to the set of all terms in \mathcal{L} using operations in the algebra $\text{RC}(X)$ as follows: $V(t \sqcup s) = V(t) \cup V(s)$; $V(t \sqcap s) = \text{Cl} \text{Int}(V(t) \cap V(s))$; $V(t^*) = \text{Cl}(X \setminus V(t))$; $V(\mathbf{0}) = \emptyset$; $V(\mathbf{1}) = X$.

Truth for atomic formulas in the topological semantics is defined as follows.

- $M \models t \leq s$ iff $V(t) \subseteq V(s)$;
- $M \models tCs$ iff $V(t) \cap V(s) \neq \emptyset$.

We extend the definition of satisfaction to complex formulas in the usual way. Thus topological models are in fact just algebraic models where the algebra in question is $\text{RC}(X)$, for some topology X . The algebraic semantics is therefore a generalization of the topological semantics.

For any topological space X , model M over X , and formula φ , we say that φ is *true in M* if $M \models \varphi$. We say that φ is *true in X* if φ is true in every model defined over X . If φ is not true in X , we say that φ is *refuted in X* .

Below, we will be interested in obtaining completeness results for specific topological spaces. To do so, we will need to transfer counterexamples to non-theorems from one class of topological spaces to another. What maps between topological spaces allow us to do this? In view of [Corollary 2.6](#) and the connection between the topological and algebraic semantics, what we need are maps between topological spaces that allow us to embed one topological contact algebra in another. Let X and Y be topological spaces. Recall that a function $f : X \rightarrow Y$ is *open* if for every open set $O \subseteq X$, $f(O)$ is open. f is *continuous* if for every open set $U \subseteq Y$, $f^{-1}(U)$ is open. Finally, f is *interior* if f is both open and continuous.

Lemma 2.16. *Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be an interior, surjective map. If $S \subseteq Y$, then*

1. $\text{Int } f^{-1}(S) = f^{-1} \text{Int}(S)$;
2. $\text{Cl } f^{-1}(S) = f^{-1} \text{Cl}(S)$.

Proof. For 1., note that by continuity, $f^{-1} \text{Int}(S)$ is an open subset of $f^{-1}(S)$. So $f^{-1} \text{Int}(S) \subseteq \text{Int } f^{-1}(S)$. By openness of f , $f(\text{Int } f^{-1}(S))$ is an open subset of $f(f^{-1}(S))$, and by surjectivity, $f(f^{-1}(S)) = S$. Therefore, $f(\text{Int } f^{-1}(S)) \subseteq \text{Int}(S)$. So $\text{Int}(f^{-1}(S)) \subseteq f^{-1} \text{Int}(S)$. We have shown that $\text{Int } f^{-1}(S) = f^{-1} \text{Int}(S)$.

For 2., note that for any set $S \subseteq Y$, $f^{-1}(Y \setminus S) = X \setminus f^{-1}(S)$. Thus,

$$\begin{aligned} f^{-1} \text{Cl}(S) &= f^{-1}(Y \setminus \text{Int}(Y \setminus S)) \\ &= X \setminus f^{-1}(\text{Int}(Y \setminus S)) \\ &= X \setminus \text{Int } f^{-1}(Y \setminus S) \\ &= X \setminus \text{Int}(X \setminus f^{-1}(S)) \\ &= \text{Cl } f^{-1}(S) \quad \square \end{aligned}$$

Lemma 2.17. *Let X , Y , and f be as in Lemma 2.16. If $S \subseteq Y$ is regular closed, then $f^{-1}(S)$ is regular closed.*

Proof. Suppose $S \subseteq Y$ is regular closed. Then $f^{-1}(S) = f^{-1}(\text{Cl } \text{Int}(S)) = \text{Cl } f^{-1} \text{Int}(S) = \text{Cl } \text{Int } f^{-1}(S)$. Therefore, $f^{-1}(S)$ is regular closed. \square

Let X and Y be topological spaces. If $f : X \rightarrow Y$ is a surjective, interior map, and $S \in \text{RC}(Y)$, then by Lemma 2.17, $f^{-1}(S) \in \text{RC}(X)$. Therefore, we can define the function $h_f : \text{RC}(Y) \rightarrow \text{RC}(X)$ by putting:

$$h_f(S) = f^{-1}(S)$$

Proposition 2.18. $h_f : \text{RC}(Y) \rightarrow \text{RC}(X)$ is an embedding.

Proof. We need to show that h_f is injective, preserves Boolean operations, and preserves contact.

- Injective.

If $S_1, S_2 \subseteq Y$ and $S_1 \neq S_2$, then WLOG there exists $y \in S_1 \setminus S_2$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. So $x \in f^{-1}(S_1)$ and $x \notin f^{-1}(S_2)$. Therefore $f^{-1}(S_1) \neq f^{-1}(S_2)$, and h_f is injective.

- Boolean operations.

$$\begin{aligned} h_f(A \vee B) &= f^{-1}(A \cup B) \\ &= f^{-1}(A) \cup f^{-1}(B) \\ &= h_f(A) \vee h_f(B) \\ h_f(-A) &= f^{-1} \text{Cl}(Y \setminus A) \\ &= \text{Cl } f^{-1}(Y \setminus A) \\ &= \text{Cl}(X \setminus f^{-1}(A)) \\ &= -h_f(A) \end{aligned}$$

- Contact.

$$\begin{aligned}
 h_f(A)C_X h_f(B) &\text{ iff } f^{-1}(A) \cap f^{-1}(B) \neq \emptyset \\
 &\text{ iff } A \cap B \neq \emptyset && \text{by surjectivity of } f \\
 &\text{ iff } AC_Y B
 \end{aligned}$$

where C_X and C_Y denote the contact relations in $RC(X)$ and $RC(Y)$, respectively. \square

Corollary 2.19. *Let X and Y be topological spaces. If $f : X \rightarrow Y$ is a surjective, interior map, then $\text{Log}(RC(X)) \subseteq \text{Log}(RC(Y))$.*

Proof. Immediate from [Corollary 2.6](#) and [Proposition 2.18](#). \square

2.4. Topological representation of contact algebras

In the previous section, we saw that each topological space X gives rise to a topological contact algebra, $RC(X)$. Such algebras are important in part because of the role they play in representation theorems for contact algebras. In this section, we recall a representation theorem given in [\[19\]](#), which shows that every contact algebra can be embedded in a topological contact algebra. The representation theorem is in some ways similar to the Stone representation theorem for Boolean algebras, but instead of taking the space of all ultrafilters in a Boolean algebra, we take the space of all clans in a contact algebra. The central notion of a clan is defined below.¹³

Definition 2.20. Let (B, C) be a contact algebra. A non-empty set $\Gamma \subseteq B$ is a *clan* if Γ satisfies:

- (C1) If $a \in \Gamma$ and $a \leq b$, then $b \in \Gamma$;
- (C2) If $a \vee b \in \Gamma$, then $a \in \Gamma$ or $b \in \Gamma$;
- (C3) If $a, b \in \Gamma$, then aCb .

Note that if Γ is a clan in (B, C) , then by (C_1) , $1 \in \Gamma$, and by (C_2) , for every $a \in B$, either $a \in \Gamma$ or $\neg a \in \Gamma$.

It is not difficult to see that every ultrafilter U in the Boolean algebra B is a clan in (B, C) , since if $a, b \in U$, then $a \wedge b \neq 0$, and by [Lemma 2.3](#), aCb . Say that two ultrafilters U and U' are *connected* if for every $a \in U$ and $b \in U'$, aCb . Then clearly if U is connected to U' , the set $U \cup U'$ is a clan. Moreover, it can be shown that each clan in (B, C) is a union of connected ultrafilters (see [\[6\]](#), Lemma 4.2). Thus clans in (B, C) are precisely unions of connected ultrafilters in B .

Recall that a collection \mathcal{B} of subsets of X is a *closed basis* for some topology on X if \mathcal{B} satisfies

- (B1) $\bigcap \mathcal{B} = \emptyset$;
- (B2) If $F, G \in \mathcal{B}$ and $x \notin F \cup G$, then there exists $H \in \mathcal{B}$ such that $F \cup G \subseteq H$ and $x \notin H$.

The closed sets of the topology determined by \mathcal{B} are arbitrary intersections of elements of \mathcal{B} . A collection \mathcal{T} of subsets of X is an *open basis* for a topology on X if \mathcal{T} satisfies

¹³ See [\[19\]](#), §2.3. A precursor to this representation of contact algebras via spaces of clans can be found in [\[6\]](#).

- (O1) $\bigcup \mathcal{T} = X$;
 (O2) If $U, V \in \mathcal{T}$ and $x \in U \cap V$, then there exists $O \in \mathcal{T}$ such that $x \in O \subseteq U \cap V$.

The open sets of the topology are arbitrary unions of basis elements. Note that if \mathcal{B} is a closed basis for a topology on X , then $\mathcal{T} = \{X \setminus F \mid F \in \mathcal{B}\}$ is an open basis for the same topology.

Let $CLAN(A)$ be the set of clans of the contact algebra $A = (B, C)$. We give topological structure to $CLAN(A)$ by specifying a closed basis. For each $a \in B$, let

$$S_a = \{\Gamma \in CLAN(A) \mid a \in \Gamma\}$$

It is easy to verify that the collection of sets $\mathcal{B} = \{S_a \mid a \in B\}$ is a closed basis for a topology on $CLAN(A)$. Indeed, for any $a, b \in B$, $S_a \cup S_b = S_{a \vee b}$. Therefore \mathcal{B} is closed under finite unions, and hence satisfies (B2). Moreover, for all $a \in B$, it's not the case that $0Ca$. Therefore $S_0 = \emptyset$, and \mathcal{B} satisfies (B1). With slight abuse of notation, we will denote by ' $CLAN(A)$ ' both the set of clans and the topological space that arises by taking \mathcal{B} as a basis.

For any $a \in B$, let $U_a = \{\Gamma \in CLAN(A) \mid a \notin \Gamma\}$. Note that $U_a = CLAN(A) \setminus S_a$. Thus the collection of sets $\{U_a \mid a \in B\}$ is an open basis for the topology on $CLAN(A)$. It follows that for any set $S \subseteq CLAN(A)$, $Int(S) = \bigcup \{U_b \mid U_b \subseteq S\}$.

Lemma 2.21. *For any $a \in B$,*

1. $Int(S_a) = U_{-a}$.
2. $Cl(U_a) = S_{-a}$.
3. S_a is a regular closed set.

Proof. 1. If $\Gamma \in U_{-a}$, then $-a \notin \Gamma$, so $a \in \Gamma$ and $\Gamma \in S_a$. Therefore U_{-a} is an open subset of S_a . So $U_{-a} \subseteq Int(S_a)$. For the reverse inclusion, suppose that $\Gamma \in Int(S_a)$. Then $\Gamma \in U_b \subseteq S_a$ for some $b \in B$. So every clan that does not contain b contains a . We claim that $-a \leq b$. (Suppose not. Then $-a \wedge -b \neq 0$. By the Ultrafilter Lemma, there is an ultrafilter Δ with $-a \wedge -b \in \Delta$. But then $-a \in \Delta$ and $-b \in \Delta$. Now Δ is a clan, since every ultrafilter is a clan. And since Δ is an ultrafilter, $a \notin \Delta$ and $b \notin \Delta$, contradicting the fact that every clan that does not contain b contains a .) But now since $\Gamma \in U_b$, $b \notin \Gamma$, so by (C1), $-a \notin \Gamma$. Therefore, $\Gamma \in U_{-a}$. We showed that $Int(S_a) \subseteq U_{-a}$.

2. is proved similarly; the details are left to the reader.

3. is an immediate consequence of 1. and 2. \square

Part 3. of Lemma 2.21 allows us to define a map h from the contact algebra $A = (B, C)$ to the contact algebra $RC(CLAN(A))$ by:

$$h(a) = S_a$$

The following proposition is proved in [19].¹⁴

Theorem 2.22. *Let $A = (B, C)$ and let $h : (B, C) \rightarrow RC(CLAN(A))$ be defined as above. Then,*

1. h is an embedding.
2. If B is a complete Boolean algebra, then h is an isomorphism.

¹⁴ See [19], Lemma 2.3.5, Corollary 2.3.8, and Proposition 2.2.3.

We will call the map h the *topological representation* of the contact algebra $A = (B, C)$. In fact, with slight abuse of notation, we will also sometimes refer to the topological space $CLAN(A)$ itself as the topological representation of A .

As a consequence of this representation theorem and [Corollary 2.9](#), we get the following weak completeness result. (The equivalence of 1. and 2. below was shown in [\[3\]](#).¹⁵ We add 3. because it will be useful in proving completeness for particular topological spaces below.)

Corollary 2.23. *The following are equivalent:*

1. φ is a theorem of \mathbb{L}_{min}^{cont} ;
2. φ is true in every topology;
3. φ is true in every finite topology (equivalently: every finite qoset).

Proof. The direction from 1. to 2. follows from [Proposition 2.4](#). The direction from 2. to 3. is obvious. For the direction from 3. to 1., suppose φ is a non-theorem of \mathbb{L}_{min}^{cont} . Then by [Corollary 2.9](#), φ is refuted in a finite frame $F = (W, R)$. By [Lemma 2.7](#), φ is refuted in $\mathcal{B}(F)$. Let $X = CLAN(\mathcal{B}(F))$. By [Theorem 2.22](#), there is an embedding h of $\mathcal{B}(F)$ into $RC(X)$. By [Corollary 2.6](#), φ is refuted in X . Moreover, since F is finite, the set X of clans in $\mathcal{B}(F)$ is finite. \square

3. The infinite binary tree

Completeness of \mathbb{L}_{min}^{cont} for the topological semantics tells us that any non-theorem of \mathbb{L}_{min}^{cont} is refuted in some topological space or other. But this leaves open the question of completeness for particular topological spaces of interest. We turn now to a study of that question. In this section, our focus is on the infinite binary tree (defined below); in the next section, we study completeness for zero-dimensional, dense-in-themselves metric spaces, including, *e.g.*, Cantor space and the rationals. After that, we turn to separable, connected, dense-in-themselves metric spaces, including, *e.g.*, the real line.

Let X be a topological space. Recall that X is *connected* if X cannot be represented as a disjoint union of two non-empty open sets. We say that X is *well-connected* if the intersection of non-empty closed subsets of X is non-empty (equivalently, there is a point that belongs to the closure of every non-empty set). Obviously every well-connected topology is connected, but the converse is not in general true; there are connected topologies that are not well-connected (*e.g.*, the real line).

Consider now Alexandroff topologies. If we view an Alexandroff topology as a qoset, then the condition that the topology is well-connected has a natural analog in terms of the quasi-order. Say that a qoset (X, \leq) is *rooted* if there exists $x \in X$ such that $x \leq y$ for all $y \in X$. In this case we call x a *root* of X .

Lemma 3.1. *An Alexandroff topology (X, τ) is well-connected iff (X, \leq_τ) is rooted.*

Proof. Let (X, τ) be an Alexandroff topology. Then,

$$\begin{aligned} X \text{ is well-connected} &\text{ iff } \exists x \in X \text{ such that } x \in Cl(S) \text{ for every non-empty } S \subseteq X \\ &\text{ iff } \exists x \in X \text{ such that } x \in Cl(\{y\}) \text{ for every } y \in X \\ &\text{ iff } \exists x \in X \text{ such that } x \leq_\tau y \text{ for every } y \in X \end{aligned}$$

¹⁵ See [\[3\]](#), Theorem 9.1, where completeness of \mathbb{L}_{min}^{cont} is proved for the class of all compact, semi-regular, T_0 topological spaces. This completeness theorem is a consequence of the fact that the topological representation of contact algebras given in §2.4 above is semi-regular, compact, and T_0 —or alternatively, that the canonical spaces considered in [\[3\]](#) have these properties (see [\[3\]](#), Lemma 9.8 and 9.9).

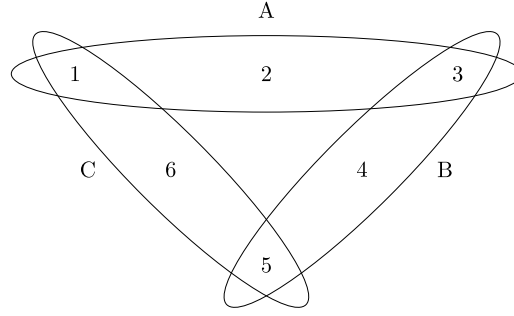


Fig. 1. The topology (Ω, τ) from Example 3.3. Three basis sets, A , B , and C , are circled.

iff $\exists x \in X$ such that x is a root of X \square

It follows from Lemma 3.1 that finite, well-connected topologies are just finite, rooted qosets.

Lemma 3.2. *Let X be a topology. If X is well-connected, then $(Univ)$ is true in X .*

Proof. Let $M = (X, V)$ be a topological model over X . If $M \models a \neq 0 \wedge b \neq 0$, then $V(a) \neq \emptyset$ and $V(b) \neq \emptyset$. Moreover, $V(a)$ and $V(b)$ are closed sets. Since X is well-connected, $V(a) \cap V(b) \neq \emptyset$. But then $M \models aCb$. \square

The converse of Lemma 3.2 is not true; there are spaces X such that $(Univ)$ is true in X , but X is not well-connected.

Example 3.3. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, and $C = \{5, 6, 1\}$. Let $\mathcal{B} = \{\Omega, \emptyset, A, B, C, A \cup B, B \cup C, C \cup A\}$. It is easy to verify that \mathcal{B} is a closed basis on the set Ω .

Let (Ω, τ) (or simply Ω) denote the topological space given by the closed basis \mathcal{B} . The reader can verify that \mathcal{B} is in fact the collection of regular closed subsets of Ω .¹⁶ Moreover, $(Univ)$ is true in Ω , since the intersection of any two non-empty basis elements is non-empty. But Ω is not well-connected. Indeed, the singletons $\{1\}$, $\{3\}$, and $\{5\}$ are closed subsets of Ω , so the intersection of all non-empty closed sets is empty. (See Fig. 1.)

Proposition 3.4. *Let $A = (B, C)$ be a contact algebra such that $(Univ)$ is true in A . Then $CLAN(A)$ is a well-connected topology.*

Proof. Suppose that $(Univ)$ is true in $A = (B, C)$. Then $\Gamma = \{a \in B \mid a \neq 0\}$ is a clan in B . Indeed, conditions (C1) and (C2) are easily verified, and since $(Univ)$ is true in (B, C) , aCb for every non-zero $a, b \in B$, so Γ satisfies (C3). Clearly $\Gamma \in S_a$ for every non-zero $a \in B$. Thus Γ belongs to every non-empty closed basis set, and hence to every non-empty closed set. It follows that $CLAN(A)$ is well-connected. \square

Proposition 3.5. *The following are equivalent:*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Univ)$.
2. φ is true in every well-connected topology.

¹⁶ The closed, non-basis sets are: $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{1, 5\}$, $\{3, 5\}$, $\{1, 3, 5\}$, $\{1, 2, 3, 5\}$, $\{3, 4, 5, 1\}$, and $\{1, 6, 5, 3\}$. If we take complements of elements of the closed basis, we get an open basis for the space: $\{\emptyset, \Omega, \{2\}, \{4\}, \{6\}, \{2, 3, 4\}, \{1, 2, 6\}, \{4, 5, 6\}\}$. Therefore none of the closed non-basis sets are regular closed. By contrast, each of the sets A , B , and C is regular closed, and therefore also the unions $A \cup B$, $A \cup C$, $B \cup C$ are regular closed. So \mathcal{B} is the collection of regular closed subsets of Ω .

3. φ is true in every finite, well-connected topology (equivalently: every finite, rooted qoset).

Proof. The direction from 1. to 2. follows from Lemma 3.2 and Proposition 2.4. The direction from 2. to 3. is obvious. For the direction from 3. to 1., suppose that φ is a non-theorem of $\mathbb{L}_{min}^{cont} + (Univ)$. By Proposition 2.15, φ is refuted in a finite universal frame F . Thus $(Univ)$ is true in $\mathcal{B}(F)$ and φ is refuted in $\mathcal{B}(F)$. Let $X = CLAN(\mathcal{B}(F))$. Clearly X is finite, and by Proposition 3.4, X is well-connected. The map h constructed in §2.4 is an embedding of $\mathcal{B}(F)$ into $RC(X)$, so φ is refuted in $RC(X)$. \square

We now want to make use of this completeness result to show that $\mathbb{L}_{min}^{cont} + (Univ)$ is complete for a particular topological space that has been of significant interest in modal logic: the infinite binary tree. Our strategy will be to transfer counterexamples to non-theorems of \mathbb{L}_{min}^{cont} from finite well-connected topological spaces to the infinite binary tree.

Let Ω be the set of all finite strings over $\{0, 1\}$, including the empty string, which we denote by Λ . For any strings s and t , let $s * t$ denote the concatenation of s and t . (Thus, e.g., $001 * 10 = 00110$.) We define the binary relation \leq over Ω by: $s \leq t$ iff $t = s * s'$ for some (possibly empty) string $s' \in \Omega$. Clearly \leq is a quasi-order. The *infinite binary tree*, \mathcal{T}_2 , is the qoset (Ω, \leq) .

The following well-known result was originally proved by Dov Gabbay and independently by Johan van Benthem; a proof can be found in, e.g., [10].¹⁷

Proposition 3.6. *If (X, \leq) is a finite rooted qoset, then there is an interior, surjective map from \mathcal{T}_2 to (X, \leq) .*

Proposition 3.7. *The following are equivalent:*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Univ)$.
2. φ is true in the infinite binary tree, \mathcal{T}_2 .

Proof. For the direction from 1. to 2., note that \mathcal{T}_2 is rooted. By Lemma 3.1, \mathcal{T}_2 is well-connected. By Proposition 3.5, every theorem of $\mathbb{L}_{min}^{cont} + (Univ)$ is true in \mathcal{T}_2 . For the reverse direction, suppose that φ is a non-theorem of $\mathbb{L}_{min}^{cont} + (Univ)$. By Proposition 3.5, φ is refuted in a finite, rooted qoset (X, \leq) . By Proposition 3.6, there is an interior, surjective map from \mathcal{T}_2 onto (X, \leq) . It follows from Corollary 2.19 that φ is refuted in \mathcal{T}_2 . \square

4. Zero-dimensional, dense-in-themselves metric spaces

Recall that a topology X is *zero-dimensional* if X has a basis of clopen sets. We say that a point $x \in X$ is *isolated* if $\{x\}$ is open in X . A topology is *dense-in-itself* if it contains no isolated points. Equivalently, X is dense-in-itself if every $x \in X$ is a point of closure of $X \setminus \{x\}$. The goal of this section is to show that \mathbb{L}_{min}^{cont} is (weakly) complete for any zero-dimensional, dense-in-itself metric space. This includes as special cases Cantor space, \mathcal{C} , the rationals, \mathbb{Q} , and the irrationals, \mathbb{P} . We begin by proving a special case of the theorem: completeness for Cantor space, where the proof is constructive and (relatively) easy to visualize. In the next section, we prove the theorem in full generality.

4.1. Cantor space

We already know that non-theorems of \mathbb{L}_{min}^{cont} are refuted in finite topologies (see Corollary 2.23). Our aim in this section is to transfer countermodels from finite topologies to the Cantor space, \mathcal{C} . To do this, we

¹⁷ See [10], Theorem 1. Goldblatt proves that there is a p-morphism from \mathcal{T}_2 onto (X, \leq) ; any p-morphism between qosets is an interior map between the corresponding topological spaces.

will construct a surjective, interior map f from \mathcal{C} onto an arbitrary finite topology. The construction of f is similar to the construction given in [1], where completeness of the modal logic $S4$ for \mathcal{C} is proved. However, instead of transferring countermodels to \mathcal{C} from finite *well-connected* topologies (i.e. rooted qosets), as in [1], we must transfer countermodels from arbitrary finite topologies. Therefore, a different construction is needed.

Recall from the construction of the infinite binary tree, $\mathcal{T}_2 = (\Omega, \leq)$, that points in Ω are finite strings over $\{0, 1\}$, including the empty string Λ . If the length of a string s is n , we say that s has *height* n . Thus, e.g., Λ has height 0, and for each $n \in \mathbb{N}$, there are 2^n nodes of height n . Let Ω^* be the set of all countably infinite strings over $\{0, 1\}$. For any $s \in \Omega$ and $t \in \Omega \cup \Omega^*$, let $s * t$ denote the concatenation of s and t . For any node $s \in \Omega$, let

$$U_s = \{x \in \Omega^* \mid x = s * s' \text{ for some } s' \in \Omega^*\}$$

In words, U_s is the set of all infinitary strings that have s as an initial segment. Consider the set

$$\mathcal{B} = \{U_s \mid s \in \Omega\}$$

Clearly \mathcal{B} is an open basis for a topology $\tau_{\mathcal{C}}$ on Ω^* . Indeed, $\bigcup \mathcal{B} = \Omega^*$, so \mathcal{B} satisfies (O1). And for any $s, t \in \Omega$, either (1) $U_s \cap U_t = U_s$, (2) $U_s \cap U_t = U_t$, or (3) $U_s \cap U_t = \emptyset$. Therefore, if for some $x \in \Omega^*$, $x \in U_s \cap U_t$, then $U_s \cap U_t$ is a basis element. So \mathcal{B} satisfies (O2). The *Cantor space*, \mathcal{C} , is homeomorphic to the topology generated by this open basis. Therefore, without any harm, we can think of Cantor space as the topology $(\Omega^*, \tau_{\mathcal{C}})$.

We now show that there is an interior, surjective map from \mathcal{C} to any finite qoset. We do this by first defining a partial function from Ω to the finite qoset, and then use this partial function to define an interior, surjective map from \mathcal{C} to the qoset. Let (X, \leq) be a finite qoset. Say that $x \in X$ is *quasi-minimal* if for all $y \in X$, $y \leq x$ implies $x \leq y$. For any $x \in X$, let $C[x] = \{y \in X \mid x \leq y \text{ and } y \leq x\}$. A *cluster* is a set of the form $C[x]$ for some $x \in X$. A cluster C is *minimal* if $C = C[x]$ for some quasi-minimal $x \in C$. Since X is finite, there are only finitely many minimal clusters in X . Let m be the number of minimal clusters in X , and enumerate them: C_1, C_2, \dots, C_m . For each $1 \leq k \leq m$, pick an arbitrary $r_k \in C_k$.

Pick $n \in \mathbb{N}$ large enough so that $m \leq 2^n$. Note that there are 2^n nodes in Ω of height n , which we will denote by s_1, \dots, s_{2^n} . We define the partial function $g : \Omega \rightarrow X$ on all strings in Ω of height $k \geq n$ as follows. Label each of the nodes of height n by the r_k 's in such a way that each r_k labels at least one node. For specificity, we can let $g(s_k) = r_k$ for $k \leq m$, and let $g(s_k) = r_1$ for $k > m$. Then label the nodes in Ω of height greater than n in the same way as introduced in [1].

To recall the construction there, for any node $s \in \Omega$, say that $s * 0$ and $s * 1$ are the *left successor* and *right successor* of s , respectively. We say that a node $s \in \Omega$ is a *left node* (*right node*) if it is a left successor (right successor) of some node.

Definition 4.1. For any $s \in \Omega$, the *t-comb* of s is the set of nodes $s, s * 0, s * 00, s * 000$, etc., plus all right successors of these nodes. (See Fig. 2.)

Since X is finite, for any $x \in X$, the set $[x]_{\leq} = \{y \in X \mid x \leq y\}$ is finite. Enumerate the elements in $[x]_{\leq}$ as follows: x_0, x_1, \dots, x_i .

For any $s \in \Omega$, if $g(s) = x$, and g has not yet been defined on the t-comb of s , put:

1. $g(s * 0) = x, g(s * 00) = x, g(s * 000) = x$, etc.
2. $g(s * 1) = x_0, g(s * 01) = x_1, g(s * 001) = x_2$, etc. In general, $g(s * 0^k 1) = x_{k \bmod i}$.

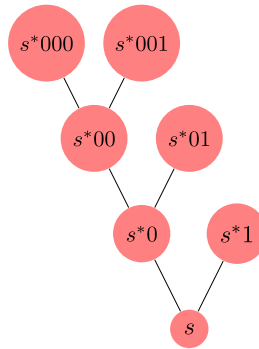


Fig. 2. The beginning of the t-comb of node s in the infinite binary tree \mathcal{T}_2 .

Thus g is defined on every node in Ω of height n , and on the t-comb of every node on which g is defined. It follows that g is defined on every node in Ω of height at least n . By contrast, g is not defined on nodes in Ω of height smaller than n .

Lemma 4.2. *For any $s, t \in \Omega$ such that g is defined on both s and t , if $s \leq t$, then $g(s) \leq g(t)$.*

Proof. For any nodes $s, t \in \Omega$, with $s \leq t$, define the *distance* between s and t as $\text{height}(t) - \text{height}(s)$. The proof of the lemma is by induction on the distance between s and t . The details are left to the reader. \square

We now define a function $f : \mathcal{C} \rightarrow X$ using the partial function g as follows. Let $t \in \Omega^*$ —i.e., t is an infinitary string. Since X is finite, there is at least one $x \in X$ such that $g(s) = x$ for infinitely many initial segments s of t . Define $f : \mathcal{C} \rightarrow X$ as follows.

$$f(t) = \begin{cases} x & \text{where } x \text{ is an arbitrarily chosen element in } X \text{ such} \\ & \text{that } g(s) = x \text{ for infinitely many initial segments } s \text{ of } t \end{cases}$$

For any $t \in \Omega^*$, let us say that t *stabilizes* on $x \in X$ if there exists some $n \in \mathbb{N}$ such that $g(s) = x$ for all initial segments s of t of height greater than n . As an example, note that if $t = s * 000 \dots$, then t stabilizes on $g(s)$. Not all infinitary strings stabilize. But note that if t *does* stabilize on some point $x \in X$, then $f(t) = x$.

Lemma 4.3. *For any $t \in \Omega^*$, if s is an initial segment of t and $g(s)$ is defined, then $g(s) \leq f(t)$.*

Proof. Since $g(s') = f(t)$ for infinitely many initial segments s' of t , there is some initial segment s' of t such that $s \leq s'$ and $g(s') = f(t)$. By Lemma 4.2, $g(s) \leq g(s') = f(t)$. \square

Lemma 4.4. *The function $f : \mathcal{C} \rightarrow (X, \leq)$ is surjective, open, and continuous.*

Proof. We take the three claims in turn.

- *Surjective.* Let $x \in X$. Note that there exists some quasi-minimal y such that $y \leq x$. Since y is quasi-minimal, y belongs to a minimal cluster, C_k . Therefore $r_k \leq y \leq x$, and for some node $s \in \Omega$ of height n , $g(s) = r_k$. Since $r_k \leq x$, $x \in [r_k]_{\leq}$. But then, by labeling of the t-comb of s , $g(t) = x$ for some node t on the t-comb of s . Consider the point $t * 000 \dots$. This infinitary string stabilizes on x . Therefore $f(t * 000 \dots) = x$. So x is in the image of f .
- *Open.* It is sufficient to show that the image under f of every basic open set U_s in \mathcal{C} is open in X . (Recall that $U_s = \{x \in \Omega^* \mid x = s * s' \text{ for some } s' \in \Omega^*\}$, where $s \in \Omega$.) Let $s \in \Omega$, and let $x \in f[U_s]$. We need

to show that $[x]_{\leq} \subseteq f[U_s]$. Let $y \in [x]_{\leq}$. Since $x \in f[U_s]$, there is some $t \in U_s$ such that $f(t) = x$. And since $t \in U_s$, s is an initial segment of t .¹⁸ By Lemma 4.3, $g(s) \leq f(t) = x \leq y$. But then by labeling of the t -comb, there is some s' in the t -comb of s such that $g(s') = y$. Consider $s' * 000 \dots$. This infinitary node stabilizes on y , so $f(s' * 000 \dots) = y$, and clearly $s' * 000 \dots \in U_s$. Therefore $y \in f[U_s]$. We have shown that $[x]_{\leq} \subseteq f[U_s]$.

- *Continuous.* It is sufficient to show that for every $x \in X$, $f^{-1}([x]_{\leq})$ is open. Suppose $t \in f^{-1}([x]_{\leq})$. Then $f(t) = y$ for some $y \geq x$. By definition of f , $g(s) = y$ for infinitely many initial segments s of t . Let s be one such initial segment. We claim that $U_s \subseteq f^{-1}([x]_{\leq})$. Indeed, if $t' \in U_s$, then by Lemma 4.3, $y = g(s) \leq f(t')$. And since $x \leq y$, we have $x \leq f(t')$. So $f(t') \in [x]_{\leq}$ and $t' \in f^{-1}([x]_{\leq})$. \square

Proposition 4.5. *The following are equivalent:*

1. φ is a theorem of \mathbb{L}_{min}^{cont} .
2. φ is true in \mathcal{C} .

Proof. The direction from 1. to 2. follows from Proposition 2.4. For the direction from 2. to 1., suppose that φ is a non-theorem of \mathbb{L}_{min}^{cont} . By Corollary 2.23, φ is refuted in some finite topology (X, τ) . Equivalently, φ is refuted in the finite qoset (X, \leq_{τ}) . By Lemma 4.4, there is an interior, surjective map $f : \mathcal{C} \rightarrow (X, \leq_{\tau})$. By Corollary 2.19, φ is refuted in \mathcal{C} . \square

4.2. The general case

We have seen that \mathbb{L}_{min}^{cont} is complete for the Cantor space \mathcal{C} ; we would like to now prove the more general result that \mathbb{L}_{min}^{cont} is complete for each zero-dimensional, dense-in-itself metric space. Our starting point is, of course, completeness of that logic for the collection of all finite qosets. Our strategy going forward will be to first prove completeness of \mathbb{L}_{min}^{cont} for a class of ‘forests’ (or disjoint union of trees) and then push counterexamples forward from the forests to the topological spaces of interest via a sequence of algebraic maps.

Let (X, \leq) be a finite qoset. As in §4.1, let m be the number of minimal clusters in (X, \leq) , and enumerate the minimal clusters: C_1, C_2, \dots, C_m . As before, we pick an arbitrary $r_k \in C_k$ for $1 \leq k \leq m$. Pick $n \in \mathbb{N}$ large enough so that $m \leq 2^n$. Then there are 2^n nodes in \mathcal{T}_2 of height n , and we denote them by s_1, s_2, \dots, s_{2^n} .

Consider the set of nodes in the infinite binary tree of height at least n :

$$\{s \in \mathcal{T}_2 \mid \text{height}(s) \geq n\}$$

together with the subspace topology. We will denote this topology by \mathcal{T}_2^n . For each $s \in \mathcal{T}_2^n$, let

$$V_s = \{t \in \mathcal{T}_2^n \mid s \leq t\}$$

Then $\mathcal{B} = \{V_s \mid s \in \mathcal{T}_2^n\}$ is an open basis for \mathcal{T}_2^n .

Note that the partial map $g : \mathcal{T}_2 \rightarrow (X, \leq)$ that was defined in §4.1 is defined on every node in \mathcal{T}_2^n . We now show that in fact g is an interior surjective map. First, the following simple lemma.

Lemma 4.6. *For any $s \in \mathcal{T}_2^n$, $g(V_s) = [g(s)]_{\leq}$.*

¹⁸ We use ‘ \leq ’ in this proof for both the quasi-order on Ω and the quasi order on X . We hope the reader will understand which is meant in each instance.

Proof. Suppose $x \in g(V_s)$. Then for some $t \in V_s$, $g(t) = x$. Since $s \leq t$, $g(s) \leq g(t)$ by Lemma 4.2, and $x \in [g(s)]_{\leq}$. Conversely, suppose $x \in [g(s)]_{\leq}$. Then by labeling of the t-comb of s , $g(t) = x$ for some t in the t-comb of s . So $t \geq s$; equivalently, $t \in V_s$ and $x \in g(V_s)$. \square

Lemma 4.7. *The function $g : \mathcal{T}_2^n \rightarrow (X, \leq)$ is continuous, open, and surjective.*

Proof. We prove each in turn.

1. *Continuous.* We show that the pre-image of any basic open set $[x]_{\leq} \subseteq X$ is open in \mathcal{T}_2^n . Suppose $x \in X$, and $s \in g^{-1}([x]_{\leq})$. Then $x \leq g(s)$. Therefore $[g(s)]_{\leq} \subseteq [x]_{\leq}$. By Lemma 4.6, $g(V_s) = [g(s)]_{\leq}$. So $g(V_s) \subseteq [x]_{\leq}$ and $V_s \subseteq g^{-1}([x]_{\leq})$. We have $s \in V_s \subseteq g^{-1}([x]_{\leq})$. Therefore, $g^{-1}([x]_{\leq})$ is open.
2. *Open.* It is sufficient to show that the image under g of any basic open set $V_s \subseteq \mathcal{T}_2^n$ is open in (X, \leq) . This is immediate from Lemma 4.6.
3. *Surjective.* For each $x \in X$, there exists $k \leq m$ such that $r_k \leq x$ (see proof of Lemma 4.4). But $g(s_k) = r_k$. By Lemma 4.6, $g(V_{s_k}) = [r_k]_{\leq}$. Therefore $x \in g(V_{s_k})$. \square

As a result of Lemma 4.7, we get the following completeness theorem.

Proposition 4.8. *The following are equivalent:*

1. φ is a theorem of \mathbb{L}_{min}^{cont} ;
2. φ is true in $RC(\mathcal{T}_2^n)$ for each $n \in \mathbb{N}$.

Proof. The direction from 1. to 2. follows from Proposition 2.4. For the direction from 2. to 1., let φ be a non-theorem of \mathbb{L}_{min}^{cont} . By Corollary 2.23, φ is refuted in a finite qoset, (X, \leq) . By Lemma 4.7, for some $n \in \mathbb{N}$ there is an interior, surjective map $g : \mathcal{T}_2^n \rightarrow (X, \leq)$. By Corollary 2.19, φ is refuted in \mathcal{T}_2^n . \square

Let $\{\mathcal{X}_i = (X_i, \tau_i) \mid i \in I\}$ be a family of topological spaces, with $X_i \cap X_j = \emptyset$ for $i \neq j$. Recall that the *disjoint union* of these spaces is the topology

$$\biguplus_{i \in I} \mathcal{X}_i = (\bigcup \{X_i \mid i \in I\}, \tau)$$

where $U \in \tau$ iff $U \cap X_i \in \tau_i$ for each $i \in I$.¹⁹ Note that if \mathcal{B}_i is a basis for the topology \mathcal{X}_i , for each $i \in I$, we can take as a basis for τ the collection of sets $\{O \in \mathcal{B}_i \mid i \in I\}$. When I is finite, we denote disjoint unions with infix notation—for example, $X \uplus Y$.

We would like to represent \mathcal{T}_2^n , for each $n \in \mathbb{N}$, as a disjoint union of 2^n copies of the infinite binary tree \mathcal{T}_2 . Let $\mathcal{T}_{2,1}, \dots, \mathcal{T}_{2,2^n}$ be pairwise disjoint copies of the infinite binary tree, \mathcal{T}_2 . More precisely, let Ω_k be the set of pairs $\{(s, k) \mid s \in \Omega\}$, and for every (s, k) and (t, k) in Ω_k , put $(s, k) \leq_k (t, k)$ iff $s \leq t$, where \leq is the quasi-order on \mathcal{T}_2 . Then $\mathcal{T}_{2,k} = (\Omega_k, \leq_k)$.

Proposition 4.9. *\mathcal{T}_2^n is homeomorphic to $\biguplus_{k \leq 2^n} \mathcal{T}_{2,k}$.*

Proof. Recall that s_1, \dots, s_{2^n} are the nodes of \mathcal{T}_2 of height n . Let $\varphi : \biguplus_{k \leq 2^n} \mathcal{T}_{2,k} \rightarrow \mathcal{T}_2^n$ be defined by:

$$(t, k) \mapsto s_k * t$$

¹⁹ Equivalently, τ is the finest topology such that the inclusion map $f_i : X_i \rightarrow X$ is continuous, for each $i \in I$.

We need to show that φ is injective, surjective, continuous, and open. For injectivity, suppose $\varphi(x) = \varphi(y)$. Let $x = (s, i)$, and let $y = (t, j)$. Then $s_i * s = s_j * t$. So the initial segment of length n of $s_i * s$ is equal to the initial segment of length n of $s_j * t$. Therefore $s_i = s_j$, and $i = j$. But then since $s_i * s = s_j * t$, also $s = t$. So $x = y$. For surjectivity, note that if $t \in \mathcal{T}_2^n$, then $t = s_k * t'$ for some $k \leq 2^n$ and $t' \in \Omega$. But then $(t', k) \in \biguplus_{k \leq 2^n} \mathcal{T}_{2,k}$ and $\varphi(t', k) = s_k * t' = t$. To see that φ is open, let O be a basic open set in $\biguplus_{k \leq 2^n} \mathcal{T}_{2,k}$. Then O is a basic open set in $\mathcal{T}_{2,k}$ for some $k \leq 2^n$ —i.e., $O = V_{(t,k)}$ for some $(t, k) \in \mathcal{T}_{2,k}$. But now we claim that

$$\varphi(V_{(t,k)}) = V_{s_k * t} \quad (1)$$

Indeed, if $y \in \varphi(V_{(t,k)})$, then $y = \varphi(x)$ for some $x \in V_{(t,k)}$. In particular, $x = (t', k)$ for some $t' \geq t$. But then $y = \varphi(x) = s_k * t'$. And since $t' \geq t$, we have $s_k * t' \geq s_k * t$, and $s_k * t' \in V_{s_k * t}$. Conversely, if $y \in V_{s_k * t}$, then $y = s_k * t'$ for some $t' \geq t$. But then $\varphi(t', k) = s_k * t' = y$. And $(t', k) \in V_{(t,k)}$. Thus $y \in \varphi(V_{(t,k)})$. Finally, to see that φ is continuous, let U be a basic open set in \mathcal{T}_2^n . Then $U = V_{s_k * t}$ for some $k \leq 2^n$ and $t \in \Omega$. By (1) and injectivity of φ , $\varphi^{-1}(U) = V_{(t,k)}$, which is open in $\biguplus_{k \leq 2^n} \mathcal{T}_{2,k}$. \square

Proposition 4.10. *Suppose X is a zero-dimensional, dense-in-itself metric space. Then $X = X_1 \uplus X_2$ for some zero-dimensional, dense-in-themselves metric spaces X_1 and X_2 . It follows that for any $n \in \mathbb{N}$, X can be written as a disjoint union*

$$\biguplus_{k=1}^n X_k$$

where each X_k is a dense-in-itself metric space.

Proof. Let X be a zero-dimensional dense-in-itself metric space. Pick $x \in X$. Since X is dense-in-itself, there exists $y \in X$ such that $y \neq x$. Since X is a metric space (hence, in particular, a T_1 space), there exists an open set O such that $x \in O$ and $y \notin O$. And since X is zero-dimensional, there is a clopen set U such that $x \in U \subseteq O$. Now we have $x \in U$ and $y \in X \setminus U$, so U and $X \setminus U$ are non-empty, clopen sets. It is easy to see that X is the disjoint union of U and $X \setminus U$, where the latter sets are equipped with the subspace topology. (This follows from the fact that both U and $X \setminus U$ are open.)

We claim that both U and $X \setminus U$ are zero-dimensional, dense-in-themselves metric spaces. Indeed, any subspace of a zero-dimensional space is zero-dimensional. (If \mathcal{B} is a basis for the space X , and S is a subspace of X , take as a basis for S the set $\{T \cap S \mid T \in \mathcal{B}\}$. This is a clopen basis for S : since T is both open and closed in X , $T \cap S$ is both open and closed in S .) And any subspace of a metric space is a metric space. So we need only check that both U and $X \setminus U$ are dense-in-themselves. This follows from the fact that both U and $X \setminus U$ are open sets in X . Indeed, let $x \in U$. Since U is open in X , any open set in U is open in X . But $\{x\}$ is not open in X , so $\{x\}$ is not open in U . Hence U is dense-in-itself. Likewise, $X \setminus U$ is dense-in-itself. \square

Let $\{A_i = (B_i, C_i) \mid i \in I\}$ be a family of contact algebras indexed by the set I . The *direct product* (or simply *product*) $\prod_{i \in I} B_i$ is a Boolean algebra whose elements are functions that assign to each $i \in I$ some element $a_i \in B_i$. We denote such functions by $(a_i)_{i \in I}$. Boolean operations in the product algebra are defined coordinatewise:

$$\begin{aligned} (a_i)_{i \in I} \vee (b_i)_{i \in I} &= (a_i \vee b_i)_{i \in I} \\ (a_i)_{i \in I} \wedge (b_i)_{i \in I} &= (a_i \wedge b_i)_{i \in I} \\ -(a_i)_{i \in I} &= (-a_i)_{i \in I} \end{aligned}$$

(Note that occurrences of the symbols ‘ $-$ ’, ‘ \vee ’, ‘ \wedge ’ on the RHS of the equations denote operations in the Boolean algebra B_i , and occurrences of those symbols on the LHS denote operations in the product algebra.) The top (bottom) element of the product algebra is the function that assigns to each $i \in I$ the top (bottom) element in B_i . We can introduce a contact relation C on $\prod_{i \in I} B_i$ by putting, for any $(a_i)_{i \in I}, (b_i)_{i \in I} \in \prod_{i \in I} B_i$:

$$(a_i)_{i \in I} C (b_i)_{i \in I} \text{ iff } a_i C_i b_i \text{ for some } i \in I$$

Lemma 4.11. C is a contact relation on $\prod_{i \in I} B_i$.

Proof. We must verify each of the conditions of Definition 2.2. With some abuse of notation, we use the same symbol, ‘0’ (‘ \vee ’, ‘ \leq ’) for the bottom element (join, partial order) in the Boolean algebras B_i and $\prod_{i \in I} B_i$.

1. Suppose $(a_i)_{i \in I} C (b_i)_{i \in I}$. Then $a_i C_i b_i$ for some $i \in I$. But then $a_i \neq 0$ and $b_i \neq 0$. So $(a_i)_{i \in I} \neq 0$ and $(b_i)_{i \in I} \neq 0$.
2. Suppose $(a_i)_{i \in I} C (b_i)_{i \in I}$ and $(b_i)_{i \in I} \leq (c_i)_{i \in I}$. Then for some $j \in I$, $a_j C_j b_j$. Moreover, for each $i \in I$, $b_i \leq c_i$, so in particular $b_j \leq c_j$. Since C_j is a contact relation, $a_j C_j c_j$. Therefore, $(a_i)_{i \in I} C (c_i)_{i \in I}$.
3. Suppose that $(a_i)_{i \in I} C ((b_i)_{i \in I} \vee (c_i)_{i \in I})$. Then for some $i \in I$, $a_i C_i (b_i \vee c_i)$. Since C_i is a contact relation, either $a_i C_i b_i$ or $a_i C_i c_i$. Thus either $(a_i)_{i \in I} C (b_i)_{i \in I}$ or $(a_i)_{i \in I} C (c_i)_{i \in I}$.
4. Suppose that $(a_i)_{i \in I} \neq 0$. Then $a_i \neq 0$ for some $i \in I$. So $a_i C_i a_i$. But then $(a_i)_{i \in I} C (a_i)_{i \in I}$.
5. Suppose that $(a_i)_{i \in I} C (b_i)_{i \in I}$. Then $a_i C_i b_i$ for some $i \in I$. Since C_i a contact relation, $b_i C_i a_i$. So $(b_i)_{i \in I} C (a_i)_{i \in I}$. \square

Thus the algebra $\prod_{i \in I} B_i$ together with the relation C is a contact algebra, which we denote by $\prod_{i \in I} A_i$. We call this the *direct product* of the A_i ’s.

Lemma 4.12. Let $\{X_i \mid i \in I\}$ be a family of topological spaces with $X_i \cap X_j = \emptyset$ for $i \neq j$, and let $X = \biguplus_{i \in I} X_i$. Let $Int_i (Cl_i)$ and $Int (Cl)$ denote the interior (closure) operators in X_i and X respectively. Then for any set $S \subseteq X$,

$$Int(S) = \bigcup_{i \in I} Int_i(S \cap X_i) \quad (2)$$

$$Cl(S) = \bigcup_{i \in I} Cl_i(S \cap X_i) \quad (3)$$

Therefore, S is a regular closed set in X if and only if $S \cap X_i$ is regular closed in X_i for all $i \in I$.

Proof. For (2) and (3) we have:

$$\begin{aligned} x \in Int(S) &\text{ iff there exists open } U \text{ in } X \text{ such that } x \in U \subseteq S \\ &\text{ iff there exist open sets } V_i \text{ in } X_i \text{ such that } x \in \bigcup_{i \in I} V_i \subseteq S \\ &\text{ iff } x \in Int_i(S \cap X_i) \text{ for some } i \in I \\ x \in Cl(S) &\text{ iff } x \in X \setminus Int(X \setminus S) \\ &\text{ iff } x \in X \setminus \bigcup_{i \in I} Int_i(X_i \setminus (S \cap X_i)) \\ &\text{ iff } x \in \bigcup_{i \in I} (X_i \setminus Int_i(X_i \setminus (S \cap X_i))) \end{aligned}$$

$$\text{iff } x \in \bigcup_{i \in I} Cl_i(S \cap X_i)$$

The final statement of the lemma follows from (2) and (3), for we have:

$$\begin{aligned} S \text{ is regular closed in } X &\text{ iff } Cl(Int(S)) = S \\ &\text{ iff } Cl\left(\bigcup_{i \in I} Int_i(S \cap X_i)\right) = S && \text{by (2)} \\ &\text{ iff } \bigcup_{i \in I} Cl_i(Int_i(S \cap X_i)) = S && \text{by (3)} \\ &\text{ iff } Cl_i(Int_i(S \cap X_i)) = S \cap X_i \text{ for all } i \in I \\ &\text{ iff } S \cap X_i \text{ is regular closed in } X_i \text{ for all } i \in I \end{aligned}$$

where the fourth ‘iff’ follows from the fact that the X_i ’s are disjoint. \square

It follows from Lemma 4.12 that if $X = \biguplus_{i \in I} X_i$, we can define a map $h : \prod_{i \in I} RC(X_i) \rightarrow RC(X)$ by putting:

$$(A_i)_{i \in I} \mapsto \bigcup_{i \in I} A_i$$

Proposition 4.13. *h is an isomorphism of contact algebras.*

Proof. We must show that h is a bijection preserving joins, complements, and contact. For ease of notation, we will use the same symbol ‘ \vee ’ (‘ $-$ ’, ‘ C ’) to denote the join (complement, contact) in the algebras $\prod_{i \in I} RC(X_i)$ and $RC(X)$.

1. *Injective.*

Suppose $h((A_i)_{i \in I}) = h((B_i)_{i \in I})$. Then $\bigcup_{i \in I} A_i = \bigcup_{i \in I} B_i$. But then $A_i = (\bigcup_{i \in I} A_i) \cap X_i = (\bigcup_{i \in I} B_i) \cap X_i = B_i$ for all $i \in I$. So $(A_i)_{i \in I} = (B_i)_{i \in I}$.

2. *Surjective.*

Let $S \in RC(X)$. By Lemma 4.12, $S \cap X_i \in RC(X_i)$ for each $i \in I$. Moreover, $h((S \cap X_i)_{i \in I}) = \bigcup_{i \in I} (S \cap X_i) = S$.

3. *Joins.*

$$\begin{aligned} h((A_i)_{i \in I} \vee (B_i)_{i \in I}) &= h((A_i \cup B_i)_{i \in I}) \\ &= \bigcup_{i \in I} (A_i \cup B_i) \\ &= \bigcup_{i \in I} A_i \cup \bigcup_{i \in I} B_i \\ &= h((A_i)_{i \in I}) \vee h((B_i)_{i \in I}) \end{aligned}$$

4. *Complements.*

$$\begin{aligned} h(-(A_i)_{i \in I}) &= h((Cl_i(X_i \setminus A_i))_{i \in I}) \\ &= \bigcup_{i \in I} Cl_i(X_i \setminus A_i) \end{aligned}$$

$$\begin{aligned}
&= Cl(X \setminus \bigcup_{i \in I} A_i) && \text{by (3) above} \\
&= -h((A_i)_{i \in I})
\end{aligned}$$

5. Contact.

$$\begin{aligned}
(A_i)_{i \in I} C (B_i)_{i \in I} &\text{ iff } A_i C_i B_i \text{ for some } i \in I \\
&\text{ iff } A_i \cap B_i \neq \emptyset \text{ for some } i \in I \\
&\text{ iff } \bigcup_{i \in I} A_i \cap \bigcup_{i \in I} B_i \neq \emptyset \\
&\text{ iff } h((A_i)_{i \in I}) C h((B_i)_{i \in I})
\end{aligned}$$

where the third ‘iff’ follows from the fact that $A_i \cap B_j = \emptyset$ for $i \neq j$. \square

It follows from Propositions 4.13 and 4.9 that for any $n \in \mathbb{N}$,

$$RC(\mathcal{T}_2^n) \cong \prod_{k=1}^{2^n} RC(T_{2,k})$$

And it follows from Propositions 4.13 and 4.10 that for any zero-dimensional, dense-in-itself metric space X , there exist dense-in-themselves metric spaces X_1, \dots, X_{2^n} such that

$$RC(X) \cong \prod_{k=1}^{2^n} RC(X_k)$$

Definition 4.14. Let B be a Boolean algebra. A unary operator I on B is an *interior operator* if it satisfies:

1. $Ia \leq a$;
2. $IIa = Ia$;
3. $I(a \wedge b) = Ia \wedge Ib$;
4. $I1 = 1$.

If B is a Boolean algebra and I is an interior operator on B , we say that (B, I) is an *interior algebra*, or *topological Boolean algebra*.

Example 4.15. Let X be a topological space, and denote by $B(X)$ the field of all subsets of X . Let I be the operator on $B(X)$ that takes a set $A \subseteq X$ to the topological interior of A . Then $(B(X), I)$ is an interior algebra, which we denote by $I(X)$.

Suppose that $\mathcal{A}_1 = (B_1, I_1)$ and $\mathcal{A}_2 = (B_2, I_2)$ are interior algebras. We say that a function $h : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a *Boolean homomorphism* if h is a homomorphism from the Boolean algebra B_1 into B_2 . We say that h is a *homomorphism of interior algebras* if h is a Boolean homomorphism that preserves the interior operator: $h(I_1 a) = I_2(h(a))$. We say that h is an *embedding of interior algebras* if h is an injective homomorphism of interior algebras.

The following proposition was proved in [11].²⁰

²⁰ See [11], Lemma 5.7.

Proposition 4.16. *If X is a dense-in-itself metric space, there is an embedding of the interior algebra $I(\mathcal{T}_2)$ into the interior algebra $I(X)$.*

Proposition 4.17. *Let X and Y be topological spaces, and let $I(X)$ and $I(Y)$ denote the interior algebras of X and Y respectively. If $h : I(Y) \rightarrow I(X)$ is an embedding of interior algebras, then the restriction of h to the regular closed subsets of Y is an embedding of the contact algebra $RC(Y)$ into the contact algebra $RC(X)$.*

Proof. First we need to check that for any regular closed set A in Y , $h(A)$ is a regular closed set in X . Note that since $h : I(Y) \rightarrow I(X)$ is an embedding of interior algebras, we have:

1. $h(Y \setminus A) = X \setminus h(A)$;
2. $h(Int_Y(A)) = Int_X(h(A))$.

It follows that:

3. $h(Cl_Y(A)) = Cl_X(h(A))$.

Now suppose that A is a regular closed subset of Y . Then

$$\begin{aligned} h(A) &= h(Cl_Y(Int_Y(A))) \\ &= Cl_X h(Int_Y(A)) \\ &= Cl_X(Int_X(h(A))) \end{aligned}$$

So $h(A)$ is a regular closed subset of X . We denote by h' the restriction of h to the regular closed subsets of Y . Then h' is an injective function from $RC(Y)$ into $RC(X)$. We have to show that h' preserves (1) joins, (2) complements, and (3) contact *as a map between regular closed algebras*.²¹

1. *Joins.*

Immediate from the fact that h preserves joins in interior algebras, and joins in regular closed algebras are the same as joins in interior algebras—just set-theoretic unions.

2. *Complements.*

$$\begin{aligned} h'(-A) &= h'(Cl_Y(Y \setminus A)) \\ &= h(Cl_Y(Y \setminus A)) \\ &= Cl_X(h(Y \setminus A)) && \text{by 3. above} \\ &= Cl_X(X \setminus h(A)) && \text{by 1. above} \\ &= Cl_X(X \setminus h'(A)) \\ &= -h'(A) \end{aligned}$$

3. *Contact.*

We denote by C_Y and C_X the contact relations in the algebras $RC(Y)$ and $RC(X)$ respectively.

²¹ Note that, e.g., complements in $RC(Y)$ are different from complements in $I(Y)$.

$$\begin{aligned}
AC_Y B &\text{ iff } A \cap B \neq \emptyset \\
&\text{ iff } h(A \cap B) \neq \emptyset \quad \text{ since } h \text{ an injective Boolean homomorphism} \\
&\text{ iff } h(A) \cap h(B) \neq \emptyset \quad \text{ since } h \text{ preserves meets of interior algebras} \\
&\text{ iff } h'(A) \cap h'(B) \neq \emptyset \\
&\text{ iff } h'(A)C_X h'(B) \quad \square
\end{aligned}$$

Remark 4.18. It is well-known that if X and Y are topological spaces and $f : X \rightarrow Y$ is an interior, surjective map, then the map $h : I(Y) \rightarrow I(X)$ defined by putting, for any $A \subseteq Y$:

$$h(A) = f^{-1}(A)$$

is an embedding of interior algebras. The restriction of h to the collection of regular closed subsets of Y is just the map $h_f : RC(Y) \rightarrow RC(X)$ defined in §2.3. Thus we have a new proof here (via Proposition 4.17) that h_f is an embedding of contact algebras. (We proved this fact about h_f directly in §2.3, without considering interior algebras.)

Corollary 4.19. *If X is a dense-in-itself metric space, there is an embedding of the contact algebra $RC(\mathcal{T}_2)$ into the contact algebra $RC(X)$.*

Proof. Immediate from Propositions 4.16 and 4.17. \square

Theorem 4.20. *Let X be any zero-dimensional, dense-in-itself metric space. Then the following are equivalent:*

1. φ is a theorem of \mathbb{L}_{min}^{cont} ;
2. φ is true in $RC(X)$.

Proof. The direction from 1. to 2. follows from Proposition 2.4. For the direction from 2. to 1., let φ be a non-theorem of \mathbb{L}_{min}^{cont} . By Proposition 4.8, φ is refuted in \mathcal{T}_2^n , for some $n \in \mathbb{N}$. Now since X is a zero-dimensional, dense-in-itself metric space, it can be written as a disjoint union $X_1 \uplus \cdots \uplus X_{2^n}$, where each X_k is a dense-in-itself metric space. By Corollary 4.19, for each $k \leq 2^n$ there is an embedding of contact algebras $h_k : RC(T_{2,k}) \rightarrow RC(X_k)$. The reader can verify that the map $h : \prod_{k=1}^{2^n} RC(T_{2,k}) \rightarrow \prod_{k=1}^{2^n} RC(X_k)$ defined by:

$$h(a_1, \dots, a_{2^n}) = (h_1(a_1), \dots, h_{2^n}(a_{2^n}))$$

is an embedding of contact algebras. But by Propositions 4.13 and 4.9, we have:

$$\prod_{k=1}^{2^n} RC(T_{2,k}) \cong RC\left(\biguplus_{k=1}^{2^n} T_{2,k}\right) \cong RC(\mathcal{T}_2^n)$$

Let $g : RC(\mathcal{T}_2^n) \rightarrow \prod_{k=1}^{2^n} RC(T_{2,k})$ be an isomorphism of contact algebras. By Propositions 4.13 and 4.10, we have:

$$\prod_{k=1}^{2^n} RC(X_k) \cong RC\left(\biguplus_{k=1}^{2^n} X_k\right) \cong RC(X)$$

Let $f : \prod_{k=1}^{2^n} RC(X_k) \rightarrow RC(X)$ be an isomorphism of contact algebras. Then the composition $f \circ h \circ g$ is an embedding of $RC(\mathcal{T}_2^n)$ into $RC(X)$. By Corollary 2.6, since φ is refuted in \mathcal{T}_2^n , φ is refuted in X . \square

Corollary 4.21. \mathbb{L}_{min}^{cont} is (weakly) complete for the rationals, \mathbb{Q} , and the irrationals, \mathbb{P} .

Proof. Immediate from Theorem 4.20 and the fact that both \mathbb{Q} and \mathbb{P} are zero-dimensional, dense-in-themselves metric spaces. \square

5. The Lebesgue measure algebra and the reals

In this section we turn to the real line, \mathbb{R} . As with all connected spaces, the axiom (Con) is true in \mathbb{R} . Below we show that $\mathbb{L}_{min}^{cont} + (Con)$ is weakly complete for the real line, and indeed for any separable, connected, dense-in-itself metric space.

The bulk of this section, however, is devoted to studying a related, non-topological model for region-based theories of space: the Lebesgue measure (contact) algebra (defined below). That algebra was first proposed as a model for region-based theories of space in [2], and further studied in that regard in [17]. Lebesgue measure is countably additive over the Lebesgue measure algebra, but as [17] shows, it is not even finitely additive over the algebra $RC(\mathbb{R})$. (For this reason, Arntzenius preferred the Lebesgue measure (contact) algebra as a model for what he calls ‘gunky’ space—or space without point-sized parts.) To our knowledge the Lebesgue measure algebra has not previously been studied in the context of formal logics for region-based theories of space. Below we show that $\mathbb{L}_{min}^{cont} + (Con)$ is complete for the Lebesgue measure (contact) algebra.

Our strategy is to focus on a smaller algebra that sits inside both the Lebesgue measure algebra and $RC(\mathbb{R})$ and which is of independent interest as a model of region-based theories of space. We first show that $\mathbb{L}_{min}^{cont} + (Con)$ is complete for that smaller algebra, and then extend completeness to the Lebesgue measure algebra.

5.1. The real line

Let us begin by recalling the following lemma proved in [9]²²:

Lemma 5.1. For any topology X , (Con) is true in X if and only if X is connected.

Proof. Suppose that X is not connected. Then there exists a clopen set $A \subseteq X$ such that $A \neq \emptyset$ and $A \neq X$. Clearly A is regular closed. Let $M = (X, V)$, where $V(a) = A$. Then $M \models a \neq 0 \wedge a \neq 1$. But $V(a^*) = Cl(X \setminus A) = X \setminus A$. So $V(a) \cap V(a^*) = \emptyset$, and $M \not\models aCa^*$.

Conversely, suppose that X is connected, and let $M = (X, V)$ be a topological model over X . If $M \models a \neq 0 \wedge a \neq 1$, then $V(a) \neq \emptyset$ and $V(a) \neq X$. If $M \not\models aCa^*$, then $V(a) \cap Cl(X \setminus V(a)) = \emptyset$. So $Cl(X \setminus V(a)) = X \setminus V(a)$ and $V(a)$ is clopen. This contradicts the fact that X is connected. Thus $M \models aCa^*$. \square

Corollary 5.2. The following are equivalent.

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Con)$.
2. φ is true in every connected topology.
3. φ is true in every finite, connected topology.

Proof. The direction from 1. to 2. follows from Lemma 5.1 and Proposition 2.4. The direction from 2. to 3. is obvious. For the direction from 3. to 1., suppose φ is a non-theorem of $\mathbb{L}_{min}^{cont} + (Con)$. By Corollary 2.12,

²² See [9], Proposition 3.7.

there is a finite, path-connected frame F such that φ is refuted in F . Since F is path-connected, (Con) is true in $B(F)$. Let $X = CLAN(\mathcal{B}(F))$. Since $\mathcal{B}(F)$ is a complete algebra, by Theorem 2.22 the embedding h of $\mathcal{B}(F)$ into $RC(X)$ is an isomorphism. Therefore, (Con) is true in X , and φ is refuted in X . By Lemma 5.1, X is connected, and clearly X is finite. \square

The next proposition was proved in [18].²³

Proposition 5.3. *Let X be a separable, connected, dense-in-itself metric space. Then every finite, connected topology is the image of X under an interior map.*

As a consequence, we have the following completeness result.

Theorem 5.4. *Let X be a separable, connected, dense-in-itself metric space. Then the following are equivalent.*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Con)$.
2. φ is true in X .

Proof. The direction from 1. to 2. follows from Lemma 5.1 and Proposition 2.4. For the reverse direction, suppose φ is a non-theorem of $\mathbb{L}_{min}^{cont} + (Con)$. By Corollary 5.2, φ is refuted in a finite, connected topology. And by Proposition 5.3, φ is refuted in X . \square

Corollary 5.5. $\mathbb{L}_{min}^{cont} + (Con)$ is (weakly) complete for the real line.

Proof. Immediate from Theorem 5.4 and the fact that the real line is a separable, connected, dense-in-itself metric space. \square

Let X be a topological space. Recall that the *boundary* of a set $S \subseteq X$ (denoted ∂S) is $Cl(S) \setminus Int(S)$. In what follows, we appeal without proof to the simple properties of boundaries listed in the following lemma.

Lemma 5.6. *For any sets $A, B \subseteq X$,*

1. $\partial(A) = \partial(X \setminus A) = Cl(A) \cap Cl(X \setminus A)$
2. $\partial(A \cup B) \subseteq \partial(A) \cup \partial(B)$
3. $\partial(A \cap B) \subseteq \partial(A) \cup \partial(B)$
4. $\partial(Cl(A)) \subseteq \partial(A)$

Consider the set of regular closed subsets of the real line, \mathbb{R} . As we know, these sets form a complete Boolean algebra, $RC(\mathbb{R})$. Many of these sets have boundaries of (Lebesgue) measure zero. (Take, for example, any finite union of closed intervals.) But some regular closed sets have boundaries with non-zero Lebesgue measure, as the following example shows.

Example 5.7. Recall the construction of a ‘fat’ Cantor set on the real unit interval, $[0, 1]$. At stage $n = 1$, remove the open middle interval of length $1/4$ from the unit interval. We are left with two intervals, $[0, 3/8]$ and $[5/8, 1]$. At stage $n = 2$, remove the open middle intervals of length $(1/4)^2$ from each of these remaining intervals. We are left with 4 remaining intervals. In general, at stage $n = k$, remove the open middle interval of length $(1/4)^k$ from each of the remaining intervals of the previous stage. Let U_k denote the union of intervals removed at stage k . Let $K = [0, 1] \setminus \bigcup_{k \in \mathbb{N}} U_k$. We refer to K as the ‘fat Cantor set.’ By adding

²³ See [18], Lemmas 13 and 17.

up the measures of the U_k 's, it is not difficult to see that the Lebesgue measure of $\bigcup_{k \in \mathbb{N}} U_k$ is $1/2$, and therefore also the measure of K is $1/2$. Let

$$\text{Odd} = \bigcup_{k \text{ odd}} U_k$$

Then $Cl(\text{Odd}) = \text{Odd} \cup K$. The closure of an open set is regular closed, so $Cl(\text{Odd})$ is regular closed. But the boundary of $Cl(\text{Odd})$ is the fat Cantor set, K , which has non-zero measure.

Lemma 5.8. *The regular closed subsets of \mathbb{R} with measure-zero boundary form a subalgebra of the Boolean algebra $RC(\mathbb{R})$.*

Proof. This follows from Lemma 5.6—in particular, part 2. guarantees closure under joins in the algebra $RC(\mathbb{R})$, and parts 1. and 4. guarantee closure under complements. \square

We denote the Boolean algebra of regular closed sets with measure-zero boundary by $RCN(\mathbb{R})$, where the ‘ N ’ is inserted to signify ‘null’ boundaries. We pause to note that $RCN(\mathbb{R})$ is of independent interest as a model of region-based theories of space, and is the subject of joint work co-authored by the author and Dana Scott. Proposition 5.15 below is part of this joint work.

Note that we can equip $RCN(\mathbb{R})$ with a contact relation by simply restricting the contact relation in $RC(\mathbb{R})$ to the sets with measure-zero boundary. We now want to show that $\mathbb{L}_{min}^{cont} + (Con)$ is complete also for $RCN(\mathbb{R})$.

Proposition 5.9. *Let X be a finite, connected topology, and let I be an open interval in the real line. Then there is an interior, surjective map $h : I \rightarrow X$ such that for any regular closed set $A \subseteq X$, $h^{-1}(A)$ is a regular closed set with null boundary.*

Proof. The proposition follows from constructions given in [4]. We view X as a finite connected qoset. [4] shows that for some $n, q \in \mathbb{N}$, there is an interior map g from some regular quasi-tree sum $\bigoplus_{i=1}^n Q_i$ of finitely many q -regular quasi-trees onto X ; and an interior map f from any open interval I onto $\bigoplus_{i=1}^n Q_i$. (For the definitions of q -regular quasi-trees and regular quasi-tree sums, see [4].) The map $h = g \circ f$ from I onto X is interior. It follows that for any regular closed set $A \subseteq X$, $h^{-1}(A)$ is a regular closed set. So we need only show that the boundary of $h^{-1}(A)$ has measure zero. Since A is closed, $g^{-1}(A)$ is closed, and therefore is a finite union of clusters, C_1, \dots, C_m , in the quasi-tree $\bigoplus_{i=1}^n Q_i$. We have $h^{-1}(A) = f^{-1}(g^{-1}(A)) = \bigcup_{k=1}^m f^{-1}(C_k)$, so $\partial(h^{-1}(A)) \subseteq \bigcup_{k=1}^m \partial(f^{-1}(C_k))$. Thus it is sufficient to show that for any cluster C in $\bigoplus_{i=1}^n Q_i$, $\partial(f^{-1}(C))$ has measure zero. This follows from construction of the map f in [4]; indeed, for any cluster C , the reader can verify that $\partial(f^{-1}(C))$ is a countable union of Cantor sets.²⁴ \square

Proposition 5.10. *There is a homeomorphism ξ from the unit interval $(0, 1)$ onto \mathbb{R} such that for any set $A \subseteq (0, 1)$, if A has measure zero then $\xi(A)$ has measure zero.*

Proof. Let $\xi : (0, 1) \rightarrow \mathbb{R}$ be the homeomorphism:

$$\xi(x) = \tan(\pi(x - 1/2))$$

We need to show that for any $A \subseteq (0, 1)$, if $\mu(A) = 0$, then $\mu(\xi(A)) = 0$.

²⁴ More precisely, $\partial(f^{-1}(C))$ is a countable union of Cantor sets in (bounded) intervals (a, b) , $[a, b)$, $(a, b]$, $[a, b]$. See Theorem 20, Theorem 19, and Theorem 13.

First note that $\xi'(x) = \sec^2(x)$ is continuous on the entire interval $(0, 1)$. Therefore for any closed interval $J = [a, b] \subseteq (0, 1)$, the function $\xi'(x)$ takes some maximum value M_J on J . It follows that for any interval $I \subseteq J$, $\mu(\xi(I)) \leq M_J \cdot \mu(I)$.

Now suppose that $B \subseteq J$ with $\mu(B) = 0$. We want to show that $\mu(\xi(B)) = 0$. It is sufficient to show that for every $\epsilon > 0$, there is a cover of $\xi(B)$ by open intervals $\{O_n \mid n \in \mathbb{N}\}$ such that $\sum_n \mu(O_n) < \epsilon$. Since $J = [a, b]$ is a closed interval in $(0, 1)$, there exist $a', b' \in (0, 1)$ such that $a' < a < b < b'$. Let $J' = [a', b']$. Since $\mu(B) = 0$, there is a cover of B by open intervals $\{U_n \mid n \in \mathbb{N}\}$ such that $\sum_n \mu(U_n) < \epsilon/M_{J'}$. WLOG we can assume that $U_n \subseteq [a', b']$ for each $n \in \mathbb{N}$. But then $\{\xi(U_n) \mid n \in \mathbb{N}\}$ is a cover of $\xi(B)$ by open intervals, and $\mu(\xi(U_n)) \leq M_{J'} \cdot \mu(U_n)$. So $\sum_n \mu(\xi(U_n)) < M_{J'} \cdot \epsilon/M_{J'} = \epsilon$. Therefore $\mu(\xi(B)) = 0$.

Finally, let $B_n = [\frac{1}{2} - \frac{n}{2n+1}, \frac{1}{2} + \frac{n}{2n+1}]$ for each $n \in \mathbb{N}$. Then $(0, 1) = \bigcup_n B_n$. Suppose that $A \subseteq (0, 1)$ with $\mu(A) = 0$. We can write $A = \bigcup_n (A \cap B_n)$, and therefore $\xi(A) = \bigcup_n \xi(A \cap B_n)$. Since A has measure zero, $\mu(A \cap B_n) = 0$ for each n . But then $\mu(\xi(A \cap B_n)) = 0$ for each n , since $A \cap B_n$ is a subset of the closed interval B_n in $(0, 1)$. By countable subadditivity, $\mu(\xi(A)) \leq \sum_n \mu(\xi(A \cap B_n)) = 0$. \square

Corollary 5.11. *Let X be a finite, connected topology. Then there is an interior, surjective map $\gamma : \mathbb{R} \rightarrow X$ such that for any regular closed set $A \subseteq X$, $\gamma^{-1}(A)$ is a regular closed set with null boundary.*

Proof. Let $A \subseteq X$ be a regular closed set. Consider $\gamma = h \circ \xi^{-1}$, where h and ξ are the maps given in Propositions 5.9 and 5.10, respectively. Since h and ξ^{-1} are interior, surjective maps, so is γ . And since A is regular closed, $\gamma^{-1}(A)$ is regular closed. By Proposition 5.9, $\partial(h^{-1}(A))$ has measure zero. By Proposition 5.10, $\xi(\partial(h^{-1}(A)))$ has measure zero. But since ξ is a homeomorphism, for any set S we have $\xi(\partial(S)) = \partial(\xi(S))$. Therefore, $\xi(\partial(h^{-1}(A))) = \partial(\xi \circ h^{-1}(A)) = \partial(\gamma^{-1}(A))$. So $\partial(\gamma^{-1}(A))$ has measure zero. Thus $\gamma : \mathbb{R} \rightarrow X$ is the map we need. \square

Let $\gamma : \mathbb{R} \rightarrow X$ be the function given in Corollary 5.11. We now define the function $h_\gamma : RC(X) \rightarrow RCN(\mathbb{R})$ by putting:

$$h_\gamma(A) = \gamma^{-1}(A)$$

Proposition 5.12. $h_\gamma : RC(X) \rightarrow RCN(\mathbb{R})$ is an embedding of contact algebras.

Proof. By Proposition 2.18, h_γ is an embedding of the contact algebra $RC(X)$ into $RC(\mathbb{R})$. By Corollary 5.11, $h_\gamma(A) \in RCN(\mathbb{R})$ for each $A \in RC(X)$. Therefore h_γ is an embedding of $RC(X)$ into the subalgebra $RCN(\mathbb{R})$ of $RC(\mathbb{R})$. \square

Proposition 5.13. *The following are equivalent:*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Con)$;
2. φ is true in $RCN(\mathbb{R})$.

Proof. The direction from 1. to 2. follows from soundness of $\mathbb{L}_{min}^{cont} + (Con)$ for $RC(\mathbb{R})$ (Proposition 2.4 and Lemma 5.1) together with the fact that the identity map on $RCN(\mathbb{R})$ is an embedding of $RCN(\mathbb{R})$ into $RC(\mathbb{R})$. For the direction from 2. to 1., suppose that φ is a non-theorem of $\mathbb{L}_{min}^{cont} + (Con)$. Then by Corollary 5.2, φ is refuted in $RC(X)$ for some finite, connected topology X . By Proposition 5.12, there is an embedding of $RC(X)$ into $RCN(\mathbb{R})$. Therefore by Corollary 2.6, φ is refuted in $RCN(\mathbb{R})$. \square

5.2. The Lebesgue measure (contact) algebra

We turn now, finally, to the Lebesgue measure (contact) algebra. Let $Borel(\mathbb{R})$ denote the Boolean σ -algebra of Borel subsets of \mathbb{R} , and let $Null$ denote the σ -ideal of Lebesgue measure zero sets. The *Lebesgue measure algebra* is the quotient Boolean algebra,

$$\mathcal{M} = Borel(\mathbb{R}) \setminus Null$$

Elements of \mathcal{M} are equivalence classes of Borel subsets of the real line. We use uppercase A, B, C , etc., to denote Borel sets, and we denote by $|A|$ the equivalence class containing the set A . Two Borel sets A and B belong to the same equivalence class if and only if the symmetric difference $A \triangle B$ has measure zero. Operations in the quotient algebra are defined in the usual way in terms of underlying sets:

$$\begin{aligned} |A| \wedge |B| &= |A \cap B| \\ |A| \vee |B| &= |A \cup B| \\ -|A| &= |\mathbb{R} \setminus A| \end{aligned}$$

Let μ henceforth denote Lebesgue measure. [2] equips the Lebesgue measure algebra with a contact relation, $C_{\mathcal{M}}$, defined as follows: $|A|C_{\mathcal{M}}|B|$ if and only if there exists $x \in \mathbb{R}$ such that for any open set O with $x \in O$,

$$\mu(A \cap O) > 0 \text{ and } \mu(B \cap O) > 0$$

Lemma 5.14. $C_{\mathcal{M}}$ is a contact relation on \mathcal{M} .

Proof. We need to show that $C_{\mathcal{M}}$ satisfies the five conditions of Definition 2.2.

1. If $|A|C_{\mathcal{M}}|B|$, then there exists $x \in \mathbb{R}$ such that for any open set O with $x \in O$, $\mu(A \cap O) > 0$ and $\mu(B \cap O) > 0$. In particular, $\mu(A) > 0$ and $\mu(B) > 0$. Therefore, $|A| \neq 0$ and $|B| \neq 0$.
2. Suppose that $|A|C_{\mathcal{M}}|B|$ and $|B| \leq |C|$. Then there exists $x \in \mathbb{R}$ such that for any open set O with $x \in O$, $\mu(A \cap O) > 0$ and $\mu(B \cap O) > 0$. Note that $|C \cap O| = |C| \wedge |O| \geq |B| \wedge |O| = |B \cap O|$. It follows that $\mu(C \cap O) > 0$. Thus $|A|C_{\mathcal{M}}|C|$.
3. Suppose that $|A|C_{\mathcal{M}}(|B| \vee |C|)$. Then there is a point $x \in \mathbb{R}$ such that for every open set O with $x \in O$, $\mu(O \cap A) > 0$ and $\mu(O \cap (B \cup C)) > 0$. Suppose (toward contradiction) that not $|A|C_{\mathcal{M}}|B|$ and not $|A|C_{\mathcal{M}}|C|$. Then there exist open sets U and V such that $x \in U$, $x \in V$, $\mu(U \cap B) = 0$, and $\mu(V \cap C) = 0$. But then $x \in U \cap V$, $U \cap V$ is open, and $\mu((U \cap V) \cap (B \cup C)) = 0$. This contradicts the fact that for every open set O with $x \in O$, $\mu(O \cap (B \cup C)) > 0$.
4. Suppose that $|A| \neq 0$. If (toward contradiction) not $|A|C_{\mathcal{M}}|A|$, then for each $x \in \mathbb{R}$, there is an open set O_x such that $x \in O_x$ and $\mu(A \cap O_x) = 0$. Clearly $A = \bigcup_{x \in \mathbb{R}} (A \cap O_x)$. Note that \mathbb{R} has a countable (open) basis, \mathcal{B} . Let $S = \{B \in \mathcal{B} \mid B \subseteq O_x \text{ for some } x \in \mathbb{R}\}$. Then $A = \bigcup_{B \in S} (A \cap B)$. But S is countable. So

$$\mu(A) \leq \sum_{B \in S} \mu(A \cap B) = 0$$

This contradicts the fact that $|A| \neq 0$. We conclude that $|A|C_{\mathcal{M}}|A|$.

5. Suppose that $|A|C_{\mathcal{M}}|B|$. Then by definition of $C_{\mathcal{M}}$, $|B|C_{\mathcal{M}}|A|$. \square

We now define the mapping $h : RCN(\mathbb{R}) \rightarrow (\mathcal{M}, C_{\mathcal{M}})$ by putting:

$$h(A) = |A|$$

for any $A \in RCN(\mathbb{R})$.

Proposition 5.15. $h : RCN(\mathbb{R}) \rightarrow (\mathcal{M}, C_{\mathcal{M}})$ is an embedding of contact algebras.²⁵

Proof. We need to show that h is injective, preserves Boolean operations, and preserves contact. We take these in turn.

1. h is injective.

Suppose $A, B \in RCN(\mathbb{R})$ and $A \neq B$. WLOG, there exists $x \in A \setminus B$. Since B is closed and $x \notin B$, there exists an open set O such that $x \in O \subseteq \mathbb{R} \setminus B$. Now since A is regular closed, x is a point of closure of $Int(A)$, and $O \cap Int(A) \neq \emptyset$. But then $O \cap Int(A)$ is a non-empty open subset of $A \setminus B$. So $\mu(A \setminus B) > 0$, and $h(A) = |A| \neq |B| = h(B)$.

2. h preserves Boolean operations.

For any $A, B \in RCN(\mathbb{R})$,

$$\begin{aligned} h(A \vee B) &= h(A \cup B) \\ &= |A \cup B| \\ &= |A| \vee |B| \\ &= h(A) \vee h(B) \\ h(-A) &= h(Cl(\mathbb{R} \setminus A)) \\ &= |Cl(\mathbb{R} \setminus A)| \\ &= |\mathbb{R} \setminus A| \\ &= -h(A) \end{aligned}$$

where the third equality in the case for complements follows from the fact that $\partial(\mathbb{R} \setminus A) = \partial A$ has measure zero.

3. h preserves contact.

We denote by C the contact relation in $RCN(\mathbb{R})$, and by $C_{\mathcal{M}}$ the contact relation in \mathcal{M} . If $A C B$, then there exists $x \in A \cap B$. Suppose O is open and $x \in O$. Since A and B are regular closed sets, x is a point of closure of $Int(A)$ and of $Int(B)$. Therefore $O \cap Int(A)$ and $O \cap Int(B)$ are non-empty open sets. Hence $\mu(O \cap A) > 0$ and $\mu(O \cap B) > 0$. This shows that $|A|C_{\mathcal{M}}|B|$. So $h(A)C_{\mathcal{M}}h(B)$.

Conversely, if $h(A)C_{\mathcal{M}}h(B)$, then there exists a point $x \in \mathbb{R}$ such that for any open set O with $x \in O$, $\mu(A \cap O) > 0$ and $\mu(B \cap O) > 0$. So $x \in Cl(A) = A$ and $x \in Cl(B) = B$. Therefore $A \cap B \neq \emptyset$, and $A C B$. \square

Lemma 5.16. The axiom (Con) is true in $(\mathcal{M}, C_{\mathcal{M}})$.

Proof. Suppose not. Then there exists a model $M = (\mathcal{M}, C_{\mathcal{M}}, V)$ such that $M \models a \neq \mathbf{0} \wedge a \neq \mathbf{1}$, but $M \not\models aCa^*$. Let $V(a) = |A|$. Then $|A| \neq 0$, $|A| \neq 1$, and not $|A|C_{\mathcal{M}}|\mathbb{R} \setminus A|$. It follows that for every point $x \in \mathbb{R}$, there is an open set O_x with $x \in O_x$ and either $\mu(A \cap O_x) = 0$ or $\mu((\mathbb{R} \setminus A) \cap O_x) = 0$. Let $S_1 = \{x \in \mathbb{R} \mid \mu(A \cap O_x) = 0\}$, and let $S_2 = \{x \in \mathbb{R} \mid \mu((\mathbb{R} \setminus A) \cap O_x) = 0\}$. Note that $S_1 \cup S_2 = \mathbb{R}$.

²⁵ As noted above, this proposition is part of joint work co-authored by the author and Dana Scott.

We claim that S_1 and S_2 are non-empty. Indeed, suppose that $S_2 = \emptyset$. Then for every $x \in \mathbb{R}$, $\mu(A \cap O_x) = 0$. The argument in the proof of [Lemma 5.14](#), part 4., shows that $\mu(A) = 0$, contradicting the fact that $|A| \neq 0$. We conclude that $S_2 \neq \emptyset$. By symmetry, also $S_1 \neq \emptyset$.

Since S_1 and S_2 are non-empty, both $\bigcup\{O_x \mid x \in S_1\}$ and $\bigcup\{O_x \mid x \in S_2\}$ are non-empty open sets, and their union is the entire real line. Since \mathbb{R} is connected, $\bigcup\{O_x \mid x \in S_1\}$ and $\bigcup\{O_x \mid x \in S_2\}$ are not disjoint. So for some $x \in S_1$ and $y \in S_2$, $O_x \cap O_y \neq \emptyset$. But $\mu(A \cap O_x \cap O_y) = 0$, since $x \in S_1$; and $\mu((\mathbb{R} \setminus A) \cap O_x \cap O_y) = 0$, since $y \in S_2$. So $\mu(O_x \cap O_y) = 0$, contradicting the fact that $O_x \cap O_y$ is a non-empty open set. \square

Theorem 5.17. *The following are equivalent:*

1. φ is a theorem of $\mathbb{L}_{min}^{cont} + (Con)$;
2. φ is true in $(\mathcal{M}, C_{\mathcal{M}})$.

Proof. The direction from 1. to 2. follows from [Lemma 5.16](#) and [Proposition 2.4](#). For the direction from 2. to 1., suppose that φ is a non-theorem of $\mathbb{L}_{min}^{cont} + (Con)$. By [Proposition 5.13](#), φ is refuted in $RCN(\mathbb{R})$. By [Proposition 5.15](#), there is an embedding of $RCN(\mathbb{R})$ into $(\mathcal{M}, C_{\mathcal{M}})$. By [Corollary 2.6](#), φ is refuted in $(\mathcal{M}, C_{\mathcal{M}})$. \square

5.3. Some additional observations

We showed above ([Theorem 5.4](#) and [Theorem 5.17](#)) that the contact algebras $RC(\mathbb{R})$ and $(\mathcal{M}, C_{\mathcal{M}})$ are both sound and (weakly) complete for the same logic, $\mathbb{L}_{min}^{cont} + (Con)$, and thus validate the same set of formulas in the language \mathcal{L} . We note in closing that despite this, there are important differences between these models of region-based theories of space. The difference can be brought out by considering the following first-order conditions on contact algebras, which were studied in several places—*e.g.*, [\[9\]](#), [\[3\]](#), and [\[19\]](#).

(Ext) If $a \neq 1$, then there exists $b \neq 0$ such that not aCb .

(Nor) If not aCb , then there exists a' and b' with $a' \vee b' = 1$ and neither aCa' nor bCb' .

Let X be a topological space. We say that X is *semi-regular* if X has a closed basis of regular closed sets. We say that X is *weakly regular* if X is semi-regular and for every non-empty open set O there exists a non-empty open set U such that $Cl(U) \subseteq O$. We say that X is κ -normal if every two disjoint regular closed sets A and B can be separated by disjoint open sets (*i.e.*, there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$).

The following proposition is proved in [\[9\]](#).²⁶

Proposition 5.18. *Let X be a semi-regular topological space. Then,*

1. X is weakly-regular if and only if $RC(X)$ satisfies (Ext);
2. X is κ -normal if and only if $RC(X)$ satisfies (Nor).

Since \mathbb{R} , \mathbb{Q} , and \mathcal{C} are metric spaces, each of them is both weakly regular and κ -normal. Therefore $RC(\mathbb{R})$, $RC(\mathbb{Q})$, and $RC(\mathcal{C})$ satisfy both (Ext) and (Nor). We now show that the Lebesgue measure contact algebra, $(\mathcal{M}, C_{\mathcal{M}})$, satisfies only the second of these conditions.

²⁶ See [\[9\]](#), Proposition 3.7.

Definition 5.19. For any Borel set $A \subseteq \mathbb{R}$, say that x is a *point of density* of A if for every open set O with $x \in O$, $\mu(A \cap O) > 0$. We denote by ‘ $PD(A)$ ’ the set of points of density of A .

Lemma 5.20. For any Borel set $A \subseteq \mathbb{R}$, $PD(A)$ is a closed set.

Proof. Suppose that x is a point of closure for $PD(A)$, and let O be an open set with $x \in O$. Then $PD(A) \cap O \neq \emptyset$. But then $\mu(O \cap A) > 0$. This shows that $x \in PD(A)$. \square

Proposition 5.21. $(\mathcal{M}, C_{\mathcal{M}})$ satisfies (Nor).

Proof. Let $a, b \in \mathcal{M}$ with $a = |A|$ and $b = |B|$, and suppose that it’s not the case that $aC_{\mathcal{M}}b$. Then $PD(A) \cap PD(B) = \emptyset$, and by Lemma 5.20, $PD(A)$ and $PD(B)$ are closed sets. Since \mathbb{R} is a normal topology, there exist disjoint open sets U and V such that $PD(A) \subseteq U$ and $PD(B) \subseteq V$. We claim that (1) not $aC_{\mathcal{M}}|Cl(V)|$ and (2) not $bC_{\mathcal{M}}|\mathbb{R} \setminus Cl(V)|$.

For (1), suppose (toward contradiction) that $aC_{\mathcal{M}}|Cl(V)|$. Then there exists $x \in PD(A)$ such that for every open set O with $x \in O$, $\mu(O \cap Cl(V)) > 0$. But U is open, and $x \in U$, so $\mu(U \cap Cl(V)) > 0$, contradicting the fact that U is disjoint from $Cl(V)$.

For (2), suppose (toward contradiction) that $bC_{\mathcal{M}}|\mathbb{R} \setminus Cl(V)|$. Then there exists $x \in PD(B)$ such that for every open set O with $x \in O$, $\mu(O \cap (\mathbb{R} \setminus Cl(V))) > 0$. But $x \in V$ and V is open. Therefore $\mu(V \cap (\mathbb{R} \setminus Cl(V))) > 0$, contradicting the fact that $V \cap (\mathbb{R} \setminus Cl(V)) = \emptyset$. \square

Proposition 5.22. $(\mathcal{M}, C_{\mathcal{M}})$ does not satisfy (Ext).

Proof. Let q_0, q_1, q_2, \dots be an enumeration of the rational numbers. For each $n \geq 0$, let O_n be the open interval centered at q_n with length $(1/2)^n$. Let $O = \bigcup_n O_n$. Then O is open, hence Borel, so $|O| \in \mathcal{M}$. Clearly $|O| \neq 1$, since $\mu(O) \leq \sum_n \mu(O_n) = \sum_n (1/2)^n = 2$. Note also that for any non-empty open set U , $O \cap U \neq \emptyset$, since U must contain some rational number. Thus for any such U , $O \cap U$ is a non-empty open set, and $\mu(O \cap U) > 0$. It follows that $PD(O) = \mathbb{R}$.

Suppose that $a = |A|$ is a non-zero element in \mathcal{M} . Then $PD(A) \neq \emptyset$. (Else, for every $x \in \mathbb{R}$, there is an open set O_x such that $x \in O_x$ and $\mu(A \cap O_x) = 0$. But then by the argument given in the proof of Lemma 5.14, part 4., $|A| = 0$.) Thus $PD(O) \cap PD(A) = \mathbb{R} \cap PD(A) \neq \emptyset$. It follows that $|O|C_{\mathcal{M}}|A|$. Thus $|O|$ is a counterexample to (Ext) in \mathcal{M} . \square

Thus, although $(\mathcal{M}, C_{\mathcal{M}})$ validates the same set of formulas as $RC(\mathbb{R})$, the first contact algebra does not satisfy (Ext), whereas the second does. The conditions (Ext) and (Nor) are, as [3] and [19] note, not expressible by a formula in the language \mathcal{L} . Therefore this difference between contact algebras does not appear at the level of which formulas in \mathcal{L} are validated.

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