Computational Metaphysics: The Virtues of Formal Proofs Beyond Maths

Christoph Benzmüller[1] — FU Berlin

BMS Fridays, Dezember 16, 2016

[1] Supported by DFG Heisenberg Fellowship BE 2501/9-1/2

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- . . .

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- Die Presse
- Wiener Zeitung
- ORF
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- DNA India
- Delhi Daily News
- India Today
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- . . .

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
- . . .

DIE WELT

FREITAG, 18. OKTOBER 2013

Zippert zappt

Teuer erkaufte Harmonie

Mindestlohn lockt SPD in dritte Ehe mit der Union

Mittwoch soll über eine große Koalition verhandelt werden. Votum des SPD-Konvents steht aus

THEMEN

Sport
Bundestrainer Löw unterschreibt beim DFB bis 2016

Wirtschaft
Der Jumbojet wird bei Boeing zum Ladenhüter

Karriere
Stress im Job
So legen Sie sich einen dicken Panzer zu

Aus aller Welt
Mietprozess in Paris: Dürfen 15 Quadratmeter 330 Euro kosten?

Dax
Im Minus

Göttliche Mathematik

Haben Forscher mit einem Computerprogramm die Existenz Gottes bewiesen?

UNION
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Holy Logic: Computer Scientists 'Prove' God Exists

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MEDIA & CULTURE

Is God Real? Scientists 'Prove' His Existence With Godel's Theory And MacBooks

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HOME / SCIENCE NEWS

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HOME / SCIENCE NEWS

Researchers say they used MacBook to prove Goedel's God theorem

God exists, say Apple fanboy scientists

With the help of just one MacBook, two Germans formalize a theorem that confirms the existence of God.

See more serious and funny news links at
`https://github.com/FormalTheology/GoedelGod/blob/master/Press/LinksToNews.md`

Presentation Structure

- **A** Introduction: Logic Zoo
- **B** **Formal Proofs I — Maths**
- **C** Higher-Order Provers as Universal Logic Reasoners
- **D** **Formal Proofs II — Beyond Maths**
- **E** Conclusion

Part A
Logic Zoo

(A) Introduction — Logic Zoo

Classical Logic, of order

0. Propositional Logic

1. First-order Logic

2. Second-order Logic

. . .

n. Higher-order Logic

Non-Classical Logics

- Intuitionistic/Constructive Logics (incl. Univalent Foundations)
- Modal Logics, Conditional Logics, Temporal Logis, Spatial Logics
- Many-valued Logics
- Paraconsistent Logics
- Free Logics, Inclusive Logics
- Logics for special applications: Ethics, Social Choice, Legal Reasoning, . . .
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Example Encodings: Maths

Set Theories:	ZF, ZFC, ...
Arithmetic:	Presburger, Peano (with induction as schemata)

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Example Encodings: Maths

Arithmetic: Peano (with induction axiom)

Identity: $\forall x.\exists F.Fx = x$

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Example Encodings: Maths

Arithmetic:	Peano (with induction axiom)
Identity:	$\forall x.\exists F.Fx = x$
Surjective Cantor theorem:	$\neg\exists F.\forall G.\exists x.Fx = G$

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Leibniz equality is same as primitive equality:	$=^L==$

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Russel's paradox:	$\{x \mid x \notin x\} \in \{x \mid x \notin x\}$

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	$(\lambda x.\neg(x\,x))\,(\lambda x.\neg(x\,x))$

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Example Encodings: Maths

Arithmetic:	Peano (with Induction Axiom)
Identity:	$\forall x_\alpha . \exists F_{\alpha\to\alpha} . Fx =_{\alpha\to\alpha\to o} x$
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Leibniz equality:	$=^L_{\alpha\to\alpha\to o} := \lambda x_\alpha . \lambda y_\alpha . \forall P_{\alpha\to o} . Px \equiv Py$
Leibniz equality is same as primitive equality:	$=^L_{\alpha\to\alpha\to o} =_{(\alpha\to\alpha\to o)\to(\alpha\to\alpha\to o)\to o} =_{\alpha\to\alpha\to o}$
Russel's paradox:	$\cancel{\{x \mid x \notin x\} \in \{x \mid x \notin x\}}$
	$\cancel{(\lambda x . \neg(x\, x))\,(\lambda x . \neg(x\, x))}$

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Henkin Semantics

Avoids the problem with Gödel's Incompleteness Theorems:

- possibly non-full function/predicate spaces $\forall F_{\alpha\to\alpha}.\ldots \quad \forall P_{\alpha\to o}.\ldots$
- more possible model structures
- fewer valid formulas

Reading: [Church,1940], [Henkin,1950], [Andrews,1971/72], [BenzmüllerEtAl,2004]

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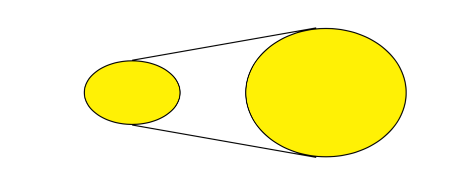
Henkin Semantics see e.g. [BenzmüllerBrownKohlhase, JSL,2004]

Standard semantics

Henkin semantics



model structures valid formulas

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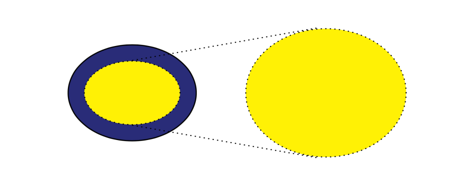

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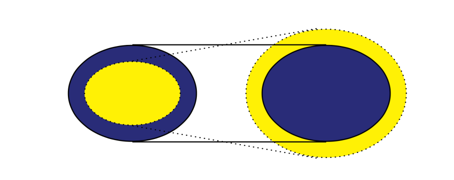

model structures valid formulas

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Example Encodings: Metaphysics/Philosophy

Necessarily, God exists: $\Box \exists x. Gx$

Kurt Gödel's definition of God: $G := \lambda x. \forall \Phi. Positive\ \Phi \rightarrow \Phi x$

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Own Research: Utilise Higher-order Logic (HOL) as Meta-Logic

This turns

- HOL into a (quite) universal logic
- HOL provers into (quite) universal reasoners

Applications in Maths, CS, AI, Philosophy, Comp. Linguistics, ...

(A) Introduction — Logic Zoo

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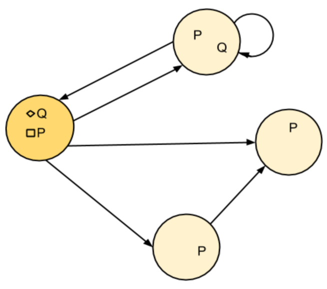
There are many different (Higher-Order) Modal Logics

$\Box\varphi$: φ holds in all (reachable) possible worlds
$\Diamond\varphi$: there is in a (reachable) possible worlds where φ holds



Properties of $\Box$ and $\Diamond$ correlated to structure of transition system between worlds

(A) Introduction — Logic Zoo

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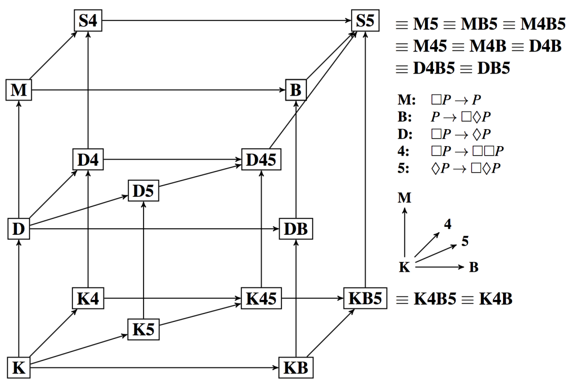
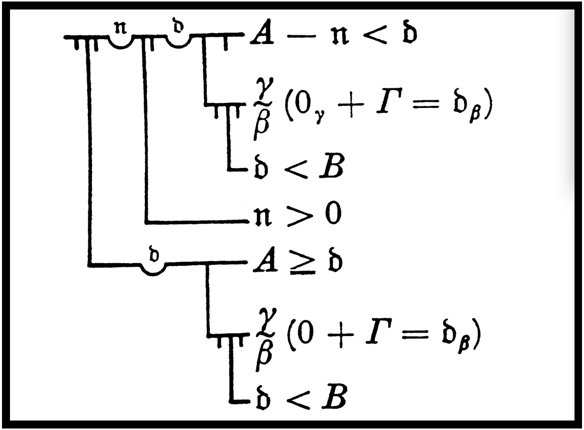
There are many different (Higher-Order) Modal Logics

Modal Logic Cube



Moreover, $\forall/\exists$-Quantifiers may have a Possibilist or Actualist reading



Part B
Formal Proofs I — Maths

(B) Formal Proofs I (Maths): Pythagorean Triples

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Two-hundred-terabyte maths proof is largest ever

A computer cracks the Boolean Pythagorean triples problem — but is it really maths?

Evelyn Lamb

26 May 2016

Contribution by:

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Can the set $N = \{1, 2, \ldots, n\}$ be divided into two parts such that no part contains a Pythagorean triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ with $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$?

$n = 10$

$$3^2 + 4^2 = 5^2$$

(choose color of the other numbers arbitrarily)

(B) Formal Proofs I (Maths): Pythagorean Triples

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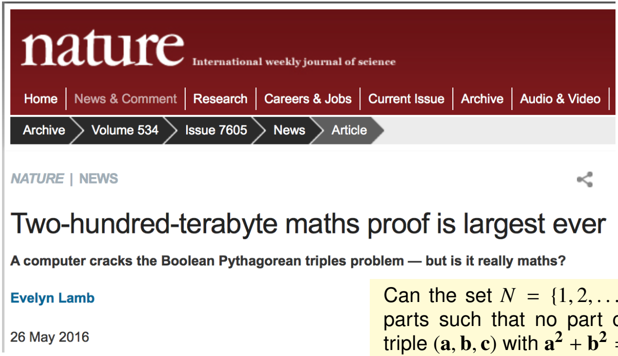
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Unicoloring forbidden!

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$n = 20$

$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$8^2 + 15^2 = 17^2$$

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$n = 30$

$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$8^2 + 15^2 = 17^2$$
$$7^2 + 24^2 = 25^2$$

(choose color of the other numbers arbitrarily)

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Can the set $N = \{1, 2, \dots, n\}$ be divided into two parts such that no part contains a Pythagorean triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ with $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2$?

$n = 40$

$$3^2 + 4^2 = 5^2$$

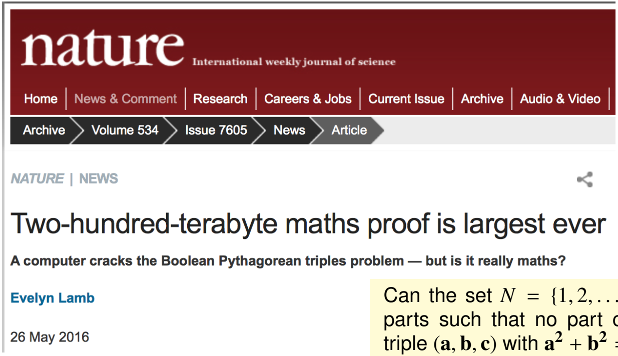
$$5^2 + \boxed{12}^2 = 13^2$$

$$8^2 + 15^2 = 17^2$$

$$7^2 + 24^2 = 25^2$$

$$20^2 + 21^2 = 29^2$$

$$\boxed{12}^2 + 35^2 = 37^2$$



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NATURE | NEWS

Two-hundred-terabyte maths proof is largest ever

A computer cracks the Boolean Pythagorean triples problem — but is it really maths?

Evelyn Lamb

26 May 2016

Contribution by:

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(B) Formal Proofs I (Maths): Pythagorean Triples

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Shown by SAT-Solver:

For $n \geq 7825$ consistent bi-coloring becomes impossible.

(B) Formal Proofs I (Maths): Remember the Logic Zoo

Classical Logic, of order

0. $\boxed{\text{Propositional Logic}}$
1. First-order Logic
2. Second-order Logic

...

n. Higher-order Logic

Non-Classical Logics

- Intuitionistic/Constructive Logics (incl. Univalent Foundations)
- Modal Logics, Conditional Logics, Temporal Logis, Spatial Logics
- Many-valued Logics
- Paraconsistent Logics
- Free Logics, Inclusive Logics
- Logics for special applications: Ethics, Social Choice, Legal Reasoning, ...
- ...

Previous example (Pythagorean Triples): Classical Propositional Logic

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Other examples: Higher-Order Logic

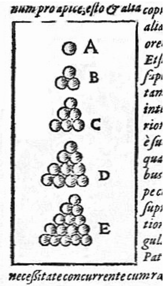
- Four-Colour Theorem: Formal verification by Gonthier in 2005 (Coq)
- Kepler's Conjecture: Formal verification by Hales in 2014 (HOL-light)

(B) Formal Proofs I (Maths): Hales' Flyspeck Project (Kepler Conjecture, 1611)

Kepler (1571-1630)

The most compact way of stacking balls of the same size in space is a pyramid.

$$V_{\_} = \frac{\pi}{\sqrt{18}} \approx 74\%$$



- Proved in 1998 by Hales: 300 page proof, with code and data
- Submitted to the Annals of Mathematics: referees gave up to verify it all
- **Flyspeck project (completed in 2014):**
  - A formal verification of the proof in HOL Light
  - 27,223 proved theorems, 228 definitions, 30 person-years
  - Large data set for heuristics, learning, and reasoning methods
- **Recent work of Cezary Kalyszik & Josef Urban:**
  Combination of **Machine Learning & Automated Theorem Proving**
  Result: more than 50% of the proofs can already be fully automated

(B) Formal Proofs I (Maths): Remember the Logic Zoo

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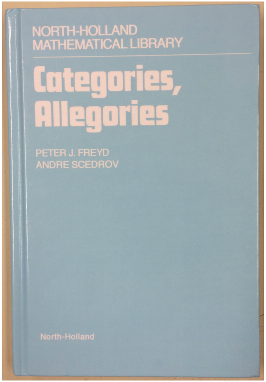
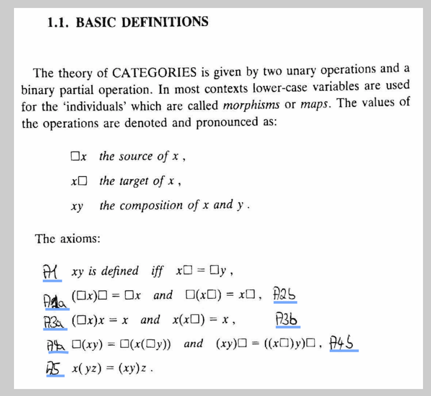
- ...

Other examples: First-order Free Logics (own jww with Dana Scott)

- Free Logics are well suited for for modeling **undefinedness and partiality**
- **Flaw detected in category theory textbook** by Freyd and Scedrov (1990)

(B) Formal Proofs I (Maths): Theory Exploration in Category Theory (Scott)





1.1. BASIC DEFINITIONS

The theory of CATEGORIES is given by two unary operations and a binary partial operation. In most contexts lower-case variables are used for the 'individuals' which are called *morphisms* or *maps*. The values of the operations are denoted and pronounced as:

$\Box x$ *the source of* x ,

$x\Box$ *the target of* x ,

xy *the composition of* x *and* y .

The axioms:

A1 xy *is defined iff* $x\Box = \Box y$,

A2a $(\Box x)\Box = \Box x$ *and* $\Box(x\Box) = x\Box$, A2b

A3a $(\Box x)x = x$ *and* $x(x\Box) = x$, A3b

A4a $\Box(xy) = \Box(x(\Box y))$ *and* $(xy)\Box = ((x\Box)y)\Box$, A4b

A5 $x(yz) = (xy)z$.

- Joint work with: Dana Scott (Berkeley) and students at FU Berlin
- Logics: Free Logic — well suited for modeling undefinedness and partiality
- **Results:**
  - development of 6 related (equivalent) axiom systems for category theory
  - from Monoids . . . via Scott's (1977) axiom system . . . to Freyd/Scedrov (1990)
  - **for Freyd and Scedrov (1990) we revealed some flaws/issues**
- Further Reading: [Benzmüller&Scott, ICMS'2016], [Benzmüller&Scott, arXiv'2016]

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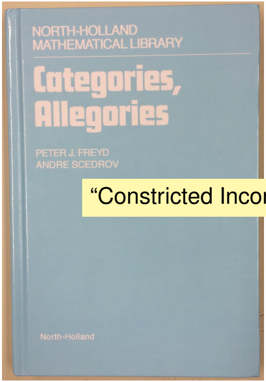

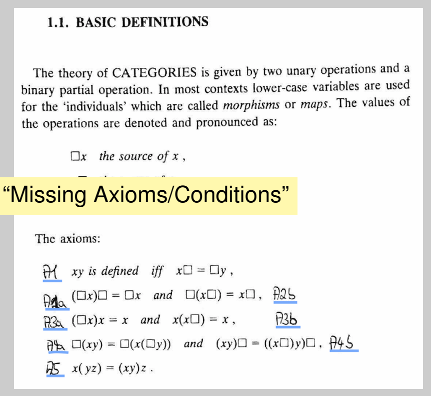

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"Constricted Inconsistency" or "Missing Axioms/Conditions"

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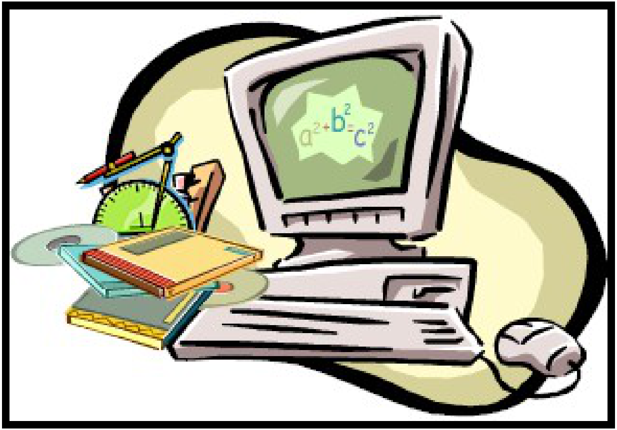
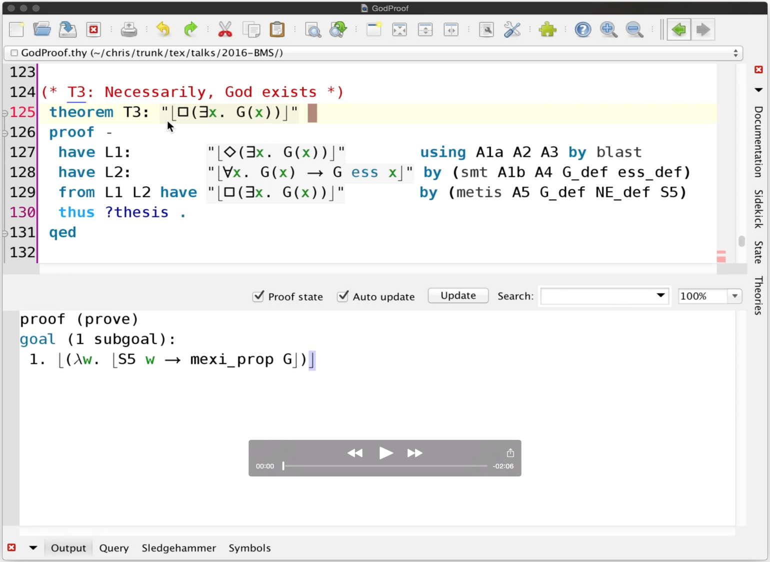


Modern Proof Assistants: Interactive and Automated Proof

Isabelle/HOL (TU Munich & Cambridge University)

(many other systems: Coq, HOL, HOL Light, PVS, Lean, NuPrL, IMPS, ACL2, . . .)

(B) Modern Proof Assistants: Interactive and Automated Proof



Part C

Higher-Order Provers as Universal Logic Reasoners

jww:
Larry Paulson (Cambridge, UK), Bruno Woltzenlogel-Paleo (ANU, Australia),
Alex Steen and Max Wisniewski (both FU Berlin)

(C) Higher-Order Provers as Universal Logic Reasoners

Classical Higher-Order Logic (HOL)
as Universal (Meta-)Logic
via Shallow Semantical Embeddings

Object-Logic L
Syntax

Object-Logic L
Semantics

Examples for L we have already studied:
Modal Logics, Conditional Logics, Intuitionistic Logics, Access Control Logics, Nominal Logics, Multivalued Logics (SIXTEEN), Logics based on Neighborhood Semantics, (Mathematical) Fuzzy Logics, Paraconsistent Logics, Free Logic . . .

Works also for (first-order & higher-order) quantifiers

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Works also for (first-order & higher-order) quantifiers

(C) Higher-Order Provers as Universal Logic Reasoners

HOL (meta-logic) $\varphi ::=$

L (object-logic) $\psi ::=$

Embedding of in

Pass this set of equations to a HOL theorem prover

(C) Higher-Order Provers as Universal Logic Reasoners

HOL $\quad s, t \quad ::= \quad c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\to o}$
(explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\to & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\, hdw \\
\exists & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, hdw \\
\Box & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \forall u_\mu\, (\neg \boldsymbol{r} wu \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \exists u_\mu\, (\boldsymbol{r} wu \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\mu\to o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

(C) Higher-Order Provers as Universal Logic Reasoners

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(C) Higher-Order Provers as Universal Logic Reasoners

Example

HOML formula	$\Diamond\exists x G x$
HOML formula embedded in HOL	valid $(\Diamond\exists x G x)$
expansion	$(\lambda\varphi\forall w_\mu \varphi\, w)(\Diamond\exists x G x)$
$\beta\eta$-normalisation	$\forall w_\mu((\Diamond\exists x G x)\, w)$
expansion	$\forall w_\mu(((\lambda\varphi\lambda w_\mu \exists u_\mu\, (rwu \wedge \varphi u))\exists x G x)\, w)$
$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu(rwu \wedge (\exists x G x) u)$
syntactic sugar	$\forall w_\mu \exists u_\mu(rwu \wedge (\exists(\lambda x G x)) u)$
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$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu(rwu \wedge \exists x G x u)$

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid φ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge (\exists x G x) u)$
syntactic sugar	$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge (\exists (\lambda x G x)) u)$
expansion	$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge ((\lambda h_{\gamma\to(\mu\to o)} \lambda w_\mu \exists d_\gamma\, h d w)(\lambda x G x)) u)$
$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge \exists x G x u)$

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid φ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

(C) Higher-Order Provers as Universal Logic Reasoners

Example

HOML formula	$\Diamond \exists x G x$
HOML formula embedded in HOL	valid $(\Diamond \exists x G x)$
expansion	$(\lambda \varphi \forall w_\mu \varphi\, w)(\Diamond \exists x G x)$
$\beta\eta$-normalisation	$\forall w_\mu ((\Diamond \exists x G x)\, w)$
expansion	$\forall w_\mu (((\lambda \varphi \lambda w_\mu \exists u_\mu\, (\boldsymbol{r} w u \wedge \varphi u)) \exists x G x)\, w)$
$\beta\eta$-normalisation	$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge (\exists x G x) u)$
syntactic sugar	$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge (\exists (\lambda x G x)) u)$
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This can be done with interactive or automated HOL theorem provers.

(C) Higher-Order Provers as Universal Logic Reasoners

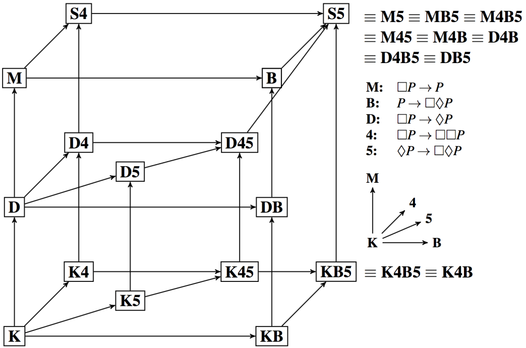
GodProof.thy (~/chris/trunk/tex/talks/2016-BMS/)

```
theory GodProof imports Main
begin
 typedecl i -- "type for possible worlds"
 typedecl μ -- "type for individuals"
 type_synonym σ = "(i⇒bool)"

(* Shallow embedding modal logic connectives in HOL *)
 abbreviation mneg ("¬_"[52]53)     where "¬φ    ≡ λw. ¬φ(w)"
 abbreviation mand (infixr"∧"51)    where "φ∧ψ   ≡ λw. φ(w)∧ψ(w)"
 abbreviation mor (infixr"∨"50)     where "φ∨ψ   ≡ λw. φ(w)∨ψ(w)"
 abbreviation mimp (infixr"→"49)    where "φ→ψ   ≡ λw. φ(w)⟶ψ(w)"
 abbreviation mequ (infixr"↔"48)    where "φ↔ψ   ≡ λw. φ(w)⟷ψ(w)"
 abbreviation mnegpred ("¬_"[52]53) where "¬Φ    ≡ λx.λw. ¬Φ(x)(w)"

(* Shallow embedding of generic box and diamnond operators *)
 abbreviation mboxgen ("□") where "□r φ ≡ λw. ∀v.  r w v ⟶ φ(v)"
 abbreviation mdiagen ("◇") where "◇r φ ≡ λw. ∃v. r w v ∧ φ(v)"

(* Shallow embedding of constant domain quantifiers in HOL *)
 abbreviation mall_const ("∀c") where "∀c Φ ≡ λw.∀x. Φ(x)(w)"
 abbreviation mallB_const (binder"∀c"[8]9) where "∀c x. φ(x) ≡ ∀c φ"
 abbreviation mexi_const ("∃c") where "∃c Φ ≡ λw.∃x. Φ(x)(w)"
 abbreviation mexiB_const (binder"∃c"[8]9) where "∃c x. φ(x) ≡ ∃c φ"
```

(C) Higher-Order Provers as Universal Logic Reasoners

What modal logic is best suited for the application at hand?

$\Diamond\Box P \rightarrow \Box P$



All these logics are well supported in our Semantic Embedding Approach

- postulate axioms (or corresponding semantic conditions)
- modify embedding of quantifiers
  $\forall \;=\; \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, h d w$ modified to $\forall \;=\; \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, (\textbf{Ex}\, d w \rightarrow h d w)$

“If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis.”

(Leibniz, 1677)

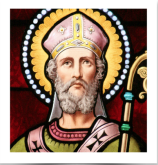
Part D
Formal Proofs II (Beyond Maths)

jww: Bruno Woltzenlogel-Paleo (ANU, Australia)

(D) Formal Proofs II (Beyond Maths)

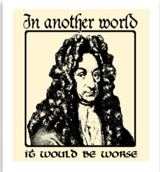
Ontological Proofs of God's Existence

A Long and Continuing Tradition in Philosophy



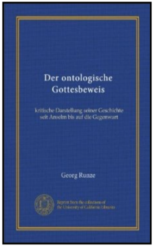
St. Anselm

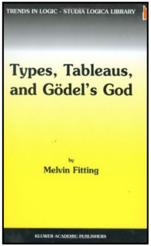
Descartes

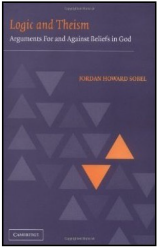


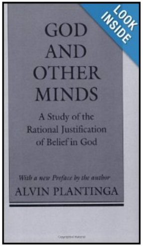
Leibniz

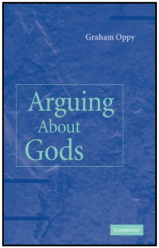
Gödel

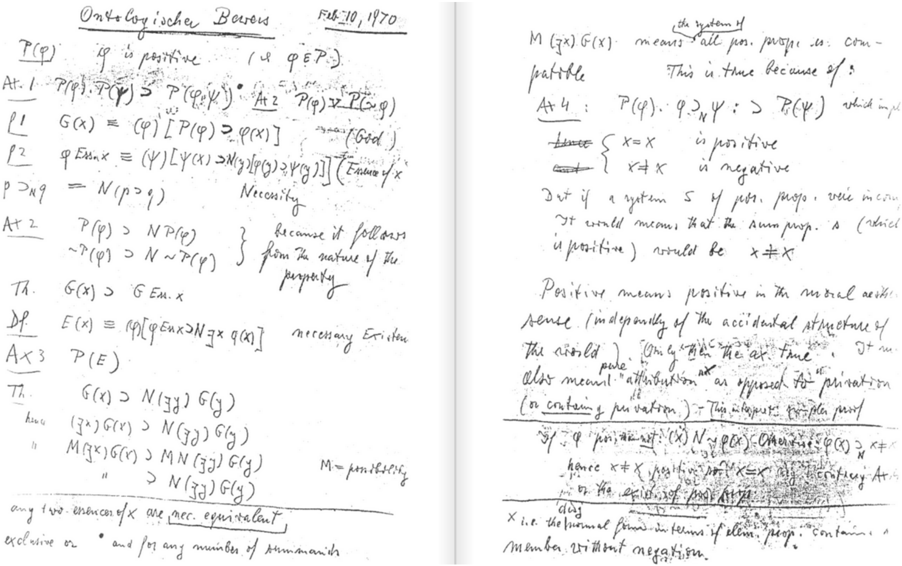












Ontologischer Beweis — Feb 10, 1970

$P(\varphi)$ — φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ — At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess. } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ — Necessity

At 2 $P(\varphi) \supset N P(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ — because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ — necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset M N(\exists y) G(y)$ — M = possibility

" $\supset N(\exists y) G(y)$

any two essences of x are nec. equivalent

exclusive or * and for any number of summands

$M(\exists x) G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi :\supset P(\psi)$ which impl.

$\{ x = x$ is positive; $x \neq x$ is negative

But if a system S of pos. props. were incon. It would mean, that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It m. also meant "attribution" as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ neg. contrary Ax. or the exist. of pos. props.

X i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

(D) Formal Proofs II (Beyond Maths): Scott's Variant of Gödel's Ontological Argument

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists x G(x)$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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second-order quantifiers

(D) Formal Proofs II (Beyond Maths): Vision of Leibniz (1646–1716) — *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
characteristica universalis and **calculus ratiocinator**

(D) Formal Proofs II (Beyond Maths): Scott's and Gödel's Variants — Demo

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Axiom A2

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Def. D1 $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 $P(G)$

Cor. C $\Diamond\exists x G(x)$

Axiom A4 $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2

$$\phi\ ess.\ x \leftrightarrow \phi(x)\ \wedge\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

Thm. T2 $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3

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Def. D3

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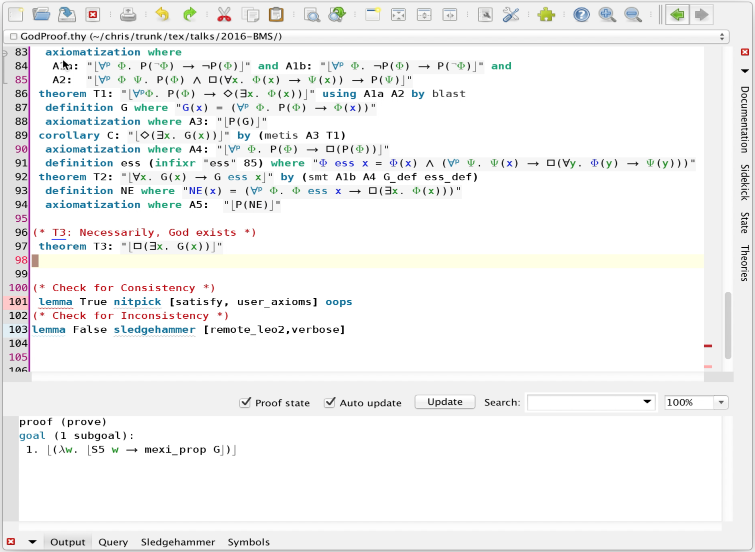
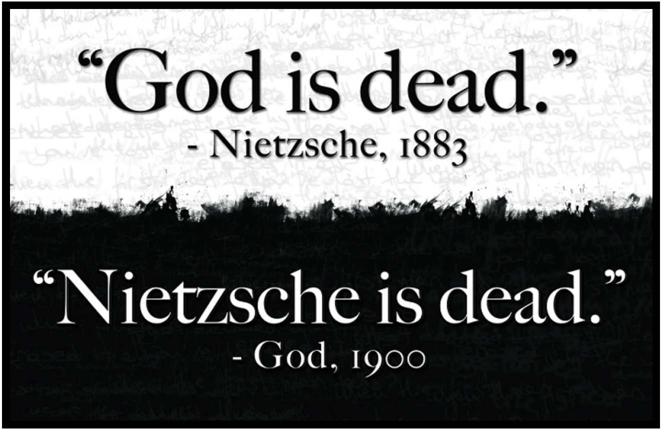
(D) Formal Proofs II (Beyond Maths): Scott's and Gödel's Variants — Demo





Results of Experiments

see e.g. [Benzmüller&WoltzenlogelPaleo, ECAI-2014]

(D) Formal Proofs II (Beyond Maths): Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.(\mathrm{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu.\phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma} X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y \dot\supset \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

(D) Formal Proofs II (Beyond Maths): Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall\psi_{\mu\to\sigma}.(p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.(\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda X_\mu.\phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.(\psi X \dot\supset \dot\Box\dot\forall Y_\mu.(\phi Y\dot\supset\psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma}X \dot\supset (ess_{(\mu\to\sigma)\to\mu\to\sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$NE_{\mu\to\sigma} = \lambda X_\mu.\dot\forall\phi_{\mu\to\sigma}.($e						
A5	$[p_{(\mu\to\sigma)\to\sigma} NE_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\dot\forall X_\mu.(g_{\mu\to\sigma} X \dot\supset ($						
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$ess_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\lambda$						
CO'	∅ (no goal, check for cons						

Automation of Scott's Variant

Summary

- Proof verified and automated
- Logic KB is sufficient (the often critised S5 not needed!)
- Possibilist & actualist quantification (for ind.)
- Exact dependencies determined
- Theorem provers found alternative Proofs
  e.g. self-identity $\lambda x(x = x)$ not needed

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	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall \phi_{\mu\to\sigma}. \dot\forall \psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu. (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu. \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall \phi_{\mu\to\sigma}. p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma} X]$						
A4	$[\dot\forall \phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot\forall \phi_{\mu\to\sigma}. (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma} X]$						
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall \phi_{\mu\to\sigma}. \dot\forall X_\mu. (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu. \dot\forall Y_\mu. (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \dot\forall \psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Consistency: Gödel vs. Scott

- **Scott's assumptions are consistent; shown by Nitpick**
- **Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result)**

(D) Formal Proofs II (Beyond Maths): Results of Experiments

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\text{.}\,\dot\neg(\phi X)) \,\dot\equiv\, \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\,\dot\forall\psi_{\mu\to\sigma}\text{.}\,(p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu\text{.}\,(\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu\text{.}\,\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu\text{.}\,\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu-}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu\text{.}\, g_{\mu\to\sigma}X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p$						
D2	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\text{.}\,\lambda$						
T2	$[\dot\forall X_\mu\text{.}\, g_{\mu\to\sigma}X \,\dot\supset\, (\text{ess}_{(\mu\to\sigma)}$						
D3	$\text{NE}_{\mu\to\sigma} = \lambda X_\mu\text{.}\,\dot\forall\phi_{\mu\to\sigma}\text{.}\,(e$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\text{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu\text{.}\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}\text{.}\,\dot\forall X_\mu\text{.}\,(g_{\mu\to\sigma}X \,\dot\supset\, (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot\supset\, \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu\text{.}\,\dot\forall Y_\mu\text{.}\,(g_{\mu\to\sigma}X \,\dot\supset\, (g_{\mu\to\sigma}Y \,\dot\supset\, X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\text{.}\,\lambda X_\mu\text{.}\,\dot\forall\psi_{\mu\to\sigma}\text{.}\,(\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu\text{.}\,(\phi Y \,\dot\supset\, \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Further Results

- **Monotheism holds**
- **Gott is flawless**
- **...**

(D) Formal Proofs II (Beyond Maths): Results of Experiments

HOL encoding

A1 $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg$

A2 $[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to}$

T1 $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists$

D1 $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu-}$

A3 $[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$

C $[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$

A4 $[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p$

D2 $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$

T2 $[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \,\dot\supset\, (\mathrm{ess}_{(\mu\to\sigma)\text{-}}$

D3 $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$

A5 $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$

T3 $[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$

Modal Collapse (detected first by Sobel)

$$\forall\varphi(\varphi \to \Box\varphi)$$

- **proved by LEO-II and Satallax**
- **already in logic KB**
- **for possibilist and actualist quantification (ind.)**

Modal Collapse can be read as:

- **There are no contingent truths**
- **Everything is determined / there is no free will**

MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \,\dot\supset\, \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \,\dot\supset\, (g_{\mu\to\sigma}Y \,\dot\supset\, X \,\dot=\, Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu.\, (\phi Y \,\dot\supset\, \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

(D) Formal Proofs II (Beyond Maths): Avoiding the Modal Collapse

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's Ontological Proof Revisited *

C. Anthony Anderson and Michael Gettings

University of California, Santa Barbara
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

Magari and others on Gödel's ontological proof

Petr Hájek
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182 07 Prague, Czech Republic
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1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes
(betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,
sondern glauben, daß ein Gott sei.
(Kant, Nachlaß)

1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

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A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

(D) Formal Proofs II (Beyond Maths): Avoiding the Modal Collapse

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Computer-supported Clarification of Controversy
1st World Congress on Logic and Religion, 2015

(D) Formal Proofs II (Beyond Maths): Avoiding the Modal Collapse

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

S/I = superfl. & indep.; R = superfl. & redund.; S/U = superfl. & unknown whether redund. or indep.; N/I = non-superfl. & indep.; P = provable; CS = counter-satisfiable

(D) Formal Proofs II (Beyond Maths): Avoiding the Modal Collapse

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	-	S/I	-	-	P (KB)	CS
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	-	S/U	-	-	P (KB)	CS
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-		-	-		
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

S/I = superfl. & indep.; R = superfl. & redund.; S/U = superfl. & unknown whether redund. or indep.; N/I = non-superfl. & indep.; P = provable; CS = counter-satisfiable

Leibniz (1646–1716)

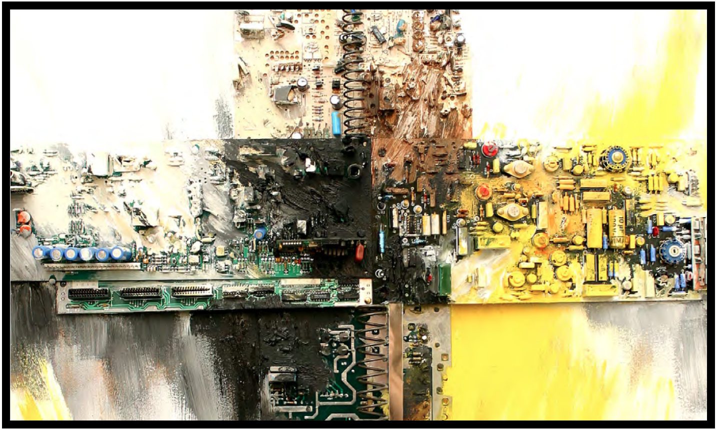
characteristica universalis *and* ***calculus ratiocinator***

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

But: **Intuitive proofs/models are needed to convince philosophers**

(D) Formal Proofs II (Beyond Maths): See our Recent Publications

- **Computer-Assisted Analysis of the Anderson-Hájek Controversy**, In Logica Universalis, 2017.
- **Analysis of an Ontological Proof Proposed by Leibniz**, Chapter in Death and Anti-Death, Volume 14, Ria University Press, 2017.
- **An Object-Logic Explanation for the Inconsistency in Gödel's Ontological Theory**, KI 2016, Springer, LNCS, 2016.
- **The Inconsistency in Gödel's Ontological Argument: A Success Story for AI in Metaphysics**, IJCAI 2016, AAAI Press, 2016.
- **The Modal Collapse as a Collapse of the Modal Square of Opposition**, Chapter in The Square of Opposition: A Cornerstone of Thought (Collection of papers related to the World Congress on the Square of Opposition IV, Vatican, 2014), Springer, Studies in Universal Logic, 2016.
- **On Logic Embeddings and Gödel's God**, WADT 2014, Springer, LNCS, 2015.
- **Experiments in Computational Metaphysics: Gödel's Proof of God's Existence**, In Science & Spiritual Quest, 9th All India Students' Conference, Bhaktivedanta Institute, Kolkata, 2015.
- **Invited Talk: On a (Quite) Universal Theorem Proving Approach and Its Application in Metaphysics**, TABLEAUX 2015, Springer, LNAI, 2015.
- **Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers**, ECAI 2014, IOS Press, Frontiers in Artificial Intelligence and Applications, 2014.
- ...
- Further papers in preparation



Rational Reconstruction of the Inconsistency of Gödel's Axioms

[Benzmüller&WoltzenlogelPaleo, IJCAI-2016]

(D) Formal Proofs II (Beyond Maths): Rational Reconstruction of Inconsistency

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \boxed{\cancel{\phi(x) \wedge}}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists x G(x)$

Difference to Gödel (who omits this conjunct)

(D) Formal Proofs II (Beyond Maths): Rational Reconstruction of Inconsistency

DemoMaterial — bash — 166×52

```
@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$true))),inference(prim_subst,[status(thm)],[66:[bind(SV11,$thf(^[SV23:mu,SV24:$i]: $false))]])).
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~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(prim_subst,[status(thm)],[66
:[bind(SV11,$thf(^[SV20:mu,SV21:$i]: (~ ((SV22@SV20)@SV21))))]])).
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SY30))))@SV4))=$false))),inference(fac_restr,[status(thm)],[57])).
 thf(87,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,
SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_equal_neg,[status(thm)],[85])).
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SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_equal_neg,[status(thm)],[86])).
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~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_neg,[status(thm)],[87])).
 thf(93,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@
SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_neg,[status(thm)],[89])).
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(~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_pos,[status(thm)],[96])).
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s(thm)],[81])).
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)).
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 thf(114,plain,($false),inference(solved_all_splits,[solved_all_splits(join,[])],[113])).
% SZS output end CNFRefutation

%***** End of derivation protocol ****
%***** no. of clauses in derivation: 97 ****
%***** clause counter: 113 ****

% SZS status Unsatisfiable for ConsistencyWithoutFirstConjunctinD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibEQ:true,rAndEQ:true,use_choice:true,use_extuni:true,use_
extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,transl
ation:fof_full)
ontoleo:DemoMaterial cbenzmueller$ 
```

(D) Formal Proofs II (Beyond Maths): Rational Reconstruction of Inconsistency

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

Def. D2* $\phi\ ess.\ x \leftrightarrow \cancel{\phi(x) \wedge}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\, (\emptyset\ ess.\ x)$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Axiom A5 $P(NE)$

- by T1, A5: $\Diamond\exists x[NE(x)]$

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

- by D3 $\Diamond\exists x[\forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y[\phi(y)]]]$
- $\Diamond\exists x[\emptyset\ ess.\ x \rightarrow \Box\exists y[\emptyset(y)]]$
- by Lemma 1 $\Diamond\exists x[\top \rightarrow \Box\exists y[\emptyset(y)]]$
- by def. of $\emptyset$ $\Diamond\exists x[\top \rightarrow \Box\bot]$
- $\Diamond\exists x[\Box\bot]$
- $\Diamond\Box\bot$

Inconsistency $\bot$

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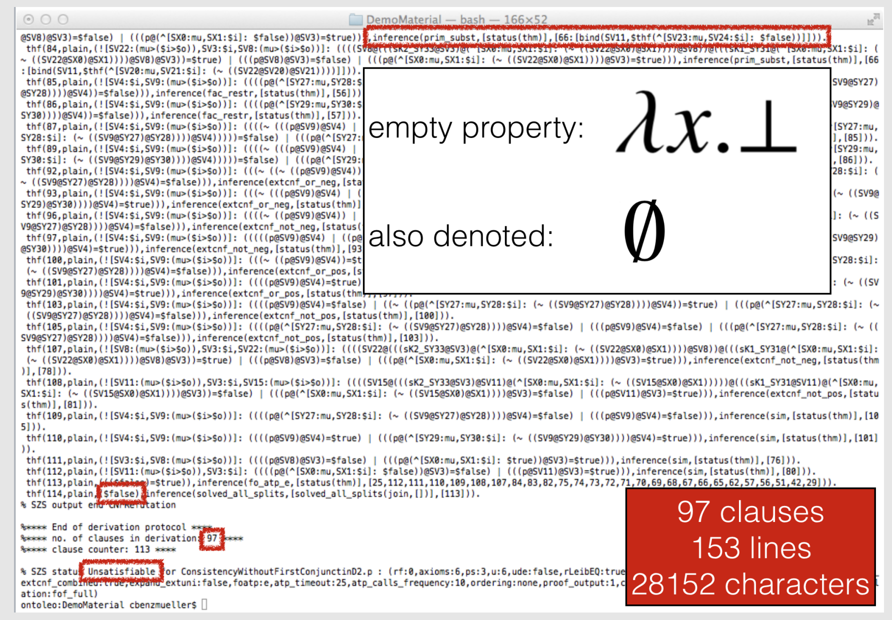
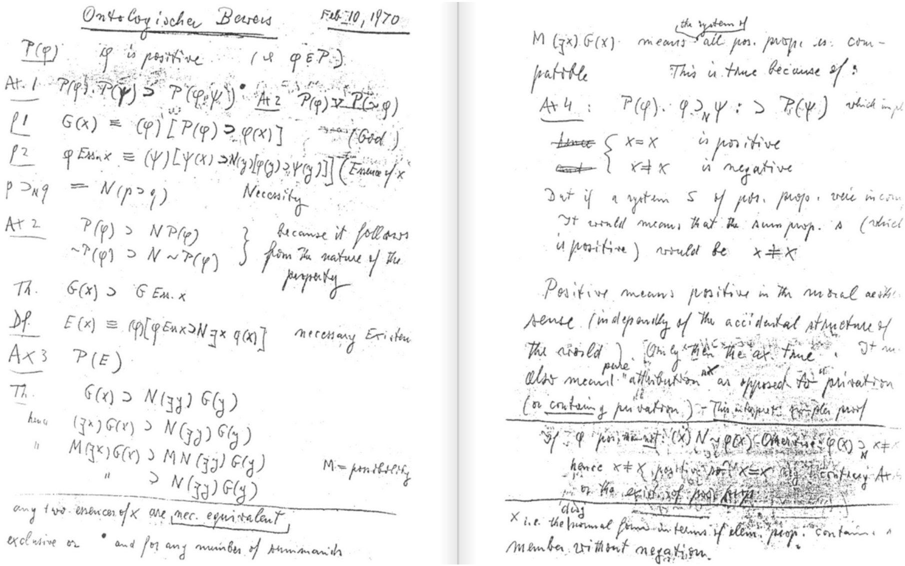
Inconsistency $\bot$

(D) Formal Proofs II (Beyond Maths): Rational Reconstruction of Inconsistency



(D) Formal Proofs II (Beyond Maths): Gödel's Manuscript — Inconsistent Axioms



Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ •   At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi\ \mathrm{Ess}\ x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N\sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G\ \mathrm{Ess.}\ x$

Df. $E(x) \equiv (\varphi)[\varphi\ \mathrm{Ess}\ x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)G(x) \supset N(\exists y)G(y)$

" $M(\exists x)G(x) \supset MN(\exists y)G(y)$ M = possibility

" $\supset N(\exists y)G(y)$

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x)G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\ \varphi \supset_N \psi : \supset P(\psi)$ which impl

~~Since~~ $x = x$ is positive; ~~and~~ $x \neq x$ is negative

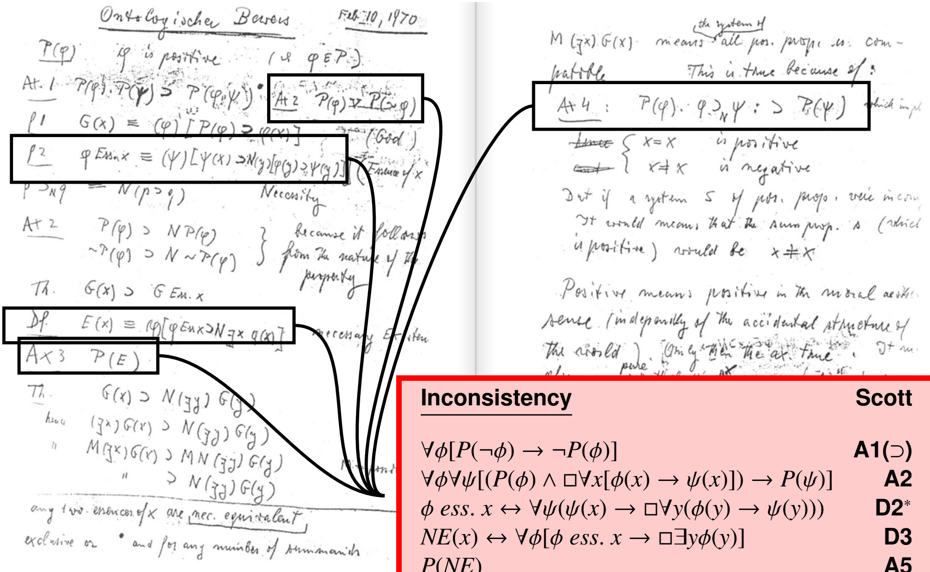
But if a system S of pos. props. were incom[patible]. It would mean, that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth[etic] sense (independently of the accidental structure of the world). Only then the ax. true. It may also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x)\, N\sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ contrary Ax … or the exist. of pos. prop.

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

(D) Formal Proofs II (Beyond Maths): Gödel’s Manuscript — Inconsistent Axioms



Inconsistency **Scott**

$\forall \phi[P(\neg\phi) \rightarrow \neg P(\phi)]$ **A1(⊃)**

$\forall \phi \forall \psi[(P(\phi) \wedge \Box \forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ **A2**

$\phi\ ess.\ x \leftrightarrow \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$ **D2***

$NE(x) \leftrightarrow \forall \phi[\phi\ ess.\ x \rightarrow \Box \exists y \phi(y)]$ **D3**

$P(NE)$ **A5**

Further Ongoing Work

jww: Ed Zalta (Stanford, US), Dana Scott (Berkeley, US),
Alex Steen and Max Wisniewski (FU Berlin), many others

(D) Formal Proofs II (Beyond Maths): Principia Metaphysica (Zalta)

–since ’83→

NOTE: This is an excerpt from an incomplete draft of the monograph *Principia Logica-Metaphysica*. The monograph currently has four parts:

Part I: Prophilosophy
Part II: Philosophy
Part III: Metaphilosophy
Part IV: Technical Appendices, Bibliography, Index

This excerpt was generated on October 18, 2016 and contains:

Principia Logico-Metaphysica

- Objective: Formalisation/Automation of Principia Metaphysica
- Joint work with: Daniel Kirchner (student of mine), Ed Zalta (Stanford)
- Main papers: soon
- Logics: Hyper-Intensional Higher-Order Modal Logic
- Results:
  - first significant part of PM has been formalised (by David)
  - excellent degree of automation
  - minor issues detected

(D) Formal Proofs II (Beyond Maths): Lecture Course on Computational Metaphysics

Metaphysics: *Foundational Branch in Philosophy*, that ...

... studies the fundamental nature of **being** and **the world** that encompasses it

... looks **beyond experience** in the real world

Flammarions Holzstich – in L'atmosphère, Paris 1888

Adresses**ultimate questions**, such as:

- What is there?
- What is it like?
- **Is there a God?**
- What can I know?

Method: **Rational Argumentation**

Lecture Course won the 2015 Central Teaching Award of FU Berlin

(jww: Alex Steen, Max Wisniewski, and others)

- MSc students in Maths, CS, Philosophy and Physics from FU, TU and HU
- Invited Lectures by Philosophers (Zalta & Lenzen) and Computer Scientists
- First course of this kind worldwide!

Conclusion

- Formal Proofs Increasingly Relevant in Maths and Beyond
- Significant Improvements in Interactive and Automated Theorem Proving
- Universal Logic Reasoning Approach via Shallow Semantic Embeddings
- **Deep Analysis of Rational Arguments on the Computer**
  - exemplary focus on the **Ontological Argument**
  - significant contribution towards a **Computational Metaphysics**
  - even **novel results** contributed by **theorem provers**
  - related work: only for Anselm's simpler argument
    - first-order ATP PROVER9 [OppenheimerZalta, 2011]
    - interactive proof assistant PVS [Rushby, 2013]
  - **approach works well in practice**: matches granularity of human arguments
  - **approach can be adapted** for other logics and logic combinations

Outlook: Many relevant other applications (but I need resources!)

- Maths, Computer Science, Artificial Intelligence
- Philosophy, Natural Language Processing
- Law, Ethical and Moral Reasoning
- Rational Argumentation in general

What about the devil?

Roman Kuznets

May 25 at 5:05pm ·

Question after the talk on Gödel's ontological proof of God: "Can such a reasoning be used to prove the non-existent of Satan?" Speaker: "I believe that is still an open problem." Pure gold.

Bruno, are you up for the challenge?

Common Beliefs regarding the Question

Hans de Nivelle I always assumed that the Anselm ontological argument applies to Satan without change. Satanity is the worst possible property. From that follows that the notion must be realized in the world, because not-realized it would be better.

Like · Reply · 2 · May 25 at 5:16pm · Edited

Roman Kuznets Roy Dyckhoff, would this be a satisfactory answer?

Like · Reply · 1 · May 25 at 6:38pm · Edited

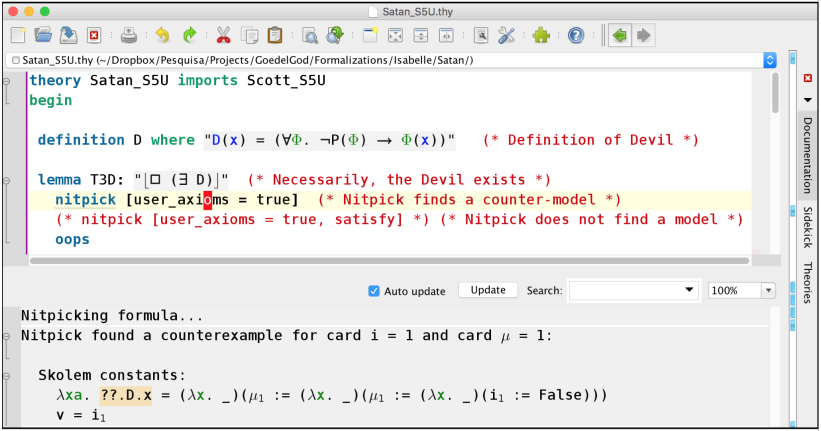
Roy Dyckhoff Worst possible for us, perhaps, but not for Satan himself...

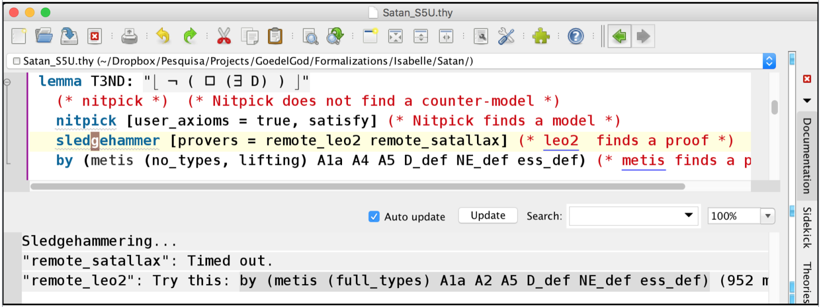
Like · Reply · 1 · May 25 at 8:04pm

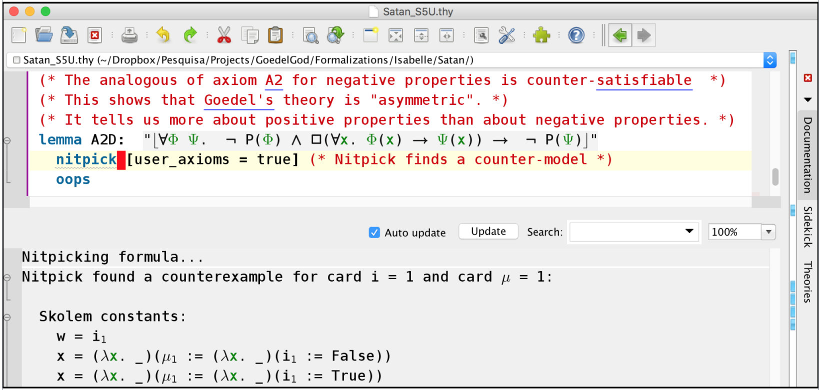
Roy Dyckhoff Non-existence is a thoroughly negative property; so, defining S as the maximally negative being, one might argue that S has the property of non-existence... (necessarily).

Like · Reply · May 25 at 9:57pm

Bruno Woltzenlogel Paleo I have just spent a few moments playing with this in Isabelle. Here is what I can tell you:


Satan_S5U.thy
Satan_S5U.thy (~/Dropbox/Pesquisa/Projects/GoedelGod/Formalizations/Isabelle/Satan/)
theory Satan_S5U imports Scott_S5U
begin
definition D where "D(x) = (∀Φ. ¬P(Φ) → Φ(x))"   (* Definition of Devil *)
lemma T3D: "⌊□ (∃ D)⌋" (* Necessarily, the Devil exists *)
nitpick [user_axioms = true] (* Nitpick finds a counter-model *)
(* nitpick [user_axioms = true, satisfy] *) (* Nitpick does not find a model *)
oops
Auto update
Update
Search:
100%
Nitpicking formula...
Nitpick found a counterexample for card i = 1 and card μ = 1:
Skolem constants:
λxa. ??.D.x = (λx. _)(μ1 := (λx. _)(μ1 := (λx. _)(i1 := False)))
v = i1
Documentation
Sidekick
Theories


Satan_S5U.thy
Satan_S5U.thy (~/Dropbox/Pesquisa/Projects/GoedelGod/Formalizations/Isabelle/Satan/)
lemma T3ND: "⌊ ¬ (□ (∃ D)) ⌋"
(* nitpick *) (* Nitpick does not find a counter-model *)
nitpick [user_axioms = true, satisfy] (* Nitpick finds a model *)
sledgehammer [provers = remote_leo2 remote_satallax] (* leo2 finds a proof *)
by (metis (no_types, lifting) A1a A4 A5 D_def NE_def ess_def) (* metis finds a p
Auto update
Update
Search:
100%
Sledgehammering...
"remote_satallax": Timed out.
"remote_leo2": Try this: by (metis (full_types) A1a A2 A5 D_def NE_def ess_def) (952 m
Documentation
Sidekick
Theories


Satan_S5U.thy
Satan_S5U.thy (~/Dropbox/Pesquisa/Projects/GoedelGod/Formalizations/Isabelle/Satan/)
(* The analogous of axiom A2 for negative properties is counter-satisfiable *)
(* This shows that Goedel's theory is "asymmetric". *)
(* It tells us more about positive properties than about negative properties. *)
lemma A2D: "⌊∀Φ Ψ. ¬ P(Φ) ∧ □(∀x. Φ(x) → Ψ(x)) → ¬ P(Ψ)⌋"
nitpick [user_axioms = true] (* Nitpick finds a counter-model *)
oops
Auto update
Update
Search:
100%
Nitpicking formula...
Nitpick found a counterexample for card i = 1 and card μ = 1:
Skolem constants:
w = i1
x = (λx. _)(μ1 := (λx. _)(i1 := False))
x = (λx. _)(μ1 := (λx. _)(i1 := True))
Documentation
Sidekick
Theories

Reading “P” as “Negative” and “G" as “Devil-like”

Scott_S5U.thy

```
theory Scott_S5U imports QML_S5U
begin
 consts P :: "(μ⇒σ)⇒σ"
 axiomatization where
  A1a: "⌊∀Φ. P(¬Φ) → ¬P(Φ)⌋" and
  A1b: "⌊∀Φ. ¬P(Φ) → P(¬Φ)⌋" and
  A2:  "⌊∀Φ Ψ. P(Φ) ∧ □(∀x. Φ(x) → Ψ(x)) → P(Ψ)⌋"
 definition G where
  "G(x) = (∀Φ. P(Φ) → Φ(x))"
 axiomatization where
  A3: "⌊P(G)⌋" and
  A4: "⌊∀Φ. P(Φ) → □(P(Φ))⌋"
 definition ess (infixr "ess" 85) where
  "Φ ess x = Φ(x) ∧ (∀Ψ. Ψ(x) → □(∀y. Φ(y) → Ψ(y)))"
 definition NE where
  "NE(x) = (∀Φ. Φ ess x → □(∃ Φ))"
 axiomatization where
  A5:  "⌊P(NE)⌋"

 theorem T3: "⌊□ (∃ G)⌋" -- {* LEO-II proves T3 in 2,5sec *}
  sledgehammer [provers = remote_leo2]
  by (metis (lifting, full_types)
     A1a A1b A2 A3 A4 A5 G_def NE_def ess_def)

 lemma True nitpick [satisfy,user_axioms,expect=genuine] oops
 -- {* Consistency is confirmed by Nitpick *}

 theorem T2: "⌊∀x. G(x) → G ess x⌋"
  sledgehammer [provers = remote_leo2]
  by (metis A1b A4 G_def ess_def)

 lemma MC: "⌊∀Φ. Φ → (□ Φ)⌋"  -- {* Modal Collapse *}
  sledgehammer [provers = remote_satallax, timeout=600]
  by (meson T2 T3 ess_def)
end
```

Do the 5 axioms still make sense?

Do they capture our intuition of negativity?

Are we willing to accept “P(NE)” ?

Are we willing to accept “P(λx.x=x)” ?

Are we willing to accept “ P(λx.T)” ?

Proof in S5

Scott's version

$$\textbf{D2: } \varphi \; ess \; x \equiv \cancel{\varphi(x)} \wedge \forall\psi.(\psi(x) \rightarrow \Box\forall x.(\varphi(x) \rightarrow \psi(x)))$$

Gödel's version did not have this conjunct !

$$\textbf{D3: } E(x) \equiv \forall\varphi.[\varphi \; ess \; x \rightarrow \Box\exists y.\varphi(y)]$$

$$\cfrac{\cfrac{}{P(G)}\,\textbf{A3} \qquad \cfrac{\cfrac{}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}\,\textbf{A2} \qquad \cfrac{}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}\,\textbf{A1a}}{\textbf{T1: } \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1: } \Diamond\exists z.G(z)}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]}\,\textbf{A1b} \qquad \cfrac{}{\forall\varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}\,\textbf{A4}}{\textbf{T2: } \forall y.[G(y) \rightarrow G \; ess \; y]} \qquad \cfrac{}{P(E)}\,\textbf{A5}}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \cfrac{}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}\,\textbf{S5}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\cfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$