

# First-Order Logic: Theory and Practice

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# First-Order Logic — From ND to Sequents and Back

- ▶ F. Pfenning: Automated Theorem Proving, Course at Carnegie Mellon University. Draft. 1999.
- ▶ A.S. Troelstra and H. Schwichtenberg: Basic Proof Theory. Cambridge. 2nd Edition 2000.
- ▶ John Byrnes: Proof Search and Normal Forms in Natural Deduction. PhD Thesis. Carnegie Mellon University. 1999.
- ▶ ... many more books on Proof Theory ...

**Frege, Russel, Hilbert** Predicate calculus and type theory as formal basis for mathematics

**Gentzen** ND as intuitive formulation of predicate calculus; introduction and elimination rules for each logical connective

*The formalization of logical deduction, especially as it has been developed by Frege, Russel, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. ... In contrast I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction (NJ for intuitionist, NK for classical predicate logic).*

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Gentzen had a pure technical motivation for sequent calculus

- ▶ Same theorems as natural deduction
- ▶ Prove of the Hauptsatz (all sequent proofs can be found with a simple strategy)
- ▶ Corollary: Consistency of formal system(s)

*The Hauptsatz says that every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not roundabout. . . .*

*In order to be able to prove the Hauptsatz in a convenient form, I had to provide a logical calculus especially for the purpose. For this the natural calculus proved unsuitable.*

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*In order to be able to prove the Hauptsatz in a convenient form, I had to provide a logical calculus especially for the purpose. For this the natural calculus proved unsuitable.* [Gentzen: Investigations into logical deduction]

**Sequent calculus** exposes many details of fine structure of proofs in a very clear manner. Therefore it is well suited to serve as a **basic representation formalism** for many automation oriented search procedures:

- ▶ Backward: tableaux, connection methods, matrix methods, some forms of resolution
- ▶ Forward: classical resolution, inverse method

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Choose the one that is most suited for you.

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Choose the one that is most suited for you.

Natural deduction rules operate on proof trees.

Example:

- ▶ Conjunction:

$$\frac{D_1 \quad D_2}{\mathbf{A} \wedge \mathbf{B}} \wedge I \quad \frac{D_1}{\mathbf{A} \wedge \mathbf{B}} \quad \frac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{A}} \wedge E_I \quad \frac{D_1}{\mathbf{A} \wedge \mathbf{B}} \quad \frac{\mathbf{A} \wedge \mathbf{B}}{\mathbf{B}} \wedge E_r$$

The presentation on the next slides treats the proof tree aspects implicit.  
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# Natural Deduction Rules Ia

► Conjunction:

$$\frac{\begin{array}{c} A \\ \hline A \wedge B \end{array}}{A \wedge B} \wedge I \quad \frac{A \wedge B}{A} \wedge E_l \quad \frac{A \wedge B}{B} \wedge E_r$$

► Disj.:

$$\frac{A}{A \vee B} \vee I_l \quad \frac{B}{A \vee B} \vee I_r \quad \frac{\begin{array}{c} A \vee B \\ \hline \begin{array}{c} C \\ \vdots \\ C \end{array} \end{array}}{\begin{array}{c} C \\ \vdots \\ C \end{array}} \vee E_r^{1,2}$$

► Implication:

$$\frac{\begin{array}{c} [A]_1 \\ \vdots \\ B \end{array}}{A \Rightarrow B} \Rightarrow I^1 \quad \frac{\begin{array}{c} A \Rightarrow B \\ A \end{array}}{B} \Rightarrow E$$

► Truth and Falsehood:

$$\top \top I \quad \frac{\perp}{C} \perp E$$

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$$[A]_1 \quad [B]_2$$

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► Negation:

$$\frac{\vdash \bot}{\neg A} \neg I \quad \frac{\neg A \quad A}{\bot} \neg E$$

► Universal Quantif.:

$$\frac{\{x/a^*\}A}{\forall x. A} \forall I \quad \frac{\forall x. A}{\{x/t\}A} \forall E$$

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For classical logic choose one of the following

- ▶ Excluded Middle

$$\frac{}{\mathbf{A} \vee \neg \mathbf{A}} XM$$

- ▶ Double Negation

$$\frac{\neg \neg \mathbf{A}}{\mathbf{A}} \neg \neg C$$

- ▶ Proof by Contradiction

$$\frac{\begin{array}{c} [\neg \mathbf{A}] \\ \vdots \\ \bot \end{array}}{\mathbf{A}} \perp_c$$

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$$\begin{array}{c} [\neg \mathbf{A}] \\ \vdots \\ \perp \\ \hline \mathbf{A} \quad \perp_c \end{array}$$

- ▶ Exchange hypotheses order is irrelevant
- ▶ Weakening hypothesis need not be used
- ▶ Contraction hypotheses can be used more than once

# Natural Deduction Proofs

$$\frac{\frac{[\mathbf{A}]_1 \quad [\mathbf{A}]_2}{\mathbf{A} \wedge \mathbf{A}} \wedge I}{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})} \Rightarrow I^2$$

$$\frac{\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A})}{\mathbf{A} \Rightarrow (\mathbf{A} \Rightarrow (\mathbf{A} \wedge \mathbf{A}))} \Rightarrow I^1$$

$$\frac{[\mathbf{A} \wedge \mathbf{B}]_1 \quad \frac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{A}} \wedge E_r}{\mathbf{B}} \wedge E_r$$

$$\frac{\frac{\mathbf{A}}{\mathbf{C} \vee \mathbf{A}} \vee I_r}{\mathbf{C} \vee \mathbf{A}} \wedge I$$

$$\frac{\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})}{(\mathbf{A} \wedge \mathbf{B}) \Rightarrow (\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A}))} \Rightarrow I^1$$

# Natural Deduction Proofs

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$$\frac{\frac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{B}} \wedge E_r \quad \frac{[\mathbf{A} \wedge \mathbf{B}]_1}{\mathbf{A}} \wedge E_l}{\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})} \wedge I$$

$$\frac{\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})}{(\mathbf{A} \wedge \mathbf{B}) \Rightarrow (\mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A}))} \Rightarrow I^1$$

**Idea:** Localizing hypotheses; explicit representation of the available assumptions for each formula occurrence in a ND proof:

$$\Gamma \vdash A$$

$\Gamma$  is a multiset of the (uncanceled) assumptions on which formula  $A$  depends.  $\Gamma$  is called context.

Example proof in context notation:

$$\frac{\frac{\frac{A_1 \vdash A \quad A_2 \vdash A}{A_1, A_2 \vdash A \wedge A} \wedge I}{A_1 \vdash A \Rightarrow (A \wedge A)} \Rightarrow I_2}{\vdash A \Rightarrow (A \Rightarrow (A \wedge A))} \Rightarrow I_1$$

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**Another Idea:** Consider sets of assumptions instead of multisets.

$$\Gamma \vdash A$$

$\Gamma$  is now a set of (uncanceled) assumptions on which formula  $A$  depends.

**Example proof:**

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## Structural properties to ensure

- ▶ Exchange (hypotheses order is irrelevant)

$$\frac{\Gamma, B, A \vdash C}{\Gamma, A, B \vdash C}$$

- ▶ Weakening (hypothesis need not be used)

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- ▶ Hypotheses:

$$\frac{}{\Gamma, A, \Delta \vdash A}$$

- ▶ Conjunction:

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$$\frac{}{\Gamma \vdash \top} \top I \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash C} \perp E$$

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- ▶ Universal Quantif.:

$$\frac{\Gamma \vdash \{x/a^*\}A}{\Gamma \vdash \forall x. A} \forall I \quad \frac{\Gamma \vdash \forall x. A}{\Gamma \vdash \{x/t\}A} \forall E$$

- ▶ Existential Quantif.:

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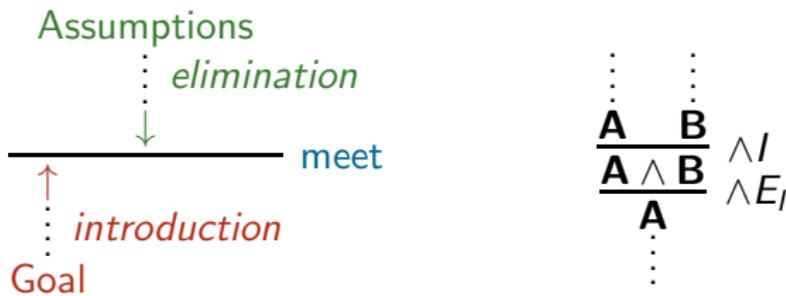
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For classical logic add:

- ▶ Proof by Contradiction:

$$\frac{\Gamma, \neg A \vdash \perp}{\Gamma \vdash A} \perp_c$$

**Idea (Prawitz, Sieg & Scheines, Byrnes & Sieg):** Detour free proofs: strictly use introduction rules bottom up (from proposed theorem to hypothesis) and elimination rules top down (from assumptions to proposed theorem). When they meet in the middle we have found a **proof in normal form**.



New annotations:

- ▶  $\mathbf{A} \uparrow$  :  $\mathbf{A}$  is obtained by an introduction derivation
- ▶  $\mathbf{A} \downarrow$  :  $\mathbf{A}$  is extracted from a hypothesis by an elimination derivation

Example:

$$\frac{\Gamma, A \vdash B \uparrow}{\Gamma \vdash_{ic} A \Rightarrow B \uparrow} \Rightarrow I \quad \frac{\Gamma \vdash_{ic} A \Rightarrow B \downarrow \quad \Gamma \vdash_{ic} A \downarrow}{\Gamma \vdash_{ic} B \downarrow} \Rightarrow E$$

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- ▶  $\mathbf{A} \downarrow$  :  $\mathbf{A}$  is extracted from a hypothesis by an elimination derivation

Example:

$$\frac{\Gamma, \mathbf{A} \vdash_{\text{ic}} \mathbf{B} \uparrow}{\Gamma \vdash_{\text{ic}} \mathbf{A} \Rightarrow \mathbf{B} \uparrow} \Rightarrow I \quad \frac{\Gamma \vdash_{\text{ic}} \mathbf{A} \Rightarrow \mathbf{B} \downarrow \quad \Gamma \vdash_{\text{ic}} \mathbf{A} \downarrow}{\Gamma \vdash_{\text{ic}} \mathbf{B} \downarrow} \Rightarrow E$$

- ▶ Hypotheses:

$$\frac{}{\Gamma, A, \Delta \vdash_{\text{ic}} A \downarrow}$$

- ▶ Conjunction:

$$\frac{\Gamma \vdash_{\text{ic}} A \uparrow \quad \Gamma \vdash_{\text{ic}} B \uparrow}{\Gamma \vdash_{\text{ic}} A \wedge B \uparrow} \wedge I \quad \frac{\Gamma \vdash_{\text{ic}} A \wedge B \downarrow}{\Gamma \vdash_{\text{ic}} A \downarrow} \wedge E_l \quad \frac{\Gamma \vdash_{\text{ic}} A \wedge B \downarrow}{\Gamma \vdash_{\text{ic}} B \downarrow} \wedge E_r$$

- ▶ Disjunction:

$$\frac{\Gamma \vdash_{\text{ic}} A \uparrow}{\Gamma \vdash_{\text{ic}} A \vee B \uparrow} \vee I_l \quad \frac{\Gamma \vdash_{\text{ic}} B \uparrow}{\Gamma \vdash_{\text{ic}} A \vee B \uparrow} \vee I_r$$

$$\frac{\Gamma \vdash_{\text{ic}} A \vee B \downarrow \quad \Gamma, A \vdash_{\text{ic}} C \uparrow \quad \Gamma, B \vdash_{\text{ic}} C \uparrow}{\Gamma \vdash C \uparrow} \vee E_r$$

- ▶ Implication:

$$\frac{\Gamma, A \vdash_{\text{ic}} B \uparrow}{\Gamma \vdash_{\text{ic}} A \Rightarrow B \uparrow} \Rightarrow I \quad \frac{\Gamma \vdash_{\text{ic}} A \Rightarrow B \downarrow \quad \Gamma \vdash_{\text{ic}} A \downarrow}{\Gamma \vdash_{\text{ic}} B \downarrow} \Rightarrow E$$

- ▶ Hypotheses:

$$\frac{}{\Gamma, A, \Delta \vdash_{ic} A \downarrow}$$

- ▶ Conjunction:

$$\frac{\Gamma \vdash_{ic} A \uparrow \quad \Gamma \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \wedge B \uparrow} \wedge I$$

$$\frac{\Gamma \vdash_{ic} A \wedge B \downarrow}{\Gamma \vdash_{ic} A \downarrow} \wedge E_l$$

$$\frac{\Gamma \vdash_{ic} A \wedge B \downarrow}{\Gamma \vdash_{ic} B \downarrow} \wedge E_r$$

- ▶ Disjunction:

$$\frac{\Gamma \vdash_{ic} A \uparrow}{\Gamma \vdash_{ic} A \vee B \uparrow} \vee I_l \quad \frac{\Gamma \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \vee B \uparrow} \vee I_r$$

$$\frac{\Gamma \vdash_{ic} A \vee B \downarrow \quad \Gamma, A \vdash_{ic} C \uparrow \quad \Gamma, B \vdash_{ic} C \uparrow}{\Gamma \vdash C \uparrow} \vee E_r$$

- ▶ Implication:

$$\frac{\Gamma, A \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \Rightarrow B \uparrow} \Rightarrow I$$

$$\frac{\Gamma \vdash_{ic} A \Rightarrow B \downarrow \quad \Gamma \vdash_{ic} A \downarrow}{\Gamma \vdash_{ic} B \downarrow} \Rightarrow E$$

# ND Intercalation Rules I

- ▶ Hypotheses:

$$\frac{}{\Gamma, A, \Delta \vdash_{ic} A \downarrow}$$

- ▶ Conjunction:

$$\frac{\Gamma \vdash_{ic} A \uparrow \quad \Gamma \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \wedge B \uparrow} \wedge I$$

$$\frac{\Gamma \vdash_{ic} A \wedge B \downarrow}{\Gamma \vdash_{ic} A \downarrow} \wedge E_l$$

$$\frac{\Gamma \vdash_{ic} A \wedge B \downarrow}{\Gamma \vdash_{ic} B \downarrow} \wedge E_r$$

- ▶ Disjunction:

$$\frac{\Gamma \vdash_{ic} A \uparrow}{\Gamma \vdash_{ic} A \vee B \uparrow} \vee I_l \quad \frac{\Gamma \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \vee B \uparrow} \vee I_r$$

$$\frac{\Gamma \vdash_{ic} A \vee B \downarrow \quad \Gamma, A \vdash_{ic} C \uparrow \quad \Gamma, B \vdash_{ic} C \uparrow}{\Gamma \vdash C \uparrow} \vee E_r$$

- ▶ Implication:

$$\frac{\Gamma, A \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \Rightarrow B \uparrow} \Rightarrow I$$

$$\frac{\Gamma \vdash_{ic} A \Rightarrow B \downarrow \quad \Gamma \vdash_{ic} A \downarrow}{\Gamma \vdash_{ic} B \downarrow} \Rightarrow E$$

# ND Intercalation Rules I

- ▶ Hypotheses:

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$$\frac{\Gamma \vdash_{ic} A \uparrow \quad \Gamma \vdash_{ic} B \uparrow}{\Gamma \vdash_{ic} A \wedge B \uparrow} \wedge I$$

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$$\frac{\Gamma \vdash_{ic} A \wedge B \downarrow}{\Gamma \vdash_{ic} B \downarrow} \wedge E_r$$

- ▶ Disjunction:

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$$\frac{\Gamma \vdash_{ic} A \Rightarrow B \downarrow \quad \Gamma \vdash_{ic} A \downarrow}{\Gamma \vdash_{ic} B \downarrow} \Rightarrow E$$

- ▶ Truth and Falsehood:

$$\frac{}{\Gamma \vdash T \uparrow} T I \quad \frac{\Gamma \vdash \perp \downarrow}{\Gamma \vdash C \uparrow} \perp E$$

- ▶ Negation:

$$\frac{\Gamma, A \vdash \perp \uparrow}{\Gamma \vdash \neg A \uparrow} \neg I \quad \frac{\Gamma \vdash \neg A \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash \perp \uparrow} \neg E$$

- ▶ Universal Quantif.:

$$\frac{\Gamma \vdash \{x/a^*\}A \uparrow}{\Gamma \vdash \forall x. A \uparrow} \forall I \quad \frac{\Gamma \vdash \forall x. A \downarrow}{\Gamma \vdash \{x/t\}A \downarrow} \forall E$$

- ▶ Existential Quantif.:

$$\frac{\Gamma \vdash \{x/t\}A \uparrow}{\Gamma \vdash \exists x. A \uparrow} \exists I \quad \frac{\Gamma \vdash \exists x. A \downarrow \quad \Gamma, \{x/a^*\}A \vdash C \uparrow}{\Gamma \vdash C \uparrow} \exists E$$

\*: parameter a must be new in context

- ▶ Truth and Falsehood:
- $$\frac{\Gamma \vdash T \uparrow}{\Gamma \vdash T \uparrow} \top I \quad \frac{\Gamma \vdash \perp \downarrow}{\Gamma \vdash C \uparrow} \perp E$$
- ▶ Negation:
- $$\frac{\Gamma, A \vdash \perp \uparrow}{\Gamma \vdash \neg A \uparrow} \neg I \quad \frac{\Gamma \vdash \neg A \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash \perp \uparrow} \neg E$$
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  - ▶ Negation: 
$$\frac{\Gamma, A \vdash \perp \uparrow}{\Gamma \vdash \neg A \uparrow} \neg I \quad \frac{\Gamma \vdash \neg A \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash \perp \uparrow} \neg E$$
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- \*: parameter a must be new in context

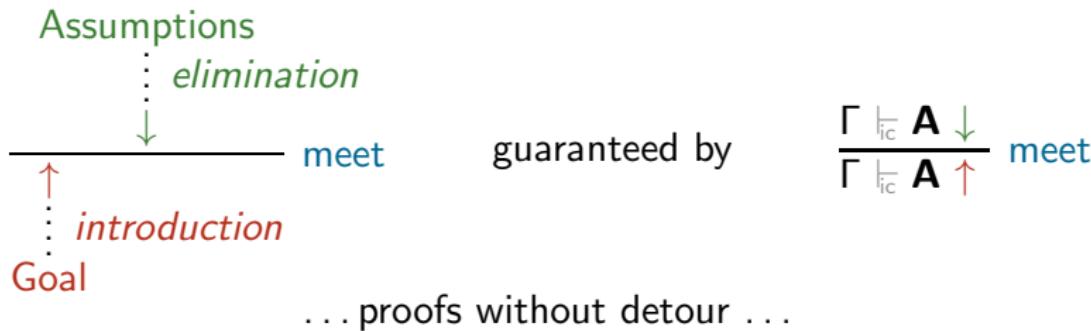
For classical logic add:

- ▶ Proof by Contradiction:

$$\frac{\Gamma, \neg A \vdash_{ic} \perp \uparrow}{\Gamma \vdash_{ic} A \uparrow} \perp_c$$

# Intercalation and ND

## Normal form proofs

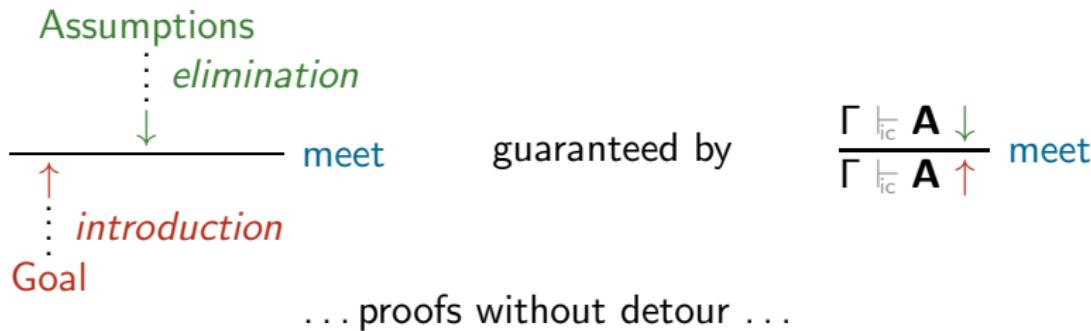


To model all ND proofs add

$$\frac{\Gamma \vdash_{ic} A \uparrow}{\Gamma \vdash_{ic} A \downarrow} \text{ roundabout}$$

# Intercalation and ND

## Normal form proofs



To model all ND proofs add

$$\frac{\Gamma \vdash_{ic} A \uparrow}{\Gamma \vdash_{ic} A \downarrow} \text{ roundabout}$$

# Example Proofs

## In normal form

$$\frac{\frac{\frac{\frac{M \wedge Q \vdash_{ic} M \wedge Q \downarrow}{M \wedge Q \vdash_{ic} Q \downarrow} \wedge E_r}{\frac{M \wedge Q \vdash_{ic} Q \uparrow}{\frac{M \wedge Q \vdash_{ic} Q \uparrow}{\frac{M \wedge Q \vdash_{ic} Q \vee S \uparrow}{\frac{M \wedge Q \vdash_{ic} (Q \vee S) \uparrow}{(M \wedge Q) \Rightarrow (Q \vee S) \uparrow}} \vee I_I}} \text{meet}}}{\vdash_{ic} (M \wedge Q) \Rightarrow (Q \vee S) \uparrow} \Rightarrow I$$

## With detour

$$\frac{\vdots \quad \vdots}{\frac{\frac{\frac{M \wedge Q \vdash_{ic} Q \uparrow \quad M \wedge Q \vdash_{ic} M \uparrow}{M \wedge Q \vdash_{ic} Q \wedge M \uparrow} \wedge I}{\frac{M \wedge Q \vdash_{ic} Q \wedge M \downarrow}{\frac{M \wedge Q \vdash_{ic} Q \downarrow}{\frac{M \wedge Q \vdash_{ic} Q \uparrow}{\vdash_{ic} (M \wedge Q) \Rightarrow Q \uparrow}} \wedge E_I}} \text{roundabout}}{\text{meet}}}$$

# Example Proofs

## In normal form

$$\frac{\frac{\frac{\frac{M \wedge Q \vdash_{ic} M \wedge Q \downarrow}{M \wedge Q \vdash_{ic} Q \downarrow} \wedge E_r}{\frac{M \wedge Q \vdash_{ic} Q \uparrow}{\frac{M \wedge Q \vdash_{ic} Q \vee S \uparrow}{\frac{M \wedge Q \vdash_{ic} (Q \vee S) \uparrow}{\vdash_{ic} (M \wedge Q) \Rightarrow (Q \vee S) \uparrow}} \vee I_I}} \text{meet}} \text{roundabout}} \Rightarrow I$$

## With detour

$$\frac{\vdots \quad \vdots}{\frac{\frac{M \wedge Q \vdash_{ic} Q \uparrow \quad M \wedge Q \vdash_{ic} M \uparrow}{\frac{M \wedge Q \vdash_{ic} Q \wedge M \uparrow}{\frac{M \wedge Q \vdash_{ic} Q \wedge M \downarrow}{\frac{M \wedge Q \vdash_{ic} Q \downarrow}{\frac{M \wedge Q \vdash_{ic} Q \uparrow}{\vdash_{ic} (M \wedge Q) \Rightarrow (Q \vee S) \uparrow}} \wedge E_I}} \text{roundabout}} \text{meet}}$$

Let  $\vdash_{\text{ic}}$  denote the intercalation calculus with rule **roundabout** and  $\vdash_{\text{ic}}$  the calculus without this rule.

- ▶ Theorem 1 (Soundness):      If  $\Gamma \vdash_{\text{ic}} A \uparrow$  then  $\Gamma \vdash A$ .
- ▶ Theorem 2 (Completeness):      If  $\Gamma \vdash A$  then  $\Gamma \vdash_{\text{ic}} A \uparrow$ .
- ▶ Is normal form proof search also complete?:  
If  $\Gamma \vdash_{\text{ic}} A \uparrow$  then  $\Gamma \vdash A \uparrow$ ?

We will investigate this question within the sequent calculus.

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If  $\Gamma \vdash_{ic} A \uparrow$  then  $\Gamma \vdash_{ic} A \uparrow ?$

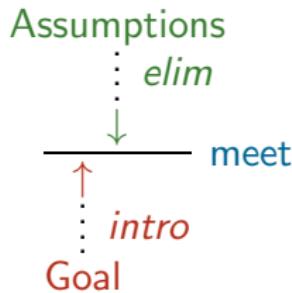
We will investigate this question within the **sequent calculus**.

Let  $\vdash_{ic}$  denote the intercalation calculus with rule **roundabout** and  $\vdash$  the calculus without this rule.

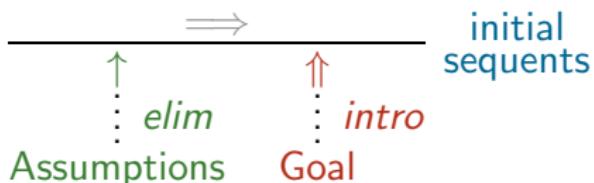
- ▶ Theorem 1 (Soundness):      If  $\Gamma \vdash_{ic} A \uparrow$  then  $\Gamma \vdash A$ .
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If  $\Gamma \vdash_{ic} A \uparrow$  then  $\Gamma \vdash_{ic} A \uparrow ?$

We will investigate this question within the **sequent calculus**.

## Normal form ND proofs

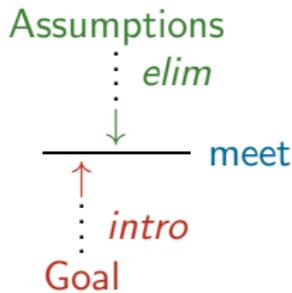


## Sequent proofs

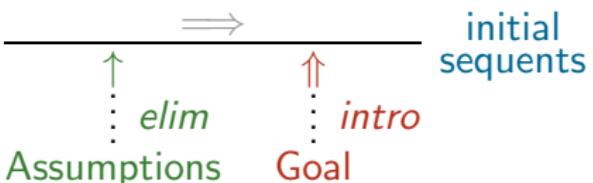


**Sequents** pair  $\langle \Gamma, \Delta \rangle$  of finite lists, multisets, or sets of formulas;  
 notation:  $\Gamma \Rightarrow \Delta$   
 Intuitive: a kind of implication,  $\Delta$  “follows from”  $\Gamma$

## Normal form ND proofs



## Sequent proofs



**Sequents** pair  $\langle \Gamma, \Delta \rangle$  of finite lists, multisets, or sets of formulas;  
 notation:  $\Gamma \Rightarrow \Delta$   
 Intuitive: a kind of implication,  $\Delta$  “follows from”  $\Gamma$

- ▶ Initial Sequents

$$\overline{\Gamma, A \Rightarrow \Delta, A} \text{ init } \quad (A \text{ atomic})$$

- ▶ Conjunction

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \wedge L \quad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \wedge R$$

- ▶ Implication

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \Rightarrow B \Rightarrow \Delta} \Rightarrow L \quad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \Rightarrow B} \Rightarrow R$$

- ▶ Truth and Falsehood

$$\overline{\Gamma, \perp \Rightarrow \Delta} \perp L \quad \overline{\Gamma \Rightarrow \Delta, \top} \top R$$

- ▶ Initial Sequents

$$\frac{}{\Gamma, \mathbf{A} \Rightarrow \Delta, \mathbf{A}} \text{ init} \quad (\mathbf{A} \text{ atomic})$$

- ▶ Conjunction

$$\frac{\Gamma, \mathbf{A}, \mathbf{B} \Rightarrow \Delta}{\Gamma, \mathbf{A} \wedge \mathbf{B} \Rightarrow \Delta} \wedge L \quad \frac{\Gamma \Rightarrow \Delta, \mathbf{A} \quad \Gamma \Rightarrow \Delta, \mathbf{B}}{\Gamma \Rightarrow \Delta, \mathbf{A} \wedge \mathbf{B}} \wedge R$$

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$$\frac{\Gamma \Rightarrow \Delta, \mathbf{A} \quad \Gamma, \mathbf{B} \Rightarrow \Delta}{\Gamma, \mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \Delta} \Rightarrow L \quad \frac{\Gamma, \mathbf{A} \Rightarrow \Delta, \mathbf{B}}{\Gamma \Rightarrow \Delta, \mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow R$$

- ▶ Truth and Falsehood

$$\frac{}{\Gamma, \perp \Rightarrow \Delta} \perp L \quad \frac{}{\Gamma \Rightarrow \Delta, \top} \top R$$

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$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \Rightarrow B \Rightarrow \Delta} \Rightarrow L \quad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \Rightarrow B} \Rightarrow R$$

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$$\frac{}{\Gamma, \mathbf{A} \Rightarrow \Delta, \mathbf{A}} \text{ init} \quad (\mathbf{A} \text{ atomic})$$

- ▶ Conjunction

$$\frac{\Gamma, \mathbf{A}, \mathbf{B} \Rightarrow \Delta}{\Gamma, \mathbf{A} \wedge \mathbf{B} \Rightarrow \Delta} \wedge L \quad \frac{\Gamma \Rightarrow \Delta, \mathbf{A} \quad \Gamma \Rightarrow \Delta, \mathbf{B}}{\Gamma \Rightarrow \Delta, \mathbf{A} \wedge \mathbf{B}} \wedge R$$

- ▶ Implication

$$\frac{\Gamma \Rightarrow \Delta, \mathbf{A} \quad \Gamma, \mathbf{B} \Rightarrow \Delta}{\Gamma, \mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \Delta} \Rightarrow L \quad \frac{\Gamma, \mathbf{A} \Rightarrow \Delta, \mathbf{B}}{\Gamma \Rightarrow \Delta, \mathbf{A} \Rightarrow \mathbf{B}} \Rightarrow R$$

- ▶ Truth and Falsehood

$$\frac{}{\Gamma, \perp \Rightarrow \Delta} \perp L \quad \frac{}{\Gamma \Rightarrow \Delta, \top} \top R$$

- ▶ Negation: 
$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \neg A \Rightarrow \Delta} \neg L \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \neg R$$

- ▶ Disjunction:

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee R \quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee L$$

- ▶ Universal Quantification:

$$\frac{\Gamma, \forall x. A, \{x/t\}A \Rightarrow \Delta}{\Gamma, \forall x. A \Rightarrow \Delta} \forall L \quad \frac{\Gamma \Rightarrow \Delta, \{x/a\}A}{\Gamma \Rightarrow \Delta, \forall x. A} \forall R$$

- ▶ Existential Quantification:

$$\frac{\Gamma, \{x/a\}A \Rightarrow \Delta}{\Gamma, \exists x. A \Rightarrow \Delta} \forall L \quad \frac{\Gamma \Rightarrow \Delta, \exists x. A, \{x/t\}A}{\Gamma \Rightarrow \Delta, \exists x. A} \forall R$$

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- ▶ Disjunction:

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee R \quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee L$$

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$$\frac{\Gamma, \forall x. A, \{x/t\}A \Rightarrow \Delta}{\Gamma, \forall x. A \Rightarrow \Delta} \forall L \quad \frac{\Gamma \Rightarrow \Delta, \{x/a\}A}{\Gamma \Rightarrow \Delta, \forall x. A} \forall R$$

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$$\frac{\Gamma, \{x/a\}A \Rightarrow \Delta}{\Gamma, \exists x. A \Rightarrow \Delta} \forall L \quad \frac{\Gamma \Rightarrow \Delta, \exists x. A, \{x/t\}A}{\Gamma \Rightarrow \Delta, \exists x. A} \forall R$$

# Example Proof

$$\begin{array}{c}
 \frac{}{\mathbf{A}, \mathbf{B} \implies \mathbf{B}} \text{ init} \\
 \frac{}{\mathbf{A} \wedge \mathbf{B} \implies \mathbf{B}} \wedge L \\
 \hline
 \frac{}{\mathbf{A} \wedge \mathbf{B} \implies \mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})} \text{ init} \\
 \frac{\mathbf{A}, \mathbf{B} \implies \mathbf{C}, \mathbf{A}}{\mathbf{A} \wedge \mathbf{B} \implies \mathbf{C}, \mathbf{A}} \wedge L \\
 \frac{}{\mathbf{A} \wedge \mathbf{B} \implies \mathbf{C} \vee \mathbf{A}} \vee R \\
 \frac{}{\mathbf{A} \wedge \mathbf{B} \implies \mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})} \wedge R \\
 \hline
 \frac{}{\implies (\mathbf{A} \wedge \mathbf{B}) \Rightarrow \mathbf{B} \wedge (\mathbf{C} \vee \mathbf{A})} \Rightarrow R
 \end{array}$$

To map natural deductions (in  $\vdash$  and  $\models_{\text{ic}}$ ) to sequent calculus derivations we add: called **cut-rule**:

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \text{Cut}$$

The question whether normal form proof search ( $\models_{\text{ic}}$ ) is complete corresponds to the question whether the cut-rule can be eliminated (is *admissible*) in sequent calculus.

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Let  $\Rightarrow^+$  denote the sequent calculus with cut-rule and  $\Rightarrow$  the sequent calculus without the cut-rule.

## Theorem 3 (Soundness)

- (a) If  $\Gamma \Rightarrow C$  then  $\Gamma \vdash_{ic}^+ C \uparrow$ .
- (b) If  $\Gamma \Rightarrow^+ C$  then  $\Gamma \vdash_{ic}^+ C \uparrow$ .

## Theorem 4 (Completeness)

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**Theorem 5 (Cut-Elimination):** Cut-elimination holds for the sequent calculus. In other words: The cut rule is *admissible* in the sequent calculus.

$$\text{If } \Gamma \Rightarrow^+ C \text{ then } \Gamma \Rightarrow C$$

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**Theorem (Normalization for ND):**

If  $\Gamma \vdash C$  then  $\Gamma \vdash_{ic} C \uparrow$ .

**Proof sketch:**

Assume  $\Gamma \vdash C$ .

Then  $\Gamma \vdash_{ic} C \uparrow$  by completeness of  $\vdash_{ic}$ .

Then  $\Gamma \Rightarrow^+ C$  by completeness of  $\Rightarrow^+$ .

Then  $\Gamma \Rightarrow C$  by cut-elimination.

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# Applications of Cut-Elimination

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# What have we done?

Natural Deduction	Intercalation	Sequent Calculus
$\vdash$ (with detours)	$\Vdash_{\text{ic}}$ (with roundabout)	$\Rightarrow^+$ (with cut)
$\rightarrow$	$\rightarrow$ $\rightarrow$	$\rightarrow$ $\downarrow$
$\leftarrow$	$\leftarrow$ $\leftarrow$	$\leftarrow$ $\leftarrow$
$\vdash$ (without detours)	$\Vdash_{\text{ic}}$ (without roundabout)	$\Rightarrow$ (without cut)

**Theorem (Consistency of ND):** There is no natural deduction derivation  $\vdash \perp$ .

**Proof sketch:**

Assume there is a proof of  $\vdash \perp$ .

Then  $\Rightarrow^+ \perp$  by completeness of  $\Rightarrow^+$  and  $\models_{\text{ic}}$ .

But  $\Rightarrow^+ \perp$  cannot be the conclusion of any sequent rule.  
Contradiction.

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We have illustrated the connection of

- ▶ natural deduction and sequent calculus
- ▶ normal form natural deductions and cut-free sequent calculus.

Fact: Sequent calculus often employed as meta-theory for specialized proof search calculi and strategies.

Question: Can these calculi and strategies be transformed to natural deduction proof search?

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