Recent Successes with a Meta-Logical Approach to Universal Logical Reasoning

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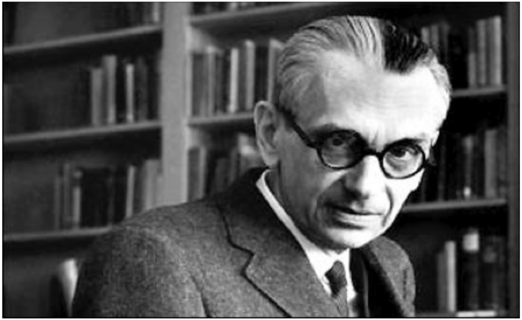
University of Luxemburg | Freie Universität Berlin | Saarland University



"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science."
- Kurt Gödel (Wang, 1996)[p. 316]

XX Brazilian Symposium on Formal Methods, Recife, 30 Dec 2017

Before I forget: — A big thanks to —

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(FU Berlin)


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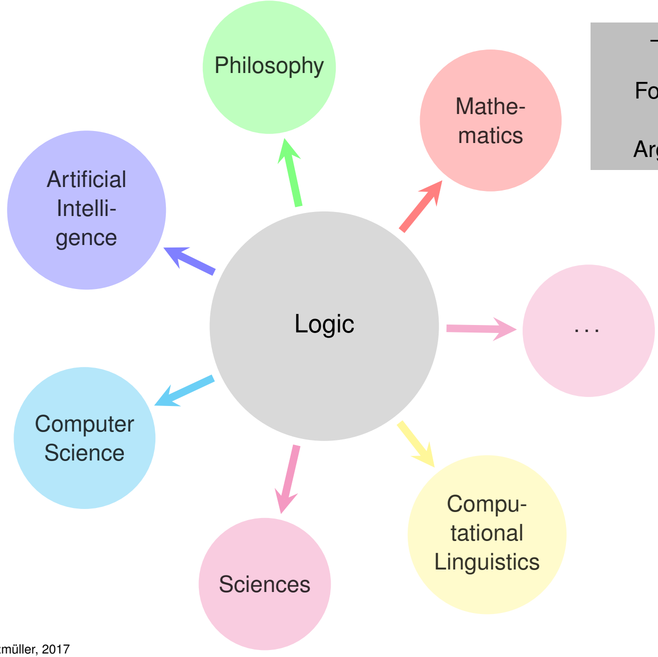
"If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis."

(Leibniz, 1677)

Logic is Relevant in Many Disciplines



— Logic —

Foundation for Rational Argumentation

Logic Zoo

Logic Zoo

Classical Logic, of order

0. Propositional Logic
1. First-order Logic
2. Second-order Logic

...

n. Higher-order Logic

Non-Classical Logics

- Intuitionistic/Constructive Logics (incl. Univalent Foundations)
- Modal Logics, Conditional Logics, Temporal Logis, Spatial Logics
- Many-valued Logics
- Paraconsistent Logics
- Free Logics, Inclusive Logics
- Logics for special applications: Ethics, Social Choice, Legal Reasoning, ...
- Separation Logic, ...

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- Separation Logic, . . .

Example Applications in Maths:

- First-order Set Theories: ZF, ZFC, . . .

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Example Applications in Maths:

- Kepler's Conjecture: Formal verification by Hales in 2014 (HOL-light)

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- Separation Logic, . . .

Free Logic:

- Elegant handling of undefinedness and partiality

Logic Zoo

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2. Second-order Logic

. . .

n. $\boxed{\text{Higher-order Logic}}$

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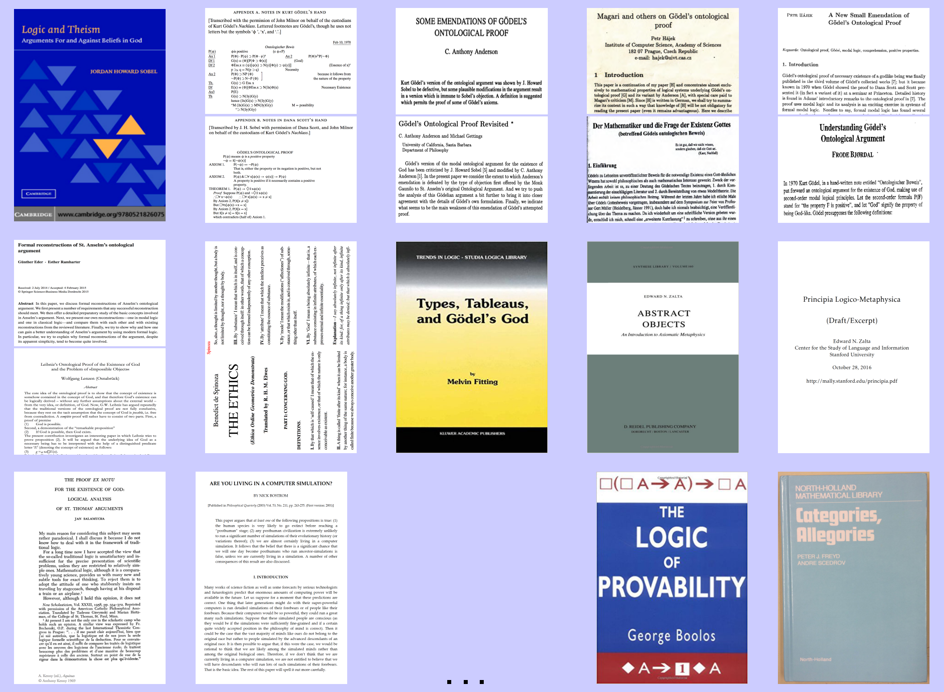
- Logics for special applications: Ethics, Social Choice, Legal Reasoning, . . .
- Separation Logic, . . .

Example Application in Metaphysics/Philosophy:

Necessarily, God exists: $\Box\exists x.Gx$

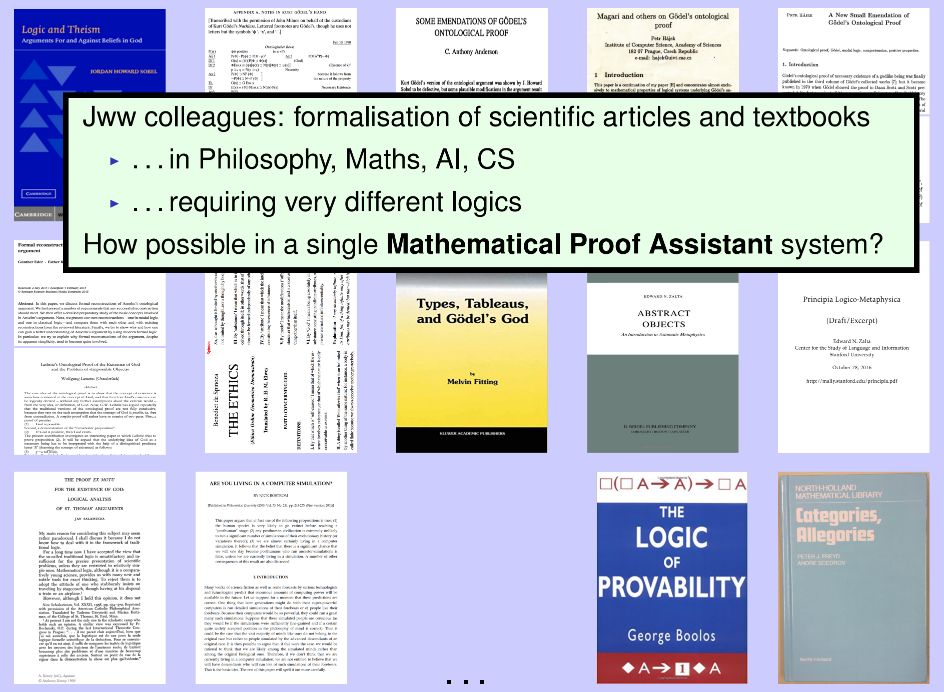
Kurt Gödel's definition of God: $G\,x \;:=\; \forall\Phi.\mathit{Positive}\;\Phi \rightarrow \Phi x$

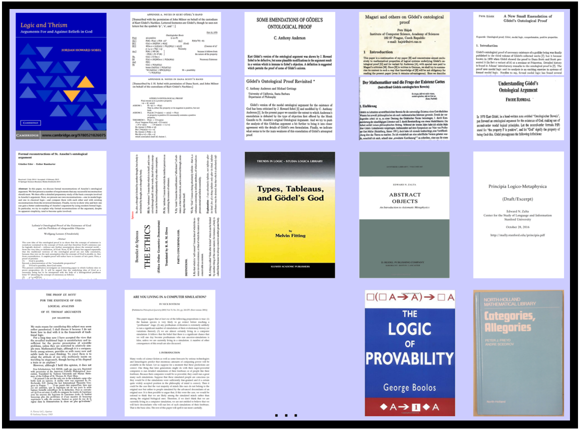

Logic and Theism
Arguments For and Against Beliefs in God
JORDAN HOWARD SOBEL
APPENDIX A. NOTES IN KURT GÖDEL'S HAND
APPENDIX B. NOTES IN DANA SCOTT'S HAND
SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF
C. Anthony Anderson
Gödel's Ontological Proof Revisited
C. Anthony Anderson and Michael Gettings
Magari and others on Gödel's ontological proof
Petr Hájek
A New Small Emendation of Gödel's Ontological Proof
Der Mathematiker und die Frage der Existenz Gottes
Understanding Gödel's Ontological Argument
FRODE BJØRDAL
Formal reconstructions of St. Anselm's ontological argument
Günther Eder · Esther Ramharter
Leibniz's Ontological Proof of the Existence of God and the Problem of »Impossible Objects«
Wolfgang Lenzen
Benedict de Spinoza
THE ETHICS
Types, Tableaus, and Gödel's God
Melvin Fitting
ABSTRACT OBJECTS
An Introduction to Axiomatic Metaphysics
EDWARD N. ZALTA
Principia Logico-Metaphysica
(Draft/Excerpt)
Edward N. Zalta
THE PROOF EX MOTU FOR THE EXISTENCE OF GOD: LOGICAL ANALYSIS OF ST. THOMAS' ARGUMENTS
JAN SALAMUCHA
ARE YOU LIVING IN A COMPUTER SIMULATION?
BY NICK BOSTROM
THE LOGIC OF PROVABILITY
George Boolos
Categories, Allegories
PETER J. FREYD
ANDRE SCEDROV

Jww colleagues: formalisation of scientific articles and textbooks

- . . . in Philosophy, Maths, AI, CS
- . . . requiring very different logics

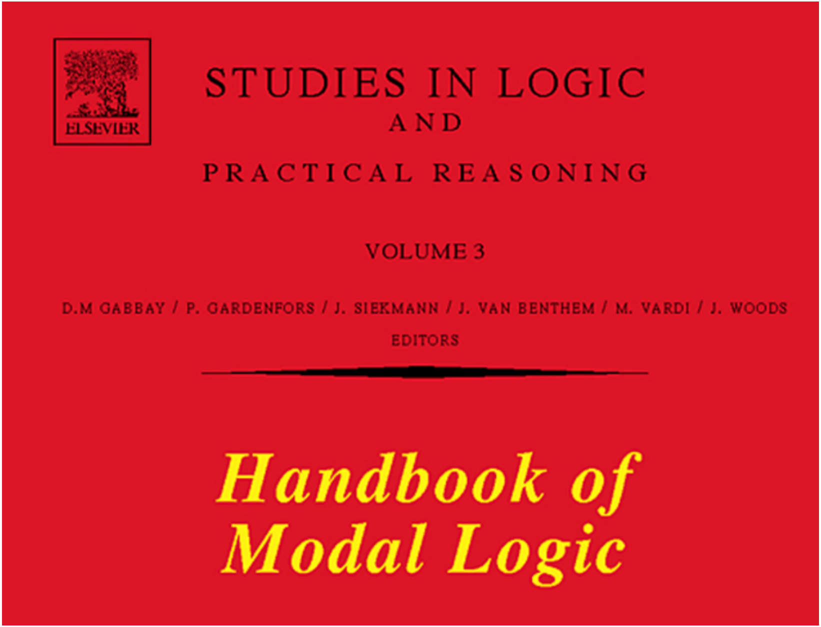
How possible in a single **Mathematical Proof Assistant** system?





Part A
Universal Logic Reasoning in Classical HO Logic (via Shallow Semantical Embeddings)

Example: Modal Logic Textbook



Example: Modal Logic Textbook

2 BASIC MODAL LOGIC

In this section we introduce the basic modal language and its relational semantics. We define basic modal syntax, introduce models and frames, and give the satisfaction definition. We then draw the reader's attention to the internal perspective that modal languages offer on relational structure, and explain why models and frames should be thought of as graphs. Following this we give the standard translation. This enables us to convert any basic modal formula into a first-order formula with one free variable. The standard translation is a bridge between the modal and classical worlds, a bridge that underlies much of the work of this chapter.

2.1 First steps in relational semantics

Suppose we have a set of proposition symbols (whose elements we typically write as p, q, r and so on) and a set of modality symbols (whose elements we typically write as m, m', m'', and so on). The choice of PROP and MOD is called the *signature* (or *similarity type*) of the language; in what follows we'll tacitly assume that PROP is denumerably infinite, and we'll often work with signatures in which MOD contains only a single element. Given a signature, we define the *basic modal language* (over the signature) as follows:

$$\varphi \quad ::= \quad p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi \leftrightarrow \psi \mid \langle m \rangle \varphi \mid [m]\varphi.$$

That is, a basic modal formula is either a proposition symbol, a boolean constant, a boolean combination of basic modal formulas, or (most interesting of all) a formula prefixed by a diamond

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Metalanguage

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Example: Modal Logic Textbook

A *model* (or *Kripke model*) $\mathfrak{M}$ for the basic modal language (over some fixed signature) is a triple $\mathfrak{M} = (W, \{R^m\}_{m \in \text{MOD}}, V)$. Here W, the *domain*, is a non-empty set, whose elements we usually call *points*, but which, for reasons which will soon be clear, are sometimes called *states*, *times*, *situations*, *worlds* and other things besides. Each R^m in a model is a binary relation on W, and V is a function (the valuation) that assigns to each proposition symbol p in PROP a subset $V(p)$ of W; think of $V(p)$ as the set of points in $\mathfrak{M}$ where p is true. The first two components $(W, \{R^m\}_{m \in \text{MOD}})$ of $\mathfrak{M}$ are called the *frame* underlying the model. If there is only one relation in the model, we typically write (W, R) for its frame, and (W, R, V) for the model itself. We encourage the reader to think of Kripke models as graphs (or to be slightly more precise, *directed graphs*, that is, graphs whose points are linked by directed arrows) and will shortly give some examples which show why this is helpful.

Suppose w is a point in a model $\mathfrak{M} = (W, \{R^m\}_{m \in \text{MOD}}, V)$. Then we inductively define the notion of a formula φ being *satisfied* (or *true*) in $\mathfrak{M}$ at point w as follows (we omit some of the clauses for the booleans):

$$
\begin{array}{rcl}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p), \\
\mathfrak{M}, w \models \top & & \text{always}, \\
\mathfrak{M}, w \models \bot & & \text{never}, \\
\mathfrak{M}, w \models \neg\varphi & \text{iff} & \text{not } \mathfrak{M}, w \models \varphi \text{ (notation: } \mathfrak{M}, w \not\models \varphi), \\
\mathfrak{M}, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi, \\
\mathfrak{M}, w \models \varphi \rightarrow \psi & \text{iff} & \mathfrak{M}, w \not\models \varphi \text{ or } \mathfrak{M}, w \models \psi, \\
\mathfrak{M}, w \models \langle m \rangle \varphi & \text{iff} & \text{for some } v \in W \text{ such that } R^m wv \text{ we have } \mathfrak{M}, v \models \varphi, \\
\mathfrak{M}, w \models [m] \varphi & \text{iff} & \text{for all } v \in W \text{ such that } R^m wv \text{ we have } \mathfrak{M}, v \models \varphi.
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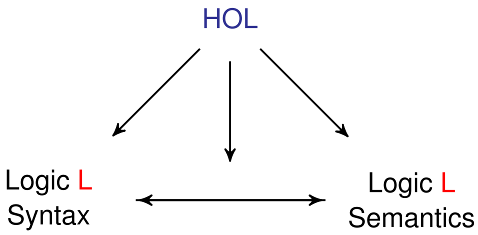
Universal Logic Reasoning in HOL



Examples for L we have already studied:
Intuitionistic Logics, (Mathematical) Fuzzy Logics, Free Logic, Modal Logics, Description Logics, Conditional Logics, Access Control Logics, Hybrid Logics, Multivalued Logics, Logics based on Neighborhood Semantics, Paraconsistent Logics, Dyadic Logic, . . .

Embedding works also for quantifiers (first-order & higher-order)

HOL provers become universal logic reasoning engines!

interactive:	Isabelle/HOL, PVS, HOL4, Hol Light, Coq/HOL, . . .
automated:	Leo-III, LEO-II, Satallax, TPS, Nitpick, Isabelle/HOL, . . .

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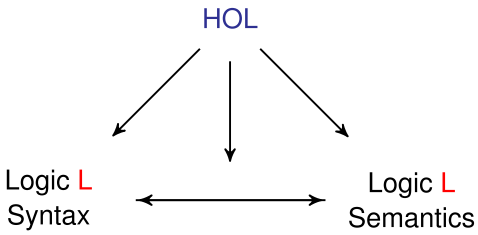

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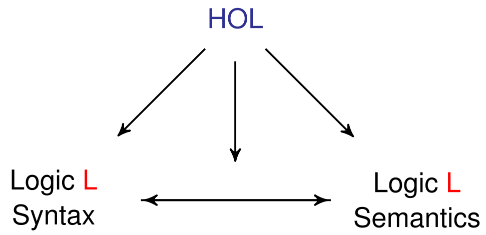

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Universal Logic Reasoning in HOL

HOL (meta-logic) $\varphi ::=$

L (object-logic) $\psi ::=$

Embedding of in

$=$

$=$

$=$

$=$

$=$

Pass this set of equations to a HOL theorem prover

HOL $s, t \quad ::= \quad c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\to o}$ (explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\to & = & \lambda\varphi_{\mu\to o}\lambda\psi_{\mu\to o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\Box & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \forall u_\mu\,(\neg \boldsymbol{r} wu \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\to o}\lambda w_\mu \exists u_\mu\,(\boldsymbol{r} wu \wedge \varphi u) \\
\forall & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \forall d_\gamma\, hdw \\
\exists & = & \lambda h_{\gamma\to(\mu\to o)}\lambda w_\mu \exists d_\gamma\, hdw \\
\text{valid} & = & [.] = \lambda\varphi_{\mu\to o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

Universal Logic Reasoning in HOL

HOL	$s, t \quad ::= $	$c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\to\beta} \mid (s_{\alpha\to\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$
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Ax (polymorphic over γ)

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Universal Logic Reasoning in HOL

Example

HOML formula $\Diamond \exists x G x$

HOML formula embedded in HOL $[\Diamond \exists x G x]$

$$(\lambda \varphi \forall w_\mu \varphi\, w)(\Diamond \exists x G x)$$

$$\forall w_\mu ((\Diamond \exists x G x)\, w)$$

$$\forall w_\mu (((\lambda \varphi \lambda w_\mu \exists u_\mu\, (\boldsymbol{r} w u \wedge \varphi u)) \exists x G x)\, w)$$

$$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge (\exists x G x) u)$$

$$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge (\exists (\lambda x G x)) u)$$

$$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge ((\lambda h_{\gamma \to (\mu \to o)} \lambda w_\mu \exists d_\gamma\, h d w)(\lambda x G x)) u)$$

$$\forall w_\mu \exists u_\mu (\boldsymbol{r} w u \wedge \exists x G x u)$$

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that $[\varphi]$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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$$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G x) u)$$

$$\forall w_\mu \exists u_\mu (rwu \wedge (\exists(\lambda x G x)) u)$$

$$\forall w_\mu \exists u_\mu (rwu \wedge ((\lambda h_{\gamma \to (\mu \to o)} \lambda w_\mu \exists d_\gamma\, hdw)(\lambda x G x)) u)$$

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What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that $[\varphi]$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Example

HOML formula $\Diamond\exists x G x$

HOML formula embedded in HOL $[\Diamond\exists x G x]$

$$(\lambda\varphi\forall w_\mu \varphi\, w)(\Diamond\exists x G x)$$

$$\forall w_\mu((\Diamond\exists x G x)\, w)$$

$$\forall w_\mu(((\lambda\varphi\lambda w_\mu\exists u_\mu\,(\boldsymbol{r}wu \wedge \varphi u))\exists x G x)\, w)$$

$$\forall w_\mu\exists u_\mu(\boldsymbol{r}wu \wedge (\exists x G x)u)$$

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Universal Logic Reasoning in HOL

Example

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Isabelle/HOL (one of various Theorem Provers for HOL)

Isabelle

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

TECHNISCHE UNIVERSITÄT MÜNCHEN

- Home
- Overview
- Installation
- Documentation

Site Mirrors:
Cambridge (.uk)
Munich (.de)
Sydney (.au)
Potsdam, NY (.us)

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle was originally developed at the University of Cambridge and Technische Universität München, but now includes numerous contributions from institutions and individuals worldwide. See the Isabelle overview for a brief introduction.

Now available: Isabelle2017 (October 2017)

Download for Mac OS X

Download for Linux - Download for Windows (32bit) - Download for Windows (64bit) - Download for Mac OS X

Some notable changes:

- Experimental support for Visual Studio Code as alternative PIDE front-end.
- Improved Isabelle/jEdit Prover IDE: management of session sources independently of editor buffers, removal of unused theories, explicit indication of theory status, more careful auto-indentation.
- Session-qualified theory imports.
- Code generator improvements: support for statically embedded computations.
- Numerous HOL library improvements.
- More material in HOL-Algebra, HOL-Computational_Algebra and HOL-Analysis (ported from HOL-Light).
- Improved Nunchaku model finder, now in main HOL.
- SQL database support in Isabelle/Scala.

See also the cumulative NEWS.

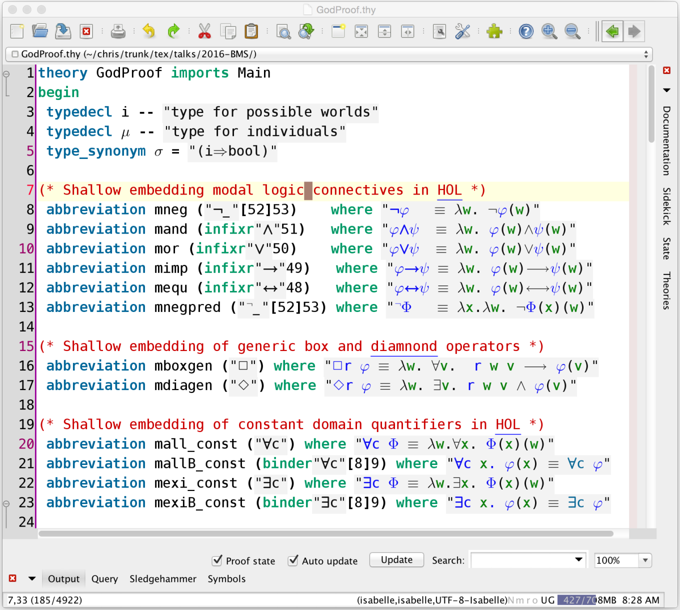
Distribution & Support

Isabelle is distributed for free under a conglomerate of open-source licenses, but the main code-base is subject to BSD-style regulations. The application bundles include source and binary packages and documentation, see the detailed installation instructions. A vast collection of Isabelle examples and applications is available from the Archive of Formal Proofs.

`https://isabelle.in.tum.de`

(many other systems: Coq, HOL, HOL Light, PVS, Lean, NuPrL, IMPS, ACL2, . . .)

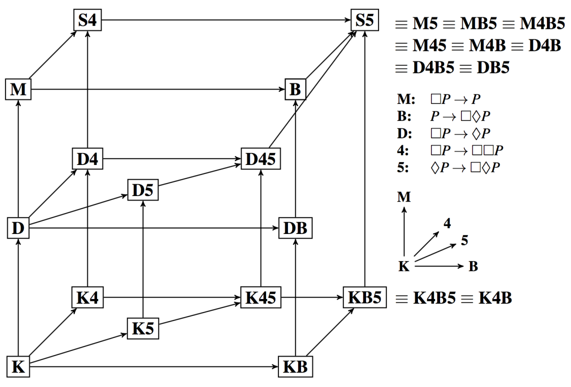
Universal Logic Reasoning in Isabelle/HOL



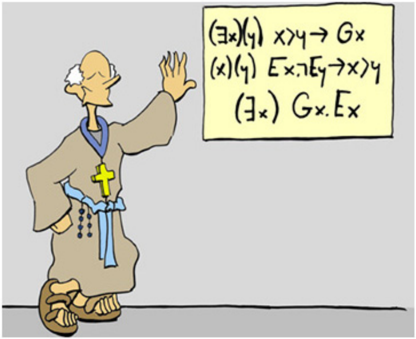
Universal Logic Reasoning in HOL

There are many different (Higher-Order) Modal Logics

Modal Logic Cube



Moreover, $\forall/\exists$-quantifiers may have a **possibilist** or **actualist** reading

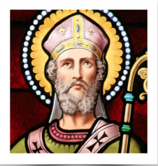


Anselm figured it all out
(or did he?)

Part B
— Ontological Argument —
Modern Variants of Kurt Gödel and Dana Scott

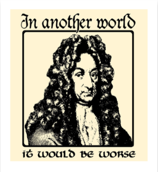
Ontological Proofs of God's Existence

A Long and Continuing Tradition in Philosophy



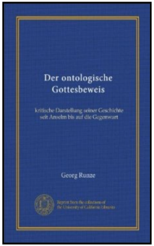
St. Anselm

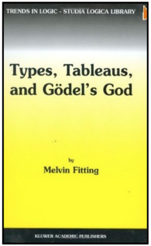
Descartes

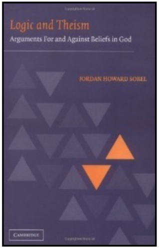


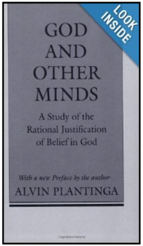
Leibniz

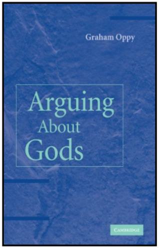
Gödel











Computational Metaphysics: Kurt Gödel's Ontological Argument

Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess. } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)\, G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x) G(x) \supset MN(\exists y)\, G(y)$ M = possibility

" $\supset N(\exists y)\, G(y)$

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x)\, G(x)$ means all [the system] ... patible This is ...

At 4: $P(\varphi).\ \varphi \supset_N \psi$ ...

{ $x = x$ is p... ; $x \neq x$ is ... }

But if a system S of ...

It would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It may also mean pure "attribution" as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x)\, N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ contrary Ax. or the existence of pos. prop.

X i.e. the normal form in terms of elem. prop. contains a member without negation.

Ontologischer Beweis — Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi) . P(\psi) \supset P(\varphi . \psi)$[•] — At 2 $P(\varphi) \vee P(\sim\varphi)$

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hence $(\exists x) G(x) \supset N(\exists y) G(y)$

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" $\supset N(\exists y) G(y)$ — M = possibility

any two essences of x are nec. equivalent

exclusive or — • and for any number of summands

$M(\exists x) G(x)$ means the system of all pos. prop. is compatible. This is true because of:

At 4: $P(\varphi) . \varphi \supset_N \psi : \supset P(\psi)$ which impl

~~true~~ { $x = x$ is positive
~~not~~ { $x \neq x$ is negative

But if a system S of pos. prop. were incom[patible] it would mean, that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth[etic] sense (independently of the accidental structure of the world). Only then the ax. true. It m[ay] also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret[ation] simpler proof

If φ pos. not: $(x) N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ neg. contrary Ax or the existence of pos. prop.

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

Computational Metaphysics: Dana Scott's Variant

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists x G(x)$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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second-order quantifiers

Computational Metaphysics: Vision of Leibniz (1646–1716) — *Calculemus!*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

Required:
characteristica universalis and **calculus ratiocinator**

Computational Metaphysics: Scott's and Gödel's Variants — Demo

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$$\forall\phi\forall\psi[(P(\phi) \land \Box\forall x[\phi(x) \to \psi(x)]) \to P(\psi)]$$

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Axiom A2

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. T1 $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 $P(G)$

Cor. C $\Diamond\exists x G(x)$

Axiom A4 $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2

$$\phi\ ess.\ x \leftrightarrow \phi(x) \ \wedge\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

Thm. T2 $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. D3

$$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$$

Axiom A5 $P(NE)$

Thm. T3 $\Box\exists x G(x)$

Computational Metaphysics: Scott's and Gödel's Variants — Demo

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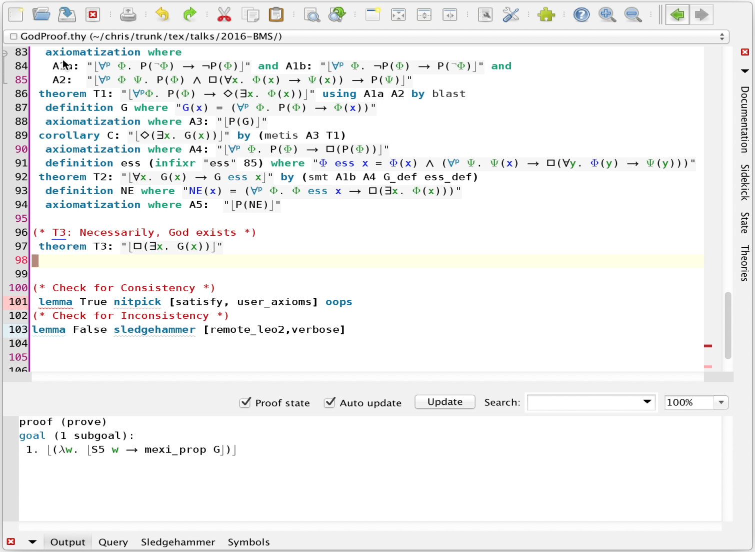

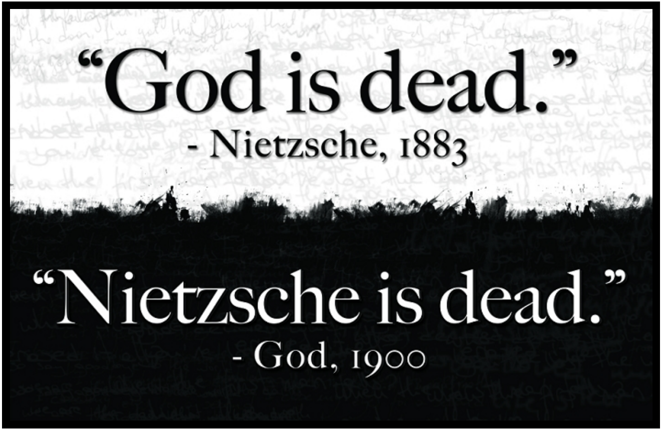

Results of our Experiments

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014 + IJCAI, 2016 + KI 2016 + . . .]

Results of our Experiments

Variant of Dana Scott

- the premises are **consistent**
- all argument steps are **logically correct** in (higher-order, extensional) modal logic
  - correct in logic **S5**
  - weaker logic **KB** is already sufficient
  - philosophical critique about use of S5 not justfied
- minimal dependencies determined by theorem provers
- alternative proofs (different from the ones in literature)

Intermediate Conclusion:

With our technology. . .
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Resultats of our Analysis

. . . we continue with Scott's version

Further corollaries we can prove

- Monotheism
- Gott is flawless (has only positive properties)
- . . .
- Modal Collapse: $\varphi \longrightarrow \Box\varphi$

  - there are no contingent truths
  - no alternative worlds
  - everything is determined
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Challenge: Can the Modal Collapse be avoided (with minimal changes)?

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$$\boxed{\varphi \rightarrow \Box \varphi}$$

— Can the modal collapse be avoided? —

Can the Modal Collapse be avoided?

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's Ontological Proof Revisited *

C. Anthony Anderson and Michael Gettings

University of California, Santa Barbara
Department of Philosophy

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Magari and others on Gödel's ontological proof

Petr Hájek
Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic
e-mail: hajek@uivt.cas.cz

1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes

(betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,
sondern glauben, daß ein Gott sei.
(Kant, Nachlaß)

1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

PETR HÁJEK

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Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

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— contributed to clarification of controversy between Hajek and Anderson —
— revealed various flaws and issues —
(Keynote: 1st World Congress on Logic and Religion, 2015)

Metaphysics: *Foundational branch in philosophy*, that ...
... studies the fundamental nature of **being** and **the world** that encompasses it
... looks **beyond experience** in the real world

Flammarions Holzstich – in L'atmosphère, Paris 1888

Adresses **ultimate questions**, such as:

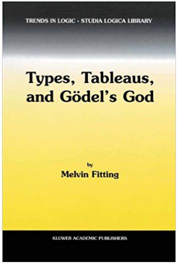
- What is there?
- What is it like?
- **Is there a God?**
- What can I know?

Method: **Rational Argumentation**

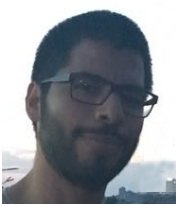
Lecture course won the 2015/16 Central Teaching Award of FU Berlin
(jww: Alex Steen, Max Wisniewski, and others)

- MSc students in Maths, CS, Philosophy and Physics from FU, TU and HU
- Invited Lectures by Philosophers (Zalta & Lenzen) and Computer Scientists
- First course of this kind worldwide!

Avoiding the Modal Collapse: Fitting's Variant



Melvin Fitting (New York)



David Fuenmayor
(Philosophy, FU Berlin)

Fitting adapts Montague/Gallin intensional logics

- $\uparrow$: intension (concept) of a given notion
- $\downarrow$: extension of a concept
- e.g. intension of 'tall-inhabitant'
  of type $ind \rightarrow (world \rightarrow o)$ resp. $\uparrow \langle ind \rangle$
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- here relevant for arguments of 'positive' & 'essence'
  positive: $\uparrow \langle\langle ind \rangle\rangle$ vs. $\uparrow \langle \uparrow \langle ind \rangle\rangle$
- moreover,
  - 'possibilist' and 'actualist' quantifiers are utilised
  - 'de dicto' and 'de re' modeling is supported

Fitting presents three variants of the argument

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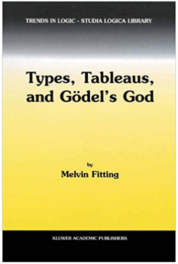

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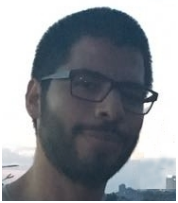

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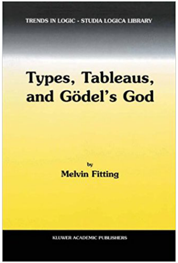

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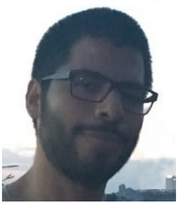

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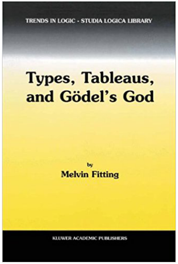

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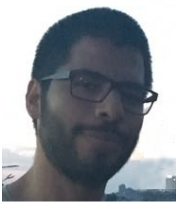

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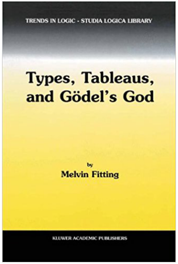
Further Experiments



Melvin Fitting (New York)

Philosophy/Metaphysics

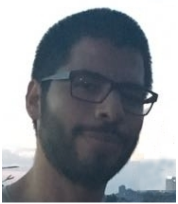
Ontological Argument
(avoids modal collapse)

Intensional higher-order modal logic

Verified (main chapters)

`https://www.isa-afp.org/entries/Types_Tableaus_and_Goedels_God.shtml`

Paper: [KI 2017]



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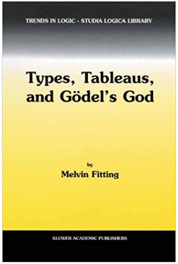

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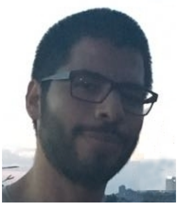

David Fuenmayor
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Ed Zalta (Stanford)

Philosophy/Metaphysics

Principia Logico-Metaphysica
(Foundations for Metaphysics & Mathematics)

Hyperintensional higher-order modal logic

Inconsistency/Paradox revealed

Emendations proposed

`https://www.isa-afp.org/entries/PLM.shtml`

[arXiv:1711.06542], submitted

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(Mathematics, FU Berlin)

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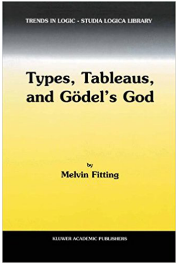

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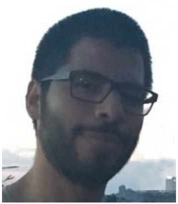

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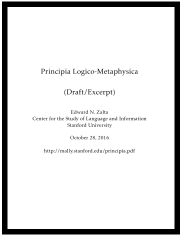
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(Draft/Excerpt)

Edward N. Zalta
Center for the Study of Language and Information
Stanford University

October 28, 2016

http://mally.stanford.edu/principia.pdf

Ed Zalta (Stanford)

Philosophy/Metaphysics

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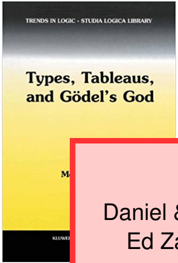

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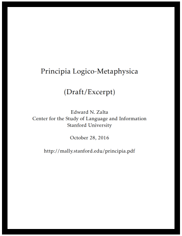
vid Fuenmayor
sophy, FU Berlin)

Kirchner Paradox

Daniel & Isabelle/HOL have become close advisors of Ed Zalta in the search for a repair of the paradox.

This is *Computational Metaphysics* par excellence!!!



Principia Logico-Metaphysica

(Draft/Excerpt)

Edward N. Zalta
Center for the Study of Language and Information
Stanford University

October 28, 2016

http://mally.stanford.edu/principia.pdf

Ed Zalta (Stanford)

Philosophy/Metaphysics

Principia Logico-Metaphysica
(Foundations for Metaphysics & Mathematics)

Hyperintensional higher-order modal logic

Inconsistency/Paradox revealed

Emendations proposed

`https://www.isa-afp.org/entries/PLM.shtml`

[arXiv:1711.06542], submitted

Daniel Kirchner
(Mathematics, FU Berlin)

Russel Paradox

- $S := \{x \mid x \notin x\}$ resp. $S := \lambda x.\neg(x\,x)$
- S := 'the set of all sets, which do not contain themselves'
- **Question:** 'Does S contain S?'
- **Way out:** types

Clark/Boolos Paradox for naive AOT

- $K := \lambda x \exists F(xF \wedge \neg Fx)$
- K := 'property of being an x that encodes a property F, x does not exemplify'
- **Question:** 'Does K exemplify K?'
- **Way out:** syntactical language restriction (restrict matrix of λ-expressions to formulas without encoding subformulas)

Kirchner Paradox for AOT

- $K' := \lambda x((\lambda y\, True)\,(\iota z(z = \exists F(xF \wedge \neg Fx))))$
- Motivation for K': exploit issue in def. of encoding subformulas in AOT (K' is actually extensionally equivalent to K)
- **Question:** Does K' exemplify K'?
- **Way out:** ... currently being explored ...

The Kirchner Paradox of Abstract Object Theory (AOT)

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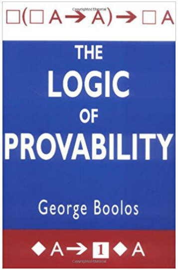
Further Experiments



Philosophy & Mathematics

Textbook on Provability Logic

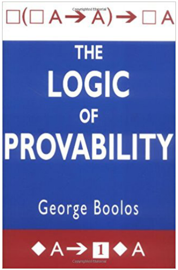
Provability Logic

Verified (most parts)

`Url:soon`

David Streit
(Mathematics, FU Berlin)

Further Experiments



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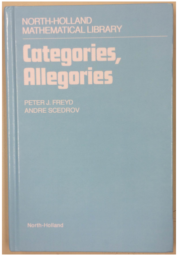


Mathematics

Textbook on Category Theory

Free first-order logic

(Constricted) Inconsistency detected

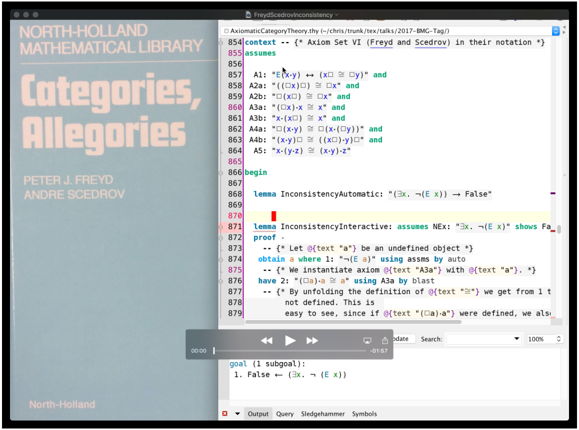
Emendation proposed

[ICMS 2016], [arXiv:1609.01493], submitted

D. Scott
(UC Berkeley)

Other Completed Studies

Variants of the Ontological Argument	jww
E. J. Lowe	David Fuenmayor Alexander Steen Max Wisniewski
G.W. Leibniz	Matthias Bentert David Streit Bruno Woltzenlogel-Paleo
G. Eder und E. Ramharter	Lukas Grätz Fabian Schütz
G. Oppy	Eberhard Höpfner Magdalena Haselsteiner Jose Antonio Akieme Rodriguez Pablo Agustin Martin Torres
C.A. Anderson, P. Hájek, F. Bjordal	Leon Weber Bruno Woltzenlogel-Paleo



Part C
Mathematics: Category Theory

Free Logic (Scott, 1967) in HOL

See our papers:
Free Logic in Isabelle/HOL, ICMS, 2016
Axiomatizing Category Theory in Free Logic, arXiv:1609.01493, 2016
Automating Free Logic in HOL, with an Experimental . . . , submitted

Free Logic: Elegant Approach to Definite Description and Undefinedness

Dana Scott. "Existence and description in formal logic." In: Bertrand Russell: Philosopher of the Century, edited by R. Schoenman. George Allen & Unwin, London, 1967, pp. 181-200. Reprinted with additions in: Philosophical Application of Free Logic, edited by K. Lambert. Oxford Universitry Press, 1991, pp. 28 - 48.

16

DANA SCOTT

Existence and Description in Formal Logic

The problem of what to do with improper descriptive phrases has bothered logicians for a long time. There have been three major suggestions of how to treat descriptions usually associated with the names of Russell, Frege and Hilbert-Bernays. The author does not consider any of these approaches really satisfactory. In many ways Russell's idea is most attractive because of its simplicity. However, on second thought one is saddened to find that the Russellian method of elimination depends heavily on the scope of the elimination.

Previous Approaches (rough sketch)

The present King of France is bald.

.

Russel (first approach)

$pkof$:= present King of France

$bald(\iota x.pkof(x))$

iff

$(\exists x.pkof(x)) \wedge (\forall x, y.((pkof(x) \wedge pkof(y)) \rightarrow x = y) \wedge (\forall x.pkof((x) \rightarrow bald(x))$

Hence, **false**.

Frege

$\iota x.pkof(x)$ does not denote; $bald(\iota x.pkof(x))$ has **no truth value**.

Hilbert-Bernays

If the existence and uniqueness conditions cannot be proved, then the term $\iota x.pkof(x)$ is **not part of the language**.

Free Logic: Elegant Approach to Definite Description and Undefinedness

Existence and Description in Formal Logic (Dana Scott), 1967

Principle 1: Bound individual variables range over domain $E \subset D$

Principle 2: Values of terms and free variables are in D, not necessarily in E only.

Principle 3: Domain E may be empty

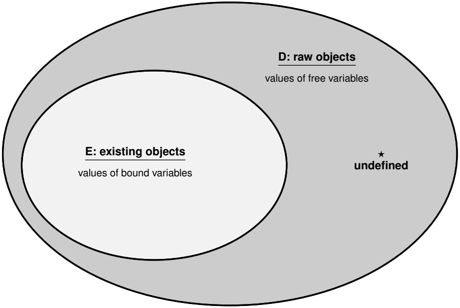
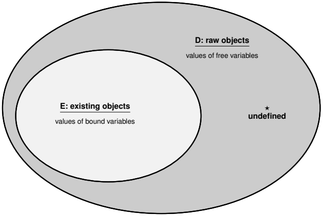
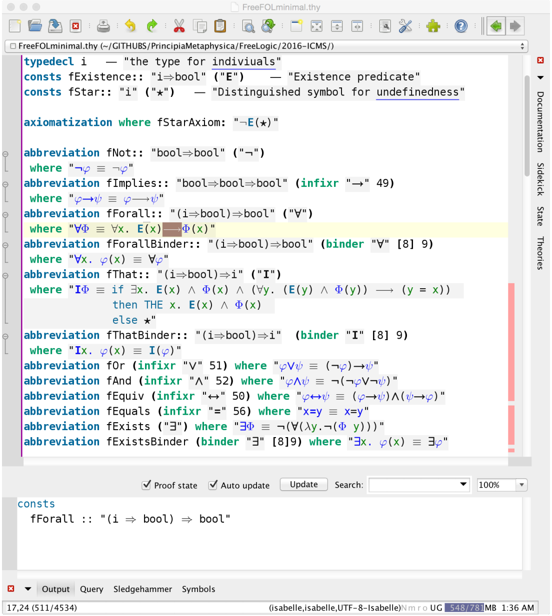


Figure: Illustration of the semantical domains of free logic

Free Logic in HOL





Exemplary Case Study: Exploration of Axioms Sets for Category Theory

Axioms Set I

Generalized Monoids

Dana Scott

Exemplary Case Study: Exploration of Axioms Sets for Category Theory

Axioms Set I

Generalized
Monoids

Axioms Set II

Dana Scott

Exemplary Case Study: Exploration of Axioms Sets for Category Theory

Axioms Set I

Generalized Monoids

Axioms Set II

Axioms Set III

Dana Scott

Exemplary Case Study: Exploration of Axioms Sets for Category Theory

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Generalized Monoids

Axioms Set II

Axioms Set III

Axioms Set IV

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Exemplary Case Study: Exploration of Axioms Sets for Category Theory

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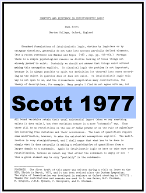
Axioms Set III

Axioms Set IV



IDENTITY AND EXISTENCE IN INTUITIONISTIC LOGIC

Dana Scott

Merton College, Oxford, England

Standard formulations of intuitionistic logic, whether by logicians or by category theorists, generally do not take into account partially defined elements. (For a recent reference see Makkai and Reyes [18], esp. pp. 144-163.) Perhaps there is a simple psychological reason: we dislike talking of those things not already proved to exist. Certainly we should not assume that things exist without making this assumption explicit. In classical logic the problem is not important, because it is always possible to split the definition (or theorem) into cases according as the object in question does or does not exist. In intuitionistic logic this way is not open to us, and the circumstance complicates many constructions, the theory of descriptions, for example. Many people I find do not agree with me, but

Scott 1977

All bound variables retain their usual existential import (when we say something exists it does exist), but free variables behave in a more "schematic" way. Thus there will be no restrictions on the use of modus ponens or on the rule of substitution involving free variables and their occurrences. The laws of quantifiers require some modification, however, to make the existential assumptions explicit. The modification is very straightforward, and I shall argue that what has to be done is simply what is done naturally in making a relativization of quantifiers from a larger domain to a subdomain. Again in intuitionistic logic we have to take care over relativization, because we cannot say that either the subdomain is empty or not — thus a given element may be only "partially" in the subdomain.

FOOTNOTE The first draft of this paper was written during a visit on leave at the ETH, Zürich in March, 1975, and it has been revised since the Durham Symposium. The style of formalization was developed in seminars at Oxford starting in 1972/73. Thanks for contributions and remarks are owed to D. van Dalen, M.P. Fourman, R. Grayson, J.M.E. Hyland, C. Smorynski, and R. Statman.

Axioms Set V

Dana Scott's Axioms from 1977

Dana Scott

Exemplary Case Study: Exploration of Axioms Sets for Category Theory

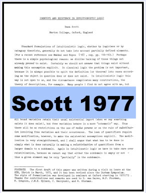
Axioms Set I

Generalized Monoids

Axioms Set II

Axioms Set III

Axioms Set IV



Axioms Set V

Dana Scott's Axioms from 1977

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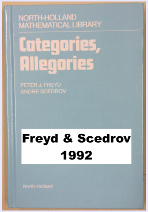


Axioms Set VI

Freyd & Scedrov's Axioms from 1992

Dana Scott

Exemplary Case Study: Exploration of Axioms Sets for Category Theory

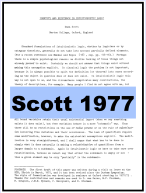
Axioms Set I

Generalized Monoids

Axioms Set II

Axioms Set III

Axioms Set IV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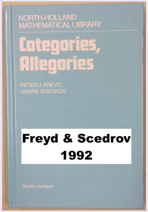
Scott 1977

Axioms Set V

Dana Scott's Axioms from 1977

—?—



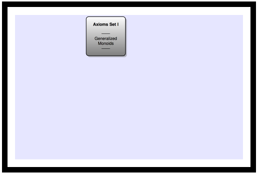
Axioms Set VI

Freyd & Scedrov's Axioms from 1992


Dana Scott

all equivalent?

Preliminaries

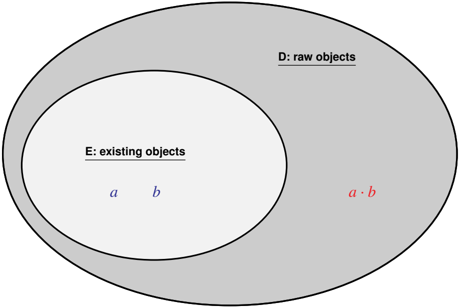


Morphisms: objects of type of i (:= raw domain D)

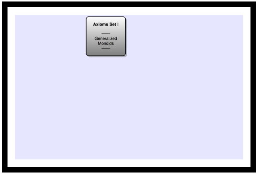
Partial functions:

domain	dom	of type $i \to i$
codomain	cod	of type $i \to i$
composition	$\cdot$	of type $i \to i \to i$ (resp. $i \times i \to i$)

Partiality of "·" handled as expected:
$a \cdot b$ may be non-existing for some existing morphisms a and b.



Preliminaries



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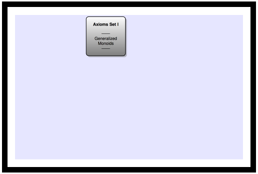

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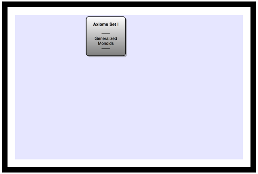
$\cong$ denotes Kleene equality: $\quad x \cong y \equiv (Ex \vee Ey) \to x = y$

(where $=$ is identity on all objects of type i, existing or non-existing)

$\cong$ is an equivalence relation: **Sledgehammer**.

Preliminaries



Morphisms: objects of type of i (:= raw domain D)

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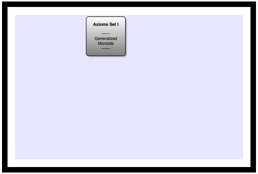
$\cong$ is an equivalence relation: **Sledgehammer**.

$\simeq$ denotes existing identity: $x \simeq y \equiv Ex \wedge Ey \wedge x = y$

$\simeq$ is symmetric and transitive, but lacks reflexivity: **Sledgehammer**, **Nitpick**.



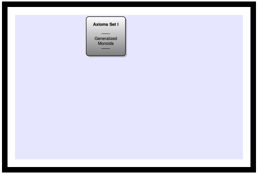
Monoid

A monoid is an algebraic structure $(S, \circ)$, where $\circ$ is a binary operator on set S, satisfying the following properties:

Closure:	$\forall a, b \in S.\ a \circ b \in S$
Associativity:	$\forall a, b, c \in S.\ a \circ (b \circ c) = (a \circ b) \circ c$
Identity:	$\exists id_S \in S.\ \forall a \in S.\ id_S \circ a = a = a \circ id_S$

That is, a monoid is a semigroup with a two-sided identity element.

From Monoids to Categories



We employ a partial, strict binary composition operation $\cdot$
Left and right identity elements are addressed in C_i, D_i, .

Categories: Axioms Set I

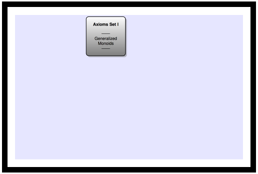
S_i	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey)$
E_i	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z.z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_i	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_i	Codomain	$\forall y.\exists i.ID(i) \wedge i \cdot y \cong y$
D_i	Domain	$\forall x.\exists j.ID(j) \wedge x \cdot j \cong x$

where I is an identity morphism predicate:

$$ID(i) \equiv (\forall x.\ E(i \cdot x) \rightarrow i \cdot x \cong x) \wedge (\forall x.\ E(x \cdot i) \rightarrow x \cdot i \cong x)$$

Experiments with Isabelle/HOL

From Monoids to Categories



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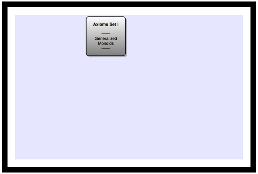
Experiments with Isabelle/HOL

- Model finder **Nitpick** confirms that this axiom set is consistent.
- Even if we assume there are non-existing objects ($\exists x.\neg(Ex)$) we get consistency.

From Monoids to Categories



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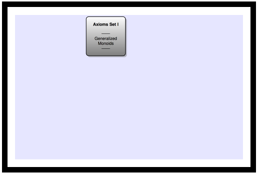
Experiments with Isabelle/HOL

- The left-to-right direction of E is implied: **Sledgehammer**.

$$E(x \cdot y) \rightarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$$

From Monoids to Categories



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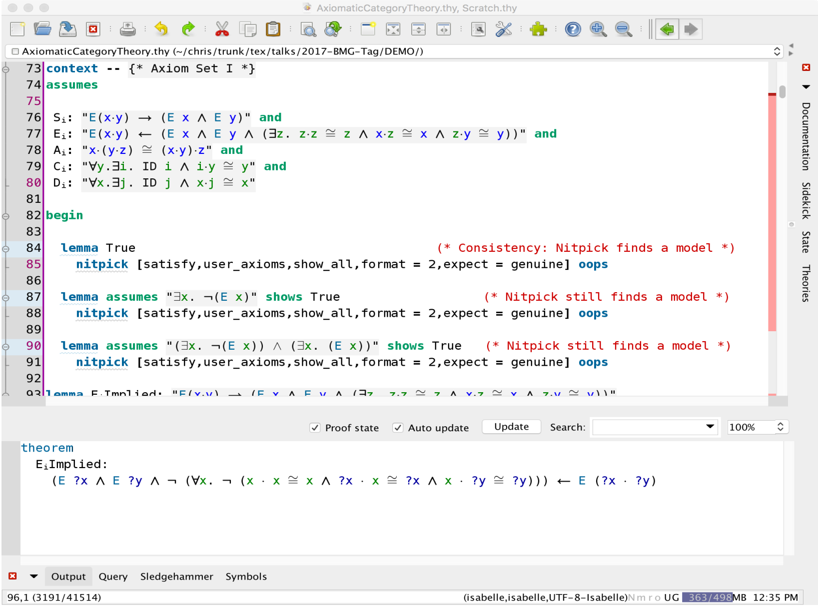
Experiments with Isabelle/HOL

- The i in axiom C is unique: **Sledgehammer**.
- The j in axiom D is unique: **Sledgehammer**.
- However, the i and j need not be equal: **Nitpick**

Demo


AxiomaticCategoryTheory.thy, Scratch.thy
AxiomaticCategoryTheory.thy (~/chris/trunk/tex/talks/2017-BMG-Tag/DEMO/)
context -- {* Axiom Set I *}
assumes
Sᵢ: "E(x·y) ⟶ (E x ∧ E y)" and
Eᵢ: "E(x·y) ⟵ (E x ∧ E y ∧ (∃z. z·z ≅ z ∧ x·z ≅ x ∧ z·y ≅ y))" and
Aᵢ: "x·(y·z) ≅ (x·y)·z" and
Cᵢ: "∀y.∃i. ID i ∧ i·y ≅ y" and
Dᵢ: "∀x.∃j. ID j ∧ x·j ≅ x"
begin
lemma True (* Consistency: Nitpick finds a model *)
nitpick [satisfy,user_axioms,show_all,format = 2,expect = genuine] oops
lemma assumes "∃x. ¬(E x)" shows True (* Nitpick still finds a model *)
nitpick [satisfy,user_axioms,show_all,format = 2,expect = genuine] oops
lemma assumes "(∃x. ¬(E x)) ∧ (∃x. (E x))" shows True (* Nitpick still finds a model *)
nitpick [satisfy,user_axioms,show_all,format = 2,expect = genuine] oops
Proof state
Auto update
Update
Search:
100%
theorem
EᵢImplied:
(E ?x ∧ E ?y ∧ ¬ (∀x. ¬ (x · x ≅ x ∧ ?x · x ≅ ?x ∧ x · ?y ≅ ?y))) ⟵ E (?x · ?y)
Output
Query
Sledgehammer
Symbols
Documentation
Sidekick
State
Theories
96,1 (3191/41514)
(isabelle,isabelle,UTF-8-Isabelle)
363/498MB 12:35 PM

Interaction: Dana – Christoph – Isabelle/HOL

Dana Scott <dana.scott@cs.cmu.edu> 8/6/16

to me

> On Aug 5, 2016, at 11:00 PM, Christoph Benzmueller <c.benzmueller@gmail.com> wrote:
>
> When we take IDD(i) as
>   (all x)[E(i.x) ==> i.x == x] &
>   (all x)[E(x.i) ==> x.i == x]
> and replace ID(i) in our SACDE-axioms by IDD(i) then I can show that
> ID(I) and IDD(i) are equivalent. See attachment New_axioms_9.png.
>
> So IDD(i) seem suited as a notion of identity morphism.

Dana

Ha! I am surprised, because I did not see how to prove:

    (all i)[IDD(i) ==> i.i == i]

I have to think about this. I hate it when computers are smarter than I am!

I guess C and D have to be used.

Christoph Benzmueller <c.benzmueller@gmail.com> 8/6/16

to Dana

Hi Dana, see the first attachment of my prvious Mail. C and S are used for this. Its called IDD-help1.

C.

Interaction: Dana – Christoph – Isabelle/HOL

Christoph Benzmueller <c.benzmueller@gmail.com> 7/23/16

to Dana

Dana,

here are the results of the experiments; doesn't look too good.

On Fri, Jul 22, 2016 at 11:43 PM, Dana Scott <dana.scott@cs.cmu.edu> wrote:

> > On Jul 21, 2016, at 9:32 AM, Christoph Benzmueller <c.benzmueller@gmail.com> wrote:
> >
> > The F-axioms are all provable from the old S-axioms.
> > But D2, D3 and E3 are not.
>
> I think I see the trouble with those D axioms. But E3 is very odd.
>
> E3: E(x.y) ==> (exist i)[Id(i) & x.(i.y) == x.y]
>
> You see, by the S-axioms, if you assume E(x.y), then E(x) & E(y) & E(cod(x))
> follows. So the "i" in the conclusion of E3 ought to be "cod(x)".
>
> Please check, therefore, whether this is provable from the S-axioms:
>
> (all x) Id(cod(x))

Apparently it isn't. See file Scott_new_axioms_4.png; the countermodel is presented in the lower window; he have:

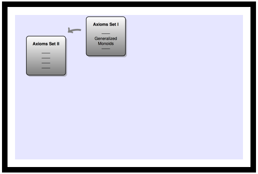
dom(i1)=i1, dom(i2)=i2, dom(i3)=i3
cod(i1)=i1, cod(i2)=i2, cod(i3)=i3
i1.i1=i1, i1.i2=i3, i1.i3=i3
i2.i1=i3, i2.i2=i2, i2.i3=i3
i3.i1=i3, i3.i2=i3, i3.i3=i3
E(i1),E(i2), ~E(i3)

Countermodel by Nitpick converted by me into a readable form

I have briefly checked it; it seems to validate each S-axiom.

> If this is OK, then E3 should have been provable.

From Monoids to Categories



Axioms Set II is developed from Axioms Set I by Skolemization of i and j in axioms C and D. We can argue semantically that every model of Axioms Set I has such functions. The strictness axiom S is extended, so that strictness is now also postulated for the new Skolem functions dom and cod.

Categories: Axioms Set II

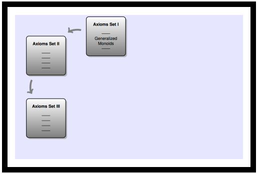
S_{ii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{ii}	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_{ii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{ii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{ii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.
- Axiom Set II implies Axioms Set I: easily proved by **Sledgehammer**.
- Axiom Set I also implies Axioms Set II (by semantical means on the meta-level)

From Monoids to Categories

In Axioms Set III the existence axiom E is simplified by taking advantage of the two new Skolem functions dom and cod.



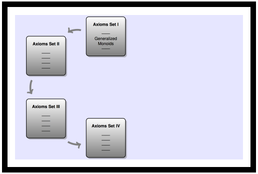
Categories: Axioms Set III

S_{iii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iii}	Existence	$E(x \cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{iii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- The left-to-right direction of existence axiom E is implied: **Sledgehammer**.
- Axioms Set III implies Axioms Set II: **Sledgehammer**.
- Axioms Set II implies Axioms Set III: **Sledgehammer**.

From Monoids to Categories



Axioms Set IV simplifies the axioms C and D. However, as it turned out, these simplifications also require the existence axiom E to be strengthened into an equivalence.

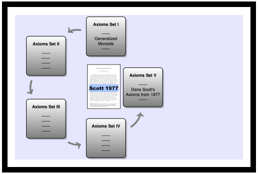
Categories: Axioms Set IV

S_{iv}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iv}	Existence	$E(x \cdot y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iv}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iv}	Codomain	$(cod\ y) \cdot y \cong y$
D_{iv}	Domain	$x \cdot (dom\ x) \cong x$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set IV implies Axioms Set III: **Sledgehammer**.
- Axioms Set III implies Axioms Set IV: **Sledgehammer**.

From Monoids to Categories



Axioms Set V simplifies axiom E (and S).
Now, strictness of $\cdot$ is implied.

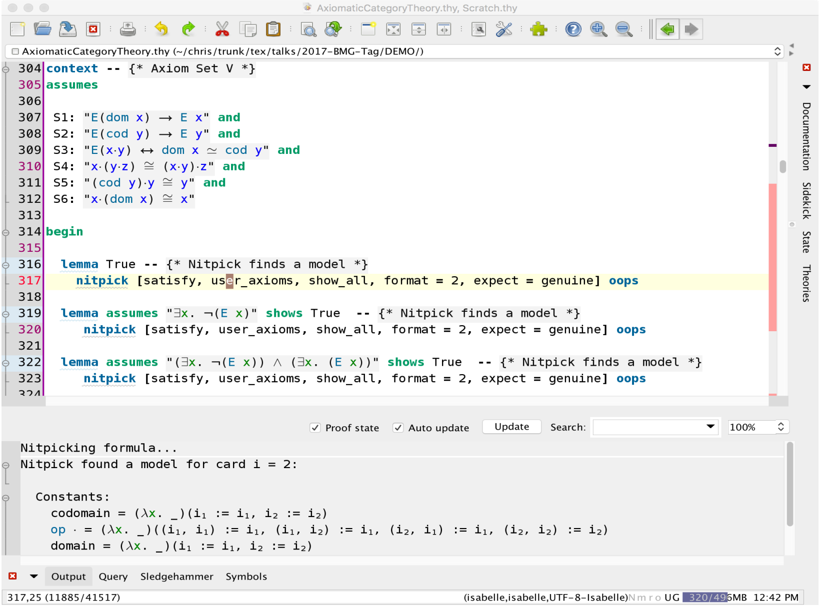
Categories: Axioms Set V (Scott, 1977)

$S1$	Strictness	$E(dom\ x) \rightarrow Ex$
$S2$	Strictness	$E(cod\ y) \rightarrow Ey$
$S3$	Existence	$E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$
$S4$	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
$S5$	Codomain	$(cod\ y) \cdot y \cong y$
$S6$	Domain	$x \cdot (dom\ x) \cong x$

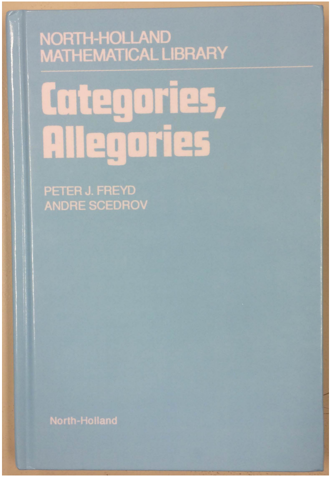
Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set V implies Axioms Set IV: **Sledgehammer**.
- Axioms Set IV implies Axioms Set V: **Sledgehammer**.

Demo


AxiomaticCategoryTheory.thy, Scratch.thy
AxiomaticCategoryTheory.thy (~/chris/trunk/tex/talks/2017-BMG-Tag/DEMO/)
context -- {* Axiom Set V *}
assumes
S1: "E(dom x) ⟶ E x" and
S2: "E(cod y) ⟶ E y" and
S3: "E(x·y) ⟷ dom x ≃ cod y" and
S4: "x·(y·z) ≅ (x·y)·z" and
S5: "(cod y)·y ≅ y" and
S6: "x·(dom x) ≅ x"
begin
lemma True -- {* Nitpick finds a model *}
nitpick [satisfy, user_axioms, show_all, format = 2, expect = genuine] oops
lemma assumes "∃x. ¬(E x)" shows True -- {* Nitpick finds a model *}
nitpick [satisfy, user_axioms, show_all, format = 2, expect = genuine] oops
lemma assumes "(∃x. ¬(E x)) ∧ (∃x. (E x))" shows True -- {* Nitpick finds a model *}
nitpick [satisfy, user_axioms, show_all, format = 2, expect = genuine] oops
Documentation
Sidekick
State
Theories
Proof state
Auto update
Update
Search:
100%
Nitpicking formula...
Nitpick found a model for card i = 2:
Constants:
codomain = (λx. _)(i1 := i1, i2 := i2)
op · = (λx. _)((i1, i1) := i1, (i1, i2) := i1, (i2, i1) := i1, (i2, i2) := i2)
domain = (λx. _)(i1 := i1, i2 := i2)
Output
Query
Sledgehammer
Symbols
317,25 (11885/41517)
(isabelle,isabelle,UTF-8-Isabelle)
UG
320/496MB
12:42 PM

Application: Cats & Alligators



1.1. BASIC DEFINITIONS

The theory of CATEGORIES is given by two unary operations and a binary partial operation. In most contexts lower-case variables are used for the 'individuals' which are called *morphisms* or *maps*. The values of the operations are denoted and pronounced as:

$\Box x$ *the source of* x ,

$x\Box$ *the target of* x ,

xy *the composition of* x *and* y .

The axioms:

A1 xy *is defined iff* $x\Box = \Box y$,

A2a $(\Box x)\Box = \Box x$ *and* $\Box(x\Box) = x\Box$, A2b

A3a $(\Box x)x = x$ *and* $x(x\Box) = x$, A3b

A4a $\Box(xy) = \Box(x(\Box y))$ *and* $(xy)\Box = ((x\Box)y)\Box$, A4b

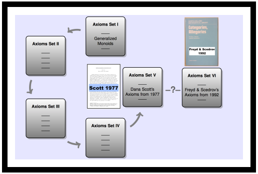
A5 $x(yz) = (xy)z$.

1.11. The ordinary equality sign = will be used only in the symmetric sense, to wit: if either side is defined then so is the other and they are equal. A theory, such as this, built on an ordered list of partial operations, the domain of definition of each given by equations in the previous, and with all other axioms equational, is called an ESSENTIALLY ALGEBRAIC THEORY.

1.12. We shall use a venturi-tube $\succeq$ for *directed equality* which means: if the left side is defined then so is the right and they are equal. The axiom that $\Box(xy) = \Box(x(\Box y))$ is equivalent, in the presence of the earlier axioms, with $\Box(xy) \succeq \Box x$ as can be seen below.

1.13. $\Box(\Box x) = \Box x$ because $\Box(\Box x) = \Box((\Box x)\Box) = (\Box x)\Box = \Box x$. Similarly $(x\Box)\Box = x\Box$.

From Monoids to Categories



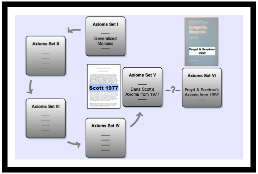
Categories: Original axiom set by Freyd and Scedrov (modulo notation)

A1	$E(x \cdot y) \leftrightarrow dom\ x \ \cong cod\ y$
A2a	$cod(dom\ x) \cong dom\ x$
A2b	$dom(cod\ y) \cong cod\ y$
A3a	$x \cdot (dom\ x) \cong x$
A3b	$(cod\ y) \cdot y \cong y$
A4a	$dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

- Consistency? — Nitpick finds a model.
- Consistency when assuming $\exists x. \neg Ex$ — Nitpick does not find a model.
- lemma $(\exists x. \neg Ex) \rightarrow False$: **Sledgehammer**. (Relevant axioms: $A1, A2a, A3a$)

From Monoids to Categories



Categories: Original axiom set by Freyd and Scedrov (modulo notation)

A1 $\quad E(x \cdot y) \leftrightarrow dom\ x \cong cod\ y$

A2a $\quad cod(dom\ x) \cong dom\ x$

A2b $\quad dom(cod\ y) \cong cod\ y$

A3a $\quad x \cdot (dom\ x) \cong x$

A3b $\quad (cod\ y) \cdot y \cong y$

A4a $\quad dom(x \cdot y) \cong dom((dom\ x) \cdot y)$

A4b $\quad cod(x \cdot y) \cong cod(x \cdot (cod\ y))$

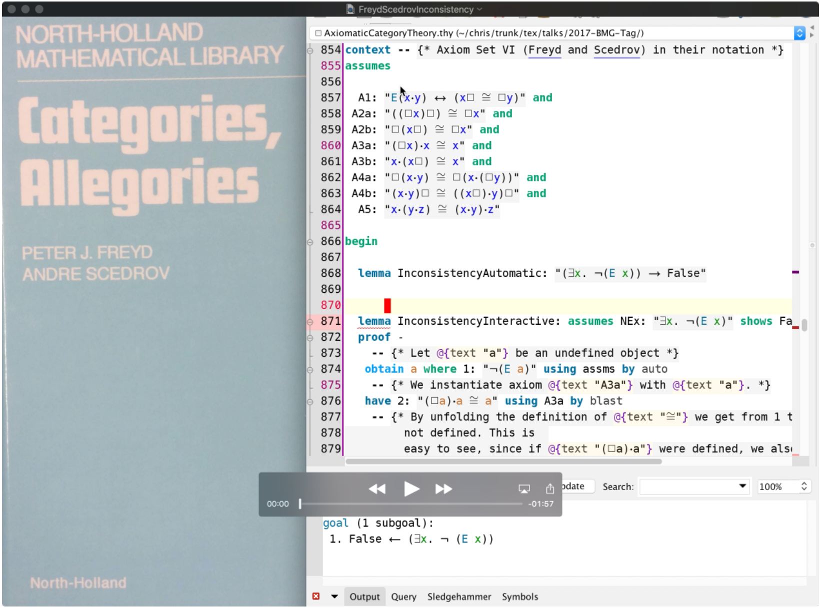
A5 $\quad x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

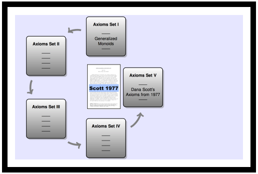
- Consistency? — Nitpick finds a model.
- Consistency when assuming $\exists x. \neg Ex$ — Nitpick does not find a model.
- lemma $(\exists x. \neg Ex) \rightarrow False$: **SLEDGEHAMMER**. (Relevant axioms: $A1, A2a, A3a$)

When interpreted in free logic, then the axioms of Freyd and Scedrov are flawed: Either all morphisms exist (i.e., $\cdot$ is total), or the axioms are inconsistent.

Demo


FreydScedrovInconsistency
NORTH-HOLLAND MATHEMATICAL LIBRARY
Categories, Allegories
PETER J. FREYD
ANDRE SCEDROV
North-Holland
AxiomaticCategoryTheory.thy (~/chris/trunk/tex/talks/2017-BMG-Tag/)
854 context -- {* Axiom Set VI (Freyd and Scedrov) in their notation *}
855 assumes
857 A1: "E(x·y) ↔ (x□ ≅ □y)" and
858 A2a: "((□x)□) ≅ □x" and
859 A2b: "□(x□) ≅ □x" and
860 A3a: "(□x)·x ≅ x" and
861 A3b: "x·(x□) ≅ x" and
862 A4a: "□(x·y) ≅ □(x·(□y))" and
863 A4b: "(x·y)□ ≅ ((x□)·y)□" and
864 A5: "x·(y·z) ≅ (x·y)·z"
866 begin
868 lemma InconsistencyAutomatic: "(∃x. ¬(E x)) ⟶ False"
871 lemma InconsistencyInteractive: assumes NEx: "∃x. ¬(E x)" shows Fa
872 proof -
873 -- {* Let @{text "a"} be an undefined object *}
874 obtain a where 1: "¬(E a)" using assms by auto
875 -- {* We instantiate axiom @{text "A3a"} with @{text "a"}. *}
876 have 2: "(□a)·a ≅ a" using A3a by blast
877 -- {* By unfolding the definition of @{text "≅"} we get from 1 t
878 not defined. This is
879 easy to see, since if @{text "(□a)·a"} were defined, we als
00:00
-01:57
Search:
100%
goal (1 subgoal):
1. False ⟵ (∃x. ¬ (E x))
Output
Query
Sledgehammer
Symbols

From Monoids to Categories



Categories: Axioms Set VI (Freyd and Scedrov, when corrected)

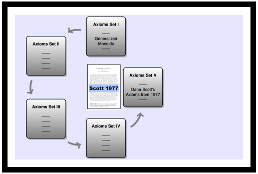
A1	$E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$
A2a	$cod(dom\ x) \cong dom\ x$
A2b	$dom(cod\ y) \cong cod\ y$
A3a	$x \cdot (dom\ x) \cong x$
A3b	$(cod\ y) \cdot y \cong y$
A4a	$dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by NITPICK.
- Axioms Set VI implies Axioms Set V: SLEDGEHAMMER.
- Axioms Set V implies Axioms Set VI: SLEDGEHAMMER.
- Redundancies:

— The $A4$-axioms are implied by the others: SLEDGEHAMMER.
— The $A2$-axioms are implied by the others: SLEDGEHAMMER.

From Monoids to Categories



Categories: Axioms Set VI (Freyd and Scedrov, when corrected)

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A2b $dom(cod\ y) \cong cod\ y$

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A3b $(cod\ y) \cdot y \cong y$

A4a $dom(x \cdot y) \cong dom((dom\ x) \cdot y)$

A4b $cod(x \cdot y) \cong cod(x \cdot (cod\ y))$

A5 $x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

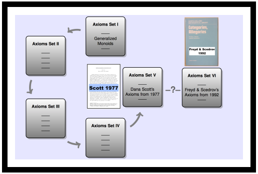
Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set VI implies Axioms Set V: **Sledgehammer**.
- Axioms Set V implies Axioms Set VI: **Sledgehammer**.
- Redundancies:

— The $A4$-axioms are implied by the others: **Sledgehammer**.

— The $A2$-axioms are implied by the others: **Sledgehammer**.

From Monoids to Categories



Maybe Freyd and Scedrov do not assume a free logic. In algebraic theories free variables often range over existing objects only. However, we can formalise this as well:

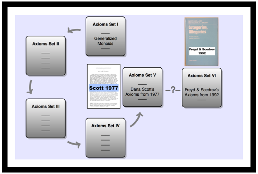
Categories: "Algebraic reading" of axiom set by Freyd and Scedrov.

A1	$\forall xy.\ E(x \cdot y) \leftrightarrow dom\ x \ \cong cod\ y$
A2a	$\forall x.\ cod(dom\ x) \cong dom\ x$
A2b	$\forall y.\ dom(cod\ y) \cong cod\ y$
A3a	$\forall x.\ x \cdot (dom\ x) \cong x$
A3b	$\forall y.\ (cod\ y) \cdot y \cong y$
A4a	$\forall xy.\ dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$\forall xy.\ cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$\forall xyz.\ x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.
- However, none of V-axioms are implied: **Nitpick**.
- For equivalence to V-axioms we need strictness of dom, cod and $\cdot$: **Sledgehammer**.

From Monoids to Categories



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A3a	$\forall x.\ x \cdot (dom\ x) \cong x$
A3b	$\forall y.\ (cod\ y) \cdot y \cong y$
A4a	$\forall xy.\ dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$\forall xy.\ cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$\forall xyz.\ x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

But: Strictness is not mentioned in Freyd and Scedrov (it could not be expressed axiomatically, when variables range over of existing objects only).
This leaves us puzzled about their axiom system.

Hence, we better prefer the Axioms Set V by Scott (from 1977).

Conclusion

HOL is a (quite) Universal Metalogic (via semantical embeddings):

- **Lean and elegant** approach to integrate and combine heterogeneous logics
- **Reuse** of existing theorem proving technology
- High degree of **automation**
- **Uniform proofs** (via the embeddings)
- Works surprisingly well in **practical applications**
- **Intuitive user interaction** at abstract level
- Approach very well suited for (interdisciplinary) **teaching** of logics

Lots of further work (<u>but I need resources!</u>)

- Maths, Computer Science, Artificial Intelligence
- Philosophy, Natural Language Processing
- **Legal-, Ethical- and Moral-Reasoning in Intelligent Machines**
- Rational Argumentation