First-Order Logic: Theory and Practice

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First-Order Logic

(this part of the lecture very closely follows Fitting's textbook)

First-Order Logic: Syntax

Example Formula: $(\forall x)((R(x,a) \wedge R(x,f(x))) \supset ((\exists y)R(x,y)))$

- Propositional Connectives: $\wedge$, $\supset$, ...
- Quantifiers: $(\exists x)\ldots$, $(\forall x)\ldots$
- Punctuation: '(' ')' ','
- Variables: countable set $\mathbf{V} = \{v_1, v_2, \ldots, x, y, z, \ldots\}$

Definition — 'Signature' of a First-Order Language **89**

- **R**: finite or countable set of relation symbols (with an arity $n \geq 0$ assigned to it)
- **F**: finite or countable set of function symbols (with an arity $n \geq 1$ assigned to it)
- **C**: finite or countable set of constant symbols

$\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ (or short **L**) is the first-order language defined by **R**, **F**, and **C**. The reuse of symbol names for different arities is sometimes useful and thus allowed.

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Definition — Terms of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ 90

The *terms* of language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ are the smallest set with:

- Any variable $x \in \mathbf{V}$ is a term of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$
- Any constant $c \in \mathbf{C}$ is a term of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$
- If $f \in \mathbf{F}$ with arity n and if $t_1, \ldots, t_n$ are terms of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ then $f(t_1, \ldots, t_n)$ is a term of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$

Example — Terms 91

$f(g(a, x)) \qquad g(f(x), g(x, y)) \qquad g(a, g(a, g(a, b)))$

First-Order Logic: Syntax

Definition — Atomic formulas of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ **92**

If $t_1, \ldots, t_n$ are terms of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ and $R \in \mathbf{R}$ with arity n then $R(t_1, \ldots, t_n)$ is an *atomic formula* of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$. Moreover, $\bot$ and $\top$ are *atomic formulas* of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$.

Definition — Formulas of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ **93**

The *formulas* of language $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ are the smallest set with:

- Any atomic formula is a formula of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$
- If A is a formula of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ then so is $\neg A$
- If A and B are formulas of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ then so is $(A \circ B)$ for any connective $\circ$
- If A is a formula of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ then so are $(\forall x)A$ and $(\exists x)A$

Example — Formulas 94

$$(\forall x)(\exists y)R(f(x,y),z)$$

$$(\forall x)(\forall y)(R(x,y) \supset (\exists z)(R(x,y) \wedge R(z,y)))$$

we are informal about parentheses

$$(\forall x)(\forall y)\{R(x,y) \supset (\exists z)(R(x,z) \wedge R(z,y))\}$$

we may use infix instead of prefix notation

$$x < y \text{ instead of } <(x,y)$$

Definition — Free-variable Occurrences 95

The *free-variable occurrences* of a formula are defined as:

1. The free-variable occurrences of an atomic formula are all variables occurrences in that formula
2. The free-variable occurrences in $\neg A$ are the free-variable occurrences in A
3. The free-variable occurrences in $A \circ B$ are the free-variable occurrences in A together with the free-variable occurrences in B
4. The free-variable occurrences in $(\forall x)A$ and $(\exists x)A$ are the free-variable occurrences in A except for the free-variable occurrences of x

An occurrence of a variable x is called *bound* if it is not a free-variable occurrence.

Exercise: Identify the free-variable occurrences in

$$(\forall x)R(x, c) \supset R(x, c)$$

$$(\forall x)(R(x, c) \supset R(x, c))$$

$$(\forall x)[(\exists y)R(f(x, y), c) \supset (\exists z)S(y, z)]$$

Definition — Sentence / Closed Formula **96**

Any formula of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ with no free-variable occurrences is called a *sentence* or *closed formula* of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$.

Substitution:

- replacement of a variable by a (possibly complex) term
- in the definitions below we assume a fixed language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$
- we call the set of terms of this fixed language $\mathbf{T}$
- all definitions are relative to $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ and $\mathbf{T}$
- substitutions are functions σ that operate on variables, terms and formulas; instead of $\sigma(t)$ we will write $t\sigma$

Definition — Substitution **97**

A *substitution* is a mapping $\sigma : \mathbf{V} \longrightarrow \mathbf{T}$ from variables to terms.

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Definition — Substitution lifted to Terms **98**

Let $\sigma : \mathbf{V} \longrightarrow \mathbf{T}$ be a substitution. We define:

- If $c \in \mathbf{C}$ then $c\sigma = c$
- $[f(t_1, \ldots, t_n)]\sigma = f(t_1\sigma, \ldots, t_n\sigma)$ for any $f \in \mathbf{F}$ and $t_1, \ldots, t_n \in \mathbf{T}$

Example — Substitution lifted to Terms **99**

...blackboard, implementation in practical exercise ...

Definition — Composition of Substitutions 100

Let σ and τ be substitutions. By the *composition* of σ and τ, denoted $\sigma\tau$, we mean that substitution such that for each variable x we have $x(\sigma\tau) = (x\sigma)\tau$.

Proposition — Substitution 101

For every $t \in \mathbf{T}$ *we have:* $t(\sigma\tau) = (t\sigma)\tau$

Proof: By structural induction on t

Proposition — Associativity of Substitution Composition 102

$(\sigma_1\sigma_2)\sigma_3 = \sigma_1(\sigma_2\sigma_3)$

Proof: Let $v \in \mathbf{V}$.

$$v(\sigma_1\sigma_2)\sigma_3 = [v(\sigma_1\sigma_2)]\sigma_3 = [(v\sigma_1)\sigma_2]\sigma_3 = (v\sigma_1)(\sigma_2\sigma_3) = v\sigma_1(\sigma_2\sigma_3)$$

Definition — Support of Substitution 103

The *support* of a substitution σ is the set of variables x for which $x\sigma \neq x$. A substitution has a *finite support* if its support set is finite.

Proposition 104

The composition of two substitutions with a finite support has again a finite support.

Proof: trivial

Remark: We are typically interested in substitutions with finite support.

Notation: Let $\{x_1 \ldots, x_n\}$ be the finite support of substitution σ. Moreover, assume that $x_i\sigma = t_i$ (for $1 \leq i \leq n$). Then, our notation for σ is: $\{x_1/t_1, \ldots, x_n/t_n\}$.

Proposition **105**

Let $\sigma_1 = \{x_1/t_1, \ldots, x_n/t_n\}$ *and* $\sigma_2 = \{y_1/u_1, \ldots, y_k/u_k\}$ *be substitutions with finite support. The composition* $\sigma_1\sigma_2$ *has notation* $\{x_1/(t_1\sigma_2), \ldots, x_n/(t_n\sigma_2), z_1/(z_1\sigma_2), \ldots, z_m/(z_m\sigma_2)\}$, *where* $z_1, \ldots, z_m$ *are those variables* y_i *that are not amongst the* x_j. *(Trivial entries* x/x *are always deleted).*

Example — Substitution **106**

$\sigma_1 = \{x/f(x,y), y/h(a), z/g(c,h(x))\}$
$\sigma_2 = \{x/b, y/g(a,x), w/z\}$

Exercise:

...implement substitutions and substitution composition yourself

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Definition — σ_x **107**

Let σ be a substitution. By σ_x we mean that substitution such that for any variable y

$$y\sigma_x = \begin{cases} y\sigma & \text{if } y \neq x \\ x & \text{if } y = x \end{cases}$$

That is, σ_x is like σ but does not change the variable x.

Definition — Substitution lifted to Formulas 108

Let σ be a substitution. *Substitution* is *lifted to formulas* as follows:

- $\top\sigma = \top$, $\bot\sigma = \bot$, $(A(t_1, \ldots, t_n))\sigma = A(t_1\sigma, \ldots, t_n\sigma)$
- $[\neg X]\sigma = \neg[X\sigma]$
- $(X \circ Y)\sigma = (X\sigma \circ Y\sigma)$ for any connective $\circ$
- $[(\forall x)\phi]\sigma = (\forall x)[\phi\sigma_x]$
- $[(\exists x)\phi]\sigma = (\exists x)[\phi\sigma_x]$

Example — Substitution lifted to Formulas 109

Let $\sigma = \{x/a, y/b\}$. What is $[(\forall x)R(x,y) \supset (\exists x)R(x,y)]\sigma$?

Definition — Substitution is free for a Formula **110**

Let σ be a substitution. We define

- σ *is free* for all atomic formulas A
- σ *is free* for $\neg X$ if σ is free for X
- σ *is free* for $(X \circ Y)$ if σ is free for X and σ is free for Y
- σ *is free* for $(\forall x)\phi$ and $(\exists x)\phi$ provided: σ_x is free for ϕ, and if y is a free variable of ϕ other than x then $y\sigma$ does not contain x.

Note: The intention of this definition is to avoid variable capture. Implementing substitution without variable capturing effectively is non-trivial.

Example **111**

. . . on blackboard . . .

Definition — Substitution lifted to Formulas (alternativ) 112

Let σ be a substitution. *Substitution* is *lifted to formulas* as follows:

- $\top\sigma = \top$, $\bot\sigma = \bot$, $(A(t_1, \ldots, t_n))\sigma = A(t_1\sigma, \ldots, t_n\sigma)$
- $[\neg X]\sigma = \neg[X\sigma]$
- $(X \circ Y)\sigma = (X\sigma \circ Y\sigma)$ for any connective $\circ$
- $[(\forall x)\phi]\sigma = (\forall v)[[\phi\{x/v\}]\sigma]$ with v fresh variable
- $[(\exists x)\phi]\sigma = (\exists v)[[\phi\{x/v\}]\sigma]$ with v fresh variable

Theorem **113**

Suppose the substitution σ is free for the formula Z, and the substitution τ is free for $Z\sigma$. Then $(Z\sigma)\tau = Z(\sigma\tau)$.

Proof: By structural induction on X.
- $Z = A$ atomic: trivial
- $Z = \neg X$: similar to $Z = (X \circ Y)$. - $Z = (X \circ Y)$: σ is free for X and Y since it is free for $(X \circ Y)$. Similarly, τ is free for $X\sigma$ and $Y\sigma$ since it is free for $(X \circ Y)\sigma$. By induction $(X\sigma)\tau = X(\sigma\tau)$ and $(Y\sigma)\tau = Y(\sigma\tau)$. Hence, $((X \circ Y)\sigma)\tau = (X\sigma \circ Y\sigma)\tau = (X\sigma)\tau \circ (Y\sigma)\tau = (X \circ Y)(\sigma\tau)$
- $Z = (\forall x)\phi$: σ_x is free for ϕ since σ is free for ϕ. τ is free for $\phi\sigma_x$ since τ is free for $[(\forall x)\phi]\sigma = (\forall x)[\phi\sigma_x]$. By induction hypothesis $(\phi\sigma_x)\tau_x = \phi(\sigma_x\tau_x)$. It is easy to verify that $\phi(\sigma_x\tau_x) = \phi(\sigma\tau)_x$. Putting things together we have: $([(\forall x)\phi]\sigma)\tau = ((\forall x)[\phi\sigma_x])\tau = (\forall x)[(\phi\sigma_x)\tau_x] = (\forall x)[\phi(\sigma_x\tau_x)] = (\forall x)[\phi(\sigma\tau)_x] = [(\forall x)\phi](\sigma\tau)_x$
- $Z = (\exists x)\phi$: similar to above

Proposition 114

Let σ and τ be substitutions that agree on the variables of term t. Then $t\sigma = t\tau$.

Proof: ... exercise ...

Proposition 115

Let σ and τ be substitutions that agree on the variables of formula ϕ. Then $\phi\sigma = \phi\tau$.

Proof: ... exercise ...

First-Order Logic: Semantics

First-order semantical structures are more complicated than what we have seen for the propositional case:

- as before we are interested in a mapping of formulas to $\{\mathtt{t}, \mathtt{f}\}$
- but, for doing so we also need to give a meaning to terms.
- in particular we need to say what objects the quantifiers quantify over
- new notion: **domain**
- moreover, we may have free variables in terms and formulas and we need to specify their meaning
- new notion: **assignment**

Definition — Model / Model Structure 116

A *model (model structure)* for the first-order language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is a pair $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ where:

- $\mathbf{D}$ is a nonempty set of objects, called the *domain* of $\mathbf{M}$
- $\mathbf{I}$ is a mapping, called *interpretation*, that associates
  - to every constant symbol $c \in \mathbf{C}$ some object $c^{\mathbf{I}} \in \mathbf{D}$
  - to every n-ary function symbol $f \in \mathbf{F}$ some n-ary function $f^{\mathbf{I}} : \mathbf{D}^n \longrightarrow \mathbf{D}$
  - to every n-ary relation symbol $P \in \mathbf{R}$ some n-ary relation $P^{\mathbf{I}} \subseteq D^n$

Definition — Assignment **117**

An *assignment* in a model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ is a mapping $\mathbf{A} : \mathbf{V} \longrightarrow \mathbf{D}$ from the set of variables to the set of objects. The image of variable v under mapping $\mathbf{A}$ is denoted by $v^{\mathbf{A}}$.

Note:

With a model and an assignment for this model for a language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ we are sufficiently equipped to calculate values for terms . . . see next slide

Definition — Denotation of Terms 118

Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ be a model for the language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$, and let $\mathbf{A}$ be an assignment in this model. To each term t of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ we associate a *value* $t^{\mathbf{I},\mathbf{A}}$ in $\mathbf{D}$ (also called the *denotation* of t in $\mathbf{M}$) as follows:

- $c^{\mathbf{I},\mathbf{A}} = c^{\mathbf{I}}$ for all constant symbols $c \in \mathbf{C}$
- $v^{\mathbf{I},\mathbf{A}} = v^{\mathbf{A}}$ for all variable symbols $v \in \mathbf{V}$
- $[f(t_1, \dots, t_n)]^{\mathbf{I},\mathbf{A}} = f^{\mathbf{I}}(t_1^{\mathbf{I},\mathbf{A}}, \dots, t_n^{\mathbf{I},\mathbf{A}})$ for all function symbols $f \in \mathbf{F}$ and terms t_i of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$

Note: This recursive definition gives a value $d \in \mathbf{D}$ to each term t of language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$. If t is closed then the value d does not depend on the assignment A; in this case we may write $t^{\mathbf{I}}$ instead of $t^{\mathbf{I},\mathbf{A}}$.

Example — Evaluation of Terms 119

Suppose the language **L** has a constant symbol 0, a one-place function symbol s, and a two-place function symbol $+$ (which is used in infix notation below).
$s(s(0) + s(x))$ and $s(x + s(x + s(0)))$ are terms of **L**. We will now evaluate these terms with respect to different models $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ and assignments $\mathbf{A}$ . . . on blackboard . . .

1. $D = \{0, 1, 2, 3, \ldots\}$, $0^{\mathbf{I}} = 0$, $s^{\mathbf{I}}$ is the successor function, $+$ is the addition operation. Moreover, let $x^{\mathbf{A}} = 3$.
2. D is the collection of all words over alphabet $\{a, b\}$, $0^{\mathbf{I}} = a$, $s^{\mathbf{I}}$ is the function that appends word a to its argument, $+$ is concatenation. Moreover, let $x^{\mathbf{A}} = aaa$
3. $D = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $0^{\mathbf{I}} = 1$, $s^{\mathbf{I}}$ is the predecessor function, $+$ is the subtraction operation. Moreover, let $x^{\mathbf{A}} = 3$.

First-Order Logic: Semantics

Definition — x-variant **120**

Let x be a variable. The assignment $\mathbf{B}$ in the model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ is an *x-variant* of assignment $\mathbf{A}$ provided that $v^{\mathbf{B}} = v^{\mathbf{A}}$ for all variables $v \neq x$.

Definition — Truth Value of Formulas 121

Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ be a model of language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$, and let $\mathbf{A}$ be an assignment in this model. To each formula Φ of $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$, we associate a *truth value* $\Phi^{\mathbf{I},\mathbf{A}}$ as follows:

- $[P(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ iff $\langle t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}} \rangle \in P^{\mathbf{I}}$
  $\top^{\mathbf{I},\mathbf{A}} = \mathtt{t}$, $\bot^{\mathbf{I},\mathbf{A}} = \mathtt{f}$
- $[\neg X]^{\mathbf{I},\mathbf{A}} = \neg[X^{\mathbf{I},\mathbf{A}}]$
- $[X \circ Y]^{\mathbf{I},\mathbf{A}} = X^{\mathbf{I},\mathbf{A}} \circ Y^{\mathbf{I},\mathbf{A}}$ for each logical connective $\circ$
- $[(\forall x)\Phi]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ iff $\Phi^{\mathbf{I},\mathbf{B}} = \mathtt{t}$ for every assignment $\mathbf{B}$ that is an x-variant of $\mathbf{A}$
- $[(\exists x)\Phi]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ iff $\Phi^{\mathbf{I},\mathbf{B}} = \mathtt{t}$ for some assignment $\mathbf{B}$ that is an x-variant of $\mathbf{A}$

Note: If Φ is closed then the truth value of Φ does not depend on the assignment A; in this case we may write $\Phi^{\mathbf{I}}$ instead of $\Phi^{\mathbf{I},\mathbf{A}}$.

Definition — Validity and Satisfiability **122**

A formula Φ of $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ is *true in the model* $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ for $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ provided that $\Phi^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ for all assignments $\mathbf{A}$ (Notation: $\mathbf{M} \models_f \Phi$).

Φ is called *valid* if Φ is true in all models for language $\mathbf{L}(\mathbf{R},\mathbf{F},\mathbf{C})$ (Notation: $\models_f \Phi$).

A set S of formulas is *satisfiable in model* $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$, provided that there is some assignment $\mathbf{A}$ (called *satisfying assignment*) such that $\Phi^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ for all $\Phi \in S$.

S is *satisfiable* if it is satisfiable in some model.

Example 123

Suppose we have a language $\mathbf{L}$ with a two-place relation symbol R and a two-place function symbol $\oplus$. Moreover, suppose we have a model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$.

- Let Φ be the formula $(\exists y)R(x, y \oplus y)$. Suppose $\mathbf{D} = \{1, 2, 3, \ldots\}$, $\oplus^{\mathbf{I}}$ is addition, and $R^{\mathbf{I}}$ the equality relation. Show/verify that $\Phi^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ iff $x^{\mathbf{A}}$ is even.
- Let Φ be the formula $(\forall x)(\forall y)(\exists z)R(x \oplus y, z)$. Let $\mathbf{D}$ and $\oplus^{\mathbf{I}}$ as above, and let $R^{\mathbf{I}}$ be the greater-than relation. Show/verify that Φ is true in $\mathbf{M}$.
- Same as above except that $R^{\mathbf{I}}$ is the grater-than-by-4-or-more-relation. Show that Φ is not true in $\mathbf{M}$ and hence not valid.

Example — cont'd **124**

- Let Φ be $(\forall x)(\forall y)\{R(x,y) \supset (\exists z)[R(x,z) \wedge R(z,y)]\}$, $\mathbf{D}$ is the set of real numbers, and $R^{\mathbf{I}}$ the greater-than relation. Show that Φ (which expresses denseness) is true in this model. If we choose $\mathbf{D}$ as the natural numbers then Φ is not valid in the model.
- Let Φ be $(\forall x)(\forall y)[R(x,y) \supset R(y,x)]$. Let $\mathbf{D} = \{7,8\}$ (non-infinite models are also fine) and $R^{\mathbf{I}}$ the relation that holds only for $\langle 7,8\rangle$. Show that Φ is not true in this model.
- Let Φ be $[(\forall x)(\forall y)R(x,y)] \supset [(\forall x)(\exists y)R(x,y)]$. Show that Φ is valid, that is, true in all models $\mathbf{M} = \langle \mathbf{D}, \mathbf{I}\rangle$.

Proposition 125

Suppose t is a closed term, Φ is a formula of the first-order language $\mathbf{L}$, and $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ is a model for $\mathbf{L}$. Let x be a variable, and let $\mathbf{A}$ be any assignment such that $x^{\mathbf{A}} = t^{\mathbf{I}}$. Then $[\Phi\{x/t\}]^{\mathbf{I},\mathbf{A}} = \Phi^{\mathbf{I},\mathbf{A}}$.
More generally, if $\mathbf{B}$ is any x-variant of $\mathbf{A}$ then $[\Phi\{x/t\}]^{\mathbf{I},\mathbf{B}} = \Phi^{\mathbf{I},\mathbf{A}}$.

Proof: ... exercise ...

Proposition 126

Suppose $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ *is a model for* $\mathbf{L}$, Φ *is a formula in this language,* $\mathbf{A}$ *is an assignment in* $\mathbf{M}$ *and* σ *is a substitution that is free for* Φ. *Moreover, let* $\mathbf{B}$ *be the assignment such that* $v^{\mathbf{B}} = (v\sigma)^{\mathbf{I},\mathbf{A}}$. *Then* $\Phi^{\mathbf{I},\mathbf{B}} = (\Phi\sigma)^{\mathbf{I},\mathbf{A}}$.

Proof:

First we show that $t^{\mathbf{I},\mathbf{B}} = (t\sigma)^{\mathbf{I},\mathbf{A}}$ for any term t of $\mathbf{L}$ ...exercise ...
Now we apply structural induction Φ.
- $\Phi := \bot$ or $\Phi := \top$: trivial
- $\Phi := P(t_1, \ldots, t_n n)$:

$[P(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{B}} = \mathtt{t}$ iff $\langle t_1^{\mathbf{I},\mathbf{B}}, \ldots, t_n^{\mathbf{I},\mathbf{B}} \rangle \in P^{\mathbf{I}}$.
$([P(t_1, \ldots, t_n)]\sigma)^{\mathbf{I},\mathbf{A}} = \mathtt{t}$ iff $\langle (t_1\sigma)^{\mathbf{I},\mathbf{A}}, \ldots, (t_n\sigma)^{\mathbf{I},\mathbf{A}} \rangle \in P^{\mathbf{I}}$. Now apply the above result on terms.

Proof (cont'd):

- $\Phi := \neg X$: similar to the case for $[X \circ Y]$ below.
- $\Phi := [X \circ Y]$:

By induction hypothesis $X^{\mathbf{I},\mathbf{B}} = (X\sigma)^{\mathbf{I},\mathbf{A}}$ and $Y^{\mathbf{I},\mathbf{B}} = (Y\sigma)^{\mathbf{I},\mathbf{A}}$ (note that the preconditions are met). Hence,
$[X \circ Y]^{\mathbf{I},\mathbf{B}} = X^{\mathbf{I},\mathbf{B}} \circ Y^{\mathbf{I},\mathbf{B}} = (X\sigma)^{\mathbf{I},\mathbf{A}} \circ (Y\sigma)^{\mathbf{I},\mathbf{A}} = [(X\sigma) \circ (Y\sigma)]^{\mathbf{I},\mathbf{A}}$.

- $\Phi := (\exists x)\varphi$: Suppose $[[(\exists x)\varphi]\sigma]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$. We show $[(\exists x)\varphi]^{\mathbf{I},\mathbf{B}} = \mathtt{t}$.

We have $[[(\exists x)\varphi]\sigma]^{\mathbf{I},\mathbf{A}} = [(\exists x)[\varphi\sigma_x]]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$. Then for some x-variant $\mathbf{A}'$ of $\mathbf{A}$, $[\varphi\sigma_x]^{\mathbf{I},\mathbf{A}'} = \mathtt{t}$. We want to apply the induction hypothesis and need to make sure that the preconditions are met: Let $\mathbf{B}'$ be the assignment such that $v^{\mathbf{B}'} = (v\sigma_x)^{\mathbf{I},\mathbf{A}'}$. σ_x is free for φ, since σ is free for $(\exists x)\varphi$. Thus, the induction hypothesis is applicable and we get $[\varphi\sigma_x]^{\mathbf{I},\mathbf{A}'} = \varphi^{\mathbf{I},\mathbf{B}'}$. Hence, $\varphi^{\mathbf{I},\mathbf{B}'} = \mathtt{t}$. Provided we can show that $\mathbf{B}'$ is an x-variant of $\mathbf{B}$ (see below) we thus get $[(\exists x)\varphi]^{\mathbf{I},\mathbf{B}} = \mathtt{t}$.

Why is $\mathbf{B}'$ is an x-variant of $\mathbf{B}$: Let $v \neq x$, we need to show $v^{\mathbf{B}'} = v^{\mathbf{B}}$.
$v^{\mathbf{B}'} = (v\sigma_x)^{\mathbf{I},\mathbf{A}'} \stackrel{(v \neq x)}{=} (v\sigma)^{\mathbf{I},\mathbf{A}'} =^* (v\sigma)^{\mathbf{I},\mathbf{A}} = v^{\mathbf{B}}$ ($*$: since $v\sigma$ cannot contain x and since $\mathbf{A}$ and $\mathbf{A}'$ agree on all variables except x). We still need to show: $[(\exists x)\varphi]^{\mathbf{I},\mathbf{B}} = \mathtt{t}$ implies $[[(\exists x)\varphi]\sigma]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$. . . . exercise . . .

- $\Phi := (\forall x)\varphi$: similar . . . exercise . . .

Exercises:

1. Show Φ is true in model $\mathbf{M}$ if and only if $(\forall x)\Phi$ is true in $\mathbf{M}$.
2. Show X is valid if and only if $\{\neg X\}$ is not satisfiable.
3. Show $X \equiv Y$ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ if and only if $X^{\mathbf{I},\mathbf{A}} = Y^{\mathbf{I},\mathbf{A}}$ for all assignments $\mathbf{A}$.
4. Let $\mathbf{L}$ and $\mathbf{L}'$ be first-order languages, with every constant, function, and relation symbol of $\mathbf{L}$ also being a symbol of $\mathbf{L}'$. Let S be a set of formulas of $\mathbf{L}$. Show S is satisfiable in some model for the language $\mathbf{L}$ if and only if S is satisfiable in some model for the language $\mathbf{L}'$.
5. Write a sentence Φ involving the two-place relation symbol R such that:
   - **5.1** Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ iff $R^{\mathbf{I}}$ is a reflexive relation on $\mathbf{D}$
   - **5.2** Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ iff $R^{\mathbf{I}}$ is a symmetric relation on $\mathbf{D}$
   - **5.3** Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ iff $R^{\mathbf{I}}$ is a transitive relation on $\mathbf{D}$
   - **5.4** Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ iff $R^{\mathbf{I}}$ is an equivalence relation on $\mathbf{D}$

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Exercises (cont'd):

6. Write a sentence Φ involving the two-place relation symbol R having both the following properties:
   - **6.1** Φ is not true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ with a one element domain $\mathbf{D}$.
   - **6.2** If $\mathbf{D}$ has two or more elements, there is some interpretation $\mathbf{I}$ such that Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$.
7. Write a sentence Φ involving the two-place relation symbol R having both the following properties:
   - **7.1** Φ is not true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ with a one or two element domain $\mathbf{D}$.
   - **7.2** If $\mathbf{D}$ has three or more elements, there is some interpretation $\mathbf{I}$ such that Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$.
8. Write a sentence Φ involving the two-place relation symbol R having both the following properties:
   - **8.1** Φ is not true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ with a finite domain $\mathbf{D}$.
   - **8.2** If $\mathbf{D}$ is infinite, there is some interpretation $\mathbf{I}$ such that Φ is true in $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$.

Exercises (cont'd):

9. In the following P and R are relation symbols and c is a constant symbol. Demonstrate the validity of the following:
   - **9.1** $(\forall x)P(x) \supset P(c)$
   - **9.2** $(\exists x)[P(x) \supset (\forall x)P(x)]$
   - **9.3** $(\exists y)(\forall x)R(x, y) \supset (\forall x)(\exists y)R(x, y)$
   - **9.4** $(\forall x)\Phi \equiv \neg(\exists x)\neg\Phi$
   - **9.5** Determine the status of $(\forall x)(\exists y)R(x, y) \supset (\exists y)(\forall x)R(x, y)$

First-Order Logic: Herbrand Models

- Did you note the similarity between assignments and substitutions?

$$\sigma : \mathbf{V} \longrightarrow \mathbf{T}$$

$$\mathbf{A} : \mathbf{V} \longrightarrow \mathbf{D}$$

- The domain **D** can be anything, so why not considering the case where **D** is the set of closed terms?

Definition — Herbrand Model **127**

A model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ for the language **L** is a *Herbrand model* if:

1. **D** is exactly the set of closed terms of **L**.
2. For each closed term t, $t^{\mathbf{I}} = t$.

Remarks:

- Herbrand models will play a special role in completeness proofs.
- Note the that in Herbrand models assignments are substitutions and vice versa, hence both $\Phi^{\mathbf{I},\mathbf{A}}$ and $\Phi\mathbf{A}$ (where $\mathbf{A}$ is used as a substitution) are meaningful.
- We will get some special (simple) variants of earlier propositions.
- Moreover, note that (for each closed formula Φ) $\Phi\mathbf{A}$ can never contain free variables. Hence the truth value for $\Phi\mathbf{A}$ depends only on the interpretation of the model (similar for terms).

Proposition 128

Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ be a Herbrand model for the language $\mathbf{L}$. For any term t of $\mathbf{L}$, not necessarily closed, we have $t^{\mathbf{I},\mathbf{A}} = (t\mathbf{A})^{\mathbf{I}}$.

Proof: Structural induction on t.
- $t := v$: $v^{\mathbf{I},\mathbf{A}} = v^{\mathbf{A}} = v\mathbf{A} =^{*} (v\mathbf{A})^{\mathbf{I}}$ (*: since $\mathbf{I}$ is the identity on closed terms)
- $t := c$: $c^{\mathbf{I},\mathbf{A}} = c^{\mathbf{I}} = (c\mathbf{A})^{\mathbf{I}}$
- $t := f(t_1, \ldots, t_n)$: $[f(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = f^{\mathbf{I}}(t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}}) \overset{\text{ind.hyp.}}{=}$
$f^{\mathbf{I}}((t_n\mathbf{A})^{\mathbf{I}}, \ldots, (t_n\mathbf{A})^{\mathbf{I}}) = [f(t_n\mathbf{A}, \ldots, t_n\mathbf{A})]^{\mathbf{I}} = [f(t_1, \ldots, t_n)\mathbf{A}]^{\mathbf{I}}$

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Proposition **129**

Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ *be a Herbrand model for the language* $\mathbf{L}$*. For a formula* Φ *of* $\mathbf{L}$ *we have* $\Phi^{\mathbf{I},\mathbf{A}} = (\Phi\mathbf{A})^{\mathbf{I}}$*.*

Proof: . . . exercise . . .

Proposition **130**

Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ *be a Herbrand model for the language* $\mathbf{L}$*. Moreover, let* Φ *be a formula of* $\mathbf{L}$*. We have:*

1. $(\forall x)\Phi$ *is true in* $\mathbf{M}$ *iff* $\Phi\{x/d\}$ *is true in* $\mathbf{M}$ *for every* $d \in \mathbf{D}$*.*
2. $(\exists x)\Phi$ *is true in* $\mathbf{M}$ *iff* $\Phi\{x/d\}$ *is true in* $\mathbf{M}$ *for some* $d \in \mathbf{D}$*.*

Proof: . . . exercise . . .

First-Order Logic: Uniform Notation

- Formulas that act universally are called γ-formulas
- Formulas that act existentially are called δ-formulas

Definition — γ- and δ-Formulas and Instances 131

Universal		Existential	
γ	$\gamma(t)$	δ	$\delta(t)$
$(\forall x)\Phi$	$\Phi\{x/t\}$	$(\exists x)\Phi$	$\Phi\{x/t\}$
$\neg(\exists x)\Phi$	$\neg\Phi\{x/t\}$	$\neg(\forall x)\Phi$	$\neg\Phi\{x/t\}$

If y is a variable that is new to γ and δ then both

- $\gamma \equiv (\forall y)\gamma(y)$
- $\delta \equiv (\forall y)\delta(y)$

are valid.

Definition — Rank of a Formula 132

(example of a recursive definition)

The *rank* $r(X)$ of a first-order formula X is defined as follows:

$$r(A) = r(\neg A) = 0 \quad \text{for atomic } A$$

$$r(\top) = r(\bot) = 0$$

$$r(\neg\top) = r(\neg\bot) = 1$$

$$r(\neg\neg Z) = r(Z) + 1$$

$$r(\alpha) = r(\alpha_1) + r(\alpha_2) + 1$$

$$r(\beta) = r(\beta_1) + r(\beta_2) + 1$$

$$r(\gamma) = r(\gamma(x)) + 1$$

$$r(\delta) = r(\delta(x)) + 1$$

Proposition — Satisfiable γ- and δ-formulas 133

Let S be a set of sentences (closed formulas), and γ and δ be sentences.

1. *If $S \cup \{\gamma\}$ is satisfiable, so is $S \cup \{\gamma, \gamma(t)\}$ for any closed term t.*
2. *If $S \cup \{\delta\}$ is satisfiable, so is $S \cup \{\delta, \delta(p)\}$ for any constant symbol p that is new to S and δ.*

Proof: (we work with closed formulas, this simplifies things)
(1.): Suppose $S \cup \{\gamma\}$ is satisfiable in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$. We show that $S \cup \{\gamma, \gamma(t)\}$ is also satisfiable in $\mathbf{M}$. γ is true in $\mathbf{M}$, hence $(\forall x)\gamma(x)$ (where variable x is new to γ) is true in $\mathbf{M}$. That means, for every assignment $\mathbf{A}$, $[\gamma(x)]^{\mathbf{I},\mathbf{A}}$ is true. In particular consider $\mathbf{A}$ with $x^{\mathbf{A}} = t^{\mathbf{I}}$. We have $[\gamma(t)]^{\mathbf{I},\mathbf{A}} =^{\text{def.}} [\gamma\{x/t\}]^{\mathbf{I},\mathbf{A}} =^{*} [\gamma(x)]^{\mathbf{I},\mathbf{A}}$ (*: by proposition 125). Hence, $S \cup \{\gamma, \gamma(t)\}$ is satisfiable.

First-Order Logic: Uniform Notation

Proof (continued): (2.): Suppose $S \cup \{\delta\}$ is satisfiable in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$, and p is a constant symbol new to S and δ. We will show that $S \cup \{\delta, \delta(p)\}$ is satisfiable, though not necessarily in the model $\mathbf{M}$.
δ is true in $\mathbf{M}$, hence so is $(\exists x)\delta(x)$ (where x is new to δ). Thus, $[\delta(x)]^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$. If we had $x^{\mathbf{A}} = p^{\mathbf{I}}$, we could complete the argument just as we did in (1.). But there is no reason to assume this is the case. However, we can construct a new model $\mathbf{M}^* = \langle \mathbf{D}, \mathbf{J} \rangle$ having the same domain as before. The interpretation $\mathbf{J}$ is chosen identical to $\mathbf{I}$ except for constant symbol p: we set $p^{\mathbf{J}} = x^{\mathbf{A}}$. Note that all sentences not containing p will evaluate exactly the same in $\mathbf{M}$ and $\mathbf{M}^*$. Thus, $S \cup \{\delta\}$ is satisfiable in $\mathbf{M}^*$ and $[\delta(x)]^{\mathbf{J},\mathbf{A}}$ is true for $\mathbf{A}$. Since $y^{\mathbf{A}} = p^{\mathbf{J}}$, we have $[\delta(p)]^{\mathbf{J},\mathbf{A}} =^{\text{def.}} [\delta\{x/p\}]^{\mathbf{J},\mathbf{A}} =^{*} [\delta(x)]^{\mathbf{J},\mathbf{A}} = t$ (*: by previous proposition, see p. 31). Hence, $S \cup \{\delta, \delta(p)\}$ is satisfiable.

In Herbrand models things become much simpler:

Proposition 134

Let $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ *be a Herbrand model for language* $\mathbf{L}$. *Moreover, let* γ *and* δ *be formulas of* $\mathbf{L}$.

1. γ *is true in* $\mathbf{M}$ *iff* $\gamma(d)$ *is true in* $\mathbf{M}$ *for every* $d \in \mathbf{D}$
2. δ *is true in* $\mathbf{M}$ *iff* $\delta(d)$ *is true in* $\mathbf{M}$ *for some* $d \in \mathbf{D}$

(Remember that each $d \in \mathbf{D}$ *is a closed term).*

Proof: ...exercise ...

Theorem — First-Order Structural Induction 135

Every formula of a first-order language **L** *has property* **Q**, *provided:*

- **Basis step:** *Every atomic formula and its negation has property* **Q**
- **Induction steps:**
  - *If X has property* **Q**, *so does $\neg\neg X$.*
  - *If α_1 and α_2 have property* **Q**, *so does α.*
  - *If β_1 and β_2 have property* **Q**, *so does β.*
  - *If $\gamma(t)$ has property* **Q** *for each term t, then γ has property* **Q**.
  - *If $\delta(t)$ has property* **Q** *for each term t, then δ has property* **Q**.

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Theorem — First-Order Structural Recursion 136

There is one, and only one, function f defined on the set of formulas of **L** *such that:*

- **Basis step:** *The value of f is specified explicitly on atomic formulas and their negations.*
- **Induction steps:**
  - *The value of f for* $\neg\neg X$ *is specified in terms of the value of f for* X.
  - *The value of f for* α *is specified in terms of the values of f for* α_1 *and* α_2.
  - *The value of f for* β *is specified in terms of the values of f for* β_1 *and* β_2.
  - *The value of f for* γ *is specified in terms of the values of f for* $\gamma(t)$.
  - *The value of f for* δ *is specified in terms of the values of f for* $\delta(t)$.

Exercises:

1. Suppose x does not occur in sentence γ. Let t be a closed term. Prove that $\gamma(t) = \gamma(x)\{x/t\}$ and that $\delta(t) = \delta(x)\{x/t\}$

Definition — First-Order Hintikka Set **137**

Let **L** be a first-order language. A set H of sentences of **L** is called a *first-order Hintikka set* (with respect to **L**), provided that:

1. For all propositional letters A, not both $A \in H$ and $\neg A \in H$
2. $\perp \notin H$ and $\neg\top \notin H$
3. if $\neg\neg Z \in H$ then $Z \in H$
4. if $\alpha \in H$ then $\alpha_1 \in H$ and $\alpha_2 \in H$
5. if $\beta \in H$ then $\beta_1 \in H$ or $\beta_2 \in H$
6. if $\gamma \in H$ then $\gamma(t) \in H$ for every closed term t of **L**
7. if $\delta \in H$ then $\delta(t) \in H$ for some closed term t of **L**

Remarks:

- The empty set is a first-order Hintikka set.
- Many finite sets are first-order Hintikka sets (exercise: examples).
- If **L** has infinitely many closed terms (all that is needed for this are a one-place function symbol f and a constant symbol c), then any Hintikka set containing a γ-formula must be infinite.
- We now present Hintikka's Lemma for first-order.

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Proposition — Hintikka's Lemma (First-Order) 138

Suppose $\mathbf{L}$ *is a language with a nonempty set of closed terms. If* H *is a first-order Hintikka set with respect to* $\mathbf{L}$*, then* H *is satisfiable in a Herbrand model.*

Proof: We construct a Herbrand model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ which satisfies H.

Let $\mathbf{D}$ be the (nonempty) set of closed terms of $\mathbf{L}$. We choose $\mathbf{I}$ so that:

- for all constant symbols c: $c^{\mathbf{I}} = c$
- for any n-ary function symbol f of $\mathbf{L}$:
  $f^{\mathbf{I}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
  We can now easily verify that: $t^{\mathbf{I}} = t$ for all closed terms t of $\mathbf{L}$
- for any n-ary relation symbol R of $\mathbf{L}$: $R^{\mathbf{I}}$ holds for $\langle t_1, \ldots, t_n \rangle$ if the sentence $R(t_1, \ldots, t_n)$ is a member of H.

Proof (cont'd): We now show (by structural induction): for each sentence (closed formula) X of $\mathbf{L}$, $X \in H$ implies X is true in M.
- $X := R(t_1, \ldots, t_n)$: We need to show that $[R(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathtt{t}$, that is, $\langle t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}} \rangle \in R^{\mathbf{I}}$. Each t_i must be closed, hence $t_i^{\mathbf{I},\mathbf{A}} = t_i^{\mathbf{I}} = t_i$ by construction of $\mathbf{I}$. Hence, we need to show that $\langle t_1, \ldots, t_n \rangle \in R^{\mathbf{I}}$. This is by construction of $\mathbf{I}$.
- $X := \bot$ and $X := \neg\top$: trivial ... exercise ...
- $X := \top$ and $X := \neg\bot$: trivial ... exercise ...
- $X := \neg R(t_1, \ldots, t_n)$: straightforward ... exercise ...
- $X := \neg\neg Z$: straightforward ... exercise ...
- $X := \alpha$: straightforward ... exercise ...
- $X := \beta$: straightforward ... exercise ...
- $X := \delta$: similar to below ... exercise ...
- $X := \gamma$: Since $\gamma \in H$, we have $\gamma(t) \in H$ for every closed term t. By induction hypothesis $\gamma(t)$ is true in $\mathbf{M}$ for every term $t \in \mathbf{D}$. Then, γ is true in $\mathbf{M}$ by proposition 134.

Exercises:

1. Suppose H is a first-order Hintikka set with respect to language $\mathbf{L}$. Let X be a sentence of $\mathbf{L}$. Then not both $X \in H$ and $\neg X \in H$.
2. Prove all open exercises in the proof of the Hintikka Lemma.

First-Order Logic: Abstract Consistency

Parameters:

- assume in a proof we have established $(\exists x)\Phi$
- often we need 'uncommitted' **constant parameters** to proceed with the proof
- remember maths classroom: *Let c be such that* Φ *holds* ...
- parameters play a role in the model existence theorem
- they also play a role in first-order proof procedures: these are applied to sentences from $\mathbf{L}$, but they may subsequently generate sentences from $\mathbf{L}^{\mathbf{par}}$

Definition — Parameters **139**

Let $\mathbf{L} := \mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ be a first-order language. Let **par** be a countable set of constant symbols that is disjoint from $\mathbf{C}$. We call the members of **par** *parameters*. Moreover, we use $\mathbf{L}^{\mathbf{par}}$ as a shorthand for the language $\mathbf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C} \cup \mathbf{par})$.

First-Order Logic: Abstract Consistency

Exercise:

1. Let p be a parameter. Show that $\Phi\{x/p\}$ is valid if and only if $(\forall x)\Phi(x)$ is valid.

First-Order Abstract Consistency:

- various versions do exist in the literature
- from an abstract perspective they are very similar and used for the same purposes
- however, they may differ regarding small and subtle details

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Definition — First-Order Consistency Property 140

Let $\mathbf{L}$ be a first-order language and let $\mathbf{L}^{\mathbf{par}}$ be an extension of $\mathbf{L}$ containing an infinite set of additional parameters. Let $\mathcal{C}$ be a collection of sets of sentences of $\mathbf{L}^{\mathbf{par}}$. $\mathcal{C}$ is a *first-order consistency property (with respect to to* $\mathbf{L}$*)* if for each $S \in \mathcal{C}$:

- ∇_c: for any $A \in \mathbf{L}^{\mathbf{par}}$, not both $A \in S$ and $\neg A \in S$
- $\nabla_\perp$: $\perp \notin S$ and $\neg\top \notin S$
- $\nabla_{\neg\neg}$: if $\neg\neg Z \in S$ then $S \cup \{Z\} \in \mathcal{C}$
- ∇_α: if $\alpha \in S$ then $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
- ∇_β: if $\beta \in S$ then $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$
- ∇_γ: if $\gamma \in S$ then $S \cup \{\gamma(t)\} \in \mathcal{C}$ for every closed term t of $\mathbf{L}^{\mathbf{par}}$.
- ∇_δ: if $\delta \in S$ then $S \cup \{\delta(p)\} \in \mathcal{C}$ for some parameter p of $\mathbf{L}^{\mathbf{par}}$.

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Theorem — First-Order Model Existence 141

If $\mathcal{C}$ is first-order consistence property with respect to $\mathbf{L}$, S is a set of sentences of $\mathbf{L}$ and $S \in \mathcal{C}$, then S is satisfiable. In fact S is satisfiable in a Herbrand model (but Herbrand model with respect to $\mathbf{L}^{\mathbf{par}}$).

Notes on the Proof:

- the proof will be developed on the next slides
- we first need some further definitions
- it follows the same idea as in the propositional case:
  - enlarge the abstract consistency property to one of finite character
  - then any member of such a class can be extended to maximal sets of sentences
  - these maximal sets of sentences are Hintikka sets
  - which we know are satisfiable (in a Herbrand model)

Definition — Subset Closed **142**

Let $\mathcal{C}$ be a first-order consistency property. $\mathcal{C}$ is *subset closed* if for all $S \in \mathcal{C}$ holds: if $S' \subseteq S$ then $S' \in \mathcal{C}$.

Lemma **143**

Every first-order consistency property can be extended to one that is subset closed.

Proof: . . . exercise . . . as before consider
$\mathcal{C}' = \{S' | S' \subseteq S \text{ and } \in \mathcal{C}\}$ . . .

Notes on the Proof (cont'd):

- the next step is to extend the consistency property to finite character

$$\mathcal{C}' = \{S | \text{ all finite subsets of S are in } S\mathcal{C}\}$$

- not yet possible; the problem is that we cannot ensure that the result will meet condition ∇_δ.
- what we need to adress is interchangeabilty of the use of parameters: whenever we used p we could have used q as well
- for this we slightly modify the notion of abstract consistency

Definition — Alternate First-Order Consistency Property 144

An *alternate first-order consistency property* is a collection $\mathcal{C}$ meeting the conditions of a first-order consistency property except that condition ∇_δ is replaced by

$\nabla_{\delta'}$: if $\delta \in S$ then $S \cup \{\delta(p)\} \in \mathcal{C}$ for every parameter p that is new to S.

Note:

- condition is stronger as before: lots of instances of δ-sentences can be added
- condition is weaker as before: if all parameters already occur in S then nothing can be added

Definition — Parameter Substitution **145**

A *parameter substitution* is a mapping π from the set of parameters to itself (not necessarily 1-1 or onto). We extend π to formulas (and sets of formulas) of $\mathbf{L}^{\mathbf{par}}$ as follows: $\Phi\pi$ means replace each parameter occurring in Φ by its image under π.

Lemma — $\mathcal{C}^+$ **146**

Suppose $\mathcal{C}$ is a first-order consistency property that is closed under subsets. We define a collection $\mathcal{C}^+$ as follows: $S \in \mathcal{C}^+$, provided $S_\pi \in \mathcal{C}$ for some parameter substitution π. Then:

1. *$\mathcal{C}^+$ extends $\mathcal{C}$*
2. *$\mathcal{C}^+$ is closed under subsets*
3. *$\mathcal{C}^+$ is an alternate first-order consistency property*

Proof: . . . exercise . . .

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Definition — Finite Character **147**

Let $\mathcal{C}$ be an alternate first-order consistency property. $\mathcal{C}$ is of *finite character* if we have: $S \in \mathcal{C}$ if and only if every finite subset S' of S is in $\mathcal{C}$.

Lemma **148**

Every subset closed alternate first-order consistency property can be extended to one that is of finite character.

Proof: ...exercise ...as before, consider
$\mathcal{C}' = \{S|$ all finite subsets of S are in $\mathcal{C}\}$ ...

Lemma — Limits **149**

Let $\mathcal{C}$ be a first-order consistency property of finite character. Moreover, let $S_1, S_2, S_3, \ldots$ be a sequence of members of $\mathcal{C}$, such that $S_1 \subseteq S_2 \subseteq S_3 \subseteq \ldots$. Then $\bigcup_n S_n \in \mathcal{C}$.

Proof: identical to the propositional case

Proof of the Model Existence Theorem: Suppose $\mathcal{C}$ is a first-order consistency property with respect to $\mathbf{L}$, S is a set of sentences of $\mathbf{L}$, and $S \in \mathcal{C}$. We construct a model in which the members of S are true.

First, we extend $\mathcal{C}$ into an alternate first-order consistency property $\mathcal{C}^*$; we then continue working with $\mathcal{C}^*$ (which also contains S).

Let $X_1, X_2, X_3, \ldots$ be an enumeration of all sentences of $\mathbf{L}^{\mathbf{par}}$ (this is possible since $\mathbf{L}^{\mathbf{par}}$ has a countable alphabet).

We define now a sequence of members $S_1, S_2, S_3, \ldots$ of $\mathcal{C}^*$ (each of which leaves unused an infinite set of parameters)

$$S_1 = S$$

$$S_{n+1} = \begin{cases} S_n & \text{if } S_n \cup \{X_n\} \notin \mathcal{C}^* \\ S_n \cup \{X_n\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^* \text{ and } X_n \text{ is not a } \delta\text{-sentence} \\ S_n \cup \{X_n\} & \text{if } S_n \cup \{X_n\} \cup \{\delta(p)\} \in \mathcal{C}^* \text{ and } X_n = \delta \\ & \text{where parameter } p \text{ is new to } S_n \cup \{X_n\} \end{cases}$$

Note that there must always be a new fresh parameter p.

Proof of the Model Existence Theorem (continued):

By construction, each $S_n \in \mathcal{C}^*$, and each S_n is a subset of S_{n+1}.
Hence, $H = \bigcup_n S_n$ (extending S) is in $\mathcal{C}^*$ (by Limits lemma)
H is maximal in $\mathcal{C}^*$ (i.e. if $H \subseteq K$ for some $K \in \mathcal{C}^*$ then $H = K$): like in propositional case.
H is a first-order Hintikka set with respect to $\mathbf{L}^{\mathbf{par}}$: The argument is the same as in the propositional case except for the δ-case. For this case we have added the extra clause in the construction of the sequence above.
By the Hintikka Lemma we have: H, and hence S, is satisfiable in a model that is Herbrand with respect to $\mathbf{L}^{\mathbf{par}}$.

Our main application of the Abstract Consistency Technique will be in proving completeness of calculi/proof procedure. However, the technique can also be used to prove some important theorems about first-order logic.

Theorem — First-Order Compactness **150**

Let S be a set of sentences of the first-order language **L**. *If every finite subset of S is satisfiable, so is S.*

Proof: Consider $\mathcal{C} = \{W \subseteq \text{sentences}(\mathbf{L}^{\mathbf{par}}) \mid (1)$ infinitely many parameters are new to W and (2) every finite subset of W is satisfiable$\}$

Easy to verify that $S \in \mathcal{C}$.

Easy to verify that $\mathcal{C}$ is an abstract consistency property.

Hence, S must be satisfiable by the First-Order Model Existence Theorem.

Corollary 151

Let **L** *be a first-order language. Any set S of sentences of* **L** *that is satisfiable in arbitrarily large finite models is also satisfiable in some infinite model.*

Proof: Remember from the exercises that we can construct formulas A_n expressing that there are at least n objects in the models.
Consider now $S^* = S \cup \{A_1, A_2, A_3, \ldots\}$.
Since S is satisfiable in arbitrarily large models, we must have that any finite subset of S^* is satisfiable.
Hence, by the Compactness Theorem, S^* is satisfiable.
But S^* cannot be satisfiable in any finite model.
Remark: This shows there is no sentence X that is true in any model with finite domain but false in any model with infinite domain.

Thus, 'being finite' cannot be captured in first-order logic!

Theorem — Löwenheim-Skolem 152

Let S be a set of sentences of first-order language **L**. *If S is satisfiable, then S is satisfiable in a countable model.*

Proof: Consider $\mathcal{C} = \{W \subseteq \text{sentences}(\mathbf{L}^{\mathbf{par}}) \mid (1) \text{ infinitely many parameters are new to } W \text{ and } (2) W \text{ is satisfiable}\}$

Easy to verify that $\mathcal{C}$ is an abstract consistency property.

$S \in \mathcal{C}$. Hence, by the Model Existence Theorem, S is satisfiable in a model **M** that is Herbrand with respect to $\mathbf{L}^{\mathbf{par}}$.

$\mathbf{L}^{\mathbf{par}}$ has a countable alphabet and hence also a countable set of closed terms.

The set of closed terms constitutes the domain of Herbrand model **M**.

Theorem — Herbrand Model **153**

1. *A set S of sentences of* $\mathbf{L}$ *is satisfiable if and only if it is satisfiable in a model that is Herbrand with respect to* $\mathbf{L}^{\mathbf{par}}$.
2. *A sentence X of* $\mathbf{L}$ *is valid if and only if X is true in models that are Herbrand with respect to* $\mathbf{L}^{\mathbf{par}}$.

Proof: . . . exercise . . .

Definition — Logical Consequence **154**

A sentence X is a *logical consequence* of a set of sentences S, provided X is true in every model $\mathbf{M}$ in which all the members of S are true. We then write $S \models_f X$.

Note: The above definition is restricted to sentences. It can be extended to arbitrary formulas (i.e. free variables may occur), but the we have a choice with respect to assignments:

1. $S \models_f Y$: if S is true in $\mathbf{M}$ (under every assignment $\mathbf{A}$) then Y is true in $\mathbf{M}$ (under every assignment $\mathbf{A}$).
2. $S \models_f Y$: if S is true in $\mathbf{M}$ under assignment $\mathbf{A}$ then also Y is true in $\mathbf{M}$ under assignment $\mathbf{A}$.

If we just consider sentences the two notions coincide.
The exercises below show that we do not loose expressive power by this restriction to sentences.

Definition — $\forall\Phi$ **155**

Let Φ be a formula with free variables $x_1, \ldots, x_n$. We write $\forall\Phi$ as shorthand for the formula $(\forall x_1) \ldots (\forall x_n)\Phi$.
Moreover, let S be a set of formulas. We write $\forall S$ as shorthand for $\{\forall\Phi \mid \Phi \in S\}$

Exercises:

- The following are equivalent
  1. X is true in every model in which the members of S are true.
  2. $\forall S \models_f \forall X$

Exercises (cont'd):

- Let X be a formula of $\mathbf{L}$ and S be a set of formulas of $\mathbf{L}$.
  Let $p_1, p_2, p_3, \ldots$ be parameters (not occurring in the formulas of $\mathbf{L}$).
  Let $v_1, v_2, v_3, \ldots$ be the list of variables of $\mathbf{L}$
  Let σ be the substitution $\{v_1/p_1, v_2/p_2, v_3/p_3, \ldots\}$.
  Note that $X\sigma$ and the members of $S\sigma$ are sentences of $\mathbf{L}^{\mathbf{par}}$.
  Now, the following are equivalent:
  1. For any model $\mathbf{M}$ and any assignment $\mathbf{A}$, if the members of S are true in $\mathbf{M}$ under assignment $\mathbf{A}$, then S is true in $\mathbf{M}$ under assignment $\mathbf{A}$.
  2. $S\sigma \models_f X\sigma$
- Exercise 5.10.3 in Fitting

The next theorem shows that only a finite amount of information of a set of sentences S is needed for establishing that sentence X is a consequence of S.

Theorem 156

Let X be a sentence of $\mathbf{L}$ and S be a set of sentences of $\mathbf{L}$.
$S \models_f X$ if and only if $S_0 \models_f X$ for some finite set $S_0 \subseteq S$.

Proof:

$\Leftarrow$: easy exercise ...

$\Rightarrow$: Suppose $S \models_f X$. Then $S \cup \{\neg X\}$ is not satisfiable. By the Compactness Theorem there is a finite subset S_0 of $S \cup \{\neg X\}$ that is not satisfiable. Then $S_0 \cup \{\neg X\}$ is not satisfiable (adding $\neg X$ does change unsatisfiability). Hence, $S_0 \models_f X$.

Definition — First-order Theory 157

Let **M** be a model for language **L**. The first-order *theory* of **M** is defined as

$$Th(\mathbf{M}) := \{X \text{ sentence of } \mathbf{L} \mid \mathbf{M} \models_f X\}$$

Problem of axiomatizability: For which models **M** can one write down a sentence A (or a recursively enumerable set A of sentences) such that

$$Th(\mathbf{M}) = \{X \mid A \models_f X\}$$

Example 158

- Presburger Arithmetic: **L** consists of the constant symbols 0, s (1-ary), and $+$ (2-ary). Domain of **M** are the naturals (or integers) and interpretation of the constant symbols is as usual.
  The theory of Presburger Arithmetic is decidable.
- Peano Arithmetic: **L** consists of the constant symbols 0, s (1-ary), $+$ (2-ary), and $*$ (2-ary). Domain of **M** are the naturals and interpretation of the constant symbols is as usual.
  The theory of Peano Arithmetik is undecidable — not even recursively enumerable.

Note: Choice of signature can have a huge impact on computational complexity of theories.