

# On a (quite) Universal Theorem Proving Approach and its Application in Metaphysics

**Christoph Benzmüller<sup>1</sup>**, FU Berlin & Stanford (CSLI/Cordura Hall)

jww: B. Woltzenlogel Paleo (& L. Paulson , C. Brown, G. Sutcliffe and many others!)

Current Issues in the Philosophy of Practice of Mathematics & Informatics  
CIMI, Toulouse, June 16 2016

```
>
> Prove-with-LEO2 Necessarily-there-exists-God.p
LEO2 tries to prove
=====
Gödel's Theorem T3: "Necessarily, there exists God"
thf(thmT3,conjecture,
  ( v
    @ ( mbox
      @ ( mexists_ind
        @ ^ [X: mu] :
          ( g @ X ) ) ) )).
Assumptions: D1, C, T2, D3, A5
. searching for proof .□
```

Url to movie: <http://www.christoph-benzmueller.de/papers/Movies-LEO2-Isabelle.mov>

<sup>1</sup>Supported by DFG Heisenberg Fellowship BE 2501/9-1/2

## Vision of Leibniz (1646–1716): *Calculemus!*



If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other ...: Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo ... dicere: calculemus.  
(Leibniz, 1684)



Required:  
**characteristica universalis** and **calculus ratiocinator**

## Talk Outline

- A: HOL as a Universal (Meta-)Logic via Semantic Embeddings**
- B: New Knowledge on the Ontological Argument from HOL ATPs**
- C: Reconstruction of the Inconsistency of Gödel's Axioms**
- D: Recent Technical Improvements**
- E: Other Recent Work: Free Logic & Application in Category Theory**
- (F: Other Recent Work: Zalta'a Theory of Abstract Object)

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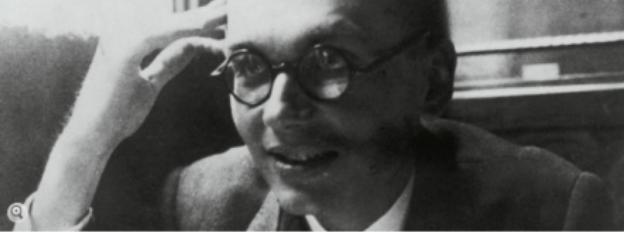
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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

## Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürtler



Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis Jahrzehntlang geheim

**Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.**

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Montag, 09.09.2013 - 12:03 Uhr

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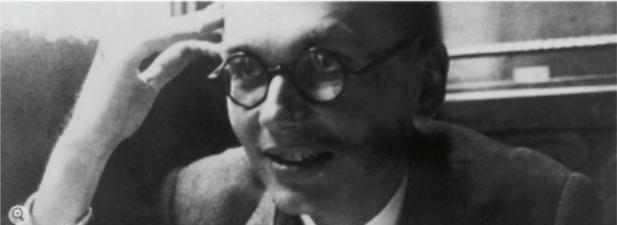
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[English Site](#) > [Germany](#) > [Science](#) > Scientists Use Computer to Mathematically Prove Gödel God Theorem

**Holy Logic: Computer Scientists 'Prove' God Exists**

By David Knight



Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

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MEDIA & CULTURE

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HOME / SCIENCE NEWS

Researchers say they used MacBook to prove Goedel's God theorem

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# **God exists, say Apple fanboy scientists**

With the help of just one MacBook, two Germans formalize a theorem that confirms the existence of God.

See more serious and funny news links at

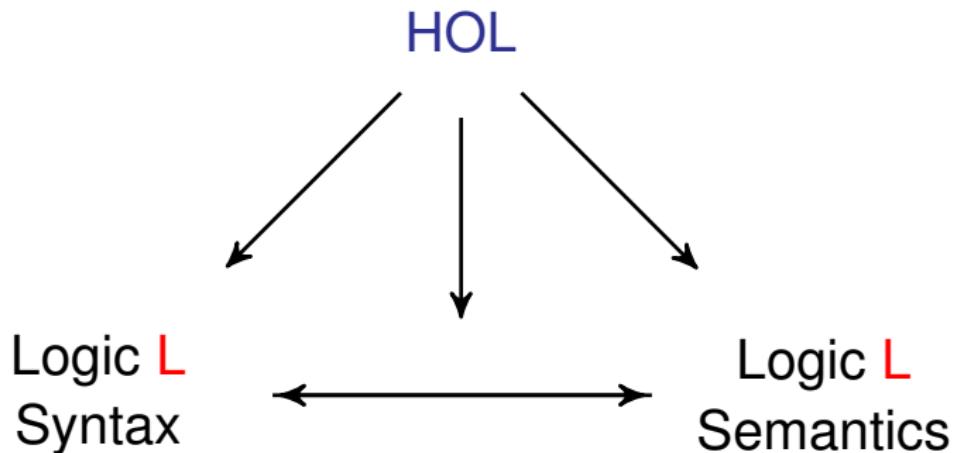
<https://github.com/FormalTheology/GödelGod/blob/master/Press/LinksToNews.md>



## Part A:

### HOL as a Universal (Meta-)Logic via Semantic Embeddings

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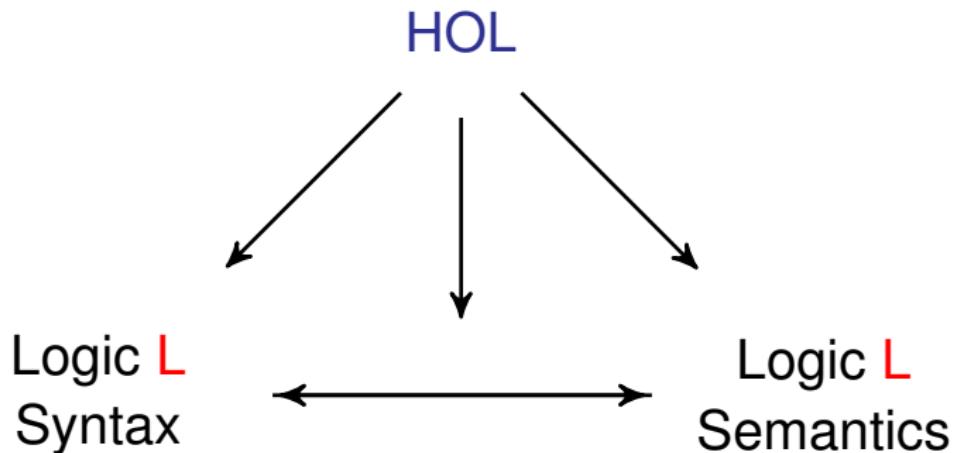


Examples for L we have already studied:

Modal Logics, Conditional Logics, Intuitionistic Logics, Access Control Logics, Nominal Logics, Multivalued Logics (SIXTEEN), Logics based on Neighborhood Semantics, (Mathematical) Fuzzy Logics, Paraconsistent Logics, Free Logic ...

Works also for (first-order & higher-order) quantifiers

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**Works also for (first-order & higher-order) quantifiers**

## Embedding Approach — Idea

HOL (meta-logic)       $\varphi ::=$  

Your-logic (object-logic)       $\psi ::=$  

Embedding of  in 

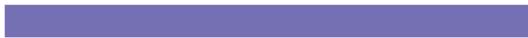
 = 

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 = 

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Embedding of meta-logical notions on  in 

*valid* = 

*satisfiable* = 

... = 

Pass this set of equations to a higher-order automated theorem prover

# Classical Higher-Order Logic (HOL)

## Simple Types

$$\alpha ::= o \mid \iota \mid \mu \mid \alpha_1 \rightarrow \alpha_2$$

HOL

$$s, t ::= c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \\ (\neg_{o \rightarrow o} s_o)_o \mid (s_o \vee_{o \rightarrow o \rightarrow o} t_o)_o \mid (\forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha t_o))_o$$

(note: binder notation  $\forall x_\alpha t_o$  as syntactic sugar for  $\forall_{(\alpha \rightarrow o) \rightarrow o} (\lambda x_\alpha t_o)$ )

HOL with Henkin semantics is (meanwhile) well understood

Origin

[Church, JSymbLog, 1940]

Henkin semantics

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Extens./Intens.

[Andrews, JSymbLog, 1971, 1972]

[BenzmüllerEtAl, JSymbLog, 2004]

[Muskens, JSymbLog, 2007]

Sound and complete provers do exists

interactive: Isabelle/HOL, PVS, HOL4, Hol Light, Coq/HOL, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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## Embedding HOML in HOL

HOML

$\varphi, \psi ::= \dots | \neg\varphi | \varphi \wedge \psi | \varphi \rightarrow \psi | \Box\varphi | \Diamond\varphi | \forall x_\gamma \varphi | \exists x_\gamma \varphi$

f

- ▶ Kripke style semantics (possible world semantics)

$M, g, s \models \neg\varphi$  iff not  $M, g, s \models \varphi$

$M, g, s \models \varphi \wedge \psi$  iff  $M, g, s \models \varphi$  and  $M, g, s \models \psi$

...

$M, g, s \models \Box\varphi$  iff  $M, g, u \models \varphi$  for all  $s$  with  $r(s, u)$

...

$M, g, s \models \forall x_\gamma \varphi$  iff  $M, [d/x]g, s \models \varphi$  for all  $d \in D_\gamma$

...

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

[Muskens, HandbookOfModalLogic, 2006]

## Embedding Approach — HOML in HOL (remember my talk at SRI in 2010!)

$$\text{HOL} \quad s, t ::= c_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha \rightarrow \beta} \mid (s_{\alpha \rightarrow \beta} t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha t_o$$

$$\text{HOML} \quad \varphi, \psi ::= \dots \mid \neg \varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box \varphi \mid \Diamond \varphi \mid \forall x_\gamma \varphi \mid \exists x_\gamma \varphi$$

**HOML in HOL:** HOML formulas  $\varphi$  are mapped to HOL predicates  $\varphi_{\mu \rightarrow o}$   
 (explicit representation of labelled formulas)

$\neg$	$= \lambda \varphi_{\mu \rightarrow o} \lambda w_\mu \neg \varphi w$
$\wedge$	$= \lambda \varphi_{\mu \rightarrow o} \lambda \psi_{\mu \rightarrow o} \lambda w_\mu (\varphi w \wedge \psi w)$
$\rightarrow$	$= \lambda \varphi_{\mu \rightarrow o} \lambda \psi_{\mu \rightarrow o} \lambda w_\mu (\neg \varphi w \vee \psi w)$
$\forall$	$= \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma hdw$
$\exists$	$= \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \exists d_\gamma hdw$
$\Box$	$= \lambda \varphi_{\mu \rightarrow o} \lambda w_\mu \forall u_\mu (\neg rwu \vee \varphi u)$
$\Diamond$	$= \lambda \varphi_{\mu \rightarrow o} \lambda w_\mu \exists u_\mu (rwu \wedge \varphi u)$
<b>valid</b>	$= \lambda \varphi_{\mu \rightarrow o} \forall w_\mu \varphi w$

Ax (polymorphic over  $\gamma$ )

The equations in Ax are given as axioms to the HOL provers!

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## Embedding HOML in HOL

### Example

HOML formula

$\diamond \exists x G(x)$

HOML formula in HOL

valid  $(\diamond \exists x G(x))_{\mu \rightarrow o}$

expansion

$(\lambda \varphi \forall w \mu \varphi w) (\diamond \exists x G(x))_{\mu \rightarrow o}$

$\beta\eta$ -normalisation

$\forall w_\mu ((\diamond \exists x G(x))_{\mu \rightarrow o} w)$

expansion

$\forall w_\mu (((\lambda \varphi_{\mu \rightarrow o} \lambda w_\mu \exists u_\mu (rwu \wedge \varphi u)) \exists x G(x))_{\mu \rightarrow o} w)$

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$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \rightarrow o} u)$

syntactic sugar

$\forall w_\mu \exists u_\mu (rwu \wedge (\exists (\lambda x G(x)))_{\mu \rightarrow o} u)$

expansion

$\forall w_\mu \exists u_\mu (rwu \wedge ((\lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \exists d_\gamma hdw) (\lambda x G(x)))_{\mu \rightarrow o} u)$

$\beta\eta$ -normalisation

$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion:

user or prover may flexibly choose expansion depth

### What are we doing?

In order to prove that  $\varphi$  is valid in HOML,

→ we instead prove that valid  $\varphi_{\mu \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

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$\forall w_\mu (((\lambda \varphi_{\mu \rightarrow o} \lambda w_\mu \exists u_\mu (rwu \wedge \varphi u)) \exists x G(x))_{\mu \rightarrow o} w)$

$\beta\eta$ -normalisation

$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \rightarrow o} u)$

syntactic sugar

$\forall w_\mu \exists u_\mu (rwu \wedge (\exists (\lambda x G(x)))_{\mu \rightarrow o} u)$

expansion

$\forall w_\mu \exists u_\mu (rwu \wedge ((\lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \exists d_\gamma hdw) (\lambda x G(x)))_{\mu \rightarrow o} u)$

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$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion:

user or prover may flexibly choose expansion depth

### What are we doing?

In order to prove that  $\varphi$  is valid in HOML,

→ we instead prove that valid  $\varphi_{\mu \rightarrow o}$  can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

## Embedding HOML in HOL

### Example

HOML formula

$\diamond \exists x G(x)$

HOML formula in HOL

valid  $(\diamond \exists x G(x))_{\mu \rightarrow o}$

expansion

$(\lambda \varphi \forall w_\mu \varphi w) (\diamond \exists x G(x))_{\mu \rightarrow o}$

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## Advantages of the Embedding Approach

1. Pragmatics and convenience:
  - implementing new provers made simple (even for not yet automated logics)
2. Availability:
  - simply reuse and adapt our existing encodings (THF, Isabelle/HOL, Coq)
3. Flexibility:
  - rapid experimentation with logic variations and logic combinations
4. Relation to labelled deductive systems:
  - extra-logical labels vs. intra-logical labels (here)
5. Relation to standard translation:
  - extra-logical translation vs. extended intra-logical translation (here)
6. Meta-logical reasoning:
  - various examples already exist, e.g. verification of modal logic cube
7. Direct calculi and user intuition:
  - possible: tactics on top of embedding, hiding of embedding
8. Soundness and completeness:
  - already proven for many non-classical logics (wrt Henkin semantics)
9. Cut-elimination:
  - generic indirect result, since HOL enjoys cut-elimination (Henkin semantics)

## Advantage: 1. Pragmatics and convenience

implementing new provers made simple (even for not yet automated logics)

### A very “Lean” Prover for HOML K

```
1  %----The base type $i (already built-in) stands here for worlds and
2  %----$o for individuals; $o (also built-in) is the type of Booleans
3  thf(mu_type,type,(mu:$Type)).
4  %----Reserved constant r for accessibility relation
5  thf(r,type,(r:$i>$i>$o)).
6  %----Modal logic operators not, or, and, implies, box, diamond
7  thf(mnot_type,type,(mnot:($i>$o)>$i>$o)).
8  thf(mnot_definition,(mnot = (^{A:$i>$o,W:$i}:~(A@W))).
9  thf(mor_type,type,(mor:($i>$o)>($i>$o)>$i>$o)).
10 thf(mor_definition,(mor = (^{A:$i>$o,Psi:$i>$o,W:$i}:(A@W)|(Psi@W))).
11 thf(mand_type,type,(mand:($i>$o)>($i>$o)>$i>$o)).
12 thf(mand_definition,(mand = (^{A:$i>$o,Psi:$i>$o,W:$i}:(A@W)&(Psi@W))).
13 thf(mimplies_type,type,(mimplies:($i>$o)>($i>$o)>$i>$o)).
14 thf(mimplies_definition,(mimplies = (^{A:$i>$o,Psi:$i>$o,W:$i}:(A@W)&(Psi@W))).
15 thf(mbox_type,type,(mbox:($i>$o)>$i>$o)).
16 thf(mbox_definition,(mbox = (^{A:$i>$o,W:$i}:[V:$i]:(~(r@W@V)|(A@V))).
17 thf(mdia_type,type,(mdia:($i>$o)>$i>$o)).
18 thf(mdia_definition,(mdia = (^{A:$i>$o,W:$i}:[V:$i]:((r@W@V)&(A@V))).
19 %----Quantifiers (constant domains) for individuals and propositions
20 thf(mforall_ind_type,type,(mforall_ind:(mu:$i>$o)>$i>$o)).
21 thf(mforall_ind_definition,(mforall_ind = (^{A:mu:$i>$o,W:$i}:[X:mu]:(A@X@W))).
22 thf(mforall_indset_type,type,(mforall_indset:((mu:$i>$o)>$i>$o)>$i>$o)).
23 thf(mforall_indset_definition,(mforall_indset = (^{A:(mu:$i>$o)}:>$i>$o,W:$i}:[X:mu:$i>$o]:(A@X@W))).
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27 thf(mexists_indset_definition,(mexists_indset = (^{A:(mu:$i>$o)}:>$i>$o,W:$i}:[X:mu:$i>$o]:(A@X@W))).
28 %----Definition of validity (grounding of lifted modal formulas)
29 thf(v_type,type,(v:($i>$o)>$o)).
30 thf(mvalid_definition,(v = (^{A:$i>$o}:[W:$i]:(A@W)))).
```

TPTP THF0 syntax:

[SutcliffeBenzmüller, J. Formalized Reasoning, 2010]

## Advantage: 1. Pragmatics and convenience

implementing new provers made simple (even for not yet automated logics)

### Approach is competitive

- ▶ First-order modal logic: see experiments in

[BenzmüllerOttenRaths, ECAI, 2012]

[BenzmüllerRaths, LPAR, 2013]

[Benzmüller, ARQNL, 2014]

- ▶ Higher-order modal logics:

There are no other systems yet!

## Advantage: 2. Availability

simply reuse and adapt our existing encodings (THF, Isabelle/HOL, Coq)

### HOML in Isabelle/HOL

The screenshot shows the Isabelle/HOL interface with the file `QML.thy` open. The code defines several abbreviations:

```
abbreviation mnnot :: "σ⇒σ" ("¬_"[52]53)
  where "¬φ ≡ λw. ¬φ(w)"
abbreviation mand :: "σ⇒σ⇒σ" (infixr "∧"51)
  where "φ∧ψ ≡ λw. φ(w) ∧ ψ(w)"
abbreviation mor :: "σ⇒σ⇒σ" (infixr "∨"50)
  where "φ∨ψ ≡ λw. φ(w) ∨ ψ(w)"
abbreviation mimp :: "σ⇒σ⇒σ" (infixr "→"49)
  where "φ→ψ ≡ λw. φ(w) → ψ(w)"
abbreviation mequ :: "σ⇒σ⇒σ" (infixr "↔"48)
  where "φ↔ψ ≡ λw. φ(w) ←→ ψ(w)"
abbreviation mall :: "('a⇒σ)⇒σ" ("∀")
  where "∀Φ ≡ λw. ∀x. Φ(x)(w)"
abbreviation mallB :: "('a⇒σ)⇒σ" (binder "∀"[8]9)
  where "∀x. φ(x) ≡ ∀φ"
abbreviation mexi :: "('a⇒σ)⇒σ" ("∃")
  where "∃Φ ≡ λw. ∃x. Φ(x)(w)"
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  where "∃x. φ(x) ≡ ∃φ"
abbreviation meq :: "μ⇒μ⇒σ" (infixr "=="52) -- "Equality"
  where "x=y ≡ λw. x = y"
abbreviation meqL :: "μ⇒μ⇒σ" (infixr "=_"52) -- "Leibniz Equality"
  where "x=_y ≡ ∀φ. φ(x)→φ(y)"
abbreviation mbox :: "σ⇒σ" ("□_"[52]53)
  where "□φ ≡ λw. ∀v. w r v → φ(v)"
abbreviation mdia :: "σ⇒σ" ("◊_"[52]53)
  where "◊φ ≡ λw. ∃v. w r v ∧ φ(v)"
```

The interface includes tabs for Output, Query, Sledgehammer, and Symbols, and a status bar at the bottom.

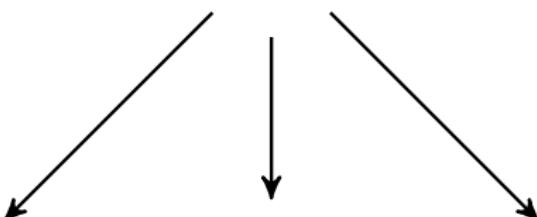
See formalisations at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations>

## Advantage: 3. Flexibility

rapid experimentation with logic variations and logic combinations

### Postulating modal axioms or semantical constraints

HOL



#### 'Syntactical' Sahlqvist axioms

M: valid  $\forall\varphi(\Box^r\varphi \rightarrow \varphi)$

$\leftrightarrow \forall x(rxx)$

(reflexivity)

B: valid  $\forall\varphi(\varphi \rightarrow \Box^r\Diamond^r\varphi)$

$\leftrightarrow \forall x\forall y(rxy \rightarrow ryx)$

(symmetry)

D: valid  $\forall\varphi(\Box^r\varphi \rightarrow \Diamond^r\varphi)$

$\leftrightarrow \forall x\exists y(rxy)$

(serial)

4: valid  $\forall\varphi(\Box^r\varphi \rightarrow \Box^r\Box^r\varphi)$

$\leftrightarrow \forall x\forall y\forall z(rxy \wedge ryz \rightarrow rxz)$

(transitivity)

5: valid  $\forall\varphi(\Diamond^r\varphi \rightarrow \Box^r\Diamond^r\varphi)$

$\leftrightarrow \forall x\forall y\forall z(rxy \wedge rxz \rightarrow ryz)$

(euclidean)

#### 'Semantical' constraints

## Advantage: 3. Flexibility

rapid experimentation with logic variations and logic combinations

### Possibilist vs. Actualist Quantification

$$\forall = \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma h d w \quad (\text{constant domains})$$

becomes

$$\forall^{va} = \lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma (\mathbf{ExInW} dw \rightarrow h d w) \quad (\text{varying domains})$$

where **ExInW** is an existence predicate

(additional axioms: non-empty domains, denotation of constants & functions)

#### **Advantage: 4. Relation to labelled deductive systems**

## extra-logical labels vs. intra-logical labels (here)



$$\diamond \exists x G(x) \text{ worldlabel } \rightarrow ((\diamond \exists x G(x))_{\mu \rightarrow o} \text{ worldlabel}_{\mu})$$

## Advantage: 5. Relation to standard translation

extra-logical translation vs. extended intra-logical translation (here)

[BenzmüllerPaulson, LogicaUniversalis, 2013]

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

### Intra-logical realisation of the standard translation

$$\begin{aligned} & (\Box\phi) \text{ a} \\ \rightarrow & ((\Box\phi)_{\mu \rightarrow o} a) \\ \rightarrow & (((\lambda\varphi_{\mu \rightarrow o} \lambda w_\mu \forall u_\mu (\neg r w u \vee \varphi u)) \phi)_{\mu \rightarrow o} a) \\ \rightarrow & (\forall u_\mu (\neg r a u \vee \phi_{\mu \rightarrow o} u)) \end{aligned}$$

We have extended this also for first-order and higher-order quantifiers!

$$\begin{aligned} & (\forall x \phi(x)) \text{ a} \\ \rightarrow & ((\forall x \phi(x))_{\mu \rightarrow o} a) \\ \rightarrow & (((\forall (\lambda x \phi(x)))_{\mu \rightarrow o} a) \\ \rightarrow & (((\lambda h_{\gamma \rightarrow (\mu \rightarrow o)} \lambda w_\mu \forall d_\gamma h d w) (\lambda x \phi(x)))_{\mu \rightarrow o} a) \\ \rightarrow & \forall d_\gamma (\phi(d)_{\mu \rightarrow o} a) \end{aligned}$$

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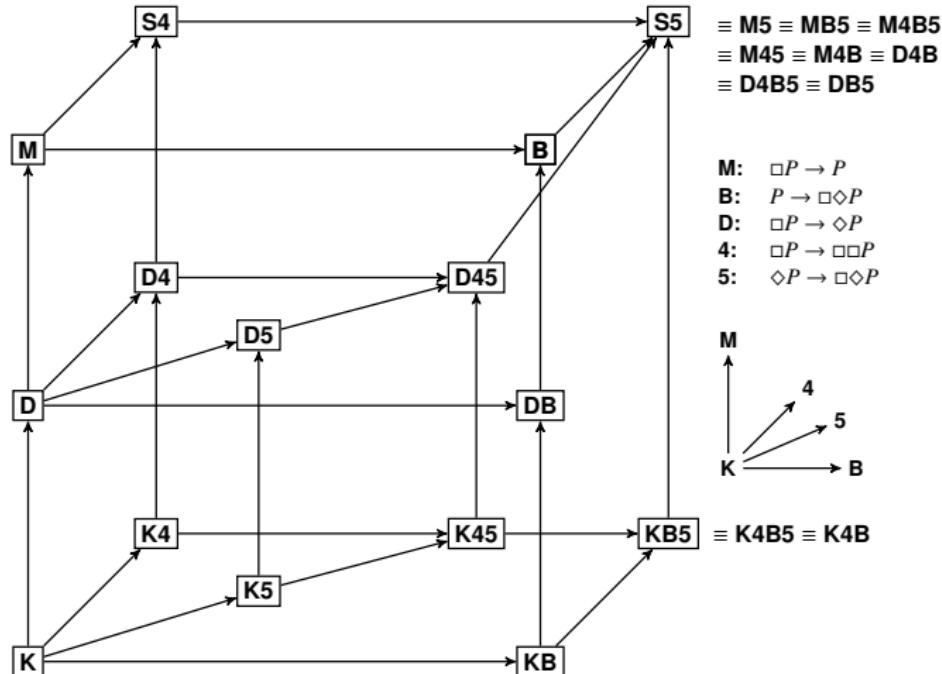
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## Advantage: 6. Meta-logical reasoning

various examples already exist, e.g. verification of modal logic cube

[Benzmüller, FestschriftWalther, 2010]

[BenzmüllerClausSultana, PxTP, 2015]



Verification of cube in less than 1 minute in Isabelle/HOL

## Advantage: 7. Direct calculi and user intuition

abstract level tactics (here in Coq) on top of embedding, hiding of embedding

[BenzmüllerWoltzenlogelPaleo, CSR'2015]

**Lemma mp\_dia:**

[mforall p, mforall q, (dia p) m-> (box (p m-> q)) m-> (dia q)].

**Proof.** mv.

intros p q H1 H2. dia\_e H1. dia\_i w0. box\_e H2 H3. apply H3. exact H1.

**Qed.**

$$\boxed{\begin{array}{c} \frac{}{\Diamond p} \stackrel{1}{\Diamond_E} \quad \frac{\Box(p \rightarrow q)}{\Box_E} \stackrel{2}{\Box_E} \\ w_0 \boxed{\frac{p \quad p \rightarrow q}{q} \rightarrow_E} \\ \frac{\frac{\frac{\Diamond q}{\Diamond_I}}{\Diamond p \rightarrow (\Box(p \rightarrow q)) \rightarrow (\Diamond q)} \rightarrow^1_I, \rightarrow^2_I}{\forall p. \forall q. \Diamond p \rightarrow (\Box(p \rightarrow q)) \rightarrow \Diamond q} \forall_I, \forall_I \end{array}}$$

## Advantage: 8. Soundness and completeness

already proven for many non-classical logics (wrt Henkin semantics)

### Soundness and Completeness

$$\models^L \varphi \text{ iff } \mathbf{Ax} \models_{\text{Henkin}}^{HOL} \text{valid } \varphi_{\mu \rightarrow o}$$

Logic L:

- ▶ Higher-order Modal Logics [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]
- ▶ First-order Multimodal Logics [BenzmüllerPaulson, LogicaUniversalis, 2013]
- ▶ Propositional Multimodal Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- ▶ Quantified Conditional Logics [Benzmüller, IJCAI, 2013]
- ▶ Propositional Conditional Logics [BenzmüllerEtAl., AMAI, 2012]
- ▶ Intuitionistic Logics [BenzmüllerPaulson, Log.J.IGPL, 2010]
- ▶ Access Control Logics [Benzmüller, IFIP SEC, 2009]
- ▶ Logic Combinations [Benzmüller, AMAI, 2011]
- ▶ ...more is on the way ... including:
  - ▶ Description Logics
  - ▶ Nominal Logics
  - ▶ Multivalued Logics (SIXTEEN)
  - ▶ Logics based on Neighborhood Semantics
  - ▶ (Mathematical) Fuzzy Logics
  - ▶ Paraconsistent Logics

## Advantage: 9. Cut-elimination

generic indirect result, since HOL enjoys cut-elimination (Henkin semantics)

### Soundness and Completeness

$$\models^L \varphi \text{ iff } \mathbf{Ax} \models_{\text{Henkin}}^{HOL} \text{valid } \varphi_{\mu \rightarrow o}$$

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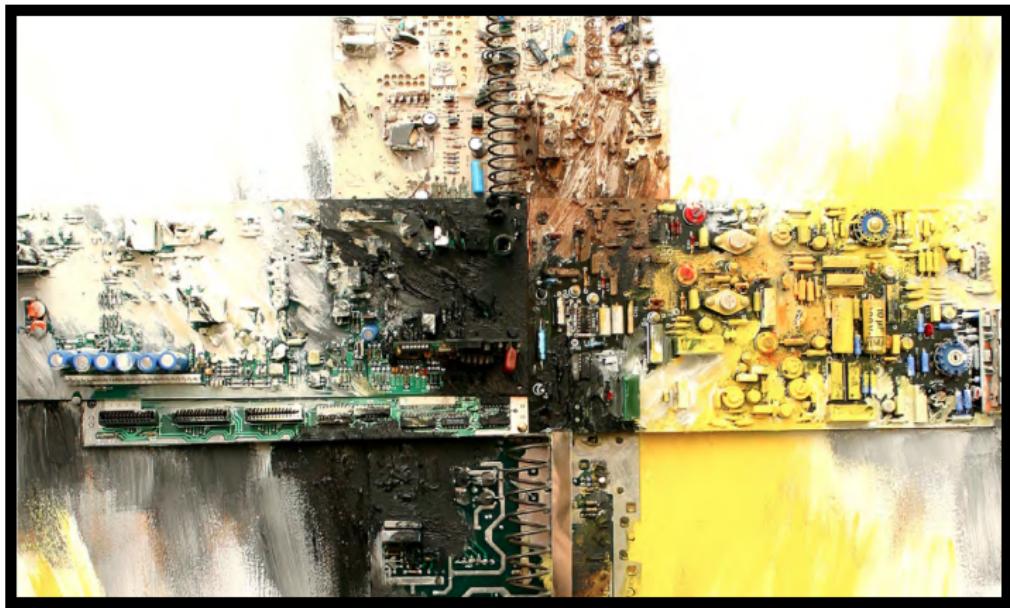
### Soundness and Completeness and Cut-elimination

$$\models^L \varphi \text{ iff } \text{Ax} \models_{\text{Henkin}}^{\text{HOL}} \text{valid } \varphi_{\mu \rightarrow o} \text{ iff } \text{Ax} \models_{\text{cut-free}}^{\text{HOL}} \text{valid } \varphi_{\mu \rightarrow o}$$

Logic L:

- ▶ Quantified Conditional Logics

[Benzmüller, J. Phil. Log., 2016]



**Part B:**  
**New Knowledge on the Ontological Argument**  
**from HOL ATPs**

## Vision of Leibniz (1646–1716): *Calculemus!*



If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other ...: Let us calculate.

(Translation by Russell)

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo ... dicere: calculemus.  
(Leibniz, 1684)



Required:  
**characteristica universalis** and **calculus ratiocinator**

## Our Contribution: Towards Computational Metaphysics

### Ontological argument for the existence of God

- ▶ Focus on Gödel's modern version in higher-order modal logic
- ▶ Experiments with HO provers and embedding approach

### Different interests in ontological arguments

- ▶ Philosophical: Boundaries of metaphysics & epistemology
- ▶ Theistic: Successful argument could convince atheists?
- ▶ Ours: Computational metaphysics (Leibniz' vision)

### Related work: only for Anselm's simpler argument

- ▶ first-order ATP PROVER9 [OppenheimerZalta, 2011]
- ▶ interactive proof assistant PVS [Rushby, 2013]

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## A Long History

pros and cons



Anselm's notion of God (Proslogion, 1078):

**“God is that, than which nothing greater can be conceived.”**

Gödel's notion of God:

**“A God-like being possesses all ‘positive’ properties.”**

To show by logical, deductive reasoning:

**“God exists.”**

$$\exists x G(x)$$

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To show by logical, deductive reasoning:

**“Necessarily, God exists.”**

$$\Box \exists x G(x)$$

# Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologischer Bereich      Feb. 10, 1970

$P(\varphi)$      $\varphi$  is positive    ( $\Leftrightarrow \varphi \in P$ )

Ax 1:  $P(p), P(\varphi) \supset P(\varphi \wedge \psi) \quad$  At 2:  $P(p) \supset P(\neg p)$

P1  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)] \quad$  (God)

P2  $\varphi_{\text{Exis}} \equiv (\psi)[\psi(x) \supset N(y)[P(y) \supset \psi(y)]] \quad$  (Existence)

$P \supset_N = N(p \supset q) \quad$  Necessity

Ax 2  $P(\varphi) \supset N P(\varphi) \quad$  } because it follows  
 $\neg P(\varphi) \supset N \neg P(\varphi) \quad$  } from the nature of the  
 property

Th.  $G(x) \supset G_{\text{Exis.}}$

Df.  $E(x) \equiv P[\varphi_{\text{Exis}} \supset N \neg x \varphi(x)] \quad$  necessary Existence

Ax 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(x) G(x) \supset M N(\exists y) G(y)$       M = possibility

"  $\supset N(\exists y) G(y)$

any two elements of  $X$  are nec. equivalent  
 exclusive or \* and for any number of them

$M(x) G(x)$  means all pos. prop. w.r.t. com-  
 patible. This is true because of:  
Ax 4:  $P(\varphi) \cdot q \supset \psi \supset P(\psi)$  which impl.  
 $\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$   
 But if a system S of pos. prop. were incon-  
 sistent it would mean that the non-prop. S (which  
 is positive) would be  $x \neq x$ .

Positive means positive in the moral aesthe-  
 tical sense (independently of the accidental structure of  
 the world). Only in the ex. frame. It is  
 also meant "Attribution" as opposed to "Platification"  
 (or containing privation). This interprets the pos. prop.

$\exists / \forall$  (privation):  $(X) N \neg P(x)$  Otherwise:  $P(x) \supset x \neq x$   
 hence  $x \neq x$  positive not  $x=x$  i.e. negation. At  
 the end of proof At 2

thus X i.e. the normal form in terms of elem. prop. contains  
 members without negation.

## Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm. T1** Positive properties are possibly exemplified:  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

**Def. D1** A *God-like* being possesses all positive properties:  $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

**Axiom A3** The property of being God-like is positive:  $P(G)$

**Cor. C** Possibly, God exists:  $\Diamond\exists xG(x)$

**Axiom A4** Positive properties are necessarily positive:  $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

**Def. D2** An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:  $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:  $\forall x[G(x) \rightarrow G \text{ ess. } x]$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:  $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$

**Axiom A5** Necessary existence is a positive property:  $P(NE)$

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Difference to Gödel (who omits this conjunct)

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Modal operators are used

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second-order quantifiers

## Gödel's God in TPTP THF

```
>
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p

Leo-II tries to prove
=====
Goedel's Theorem T3: "Necessarily, God exists"
thf(thmT3,conjecture,
  ( v
    @ ( mbox
      @ ( mexists_ind
        @ ^ [X: mu] :
          ( g @ X ) ) ) )).

Assumptions: D1, C, T2, D3, A5

. searching for proof ..

*****
* Proof found *
*****
% SWS status Theorem for Notwendigerweise-existiert-Gott.p

. generating proof object □
```

Url to movie: <http://www.christoph-benzmueller.de/papers/Movie1.mov>

# Gödel's God in Isabelle/HOL

The screenshot shows the Isabelle/HOL IDE interface with the file `Scott_SSU.thy` open. The code defines a theory `Scott_SSU` that imports `QML_SSU`. It includes several axioms (A1-A5) and definitions (G, ess, NE) related to propositional logic and modal operators. A theorem is stated at the bottom.

```
theory Scott_SSU imports QML_SSU
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  axiomatization where
    A1: "[∀Φ. P(¬Φ) ↔ ¬P(Φ)]" and
    A2: "[∀Φ. ∀Ψ. (P(Φ) ∧ □(∀x. Φ(x) → Ψ(x))) → P(Ψ)]"
    definition G where
      D1: "G(x) = (¬Φ. P(Φ) → Φ(x))"
    axiomatization where
      A3: "[P(G)]" and
      A4: "[∀Φ. P(Φ) → □(P(Φ))]"
    definition ess (infixr "ess" 85) where
      D2: "Φ ess x = Φ(x) ∧ (¬Ψ. Ψ(x) → □(¬Υ. Φ(Υ) → Ψ(Υ)))"
    definition NE where
      D3: "NE(x) = (¬Φ. Φ ess x → □(¬Υ. Φ(Υ) → Ψ(Υ)))"
    axiomatization where
      A5: "[P(NE)]"

  theorem (* Notwendigerweise, existiert Gott *)

```

# Gödel's God in Coq

The screenshot shows the CoqIDE interface with a menu bar (File, Edit, Navigation, Try Tactics, Templates, Queries, Display, Compile, Windows, Help) and a toolbar with icons for file operations. The workspace contains several files: "scratch", "Modal.v", "ModalClassical.v", and "GoedelGod-Scott.v". The "GoedelGod-Scott.v" file is open and displays a Coq proof script. The script includes comments, parameters, axioms, and a theorem with its proof. The proof involves various tactics like intro, assert, and exact. On the right side of the interface, there is a proof state window showing two subgoals and their progress. The first subgoal is labeled (1/2) and the second (2/2). The proof state also shows the current tactic being used.

```
(* Constant predicate that distinguishes positive properties *)
Parameter Positive: (u -> o) -> o.

(* Axiom A1: either a property or its negation is positive, but not both *)
Axiom axiom1a : V (mforall p, (Positive (fun x: u -> m~(p x))) m-> (m~ (Positive p))). 
Axiom axiom1b : V (mforall p, (m~ (Positive p)) m-> (Positive (fun x: u -> m~(p x)))). 

(* Axiom A2: a property necessarily implied by a positive property is positive *)
Axiom axiom2: V (mforall p, mforall q, Positive p m/\ (box (mforall x, (p x) m-> (q x)))). 

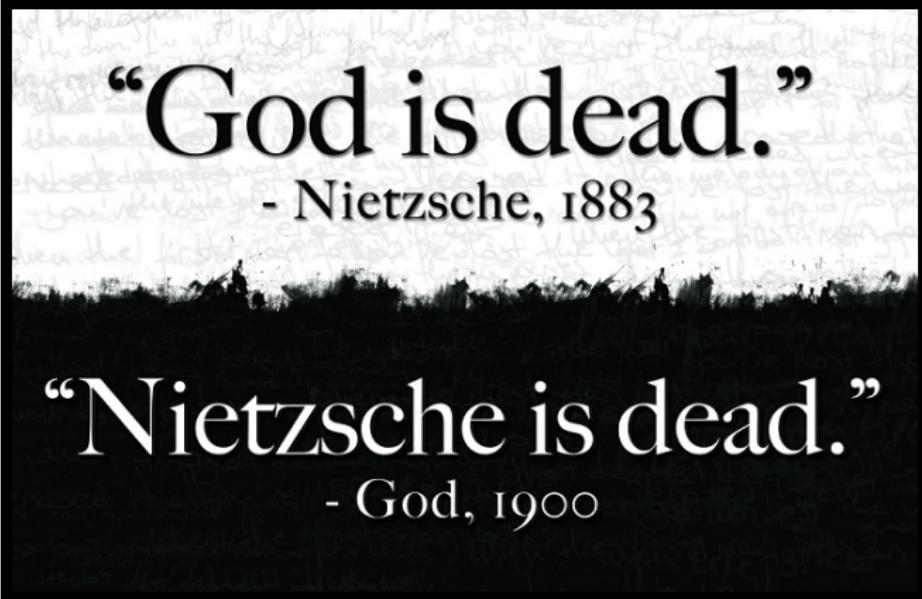
(* Theorem T1: positive properties are possibly exemplified *)
Theorem theorem1: V (mforall p, (Positive p) m-> dia (mexists x, p x)).
Proof.
intro.
intro p.
intro H1.
proof_by_contradiction H2.
apply not_dia_box_not_in H2.
assert (H3: ((box (mforall x, m~ (p x))) w)). (* Lemma from Scott's notes *)
  box_intro wl R1.
  intro x.
  assert (H4: ((m~ (mexists x : u, p x)) wl)).
  box_elim H2 wl R1 G2.
  exact G2.

  clear H2 R1 H1 w.
  intro H5.
  apply H4.
  exists x.
  exact H5.

assert (H6: ((box (mforall x, (p x) m-> m~ (x m= x))) w)). (* Lemma from Scott's notes *)
  box_intro wl R1.
  intro x.
  intro H7.
  intro H8.
  box_elim H3 wl R1 G3.
  assert G3 with /v .=_ v.
```

See verifiable Coq document at:

<https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq>



“God is dead.”

- Nietzsche, 1883

“Nietzsche is dead.”

- God, 1900

### Findings from our study

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

# Main Findings [BenzmüllerWolzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \exists X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1 0.1/0.1	0.0/0.0 0.0/5.2	—/— —/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Diamond} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0 0.0/0.0	0.0/0.0 5.2/31.3	—/— —/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
			K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \exists Y_\mu^* \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
			KB	THM	—/—	—/—	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/— —/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(○), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB KB	UNS UNS	7.5/7.8 —/—	—/— —/—	—/— —/—

# Main Findings [BenzmüllerWolzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II concl/verif	Satallax concl/verif	Nitpick concl/verif
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p\phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma} \dot{\forall} \psi_{\mu \rightarrow \sigma} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu. (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p\psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Diamond} \exists X_\mu. \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Diamond} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p\phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma} (\text{ess } \phi X \dot{\wedge} \dot{\Box} \exists Y_\mu. \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	17.9/—	3.3/3.2	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma} \dot{\forall} X_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	16.5/—	0.0/0.0	—/—
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_\mu. (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{=} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
CO'	∅ (no goal, check for consistency)	A1(○), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
			KB	UNS	—/—	—/—	—/—

# Main Findings [BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\neg} \psi_{\mu^*} (\phi Y \dot{\wedge} \dot{\neg} \psi Y) \dot{\wedge} p \psi)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\phi} \exists X_\mu. \phi X$	A1( $\supset$ ), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \psi \dot{\wedge} \psi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\phi} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\neg} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\neg} \dot{\forall} Y_{\mu^*} (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
Q3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\dots)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\phi} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{\wedge} \Box s_\sigma]$						
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu. (s_\mu \dot{\wedge} X \dot{\wedge} (s_\mu \dot{\wedge} \Box s_\mu))$						
MT	$\dot{\forall} X_\mu. \dot{\forall} Y_{\mu^*} (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu^* \rightarrow \sigma^*} Y \dot{\wedge} \Box g_{\mu \rightarrow \sigma} X))$						
CO	0 (no goal, check for const.)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma}. \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\neg} \dot{\forall} Y_{\mu^*} (\phi Y \dot{\wedge} \psi Y))$						
CO'	0 (no goal, check for const.)						

## Automating Scott's proof script

**T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax**

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ A1 and A2
  - ▶ A1( $\supset$ ) and A2
- ▶ notion of quantification
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

## Main Findings [BenzmüllerWolzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)]$						
A2	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi]$						
T1	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \exists X_\mu^* \phi X]$	A1(2), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[n_\mu \rightarrow \sigma \quad g_\mu \quad 1]$						
C	$[\dot{\phi} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$[\dot{\forall} \psi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \psi \dot{\wedge} \dot{\Box} p \psi]$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$[\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)]$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi X \dot{\wedge} \phi X)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{\rightarrow} \dot{\Box} s_\sigma]$						
FG	$[\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (s_\sigma \dot{\rightarrow} X \dot{\wedge} (s_\sigma \dot{\rightarrow} \dot{\Box} X))]$						
MT	$[\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\rightarrow} X))]$						
CO	0 (no goal, check for const.)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y)$						
CO'	0 (no goal, check for const.)						

### Automating Scott's proof script

C: "Possibly, God exists"  
proved by LEO-II and Satallax

- ▶ in logic: K
- ▶ from assumptions:
  - ▶ T1, D1, A3
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\neg} \dot{\forall} X_\mu^* (\phi X \dot{\neg} \psi X) \dot{\neg} p \psi)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\neg} \dot{\exists} X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\neg} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\exists} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\neg} \dot{\forall} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \sigma} = \lambda \phi. \lambda Y_\mu^*. \phi X \wedge \dot{\forall} Y_\mu^* (\phi Y \dot{\neg} \lambda(Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\neg} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \sigma} \psi X \dot{\neg} \lambda(Y))$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\exists} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						

- MC  $\vdash \dot{\neg} \dot{\exists} s_\sigma$
- FG  $\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\neg} (g_\mu$
- MT  $\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\neg} (g_\mu$
- CO'  $\emptyset$  (no goal, check for const)
- D2'  $\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$
- CO'  $\emptyset$  (no goal, check for const)

## Automating Scott's proof script

**T2: "Being God-like is an ess. of any God-like being"**  
**proved by LEO-II and Satallax**

- ▶ **in logic: K**
- ▶ **from assumptions:**
  - ▶ A1, D1, A4, D2
- ▶ **for domain conditions:**
  - ▶ **possibilist quantifiers (constant dom.)**
  - ▶ **actualist quantifiers for individuals (varying dom.)**

# Main Findings [BenzmüllerWolzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\neg} \psi_{\mu^*} (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\exists} X_\mu. \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\neg} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\neg} \psi Y_\mu. (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu. g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\phi$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\forall} \exists X_\mu. g_{\mu \rightarrow \sigma} X]$						

## Automating Scott's proof script

**T3: "Necessarily, God exists"**  
proved by LEO-II and Satallax

- ▶ in logic: KB
- ▶ from assumptions:
  - ▶ D1, C, T2, D3, A5
- ▶ for domain conditions:
  - ▶ possibilist quantifiers (constant dom.)
  - ▶ actualist quantifiers for individuals (varying dom.)

For logic K we got a **countermodel** by Nitpick

## Main Findings [BenzmüllerWolzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II concl/very	Satallax concl/very	Nitpick concl/very
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{=} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} \exists X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$		K	THM	0.1/0.1	0.0/5.2	—/—
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\Box} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi X \dot{\wedge} \dot{\Box} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$		K	THM	12.9/14.0	0.0/0.0	—/—
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\Box} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
MC	$[s_\sigma \dot{\wedge} \dot{\Box} s_\sigma]$						
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (s_\sigma \dot{\wedge} X \dot{\wedge} (s_\sigma \dot{\wedge} \phi X))$						
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X))$						
CO	0 (no goal, check for consistency)						
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y)$						
CO'	0 (no goal, check for consistency)						

### Automating Scott's proof script

#### Summary

- ▶ proof verified and automated
- ▶ KB is sufficient (criticized logic S5 not needed!)
- ▶ possibilist and actualist quantifiers (individuals)
- ▶ exact dependencies determined experimentally
- ▶ ATPs have found alternative proofs  
e.g. self-identity  $\lambda x(x = x)$  is not needed

# Main Findings [BenzmüllerWolzenlogelPaleo, ECAI, 2014]

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$		K	THM	0.1/0.1	0.0/0.0	—/—
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\square} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X)) \dot{\wedge} p \psi$	A1(?) A2					
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X$						
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} p$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} X)$						
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^*. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} \phi)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
<b>Consistency check: Gödel vs. Scott</b>							
<ul style="list-style-type: none"> <li>▶ Scott's assumptions are consistent; shown by Nitpick</li> <li>▶ Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result?)</li> </ul>							
MC	$[s_\sigma \dot{\wedge} \dot{\square} s_\sigma]$	A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K KB KB	CSA THM THM	—/— 0.0/0.1 —/—	—/— 0.1/5.3 —/—	8.2/7.5 —/— —/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	17.9/— 16.5/—	3.3/3.2 0.0/0.0	—/— —/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))$	A1, D1, A3, A4, D2, D3, A5 D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB KB	THM THM THM	12.8/15.1 —/— —/—	0.0/5.4 0.0/3.3 —/—	—/— —/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma} \lambda X_\mu^*. \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\square} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$	A1(?) A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p \phi)$						
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \wedge \dot{\square} \dot{\forall} X_\mu^* (\phi X \dot{\wedge} \psi X) \dot{\wedge} p \psi)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} \exists X_\mu^* \phi X$	A1(○), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} \exists X_\mu^* \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$						
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\square} p$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$						
T2	$\dot{\forall} X_\mu^* p_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma}) X$						
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (e$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$	A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
T2'	$[\dot{\square} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot{\wedge} \dot{\square} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	17.9/— /	3.3/3.2 /	—/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (\dot{\neg}(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \dot{\wedge} \dot{\neg}(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— 12.8/15.1	0.0/0.0 0.0/5.4	—/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\wedge} (g_{\mu \rightarrow \sigma} Y \dot{\wedge} X \dot{\equiv} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	—/— —/—	0.0/3.3 —/—	—/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\square} \dot{\forall} Y_\mu^* (\phi Y \dot{\wedge} \psi Y))$	A1(○), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

## Further Results

- ▶ Monotheism holds
- ▶ God is flawless

HOL encoding	
A1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu \dot{\exists}$
A2	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \dot{\exists}$
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\exists} \dot{\exists}$
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu \dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \dot{\exists} \dot{\forall} p_\sigma$
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$
C	$[\dot{\exists} \dot{\exists} X_\mu^* g_{\mu \rightarrow \sigma} X]$
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\exists} \dot{\forall} p_\sigma$
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda$
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\exists} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma})$
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu \dot{\forall} \phi_{\mu \rightarrow \sigma^*} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma})$
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$
T3	$[\dot{\exists} \dot{\exists} X_\mu^* g_{\mu \rightarrow \sigma} X]$

## Modal Collapse (Sobel)

$$\forall \varphi (\varphi \supset \Box \varphi)$$

- ▶ proved by LEO-II and Satallax
- ▶ for possibilist and actualist quantification (ind.)

Main critique on Gödel's ontological proof:

- ▶ there are no contingent truths
- ▶ everything is determined / no free will

MC	$[s_\sigma \dot{\exists} \dot{\forall} s_\sigma]$	D2, T2, T3 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	17.9/— —/—	3.3/3.2 —/—	—/— —/—
FG	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} X_\mu^* (g_{\mu \rightarrow \sigma} X \supset (\neg(p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi) \supset \neg(\phi X)))$	A1, D1 A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	16.5/— —/—	0.0/0.0 —/—	—/— —/—
MT	$\dot{\forall} X_\mu^* \dot{\forall} Y_\mu^* (g_{\mu \rightarrow \sigma} X \dot{\exists} (g_{\mu \rightarrow \sigma} Y \dot{\exists} X \dot{=} Y))$	D1, FG A1, A2, D1, A3, A4, D2, D3, A5	KB KB	THM THM	12.8/15.1 —/—	0.0/5.4 0.0/3.3	—/— —/—
CO	∅ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\exists} \dot{\forall} Y_\mu^* (\phi Y \dot{\exists} \psi Y))$	A1, A2, D1, A3, A4, D2, D3, A5	KB	UNDEF	7.5/7.8	—/—	—/—
CO'	∅ (no goal, check for consistency)	A1(∅), A2, D2', D3, A5 A1, A2, D1, A3, A4, D2', D3, A5	KB	UNDEF	—/—	—/—	—/—

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$\dot{\vee} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} (\lambda X_\mu. \dot{\vdash} (\phi X)) \dot{\equiv} \dot{\neg} (p \phi)$						
A2	$\dot{\vee} \phi_{\mu \rightarrow \sigma^*} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} \dot{\forall} X_\mu^* (\phi X \dot{\vdash} \psi X) \dot{\vdash} p \psi)$						
T1	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} \exists X_\mu^* \phi X$	A1(○), A2 A1, A2	K	THM	0.1/0.1 0.1/0.1	0.0/0.0 0.0/5.2	—/— —/—
D1	$g_{\mu \rightarrow \sigma} = \lambda X_\mu. \dot{\vee} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \phi X$						
A3	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} g_{\mu \rightarrow \sigma}]$						
C	$[\dot{\forall} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	T1, D1, A3 A1, A2, D1, A3	K	THM	0.0/0.0 0.0/0.0	0.0/0.0 5.2/31.3	—/— —/—
A4	$\dot{\forall} \phi_{\mu \rightarrow \sigma^*} p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \phi \dot{\wedge} \dot{\forall} p \phi$						
D2	$\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} = \lambda \phi_{\mu \rightarrow \sigma^*} \lambda X_\mu^* \phi X \dot{\wedge} \dot{\forall} \psi_{\mu \rightarrow \sigma^*} (\psi X \dot{\wedge} \dot{\forall} \dot{\forall} Y_\mu^* (\phi Y \dot{\vdash} \psi Y))$						
T2	$\dot{\forall} X_\mu^* g_{\mu \rightarrow \sigma} X \dot{\wedge} (\text{ess}_{(\mu \rightarrow \sigma) \rightarrow \mu \rightarrow \sigma} g X)$	A1, D1, A4, D2 A1, A2, D1, A3, A4, D2	K	THM	19.1/18.3 12.9/14.0	0.0/0.0 0.0/0.0	—/— —/—
D3	$\text{NE}_{\mu \rightarrow \sigma} = \lambda X_\mu^* \dot{\vee} \phi_{\mu \rightarrow \sigma^*} (\text{ess } \phi X \dot{\wedge} \dot{\forall} \exists Y_\mu^* \phi Y)$						
A5	$[p_{(\mu \rightarrow \sigma) \rightarrow \sigma} \text{NE}_{\mu \rightarrow \sigma}]$						
T3	$[\dot{\forall} \exists X_\mu^* g_{\mu \rightarrow \sigma} X]$	D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5 D1, C, T2, D3, A5 A1, A2, D1, A3, A4, D2, D3, A5	K K KB KB	CSA CSA THM THM	—/— —/— 0.0/0.1 —/—	—/— —/— 0.1/5.3 —/—	3.8/6.2 8.2/7.5 —/— —/—
<b>Observation</b>							
▶ good performance of ATPs							
▶ excellent match between argumentation granularity in papers and the reasoning strength of the ATPs							
			3, A5	KB	THM	17.9/—	3.3/3.2
			3, A5	KB	THM	—/—	—/—
			3, A5	KB	THM	16.5/—	0.0/0.0
			3, A5	KB	THM	12.8/15.1	0.0/5.4
			3, A5	KB	THM	—/—	0.0/3.3
			3, A5	KB	SAT	—/—	—/—
			3, A5	KB	UNS	7.5/7.8	7.3/7.4
			3, A5	KB	UNS	—/—	—/—

# Avoiding the Modal Collapse: Recent Variants

## SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

## Der Mathematiker und die Frage der Existenz Gottes (betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nichts wissen,  
wenn wir glauben, daß ein Gott sei.  
(Kant, Nachleß)

### 1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. In der vorliegenden Arbeit ist er, zu einer Deutung des Beweises, durch Bereitstellung von etwas Modelltheoretischer Arbeit und einer detaillierten Begründung, durch Bereitstellung von etwas Modelltheoretischer Arbeit und einer detaillierten Begründung, im besonderen auf dem Symposium zur Peter von Professor Gert Müller (Heidelberg, Januar 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung“<sup>1</sup> zu schreiben, ohne aus ihr einen

### Gödel's Ontological Proof Revisited \*

C. Anthony Anderson and Michael Gettings  
University of California, Santa Barbara  
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

PETR HÁJEK

## A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

### 1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needless to say, formal modal logic has found several

### Magari and others on Gödel's ontological proof

Petr Hájek  
Institute of Computer Science, Academy of Sciences  
182 07 Prague, Czech Republic  
e-mail: hajek@uivt.cas.cz

### 1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variants by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

### Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula  $P(F)$  stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

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## Results Obtained with Fully Automated Reasoners

A controversy between Magari, Hájek and Anderson regarding the redundancy of some axioms

Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	S/I	-	-	P (KB)	CS	
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	S/U	-	-	P (KB)	CS	
Hájek AOE'' (var)	-	-		-	-	S/I	-	-	S/I	-	P (KB)	CS	
Anderson (simp) (var)	-	R	R		R (K4B)	-	-	-	-	-			
Bjørdal (const)	R (K4)	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R	-	R (KT)	-	-	N/I	-	-	P (KB)	CS

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Proof	D1'	A:A1'	A2'	A3'	A4	A4'	H:A4	A5	A5'	H:A5	T3	T3'	MC
Scott (const)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Scott (var)	-	-	-	-	N/I	-	-	N/I	-	-	P	-	P
Anderson (const)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (var)	-	-	-	-	R (K4B)	-	-	-	R	-	-	P	CS
Anderson (mix)	-	-	-	-	R (K4B)	-	-	-	I	-	-	CS	CS
Hájek AOE' (var)	-	-	CS		S/I	-	-	S/I	-	-	P (KB)	CS	
Hájek AOE'_0 (var)	-	-	-	CS	R	-	-	S/U	-	-	P (KB)	CS	
Hájek AOE'' (var)	-	-			-	-	S/I	-	-	S/I	-	P (KB)	CS
Anderson (simp) (var)	-	R	R			R (K4B)	-	-	-	-	-	-	
Bjørdal (const)	R (K4)	-	R	R		R (KT)	-	-	N/I	-	-	P (KB)	CS
Bjørdal (var)	CS	-	R	R		R (KT)	-	-	N/I	-	-	P (KB)	CS

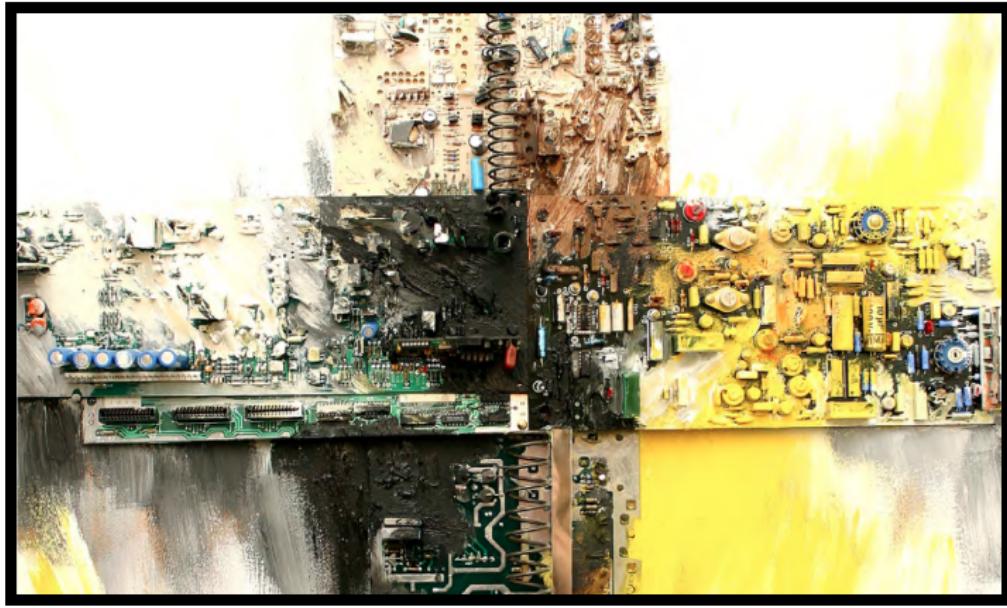


Leibniz (1646–1716)

*characteristica universalis* and *calculus ratiocinator*

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.

But: Intuitive proofs/models are needed to convince philosophers



## Part C: Reconstruction of the Inconsistency of Gödel's Axioms

## Scott's Version of Gödel's Axioms, Definitions and Theorems

**Axiom A1** Either a property or its negation is positive, but not both:  $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

**Axiom A2** A property necessarily implied by a positive property is positive:

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm. T1** Positive properties are possibly exemplified:

$$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$$

**Def. D1** A *God-like* being possesses all positive properties:

$$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$$

**Axiom A3** The property of being God-like is positive:

$$P(G)$$

**Cor. C** Possibly, God exists:

$$\Diamond\exists xG(x)$$

**Axiom A4** Positive properties are necessarily positive:

$$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$$

**Def. D2** An essence of an individual is a property possessed by it and necessarily implying any of its properties:  $\phi\text{ ess. }x \leftarrow \boxed{\phi(x)} \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

**Thm. T2** Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G\text{ ess. }x]$$

**Def. D3** Necessary existence of an individual is the necessary exemplification of all its essences:  $NE(x) \leftrightarrow \forall\phi[\phi\text{ ess. }x \rightarrow \Box\exists y\phi(y)]$

**Axiom A5** Necessary existence is a positive property:

$$P(NE)$$

**Thm. T3** Necessarily, God exists:

$$\Box\exists xG(x)$$

Difference to Gödel (who omits this conjunct)

# Inconsistency (Gödel): Proof by LEO-II in KB

```

DemoMaterial — bash — 166x52

@SV8)@SV3)=$false) | (((p(@[^SX0:mu,SX1:$i]: $false))@SV3)=$true))), inference(prim_subst,[status(thm)], [66:[bind(SV11,$thf(^SV23:mu,SV24:$i): $false))]]))).
    thf(84,plain,!([SV22:(mu($i>$o)),SV3:$i,SV8:(mu($i>$o))]: (((SV0@(^sk2_SK33@SV3)@(^SX0:mu,SX1:$i): (~((SV22@SX0)@SX1)))@SV8)@((($sk1_SK31@(^SX0:mu,SX1:$i): (~((SV22@SX0)@SX1)))@SV3)@strue)) | (((p@SV8)@SV3)=$false) | (((p(@[^SX0:mu,SX1:$i]: (~((SV22@SX0)@SX1)))@SV3)@strue))).
    thf(85,plain,!([SV4:$i,SV9:(mu($i>$o))]: (~((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4) = ((p(@[^SY27:mu,SY28:$i]: (~((SV9@SY27)@SY28))))@SV4) = ((p@[^SY29:mu,SY30:$i]: (~((SV9@SY29)@SY30))))@SV4))= $false)), inference(fac_restr,[status(thm)], [57])).
    thf(86,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4)=$true) | (((p@SV9)@SV4) = ((p@[^SY29:mu,SY30:$i]: (~((SV9@SY29)@SY30))))@SV4))= $false)), inference(fac_restr,[status(thm)], [57])).
    thf(87,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | ((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4)) | (~((~((p@SV9)@SV4)) | (~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))))= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_equal_neg,[status(thm)], [85])).
    thf(88,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | ((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4)) | (~((~((p@SV9)@SV4)) | (~((~((p@SV9)@SV4) | (~((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true)), inference(extcnf_equal_neg,[status(thm)], [86])).
    thf(89,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | (~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))))= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_or_neg,[status(thm)], [87])).
    thf(90,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | ((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $false) | (((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true), inference(extcnf_or_neg,[status(thm)], [89])).
    thf(91,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | (~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_or_neg,[status(thm)], [91])).
    thf(92,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | (~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))))= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_or_neg,[status(thm)], [92])).
    thf(93,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4) | ((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $false) | (((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true), inference(extcnf_or_neg,[status(thm)], [93])).
    thf(94,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_or_pos,[status(thm)], [94])).
    thf(95,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true) | (((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true), inference(extcnf_or_pos,[status(thm)], [95])).
    thf(96,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true)), inference(extcnf_or_pos,[status(thm)], [96])).
    thf(97,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true) | (((p@(^SY29:mu,SY30:$i): (~((SV9@SY29)@SY30))))@SV4))= $true)), inference(extcnf_or_pos,[status(thm)], [97])).
    thf(98,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $false) | (~((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [98])).
    thf(99,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $false) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [99])).
    thf(100,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true)), inference(extcnf_not_pos,[status(thm)], [100])).
    thf(101,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true)), inference(extcnf_not_pos,[status(thm)], [101])).
    thf(102,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true)), inference(extcnf_not_pos,[status(thm)], [102])).
    thf(103,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true)), inference(extcnf_not_pos,[status(thm)], [103])).
    thf(104,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [104])).
    thf(105,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [105])).
    thf(106,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [106])).
    thf(107,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [107])).
    thf(108,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [108])).
    thf(109,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $false)), inference(extcnf_not_pos,[status(thm)], [109])).
    thf(110,plain,!([SV4:$i,SV9:(mu($i>$o))]: (((~((p@SV9)@SV4))= $true) | ((~((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true) | (((p@SV9)@SV4)= $false) | (((p@(^SY27:mu,SY28:$i): (~((SV9@SY27)@SY28))))@SV4))= $true)), inference(extcnf_not_pos,[status(thm)], [110])).
    thf(111,plain,!([SV3:$i,SV8:(mu($i>$o))]: (((~((p@SV8)@SV3))= $false) | ((~((p@(^SX0:mu,SX1:$i): $true))@SV3))= $true))), inference(sim,[status(thm)], [76])).
    thf(112,plain,!([SV1:$i,SV2:(mu($i>$o))]: (((~((p@(^SX0:mu,SX1:$i): $false))@SV3))= $false) | (((p@(^SX0:mu,SX1:$i): (~((SV1@SX0)@SX1)))@SV3))= $false) | (((p@(^SX0:mu,SX1:$i): (~((SV1@SX0)@SX1)))@SV3))= $true))), inference(sim,[status(thm)], [80])).
    thf(113,plain,!([$false],inference(fa_atp_e,[status(thm)], [25,112,111,118,109,108,107,84,83,82,75,74,73,72,71,70,69,68,67,66,65,62,57,56,51,42,29])).
    thf(114,plain,![$false],inference(solved_all_splits,[solved_all_splits(join,[])]), [113])).
% S25 output end CNFRefutation

***** End of derivation protocol *****
***** no. of clauses in derivation: 97 *****
***** clause counter: 113 *****
% S25 status Unsatisfiable for ConsistencyWithoutFirstConjunctionD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibE0:true,rAndE0:true,use_choice:true,use_extuni:true,use_extcnf_combined:true,expand_extuni:false,faotp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,faotp_calls:2,translati
on:of_full)
ontoleo:DemoMaterial cbenzmueller$ 

```

# Inconsistency (Gödel): Verification in Isabelle/HOL (KB)

The screenshot shows the Isabelle/HOL interface with the following details:

- Title Bar:** GoedelGodWithoutConjunctInEss\_KB.thy
- Toolbar:** Standard file operations (New, Open, Save, Print, etc.) and navigation icons.
- Text Area:** The code for the theory GoedelGodWithoutConjunctInEss\_KB.thy. The code defines various constants, axioms, theorems, and definitions related to Gödel's incompleteness theorem.
- Right Panel:** A vertical sidebar with tabs: Documentation, Sidekick, and Theories.
- Status Bar:** Shows the file size (11,1 (477/1095)), the operating system (isabelle,sidekick,UTF-8-Isabelle), the number of proofs (Nr 0 UG 263/347), and the current time (MB 17:18).

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
  consts P :: " $(\mu \Rightarrow \sigma) \Rightarrow \sigma$ "
  axiomatization where A1a: "[\forall(\lambda\Phi. P (\lambda x. m \multimap (\Phi x)) \multimap m \multimap (P \Phi))]"
    and A2: "[\forall(\lambda\Phi. \forall(\lambda\Psi. (P \Phi \multimap \square (\forall(\lambda x. \Phi x \multimap \Psi x)) \multimap P \Psi)))]"
  -- {* Positive properties are possibly exemplified. *}
  theorem T1: "[\forall(\lambda\Phi. P \Phi \multimap \diamond (\exists \Phi))]" by (metis A1a A2)
  definition ess (infixr "ess" 85) where "\Phi ess x = \forall(\lambda\Psi. \Psi x \multimap \square (\forall(\lambda y. \Phi y \multimap \Psi y)))"
  -- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(\forall(\lambda x. (\lambda y. \text{False}) ess x))]" by (metis ess_def)
  definition NE where "NE x = \forall(\lambda\Phi. \Phi ess x \multimap \square (\exists \Phi))"
  axiomatization where sym: "x r y \longrightarrow y r x"
  -- {* Exemplification of necessary existence is not possible. *}
  lemma Lemma2: "[m \multimap (\diamond (\exists NE))]" by (metis sym Lemma1 NE_def)
  axiomatization where A5: "[P NE]"
  -- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
  lemma False by (metis A5 T1 Lemma2)
end
```

# Inconsistency (Gödel): Verification in Isabelle/HOL (K)

The screenshot shows the Isabelle/HOL interface with the following details:

- Title Bar:** GoedelGodWithoutConjunctInEss\_K.thy
- Toolbar:** Standard Isabelle/HOL icons for file operations, search, and navigation.
- Text Area:** The code for the theory GoedelGodWithoutConjunctInEss\_K.thy. The code defines constants P, ess, and NE, and axiomatizations A1a and A2. It includes proofs for theorems T1 and Lemmal, and an axiomatization A5. A note indicates the inconsistency follows from A5, Lemmal, NE\_def, and T1. A sledgehammer proof is also shown.
- Right Panel:** Navigation links: Documentation, Sidekick, Theories.
- Status Bar:** Output, Query, Sledgehammer, Symbols; 21,7 (980/982); (isabelle,sidekick,UTF-8-Isabelle)Nr o UG 302/343MB 11:37

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: " $(\mu \Rightarrow \sigma) \Rightarrow \sigma$ "
  definition ess (infixr "ess" 85) where " $\Phi \text{ ess } x = \forall(\lambda\Psi. \Psi x \rightarrow \square (\forall(\lambda y. \Phi y \rightarrow \Psi y)))$ "
  definition NE where " $\text{NE } x = \forall(\lambda\Phi. \Phi \text{ ess } x \rightarrow \square (\exists \Phi))$ "
  axiomatization where A1a: "[ $\forall(\lambda\Phi. P (\lambda x. m \rightarrow (\Phi x)) \rightarrow m \rightarrow (P \Phi))$ ]"
    and A2: "[ $\forall(\lambda\Phi. \forall(\lambda\Psi. (P \Phi \wedge \square (\forall(\lambda x. \Phi x \rightarrow \Psi x)) \rightarrow m \rightarrow P \Psi))$ ]"

  -- {* Positive properties are possibly exemplified. *}
  theorem T1: "[ $\forall(\lambda\Phi. P \Phi \rightarrow \diamond (\exists \Phi))$ ]" by (metis A1a A2)

  -- {* The empty property is an essence of every individual. *}
  lemma Lemmal: "[ $\forall(\lambda x. (\lambda y. \lambda w. \text{False}) \text{ ess } x))$ ]" by (metis ess_def)

  axiomatization where A5: "[P NE]"

  -- {* Now the inconsistency follows from A5, Lemmal, NE_def and T1 *}
  lemma False
  -- {* sledgehammer [remote_leo2] *}
  by (metis A5 Lemmal NE_def T1)

end
```

## Inconsistency (Gödel): Informal Argument (in KB and K)

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

**Def. D2\***

$$\phi \text{ ess. } x \leftrightarrow \cancel{\phi(x)} \forall \psi(\psi(x) \rightarrow \square \forall y(\phi(y) \rightarrow \psi(y)))$$

## Inconsistency (Gödel): Informal Argument (in KB and K)

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

**Def. D2\***

$$\phi \text{ ess. } x \leftrightarrow \cancel{\exists \psi} \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$$

**Lemma 1** The empty property is an essence of every entity.  $\forall x (\emptyset \text{ ess. } x)$

## Inconsistency (Gödel): Informal Argument (in KB and K)

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**Theorem 1** Positive Properties are possibly exemplified.  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

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**Axiom A5**

$$P(NE)$$

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[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

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$$\phi \text{ ess. } x \leftrightarrow \cancel{\Diamond(\phi)} \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$$

**Lemma 1** The empty property is an essence of every entity.  $\forall x(\emptyset \text{ ess. } x)$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall \phi[P(\phi) \rightarrow \Diamond \exists x \phi(x)]$

**Axiom A5**

$$P(NE)$$

► by T1, A5:  $\Diamond \exists x[NE(x)]$

## Inconsistency (Gödel): Informal Argument (in KB and K)

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

**Def. D2\***

$$\phi \text{ ess. } x \leftrightarrow \cancel{\exists(\forall)} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

**Lemma 1** The empty property is an essence of every entity.  $\forall x(\emptyset \text{ ess. } x)$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

**Axiom A5**

$$P(NE)$$

► by T1, A5:  $\Diamond\exists x[NE(x)]$

**Def. D3**

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$$

## Inconsistency (Gödel): Informal Argument (in KB and K)

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

**Def. D2\***

$$\phi \text{ ess. } x \leftrightarrow \cancel{\exists \psi} \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$$

**Lemma 1** The empty property is an essence of every entity.  $\forall x(\emptyset \text{ ess. } x)$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall \phi[P(\phi) \rightarrow \Diamond \exists x \phi(x)]$

**Axiom A5**

$$P(NE)$$

► by T1, A5:  $\Diamond \exists x[NE(x)]$

**Def. D3**

$$NE(x) \leftrightarrow \forall \phi[\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)]$$

►  $\Diamond \exists x[\forall \phi[\phi \text{ ess. } x \rightarrow \Box \exists y[\phi(y)]]]$

## Inconsistency (Gödel): Informal Argument (in KB and K)

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

**Def. D2\***

$$\phi \text{ ess. } x \leftrightarrow \cancel{\exists \psi} \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$$

**Lemma 1** The empty property is an essence of every entity.  $\forall x(\emptyset \text{ ess. } x)$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall \phi[P(\phi) \rightarrow \Diamond \exists x \phi(x)]$

**Axiom A5**

► by T1, A5:  $\Diamond \exists x[NE(x)]$

**Def. D3**

$$NE(x) \leftrightarrow \forall \phi[\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)]$$

►  $\Diamond \exists x[\forall \phi[\phi \text{ ess. } x \rightarrow \Box \exists y[\phi(y)]]]$

►  $\Diamond \exists x[\emptyset \text{ ess. } x \rightarrow \Box \exists y[\emptyset(y)]]$

## Inconsistency (Gödel): Informal Argument (in KB and K)

[BenzmüllerWoltzenlogelPaleo, IJCAI, 2016]

**Def. D2\***

$$\phi \text{ ess. } x \leftrightarrow \cancel{\forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))}$$

**Lemma 1** The empty property is an essence of every entity.  $\forall x(\emptyset \text{ ess. } x)$

**Theorem 1** Positive Properties are possibly exemplified.  $\forall \phi[P(\phi) \rightarrow \Diamond \exists x \phi(x)]$

**Axiom A5**

► by T1, A5:  $\Diamond \exists x[NE(x)]$

**Def. D3**

$$NE(x) \leftrightarrow \forall \phi[\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)]$$

►  $\Diamond \exists x[\forall \phi[\phi \text{ ess. } x \rightarrow \Box \exists y[\phi(y)]]]$

►  $\Diamond \exists x[\emptyset \text{ ess. } x \rightarrow \Box \exists y[\emptyset(y)]]$

► by L1  $\Diamond \exists x[\top \rightarrow \Box \exists y[\emptyset(y)]]$

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$$P(NE)$$

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► by def. of  $\emptyset$   $\Diamond \exists x[\top \rightarrow \Box \perp]$

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►  $\diamond \exists x[\square \perp]$

►  $\diamond \square \perp$

**Inconsistency**

$\perp$

# Gödel's Manuscript: Identifying the Inconsistent Axioms

Ontologischer Bereich      Feb. 10, 1970

$P(\varphi)$      $\varphi$  is positive    ( $\Leftrightarrow \varphi \in P$ )

Ax 1:  $P(p), P(\varphi) \supset P(\varphi \wedge \psi) \quad$  At 2:  $P(p) \supset P(x \neq y)$

P1  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)] \quad$  (God)

P2  $\varphi_{\text{Exn}x} \equiv (\psi)[\psi(x) \supset N(y)[P(y) \supset \psi(y)]] \quad$  (Existence)

$P \supset_N q = N(p \supset q) \quad$  Necessity

Ax 2  $P(\varphi) \supset N P(\varphi) \quad$  } because it follows  
 $\neg P(\varphi) \supset N \neg P(\varphi) \quad$  } from the nature of the  
 property

Th.  $G(x) \supset G_{\text{Exn}x}$

Df.  $E(x) \equiv P[\varphi_{\text{Exn}x} \supset N \neg x \cdot \varphi(x)] \quad$  necessary Existence

Ax 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(x) G(x) \supset M N(\exists y) G(y) \quad$  M = possibility

"  $\supset N(\exists y) G(y)$

any two elements of  $X$  are nec. equivalent  
 exclusive or \* and for any number of them

$M(x) G(x)$  means all pos. prop. w.r.t. com-  
 patible. This is true because of:  
Ax 4:  $P(\varphi) \cdot q \supset \psi \supset P(\psi)$  which impl.  
 $\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$   
 But if a system S of pos. prop. were incons.  
 It would mean that the non-prop. S (which  
 is positive) would be  $x \neq x$ .

Positive means positive in the moral aest.  
 sense (independently of the accidental structure of  
 the world). Only in the ex. True. It is  
 also meant "Attribution" as opposed to "Platization"  
 (or containing negation). This interprets the word

of "positive":  $(x) N \neg p(x)$  Otherwise:  $P(x) \supset x \neq x$   
 hence  $x \neq x$  positive not  $x=x$  i.e. negation. At  
 the end of proof Ax 4  
 also X i.e. the normal form in terms of elem. prop. contains  
 members without negation.

# Gödel's Manuscript: Identifying the Inconsistent Axioms

Ontologischer Beweis      Feb. 10, 1970

$P(\phi)$      $\phi$  is positive    ( $\Leftrightarrow \phi \in P$ )

Ax 1:  $P(p), P(\neg p) \supset P(\phi \wedge \neg p)$     Ax 2:  $P(p) \supset P(\neg \neg p)$

p1  $G(x) \equiv (\phi)[P(\phi) \supset \phi(x)]$     (God)

p2  $\phi \text{ Em. } x \equiv (\psi)[\psi(x) \supset N(y)[P(y) \supset \psi(y)]]$     (Em. of  $x$ )

$P \supset_N q = N(P \supset q)$     Necessity

Ax 2     $P(\phi) \supset N P(\phi)$      $\neg P(\phi) \supset N \neg P(\phi)$     } because it follows from the nature of the property

Th.  $G(x) \supset G \text{ Em. } x$

Df.  $E(x) \equiv P[\phi \text{ Em. } x \supset N \exists y \phi(x)]$     necessary Existence

Ax 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

"     $M(x) G(x) \supset M N(\exists y) G(y)$

"     $\supset N(\exists y) G(y)$

any two instances of  $x$  are nec. equivalent

exclusive or    and for any number of them

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Th.  $\begin{cases} x = x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. prop. were incons. it would mean that the non-prop. S (which is positive) would be  $x \neq x$ .

Positive means positive in the moral aesthet. sense (independently of the accidental structure of the world). Only  $\exists$  in the ax. frame. It is also pure.

## Inconsistency

$$\forall \phi [P(\neg \phi) \rightarrow \neg P(\phi)]$$

$$\forall \phi \forall \psi [(P(\phi) \wedge \square \forall x [\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

$$\phi \text{ ess. } x \leftrightarrow \forall \psi (\psi(x) \rightarrow \square \forall y (\phi(y) \rightarrow \psi(y)))$$

$$NE(x) \leftrightarrow \forall \phi [\phi \text{ ess. } x \rightarrow \square \exists y \phi(y)]$$

$$P(NE)$$

Scott

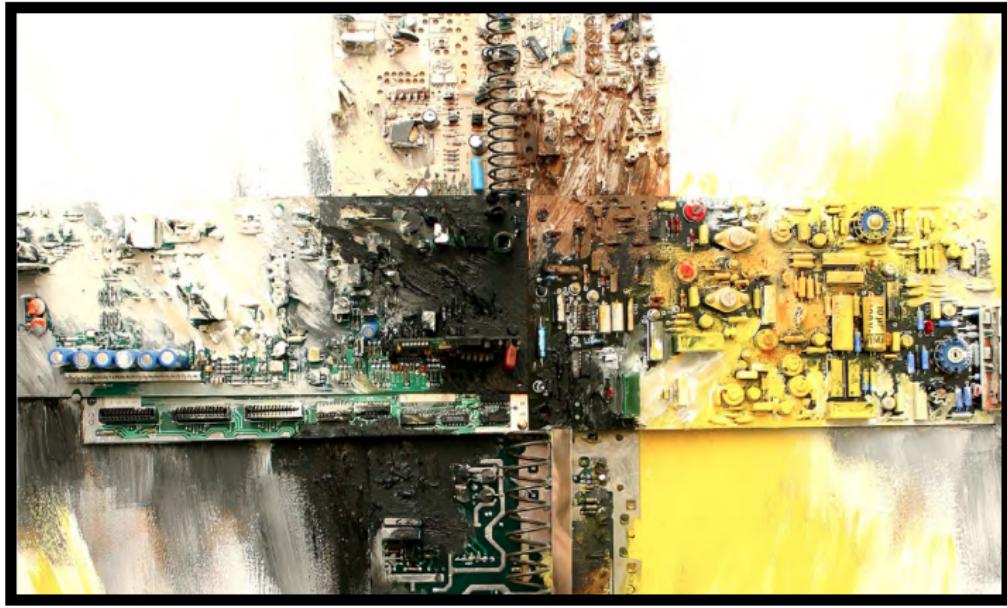
A1( $\supset$ )

A2

D2\*

D3

A5



## Part D: Recent Technical Improvements

## Usability: More Intuitive Syntax for Embedded Logics in Isabelle

```
definition ess :: " $(\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$ " (infixr "ess" 85) where  
"Φ ess x = Φ x m \wedge \forall(\lambda\Psi. \Psi x m \rightarrow \Box(\forall(\lambda y. \Phi y m \rightarrow \Psi y)))"
```

```
definition ess (infixr "ess" 85) where  
"Φ ess x = Φ(x) \wedge (\forall\Psi. \Psi(x) \rightarrow \Box(\forall y. \Phi(y) \rightarrow \Psi(y)))"
```

## Improved Embedding of Modal Logic S5: S5U

### Modal Logic S5

- ▶ Reflexivity:  $\forall x.(r x x)$
- ▶ Symmetry:  $\forall x.\forall y.(r x y) \rightarrow (r y x)$
- ▶ Transitivity:  $\forall x.\forall y.\forall z.(r x y) \wedge (r y z) \rightarrow (r x z)$

### Modal Logic S5U: with universal accessibility

- ▶ Universality:  $\forall x.\forall y.(r x y)$

### S5

$$\Box\varphi \equiv \lambda w.\forall v.\cancel{r(v,w)} \rightarrow \varphi(v) \quad \text{and} \quad \Diamond\varphi \equiv \lambda w.\exists v.\cancel{r(w,v)} \wedge \varphi(v)$$

### S5U

$$\Box\varphi \equiv \lambda w.\forall v.\varphi(v) \quad \text{and} \quad \Diamond\varphi \equiv \lambda w.\exists v.\varphi(v)$$

# Gödel's God in Isabelle/HOL

The screenshot shows the Isabelle/HOL IDE interface with the file `Scott_SSU.thy` open. The code defines a theory `Scott_SSU` that imports `QML_SSU`. It includes several axioms (A1-A5) and definitions (G, ess, NE) related to propositional logic and modal operators. A theorem at the bottom states "Notwendigerweise, existiert Gott".

```
theory Scott_SSU imports QML_SSU
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  axiomatization where
    A1: "[∀Φ. P(¬Φ) ↔ ¬P(Φ)]" and
    A2: "[∀Φ. ∀Ψ. (P(Φ) ∧ □(∀x. Φ(x) → Ψ(x))) → P(Ψ)]"
    definition G where
      D1: "G(x) = (¬Φ. P(Φ) → Φ(x))"
    axiomatization where
      A3: "[P(G)]" and
      A4: "[∀Φ. P(Φ) → □(P(Φ))]"
      definition ess (infixr "ess" 85) where
        D2: "Φ ess x = Φ(x) ∧ (¬Ψ. Ψ(x) → □(¬Y. Φ(y) → Ψ(y)))"
      definition NE where
        D3: "NE(x) = (¬Φ. Φ ess x → □(¬x. Φ(x)))"
    axiomatization where
      A5: "[P(NE)]"

  theorem (* Notwendigerweise, existiert Gott *)

```



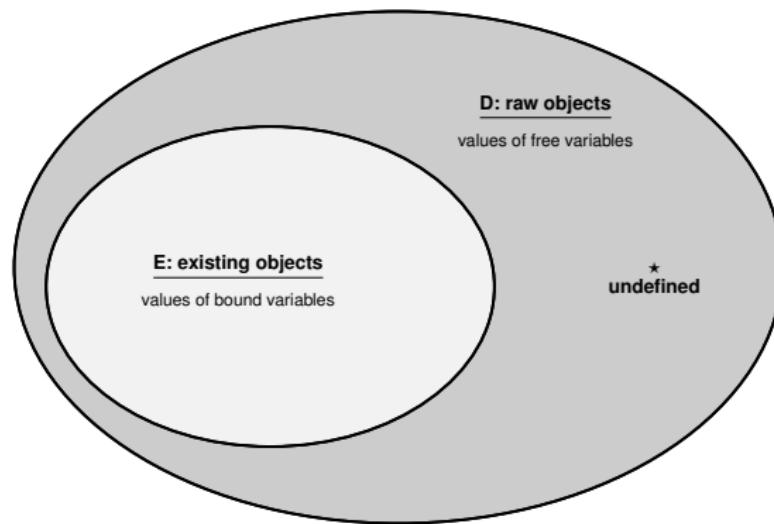
**Part E:**  
**Free Logic (Scott) & Application in Category Theory**  
jww: Dana Scott

## Free Logic: Elegant Approach to Deal with Undefinedness

**Principle 1:** Bound individual variables range over domain  $E \subset D$

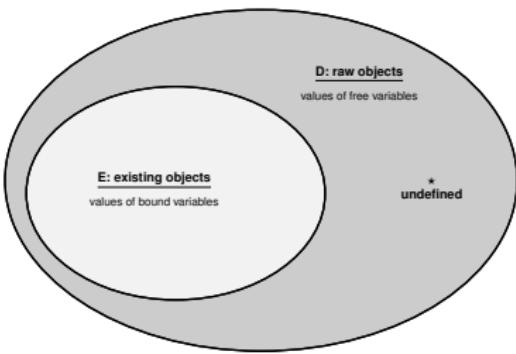
**Principle 2:** The domain  $E$  may be empty.

**Principle 3:** The values of terms and free variables are in  $D$ , not necessarily in  $E$  only.



**Figure:** Illustration of the Semantical Domains of Free Logic

# Free Logic in HOL: Application in Category Theory (jww Dana Scott)



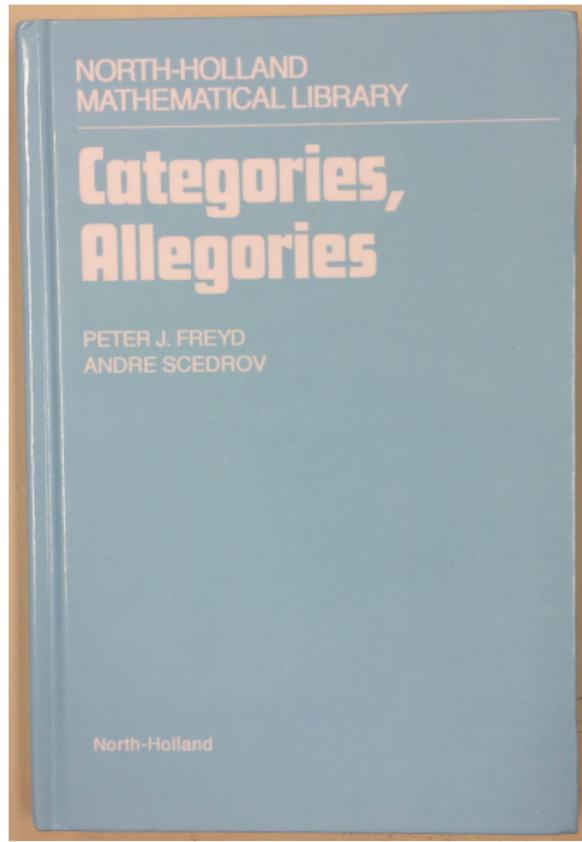
FreeFOLminimal.thy

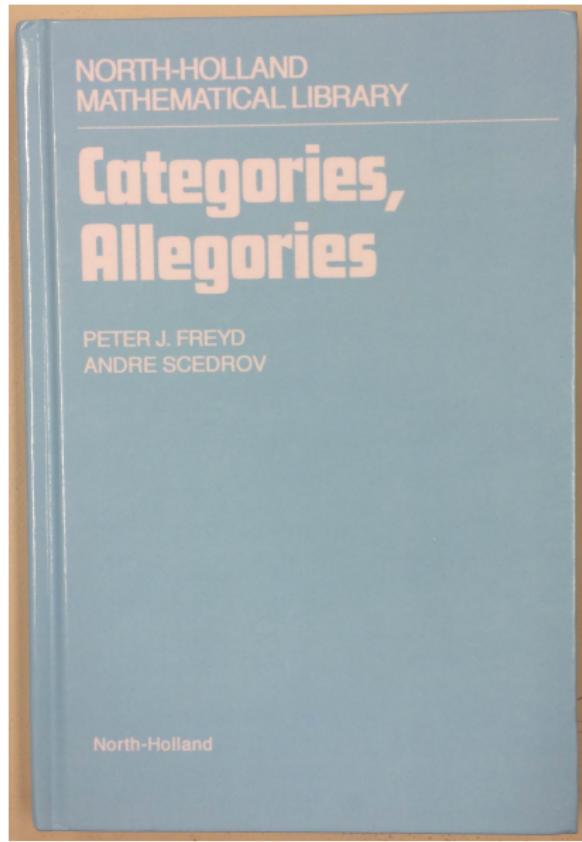
```
typedcl i -- "the type for individuals"
consts fExistence:: "i⇒bool" ("E") -- "Existence predicate"
consts fStar:: "i*" ("*") -- "Distinguished symbol for undefinedness"

axiomatization where fStarAxiom: "¬E(*)"

abbreviation fNot:: "bool⇒bool" ("¬")
where "¬φ ≡ ¬φ"
abbreviation fImplies:: "bool⇒bool⇒bool" (infixr "→" 49)
where "φ→ψ ≡ φ→ψ"
abbreviation fForall:: "(i⇒bool)⇒bool" ("∀")
where "∀Φ ≡ ∀x. E(x) → Φ(x)"
abbreviation fForallBinder:: "(i⇒bool)⇒bool" (binder "∀" [8] 9)
where "∀x. φ(x) ≡ ∀φ"
abbreviation fThat:: "(i⇒bool)=i" ("I")
where "IΦ ≡ if ∃x. E(x) ∧ Φ(x) ∧ (∀y. (E(y) ∧ Φ(y)) → (y = x))
      then THE x. E(x) ∧ Φ(x)
      else *"
abbreviation fThatBinder:: "(i⇒bool)=i" (binder "I" [8] 9)
where "Ix. φ(x) ≡ I(φ)"
abbreviation fOr (infixr "∨" 51) where "φ∨ψ ≡ (¬φ)→ψ"
abbreviation fAnd (infixr "∧" 52) where "φ∧ψ ≡ ¬(φ∨¬ψ)"
abbreviation fEquiv (infixr "↔" 50) where "φ↔ψ ≡ (φ→ψ)∧(ψ→φ)"
abbreviation fEquals (infixr "≡" 56) where "x≡y ≡ x=y"
abbreviation fExists ("∃") where "∃b ≡ ¬(∀(λy. ¬(Φ y)))"
abbreviation fExistsBinder (binder "∃" [8] 9) where "∃x. φ(x) ≡ ∃φ"

consts
fForall :: "(i ⇒ bool) ⇒ bool"
```





## 1.1. BASIC DEFINITIONS

The theory of CATEGORIES is given by two unary operations and a binary partial operation. In most contexts lower-case variables are used for the ‘individuals’ which are called *morphisms* or *maps*. The values of the operations are denoted and pronounced as:

- $\square x$  the source of  $x$ ,
- $x\square$  the target of  $x$ ,
- $xy$  the composition of  $x$  and  $y$ .

The axioms:

- $\text{A1: } xy \text{ is defined iff } x\square = \square y,$
- $\text{A2a: } (\square x)\square = \square x \text{ and } \square(x\square) = x\square, \quad \text{A2b}$
- $\text{A3a: } (\square x)x = x \text{ and } x(\square x) = x, \quad \text{A3b}$
- $\text{A4a: } \square(xy) = \square(x(\square y)) \text{ and } (xy)\square = ((x\square)y)\square, \quad \text{A4b}$
- $\text{A5: } x(yz) = (xy)z.$

**1.11.** The ordinary equality sign  $=$  will be used only in the symmetric sense, to wit: if either side is defined then so is the other and they are equal. A theory, such as this, built on an ordered list of partial operations, the domain of definition of each given by equations in the previous, and with all other axioms equational, is called an ESSENTIALLY ALGEBRAIC THEORY.

**1.12.** We shall use a venturi-tube  $\geq$  for *directed equality* which means: if the left side is defined then so is the right and they are equal. The axiom that  $\square(xy) = \square(x(\square y))$  is equivalent, in the presence of the earlier axioms, with  $\square(xy) \geq \square x$  as can be seen below.

**1.13.**  $\square(\square x) = \square x$  because  $\square(\square x) = \square((\square x)\square) = (\square x)\square = \square x$ . Similarly  $(x\square)\square = x\square$ .

```

FreydScedrovMinimal.thy (~ /GITHUBS/PrincipiaMetaphysica/FreeLogic/2016-ICMS/)

consts source:: "i=i" ("□_"
target:: "i=i" ("_□" [110] 111)
composition:: "i→i=i" (infix "◦" 110)

abbreviation OrdinaryEquality:: "i→i=bool" (infix "≈" 60)
where "x ≈ y ≡ ((E x) ↔ (E y)) ∧ x = y"

axiomatization FreydsAxiomSystemReduced where
A1: "E(x·y) ↔ ((x□) ≈ (y□))" and
A2a: "((□x)□) ≈ □x" and
A3a: "(□x)·x ≈ x" and
A3b: "x·(□x) ≈ x" and
A5: "x·(y·z) ≈ (x·y)·z"

lemma B2b: "□(x□) ≈ □x" by (metis A1 A2a A3a)
lemma B4a: "□(x·y) ≈ □(x·(□y))" by (metis A1 A2a A3a)
lemma B4b: "(x·y)□ ≈ ((x□)·y)□" sledgehammer [A1 A2a A3a A3b A5]

Sledgehammer...
"spass": Try this: by (metis A1 A2a A3a) (44 ms).
"e": Try this: by (metis A1 A2a A3a) (24 ms).
"z3": Try this: by (metis A1 A2a A3a) (83 ms).
"cvc4": Try this: by (metis A1 A2a A3a) (21 ms).

Output Query Sledgehammer Symbols
23.43 (635/4324) (isabelle,isabelle,UTF-8-isabelle) 61MB 2:00 AM

```

## Conclusion

### Overall Achievements

- ▶ significant contribution towards a **Computational Metaphysics**
- ▶ **novel results** contributed by **HOL-ATPs**
- ▶ infrastructure can be adapted for **other logics and logic combinations**
- ▶ **basic technology works well**; however, improvements still needed

### Relevance (wrt foundations and applications)

- ▶ Philosophy, AI, Computer Science, Computational Linguistics, Maths

### Related work: only for Anselm's simpler argument

- ▶ first-order ATP PROVER9 [OppenheimerZalta, 2011]
- ▶ interactive proof assistant PVS [Rushby, 2013]

### Ongoing/Future work

- ▶ (Awarded) Lecture course **Computational Metaphysics** at FU Berlin
- ▶ Landscape of verified/falsified ontological arguments
- ▶ You may contribute: <https://github.com/FormalTheology/GoedelGod.git>