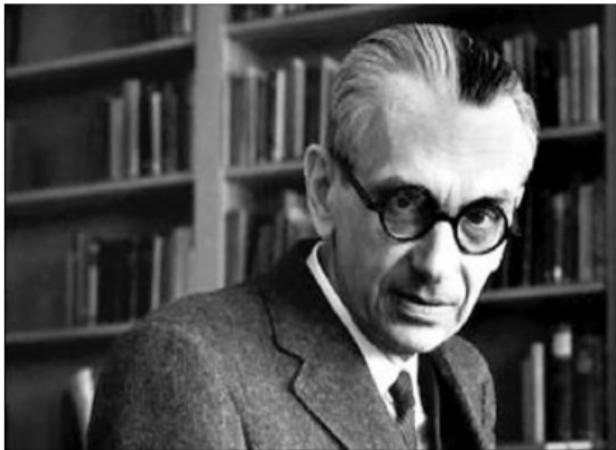


# Can computers help to sharpen our understanding of ontological arguments?

Christoph Benzmüller (jww D. Fuenmayor)

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“There is a scientific (exact) philosophy and theology,  
which deals with concepts of the highest abstractness;  
and this is also most highly fruitful for science.”

- Kurt Gödel (Wang, 1996)[p. 316]

## Presentation Outline

### A Ontological Argument of Gödel & Scott on the Computer

- ▶ Recap (AISSQ-2015 talk) of Methodology and Main Findings

### B Relevant Notions for this Talk:

- ▶ Intension vs. extension of properties (philosophy of language)
- ▶ Ultrafilter (mathematics)

### C Comparative Analysis on the Computer:

- ▶ Gödel/Scott (1972) variant
- ▶ Anderson's (1990) variant
- ▶ Fitting's (2002) variant

### D Discussion: Metaphysics, Mathematics and Reality



## Part A

### — Computational Metaphysics (recap) —

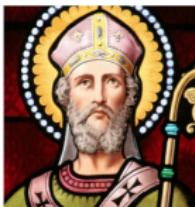
### Ontological Argument by Gödel & Scott on the Computer

Related work:

- ▶ Ed Zalta (& co) with PROVER9 at Stanford [AJP 2011, CADE 2015]
- ▶ John Rushby with PVS at SRI [CAV-WS 2013, JAL 2018]

# Ontological Proofs of God's Existence

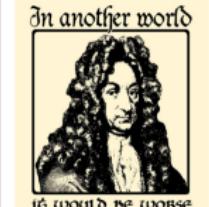
## A Long and Continuing Tradition in Philosophy



St. Anselm



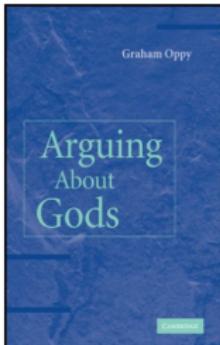
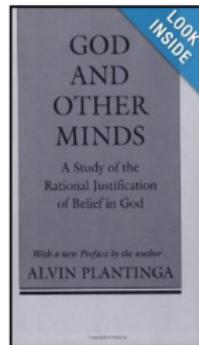
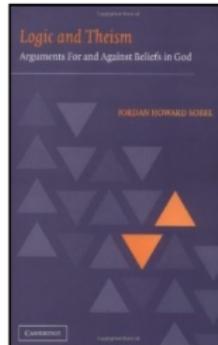
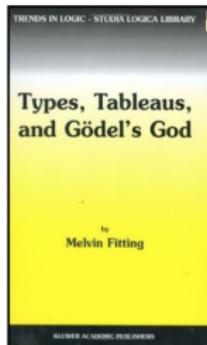
Descartes



Leibniz



Gödel



# Computational Metaphysics: Kurt Gödel's Ontological Argument (1970)

Ontologischer Beweis      Feb 10, 1970

P(φ)     $\varphi$  is positive      ( $\Leftrightarrow \varphi \in P$ )

At 1  $P(\varphi), P(\psi) \vdash P(\varphi \wedge \psi)$       At 2  $P(\varphi) \vdash P(\neg \varphi)$

P1  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$       (God)

P2  $\varphi \text{ Emx} \equiv (\psi)[\forall x(\varphi(x) \supset N(y)[\varphi(y) \supset \psi(y)])]$       (Existence)

$P \supset_N q = N(p \supset q)$       Necessity

At 2  $\begin{array}{l} P(\varphi) \supset N P(\varphi) \\ \sim P(\varphi) \supset N \sim P(\varphi) \end{array} \left. \begin{array}{l} \text{because it follows} \\ \text{from the nature of the} \\ \text{property} \end{array} \right\}$

Th.  $G(x) \supset G \text{ Em. } x$

Df.  $E(x) \equiv (\varphi)[\varphi \text{ Emx} \supset N \exists x \varphi(x)]$       necessary Existence

At 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

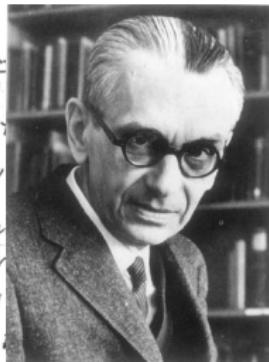
thus  $(\exists x) G(x) \supset N(\exists y) G(y)$

"  $M(\exists x) G(x) \supset M N(\exists y) G(y)$

"  $\supset N(\exists y) G(y)$       M = permuting

any two instances of  $x$  are met. equivalent  
exclusive or      and for any number of numerants

$M(x) G(x)$  means "all  
possible This is:  
At 4:  $P(\varphi), \varphi \supset \psi$   
 $\begin{cases} x=x & \text{is pr.} \\ x \neq x & \text{is not pr.} \end{cases}$   
 But if a system S is  
it would mean, that  
(if positive) would be  $x \neq x$



Positive means positive in the moral aesthetic sense (independently of the accidental structure of the world). Only ~~the~~ <sup>in</sup> the at time. It may also mean "affirmation" as opposed to "privation" (or crushing privation). This supports the platonist

$\neg \varphi$  is negative  $\neg (\varphi \supset \psi)$  Otherwise  $\varphi \supset \psi \supset x \neq x$   
hence  $x \neq x$  (positive)  $\neg x \neq x$  (negative) At the end of proof At

i.e. the formal form in terms of elem. prop. contains a member without negation.

# Computational Metaphysics: Kurt Gödel's Ontological Argument (1970)

Ontologischer Beweis      FEB 10, 1970

P( $\varphi$ )     $\varphi$  is positive    ( $\Leftrightarrow \varphi \in P$ )

At. 1:  $P(\varphi), P(\psi) \vdash P(\varphi \wedge \psi) \quad \vdash P(\varphi) \wedge P(\neg \varphi)$

P1  $G(x) \equiv (\varphi) [P(\varphi) \supset \varphi(x)] \quad (\text{God})$

P2  $\varphi \text{ FM } x = (\forall y) E_{\text{Ex. } x} y [P(y) \supset \varphi(y)] \quad (\text{Ex. of } x)$

$P \supset_N q = N(P \supset q) \quad \text{Necessity}$

At. 2     $\begin{array}{l} P(\varphi) \supset N P(\varphi) \\ \sim P(\varphi) \supset N \sim P(\varphi) \end{array} \quad \left. \begin{array}{l} \text{because it follows} \\ \text{from the nature of the} \\ \text{property} \end{array} \right.$

Th.  $G(x) \supset E_{\text{Ex. } x}$

Df.  $E(x) \equiv (\forall y) [P(\text{Ex. } x \supset N \exists z \varphi(z))]$     necessary Existence

At. 3  $P(E)$

Th.  $G(x) \supset N \exists z \varphi(z)$

$M(x) F(x)$ : means all pos. prop. w.r.t. com-  
patible      <sup>the system of</sup>  
This is true because of:

At 4:  $P(\varphi), \varphi \supset \psi \vdash P(\psi)$  which impl.  
~~True~~ {  $x=x$  is positive  
~~False~~ {  $x \neq x$  is negative

But if a system S of pos. prop. were incon-  
sistent it would mean, that the non-prop. S (which  
is positive) would be  $x \neq x$

Positive means positive in the moral aesth.  
sense (independently of the accidental structure of  
the world). Only ~~in the~~ the at. time. It re-  
fers also to "attribution" as opposed to "privatism"

## Notion of "Godlike":

- Being Godlike is equivalent to having all positive properties.

Note: this definition is "second-order".

# Computational Metaphysics: Kurt Gödel's Ontological Argument (1970)

Ontologischer Beweis

Feb 10, 1970

$M(\exists x)F(x)$  means all non-empty sets in the system of

## In the end we prove

- Necessarily (N), there exists God.

Note: we need to formalize "necessity" and "possibility".

Th.  $G(x) \supset G \text{ Em. } x$

Df.  $E(x) \equiv \exists p(p \in x \wedge \forall y \exists q(q \in y \rightarrow q \in x))$  necessary Existence

Ax 3  $\mathcal{P}(E)$

Th.  $G(x) \supset N(\exists y)G(y)$

hence  $(\exists x)G(x) \supset N(\exists y)G(y)$

"  $M(\exists x)G(x) \supset MN(\exists y)G(y)$

"  $\supset N(\exists y)G(y)$  M-penitent

any two elements of  $M$  are not equivalent,  
exclusive or \* and for any number of members in  $M$

a positive would be  $x \neq x$

Positive means positive in the moral aesthetic sense (independently of the accidental structure of the world). Only ~~the~~ <sup>the</sup> at time. It may also mean "affirmation" as opposed to "privation" (or crushing <sup>destruction</sup> penitence). This supports the plausibility of

$\exists y \supset p(y) \wedge (\forall x)(N(\neg p(x)) \supset G(x) \supset x \neq x)$

hence  $x \neq x$  (positive)  $\wedge x = x$  (negative) At the epiphany of God (At)

i.e. the formal formulation of elem. prop. contains a member without negation.

# Computational Metaphysics: Gödel's (1970) and Scott's Variants (1972)

Onkelogischer Bereich      Feb 10, 1970

$P(\varphi)$      $\varphi$  is positive      ( $\Leftrightarrow \varphi \in P$ )

At 1  $P(\varphi), P(\psi) \supset P(\varphi \wedge \psi)$       At 2  $P(\varphi) \supset P(\neg \varphi)$

P1  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$       (Good)

P2  $\varphi \text{ Emx} \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$       (Emx  $\forall x$ )

$P \supset_N q = N(p \supset q)$       Necessity

At 2  $\begin{array}{l} P(\varphi) \supset N P(\varphi) \\ \sim P(\varphi) \supset N \sim P(\varphi) \end{array} \left. \begin{array}{l} \text{because it follows} \\ \text{from the nature of the} \\ \text{property} \end{array} \right\}$

Th.  $G(x) \supset G \text{ Em. } x$

Df.  $E(x) \equiv (\varphi)[\varphi \text{ Emx} \supset N \exists x \varphi(x)]$       necessary Existence

At 3  $P(E)$

Th.  $G(x) \supset N(\exists y) G(y)$

    hence  $(\exists x) G(x) \supset N(\exists y) G(y)$

    "       $M(\exists x) G(x) \supset M N(\exists y) G(y)$

    "       $\supset N(\exists y) G(y)$       M = permitting

any two instances of  $x$  are mech. equivalent

exclusive or      and for any number of nonnumericals

$M(\exists x) G(x)$  means "all  
possible This is:  
At 4:  $P(\varphi) \cdot q \supset \psi$   
~~the~~  $\left\{ \begin{array}{l} x=x \text{ is pr} \\ x \neq x \text{ is } \end{array} \right.$   
But if a system S has  
it would mean, that the num.prop. is (which  
is positive) would be  $x \neq x$



Positive means positive in the moral aesthetic sense (independently of the accidental structure of the world). Only ~~the~~ the at time. It may also mean "Attribution" as opposed to "privatization (or creation of privation)." This supports the plausibility

$\neg \varphi \text{ positive } \neg (\varphi \supset \psi) \text{ otherwise } \varphi \supset \psi \supset x \neq x$   
hence  $x \neq x$  positive and  $x = x$  negative At the end of prof. At

i.e. the formal form in terms of elem. prop. contains a member without negation.

# Computational Metaphysics: Gödel's (1970) and Scott's Variants (1972)

Onkologischer Beweis      Feb 10, 1970

P(φ)     $\varphi$  is positive      ( $\Leftrightarrow \varphi \in P$ )

At 1  $P(\varphi), P(\psi) \vdash P(\varphi \wedge \psi)$       At 2  $P(\varphi) \vdash P(\neg \varphi)$

P1  $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$       (Göd.)

P2  $\varphi \text{ Emax } x \equiv (\psi)[\forall y(x \supset N(y) \supset \varphi(y) \supset \psi(y))]$       (Emax  $\forall x$ )

$P \supset_N q = N(p \supset q)$       Necessity

At 2  $\begin{array}{l} P(\varphi) \supset N P(\varphi) \\ \sim P(\varphi) \supset N \sim P(\varphi) \end{array} \left. \begin{array}{l} \text{because it follows} \\ \text{from the nature of the} \\ \text{property} \end{array} \right\}$

Th.  $G(x) \supset G_{\text{max}}$

Df.  $E(x) \equiv (\varphi)[\varphi \text{ Emax } \exists x \varphi(x)]$       necessary Existence

A (3)  $P(E)$

Th.  $G(x) \supset N(\forall y) G(y)$

$M(x) G(x)$ : means "all pos. propo. in: com-  
patible"      the system of  
This is true because of:  
At 4:  $P(\varphi), \varphi \supset \psi \vdash P(\psi)$  which impl.  
 $\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$   
But if a system S of pos. propo. were incon-  
sistent it would mean, that the num.prop. s (which  
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Positive means positive in the moral aesth.  
sense. (independently of the accidental structure of  
the world). [Only ~~in the at. time~~ It me-  
ans also "Attribution" as opposed to "privation"  
(or contains negation). This is important and

## (Main) Difference between Gödel and Scott

- **Gödel:** Property E is essence of x iff all of x's properties are entailed by E.
- **Scott:** Property E is essence of x iff **x has E and** all of x's properties are entailed by E.

## (Higher-Order) Modal Logic

$\Box P$

P is necessary, P is obligatory, P is known/believed, . . .

$\Diamond P$

P is possible, P is permissible, P is epistemically/doxastic. possible, . . .

$\Box$  and  $\Diamond$  are not truth-functional

Higher-Order Logic can be extended by  $\Box P$  and  $\Diamond P$

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## (Higher-Order) Modal Logic

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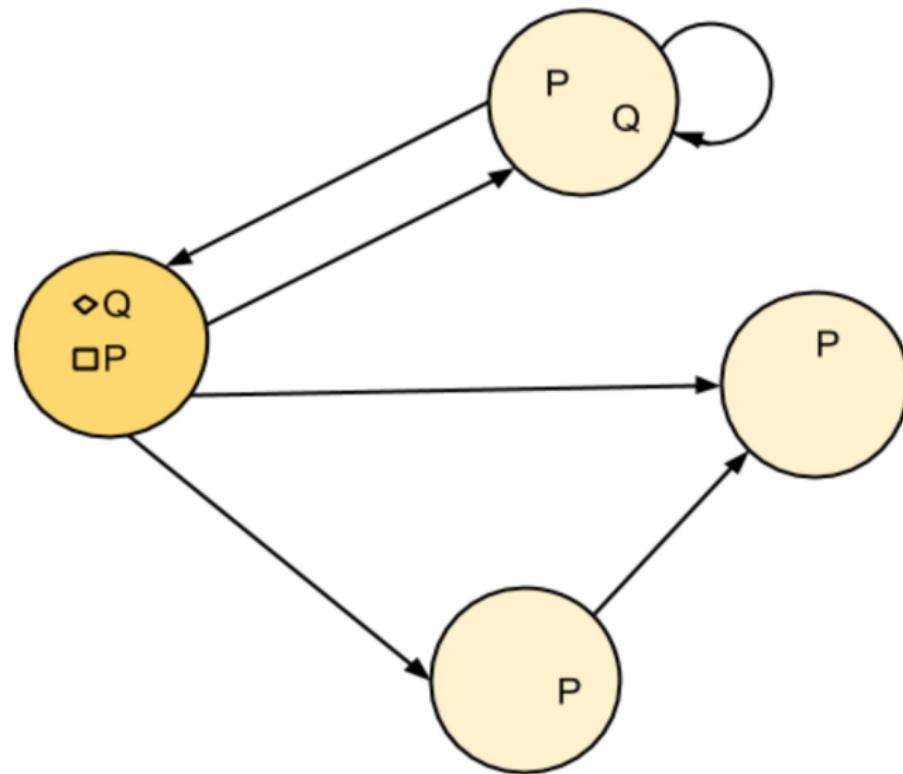
$\Diamond P$

**P is possible**, P is permissible, P is epistemically/doxastic. possible, ...

$\Box$  and  $\Diamond$  are not truth-functional

Higher-Order Logic can be extended by  $\Box P$  and  $\Diamond P$

## (Higher-Order) Modal Logics: Kripke-style Semantics - Possible Worlds



## Computational Metaphysics: Scott's and Gödel's Variants — Demo

**Axiom** Either a property or its negation is positive, but not both:  $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

**Axiom** A property necessarily implied by a positive property is positive:

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

**Thm.** Positive properties are possibly exemplified:  $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

**Def.** A Godlike being possesses all positive properties:  $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

**Axiom** The property of being Godlike is positive:  $P(G)$

**Cor.** Possibly, God exists:  $\Diamond\exists xG(x)$

**Axiom** Positive properties are necessarily positive:  $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

**Def.** An essence of an individual is a **property possessed by it and necessarily implying any of its properties**:  $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

**Thm.** Being Godlike is an essence of any Godlike being:  $\forall x[G(x) \rightarrow G \text{ ess. } x]$

**Def.** Necessary existence of an individual is the necessary exemplification of all its essences:  $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$

**Axiom** Necessary existence is a positive property:  $P(NE)$

**Thm.** Necessarily, God exists:  $\Box\exists xG(x)$

# Computational Metaphysics: Scott's and Gödel's Variants — Demo

Axiom

$$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$$

Axiom

$$\forall\phi\forall\psi[(P(\phi) \wedge \square\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm.

$$\forall\phi[P(\phi) \rightarrow \diamond\exists x\phi(x)]$$

Def.

$$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$$

Axiom

$$P(G)$$

Cor.

$$\diamond\exists xG(x)$$

Axiom

$$\forall\phi[P(\phi) \rightarrow \square P(\phi)]$$

Def.

$$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \square\forall y(\phi(y) \rightarrow \psi(y)))$$

Thm.

$$\forall x[G(x) \rightarrow G \text{ ess. } x]$$

Def.

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \square\exists y\phi(y)]$$

Axiom

$$P(NE)$$

Thm.

$$\square\exists xG(x)$$

## Computational Metaphysics: Scott's and Gödel's Variants — Demo

Axiom

$$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$$

Axiom

$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Def.

$$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$$

Axiom

$$P(G)$$

Axiom

$$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$$

Def.

$$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

Def.

$$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$$

Axiom

$$P(NE)$$

Thm.

$$\Box\exists xG(x)$$

# Computational Metaphysics: Scott's and Gödel's Variants — Demo

The screenshot shows a computer screen with a window titled "UltrafilterMovie1". Inside the window, there is a code editor displaying a file named "GoedelProof.thy". The code is written in Isabelle/HOL, a formal proof system. The code defines various logical constructs and proves theorems related to Gödel's incompleteness theorem. The interface includes a vertical toolbar on the right with buttons for "Documentation", "Sidekick", "State", and "Theories". At the bottom of the window, there is a toolbar with buttons for "Output", "Query", "Sledgehammer", and "Symbols". A status bar at the very bottom shows the text "theorem U3:  $\mathcal{P}' \subseteq \mathcal{P} \wedge \mathcal{P} \subseteq$ " and "Undefined fact: 'T6'". Below the status bar, there is a media control bar with buttons for back, forward, and volume, and a progress bar indicating the video is at 00:00 of 02:26.

```
theory GoedelProof imports IHOML      (* This formalization follows Fitting's textbook *)
begin

(*Positiveness/perfection: uninterpreted constant symbol*)
consts positiveProperty::"(e⇒i⇒bool)⇒i⇒bool" ("P")
(*Some auxiliary definitions needed to formalise A3*)
definition h1 ("pos")    where "pos Z ≡ ∀X. Z X → P X"
definition h2 (infix "∩" 60) where "X ∩ Z ≡ □(∀x.(X x ↔ (¬(Y z) → (Y x))))"
definition h3 (infix "⇒" 60) where "X ⇒ Y ≡ □(∀z. X z → Y z)"

(**Part I**)
(*D1*) definition G ("G") where "G ≡ (λx. ∀Y. P Y → Y x)"
(*A1*) axiomatization where Ala: "[|X. P (¬X) → ¬(P X)|]" and Alb:"[|X. ¬(P X) → P (¬X)|]"
(*A2*) axiomatization where A2: "[|X Y. (P X ∧ (X ⇒ Y)) → P Y|]"
(*A3*) axiomatization where A3: "[|Z X. (pos Z ∧ X ∩ Z) → P X|]"
(*T1*) theorem T1: "[|X. P X → ◇E X|]" by (metis Ala A2 h3_def)
(*T2*) theorem T2: "[|P G|]" proof -
  {have 1: "¬∀w. ∃Z X. (P G ∨ pos Z ∧ X ∩ Z ∧ ¬P X) w" by (metis(full_types) G_def h1_def h2_def)
   have 2: "[|Z X. (pos Z ∧ X ∩ Z) → P X|] → [|P G|]" using 1 by auto}
  thus ?thesis using A3 by blast qed
(*T3*) theorem T3: "[|◇E G|]" using T1 T2 by simp

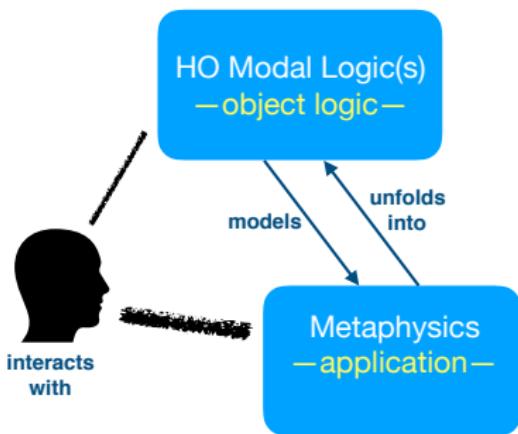
(**Part II**)
(*Logic VDK*) axiomatization where Gamma = "symmetric & open"
```

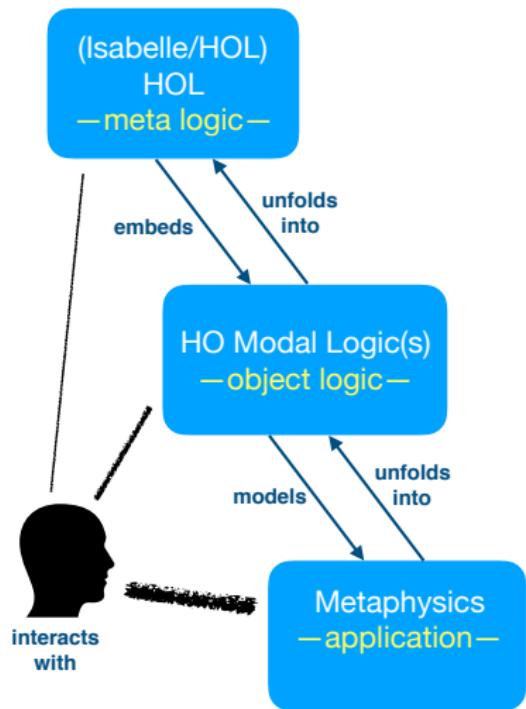


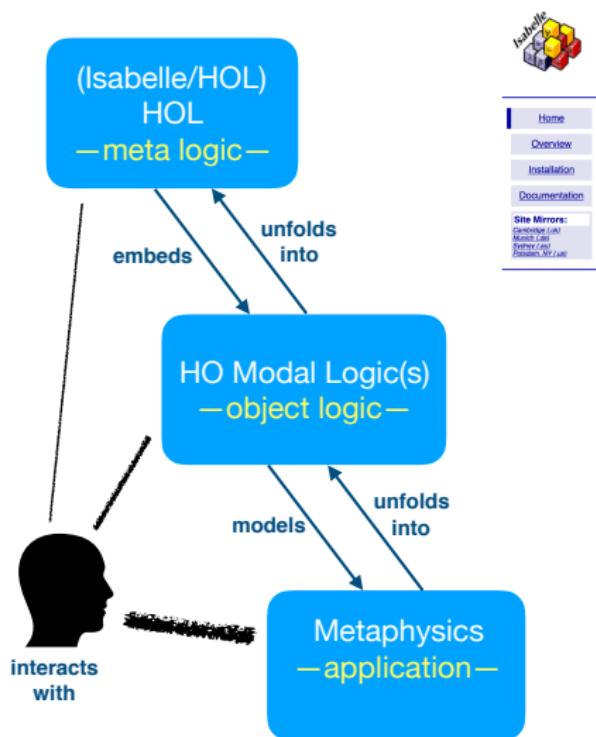
interacts  
with



Metaphysics  
—application—







Home  
Overview  
Installation  
Documentation

Site Mirrors:  
Cambridge (UK)  
Munich (Germany)  
Dresden (Germany)  
Pittsburgh (USA)

## Isabelle

UNIVERSITY OF CAMBRIDGE  
Computer Laboratory  
**TUM**  
Technische Universität München

### What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulae to be expressed in a formal language and provides tools for proving those formulae in a logical calculus. Isabelle was originally developed at the University of Cambridge and Technische Universität München, but now includes numerous contributions from institutions and individuals worldwide. See the [Isabelle overview](#) for a brief introduction.

### Now available: Isabelle2017 (October 2017)



[Download for Linux](#) - [Download for Windows \(32bit\)](#) - [Download for Windows \(64bit\)](#) - [Download for Mac OS X](#)

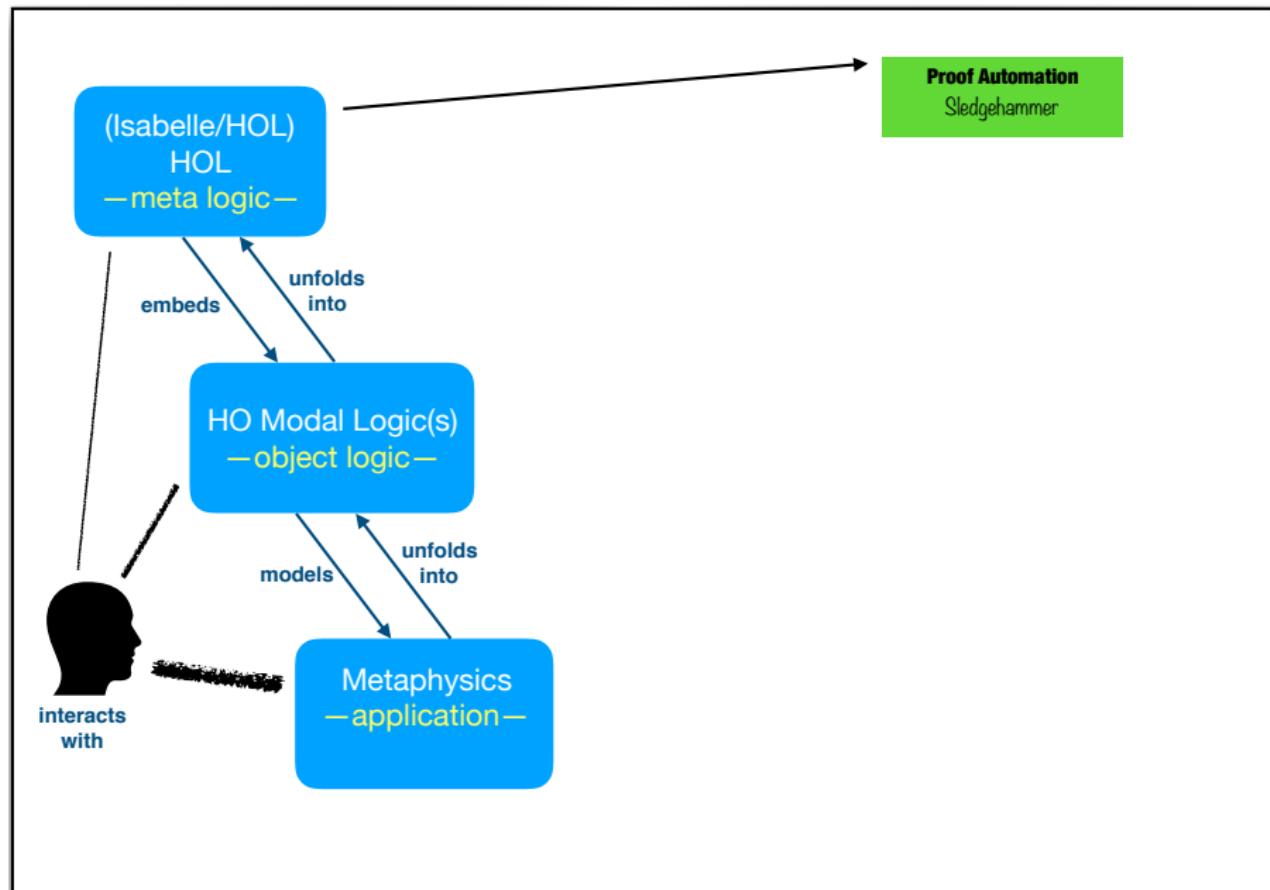
### Some notable changes:

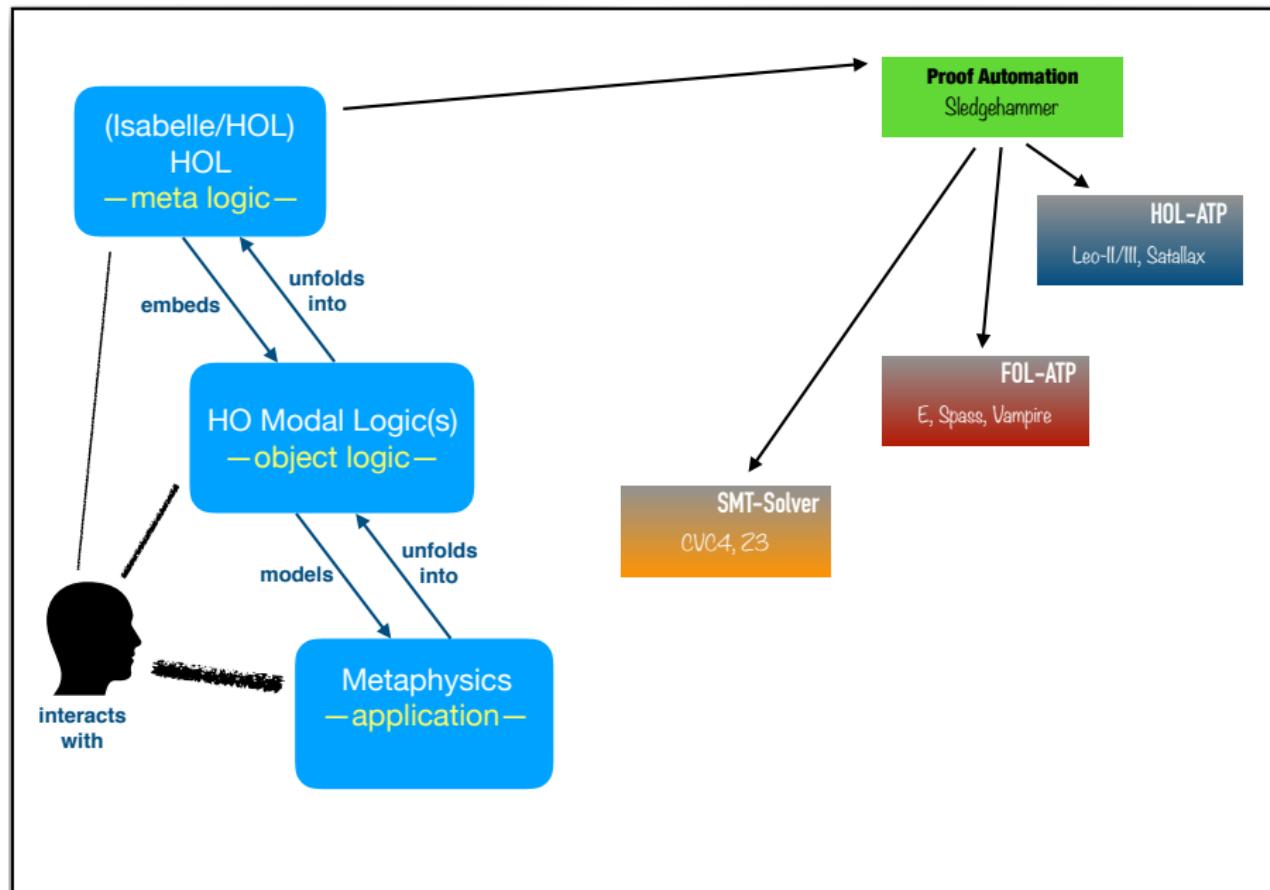
- Experimental support for Visual Studio Code as alternative PIDE front-end.
- Improved Isabelle/Edits Prover IDE: management of session sources independently of editor buffers, removal of unused theories, explicit indication of theory status, more careful auto-indentation.
- Session-qualified theory imports.
- Code navigation improvements: support for statically embedded computations.
- Numerous HOL library improvements.
- More material in HOL-Algebra, HOL-Computational\_Algebra and HOL-Analysis (ported from HOL-Light).
- Improved Nunchaku model finder, now in main HOL.
- SML database support in Isabelle/Scala.

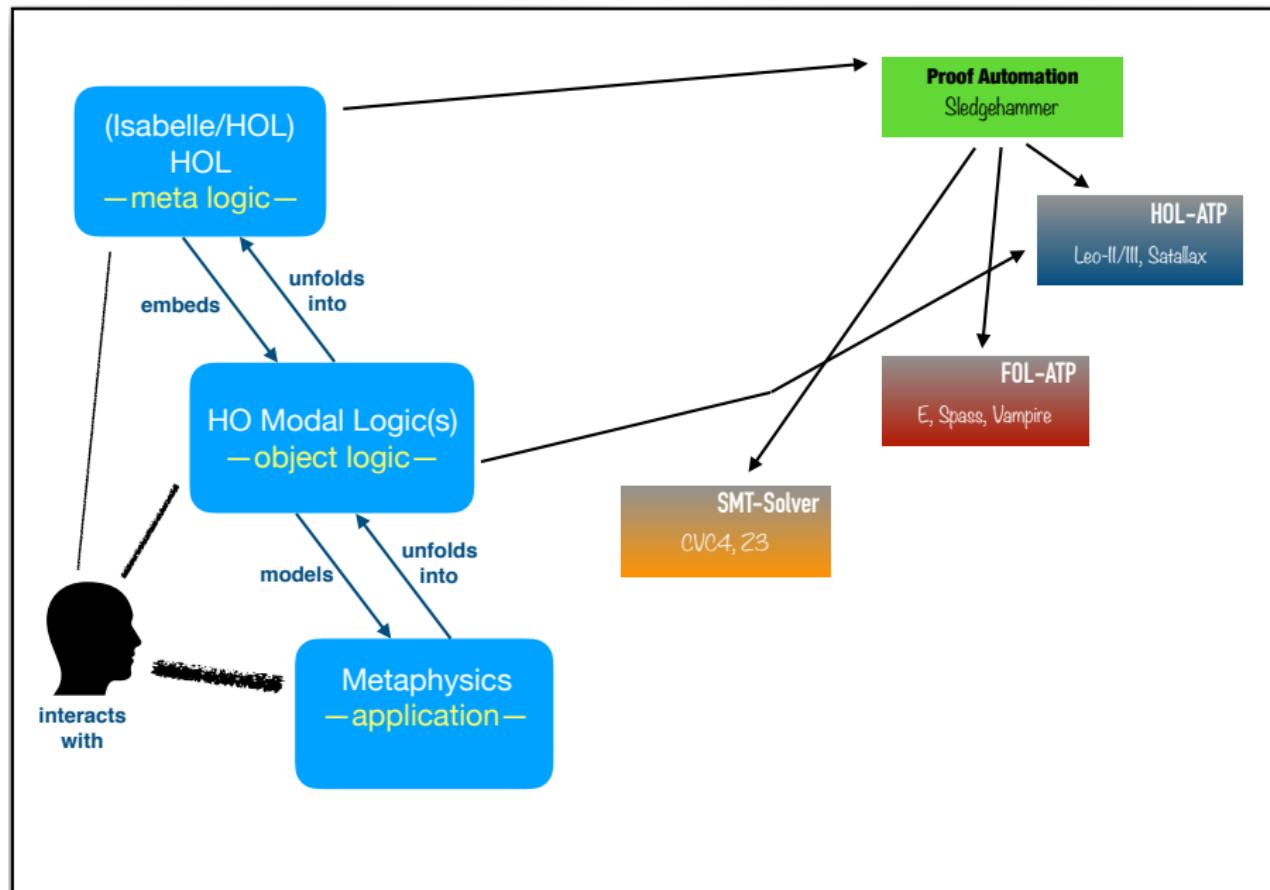
See also the cumulative [NEWS](#).

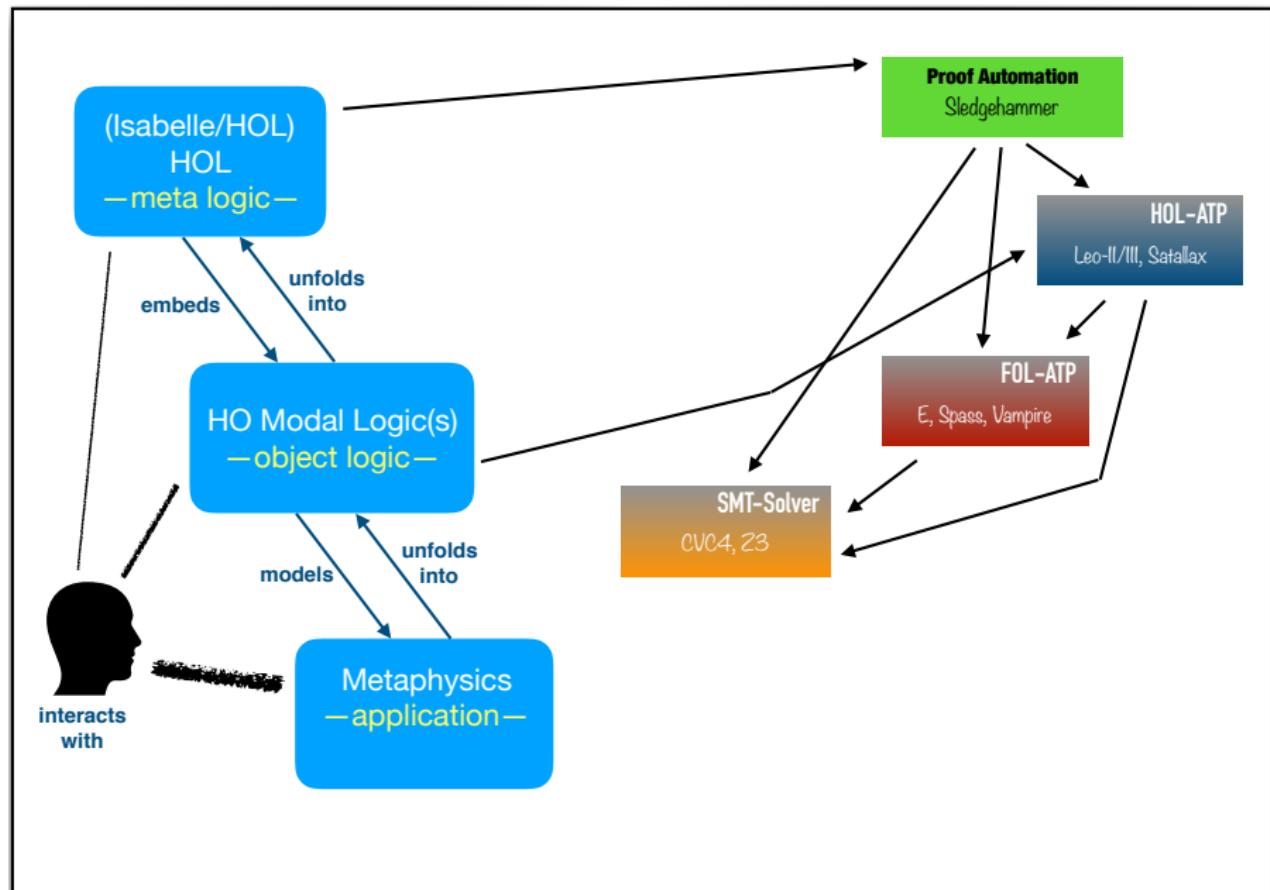
### Distribution & Support

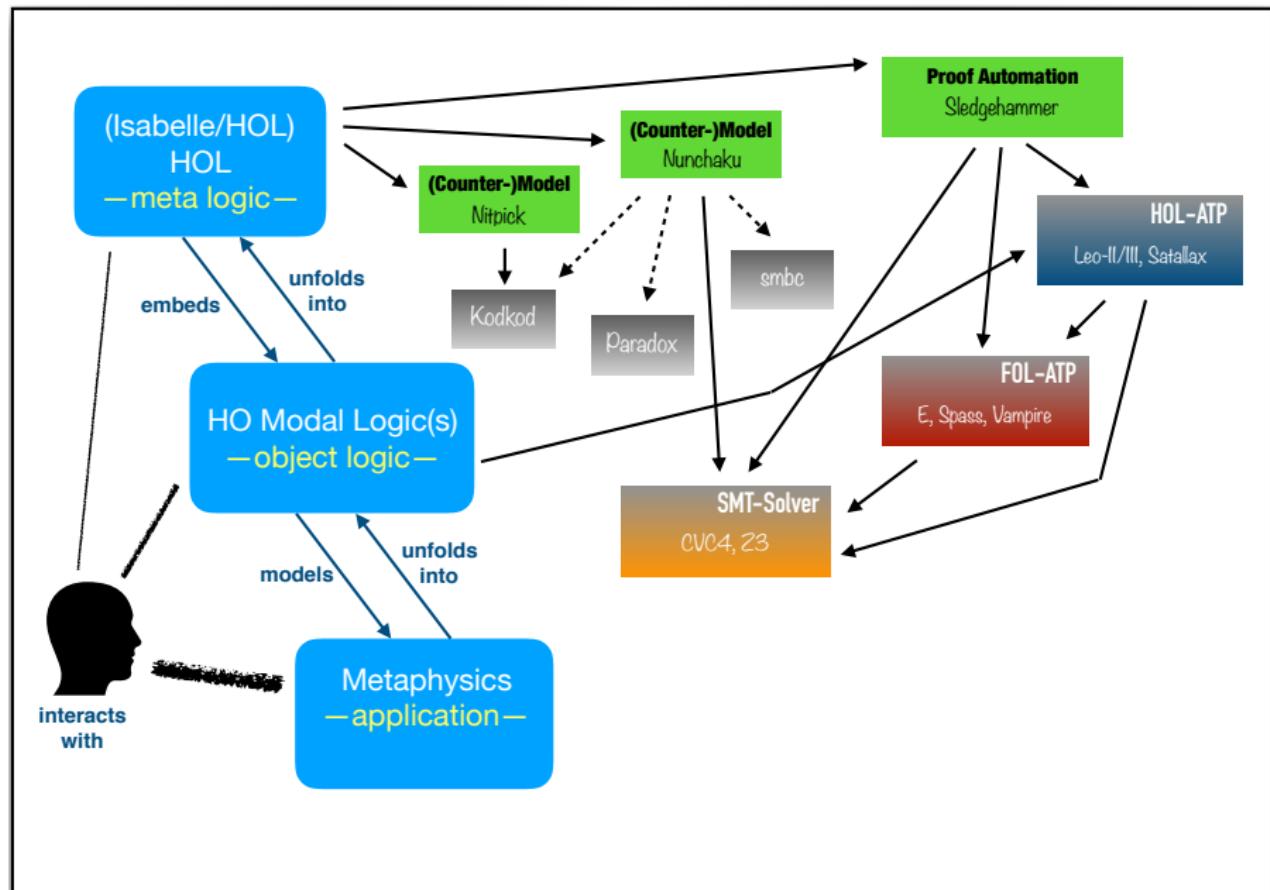
Isabelle is distributed for free under a conglomeration of open-source licenses, but the main code-base is subject to BSD-style regulations. The application bundles include source and binary packages and documentation, see the detailed [installation instructions](#). A vast collection of Isabelle examples and applications is available from the [Archive of Formal Proofs](#).

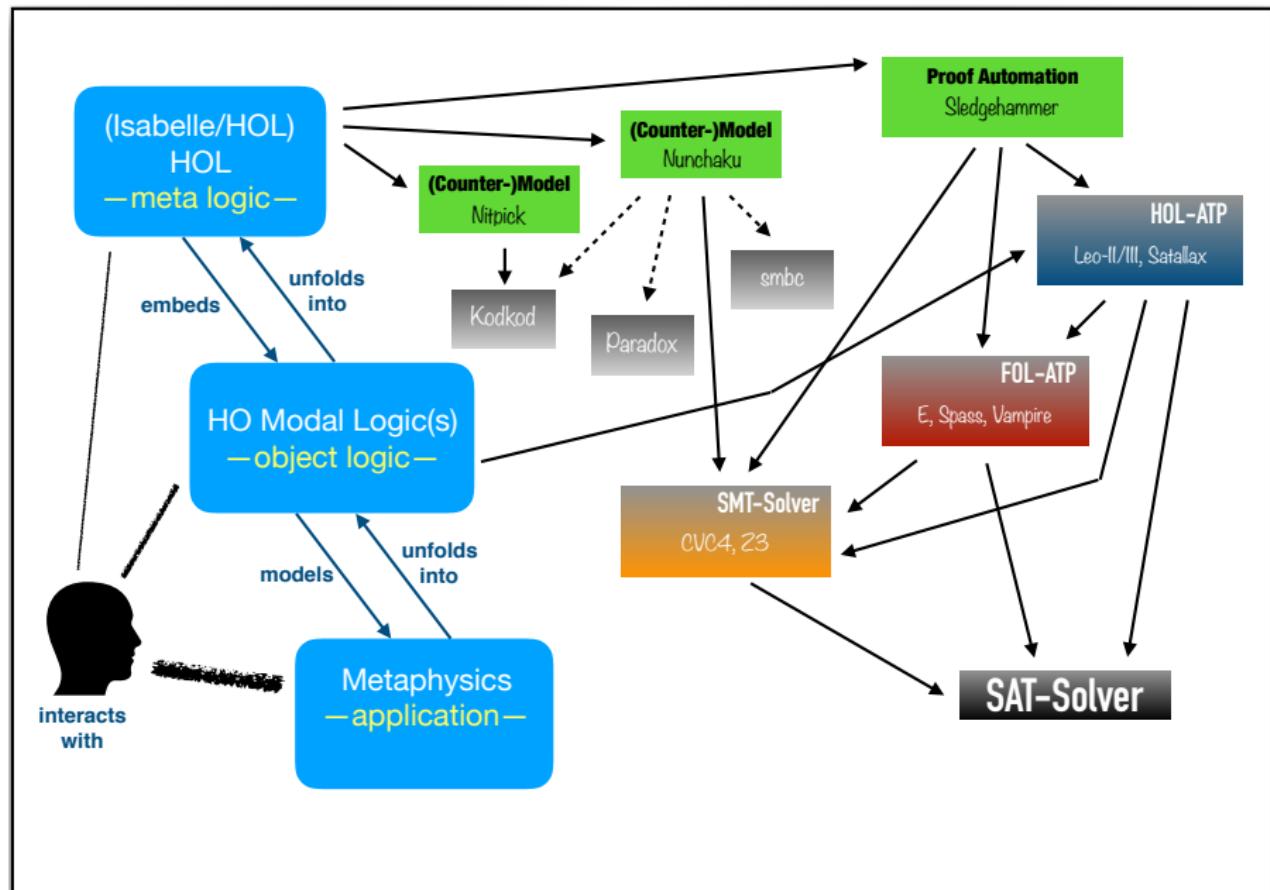


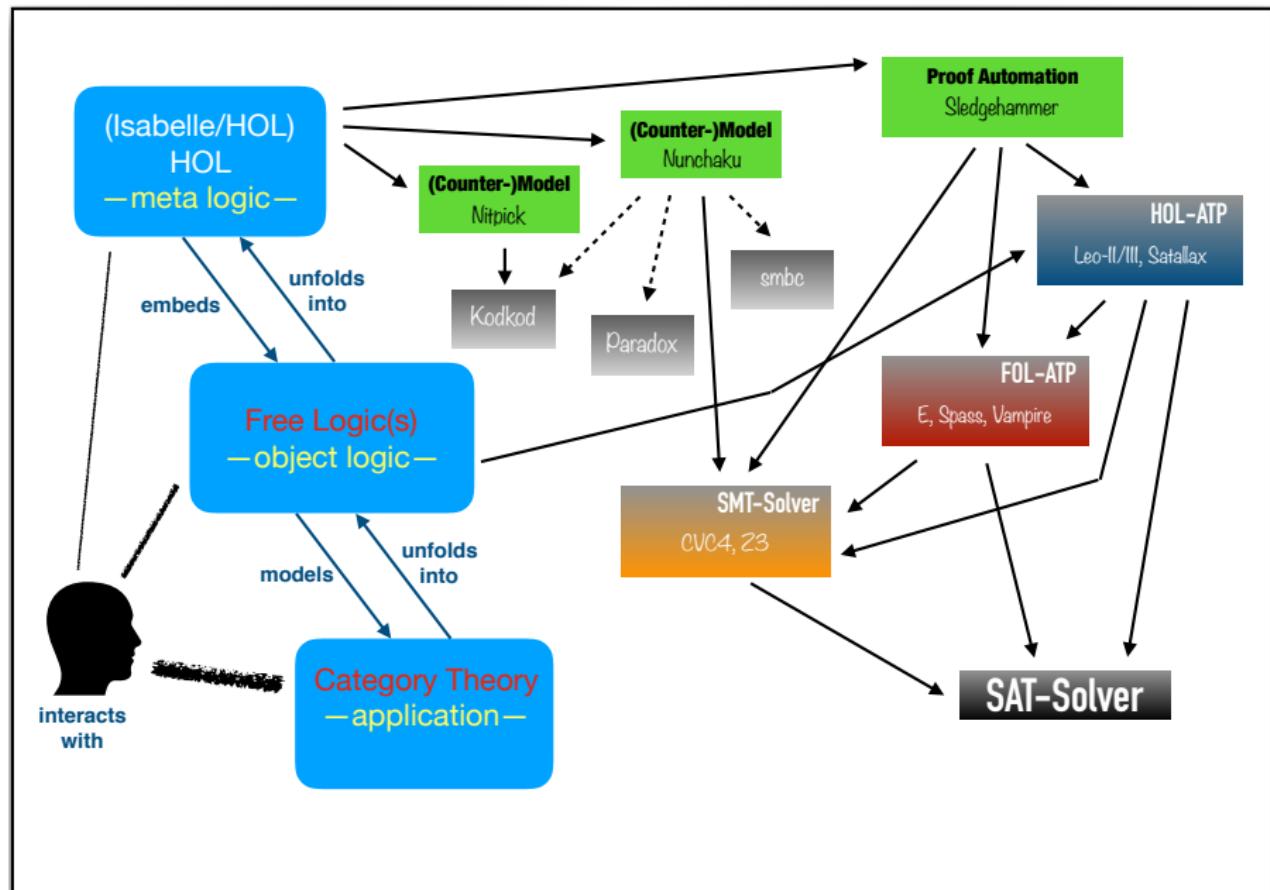


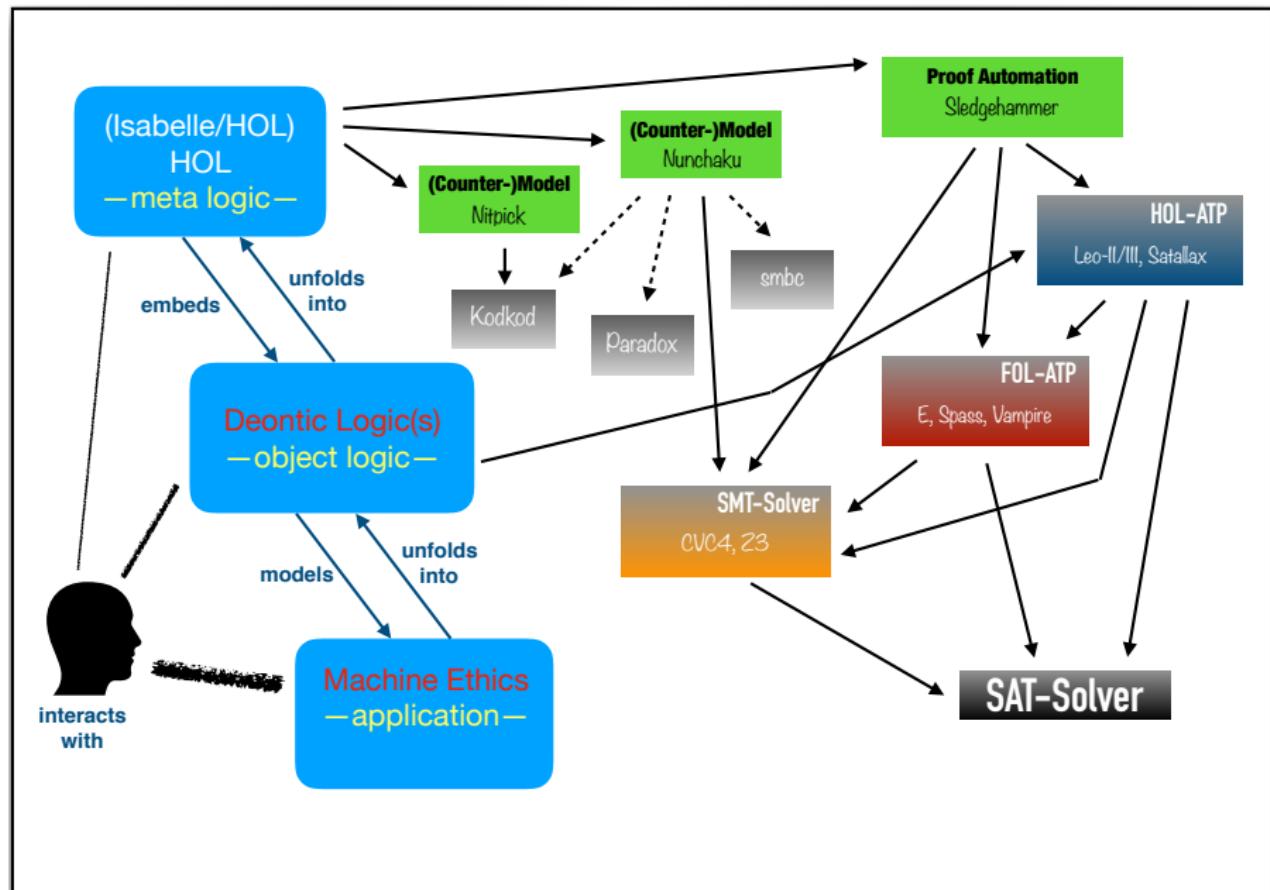


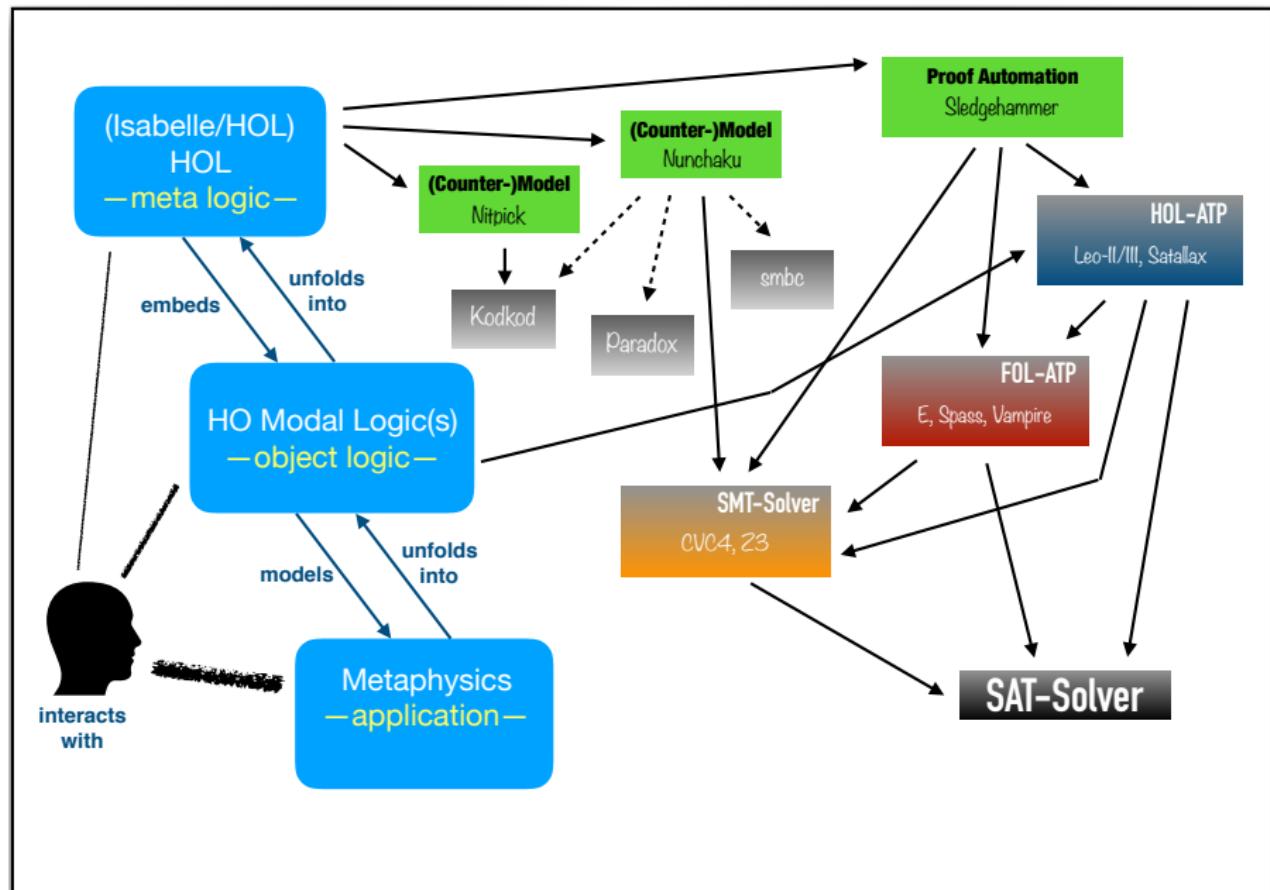


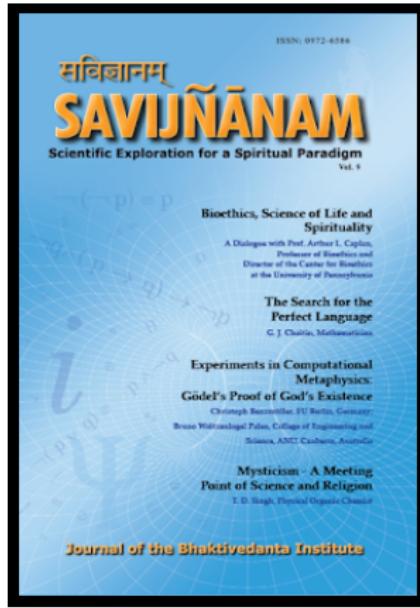












## Results of our Experiments (jww B. Woltzenlogel-Paleo) (see also [Savijnanam 2017] and [AISSQ 2015] talk)

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### Variant of Dana Scott (1972)

- ▶ the premises are **consistent**
- ▶ all argument steps are **logically correct** in (higher-order, extensional) modal logic
  - correct in logic **S5**
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... we continue with Scott's version

Further corollaries we can prove

- ▶ Monotheism
- ▶ Gott is flawless (has only positive properties)
- ▶ ...
- ▶ Modal Collapse:  $\varphi \rightarrow \Box \varphi$

- ▶ there are no contingent truths
- ▶ no alternative worlds
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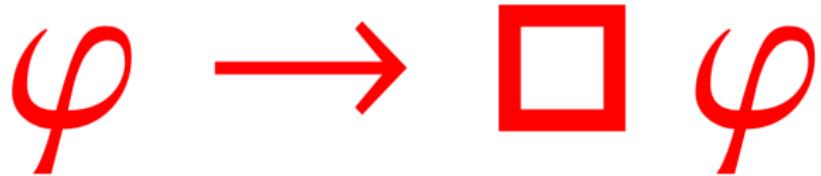
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— Can the modal collapse be avoided? —

## Remainder of this Talk

We will have a closer look at

- ▶ Gödel/Scott (1972) modal collapse
- ▶ C. Anthony Anderson (1990) avoids modal collapse
- ▶ Melvin Fitting (2002) avoids modal collapse

### Questions:

- ▶ How do Anderson and Fitting the avoid modal collapse?
- ▶ Are their solutions related?

To answer this questions we will apply some notions from

- ▶ mathematics: ultrafilters
- ▶ philosophy of language: extension and intension of predicates

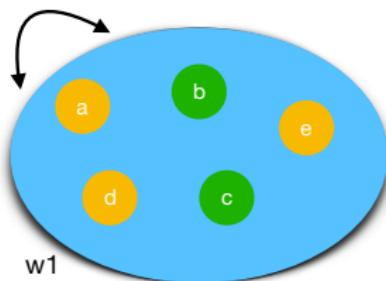


## Part B

### Some Relevant Pillar Stones for this Talk

## Intension vs. Extension of a Predicate (Philosophy of Language)

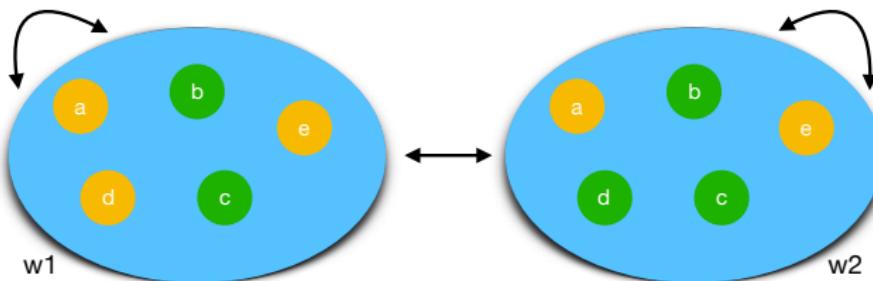
Example predicate: **IsChessGrandmaster**



- Intensional Predicate **IsChessGrandmaster (ICG)**
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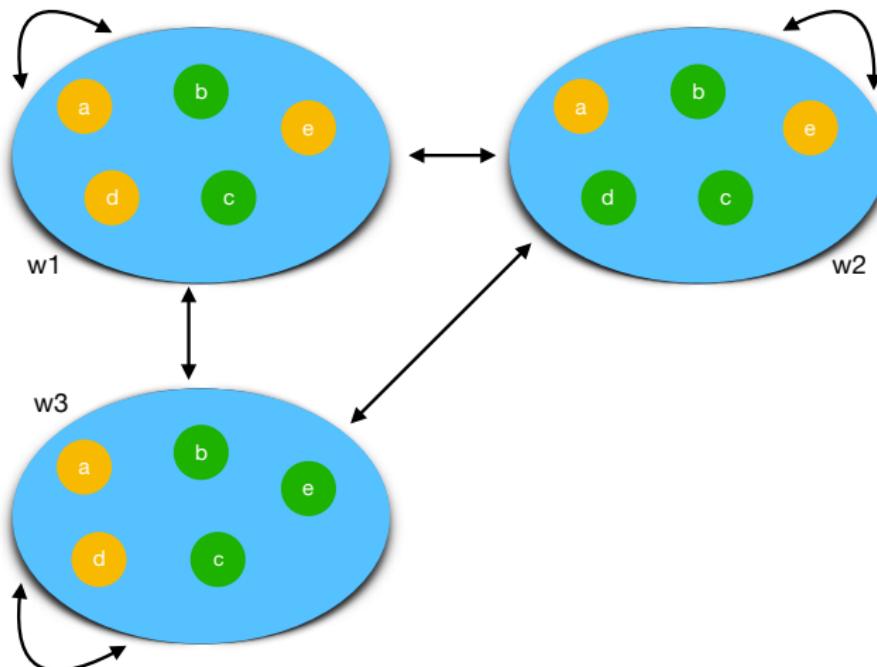
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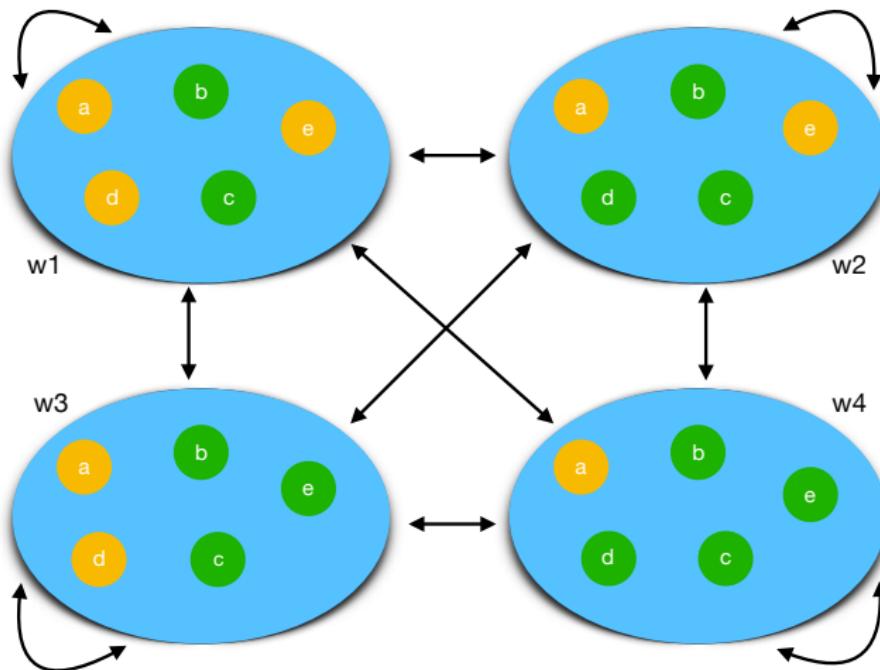
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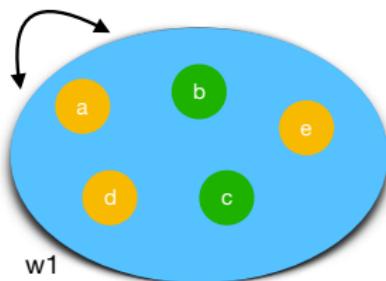
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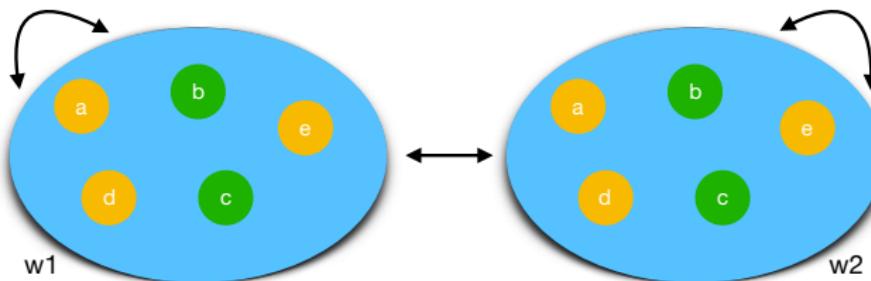
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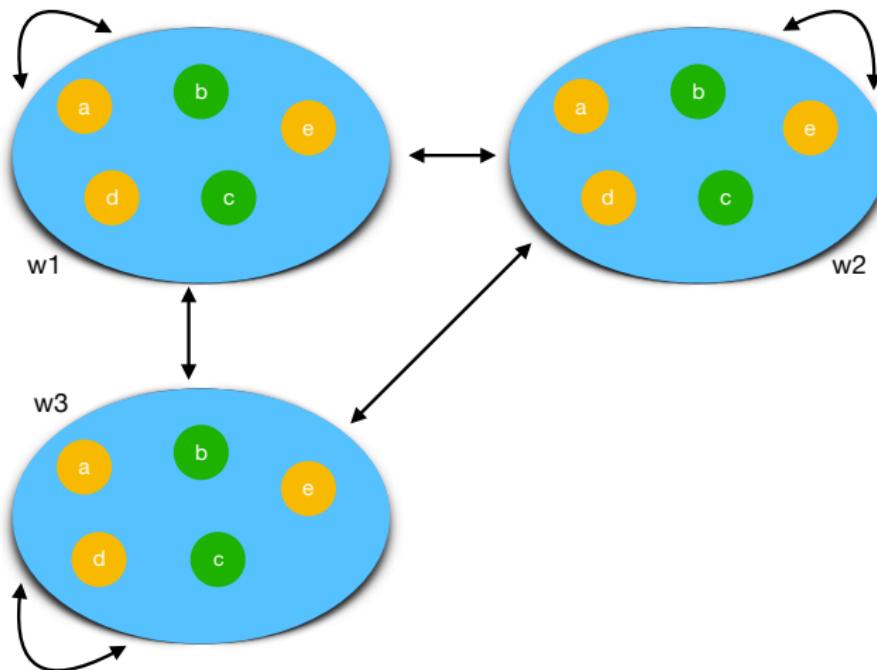
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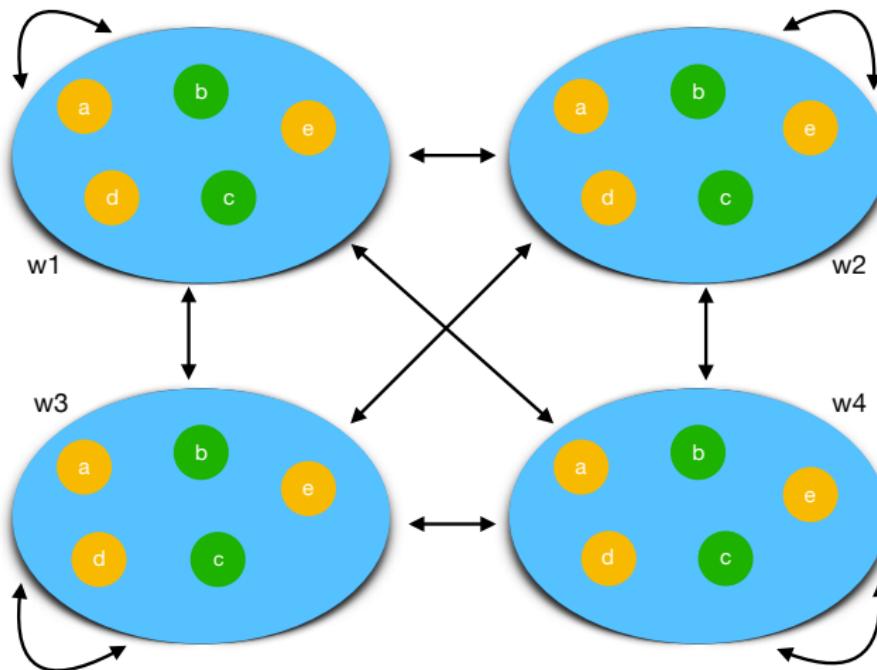
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### Example:

$$X = \{1, 2, 3, 4\}$$

$$\begin{aligned}\mathcal{P}(X) = & \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ & \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}\end{aligned}$$

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## Ultrafilter (Mathematics)

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$$U^1 = \{ \quad \quad \quad \{1, 4\}, \quad \quad \quad \}$$

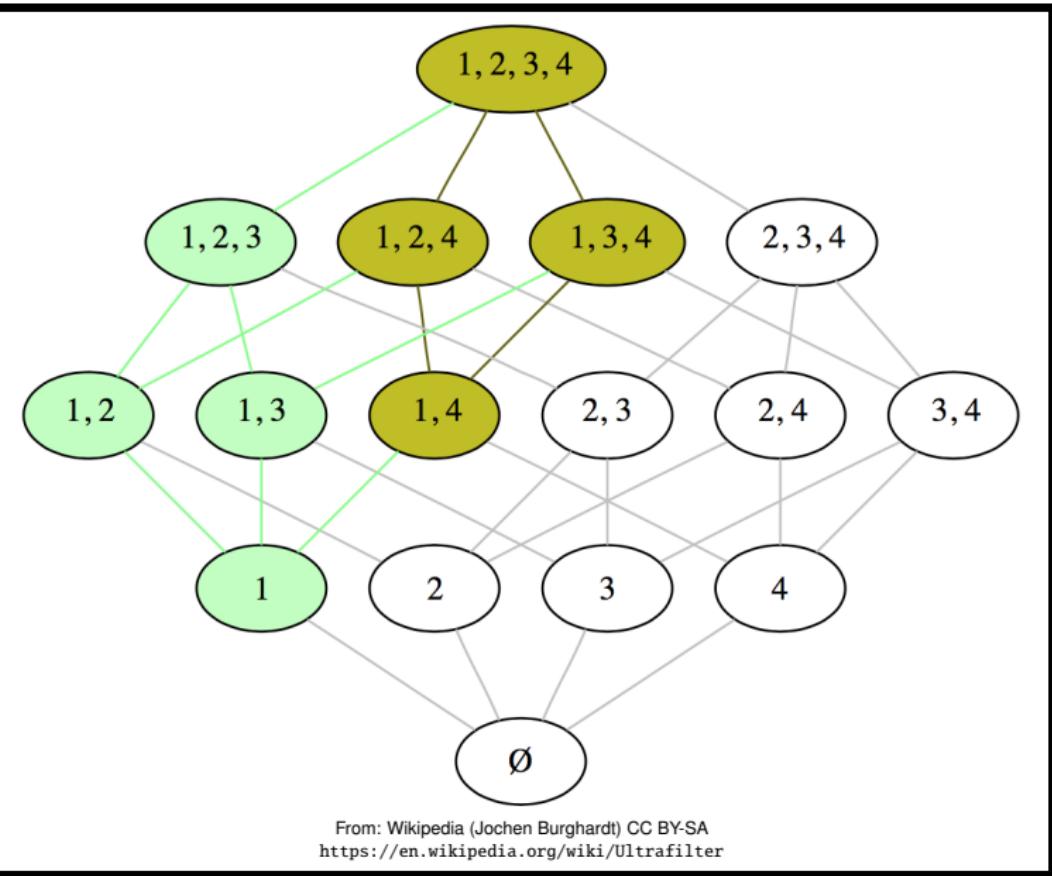
$$U^2 = \{ \quad \quad \quad \{1, 4\}, \quad \quad \quad \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\} \}$$

$$U^3 = \{\{1\}, \quad \quad \quad \{1, 4\}, \quad \quad \quad \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$$

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**1** is element of all sets in  $U$  (**1** has all properties of  $U$ )

## Ultrafilter (Mathematics)





**Part C**  
**— Comparative Analysis —**  
**Variants of Gödel/Scott, Anderson and Fitting**

## Ontological Argument: Variant by Gödel/Scott

### Part I - Proving that God's existence is possible

**D1** Being Godlike is equivalent to having all positive properties.

**A1** Exactly one of a property or its negation is positive.

**A2** Any property entailed by a positive property is positive.

**A3** The combination of any collection of positive properties is itself positive.

**From A1 and A2 follows theorem T1:**

**T1** Every positive property is possibly instantiated.

**From D1 and A3 follows:**

**T2** Being Godlike is a positive property.

**From T1 and T2 follows:**

**T3** Being Godlike is possibly instantiated.

## Ontological Argument: Variant by Gödel/Scott

### Part I - Proving that God's existence is possible

#### D1 Part II - Proving that God's existence is necessary, if possible

A1 D2 A property E is the essence of an individual x iff x has E and all of x's properties are entailed by E.<sup>a</sup>

A3 A4 Positive properties are necessarily positive.

From A1 and A4 (using definitions D1 and D2) follows:

T1 T4 Being Godlike is an essential property of any Godlike individual.

From T1 and D2 follows:

D3 Necessary existence of an individual is the necessary instantiation of all its essences.

From T2 and D3 follows:

A5 Necessary existence is a positive property.

From T4 and A5 (using D1, D2, D3) follows:

T3 T5 Being Godlike, if instantiated, is necessarily instantiated.

And finally from T3, T5 (together with some implicit modal axioms, e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

---

<sup>a</sup>The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

## Ontological Argument: Variant by Gödel/Scott

### Part I - Proving that God's existence is possible

#### D1 Part II - Proving that God's existence is necessary, if possible

A1 D2 A property  $F$  is the essence of an individual  $x$  iff  $x$  has  $F$  and all of  $x$ 's

A2 p "Modal Collapse" is implied by these axioms:  $\varphi \supset \Box\varphi$

A3 A4 P ▷ determinism

From From "positive properties ( $\mathcal{P}$ )" are applied here to intensional properties.

T1 T4 E We can prove:

From D3 M ▷  $\mathcal{P}$  is an ultrafilter all

T2 i

From A5 M Let  $\mathcal{P}'$  be the set of "rigidly intensionalised extensions" of positive properties. We can prove:

From T3 T5 E ▷  $\mathcal{P}'$  is an ultrafilter

And ▷  $\mathcal{P} = \mathcal{P}'$

ms,

ms,

e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

<sup>a</sup>The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

## Ontological Argument: Variant by Gödel/Scott

```
1 theory GoedelProof imports IHOML      (* This formalization follows Fitting's textbook *)
2 begin
3 (*Positiveness/perfection: uninterpreted constant symbol*)
4 consts positiveProperty::"(e⇒i⇒bool)⇒i⇒bool" ("P")
5 (*Some auxiliary definitions needed to formalise A3*)
6 definition h1 ("pos")    where "pos Z ≡ ∀X. Z X → P X"
7 definition h2 (infix "⊓" 60) where "X ⊓ Z ≡ □(∀x.(X x ↔ (∀Y.(Z Y) → (Y x))))"
8 definition h3 (infix "⇒" 60) where "X ⇒ Y ≡ □(∀z. X z → Y z)"
9
10 (**Part I**)
11 (*D1*) definition G ("G") where "G ≡ (λx. ∀Y. P Y → Y x)"
12 (*A1*) axiomatization where Ala: "[∀X. P (¬X) → ¬(P X)]" and Alb:"[∀X. ¬(P X) → P (¬X)]"
13 (*A2*) axiomatization where A2: "[∀X Y. (P X ∧ (X ⇒ Y)) → P Y]"
14 (*A3*) axiomatization where A3: "[∀Z X. (pos Z ∧ X ⊓ Z) → P X]"
15 (*T1*) theorem T1: "[∀X. P X → ◊∃E X]" by (metis Ala A2 h3_def)
16 (*T2*) theorem T2: "[P G]" proof -
17   {have 1: "∀w. ∃Z X. (P G ∨ pos Z ∧ X ⊓ Z ∧ ¬P X) w" by (metis(full_types) G_def h1_def h2_def)
18   have 2: "[∀Z X. (pos Z ∧ X ⊓ Z) → P X] → [P G]" using 1 by auto}
19   thus ?thesis using A3 by blast qed
20 (*T3*) theorem T3: "[◊∃E G]" sledgehammer using T1 T2 by simp
21
```

# Ontological Argument: Variant by Gödel/Scott

```
21 (**Part II**)
22 (*Logic KB*) axiomatization where symm: "symmetric aRel"
23 (*A4*) axiomatization where A4: "[ $\forall x. P x \rightarrow \square(P x)$ ]"
24 (*D2*) definition ess ("E") where " $E Y x \equiv (Y x) \wedge (\forall z. z x \rightarrow Y \Rightarrow z)$ "
25 (*T4*) theorem T4: "[ $\forall x. G x \rightarrow (E G x)$ ]" by (metis Alb A4 G_def h3_def ess_def)
26 (*D3*) definition NE ("NE") where " $NE x \equiv (\lambda w. (\forall y. E y x \rightarrow \square \exists^E y) w)$ "
27 (*A5*) axiomatization where A5: "[P NE]"
28 (*T5*) theorem T5: "[ $(\Diamond \exists^E G) \rightarrow \square \exists^E G$ ]" by (metis A5 G_def NE_def T4 symm)
29 (*T6*) theorem T6: "[ $\square \exists^E G$ ]" using T3 T5 by blast
30
31 (**Consistency**)
32 lemma True nitpick[satisfy] oops (*Model found by Nitpick: the axioms are consistent*)
33
34 (**Modal Collapse**)
35 lemma ModalCollapse: "[ $\forall \Phi. (\Phi \rightarrow (\square \Phi))$ ]" proof -
36   {fix w fix Q
37     have " $\forall x. G x w \longrightarrow (\forall z. z x \rightarrow \square(\forall^E z. G z \rightarrow z z)) w$ " by (metis Alb A4 G_def)
38     hence 1: " $(\exists x. G x w) \longrightarrow ((Q \rightarrow \square(\forall^E z. G z \rightarrow Q)) w)$ " by force
39     have " $\exists x. G x w$ " using T3 T6 symm by blast
40     hence " $(Q \rightarrow \square Q) w$ " using 1 T6 by blast
41   } thus ?thesis by auto qed
42
43 (**Some Corollaries**)
44 (*C1*) theorem C1: "[ $\forall E P x. ((E E x) \wedge (P x)) \rightarrow (E \Rightarrow P)$ ]" by (metis ess_def)
45 (*C2*) theorem C2: "[ $\forall X. \neg P X \rightarrow \square(\neg P X)$ ]" using A4 symm by blast
46   definition h4 ("N") where " $N X \equiv \neg P X$ "
47 (*C3*) theorem C3: "[ $\forall X. N X \rightarrow \square(N X)$ ]" by (simp add: C2 h4_def)
```

# Ontological Argument: Variant by Gödel/Scott

```
49 (**Positive Properties and Ultrafilters**)
50 abbreviation emptySet ("∅") where "∅ ≡ λx w. False"
51 abbreviation entails (infixr "C" 51) where "φ C ψ ≡ ∀x w. φ x w → ψ x w"
52 abbreviation andPred (infixr "Π" 51) where "φ Π ψ ≡ λx w. φ x w ∧ ψ x w"
53 abbreviation negpred ("¬" [52]53) where "¬ψ ≡ λx w. ¬ψ x w"
54 abbreviation "ultrafilter Φ cw ≡
55   ¬(Φ ∅ cw)
56   ∧ (∀φ. ∀ψ. (Φ φ cw ∧ Φ ψ cw) → (Φ (φ Π ψ) cw))
57   ∧ (∀φ::e⇒i=bool. ∀ψ::e⇒i=bool. (Φ φ cw ∨ Φ (¬φ) cw) ∧ ¬(Φ φ cw ∧ Φ (¬φ) cw))
58   ∧ (∀φ::e⇒i=bool. ∀ψ::e⇒i=bool. (Φ φ cw ∧ φ ⊆ ψ) → Φ ψ cw)"
59 lemma helpA: "∀w. ¬(P ∅ w)" using T1 by auto
60 lemma helpB: "∀φ ψ w. (P φ w ∧ P ψ w) → (P (φ Π ψ) w)" by (smt Alb G_def T3 T6 symm)
61 lemma helpC: "∀φ ψ w. (P φ w ∨ P (¬φ) w) ∧ ¬(P φ w ∧ P (¬φ) w)" using Ala Alb by blast
62 lemma helpD: "∀φ ψ w. (P φ w ∧ (φ ⊆ ψ)) → P ψ w" by (metis Alb A4 G_def T1 T6)
63
64 (*U1*) theorem U1: "∀w. ultrafilter P w" using helpA helpB helpC helpD by simp
65
66 (*(φ) converts an extensional object φ into 'rigid' intensional one*)
67 abbreviation trivialConversion ("(L)") where "(φ) ≡ (λw. φ)"
68
69 (*Q ↓φ: the extension of a (possibly) non-rigid predicate φ is turned into a rigid intensional one,
70 then Q is applied to the latter; ↓φ can be read as "the rigidly intensionalised predicate φ"*)
71 abbreviation mextPredArg (infix "↓" 60) where "Q ↓φ ≡ λw. Q (λx. (φ x w)) w"
72 lemma "∀Q φ. Q φ = Q ↓φ" nitpick oops (*countermodel: the two notions are not the same*)
73
74 lemma helpE: "∀w. ¬((P ↓∅) w)" using T1 by blast
75 lemma helpF: "∀φ ψ w. ((P ↓φ) w ∧ (P ↓ψ) w) → ((P ↓(φ Π ψ)) w)" by (smt Alb C2 G_def T3 symm)
76 lemma helpG: "∀w. ((P ↓φ) w ∨ (P ↓(¬φ)) w) ∧ ¬((P ↓φ) w ∧ (P ↓(¬φ)) w)" using Ala Alb by blast
77 lemma helpH: "∀w. ((P ↓φ) w ∧ φ ⊆ ψ) → (P ↓ψ) w" by (metis Alb A5 G_def NE_def T3 T4 symm)
78
79 abbreviation "P' φ ≡ (P ↓φ)" (*P': the set of all rigidly intensionalised positive properties*)
80
81 (*U2*) theorem U2: "∀w. ultrafilter P' w" using helpE helpF helpG helpH by simp
82 (*U3*) theorem U3: "(P' ⊆ P) ∧ (P ⊆ P')" by (smt Alb G_def T1 T6 symm) (*P' and P are equal*)
83
```

# Ontological Argument: Variant by Gödel/Scott

The screenshot shows the Isabelle/HOL proof assistant interface with the file `GoedelProof.thy` open. The code implements the Gödel/Scott variant of the ontological argument. The interface includes a vertical toolbar on the right with tabs for Documentation, Sidekick, State, and Theories, and a status bar at the bottom.

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theory GoedelProof imports IHOML      (* This formalization follows Fitting's textbook *)
begin

(*Positiveness/perfection: uninterpreted constant symbol*)
consts positiveProperty::"(e⇒i⇒bool)⇒i⇒bool" ("P")
(*Some auxiliary definitions needed to formalise A3*)
definition h1 ("pos")    where "pos Z ≡ ∀X. Z X → P X"
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(*D1*) definition G ("G") where "G ≡ (λx. ∀Y. P Y → Y x)"
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  {have 1: "¬∀w. ∃Z X. (P G ∨ pos Z ∧ X ∩ Z ∧ ¬P X) w" by (metis(full_types) G_def h1_def h2_def)
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  thus ?thesis using A3 by blast qed
(*T3*) theorem T3: "[¬∃E G]" using T1 T2 by simp

(**Part II**)
(*Logic KB*) axiomatization where symm: "symmetric apol"
```

At the bottom, a timeline shows the current position at 00:00 of a video titled "UltrafilterMovie1". The timeline bar is red, indicating the video is playing.

Output Query Sledgehammer Symbols

## SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

[Faith and Philosophy 1990]

## Ontological Argument: Variant by Anderson

### Part I - Proving that God's existence is possible

**D1** Being Godlike is equivalent to having all positive properties.

**A1** Exactly one of a property or its negation is positive.

**A2** Any property entailed by a positive property is positive.

**A3** The combination of any collection of positive properties is itself positive.

**From A1 and A2 follows theorem T1:**

**T1** Every positive property is possibly instantiated.

**From D1 and A3 follows:**

**T2** Being Godlike is a positive property.

**From T1 and T2 follows:**

**T3** Being Godlike is possibly instantiated.

## Ontological Argument: Variant by Anderson

### Part I - Proving that God's existence is possible

**D1** Being Godlike is equivalent to having all positive properties.

**A1a** If a property is positive, then its negation is not positive.

**A1b** If the negation of a property is not positive, then the property is positive.

**A2** Any property entailed by a positive property is positive.

**A3** The combination of any collection of positive properties is itself positive.

**From A1 and A2 follows theorem T1:**

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### Part I - Proving that God's existence is possible

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**A1b** ~~If the negation of a property is not positive, then the property is positive.~~

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**From A1 and A2 follows theorem T1:**

**T1** Every positive property is possibly instantiated.

**From D1 and A3 follows:**

**T2** Being Godlike is a positive property.

**From T1 and T2 follows:**

**T3** Being Godlike is possibly instantiated.

## Ontological Argument: Variant by Anderson

### Part I - Proving that God's existence is possible

**D1'** Being Godlike is equivalent to having all and only the positive properties as necessary properties.

**A1a** If a property is positive, then its negation is not positive.

**A1b** ~~If the negation of a property is not positive, then the property is positive.~~

**A2** Any property entailed by a positive property is positive.

**A3** The combination of any collection of positive properties is itself positive.

**From A1 and A2 follows theorem T1:**

**T1** Every positive property is possibly instantiated.

**From D1 and A3 follows:**

**T2** Being Godlike is a positive property.

**From T1 and T2 follows:**

**T3** Being Godlike is possibly instantiated.

## Ontological Argument: Variant by Anderson

### Part I - Proving that God's existence is possible

#### D1' Part II - Proving that God's existence is necessary, if possible

A1a D2 A property E is the essence of an individual x iff x has E and all of x's properties are entailed by E.

A1b A4 Positive properties are necessarily positive.

**From A1 and A4 (using definitions D1 and D2) follows:**

A2 T4 Being Godlike is an essential property of any Godlike individual.

A3 D3 Necessary existence of an individual is the necessary instantiation of all its essences.

T1 A5 Necessary existence is a positive property.

**From T4 and A5 (using D1, D2, D3) follows:**

T2 T5 Being Godlike, if instantiated, is necessarily instantiated.

**From T5 and A2 (using D1, D2, D3) follows:**  
**And finally from T3, T5 (together with some implicit modal axioms, e.g. S5) the existence of (at least a) God follows:**

T6 Being Godlike is necessarily instantiated.

## Ontological Argument: Variant by Anderson

### Part I - Proving that God's existence is possible

D1' Part II - Proving that God's existence is necessary, if possible

A1a D2' A property E is an essence ( $\mathcal{E}^A$ ) of an individual x if and only if all of  
x's necessary properties are entailed by E and (conversely) all  
properties entailed by E are necessary properties of x.

A1b A4 Positive properties are necessarily positive.

A2 From A1 and A4 (using definitions D1 and D2) follows:

A3 T4 Being Godlike is an essential property of any Godlike individual.

From D3 Necessary existence of an individual is the necessary instantiation of all  
its essences.

T1 From A5 Necessary existence is a positive property.

T2 From T4 and A5 (using D1, D2, D3) follows:

T3 From T5 Being Godlike, if instantiated, is necessarily instantiated.

And finally from T3, T5 (together with some implicit modal axioms,  
e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

## Ontological Argument: Variant by Anderson

### Part I - Proving that God's existence is possible

### D1' Part II - Proving that God's existence is necessary, if possible

D2'

x “Modal Collapse” is \*not\* implied by these axioms

A1a

A1b

A2

A4

A2

A3

From

- ▶ no determinism

T4 E “positive properties ( $\mathcal{P}$ )” are applied here to intensional properties.

From

D3 N We have:

- ▶  $\mathcal{P}$  is \*not\* an ultrafilter (has countermodel)

From

A5 N

Let  $\mathcal{P}'$  be the set of all “rigidly intensionalised extensions” of positive properties. We can prove:

From

T2 E

T5 E

And e.g. S

▶  $\mathcal{P}'$  is an ultrafilter

▶  $\mathcal{P} \neq \mathcal{P}'$

T6 Being Godlike is necessarily instantiated.

## Ontological Argument: Variant by Anderson

```
1 theory AndersonProof imports IHOML
2 begin
3 (*Positiveness/perfection: uninterpreted constant symbol*)
4 consts positiveProperty::"(e⇒i⇒bool)⇒i⇒bool" ("P")
5 (*Some auxiliary definitions*)
6 definition h3 (infix "⇒" 60) where "X ⇒ Y ≡ □(∀z. X z → Y z)"
7
8 (**Part I**)
9 (*D1*) definition GA ("GA") where "GA ≡ λx. ∀Y. (P Y) ↔ □(Y x)"
10 (*Ala*) axiomatization where Ala:"[∀X. P (→X) → ¬(P X)]"
11 (*A2*) axiomatization where A2: "[∀X Y. (P X ∧ (X ⇒ Y)) → P Y]"
12 (*T1*) theorem T1: "[∀X. P X → ◊∃E X]" using Ala A2 h3_def by metis
13 (*T2*) axiomatization where T2: "[P GA]" (*here we postulate T2 instead of proving it*)
14 (*T3*) theorem T3: "[◊∃E GA]" using T1 T2 h3_def by blast
15
```

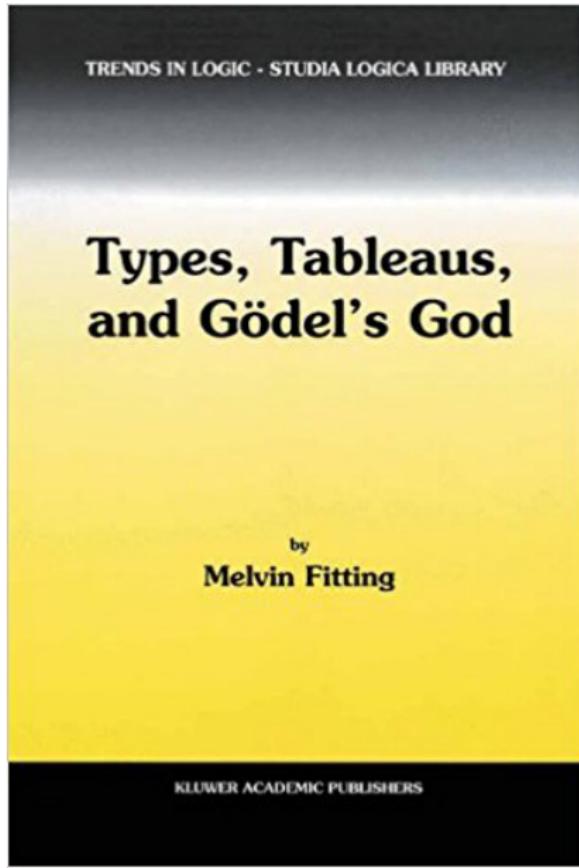
# Ontological Argument: Variant by Anderson

```
1 theory AndersonProof imports IHOML
2
3 16 (**Part II**)
4 17 (*Logic KB*) axiomatization where symm: "symmetric aRel"
5 18 (*A4*) axiomatization where A4: "[ $\forall X. P X \rightarrow \square(P X)$ ]"
6 19 (*D2'*) abbreviation essA (" $E^A$ ") where " $E^A Y x \equiv (\forall Z. \square(Z x) \leftrightarrow Y \Rightarrow Z)$ "
7 20 (*T4*) theorem T4: "[ $\forall x. G^A x \rightarrow (E^A G^A x)$ ]" by (metis A2 GA_def T2 symm h3_def)
8 21 (*D3*) abbreviation NEA (" $NE^A$ ") where " $NE^A x \equiv (\lambda w. (\forall Y. E^A Y x \rightarrow \square \exists^E Y) w)$ "
9 22 (*A5*) axiomatization where A5: "[ $P NE^A$ ]"
10 23 (*T5*) theorem T5: "[ $\square \exists^E G^A \longrightarrow \square \exists^E G^A$ ]" by (metis A2 GA_def T2 symm h3_def)
11 24 (*T6*) theorem T6: "[ $\square \exists^E G^A$ ]" using T3 T5 by blast
12 25
13 26 (**Modal collapse is countersatisfiable**)
14 27 lemma "[ $\forall \Phi. (\Phi \rightarrow (\square \Phi))$ ]" nitpick oops (*Countermodel found by Nitpick*)
15 28
16 29 (**Consistency**)
17 30 lemma True nitpick[satisfy] oops (*model found by Nitpick: the axioms are consistent*)
18 31
19 32 (**Some Corollaries**)
20 33 (*C1*) theorem C1: "[ $\forall E x. ((E^A E x) \wedge (P x)) \rightarrow (E \Rightarrow P)$ ]" nitpick oops (*countermodel*)
21 34 (*C2*) theorem C2: "[ $\forall X. \neg P X \rightarrow \square(\neg P X)$ ]" using A4 symm by blast
22 35 definition h4 (" $\mathcal{N}$ ") where " $\mathcal{N} X \equiv \neg P X$ "
23 36 (*C3*) theorem C3: "[ $\forall X. \mathcal{N} X \rightarrow \square(\mathcal{N} X)$ ]" by (simp add: C2 h4_def)
```

# Ontological Argument: Variant by Anderson

```
1 theory AndersonProof imports IHOML
2
3 (*Positive Properties and Ultrafilters*)
4 abbreviation emptySet ("∅") where "∅ ≡ λx w. False"
5 abbreviation entails (infixr "⊆" 51) where "φ ⊆ ψ ≡ ∀x w. φ x w → ψ x w"
6 abbreviation andPred (infixr "Π" 51) where "φ Π ψ ≡ λx w. φ x w ∧ ψ x w"
7 abbreviation negpred ("¬" [52] 53) where "¬ψ ≡ λx w. ¬ψ x w"
8 abbreviation "ultrafilter" Φ cw ≡
9   ¬(Φ ∅ cw)
10  ∧ ( ∀φ. ∀ψ. (Φ φ cw ∧ Φ ψ cw) → (Φ (φ Π ψ) cw))
11  ∧ ( ∀ψ::e⇒i⇒bool. ∀ψ::e⇒i⇒bool. (Φ φ cw ∨ Φ (¬φ) cw) ∧ ¬(Φ φ cw ∧ Φ (¬φ) cw))
12  ∧ ( ∀ψ::e⇒i⇒bool. ∀ψ::e⇒i⇒bool. (Φ φ cw ∧ φ ⊆ ψ) → Φ ψ cw)
13
14 (*U1*) theorem U1: "∀w. ultrafilter P w" nitpick[user_axioms,format=2,show_all] oops (*countermodel*)
15 lemma helpC: "∀φ ψ w. (P φ w ∨ P (¬φ) w) ∧ ¬(P φ w ∧ P (¬φ) w)" nitpick oops (*countermodel*)
16
17 (*(φ) converts an extensional object φ into 'rigid' intensional one*)
18 abbreviation trivialConversion ("(φ)") where "(φ) ≡ (λw. φ)"
19 (*Q ↓φ: the extension of a (possibly) non-rigid predicate φ is turned into a rigid intensional one,
20 then Q is applied to the latter; ↓φ can be read as "the rigidly intensionalised predicate φ"*)
21 abbreviation mextPredArg (infix "↓" 60) where "Q ↓φ ≡ λw. Q (λx. (φ x w)) w"
22 lemma "∀Q φ. Q φ = Q ↓φ" nitpick oops (*countermodel*: the two notions are not the same*)
23
24 lemma helpE: "∀w. ¬((P ↓∅) w)" using T1 by blast
25 lemma helpF: "∀φ ψ w. ((P ↓φ) w ∧ (P ↓ψ) w) → ((P ↓(φ Π ψ)) w)" by (smt GA_def T3 T5 symm)
26 lemma helpG: "∀w. ((P ↓φ) w ∨ (P ↓(¬φ)) w) ∧ ¬((P ↓φ) w ∧ (P ↓(¬φ)) w)" by (smt GA_def T3 T5 symm)
27 lemma helpH: "∀w. ((P ↓φ) w ∧ φ ⊆ ψ) → (P ↓ψ) w" by (metis (no_types, lifting) A4 C2 GA_def T3)
28
29 abbreviation "P' φ ≡ (P ↓φ)" (*P': the set of all rigidly intensionalised positive properties*)
30
31 (*U2*) theorem U2: "∀w. ultrafilter P' w" using helpE helpF helpG helpH by simp
32 (*U3*) theorem U3: "(P' ⊆ P) ∧ (P ⊆ P')" nitpick oops (*countermodel*: P' and P are not equal*)
```

## Ontological Argument: Variant by Fitting (2002)



## Ontological Argument: Variant by Fitting (2002)

### Part I - Proving that God's existence is possible

**D1** Being Godlike is equivalent to having all positive properties.

**A1** Exactly one of a property or its negation is positive.

**A2** Any property entailed by a positive property is positive.

**A3** The combination of any collection of positive properties is itself positive.

**From A1 and A2 follows theorem T1:**

**T1** Every positive property is possibly instantiated.

**From D1 and A3 follows:**

**T2** Being Godlike is a positive property.

**From T1 and T2 follows:**

**T3** Being Godlike is possibly instantiated.

Fully analogous to Gödel/Scott.

But: “positive properties” applied to extensions of properties only!

## Ontological Argument: Variant by Fitting (2002)

### Part I - Proving that God's existence is possible

#### D1 Part II - Proving that God's existence is necessary, if possible

A1 D2 A property E is the essence of an individual x iff x has E and all of x's properties are entailed by E.<sup>a</sup>

A2 A3 Positive properties are necessarily positive.

From A1 and A4 (using definitions D1 and D2) follows:

T1 T4 Being Godlike is an essential property of any Godlike individual.

From T1 and D3 (using definition D2) follows:

T2 D3 Necessary existence of an individual is the necessary instantiation of all its essences.

From T2 and A5 (using D1, D2, D3) follows:

T3 A5 Necessary existence is a positive property.

From T3 and T4 (using D1, D2, D3) follows:

T5 T5 Being Godlike, if instantiated, is necessarily instantiated.

And finally from T3, T5 (together with some implicit modal axioms, e.g. S5) the existence of (at least a) God follows:

T6 T6 Being Godlike is necessarily instantiated.

---

<sup>a</sup>The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

## Ontological Argument: Variant by Fitting (2002)

### Part I - Proving that God's existence is possible

#### D1 Part II - Proving that God's existence is necessary, if possible

A1 D2 A property E is the essence of an individual x iff x has E and all of x's properties are entailed by E.<sup>a</sup>

A2 A3 Positive properties are necessarily positive.

From

“Modal Collapse” is \*not\* implied by these axioms

T1 E

From

D3 N  
i

$$\varphi \supset \Box\varphi \quad (\text{has countermodel})$$

T2

From

A5 N  
We can prove that these “positive property extensions” (which corresponds to  $\mathcal{P}'$  from before) form an ultrafilter.

T3 E

And

e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

<sup>a</sup>The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

## Ontological Argument: Variant by Fitting (2002)

```
1 theory FittingProof imports IHOML
2 begin
3 (*Positiveness/perfection: uninterpreted constant symbol*)
4 consts Positiveness::"(e⇒bool)⇒i⇒bool" ("P")
5 (*Some auxiliary definitions*)
6 (* $\varphi$ ) converts an extensional object  $\varphi$  into `rigid' intensional one*)
7 abbreviation trivialConversion ("(_)"") where " $\varphi$ " ≡  $(\lambda w. \varphi)$ "
8 abbreviation Entails (infix " $\Rightarrow$ " 60) where " $X \Rightarrow Y$ " ≡  $\Box(\forall z. (X z) \rightarrow (Y z))$ "
9 (* $\varphi$ 's argument is a relativized term (of extensional type) derived from an intensional predicate.*)
10 abbreviation extPredArg (infix " $\Downarrow$ " 60) where " $\varphi \Downarrow P$ " ≡  $\lambda w. \varphi (\lambda x. P x w) w$ "
11 (*A variant of the latter where  $\varphi$  takes intensional terms as argument.*)
12 abbreviation mextPredArg (infix " $\Downarrow$ " 60) where " $\varphi \Downarrow P$ " ≡  $\lambda w. \varphi (\lambda x. (P x w)) w$ "
13 (*Another variant where  $\varphi$  has two arguments (the first one being relativized).*)
14 abbreviation extPredArg1 (infix " $\Downarrow_1$ " 60) where " $\varphi \Downarrow_1 P$ " ≡  $\lambda z. \lambda w. \varphi (\lambda x. P x w) z w$ "
15
16 (**Part I**)
17 (*D1*) abbreviation God ("G") where "G" ≡  $(\lambda x. \forall y. P Y \rightarrow (Y x))$ "
18 (*A1*) axiomatization where Ala:"[ \forall X. P (\neg X) \rightarrow \neg(P X) ]" and Alb:"[ \forall X. \neg(P X) \rightarrow P (\neg X) ]"
19 (*A2*) axiomatization where A2: "[ \forall X Y. (P X \wedge (X \Rightarrow Y)) \rightarrow P Y ]"
20 (*T1*) theorem T1: "[ \forall X. P X \rightarrow \Diamond(\exists z. (X z)) ]" using Ala A2 by blast
21 (*T2*) axiomatization where T2: "[ P \Downarrow G ]"
22 (*T3*) theorem T3deRe: "[ (\lambda X. \Diamond \exists^E X) \Downarrow G ]" using T1 T2 by simp
23 theorem T3deDicto: "[ \Diamond \exists^E \Downarrow G ]" nitpick oops (*countermodel*)
24
```

# Ontological Argument: Variant by Fitting (2002)

```
1 theory FittingProof imports IHOML
2 begin
3 (*Positiveness/perfection: uninterpreted constant symbol*)
4 const Positiveness :: "(o::bool) -> i::bool" ("P")
5
6 (**Part II*)
7 (*Logic KB*) axiomatization where symm: "symmetric aRel"
8 (*A4*) axiomatization where A4: "[|X. P X → □(P X)|]"
9 (*D2*) abbreviation Essence ("E") where "E Y x ≡ (Y x) ∧ (VZ.(Z x)→(Y⇒Z))"
10 (*T4*) theorem T4: "[|X. G x → ((E ↓_1 G) x)|]" using Alb by metis
11 (*D3*) definition NE ("NE") where "NE x ≡ λw. (VY. E Y x → □(EZ. (Y z))) w"
12 (*A5*) axiomatization where A5: "[|P |NE|"
13   lemma help1: "|[|E ↓ G → □EZ ↓ G|]| sorry (*longer interactive proof, omitted here*)
14   lemma help2: "|[|E ↓ G → ((λX. □EZ X) ↓ G)|]| by (metis A4 help1)
15 (*T5*) theorem T5deDicto: "|[|◇EZ ↓ G|] → [|EZ ↓ G|]|" using T3deRe help1 by blast
16   theorem T5deRe: "|[|λX. ◇EZ X) ↓ G|] → [|λX. EZ X) ↓ G|]|" by (metis A4 help2)
17 (*T6*) theorem T6deDicto: "|[|EZ ↓ G|]|" using T3deRe help1 by blast
18   theorem T6deRe: "|[|λX. EZ X) ↓ G|]|" using T3deRe help2 by blast
19
20 (**Consistency**)
21 40 lemma True nitpick[satisfy] oops (*Model found by Nitpick: the axioms are consistent*)
22
23 (**Modal Collapse**)
24 43 lemma ModalCollapse: "|[|Φ. (Φ → (□ Φ))|]| nitpick oops (*countermodel*)"
25
26 (**Some Corollaries**)
27 (* Todo (*C1*) theorem C1: "|[|VE P x. ((E E x) ∧ (P x)) → (E ⇒ P)|]|" by (metis ess_def) *)
28 (*C2*) theorem C2: "|[|VX. ¬P X → □(¬P X)|]|" using A4 symm by blast
29   definition h4 ("N") where "N X ≡ ¬P X"
30 (*C3*) theorem C3: "|[|VX. N X → □(N X)|]|" by (simp add: C2 h4_def)
31   definition "rigid φ ≡ ∀x. φ x → □(φ x)"
32 (*C4*) theorem "|[|Vφ. P φ → rigid (λx. (φ x))|]|" by (simp add: rigid_def)
33 (*C5*) theorem "|[|rigid P|]|" by (simp add: A4 rigid_def)
```

# Ontological Argument: Variant by Fitting (2002)

```
1 theory FittingProof imports IHOML
2 begin
3 (*Positiveness/perfection: uninterpreted constant symbol*)
4 const Positiveness :: "(o::bool) -> i::bool" ("P")
5
6 (**Part II*)
7 (*Logic KB*) axiomatization where symm: "symmetric aRel"
8 (*A4*) axiomatization where A4: "[|X. P X → □(P X)|]"
9 (*D2*) abbreviation Essence ("E") where "E Y x ≡ (Y x) ∧ (VZ. (Z x) → (Y ⇒ Z))"
10 (*T4*) theorem T4: "[|X. G x → ((E ↓G) x)|]" using Alb by metis
11 (*D3*) definition NE ("NE") where "NE x ≡ λw. (VY. E Y x → □(∃z. (Y z))) w"
12 (*A5*) theorem A5: "[|X. G x → ((E ↓G) x)|]" using T4, NE by metis
13
14 (**Positive Properties and Ultrafilters**)
15 abbreviation empty ("∅") where "∅ ≡ λx. False"
16 abbreviation intersect (infix "∩" 52) where "φ ∩ ψ ≡ (λx. φ x ∧ ψ x)"
17 abbreviation nnegpred ("¬_" "[52]53) where "¬ψ ≡ λx. ¬ψ(x)"
18 abbreviation entail (infixr "⊆" 51) where "φ ⊆ ψ ≡ ∀x. φ x → ψ x"
19 abbreviation "ultrafilter" Φ cw ≡
20   ¬(Φ ∅ cw) (* The empty set is not an element of U *)
21   ∧ (∀ψ::(e⇒bool). ∀ψ::(e⇒bool). (Φ φ cw ∧ Φ ψ cw) → (Φ (φ ∩ ψ) cw))
22   ∧ (∀ψ::(e⇒bool). ∀ψ::(e⇒bool). (Φ φ cw ∨ Φ (¬φ) cw) ∧ ¬(Φ φ cw ∧ Φ (¬φ) cw))
23   ∧ (∀ψ::(e⇒bool). ∀ψ::(e⇒bool). (Φ φ cw ∧ φ ⊆ ψ) → Φ ψ cw)
24 lemma lemmaA: "∀w. ¬(P ∅ w)" using T1 by blast
25 lemma lemmaB: "∀w. (P φ w ∧ P ψ w) → (P (φ ∩ ψ) w)" by (metis Alb T3deRe)
26 lemma lemmaC: "∀w. (P φ w ∨ P (¬φ) w) ∧ ¬(P φ w ∧ P (¬φ) w)" using Ala Alb by blast
27 lemma lemmaD: "∀w. (P φ w ∧ φ ⊆ ψ) → P ψ w" by (smt A2)
28
29 (*U1*) theorem "∀w. ultrafilter P w" by (smt lemmaA lemmaB lemmaC lemmaD)
30
31 (*C4*) theorem "[|φ. P φ → rigid (λx. (φ x))|]" by (simp add: rigid_def)
32 (*C5*) theorem "[|rigid P|]" by (simp add: A4 rigid_def)
```

## Summary of Results

- ▶ “Godlike” has been defined in terms of “positive properties”
- ▶ “positive properties” has been linked with the notion of “ultrafilter”.
- ▶ In our experiments we then distinguished between
  - $\mathcal{P}$ : positive intensional properties
  - $\mathcal{P}'$ : positive (“rigidly intensionalised”) extensions of properties
- ▶ Gödel/Scott variant axiomatises  $\mathcal{P}$ :  $\mathcal{P} = \mathcal{P}'$  is an ultrafilter
- ▶ Anderson’s variant axiomatises  $\mathcal{P}$ :  $\mathcal{P} \neq \mathcal{P}'$ ; only  $\mathcal{P}'$  is an ultrafilter
- ▶ Fitting’s variant axiomatises only  $\mathcal{P}'$ :  $\mathcal{P}'$  is an ultrafilter

Modal collapse holds for Gödel/Scott variant, but not for Anderson’s & Fitting’s!

They achieve this in seemingly different ways.

Mathematically, however, their solutions appear closely related.

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Mathematically, however, their solutions appear closely related.



**Part D**  
**— Discussion —**  
**Metaphysics, Mathematics and Reality**

## Discussion: Metaphysics, Mathematics and Reality

- ▶ There are consistent theistic theories which
  - ▶ imply the necessary existence of a Godlike (superior) being
  - ▶ support different philosophical positions: determinism / non-determinism
- ▶ Theistic belief (at least in an abstract sense) not necessarily irrational
- ▶ By adopting the notion of “ultrafilter” these theistic theories were mapped here to mathematical theories

### Question

- ▶ Relevance of existence results for the real world?
- ▶ Existence results in metaphysics vs. mathematics — difference?

## Discussion: Metaphysics, Mathematics and Reality

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  - ▶ support different philosophical positions: determinism / non-determinism
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## Conclusion

- ▶ Experiments in Computational Metaphysics: Ontological Argument
- ▶ Universal Logical Reasoning Approach
- ▶ Further developed and applied since AISSQ 2015
- ▶ Interesting new results
- ▶ Approach has other relevant and pressing applications (e.g., machine ethics)

### Evidence provided for central claim of this talk

- ▶ Computers may help to sharpen our understanding of arguments
- ▶ Universal (meta-)logical reasoning approach particularly well suited

### Related work

- ▶ Ed Zalta (& co) with PROVER9 at Stanford [AJP 2011, CADE 2015]
- ▶ John Rushby with PVS at SRI [CAV-WS 2013, JAL 2018]