Findings from a Computer-supported Analysis of Variants of Gödel's Ontological Proof

Christoph Benzmüller[1], Freie Universität Berlin
jww: Bruno Woltzenlogel-Paleo, TU Vienna

```
>
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p

Leo-II tries to prove
=====================

  Goedel's Theorem T3: "Necessarily, God exists"
  thf(thmT3,conjecture,
      ( v
      @ ( mbox
        @ ( mexists_ind
          @ ^ [X: mu] :
              ( g @ X ) ) ) )).

  Assumptions: D1, C, T2, D3, A5

 . searching for proof ..

 ********************
 *   Proof found    *
 ********************
 % SZS status Theorem for Notwendigerweise-existiert-Gott.p

 . generating proof object
```

[1] Supported by DFG Heisenberg Fellowship BE 2501/9-1/2

Vision of Leibniz (1646–1716): *Calculemus!*

Freie Universität Berlin

Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo . . . dicere: calculemus. (Leibniz, 1684)

If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other . . . : Let us calculate.
(Translation by Russell)

Required:
characteristica universalis and **calculus ratiocinator**

Ontological argument for the existence of God

Focus on Gödel's modern version in higher-order modal logic

Automation with provers for higher-order classical logic (HOL)

- verification (or falsification) of known results
- some novel results (by HOL-ATPs)
- HOL as a universal metalogic via logic embeddings (characteristica universalis?)

Different Interests in Ontological Arguments

- Philosophical: Boundaries of Metaphysics & Epistemology
- Theistic: Successful argument could convince atheists?
- Ours: Can computers (theorem provers) be used ...

  ... to formalize the definitions, axioms and theorems?
  ... to verify/falsify the arguments step-by-step?
  ... to automate (sub-)arguments?
  ... to generate new knowledge?

A Long History

pros and cons

... Anselm v. C. Gaunilo ... Th. Aquinas ...... Descartes Spinoza Leibniz ... Hume Kant ... Hegel ... Frege ... Hartshorne Malcolm Lewis Plantinga Gödel ...

Anselm's notion of God (Proslogion, 1078):

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical, deductive reasoning:

"God exists."

$$\exists x G(x)$$

A Long History

pros and cons

… Anselm v. C. Gaunilo … Th. Aquinas …… Descartes Spinoza Leibniz … Hume Kant … Hegel … Frege … Hartshorne Malcolm Lewis Plantinga Gödel …

Anselm's notion of God (Proslogion, 1078):

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical, deductive reasoning:

"Necessarily, God exists."

$$\Box \exists x G(x)$$

Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess.} x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)\, G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)G(x) \supset MN(\exists y)\, G(y)$ M = possibility

" $\supset N(\exists y)\, G(y)$

any two essences of x are nec. equivalent

exclusive or * and for any number of summands

$M(\exists x)G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl

$x = x$ is positive

$x \neq x$ is negative

But if a system S of pos. props. were incom it would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the at. true. It m. also meant pure "attribution"* as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ contrary At... or the exist. of pos. prop.

X i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall \phi[P(\neg \phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$\forall \phi \forall \psi[(P(\phi) \wedge \Box \forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall \phi[P(\phi) \rightarrow \Diamond \exists x \phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall \phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond \exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall \phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \ ess \ x \leftrightarrow \phi(x) \wedge \forall \psi(\psi(x) \rightarrow \Box \forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G \ ess \ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall \phi[\phi \ ess \ x \rightarrow \Box \exists y \phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box \exists x G(x)$

Scott's Version of Gödel's Axioms, Definitions and Theorems

Freie Universität Berlin

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess\ x \leftrightarrow \boxed{\phi(x)\ \wedge} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

Difference to Gödel (who omits this conjunct)

Freie Universität Berlin

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

Modal operators are used

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive: $\boxed{\forall\phi\forall\psi}(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \leftrightarrow \boxed{\forall\phi}[P(\phi) \rightarrow \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi\ ess\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

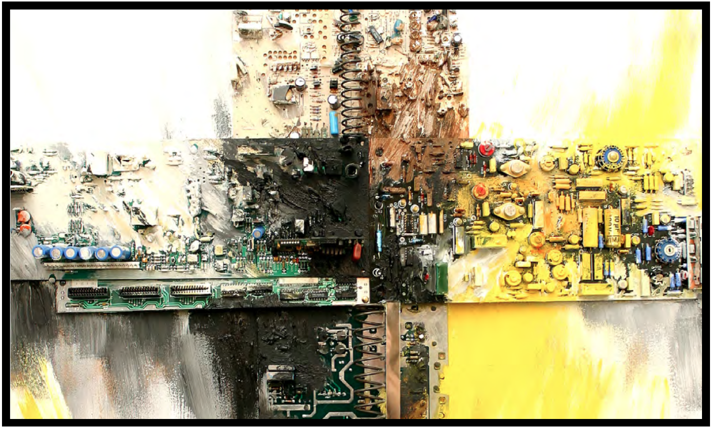
Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \rightarrow G\ ess\ x]$

Def. D3 *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

second-order quantifiers



How to automate Higher-Order Modal Logic?

Challenge: No provers for *Higher-order Modal Logic* (HOML)

Church's Simple Type Theory

Our solution: Embedding in **Higher-order Classical Logic** (HOL)

Then use existing HOL theorem provers for reasoning in HOML

[BenzmüllerPaulson, Logica Universalis, 2013]

Assumption: Henkin semantics for both HOML and HOL

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

Embedding HOML in HOL

HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $s, t \quad ::= \quad C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\rightarrow o}$ (explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\rightarrow & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, h d w \\
\exists & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \exists d_\gamma\, h d w \\
\Box & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \forall u_\mu\, (\neg r w u \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\mu\rightarrow o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

Embedding HOML in HOL

HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $s, t \quad ::= \quad C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\rightarrow o}$ (explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\rightarrow & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, h d w \\
\exists & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \exists d_\gamma\, h d w \\
\Box & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \forall u_\mu\, (\neg r w u \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \exists u_\mu\, (r w u \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\mu\rightarrow o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

Embedding HOML in HOL

HOML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x_\gamma\, \varphi \mid \exists x_\gamma\, \varphi$

HOL $s, t \quad ::= \quad C_\alpha \mid x_\alpha \mid (\lambda x_\alpha s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid \neg s_o \mid s_o \vee t_o \mid \forall x_\alpha\, t_o$

HOML in HOL: HOML formulas φ are mapped to HOL predicates $\varphi_{\mu\rightarrow o}$ (explicit representation of labelled formulas)

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \neg\varphi w \\
\wedge & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\varphi w \wedge \psi w) \\
\rightarrow & = & \lambda\varphi_{\mu\rightarrow o}\lambda\psi_{\mu\rightarrow o}\lambda w_\mu(\neg\varphi w \vee \psi w) \\
\forall & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \forall d_\gamma\, hdw \\
\exists & = & \lambda h_{\gamma\rightarrow(\mu\rightarrow o)}\lambda w_\mu \exists d_\gamma\, hdw \\
\Box & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \forall u_\mu\, (\neg rwu \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\mu\rightarrow o}\lambda w_\mu \exists u_\mu\, (rwu \wedge \varphi u) \\
\text{valid} & = & \lambda\varphi_{\mu\rightarrow o}\forall w_\mu \varphi w
\end{array}
$$

Ax (polymorphic over γ)

The equations in Ax are given as axioms to the HOL provers!

Example

HOML formula $\Diamond \exists x G(x)$

HOML formula in HOL valid $(\Diamond \exists x G(x))_{\mu \to o}$

expansion, $\beta\eta$-conversion $\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$

expansion, $\beta\eta$-conversion $\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \to o} u)$

expansion, $\beta\eta$-conversion $\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Freie Universität Berlin

Example

HOML formula	$\Diamond \exists x G(x)$
HOML formula in HOL	valid $(\Diamond \exists x G(x))_{\mu\rightarrow o}$
expansion, $\beta\eta$-conversion	$\forall w_{\mu} (\Diamond \exists x G(x))_{\mu\rightarrow o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_{\mu} \exists u_{\mu} (rwu \wedge (\exists x G(x))_{\mu\rightarrow o} u)$
expansion, $\beta\eta$-conversion	$\forall w_{\mu} \exists u_{\mu} (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu\rightarrow o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Example

HOML formula	$\Diamond \exists x G(x)$
HOML formula in HOL	valid $(\Diamond \exists x G(x))_{\mu \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Example

HOML formula	$\Diamond \exists x G(x)$
HOML formula in HOL	valid $(\Diamond \exists x G(x))_{\mu \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Freie Universität Berlin

Example

HOML formula	$\Diamond \exists x G(x)$
HOML formula in HOL	valid $(\Diamond \exists x G(x))_{\mu\rightarrow o}$
expansion, $\beta\eta$-conversion	$\forall w_\mu (\Diamond \exists x G(x))_{\mu\rightarrow o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu\rightarrow o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu\rightarrow o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Example

HOML formula	$\Diamond\exists x G(x)$
HOML formula in HOL	valid $(\Diamond\exists x G(x))_{\mu\to o}$
expansion, $\beta\eta$-conversion	$\forall w_\mu (\Diamond\exists x G(x))_{\mu\to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu\to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge \exists x Gxu)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu\to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Example

HOML formula	$\Diamond \exists x G(x)$
HOML formula in HOL	valid $(\Diamond \exists x G(x))_{\mu \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\mu (\Diamond \exists x G(x))_{\mu \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge (\exists x G(x))_{\mu \to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\mu \exists u_\mu (rwu \wedge \exists x G x u)$

Expansion: user or prover may flexibly choose expansion depth

What are we doing?

In order to prove that φ is valid in HOML,
–> we instead prove that valid $\varphi_{\mu \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Freie Universität Berlin

Modal logic axioms	Semantical conditions
M: valid $\forall\varphi(\Box\varphi \rightarrow \varphi)$	$\forall x(rxy)$
B: valid $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$	$\forall x\forall y(rxy \rightarrow ryx)$
D: valid $\forall\varphi(\Box\varphi \rightarrow \Diamond\varphi)$	$\forall x\exists y(rxy)$
4: valid $\forall\varphi(\Box\varphi \rightarrow \Box\Box\varphi)$	$\forall x\forall y\forall z(rxy \land ryz \rightarrow rxz)$
5: valid $\forall\varphi(\Diamond\varphi \rightarrow \Box\Diamond\varphi)$	$\forall x\forall y\forall z(rxy \land rxz \rightarrow ryz)$

$\forall \;\; = \lambda h_{\gamma\to(\mu\to o)} \lambda w_\mu \forall d_\gamma \, hdw$ (possibilist / constant dom.)

becomes

$\forall^{va} = \lambda h_{\gamma\to(\mu\to o)} \lambda w_\mu \forall d_\gamma \, (\mathbf{ExInW} dw \to hdw)$ (actualist / varying dom.)

where **ExInW** is an existence predicate.

Additional axioms:

- domains are non-empty $\forall w_\mu \exists x_\mu \mathbf{exInW} xw$
- denotation (constants & functions) $\forall w_\mu \mathbf{exInW} cw$

  $\forall w_\mu (\mathbf{exInW} t^1 w \wedge \ldots \wedge \mathbf{exInW} t^n w \supset \mathbf{exInW}(f\, t^1 \ldots t^n) w)$

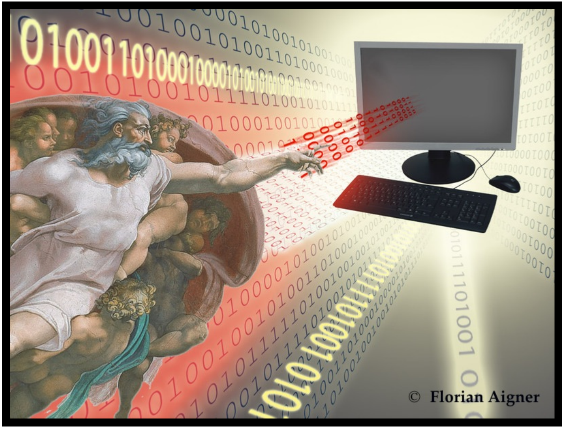
Cumulative domains: $\forall x \forall v \forall w (\mathbf{exInW} xv \wedge rvw \supset \mathbf{exInW} xw)$

Soundness and Completeness (and Cut-elimination)

$$\models^{L} \varphi \quad \text{iff} \quad Ax \models^{HOL}_{\text{Henkin}} \mathit{valid}\, \varphi_{\mu\rightarrow o} \quad (\text{iff} \quad Ax \vdash^{\text{HOL}}_{\text{cut-free}} \mathit{valid}\, \varphi_{\mu\rightarrow o})$$

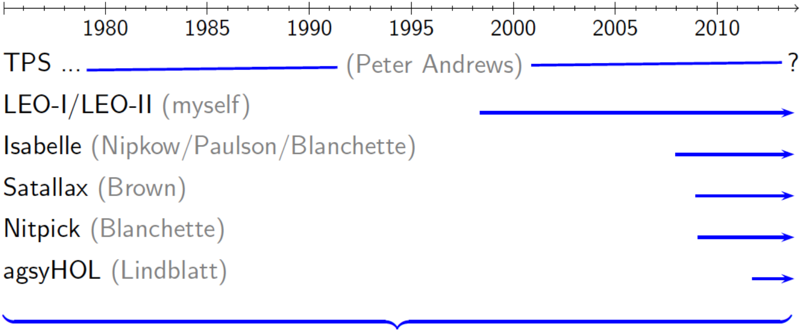
Logic L:

- Higher-order Modal Logics
- First-order Multimodal Logics
- Propositional Multimodal Logics
- Quantified Conditional Logics
- Propositional Conditional Logics
- Intuitionistic Logics
- Access Control Logics
- Logic Combinations
- ...more is on the way ... including:
  - Description Logics
  - Nominal Logics
  - Multivalued Logics (SIXTEEN)
  - Logics based on Neighborhood Semantics
  - (Mathematical) Fuzzy Logics
  - Paraconsistent Logics



Automated and Interactive Theorem Provers for HOL

Automated Theorem Provers and Model Finders for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
  - can be called remotely via SystemOnTPTP at Miami
  - they significantly gained in strength over the last years
  - they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —

Proof Automation with LEO-II

```
>
>
> Beweise-mit-Leo2 Notwendigerweise-existiert-Gott.p

Leo-II tries to prove
=====================

  Goedel's Theorem T3: "Necessarily, God exists"
  thf(thmT3,conjecture,
      ( v
      @ ( mbox
        @ ( mexists_ind
          @ ^ [X: mu] :
              ( g @ X ) ) ) )).

  Assumptions: D1, C, T2, D3, A5

 . searching for proof ..

 ********************
 *   Proof found    *
 ********************
 % SZS status Theorem for Notwendigerweise-existiert-Gott.p

 . generating proof object
```

Provers can be called remotely in Miami — no local installation needed!

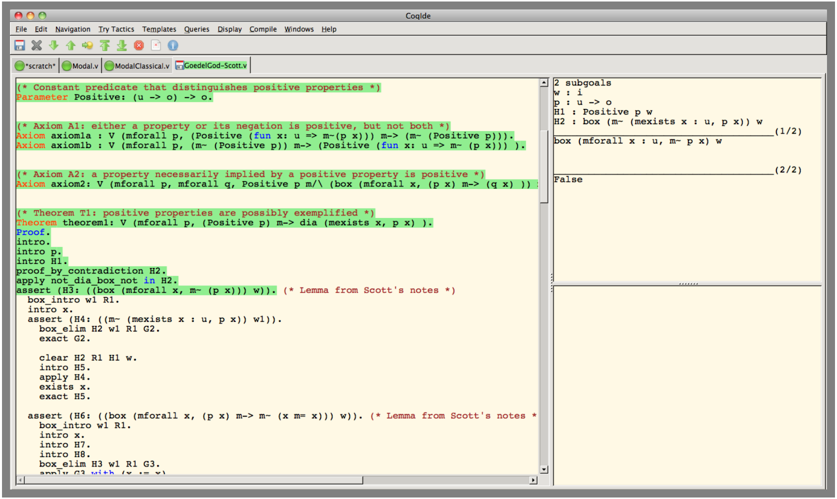
Download our experiments from

`https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/THF`

Freie Universität Berlin

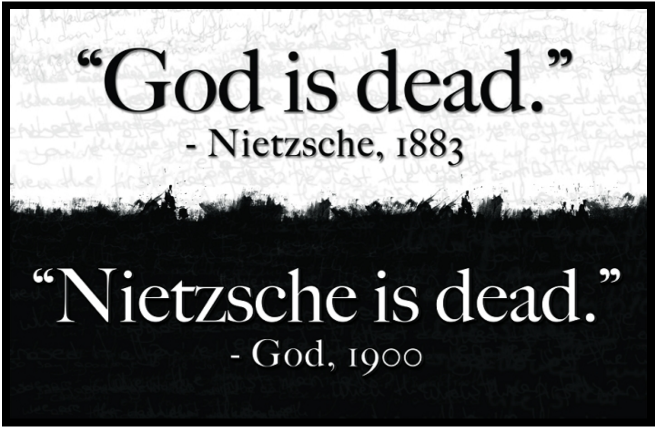
See verifiable Isabelle/HOL document (Archive of Formal Proofs) at:

`http://afp.sourceforge.net/entries/GoedelGod.shtml`



See verifiable Coq document at:

https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Coq



Findings from our study

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014]

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}. \dot\forall\psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu. (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu. \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu. g_{\mu\to\sigma} X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu. g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot\forall\phi_{\mu\to\sigma}. (\mathrm{ess}\,\phi X \dot\supset \dot\Box\dot\exists Y_\mu. \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma} \mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu. g_{\mu\to\sigma} X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}. \dot\forall X_\mu. (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu. \dot\forall Y_\mu. (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \dot\forall\psi_{\mu\to\sigma}. (\psi X \dot\supset \dot\Box\dot\forall Y_\mu. (\phi Y \dot\supset \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_{\mu}.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_{\mu}.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_{\mu}.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_{\mu}.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_{\mu}.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_{\mu}.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_{\mu}.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_{\mu}.\, g_{\mu\to\sigma}X \dot{\supset} (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_{\mu}.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathrm{ess}\,\phi X \dot{\supset} \dot{\Box}\dot{\exists}Y_{\mu}.\, \phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_{\mu}.\, g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_{\sigma} \dot{\supset} \dot{\Box}s_{\sigma}]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_{\mu}.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_{\mu}.\, \dot{\forall}Y_{\mu}.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2’	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_{\mu}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_{\mu}.\, (\phi Y \dot{\supset} \psi Y))$						
CO’	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2’, D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2’, D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

Freie Universität Berlin

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall \phi_{\mu\to\sigma}.\, \dot\forall \psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall \phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall \psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall \phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall \phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	∅ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	∅ (no goal, check for cons						

Automating Scott's proof script

T1: "Positive properties are possibly exemplified" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **A1 and A2**
  - **A1(⊃) and A2**
- **notion of quantification**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \doteq \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\wedge\, \dot\Box\dot\forall X_\mu.(\phi X \,\dot\supset\, \psi X)) \,\dot\supset\, p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Diamond\dot\exists X_\mu.\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \,\dot\supset\, \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot\wedge\, \dot\forall\psi_{\mu\to\sigma}.(\psi X \,\dot\supset\, \dot\Box\dot\forall Y_\mu.(\phi Y \,\dot\supset\, \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \,\dot\supset\, (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.(\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \,\dot\supset\, \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.(g_{\mu\to\sigma}X \,\dot\supset\, ($						
MT	$[\dot\forall X_\mu.\dot\forall Y_\mu.(g_{\mu\to\sigma}X \,\dot\supset\, (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

C: "Possibly, God exists"
proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **T1, D1, A3**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu. \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}. \dot{\forall}\psi_{\mu\to\sigma}. (p_{(\mu\to\sigma)\to\sigma}\phi \dot{\wedge} \dot{\Box}\dot{\forall}X_\mu. (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu. \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu. \dot{\forall}\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu. g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}. p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda X_\mu. \phi X \dot{\wedge} \dot{\forall}\psi_{\mu\to\sigma}. (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu. (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu. g_{\mu\to\sigma}X \dot{\supset} (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} g X)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu. \dot{\forall}\phi_{\mu\to\sigma}. (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu. g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}. \dot{\forall}X_\mu. (g_{\mu\to\sigma}X \dot{\supset} ($						
MT	$[\dot{\forall}X_\mu. \dot{\forall}Y_\mu. (g_{\mu\to\sigma}X \dot{\supset} (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}. \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

T2: "Being God-like is an ess. of any God-like being" proved by LEO-II and Satallax

- **in logic: K**
- **from assumptions:**
  - **A1, D1, A4, D2**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \dot\wedge \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma}X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\, \mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$						
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma}X \dot\supset ($						
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma}X \dot\supset (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO'	$\emptyset$ (no goal, check for cons						

Automating Scott's proof script

T3: "Necessarily, God exists" proved by LEO-II and Satallax

- **in logic: KB**
- **from assumptions:**
  - **D1, C, T2, D3, A5**
- **for domain conditions:**
  - **possibilist quantifiers (constant dom.)**
  - **actualist quantifiers for individuals (varying dom.)**

For logic K we got a countermodel by Nitpick

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \,\dot{\wedge}\, \dot{\Box}\dot{\forall}X_\mu.\, (\phi X \dot{\supset} \psi X)) \dot{\supset} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}X_\mu.\, \phi X]$	A1($\supset$), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p\phi]$						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \phi X \,\dot{\wedge}\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
T2	$[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma}\, gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathrm{e}$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$						
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$						
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} ($						
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_\mu$						
CO	$\emptyset$ (no goal, check for cons						
D2’	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$						
CO’	$\emptyset$ (no goal, check for cons						

Automating Scott’s proof script

Summary

- proof verified and automated
- KB is sufficient (critisized logic S5 not needed!)
- possibilist and actualist quantifiers (individuals)
- exact dependencies determined experimentally
- ATPs have found alternative proofs
  e.g. self-identity $\lambda x(x = x)$ is not needed

Main Findings

Freie Universität Berlin

Consistency check: Gödel vs. Scott

- **Scott's assumptions are consistent; shown by Nitpick**
- **Gödel's assumptions are inconsistent; shown by LEO-II (new philosophical result?)**

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot\neg(\phi X)) \dot\equiv \dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to\sigma}\phi \dot\wedge \dot\Box\dot\forall X_\mu.\, (\phi X \dot\supset \psi X)) \dot\supset p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Diamond\dot\exists X_\mu.\, \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\ldots}$						
A3	$[p_{(\mu\to\sigma)\to\sigma}\, g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
A4	$[\dot\forall\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot\supset \dot\Box p\ldots$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda\ldots$						
T2	$[\dot\forall X_\mu.\, g_{\mu\to\sigma} X \dot\supset (\mathrm{ess}_{(\mu\to\sigma)\ldots}$						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot\forall\phi_{\mu\to\sigma}.\, (\mathrm{e}\ldots$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu.\, g_{\mu\to\sigma} X]$						
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \dot\supset \dot\Box s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot\forall\phi_{\mu\to\sigma}.\, \dot\forall X_\mu.\, (g_{\mu\to\sigma} X \dot\supset (\dot\neg(p_{(\mu\to\sigma)\to\sigma}\phi) \dot\supset \dot\neg(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot\forall X_\mu.\, \dot\forall Y_\mu.\, (g_{\mu\to\sigma} X \dot\supset (g_{\mu\to\sigma} Y \dot\supset X \dot= Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot\forall\psi_{\mu\to\sigma}.\, (\psi X \dot\supset \dot\Box\dot\forall Y_\mu.\, (\phi Y \dot\supset \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\centerdot \dot{\neg}(\phi X)) \dot{\equiv} \dot{\neg}(p\phi)]$						
A2	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot \dot{\forall}\psi_{\mu\to\sigma}\centerdot (p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot{\wedge}} \dot{\Box}\dot{\forall}X_\mu\centerdot(\phi X \mathbin{\dot{\supset}} \psi X)) \mathbin{\dot{\supset}} p\psi]$						
T1	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot{\supset}} \dot{\Diamond}\dot{\exists}X_\mu\centerdot \phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma} = \lambda X_\mu\centerdot \dot{\forall}\phi_{\mu\to\sigma}\centerdot p_{(\mu-}$ … $\phi \mathbin{\dot{\supset}} \phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$						
C	$[\dot{\Diamond}\dot{\exists}X_\mu\centerdot g_{\mu\to\sigma}X]$						
A4	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi \mathbin{\dot{\supset}} \dot{\Box}p$ …						
D2	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to}$ … $\lambda\phi_{\mu\to\sigma}\centerdot \lambda$ …						
T2	$[\dot{\forall}X_\mu\centerdot g_{\mu\to\sigma}X \mathbin{\dot{\supset}} (\mathbf{ess}_{(\mu\to\sigma)}$ …						
D3	$\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu\centerdot \dot{\forall}\phi_{\mu\to\sigma}\centerdot (\mathrm{e}$ …						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot{\Box}\dot{\exists}X_\mu\centerdot g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
MC	$[s_\sigma \mathbin{\dot{\supset}} \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}\centerdot \dot{\forall}X_\mu\centerdot (g_{\mu\to\sigma}X \mathbin{\dot{\supset}} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \mathbin{\dot{\supset}} \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu\centerdot \dot{\forall}Y_\mu\centerdot (g_{\mu\to\sigma}X \mathbin{\dot{\supset}} (g_{\mu\to\sigma}Y \mathbin{\dot{\supset}} X \doteq Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathbf{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}\centerdot \lambda X_\mu\centerdot \dot{\forall}\psi_{\mu\to\sigma}\centerdot (\psi X \mathbin{\dot{\supset}} \dot{\Box}\dot{\forall}Y_\mu\centerdot(\phi Y \mathbin{\dot{\supset}} \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1(⊃), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Further Results

- **Monotheism holds**
- **God is flawless**

Main Findings

HOL encoding

A1 $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu.\, \dot{\neg}$

A2 $[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (p_{(\mu\to\sigma)\to}$

T1 $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Diamond}\dot{\exists}$

D1 $g_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu-}$

A3 $[p_{(\mu\to\sigma)\to\sigma} g_{\mu\to\sigma}]$

C $[\dot{\Diamond}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$

A4 $[\dot{\forall}\phi_{\mu\to\sigma}.\, p_{(\mu\to\sigma)\to\sigma}\phi \dot{\supset} \dot{\Box}p$

D2 $\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda$

T2 $[\dot{\forall}X_\mu.\, g_{\mu\to\sigma}X \dot{\supset} (\mathrm{ess}_{(\mu\to\sigma)}$

D3 $\mathrm{NE}_{\mu\to\sigma} = \lambda X_\mu.\, \dot{\forall}\phi_{\mu\to\sigma}.\, (\mathrm{e}$

A5 $[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$

T3 $[\dot{\Box}\dot{\exists}X_\mu.\, g_{\mu\to\sigma}X]$

Modal Collapse (Sobel)

$$\forall\varphi(\varphi \supset \Box\varphi)$$

- proved by LEO-II and Satallax
- for possibilist and actualist quantification (ind.)

Main critique on Gödel's ontological proof:

- there are no contingent truths
- everything is determined / no free will

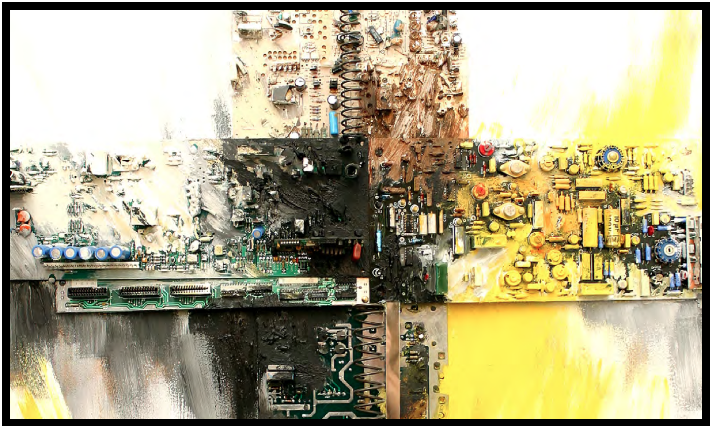
MC	$[s_\sigma \dot{\supset} \dot{\Box}s_\sigma]$	D2, T2, T3	KB	THM	17.9/—	3.3/3.2	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
FG	$[\dot{\forall}\phi_{\mu\to\sigma}.\, \dot{\forall}X_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (\dot{\neg}(p_{(\mu\to\sigma)\to\sigma}\phi) \dot{\supset} \dot{\neg}(\phi X)))]$	A1, D1	KB	THM	16.5/—	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
MT	$[\dot{\forall}X_\mu.\, \dot{\forall}Y_\mu.\, (g_{\mu\to\sigma}X \dot{\supset} (g_{\mu\to\sigma}Y \dot{\supset} X \dot{=} Y))]$	D1, FG	KB	THM	—/—	0.0/3.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
CO	$\emptyset$ (no goal, check for consistency)	A1, A2, D1, A3, A4, D2, D3, A5	KB	SAT	—/—	—/—	7.3/7.4
D2'	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma} = \lambda\phi_{\mu\to\sigma}.\, \lambda X_\mu.\, \dot{\forall}\psi_{\mu\to\sigma}.\, (\psi X \dot{\supset} \dot{\Box}\dot{\forall}Y_\mu.\, (\phi Y \dot{\supset} \psi Y))$						
CO'	$\emptyset$ (no goal, check for consistency)	A1($\supset$), A2, D2', D3, A5	KB	UNS	7.5/7.8	—/—	—/—
		A1, A2, D1, A3, A4, D2', D3, A5	KB	UNS	—/—	—/—	—/—

Main Findings

	HOL encoding	dependencies	logic	status	LEO-II const/vary	Satallax const/vary	Nitpick const/vary
A1	$[\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}(\lambda X_\mu\centerdot\dot\neg(\phi X))\dot\equiv\dot\neg(p\phi)]$						
A2	$[\dot\forall\phi_{\mu\to\sigma}\centerdot\dot\forall\psi_{\mu\to\sigma}\centerdot(p_{(\mu\to\sigma)\to\sigma}\phi\,\dot\wedge\,\dot\Box\dot\forall X_\mu\centerdot(\phi X\,\dot\supset\,\psi X))\,\dot\supset\,p\psi]$						
T1	$[\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi\,\dot\supset\,\dot\Diamond\dot\exists X_\mu\centerdot\phi X]$	A1(⊃), A2	K	THM	0.1/0.1	0.0/0.0	—/—
		A1, A2	K	THM	0.1/0.1	0.0/5.2	—/—
D1	$g_{\mu\to\sigma}=\lambda X_\mu\centerdot\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi\,\dot\supset\,\phi X$						
A3	$[p_{(\mu\to\sigma)\to\sigma}g_{\mu\to\sigma}]$						
C	$[\dot\Diamond\dot\exists X_\mu\centerdot g_{\mu\to\sigma}X]$	T1, D1, A3	K	THM	0.0/0.0	0.0/0.0	—/—
		A1, A2, D1, A3	K	THM	0.0/0.0	5.2/31.3	—/—
A4	$[\dot\forall\phi_{\mu\to\sigma}\centerdot p_{(\mu\to\sigma)\to\sigma}\phi\,\dot\supset\,\dot\Box p\phi]$						
D2	$\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}=\lambda\phi_{\mu\to\sigma}\centerdot\lambda X_\mu\centerdot\phi X\,\dot\wedge\,\dot\forall\psi_{\mu\to\sigma}\centerdot(\psi X\,\dot\supset\,\dot\Box\dot\forall Y_\mu\centerdot(\phi Y\,\dot\supset\,\psi Y))$						
T2	$[\dot\forall X_\mu\centerdot g_{\mu\to\sigma}X\,\dot\supset\,(\mathrm{ess}_{(\mu\to\sigma)\to\mu\to\sigma}gX)]$	A1, D1, A4, D2	K	THM	19.1/18.3	0.0/0.0	—/—
		A1, A2, D1, A3, A4, D2	K	THM	12.9/14.0	0.0/0.0	—/—
D3	$\mathrm{NE}_{\mu\to\sigma}=\lambda X_\mu\centerdot\dot\forall\phi_{\mu\to\sigma}\centerdot(\mathrm{ess}\,\phi X\,\dot\supset\,\dot\Box\dot\exists Y_\mu\centerdot\phi Y)$						
A5	$[p_{(\mu\to\sigma)\to\sigma}\mathrm{NE}_{\mu\to\sigma}]$						
T3	$[\dot\Box\dot\exists X_\mu\centerdot g_{\mu\to\sigma}X]$	D1, C, T2, D3, A5	K	CSA	—/—	—/—	3.8/6.2
		A1, A2, D1, A3, A4, D2, D3, A5	K	CSA	—/—	—/—	8.2/7.5
		D1, C, T2, D3, A5	KB	THM	0.0/0.1	0.1/5.3	—/—
		A1, A2, D1, A3, A4, D2, D3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	17.9/—	3.3/3.2	—/—
		3, A5	KB	THM	—/—	—/—	—/—
			KB	THM	16.5/—	0.0/0.0	—/—
		3, A5	KB	THM	12.8/15.1	0.0/5.4	—/—
			KB	THM	—/—	0.0/3.3	—/—
		3, A5	KB	THM	—/—	—/—	—/—
		3, A5	KB	SAT	—/—	—/—	7.3/7.4
			KB	UNS	7.5/7.8	—/—	—/—
		D3, A5	KB	UNS	—/—	—/—	—/—

Observation

- good performance of ATPs
- excellent match between argumentation granularity in papers and the reasoning strength of the ATPs



Reconstruction of the Inconsistency of Gödel's Axioms

Inconsistency (Gödel): Proof by LEO-II in KB

Freie Universität Berlin

DemoMaterial — bash — 166×52

```
@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$true))),inference(prim_subst,[status(thm)],[66:[bind(SV11,$thf(^[SV23:mu,SV24:$i]: $false))]])).
 thf(84,plain,(![SV22:(mu>($i>$o)),SV3:$i,SV8:(mu>($i>$o))]: ((((SV8@(((sK2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV8))@(((sK1_SY31@(^[SX0:mu,SX1:$i]: (
~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(prim_subst,[status(thm)],[66
:[bind(SV11,$thf(^[SV20:mu,SV21:$i]: (~ ((SV22@SV20)@SV21))))]])).
 thf(85,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | ((((p@SV9)@SV4) = ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)
@SY28))))@SV4))=$false))),inference(fac_restr,[status(thm)],[56])).
 thf(86,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true) | ((((p@SV9)@SV4) = ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@
SY30))))@SV4))=$false))),inference(fac_restr,[status(thm)],[57])).
 thf(87,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,
SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_equal_neg,[status(thm)],[85])).
 thf(89,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4))) | (~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY29:mu,
SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_equal_neg,[status(thm)],[86])).
 thf(92,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ ((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))))=$false) | (((p@(^[SY27:mu,SY28:$i]: (
~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_neg,[status(thm)],[87])).
 thf(93,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ (((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)))=$false) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@
SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_neg,[status(thm)],[89])).
 thf(96,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((~ ((p@SV9)@SV4)) | (~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)))=$true) | (((p@(^[SY27:mu,SY28:$i]: (~ ((S
V9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_neg,[status(thm)],[92])).
 thf(97,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((((p@SV9)@SV4) | ((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4))=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)
@SY30))))@SV4)=$true))),inference(extcnf_not_neg,[status(thm)],[93])).
 thf(100,plain,(![SV4:$i,SV9:(mu>($i>$o))]: (((~ ((p@SV9)@SV4))=$true) | ((~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))=$true) | (((p@(^[SY27:mu,SY28:$i]:
(~ ((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_or_pos,[status(thm)],[96])).
 thf(101,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV
9@SY29)@SY30))))@SV4)=$true))),inference(extcnf_or_pos,[status(thm)],[97])).
 thf(103,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$false) | ((~ ((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4))=$true) | (((p@(^[SY27:mu,SY28:$i]: (~
((SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_pos,[status(thm)],[100])).
 thf(105,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4)=$false) | (((p@(^[SY27:mu,SY28:$i]: (~ ((
SV9@SY27)@SY28))))@SV4)=$false))),inference(extcnf_not_pos,[status(thm)],[103])).
 thf(107,plain,(![SV8:(mu>($i>$o)),SV3:$i,SV22:(mu>($i>$o))]: ((((SV22@(((sK2_SY33@SV3)@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV8))@(((sK1_SY31@(^[SX0:mu,SX1:$i]:
(~ ((SV22@SX0)@SX1))))@SV8)@SV3))=$true) | (((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV22@SX0)@SX1))))@SV3)=$true))),inference(extcnf_not_neg,[status(thm
)],[78])).
 thf(108,plain,(![SV11:(mu>($i>$o)),SV3:$i,SV15:(mu>($i>$o))]: ((((SV15@(((sK2_SY33@SV3)@SV11)@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1)))))@(((sK1_SY31@SV11)@(^[SX0:mu,
SX1:$i]: (~ ((SV15@SX0)@SX1))))@SV3))=$false) | (((p@(^[SX0:mu,SX1:$i]: (~ ((SV15@SX0)@SX1))))@SV3)=$false) | (((p@SV11)@SV3)=$true))),inference(extcnf_not_pos,[statu
s(thm)],[81])).
 thf(109,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@(^[SY27:mu,SY28:$i]: (~ ((SV9@SY27)@SY28))))@SV4)=$false) | (((p@SV9)@SV4)=$false))),inference(sim,[status(thm)],[10
5])).
 thf(110,plain,(![SV4:$i,SV9:(mu>($i>$o))]: ((((p@SV9)@SV4)=$true) | (((p@(^[SY29:mu,SY30:$i]: (~ ((SV9@SY29)@SY30))))@SV4)=$true))),inference(sim,[status(thm)],[101]
)).
 thf(111,plain,(![SV3:$i,SV8:(mu>($i>$o))]: ((((p@SV8)@SV3)=$false) | (((p@(^[SX0:mu,SX1:$i]: $true))@SV3)=$true))),inference(sim,[status(thm)],[76])).
 thf(112,plain,(![SV11:(mu>($i>$o)),SV3:$i]: ((((p@(^[SX0:mu,SX1:$i]: $false))@SV3)=$false) | (((p@SV11)@SV3)=$true))),inference(sim,[status(thm)],[80])).
 thf(113,plain,((($false)=$true)),inference(fo_atp_e,[status(thm)],[25,112,111,110,109,108,107,84,83,82,75,74,73,72,71,70,69,68,67,66,65,62,57,56,51,42,29])).
 thf(114,plain,($false),inference(solved_all_splits,[solved_all_splits(join,[])],[113])).
% SZS output end CNFRefutation

%***** End of derivation protocol ****
%***** no. of clauses in derivation: 97 ****
%***** clause counter: 113 ****

% SZS status Unsatisfiable for ConsistencyWithoutFirstConjunctinD2.p : (rf:0,axioms:6,ps:3,u:6,ude:false,rLeibEQ:true,rAndEQ:true,use_choice:true,use_extuni:true,use_
extcnf_combined:true,expand_extuni:false,foatp:e,atp_timeout:25,atp_calls_frequency:10,ordering:none,proof_output:1,clause_count:113,loop_count:0,foatp_calls:2,transl
ation:fof_full)
ontoleo:DemoMaterial cbenzmueller$
```

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1(⊃) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1(⊃) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

by A1(⊃), A2

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1($\supset$) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

by A1($\supset$), A2

Def. D2*

$$\phi \; ess \; x \leftrightarrow \cancel{\phi(x) \wedge} \; \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$$

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1($\supset$) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

by A1($\supset$), A2

Def. D2* $\phi\ ess\ x \leftrightarrow \cancel{\phi(x) \wedge} \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset\ ess\ x)$

by D2*

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1($\supset$) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

by A1($\supset$), A2

Def. D2* $\phi \; ess \; x \leftrightarrow \cancel{\phi(x) \wedge} \; \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset \; ess \; x)$

by D2*

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi \; ess \; x \rightarrow \Box\exists y\phi(y)]$

Axiom B $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$ (resp. $\forall x\forall y(rxy \rightarrow ryx)$)

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1(⊃) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

by A1(⊃), A2

Def. D2* $\phi \; ess \; x \leftrightarrow \cancel{\phi(x) \wedge} \; \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset \; ess \; x)$

by D2*

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi \; ess \; x \rightarrow \Box\exists y\phi(y)]$

Axiom B $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$ (resp. $\forall x\forall y(rxy \rightarrow ryx)$)

Lemma 2 Exemplification of necessary existence is not possible. $\neg\Diamond\exists x\,NE(x)$

by B, D3, Lemma1

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1(⊃) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$
by A1(⊃), A2

Def. D2* $\phi\ ess\ x \leftrightarrow \cancel{\phi(x) \wedge}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset\ ess\ x)$
by D2*

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$

Axiom B $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$ (resp. $\forall x\forall y(rxy \rightarrow ryx)$)

Lemma 2 Exemplification of necessary existence is not possible. $\neg\Diamond\exists x\,NE(x)$
by B, D3, Lemma1

Axiom A5 $P(NE)$

Inconsistency (Gödel): Reconstruction of Informal Argument (KB)

(special thanks to Chad Brown for a fruitful discussion)

Axiom A1(⊃) $\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$

Axiom A2 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$

Theorem 1 Positive Properties are possibly exemplified. $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

by A1(⊃), A2

Def. D2* $\phi\ ess\ x \leftrightarrow \cancel{\phi(x) \wedge}\ \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Lemma 1 The empty property is an essence of every entity. $\forall x\,(\emptyset\ ess\ x)$

by D2*

Def. D3 $NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$

Axiom B $\forall\varphi(\varphi \rightarrow \Box\Diamond\varphi)$ (resp. $\forall x\forall y(rxy \rightarrow ryx)$)

Lemma 2 Exemplification of necessary existence is not possible. $\neg\Diamond\exists x\, NE(x)$

by B, D3, Lemma1

Axiom A5 $P(NE)$

Inconsistency $\bot$

by A5, T1, Lemma2

Inconsistency (Gödel): Verification in Isabelle/HOL (KB)

Freie Universität Berlin

GoedelGodWithoutConjunctInEss_KB.thy

GoedelGodWithoutConjunctInEss_KB.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_KB imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
                   and A2:  "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"

-- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (∃ Φ))]"                          by (metis A1a A2)

  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"

-- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(∀(λx.( (λy.λw. False) ess x)))]"               by (metis ess_def)

  definition NE  where "NE x = ∀(λΦ. Φ ess x m→ □ (∃ Φ))"
  axiomatization where sym: "x r y ⟶ y r x"

-- {* Exemplification of necessary existence is not possible. *}
  lemma Lemma2: "[m¬ (◇ (∃ NE))]"                        by (metis sym Lemma1 NE_def)

  axiomatization where A5:  "[P NE]"

-- {* Now the inconsistency follows from A5, T1 and Lemma2 *}
  lemma False                                               by (metis A5 T1 Lemma2)
end
```

Documentation Sidekick Theories

Output Query Sledgehammer Symbols

11,1 (477/1095) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 263/347MB 17:18

Inconsistency (Gödel): Verification in Isabelle/HOL (K)

Freie Universität Berlin

GoedelGodWithoutConjunctInEss_K.thy

GoedelGodWithoutConjunctInEss_K.thy (~/GoedelGod/Talks/2015/LogicAndReligion/DemoMaterial/)

```
theory GoedelGodWithoutConjunctInEss_K imports QML
begin
  consts P :: "(μ ⇒ σ) ⇒ σ"
  definition ess (infixr "ess" 85) where "Φ ess x = ∀(λΨ. Ψ x m→ □ (∀(λy. Φ y m→ Ψ y)))"
  definition NE  where                   "NE x     = ∀(λΦ. Φ ess x m→ □ (∃ Φ))"
  axiomatization where A1a: "[∀(λΦ. P (λx. m¬ (Φ x)) m→ m¬ (P Φ))]"
                   and A2:  "[∀(λΦ. ∀(λΨ. (P Φ m∧ □ (∀(λx. Φ x m→ Ψ x))) m→ P Ψ))]"

-- {* Positive properties are possibly exemplified. *}
  theorem T1: "[∀(λΦ. P Φ m→ ◇ (∃ Φ))]"                          by (metis A1a A2)

-- {* The empty property is an essence of every individual. *}
  lemma Lemma1: "[(∀(λx.((λy.λw. False) ess x)))]"               by (metis ess_def)

  axiomatization where A5:  "[P NE]"

-- {* Now the inconsistency follows from A5, Lemma1, NE_def and T1 *}
  lemma False
  -- {* sledgehammer [remote_leo2] *}
                                              by (metis A5 Lemma1 NE_def T1)
end
```

Documentation Sidekick Theories

Output Query Sledgehammer Symbols

21,7 (980/982) (isabelle,sidekick,UTF-8-Isabelle)Nmro UG 302/343MB 11:37

Gödel's Manuscript

Ontologischer Beweis — Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ • At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess.} x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess.} x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x) G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x) G(x) \supset MN(\exists y)\, G(y)$

" $\supset N(\exists y)\, G(y)$ — M = possibility

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x) G(x)$ means the system of all pos. prop. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl

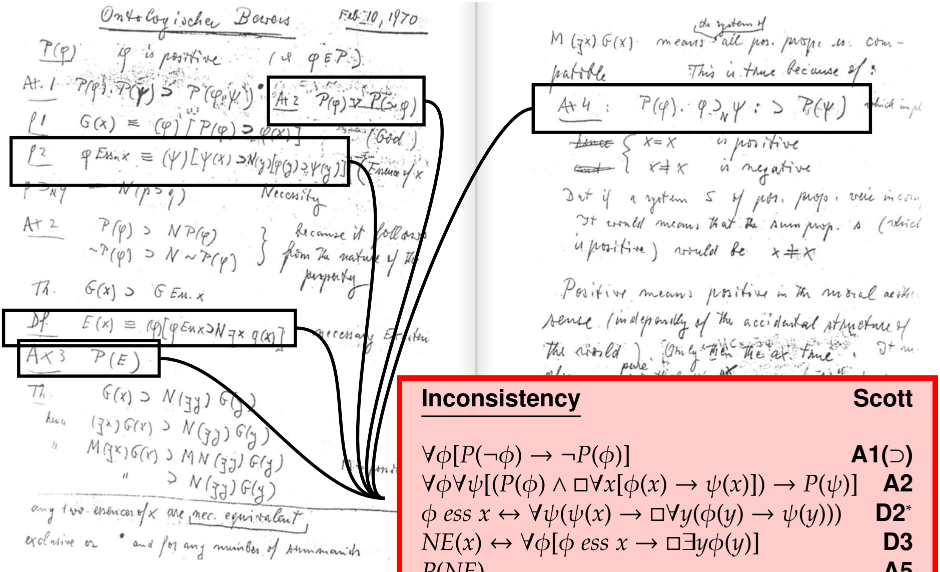
{ $x = x$ is positive; $x \neq x$ is negative

But if a system S of pos. prop. were incom it would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It may also mean pure "attribution"^x as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim\varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ neg. contrary At... or the existence of pos. prop.

x i.e. the disj. normal form in terms of elem. prop. contains a member without negation.



Inconsistency **Scott**

$\forall\phi[P(\neg\phi) \rightarrow \neg P(\phi)]$	**A1(⊃)**
$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$	**A2**
$\phi\ ess\ x \leftrightarrow \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$	**D2***
$NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \rightarrow \Box\exists y\phi(y)]$	**D3**
$P(NE)$	**A5**

Summary: Results of Experiments

Gödel's version	K constant	K varying	KB constant	KB varying	S5 constant	S5 varying
Consistency	×	×	×	×	×	×
T1	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
C	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T2	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
T3	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Flawless God	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Monotheism	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete
Modal Collapse	obsolete	obsolete	obsolete	obsolete	obsolete	obsolete

Further logic details

- Henkin semantics
- full comprehension
- rigid constant symbols

Question: **Has this inconsistency be reported before?**

If not, then LEO-II deserves (part of) the credit!

Summary: Results of Experiments

Scott's version	K		KB		S5	
	constant	varying	constant	varying	constant	varying
Consistency	✓	✓	✓	✓	✓	✓
T1	✓	✓	✓	✓	✓	✓
C	✓	✓	✓	✓	✓	✓
T2	✓	✓	✓	✓	✓	✓
T3	×	×	✓	✓	✓	✓
Flawless God	✓	✓	✓	✓	✓	✓
Monotheism	✓	✓	✓	✓	✓	✓
Modal Collapse	✓	✓	✓	✓	✓	✓

Further logic details

- Henkin semantics
- full comprehension
- rigid constant symbols

Avoiding the Modal Collapse: Recent Variants

Freie Universität Berlin

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's Ontological Proof Revisited *

C. Anthony Anderson and Michael Gettings

University of California, Santa Barbara
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

Magari and others on Gödel's ontological proof

Petr Hájek
Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic
e-mail: hajek@uivt.cas.cz

1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes

(betreffend Gödels ontologischen Beweis)

Es ist gut, daß wir nicht wissen,
sondern glauben, daß ein Gott sei.
(Kant, Nachlaß)

1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung“[1] zu schreiben, ohne aus ihr einen

PETR HÁJEK

A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

1. Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

Understanding Gödel's Ontological Argument

FRODE BJØRDAL

In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

Avoiding the Modal Collapse: Some Emendations

Freie Universität Berlin

> **SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF**
>
> C. Anthony Anderson
>
> Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms

> **Gödel's Ontological Proof Revisited** *
>
> C. Anthony Anderson and Michael Gettings
>
> University of California, Santa Barbara
> Department of Philosophy
>
> Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

> **Magari and others on Gödel's ontological proof**
>
> Petr Hájek
>
> Institute of Computer Science, Academy of Sciences
> 182 07 Prague, Czech Republic
> e-mail: hajek@uivt.cas.cz
>
> **1 Introduction**
>
> This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

> **Der Mathematiker und die Frage der Existenz Gottes**
> **(betreffend Gödels ontologischen Beweis)**
>
> Es ist gut, daß wir nicht wissen, sondern glauben, daß ein Gott sei. (Kant, Nachlaß)
>
> **1. Einführung**
>
> Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

> PETR HÁJEK **A New Small Emendation of Gödel's Ontological Proof**
>
> *Keywords*: Ontological proof, Gödel, modal logic, comprehension, positive properties.
>
> **1. Introduction**
>
> Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known in 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adams' introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several

> **Understanding Gödel's Ontological Argument**
>
> FRODE BJØRDAL
>
> In 1970 Kurt Gödel, in a hand-written note entitled "Ontologischer Beweis", put forward an ontological argument for the existence of God, making use of second-order modal logical principles. Let the second-order formula P(F) stand for "the property F is positive", and let "God" signify the property of being God-like. Gödel presupposes the following definitions:

Computer-supported Clarification of Controversy
see next talk by Bruno Woltzenlogel-Paleo

Conclusion

Overall Achievements

- significant contribution towards a **Computational Metaphysics**
- **novel results** contributed by **HOL-ATPs**
- infrastructure can be adapted for **other logics and logic combinations**
- **our technology is sufficiently mature** for use by philosophers

Relevance (wrt foundations and applications)

- Philosophy, AI, Computer Science, Computational Linguistics, Maths

Little related work: only for Anselm's simpler argument

- first-order ATP PROVER9 [OppenheimerZalta, 2011]
- interactive proof assistant PVS [Rushby, 2013]

Ongoing/Future work **see next talk by Bruno Woltzenlogel-P.**

- Landscape of verified/falsified ontological arguments
- You may consider to contribute:
  `https://github.com/FormalTheology/GoedelGod.git`

Freie Universität Berlin

(Interim) Culmination of two decades of related own research

- Theory of classical higher-order logic (HOL) (since 1995)
- Automation of HOL / own LEO provers (since 1998)
- Integration of interactive and automated theorem proving (since 1999)
- International TPTP infrastructure for HOL (since 2006)
- HOL as a universal logic via semantic embeddings (since 2008)
- jww Bruno Woltzenlogel-Paleo:
  Application in Metaphysics: Ontological Argument (since 2013)

... success story (despite strong criticism/opposition on the way!) ...
... huge media attention ...

(Interim) Own standpoint

- I am not fully convinced (yet) by the ontological argument.
- However, it seems to me that **the belief in a (God-like) supreme being is at least not necessarily irrational/inconsistent**.

SPIEGEL ONLINE WISSENSCHAFT

Login | Registrierung

Politik | Wirtschaft | Panorama | Sport | Kultur | Netzwelt | Wissenschaft | Gesundheit | einestages | Karriere | Uni | Schule | Reise | Auto

Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- . . .

Austria
- Die Presse
- Wiener Zeitung
- ORF
- . . .

Italy
- Repubblica
- Ilsussidario
- . . .

India
- DNA India
- Delhi Daily News
- India Today
- . . .

US
- ABC News
- . . .

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
- . . .

Home | Video | Themen | Forum | English | DER SPIEGEL | SPIEGEL TV | Abo | Shop   RSS | Mobile | Newsletter

Sign in | Register

SPIEGEL ONLINE INTERNATIONAL

Front Page World Europe Germany Business Zeitgeist Newsletter

English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- . . .

Austria
- Die Presse
- Wiener Zeitung
- ORF
- . . .

Italy
- Repubblica
- Ilsussidario
- . . .

India
- DNA India
- Delhi Daily News
- India Today
- . . .

US
- ABC News
- . . .

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
- . . .


DIE WELT

FREITAG, 18. OKTOBER 2013

Zippert zappt

Teuer erkaufte Harmonie

Mindestlohn lockt SPD in dritte Ehe mit der Union

Mittwoch soll über eine große Koalition verhandelt werden. Votum des SPD-Konvents steht aus

THEMEN

Sport
Bundestrainer Löw unterschreibt beim DFB bis 2016

Wirtschaft
Der Jumbojet wird bei Boeing zum Ladenhüter

Karriere
Stress im Job: So legen Sie sich einen dicken Panzer zu

Aus aller Welt
Mietprozess in Paris: Dürfen 15 Quadratmeter 330 Euro kosten?

Dax
Im Minus

Göttliche Mathematik

Haben Forscher mit einem Computerprogramm die Existenz Gottes bewiesen?

UNION GLASHÜTTE/SA

Austria
- Die Presse
- Wiener Zeitung
- ORF
- . . .

Italy
- Repubblica
- Ilsussidario
- . . .

India
- DNA India
- Delhi Daily News
- India Today
- . . .

US
- ABC News
- . . .

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
- . . .

SCIENCE NEWS

HOME / SCIENCE NEWS / RESEARCHERS SAY THEY USED MACBOOK TO PROVE GOEDEL'S GOD THEOREM

Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | 1 comments

See more serious and funny news links at
`https://github.com/FormalTheology/GoedelGod/tree/master/Press`