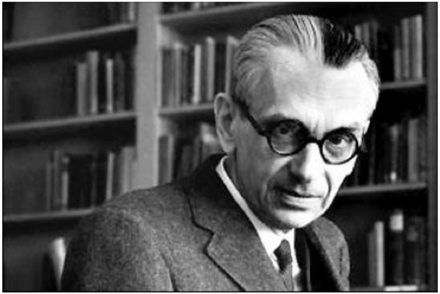
On Universal Logical Reasoning & Gödel's Ontological Argument

Christoph Benzmüller – Freie Universität Berlin | University of Luxembourg



"There is a scientific (exact) philosophy and theology,
which deals with concepts of the highest abstractness;
and this is also most highly fruitful for science."
- Kurt Gödel (Wang, 1996)[p. 316]

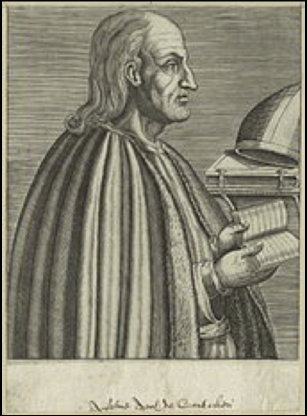
Flammarions Holzstich – in L'atmosphère, Paris 1888.

— Computational Metaphysics —
Ontological Argument on the Computer

Related work (on earlier variants):

- Ed Zalta (& co) with PROVER9 at Stanford [AJP 2011, CADE 2015]
- John Rushby with PVS at SRI [CAV-WS 2013, JAL 2018]

Ontological Argument — A Long Tradition



Anselm v. Canterbury (1033-1109)

Rational Argument for Existence of God

God is . . . that being than which nothing greater can be conceived.

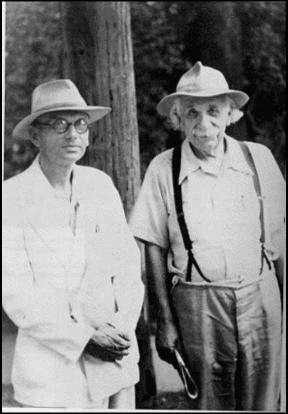
(Proslogion, 1077/78)

It follows:

God exists!?

Anselm v. C.
Gaunilo
Th. Aquinas
Descartes
Spinoza
Leibniz
Hume
Kant
Hegel
Frege
Hartshorne
Malcolm
Lewis
Plantinga
Gödel

Ontological Argument — A Long Tradition



Kurt Gödel (1914-1976) with Einstein

Rational Argument for Existence of God

Definition:
A *Godlike* being possesses all positive properties.

(plus further axioms and definitions)

Theorem:
God exists necessarily.

Anselm v. C.
Gaunilo
Th. Aquinas
Descartes
Spinoza
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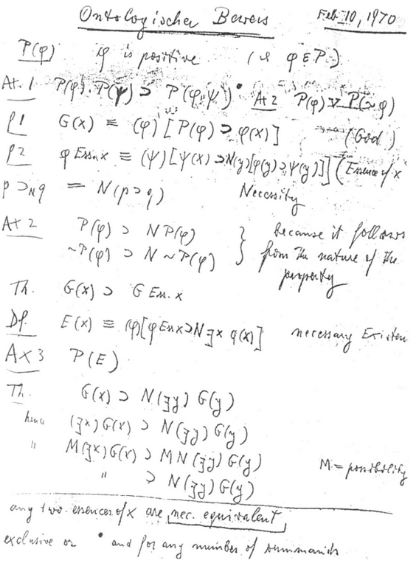
Computational Metaphysics: Kurt Gödel's Ontological Argument (1970)



Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive   (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess. } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset MN(\exists y) G(y)$ M = possibility

" $\supset N(\exists y) G(y)$

any two essences of x are nec. equivalent

exclusive or   • and for any number of summands

$M(\exists x)G(x)$ means all [the system] …

patible   This is …

At 4: $P(\varphi).\, \varphi \supset_N \psi$ …

$x = x$ is p…

$x \neq x$ is …

But if a system S of …

It would mean that …

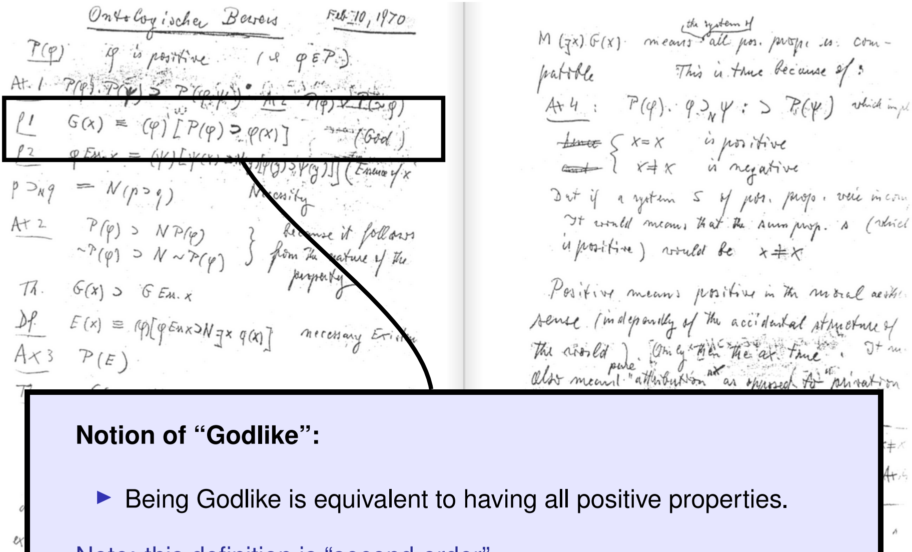
is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It m… also means pure "attribution" as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ positive: $(x) N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive so $x = x$ … contrary At … or the exist. of pos. prop.

x i.e. the normal form in terms of elem. prop. contains a member without negation.

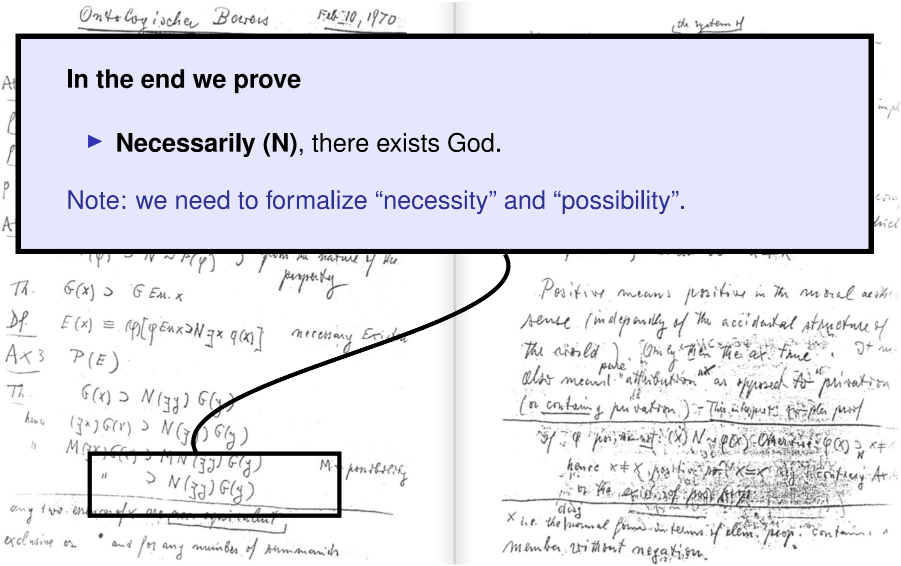
Computational Metaphysics: Kurt Gödel’s Ontological Argument (1970)



Notion of “Godlike”:

- Being Godlike is equivalent to having all positive properties.

Note: this definition is “second-order”.



In the end we prove

- **Necessarily (N)**, there exists God.

Note: we need to formalize “necessity” and “possibility”.

Presentation Outline

A Universal Logical Reasoning (in Meta-Logic HOL)

B Ontological Argument of Gödel & Scott on the Computer

- Recap of Methodology and Main Findings

C Relevant Notions for this Talk:

- Intension vs. Extension of Properties
- Ultrafilter

D Comparative Analysis:

- Gödel/Scott (1972) variant
- Anderson's (1990) variant
- Fitting's (2002) variant

E Further Applications

- Principia Logico-Metaphysica
- Category Theory
- Normative Reasoning and Machine Ethics

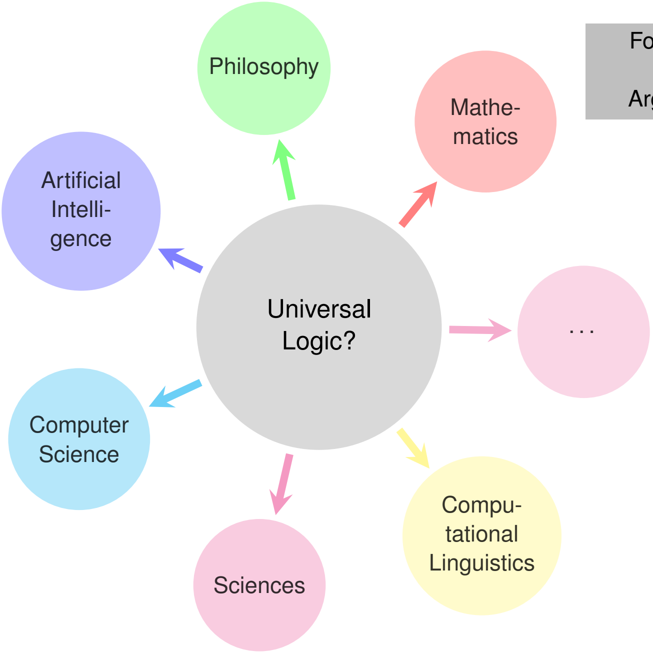
E Conclusion

"If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis."

(Leibniz, 1677)

Part A
Universal Logical Reasoning in Meta-Logic HOL

Foundation for Rational Argumentation



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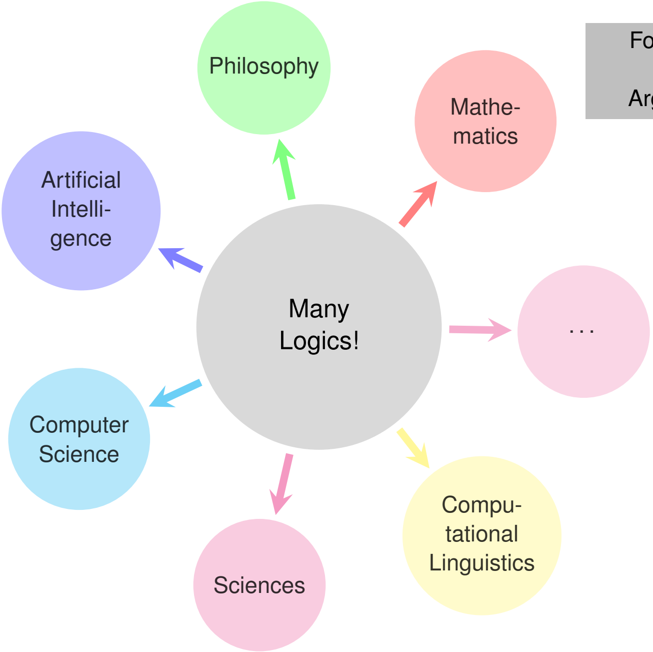


Logic Zoo

Logic Zoo

Classical Logic, of order

0. Propositional Logic
1. First-order Logic
2. Second-order Logic

. . .

n. Higher-order Logic

Non-Classical Logics

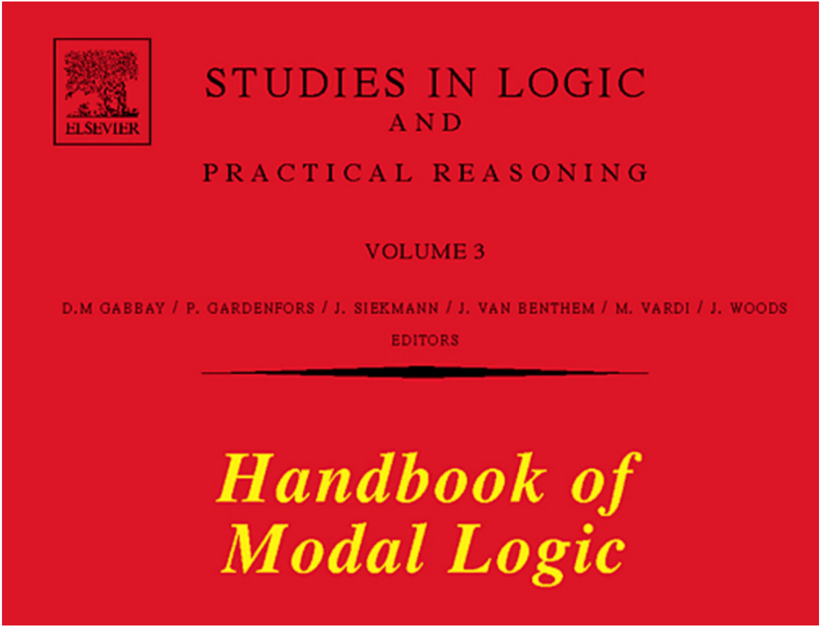
- Intuitionistic/Constructive Logics (incl. Univalent Foundations)
- Modal Logics, Conditional Logics, Temporal Logis, Spatial Logics
- Many-valued Logics
- Paraconsistent Logics
- Free Logics, Inclusive Logics
- Logics for special applications: Ethics, Social Choice, Legal Reasoning, . . .
- Separation Logic, . . .

Example Application in Metaphysics/Philosophy:

Necessarily, God exists: $\Box \exists x. Gx$

Kurt Gödel's definition of God: $G x \ := \ \forall \Phi. \mathit{Positive}\ \Phi \rightarrow \Phi x$

Example: Modal Logic Textbook



Example: Modal Logic Textbook

2 BASIC MODAL LOGIC

In this section we introduce the basic modal language and its relational semantics. We define basic modal syntax, introduce models and frames, and give the satisfaction definition. We then draw the reader's attention to the internal perspective that modal languages offer on relational structure, and explain why models and frames should be thought of as graphs. Following this we give the standard translation. This enables us to convert any basic modal formula into a first-order formula with one free variable. The standard translation is a bridge between the modal and classical worlds, a bridge that underlies much of the work of this chapter.

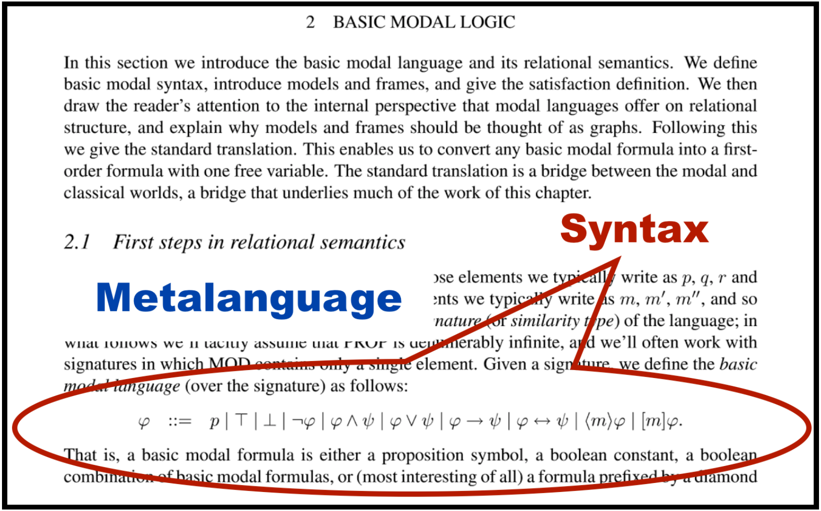
2.1 First steps in relational semantics

Suppose we have a set of proposition symbols (whose elements we typically write as p, q, r and so on) and a set of modality symbols (whose elements we typically write as m, m', m'', and so on). The choice of PROP and MOD is called the *signature* (or *similarity type*) of the language; in what follows we'll tacitly assume that PROP is denumerably infinite, and we'll often work with signatures in which MOD contains only a single element. Given a signature, we define the *basic modal language* (over the signature) as follows:

$$\varphi \quad ::= \quad p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi \mid \varphi \leftrightarrow \psi \mid \langle m \rangle \varphi \mid [m]\varphi.$$

That is, a basic modal formula is either a proposition symbol, a boolean constant, a boolean combination of basic modal formulas, or (most interesting of all) a formula prefixed by a diamond

Example: Modal Logic Textbook



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That is, a basic modal formula is either a proposition symbol, a boolean constant, a boolean combination of basic modal formulas, or (most interesting of all) a formula prefixed by a diamond

Example: Modal Logic Textbook

A *model* (or *Kripke model*) $\mathfrak{M}$ for the basic modal language (over some fixed signature) is a triple $\mathfrak{M} = (W, \{R^m\}_{m \in \text{MOD}}, V)$. Here W, the *domain*, is a non-empty set, whose elements we usually call *points*, but which, for reasons which will soon be clear, are sometimes called *states*, *times*, *situations*, *worlds* and other things besides. Each R^m in a model is a binary relation on W, and V is a function (the valuation) that assigns to each proposition symbol p in PROP a subset $V(p)$ of W; think of $V(p)$ as the set of points in $\mathfrak{M}$ where p is true. The first two components $(W, \{R^m\}_{m \in \text{MOD}})$ of $\mathfrak{M}$ are called the *frame* underlying the model. If there is only one relation in the model, we typically write (W, R) for its frame, and (W, R, V) for the model itself. We encourage the reader to think of Kripke models as graphs (or to be slightly more precise, *directed graphs*, that is, graphs whose points are linked by directed arrows) and will shortly give some examples which show why this is helpful.

Suppose w is a point in a model $\mathfrak{M} = (W, \{R^m\}_{m \in \text{MOD}}, V)$. Then we inductively define the notion of a formula φ being *satisfied* (or *true*) in $\mathfrak{M}$ at point w as follows (we omit some of the clauses for the booleans):

$$
\begin{array}{rll}
\mathfrak{M}, w \models p & \text{iff} & w \in V(p), \\
\mathfrak{M}, w \models \top & & \text{always}, \\
\mathfrak{M}, w \models \bot & & \text{never}, \\
\mathfrak{M}, w \models \neg\varphi & \text{iff} & \text{not } \mathfrak{M}, w \models \varphi \text{ (notation: } \mathfrak{M}, w \not\models \varphi), \\
\mathfrak{M}, w \models \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi, \\
\mathfrak{M}, w \models \varphi \to \psi & \text{iff} & \mathfrak{M}, w \not\models \varphi \text{ or } \mathfrak{M}, w \models \psi, \\
\mathfrak{M}, w \models \langle m \rangle \varphi & \text{iff} & \text{for some } v \in W \text{ such that } R^m wv \text{ we have } \mathfrak{M}, v \models \varphi, \\
\mathfrak{M}, w \models [m] \varphi & \text{iff} & \text{for all } v \in W \text{ such that } R^m wv \text{ we have } \mathfrak{M}, v \models \varphi.
\end{array}
$$

Example: Modal Logic Textbook

A *model* (or *Kripke model*) $\mathfrak{M}$ for the basic modal language (over some fixed signature) is a triple $\mathfrak{M} = (W, \{R^m\}_{m \in \text{MOD}}, V)$. Here W, the *domain*, is a non-empty set, whose elements we usually call *points*, but which, for reasons which will soon be clear, are sometimes called *states*, *times* … in a model is a binary relation on W, and … position symbol p in PROP a subset $V(p)$ … p is true. The first two components $(W, \{$ … model. If there is only one relation in the … (W, R, V) for the model itself. We encourage the reader to think of Kripke models as graphs (or to be slightly more precise, *directed graphs*, that is, graphs whose points are linked by directed arrows) and will shortly give some examples which show why this is helpful.

Metalanguage

Suppose w is a point in a model $\mathfrak{M} = (W, \{R^m\}_{m \in \text{MOD}}, V)$. Then we inductively define the notion of a formula φ being *satisfied* (or *true*) in $\mathfrak{M}$ at point w as follows (we omit some of the clauses for the booleans):

Semantics

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\mathfrak{M}, w \models \langle m \rangle \varphi & \text{iff} & \text{for some } v \in W \text{ such that } R^m wv \text{ we have } \mathfrak{M}, v \models \varphi, \\
\mathfrak{M}, w \models [m]\varphi & \text{iff} & \text{for all } v \in W \text{ such that } R^m wv \text{ we have } \mathfrak{M}, v \models \varphi.
\end{array}
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Kripke Style Semantics

$M, g, s \models P$ if and only if $s \in g(P)$

$M, g, s \models \neg \varphi$ if and only if $M, g, s \not\models \varphi$

$M, g, s \models \varphi \vee \psi$ if and only if $M, g, s \models \varphi$ or $M, g, s \models \psi$

$M, g, s \models \Box \varphi$ if and only if for all t with sRt we have $M, g, t \models \varphi$

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Standard Translation for Propositional Fragment (encoded in HOL)

- "lifted" predicate $P_{i \to o}$
- $\neg_{(i\to o)\to(i\to o)}\ \varphi_{i\to o}\ w_i = \neg(\varphi w)$
- $\vee_{(i\to o)\to(i\to o)\to(i\to o)}\ \varphi_{i\to o}\ \psi_{i\to o}\ w_i = \varphi w \vee \psi w$
- $\Box_{(i\to o)\to(i\to o)}\ \varphi_{i\to o}\ w_i = \forall v_i\ Rwv \to \varphi v$

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- $\Box_{(i\to o)\to(i\to o)} = \lambda\varphi_{i\to o}\, \lambda w_i\, \forall v_i\, Rwv \to \varphi v$

Validity

- $[\varphi_{i\to o}] = \forall w_i\, \varphi\, w$ resp. $[.] = \lambda\varphi_{i\to o}\, \forall w_i\, \varphi\, w$

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$M, g, s \models P$ if and only if $s \in g(P)$

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Modal Logic (in fact, Hybrid Logic) as a Fragment of HOL

Kripke Style Semantics

$M, g, s \models \forall x\, \varphi$ if and only if for all $d \in D$ we have $M, ([d/x]g), s \models \varphi$

Standard Translation extended for Quantifiers (and encoded in HOL)

- in HOL $\forall x_\alpha\, \varphi x$ is shorthand for $\Pi_{(\alpha\to o)\to o}(\lambda x_\alpha\, \varphi x)$ —no binder needed!!!
- $\Pi_{(\alpha\to(i\to o))\to(i\to o)} = \lambda\Phi_{\alpha\to(i\to o)}\, \lambda w_i\, \Pi_{(\alpha\to o)\to o}(\lambda x_\alpha\, \Phi x w)$

Example (compositionality and λ-conversion at work)

$$
\begin{aligned}
[\Box\forall x\, Px] \quad &\equiv \quad [\Box\Pi(\lambda x\, Px)] \quad \equiv \quad [\Box\Pi(\lambda x\, \lambda w\, Pxw)]\\
&\equiv \quad [\Box((\lambda\Phi\, \lambda w\, \Pi(\lambda x\, \Phi x w))(\lambda x\, \lambda w\, Pxw))]\\
&\equiv \quad [\Box(\lambda w\, \Pi(\lambda x\, (\lambda x\, \lambda w\, Pxw)xw))]\\
&\equiv \quad [\Box(\lambda w\, \Pi(\lambda x\, Pxw))]\\
&\equiv \quad [(\lambda\varphi\, \lambda w\, \Pi(\lambda v\, Rwv \to \varphi v))(\lambda w\, \Pi(\lambda x\, Pxw))]\\
&\equiv \quad [(\lambda w\, \Pi(\lambda v\, Rwv \to (\lambda w\, \Pi(\lambda x\, Pxw))v))]\\
&\equiv \quad [\lambda w\, \Pi(\lambda v\, Rwv \to \Pi(\lambda x\, Pxv))]\\
&\equiv \quad [\lambda w\, \forall v\, (Rwv \to \forall x\, Pxv)]\\
&\equiv \quad \forall w\, \forall v\, (Rwv \to \forall x\, Pxv)
\end{aligned}
$$

- above: possibilist quantification
- actualist quantification: $\Pi = \lambda\Phi\, \lambda w\, \Pi(\lambda x$ **existsAt x w** $\to \Phi x w)$
- also supported: local and global validity and consequence

Kripke Style Semantics

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&\equiv [\Box(\lambda w\, \Pi(\lambda x\, Pxw))]\\
&\equiv [(\lambda\varphi\, \lambda w\, \Pi(\lambda v\, Rwv \to \varphi v))(\lambda w\, \Pi(\lambda x\, Pxw))]\\
&\equiv [(\lambda w\, \Pi(\lambda v\, Rwv \to (\lambda w\, \Pi(\lambda x\, Pxw))v))]\\
&\equiv [\lambda w\, \Pi(\lambda v\, Rwv \to \Pi(\lambda x\, Pxv))]\\
&\equiv [\lambda w\, \forall v\, (Rwv \to \forall x\, Pxv)]\\
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Example (compositionality and λ-conversion at work)

$$\begin{aligned}
[\Box\forall x\, Px] &\equiv [\Box\Pi(\lambda x\, Px)] \equiv [\Box\Pi(\lambda x\, \lambda w\, Pxw)] \\
&\equiv [\Box((\lambda\Phi\, \lambda w\, \Pi(\lambda x\, \Phi x w))(\lambda x\, \lambda w\, Pxw))] \\
&\equiv [\Box(\lambda w\, \Pi(\lambda x\, (\lambda x\, \lambda w\, Pxw)xw))] \\
&\equiv [\Box(\lambda w\, \Pi(\lambda x\, Pxw))] \\
&\equiv [(\lambda\varphi\, \lambda w\, \Pi(\lambda v\, Rwv \to \varphi v))(\lambda w\, \Pi(\lambda x\, Pxw))] \\
&\equiv [(\lambda w\, \Pi(\lambda v\, Rwv \to (\lambda w\, \Pi(\lambda x\, Pxw))v))] \\
&\equiv [\lambda w\, \Pi(\lambda v\, Rwv \to \Pi(\lambda x\, Pxv))] \\
&\equiv [\lambda w\, \forall v\, (Rwv \to \forall x\, Pxv)] \\
&\equiv \forall w\, \forall v\, (Rwv \to \forall x\, Pxv)
\end{aligned}$$

- above: possibilist quantification
- actualist quantification: $\Pi = \lambda\Phi\, \lambda w\, \Pi(\lambda x$ **existsAt x w** $\to \Phi x w)$
- also supported: local and global validity and consequence

Kripke Style Semantics

$M, g, s \models \forall x\, \varphi$ if and only if for all $d \in D$ we have $M, ([d/x]g), s \models \varphi$

Standard Translation extended for Quantifiers (and encoded in HOL)

- in HOL $\forall x_\alpha\ \varphi x$ is shorthand for $\Pi_{(\alpha\to o)\to o}(\lambda x_\alpha\ \varphi x)$ —no binder needed!!!
- $\Pi_{(\alpha\to(i\to o))\to(i\to o)} = \lambda\Phi_{\alpha\to(i\to o)}\ \lambda w_i\ \Pi_{(\alpha\to o)\to o}(\lambda x_\alpha\ \Phi x w)$

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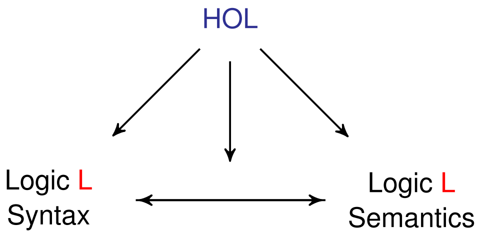
Universal Reasoning in Meta-Logic HOL



Examples for L we have already studied:
Intuitionistic Logics, (Mathematical) Fuzzy Logics, Free Logic, Modal Logics, Description Logics, Conditional Logics, Access Control Logics, Hybrid Logics, Multivalued Logics, Logics with Neighborhood Semantics, Paraconsistent Logics, Deontic Logics, . . .

Embedding works also for quantifiers (first-order & higher-order)

HOL provers become universal logic reasoning engines!

interactive:	Isabelle/HOL, PVS, HOL4, Hol Light, Coq/HOL, . . .
automated:	Leo-III, LEO-II, Satallax, TPS, Nitpick, Isabelle/HOL, . . .

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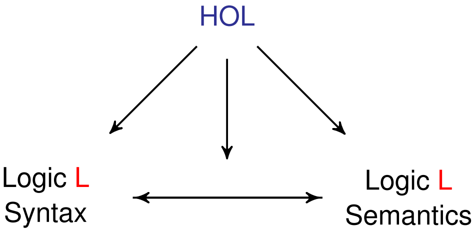

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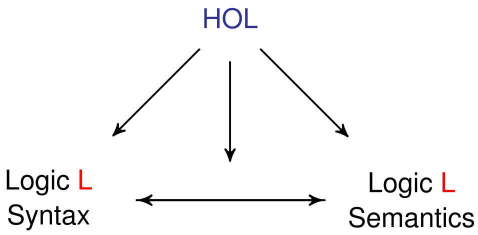

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Universal Reasoning in Isabelle/HOL

GodProof.thy

GodProof.thy (~/chris/trunk/tex/talks/2018-DEON/DEMO/)

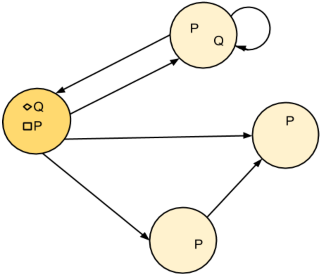
```
theory GodProof imports Main
begin
 typedecl i -- "type for possible worlds"
 typedecl μ -- "type for individuals"
 type_synonym σ = "(i⇒bool)"

(* Shallow embedding modal logic connectives in HOL *)
 abbreviation mneg ("¬_"[52]53)    where "¬φ   ≡ λw. ¬φ(w)"
 abbreviation mand (infixr"∧"51)   where "φ∧ψ  ≡ λw. φ(w)∧ψ(w)"
 abbreviation mor (infixr"∨"50)    where "φ∨ψ  ≡ λw. φ(w)∨ψ(w)"
 abbreviation mimp (infixr"→"49)   where "φ→ψ ≡ λw. φ(w)⟶ψ(w)"
 abbreviation mequ (infixr"↔"48)   where "φ↔ψ ≡ λw. φ(w)⟷ψ(w)"
 abbreviation mnegpred ("¬_"[52]53) where "¬Φ   ≡ λx.λw. ¬Φ(x)(w)"

(* Generic box and diamond operators *)
 abbreviation mboxgen ("□") where "□r φ ≡ λw. ∀v.  r w v ⟶ φ(v)"
 abbreviation mdiagen ("◇") where "◇r φ ≡ λw. ∃v. r w v ∧ φ(v)"

(* Shallow embedding of constant domain quantifiers in HOL *)
 abbreviation mall_const ("∀c") where "∀c Φ ≡ λw.∀x. Φ(x)(w)"
 abbreviation mallB_const (binder"∀c"[8]9) where "∀c x. φ(x) ≡ ∀c φ"
 abbreviation mexi_const ("∃c") where "∃c Φ ≡ λw.∃x. Φ(x)(w)"
 abbreviation mexiB_const (binder"∃c"[8]9) where "∃c x. φ(x) ≡ ∃c φ"

(* Global validity: truth in all possible worlds *)
 abbreviation mvalid :: "σ ⇒ bool" ("⌊_⌋"[7]110) where "⌊p⌋ ≡ ∀w. p w"

(* Shallow embedding of varying domain quantifiers in HOL *)
```

Documentation Sidekick State Theories

Output Query Sledgehammer Symbols

Universal Logic Reasoning in Isabelle/HOL

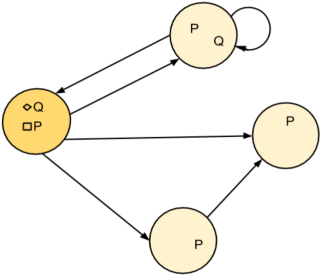
Properties of $\Box$ and $\Diamond$ correlated to structure of transition system between worlds



- Logic K: — (no restrictions, any structure)
- Logic M: reflexiv transition relation, $[\forall P\ \Box P \rightarrow P]$
- Logic KB: symmetric transition relation, $[\forall P\ P \rightarrow \Box\Diamond P]$
- Logic S5: equivelance relation as transition system, add $[\forall P\ \Box P \rightarrow \Box\Box P]$

Universal Logic Reasoning in Isabelle/HOL

Properties of $\Box$ and $\Diamond$ correlated to structure of transition system between worlds



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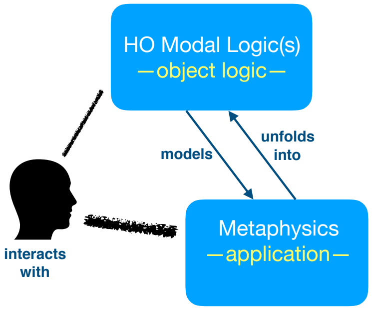
- Logic S5: equivelance relation as transition system, add $[\forall P\ \Box P \to \Box\Box P]$

We e.g. have the choice between $[\forall P\ \Box P \to P]$ **or** $\forall x\ Rxx$

Universal (Meta-)Logical Reasoning Framework [SCP, 2019], [Open Philosophy,, 2019]

Universal (Meta-)Logical Reasoning Framework [SCP, 2019], [Open Philosophy,, 2019]



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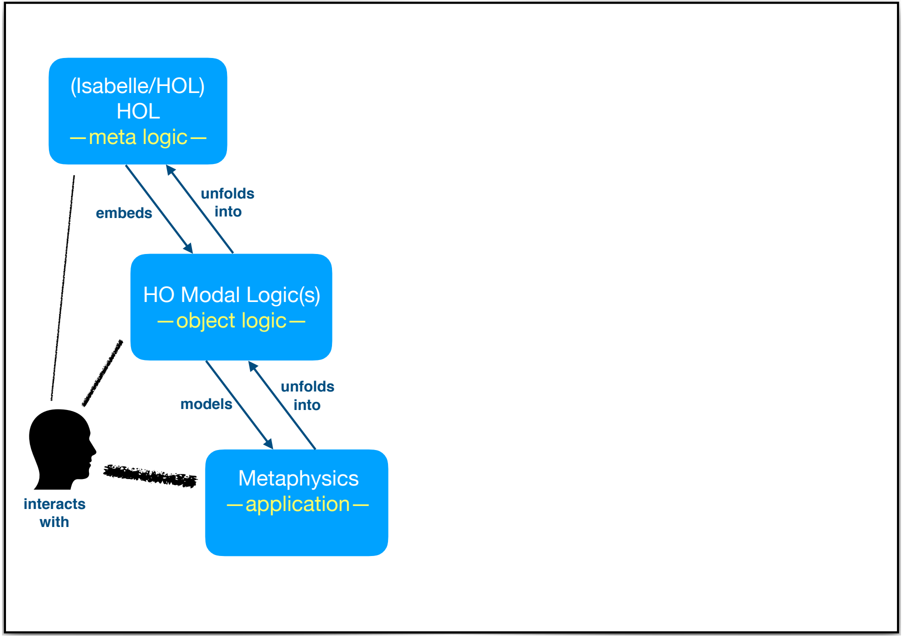

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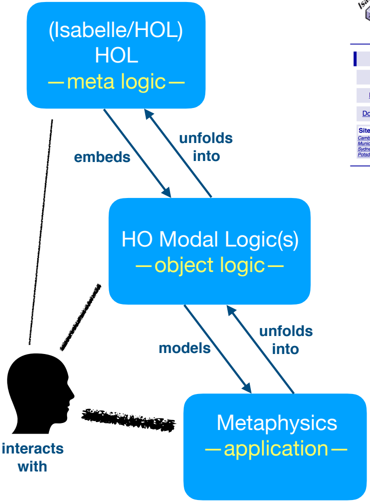

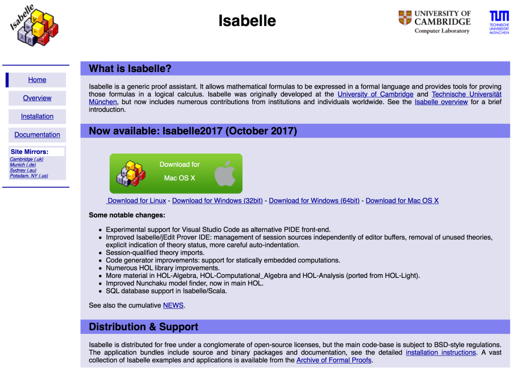

Isabelle

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

Home
Overview
Installation
Documentation

Site Mirrors:
Cambridge (.uk)
Munich (.de)
Sydney (.au)
Potsdam, NY (.us)

What is Isabelle?

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. Isabelle was originally developed at the University of Cambridge and Technische Universität München, but now includes numerous contributions from institutions and individuals worldwide. See the Isabelle overview for a brief introduction.

Now available: Isabelle2017 (October 2017)

Download for Mac OS X

Download for Linux - Download for Windows (32bit) - Download for Windows (64bit) - Download for Mac OS X

Some notable changes:

- Experimental support for Visual Studio Code as alternative PIDE front-end.
- Improved Isabelle/jEdit Prover IDE: management of session sources independently of editor buffers, removal of unused theories, explicit indication of theory status, more careful auto-indentation.
- Session-qualified theory imports.
- Code generator improvements: support for statically embedded computations.
- Numerous HOL library improvements.
- More material in HOL-Algebra, HOL-Computational_Algebra and HOL-Analysis (ported from HOL-Light).
- Improved Nunchaku model finder, now in main HOL.
- SQL database support in Isabelle/Scala.

See also the cumulative NEWS.

Distribution & Support

Isabelle is distributed for free under a conglomerate of open-source licenses, but the main code-base is subject to BSD-style regulations. The application bundles include source and binary packages and documentation, see the detailed installation instructions. A vast collection of Isabelle examples and applications is available from the Archive of Formal Proofs.

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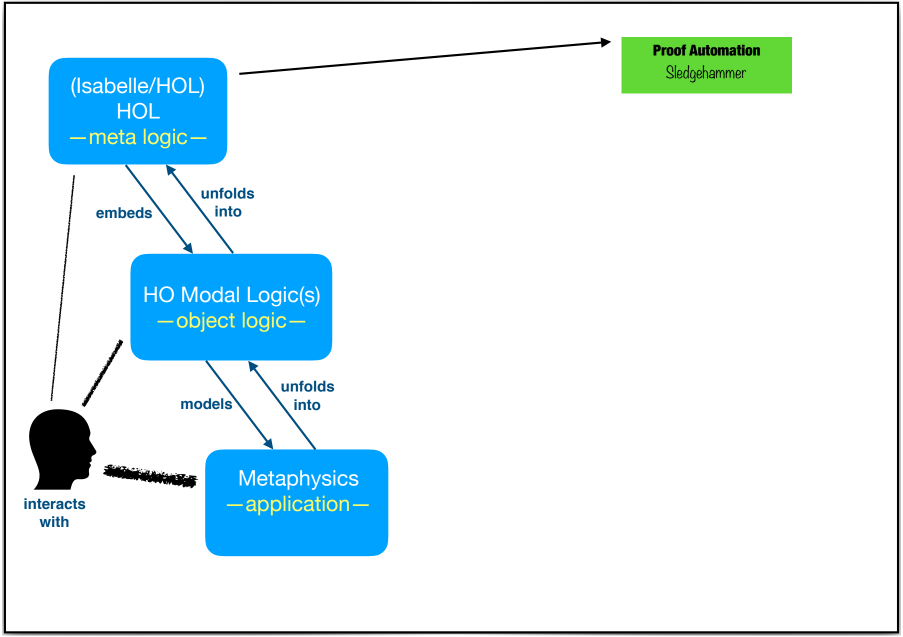

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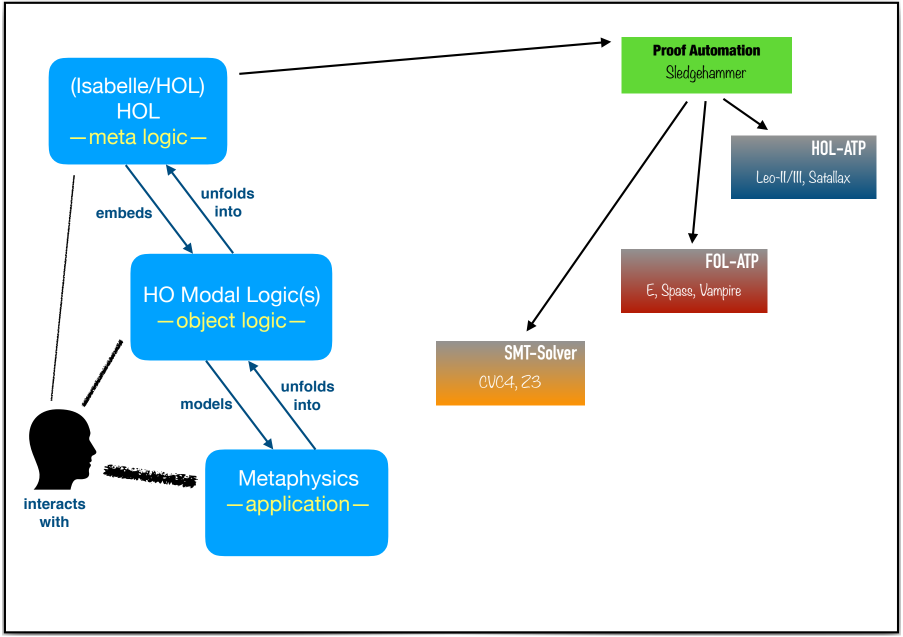

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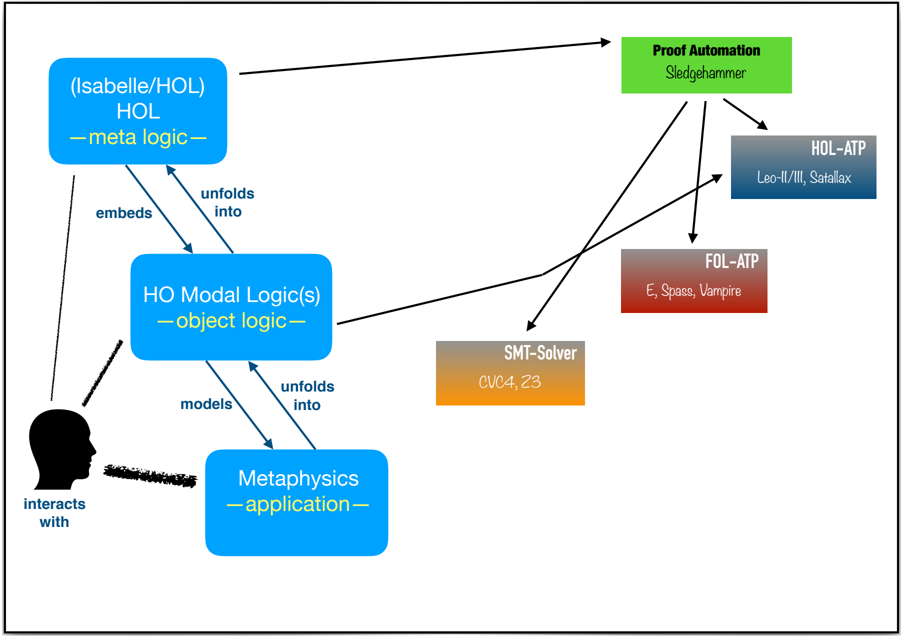

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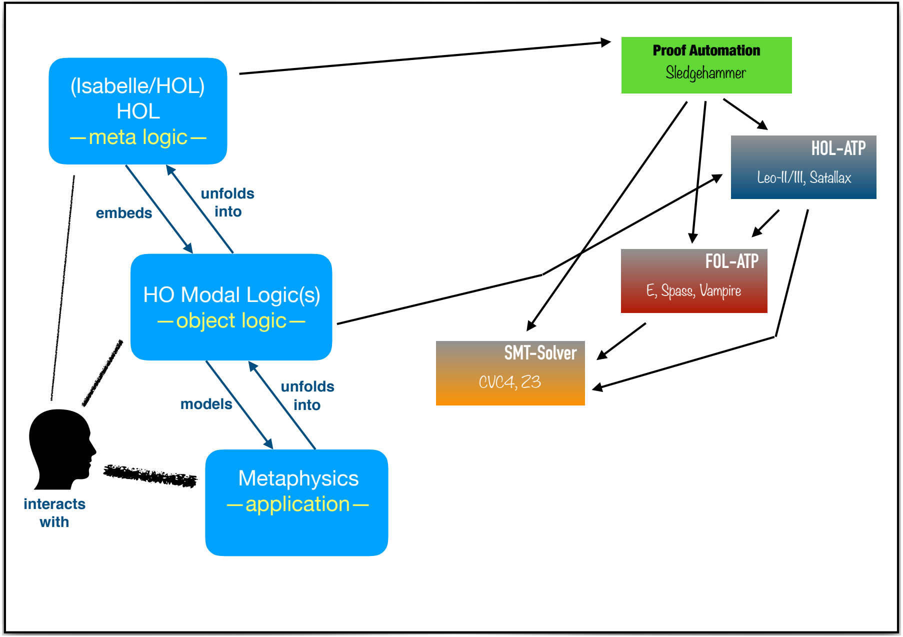

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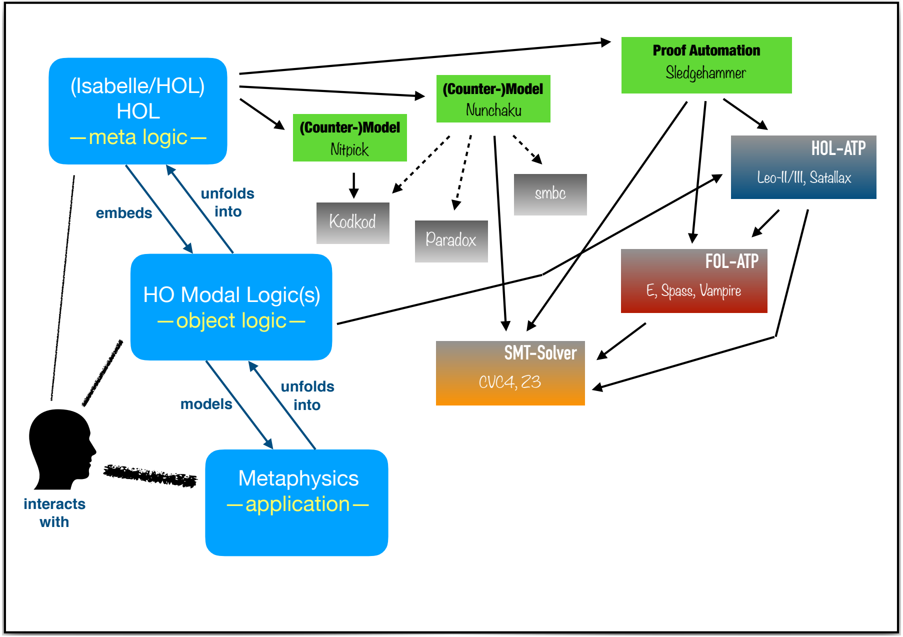

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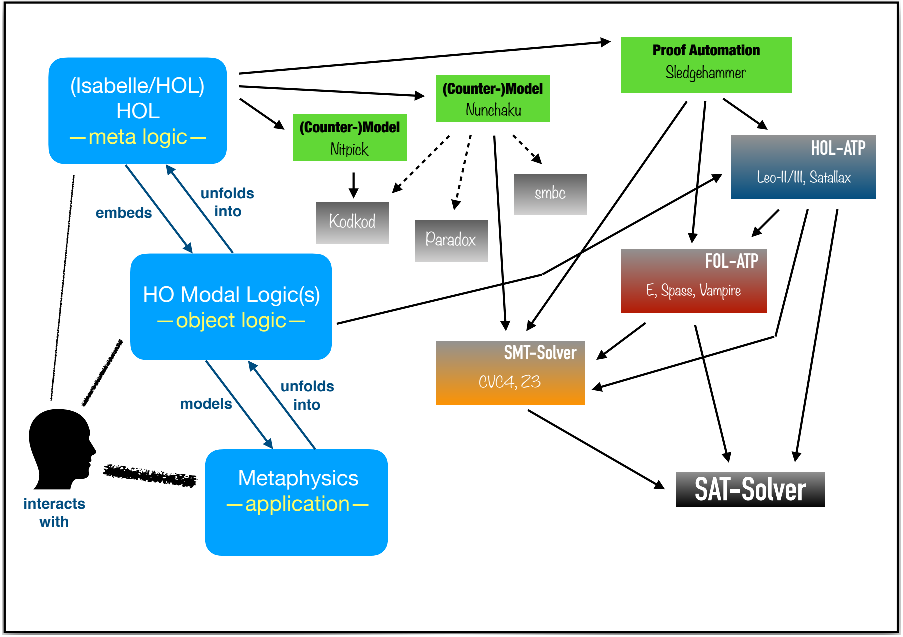

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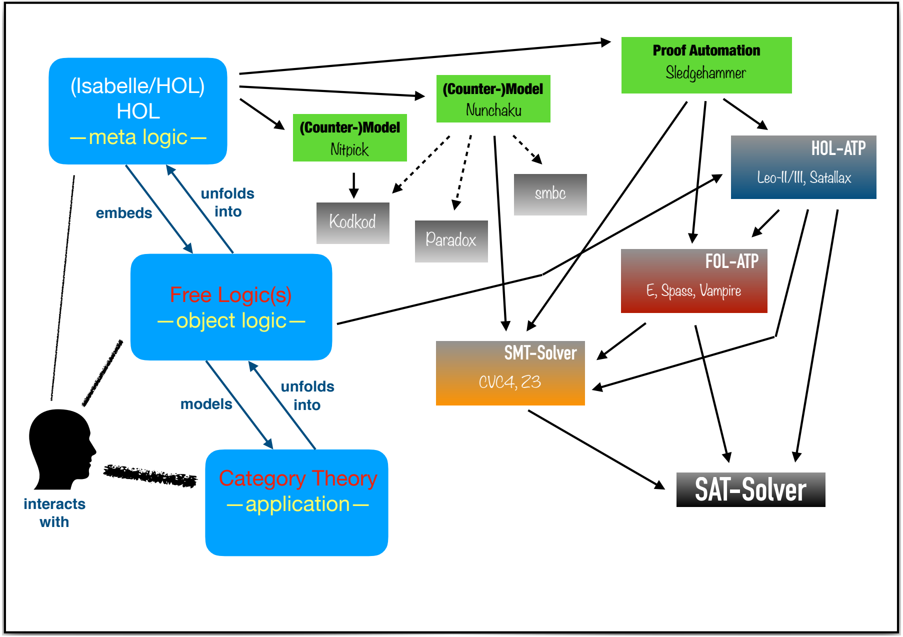

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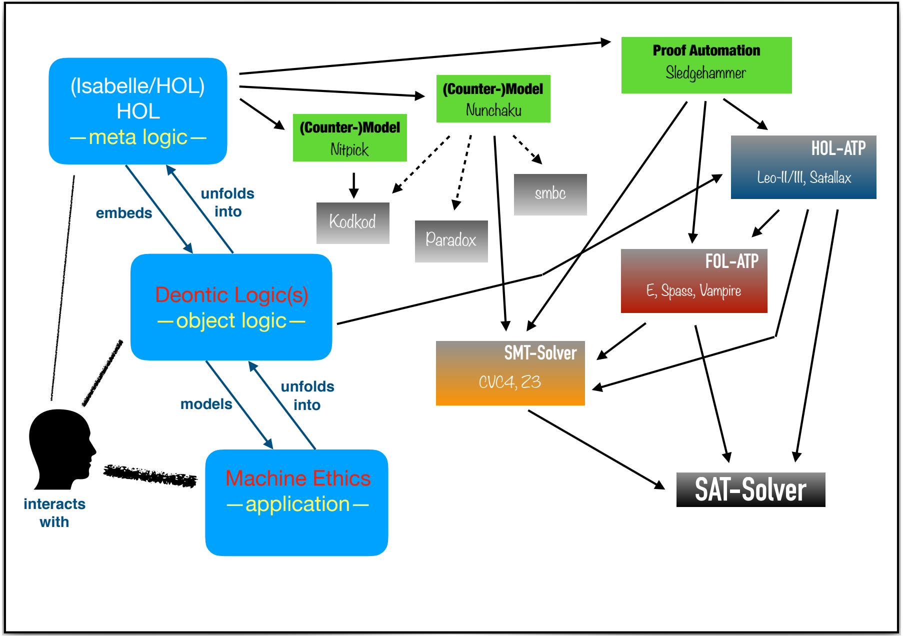

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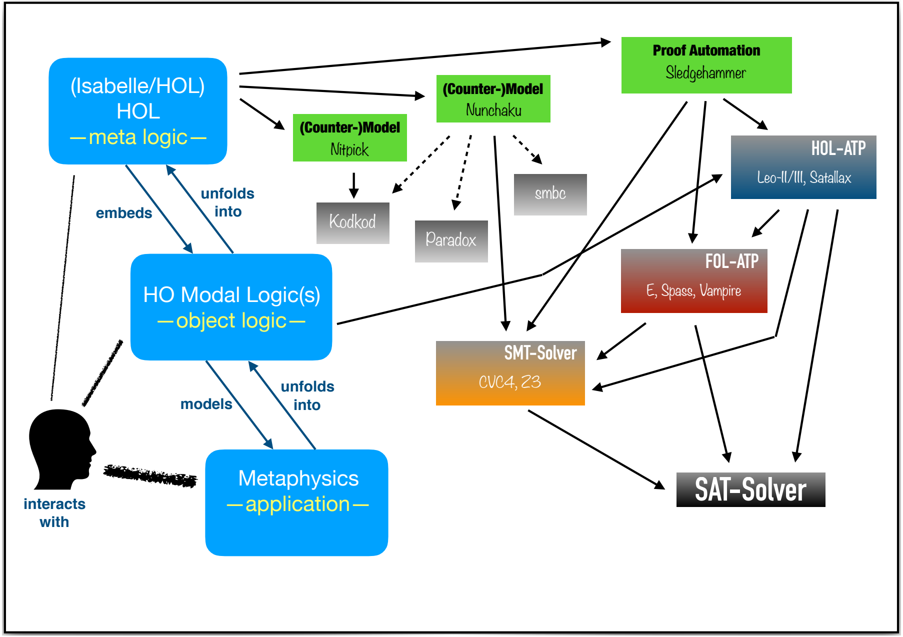
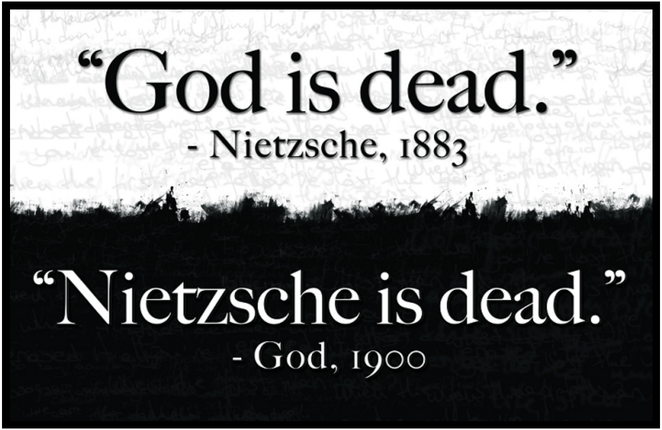
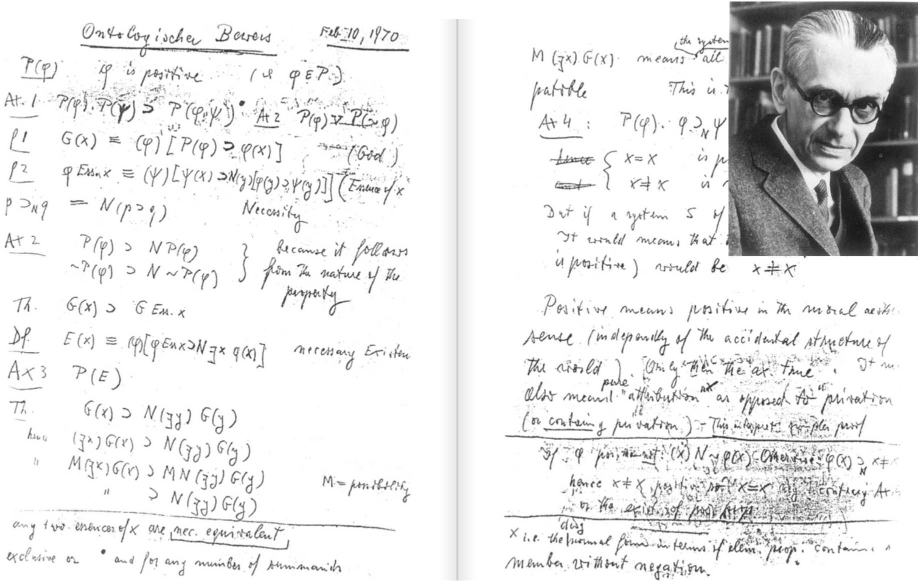




Part B
Ontological Argument of Gödel & Scott on the Computer
(Recap)

[BenzmüllerWoltzenlogelPaleo, ECAI, 2014 + IJCAI, 2016 + KI 2016 + . . .]

Computational Metaphysics: Kurt Gödel's Ontological Argument (1970)



Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

Df 1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

Df 2 $\varphi \text{ Ess.} x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess.} x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess} x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)G(x) \supset MN(\exists y)\, G(y)$

" $\supset N(\exists y)\, G(y)$

M = possibility

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x)G(x)$ means all [the system] ...

patible This is ...

At 4: $P(\varphi).\varphi \supset_N \psi$ ...

{ $x = x$ is p... ; $x \neq x$ is ...

But if a system S of ...

It would mean that ...

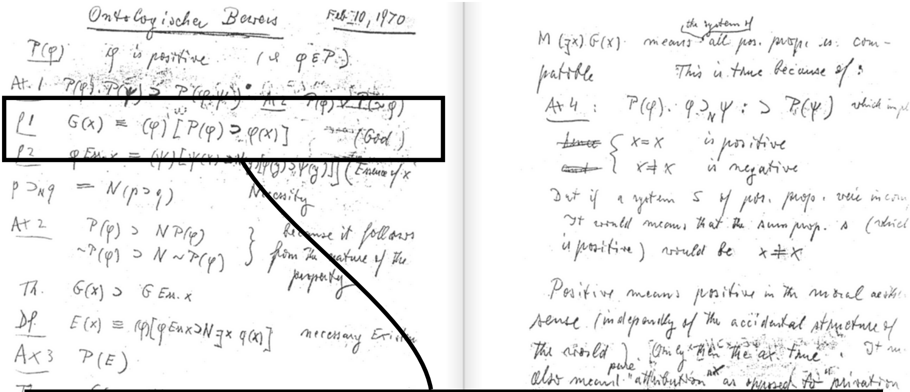
is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It m... also means pure "attribution"* as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ ... contrary Ax ... or the exist of pos prop

* i.e. the normal form in terms of elem. prop. contains a member without negation.

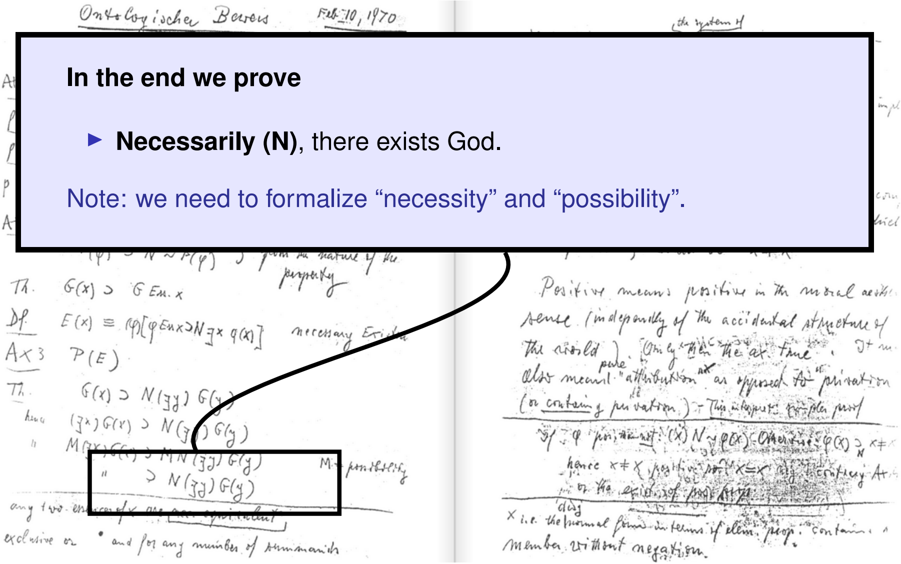
Computational Metaphysics: Kurt Gödel's Ontological Argument (1970)



Notion of "Godlike":

- Being Godlike is equivalent to having all positive properties.

Note: this definition is "second-order".



In the end we prove

- **Necessarily (N)**, there exists God.

Note: we need to formalize "necessity" and "possibility".

Computational Metaphysics: Gödel's (1970) and Scott's Variants (1972)

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~~true~~ { $x = x$ is p...

~~false~~ $x \neq x$ is ...

But if a system S of ...

It would mean, that the sum prop. s (which is positive) would be $x \neq x$

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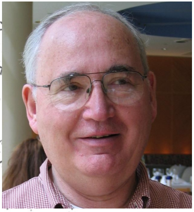
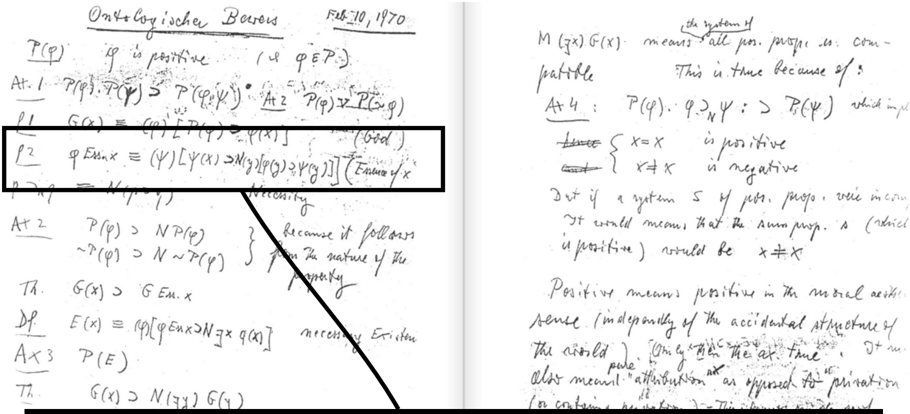


Computational Metaphysics: Gödel's (1970) and Scott's Variants (1972)



(Main) Difference between Gödel and Scott: Def. of "Essence (Ess.)"

Gödel:	Property E is Ess. of x iff all of x's properties are nec. entailed by E.
Scott:	Property E is Ess. of x iff **x has E and** all of x's properties are nec. entailed by E.

Computational Metaphysics: Dana Scott's Variant

Axiom Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A property necessarily implied by a positive property is positive:
$$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Thm. Positive properties are possibly exemplified: $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$

Def. **A *Godlike* being possesses all positive properties**: $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$

Axiom The property of being Godlike is positive: $P(G)$

Cor. Possibly, God exists: $\Diamond\exists x G(x)$

Axiom Positive properties are necessarily positive: $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$

Def. An *essence* of an individual is a **property possessed by it and** necessarily implying any of its properties: $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$

Thm. Being Godlike is an essence of any Godlike being: $\forall x[G(x) \rightarrow G\ ess.\ x]$

Def. *Necessary existence* of an individual is the necessary exemplification of all its essences: $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$

Axiom Necessary existence is a positive property: $P(NE)$

Thm. Necessarily, God exists: $\Box\exists x G(x)$

Computational Metaphysics: Scott's Variant

Axiom Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$

Axiom A property necessarily implied by a positive property is positive: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \to \psi(x)]) \to P(\psi)]$

Thm. Positive properties are possibly exemplified: $\forall\phi[P(\phi) \to \Diamond\exists x\phi(x)]$

Def. **A *Godlike* being possesses all positive properties**: $G(x) \leftrightarrow \forall\phi[P(\phi) \to \phi(x)]$

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Difference to Gödel (who omits this conjunct)

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Computational Metaphysics: Scott's and Gödel's Variants — Demo

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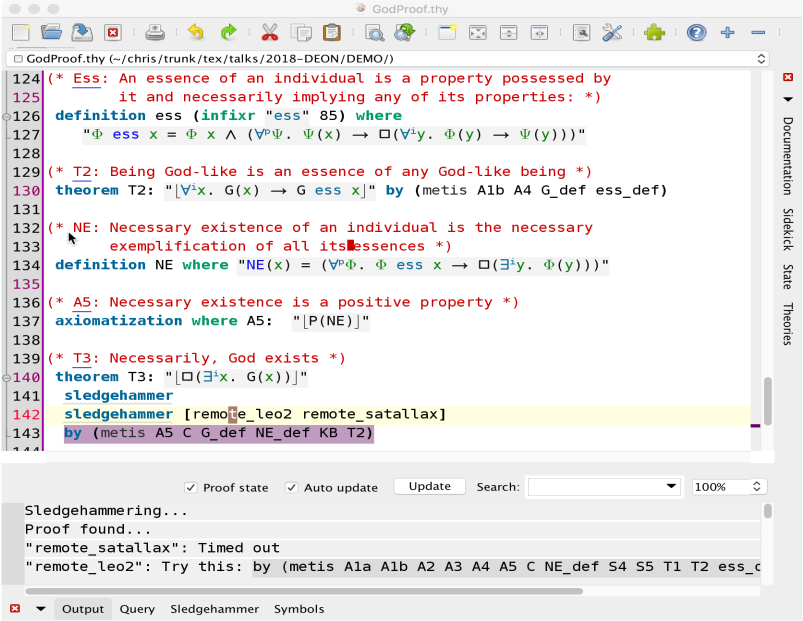

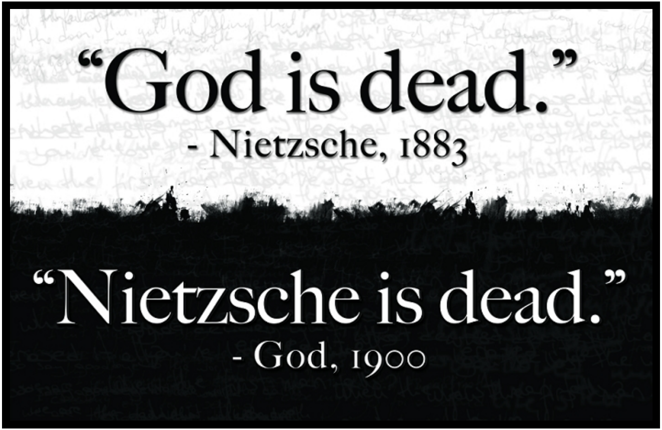

Results of our Experiments (jww B. Woltzenlogel-Paleo)

(see also [Savijnanam 2017], [IJCAI 2016], [ECAI 2014])

Results of our Experiments

Variant of Dana Scott (1972)

- the premises are **consistent**
- all argument steps are **logically correct** in (higher-order, extensional) modal logic

  - correct in logic **S5**
  - weaker logic **KB** is already sufficient
  - critique about use of S5 not justfied

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Results of our Analysis

. . . we continue with Scott's version

Further corollaries we can prove

- Monotheism
- Gott is flawless (has only positive properties)
- . . .
- Modal Collapse: $\varphi \longrightarrow \Box\varphi$

  - there are no contingent truths
  - no alternative worlds
  - everything is determined
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Challenge: Can the Modal Collapse be avoided (with minimal changes)?

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News and . . .


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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- . . .

Austria
- Die Presse
- Wiener Zeitung
- ORF
- . . .

Italy
- Repubblica
- Ilsussidario
- . . .

India
- DNA India
- Delhi Daily News
- India Today
- . . .

US
- ABC News
- . . .

International
- Spiegel International
- Yahoo Finance
- United Press Intl.
- . . .

News and Fake News

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God exists, say Apple fanboy scientists

With the help of just one MacBook, two Germans formalize a theorem that confirms the existence of God.

Fake news by award winning journalist *Chris Matyszczyk (c|net)*

Can the Modal Collapse be avoided?

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

Gödel's Ontological Proof Revisited *

C. Anthony Anderson and Michael Gettings

University of California, Santa Barbara
Department of Philosophy

Gödel's version of the modal ontological argument for the existence of God has been criticized by J. Howard Sobel [5] and modified by C. Anthony Anderson [1]. In the present paper we consider the extent to which Anderson's emendation is defeated by the type of objection first offered by the Monk Gaunilo to St. Anselm's original Ontological Argument. And we try to push the analysis of this Gödelian argument a bit further to bring it into closer agreement with the details of Gödel's own formulation. Finally, we indicate what seems to be the main weakness of this emendation of Gödel's attempted proof.

Magari and others on Gödel's ontological proof

Petr Hájek
Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic
e-mail: hajek@uivt.cas.cz

1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe

Der Mathematiker und die Frage der Existenz Gottes

(betreffend Gödels ontologischen Beweis)

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sondern glauben, daß ein Gott sei.
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1. Einführung

Gödels zu Lebzeiten unveröffentlichter Beweis für die notwendige Existenz eines Gott-ähnlichen Wesens hat sowohl philosophisches als auch mathematisches Interesse geweckt. Zweck der vorliegenden Arbeit ist es, zu einer Deutung des Gödelschen Textes beizutragen, 1. durch Kommentierung der einschlägigen Literatur und 2. durch Bereitstellung von etwas Modelltheorie. Die Arbeit enthält keinen philosophischen Beitrag. Während der letzten Jahre habe ich etliche Male über Gödels Gottesbeweis vorgetragen, insbesondere auf dem Symposium zur Feier von Professor Gert Müller (Heidelberg, Jänner 1991), doch habe ich niemals beabsichtigt, eine Veröffentlichung über das Thema zu machen. Da ich wiederholt um eine schriftliche Version gebeten wurde, entschloß ich mich, schnell eine „erweiterte Kurzfassung"[1] zu schreiben, ohne aus ihr einen

PETR HÁJEK

A New Small Emendation of Gödel's Ontological Proof

Keywords: Ontological proof, Gödel, modal logic, comprehension, positive properties.

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— contributed to clarification of controversy —

— revealed various flaws and issues —

[Logica Universalis, 2017]

Remainder of this Talk

We will have a closer look at

- Gödel/Scott (1972) modal collapse
- C. Anthony Anderson (1990) avoids modal collapse
- Melvin Fitting (2002) avoids modal collapse

Questions:

- How do Anderson and Fitting the avoid modal collapse?
- Are their solutions related?

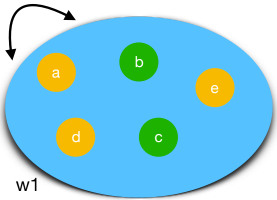
To answer this questions we will apply some notions from

- mathematics: **ultrafilters**
- philosophy of language: **extension and intension of predicates**

Part C
Relevant Notions for this Talk

Intension vs. Extension of a Predicate (following [Fitting, 2002]))

Example predicate: IsChessGrandmaster



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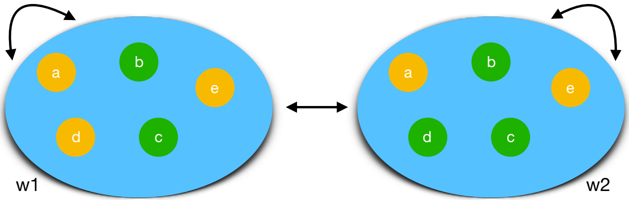

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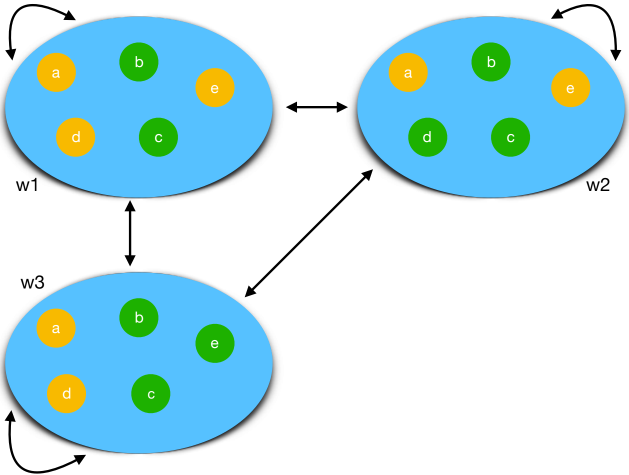

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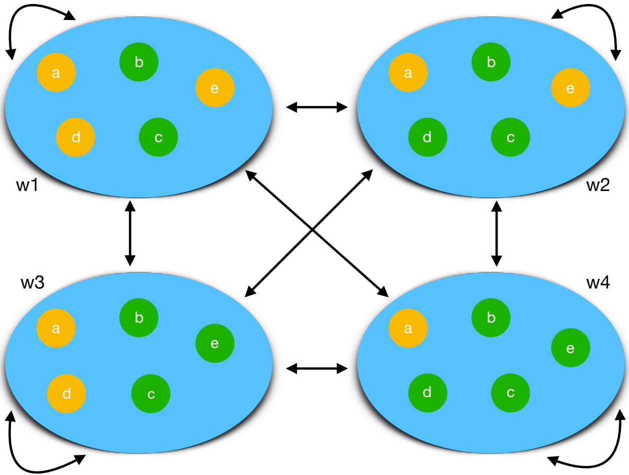

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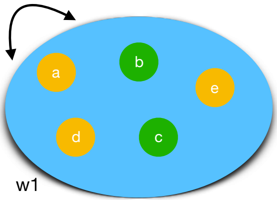

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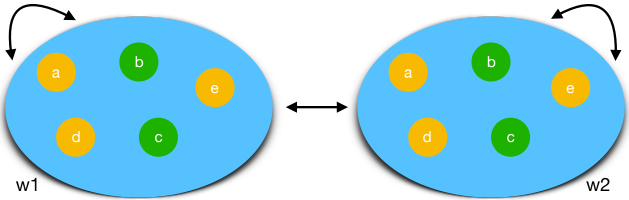

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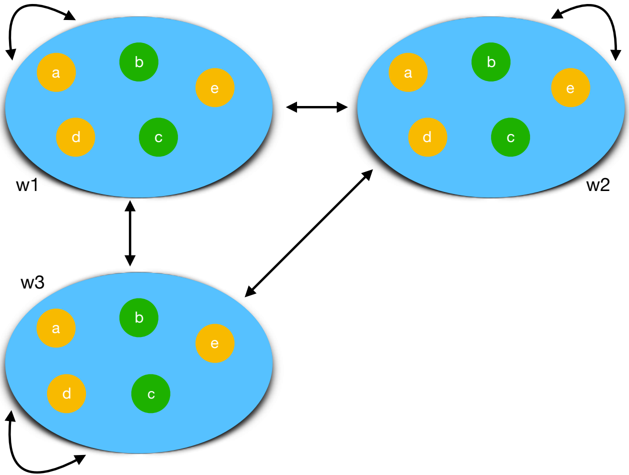

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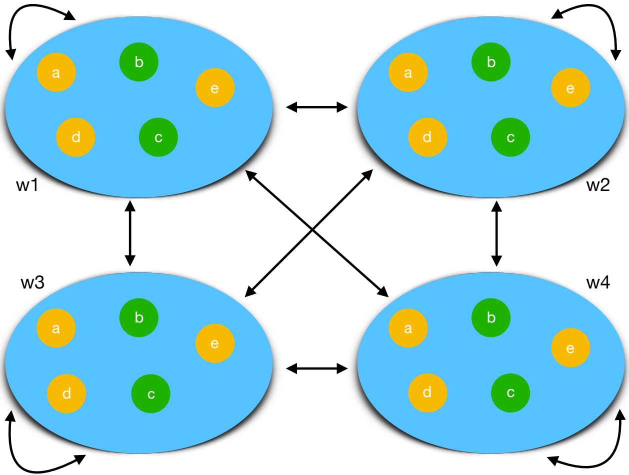

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Ultrafilter

Definition of Ultrafilter:
Given an arbitrary set X. An ultrafilter U on the powerset $\mathcal{P}(X)$ is a subset of $\mathcal{P}(X)$ such that (where $A, B \in \mathcal{P}(X)$):

1. $\emptyset$ is not an element of U.
2. If A is subset of B and A is element of U, then B is also element of U.
3. If A and B are elements of U, then so is their intersection.
4. Either A or its relative complement $X \setminus A$ is an element of U.

Ultrafilter

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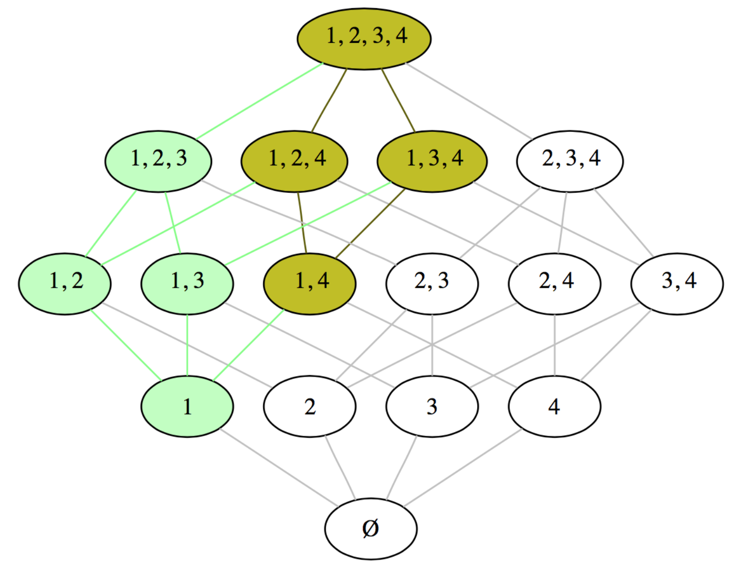
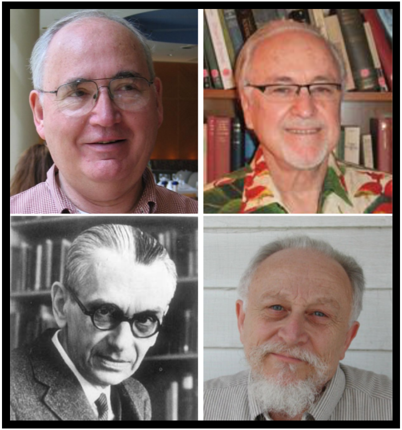
$\mathbf{1}$ is element of all sets in U ($\mathbf{1}$ has all properties of U)

Ultrafilter



From: Wikipedia (Jochen Burghardt) CC BY-SA
`https://en.wikipedia.org/wiki/Ultrafilter`



Part D
— Comparative Analysis —
Variants of Gödel/Scott, Anderson and Fitting

Ontological Argument: Variant by Gödel/Scott

Part I - Proving that God's existence is possible

- **D1** **Being Godlike is equivalent to having all positive properties.**
- **A1** Exactly one of a property or its negation is positive.
- **A2** Any property entailed by a positive property is positive.
- **A3** The combination of any collection of positive properties is itself positive.

From A1 and A2 follows theorem T1:

- **T1** Every positive property is possibly instantiated.

From D1 and A3 follows:

- **T2** Being Godlike is a positive property.

From T1 and T2 follows:

- **T3** Being Godlike is possibly instantiated.

Ontological Argument: Variant by Gödel/Scott

Part I - Proving that God's existence is possible

D1

A1

A2

A3

Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

D2 A property E is the essence of an individual x iff x has E and all of x's properties are nec. entailed by E.[a]

A4 Positive properties are necessarily positive.

From A1 and A4 (using definitions D1 and D2) follows:

T4 Being Godlike is an essential property of any Godlike individual.

D3 Necessary existence of an individual is the necessary instantiation of all its essences.

A5 Necessary existence is a positive property.

From T4 and A5 (using D1, D2, D3) follows:

T5 Being Godlike, if instantiated, is necessarily instantiated.

And finally from T3, T5 (together with some implicit modal axioms, e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

[a] The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

Ontological Argument: Variant by Gödel/Scott

Part I - Proving that God's existence is possible

D1

A1

A2

A3

Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

D2 A property E is the essence of an individual x iff x has E and all of x's p

A4 F

From

T4 E

D3 N ... all i

A5 N

From

T5 E

And ... **ms,** **e.g. S**

T6 Being Godlike is necessarily instantiated.

“Modal Collapse” is implied by these axioms: $\varphi \supset \Box\varphi$

- determinism

“positive properties ($\mathcal{P}$)” applied here to (non-rigid) intensions of properties. We can prove:

- $\mathcal{P}$ is an ultrafilter

Let $\mathcal{P}'$ be the set of “rigidly intensionalised extensions” of positive properties. ($\mathcal{P}'\varphi := \mathcal{P}\,(\downarrow \varphi)$)

We can prove:

- $\mathcal{P}'$ is an ultrafilter
- $\mathcal{P} = \mathcal{P}'$

---

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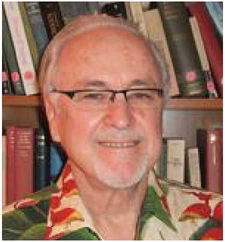
Ontological Argument: Variant by Anderson (1990)

SOME EMENDATIONS OF GÖDEL'S ONTOLOGICAL PROOF

C. Anthony Anderson

Kurt Gödel's version of the ontological argument was shown by J. Howard Sobel to be defective, but some plausible modifications in the argument result in a version which is immune to Sobel's objection. A definition is suggested which permits the proof of some of Gödel's axioms.

[Faith and Philosophy 1990]



Ontological Argument: Variant by Anderson

Part I - Proving that God's existence is possible

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From T1 and T2 follows:

T3 Being Godlike is possibly instantiated.

Ontological Argument: Variant by Anderson

Part I - Proving that God's existence is possible

D1 Being Godlike is equivalent to having all positive properties.

A1a **If a property is positive, then its negation is not positive.**

A1b **If the negation of a property is not positive, then the property is positive.**

A2 Any property entailed by a positive property is positive.

A3 The combination of any collection of positive properties is itself positive.

From A1 and A2 follows theorem T1:

T1 Every positive property is possibly instantiated.

From D1 and A3 follows:

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From T1 and T2 follows:

T3 Being Godlike is possibly instantiated.

Part I - Proving that God's existence is possible

D1 Being Godlike is equivalent to having all positive properties.

A1a **If a property is positive, then its negation is not positive.**

~~**A1b** **If the negation of a property is not positive, then the property is positive.**~~

A2 Any property entailed by a positive property is positive.

A3 The combination of any collection of positive properties is itself positive.

From A1 and A2 follows theorem T1:

T1 Every positive property is possibly instantiated.

From D1 and A3 follows:

T2 Being Godlike is a positive property.

From T1 and T2 follows:

T3 Being Godlike is possibly instantiated.

Ontological Argument: Variant by Anderson

Part I - Proving that God's existence is possible

D1' **Being Godlike is equivalent to having all and only the positive properties as necessary properties.**

A1a **If a property is positive, then its negation is not positive.**

~~**A1b** **If the negation of a property is not positive, then the property is positive.**~~

A2 Any property entailed by a positive property is positive.

A3 The combination of any collection of positive properties is itself positive.

From A1 and A2 follows theorem T1:

T1 Every positive property is possibly instantiated.

From D1 and A3 follows:

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Ontological Argument: Variant by Anderson

Part I - Proving that God's existence is possible

D1'

A1a

~~**A1b**~~

A2

A3

Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

D2 A property E is the essence of an individual x iff <u>x has E and</u> all of x's properties are nec. entailed by E.

A4 Positive properties are necessarily positive.

From A1 and A4 (using definitions D1 and D2) follows:

T4 Being Godlike is an essential property of any Godlike individual.

D3 Necessary existence of an individual is the necessary instantiation of all its essences.

A5 Necessary existence is a positive property.

From T4 and A5 (using D1, D2, D3) follows:

T5 Being Godlike, if instantiated, is necessarily instantiated.

And finally from T3, T5 (together with some implicit modal axioms, e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

Ontological Argument: Variant by Anderson

Part I - Proving that God's existence is possible

D1'

A1a

~~A1b~~

A2

A3

Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

D2' **A property E is an essence ($\mathcal{E}^A$) of an individual x if and only if all of x's necessary properties are nec. entailed by E and (conversely) all properties nec. entailed by E are necessary properties of x.**

A4 Positive properties are necessarily positive.

From A1 and A4 (using definitions D1 and D2) follows:

T4 Being Godlike is an essential property of any Godlike individual.

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A1a

~~A1b~~

A2

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Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

D2' A … l of
x … all
p

A4 F

From

T4 E

D3 N … all
it

A5 N

From

T5 E

And … ms,
e.g. S

"Modal Collapse" is *not* implied by these axioms

$$\varphi \supset \Box\varphi \qquad \text{(has countermodel)}$$

- no determinism

"positive properties ($\mathcal{P}$)" are applied here to intensional properties. We have:

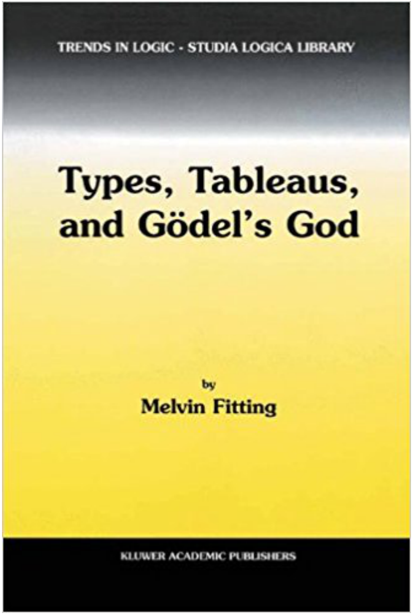
- $\mathcal{P}$ is *not* an ultrafilter (has countermodel)

Let $\mathcal{P}'$ be the set of all "rigidly intensionalised extensions" of positive properties. We can prove:

- $\mathcal{P}'$ is an ultrafilter
- $\mathcal{P} \neq \mathcal{P}'$

T6 Being Godlike is necessarily instantiated.

Ontological Argument: Variant by Fitting (2002)



Ontological Argument: Variant by Fitting (2002)

Part I - Proving that God's existence is possible

D1 Being Godlike is equivalent to having all positive properties.

A1 Exactly one of a property or its negation is positive.

A2 Any property entailed by a positive property is positive.

A3 The combination of any collection of positive properties is itself positive.

From A1 and A2 follows theorem T1:

T1 Every positive property is possibly instantiated.

From D1 and A3 follows:

T2 Being Godlike is a positive property.

From T1 and T2 follows:

T3 Being Godlike is possibly instantiated.

Fully analogous to Gödel/Scott.

But: "positive properties" applied to extensions of properties only!

Part I - Proving that God's existence is possible

D1

A1

A2

A3

Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

D2 A property E is the essence of an individual x iff x has E and all of x's properties are nec. entailed by E.[a]

A4 Positive properties are necessarily positive.

From A1 and A4 (using definitions D1 and D2) follows:

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---

[a] The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

Ontological Argument: Variant by Fitting (2002)

Part I - Proving that God's existence is possible

D1

A1

A2

A3

Fro

T1

Fro

T2

Fro

T3

Part II - Proving that God's existence is necessary, if possible

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A4 Positive properties are necessarily positive.

From

T4 E

D3 N i

A5 N

From

T5 E

And ...

e.g. S5) the existence of (at least a) God follows:

T6 Being Godlike is necessarily instantiated.

[a] The underlined part in definition D2 has been added by Scott. Gödel originally omitted this part.

"Modal Collapse" is *not* implied by these axioms

$$\varphi \supset \Box\varphi \qquad \text{(has countermodel)}$$

We can prove that these "positive property extensions" (which corresponds to $\mathcal{P}'$ from before) form an ultrafilter.

Summary of Results

- "Godlike" has been defined in terms of "positive properties"
- "positive properties" has been linked with the notion of "ultrafilter".
- In our experiments we then distinguished between
  - $\mathcal{P}$: positive intensional properties
  - $\mathcal{P}'$: positive ("rigidly intensionalised") extensions of properties
- Gödel/Scott variant axiomatises $\mathcal{P}$: $\mathcal{P} = \mathcal{P}'$ is an ultrafilter
- Anderson's variant axiomatises $\mathcal{P}$: $\mathcal{P} \neq \mathcal{P}'$; only $\mathcal{P}'$ is an ultrafilter
- Fitting's variant axiomatises only $\mathcal{P}'$: $\mathcal{P}'$ is an ultrafilter

Modal collapse holds for Gödel/Scott variant, but not for Anderson's & Fitting's!

They achieve this in seemingly different ways.

Mathematically, however, their solutions appear closely related.

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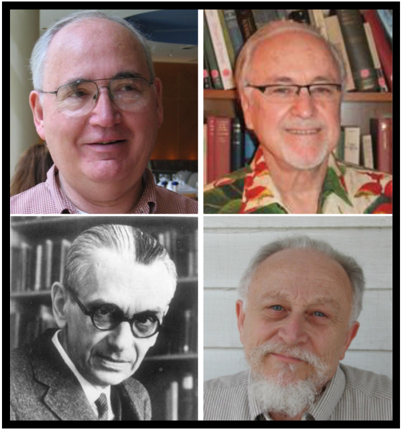


Part D
— Further Applications —

Further Applications

Ed Zalta (Stanford)

Principia Logico-Metaphysica

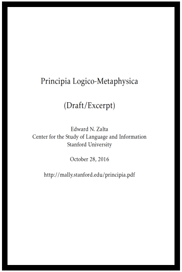
Hyperintensional higher-order modal logic

Inconsistency/Paradox detected & fixed

[Open Philosophy, 2019]


Daniel Kirchner
(Mathematics, FU Berlin)

Further Applications



Principia Logico-Metaphysica

(Draft/Excerpt)

Edward N. Zalta
Center for the Study of Language and Information
Stanford University

October 28, 2016

http://mally.stanford.edu/principia.pdf

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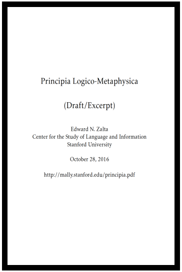
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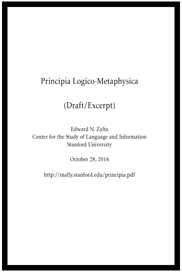
Kirchner Paradox

Daniel & Isabelle/HOL have become close advisors of Ed Zalta in the search for a repair

Computational Metaphysics par excellence!!!

Further Applications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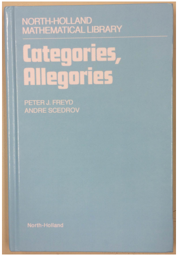


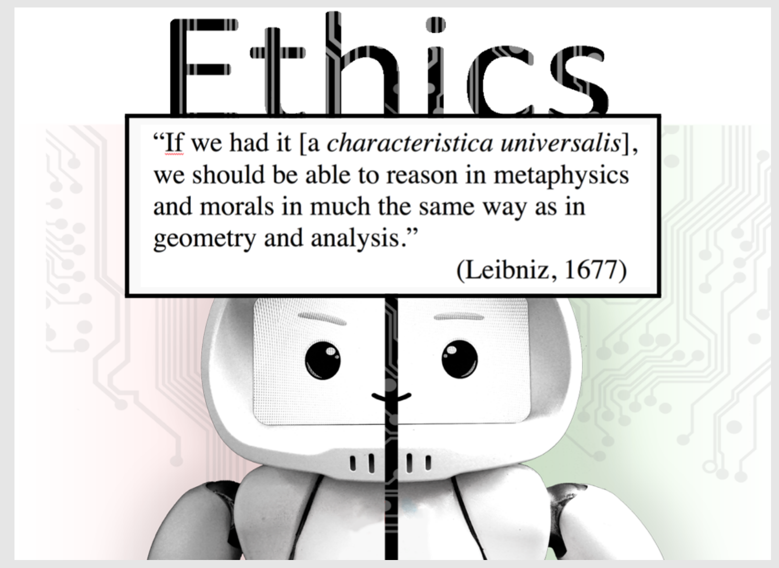
Category Theory

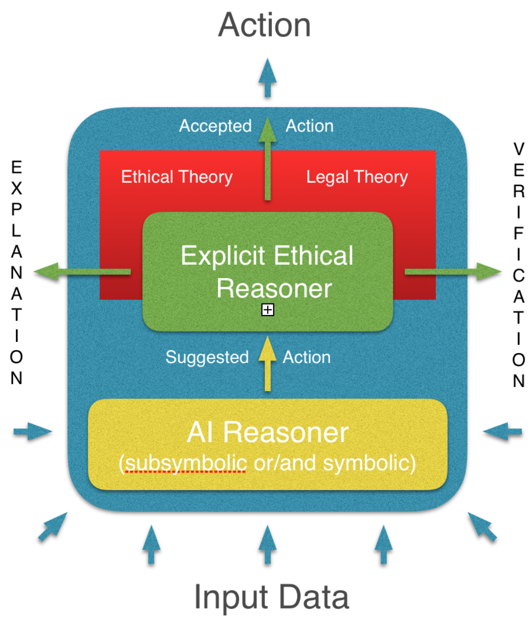
Free first-order logic

(Constricted) Inconsistency detected & fixed

[JAR, 2019]

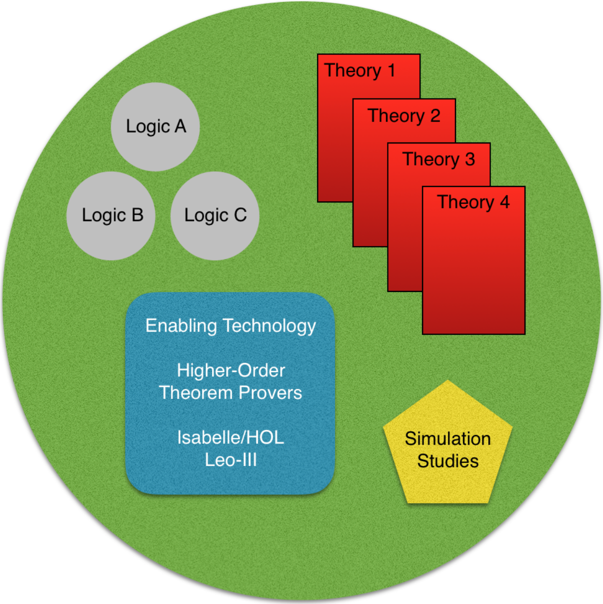
D. Scott
(UC Berkeley)


Ethics
“If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis.”
(Leibniz, 1677)



Action
Accepted Action
Ethical Theory
Legal Theory
Explicit Ethical Reasoner
EXPLANATION
VERIFICATION
Suggested Action
AI Reasoner
(subsymbolic or/and symbolic)
Input Data

Normative Reasoning Experimentation Platform



Conclusion

- Universal Logical Reasoning Approach
- Experiments in Computational Metaphysics: Ontological Argument
- Interesting new Results (Cross-Fertilisation)

Computers may help to Sharpen our Understanding of Arguments

Many further Applications

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Gewirth's Principle of Generic Consistency (PGC) in Isabelle/HOL

> "Act in accord with the generic rights of your recipients as well as of yourself. I shall call this the Principle of Generic Consistency (PGC), since it combines the formal consideration of consistency with the material consideration of rights to the generic features or goods of action." **(Alan Gewirth, Reason and Morality, 1978)**

- **Gewirth's PGC has**
  - stirred much controversy in moral philosophy
  - been discussed as means to bound the impact of artificial general intelligence (AGI)
- **Idea (in a nutshell):**
  - devise a safety mechanism of a mathematical (deductive) nature
  - to ensure that an AGI respects human's freedom and well-being
  - mechanism is based on assumption that it is able to recognize itself, as well as us humans, as agents (prospective purposive agents, PPA) which
    - act voluntarily on self-chosen purposes, and
    - reason rationally
- **References**

  - A. Gewirth. Reason and morality. U of Chicago Press, 1978.
  - D. Beyleveld. The dialectical necessity of morality: An analysis and defense of Alan Gewirth's argument to the principle of generic consistency. U of Chicago Press, 1991.
  - A. Kornai. Bounding the impact of AGI. J. Experimental & Theoretical AI, 2014.

— Brief Discussion —

Discussion

- There are consistent theistic theories which
  - imply the necessary existence of a supreme being
  - support different philosophical positions: determinism / non-determinism
- Theistic belief (at least in an abstract sense) not necessarily irrational
- By adopting the notion of “ultrafilter” these
  theistic theories were mapped here to mathematical theories

Question

- Existence results in metaphysics vs. mathematics — difference?

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Native Support for Deontic Logic(s) in Leo-III

∀P.Leo III

Leo III - A MASSIVELY PARALLEL HIGHER-ORDER THEOREM PROVER

What is Leo-III?

- ATP for classical HOL (A. Steen, C. Benzmüller)
- Integrates SMT- & FO-ATPs: **competitive on (decidable) fragments of HOL**
- native support for **more than 120 logics** (all normal quantified modal logics)
- including native support for **quantified SDL and DDL**
- Website: `http://page.mi.fu-berlin.de/lex/leo3/`

Recent Independent Evaluation:

- GRUNGE: A Grand Unified ATP Challenge [arXiv:1903.02539]

⇒ **Leo-III is the world's leading (and most expressive) ATP**

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DEMO: Anderson's Variant

Isabelle2018/HOL - AndersonProof.thy (modified)

AndersonProof.thy (~/GITHUBS/chrisgitlab/talks/2019-Dubrovnik/)

```
theory AndersonProof imports IHOML
begin
(*Positiveness/perfection: uninterpreted constant symbol*)
  consts positiveProperty::"(e⇒i⇒bool)⇒i⇒bool" ("𝒫")
(*Some auxiliary definitions*)
  definition h3 (infix "⇛" 60) where "X ⇛ Y ≡  □(∀ᴱz. X z ⟶ Y z)"
(**Part I**)
 (*D1'*)  definition GA ("Gᴬ") where "Gᴬ ≡ λx. ∀Y. (𝒫 Y) ↔ □(Y x)"
 (*A1a*) axiomatization where A1a:"⌊∀X. 𝒫 (⇁X) ⟶ ¬(𝒫 X) ⌋"
 (*A2*)  axiomatization where A2: "⌊∀X Y. (𝒫 X ∧ (X ⇛ Y)) ⟶ 𝒫 Y⌋"
 (*T1*)  theorem T1: "⌊∀X. 𝒫 X ⟶ ◇∃ᴱ X⌋" using A1a A2 h3_def by metis
 (*T2*)  axiomatization where T2: "⌊𝒫 Gᴬ⌋"  (*here we postulate T2 instead of proving it*)
 (*T3*)  theorem T3: "⌊◇∃ᴱ Gᴬ⌋" using T1 T2 h3_def by blast
(**Part II**)
 (*Logic KB*) axiomatization where symm: "∀x y. x r y ⟶ y r x"
 (*A4*)  axiomatization where A4: "⌊∀X. 𝒫 X ⟶ □(𝒫 X)⌋"
 (*D2'*) abbreviation essA ("ℰᴬ") where "ℰᴬ Y x ≡ (∀Z. □(Z x) ↔ Y ⇛ Z)"
 (*T4*)  theorem T4: "⌊∀x. Gᴬ x ⟶ (ℰᴬ Gᴬ x)⌋" by (metis A2 GA_def T2 symm h3_def)
 (*D3*)  abbreviation NEA ("NEᴬ") where "NEᴬ x  ≡ (λw. (∀Y. ℰᴬ Y x ⟶ □∃ᴱ Y) w)"
 (*A5*)  axiomatization where A5: "⌊𝒫 NEᴬ⌋"
 (*T5*)  theorem T5: "⌊◇∃ᴱ Gᴬ⌋ ⟶ ⌊□∃ᴱ Gᴬ⌋" by (metis A2 GA_def T2 symm h3_def)
 (*T6*)  theorem T6: "⌊□∃ᴱ Gᴬ⌋" using T3 T5 by blast

(**Modal collapse is countersatisfiable**)
lemma "⌊∀Φ.(Φ ⟶ (□ Φ))⌋" nitpick[user_axioms] oops (*Countermodel found by Nitpick*)
```

Documentation Sidekick State Theories

☑ Proof state ☑ Auto update Update Search: 100%

Output Query Sledgehammer Symbols

28,53 (1529/6107) (isabelle,isabelle,UTF-8-Isabelle) nmroUG 195/587MB 6:25 PM

DEMO: Anderson's Variant

```
Isabelle2018/HOL - AndersonProof.thy
AndersonProof.thy (~/GITHUBS/chrisgitlab/talks/2019-Dubrovnik/)

(**Positive Properties and Ultrafilters**)
  abbreviation emptySet ("∅") where "∅ ≡ λx w. False"
  abbreviation entails (infixr"⊆"51) where "φ⊆ψ  ≡ ∀x w. φ x w ⟶ ψ x w"
  abbreviation andPred (infixr"⊓"51) where "φ⊓ψ  ≡ λx w. φ x w ∧ ψ x w"
  abbreviation negpred ("¯_"[52]53) where "¯ψ   ≡ λx w. ¬ψ x w"
  abbreviation "ultrafilter Φ cw ≡
       ¬(Φ ∅ cw)
    ∧  (∀φ. ∀ψ. (Φ φ cw ∧ Φ ψ cw) ⟶ (Φ (φ ⊓ ψ) cw))
    ∧  (∀φ::e⇒i⇒bool. ∀ψ::e⇒i⇒bool. (Φ φ cw ∨ Φ (¯φ) cw) ∧ ¬(Φ φ cw ∧ Φ (¯φ) cw))
    ∧  (∀φ::e⇒i⇒bool. ∀ψ::e⇒i⇒bool.  (Φ φ cw ∧ φ ⊆ ψ) ⟶ Φ ψ cw)"

(*U1*) theorem U1: "∀w. ultrafilter 𝒫 w" nitpick[user_axioms,format=2,show_all] oops (*counterm.*)

Proof state  Auto update  Update  Search:  100%

  ψ = (λx. _)((e₁, i₁) := False, (e₁, i₂) := False)
  w = i₁
Constants:
  𝒫 = (λx. _)
      (((λx. _)((e₁, i₁) := True, (e₁, i₂) := True), i₁) := True,
       ((λx. _)((e₁, i₁) := True, (e₁, i₂) := True), i₂) := True,
       ((λx. _)((e₁, i₁) := True, (e₁, i₂) := False), i₁) := False,
       ((λx. _)((e₁, i₁) := True, (e₁, i₂) := False), i₂) := False,
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       ((λx. _)((e₁, i₁) := False, (e₁, i₂) := True), i₂) := False,
       ((λx. _)((e₁, i₁) := False, (e₁, i₂) := False), i₁) := False,
       ((λx. _)((e₁, i₁) := False, (e₁, i₂) := False), i₂) := False)
  existsAt = (λx. _)((e₁, i₁) := True, (e₁, i₂) := True)
  (r) = (λx. _)((i₁, i₁) := True, (i₁, i₂) := True, (i₂, i₁) := True, (i₂, i₂) := True)

Output  Query  Sledgehammer  Symbols
41,46 (2225/5923)    (isabelle,isabelle,UTF-8-Isabelle) nmro UG  200/502MB  6:38 PM
```
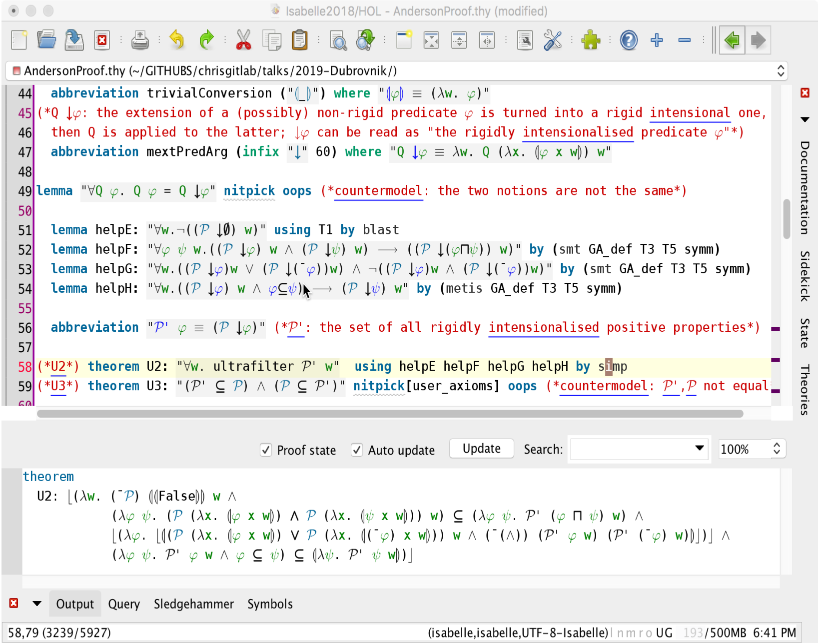

DEMO: Anderson's Variant



DEMO: Anderson's Variant

Isabelle2018/HOL - AndersonProof.thy

AndersonProof.thy (~/GITHUBS/chrisgitlab/talks/2019-Dubrovnik/)

```
 abbreviation trivialConversion ("⦇_⦈") where "⦇φ⦈ ≡ (λz::i. φ)"
(*Q ↓φ: the extension of a (possibly) non-rigid predicate φ is turned into a rigid intensional one,
 then Q is applied to the latter; ↓φ can be read as "the rigidly intensionalised predicate φ"*)
 abbreviation mextPredArg (infix "↓" 60) where "Q ↓φ ≡ λw. Q (λx. ⦇φ x w⦈) w"

lemma "∀Q φ. Q φ = Q ↓φ" nitpick oops (*countermodel: the two notions are not the same*)

 lemma helpE: "∀w.¬((𝒫 ↓∅) w)" using T1 by blast
 lemma helpF: "∀φ ψ w.((𝒫 ↓φ) w ∧ (𝒫 ↓ψ) w) ⟶ ((𝒫 ↓(φ⊓ψ)) w)" by (smt GA_def T3 T5 symm)
 lemma helpG: "∀w.((𝒫 ↓φ)w ∨ (𝒫 ↓(¯φ))w) ∧ ¬((𝒫 ↓φ)w ∧ (𝒫 ↓(¯φ))w)" by (smt GA_def T3 T5 symm)
 lemma helpH: "∀w.((𝒫 ↓φ) w ∧ φ⊆ψ) ⟶ (𝒫 ↓ψ) w" by (metis GA_def T3 T5 symm)

abbreviation "𝒫' φ ≡ (𝒫 ↓φ)" (*𝒫': the set of all rigidly intensionalised positive properties*)

(*U2*) theorem U2: "∀w. ultrafilter 𝒫' w"  using helpE helpF helpG helpH by simp
(*U3*) theorem U3: "(𝒫' ⊆ 𝒫) ∧ (𝒫 ⊆ 𝒫')" nitpick[user_axioms] oops (*countermodel: 𝒫',𝒫 not equal*)
```

☑ Proof state ☑ Auto update Update Search: 100%

```
theorem
  U2: ⌊(λw. (¯𝒫) ∅ w ∧
           (λφ ψ. (𝒫 (λx. ⦇φ x w⦈) ∧ 𝒫 (λx. ⦇ψ x w⦈)) w) ⊆ (λφ ψ. 𝒫' (φ ⊓ ψ) w) ∧
           ⌊(λφ. ⌊(λψ. (𝒫 (λx. ⦇φ x w⦈) ∨ 𝒫 (λx. ⦇(¯φ) x w⦈)) w ∧ (¯(∧)) (𝒫' φ w) (𝒫' (¯φ) w))⌋)⌋ ∧
           (λφ ψ. 𝒫' φ w ∧ φ ⊆ ψ) ⊆ (λφ ψ. 𝒫' ψ w))⌋
```

Output Query Sledgehammer Symbols

58,80 (3243/5930) (isabelle,isabelle,UTF-8-Isabelle) nmro UG 314/501MB 6:44 PM

Documentation Sidekick State Theories

DEMO: Anderson's Variant

Isabelle2018/HOL - AndersonProof.thy

AndersonProof.thy (~/GITHUBS/chrisgitlab/talks/2019-Dubrovnik/)

```
(**Modal logic S5: Consistency and Modal Collapse**)
 axiomatization where refl: "∀x. x r x" and trans: "∀x y z. x r y ∧ y r z ⟶ x r z"
 lemma True nitpick[satisfy] oops  (*Model found by Nitpick: the axioms are consistent*)
 lemma ModalCollapse: "⌊∀Φ.(Φ → (□ Φ))⌋" nitpick[user_axioms,show_all,format=2] oops (*counterm.

(**Barcan and Converse Barcan Formula for Individuals (type e)**)
 lemma BarcanInd1: "⌊(∀ᴱx::e. (□ (φ(x)))) → (□ (∀ᴱx::e. φ(x)))⌋" nitpick oops (*countermodel*)
 lemma ConvBarcanInd1: "⌊(□ (∀ᴱx::e. φ(x))) → (∀ᴱx::e. (□ (φ(x))))⌋" nitpick oops (*countermodel*
(**Barcan and Converse Barcan Formula for Properties (type e⇒i⇒bool)**)
 lemma BarcanPred1: "⌊(∀x::e⇒i⇒bool. (□ (φ(x)))) → (□ (∀x::e⇒i⇒bool. φ(x)))⌋" by simp
 lemma ConvBarcanPred1: "⌊(□ (∀x::e⇒i⇒bool. φ(x))) → (∀x::e⇒i⇒bool. (□ (φ(x))))⌋" by simp
```

Proof state   Auto update   Update   Search: 100%

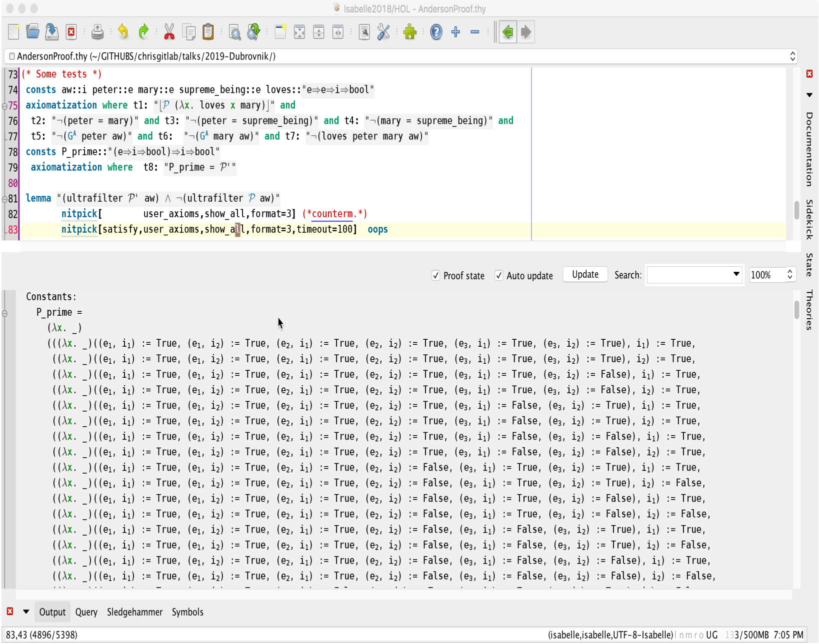
```
Nitpicking formula...
Nitpick found a counterexample for card e = 1 and card i = 2:

  Skolem constants:
    v = i₁
    w = i₂
    x = (λx. _)(i₁ := False, i₂ := True)
  Constants:
    𝓟 = (λx. _)
         (((λx. _)((e₁, i₁) := True, (e₁, i₂) := True), i₁) := True,
          ((λx. _)((e₁, i₁) := True, (e₁, i₂) := True), i₂) := True,
          ((λx. _)((e₁, i₁) := True, (e₁, i₂) := False), i₁) := False,
          ((λx. _)((e₁, i₁) := True, (e₁, i₂) := False), i₂) := False,
          ((λx. _)((e₁, i₁) := False, (e₁, i₂) := True), i₁) := False,
          ((λx. _)((e₁, i₁) := False, (e₁, i₂) := True), i₂) := False,
          ((λx. _)((e₁, i₁) := False, (e₁, i₂) := False), i₁) := False,
          ((λx. _)((e₁, i₁) := False, (e₁, i₂) := False), i₂) := False)
    existsAt = (λx. _)((e₁, i₁) := True, (e₁, i₂) := True)
    (r) = (λx. _)((i₁, i₁) := True, (i₁, i₂) := True, (i₂, i₁) := True, (i₂, i₂) := True)
```

Output   Query   Sledgehammer   Symbols

64,49 (3629/5788)   (isabelle,isabelle,UTF-8-Isabelle) nmro UG 320/571MB 6:54 PM

Documentation   Sidekick   State   Theories

DEMO: Anderson’s Variant



DEMO: Anderson's Variant