Hasso Plattner Institut

Introduction to Probabilistic Machine Learning

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Tutorial 6 – Recap Theory Unit 6

Overview

1. **Questions and Updates**
2. Recap: Main Concepts of Unit 6
3. Example: TrueSkill 1 vs 1
4. Hints for Exercise 3 (to be handed in Monday June 2)

Course Overview

Week	Topic Lecture	Tutorial	Exercises
07.04. & 08.04.	1 Probability Theory	Intro Julia	
14.04. & 15.04.	2 Inference Methods and Decision-Making	no tutorial	Exercise 1
21.04. & 22.04.	**no lecture**	Theory Unit 1 & 2	(14.04. – 08.05.)
28.04. & 29.04.	3 Graphical Models: Independence	Theory Unit 3	
05.05. & 06.05.	4 Graphical Models: Exact Inference	Theory Unit 4	Exercise 2
12.05. & 13.05.	5 Graphical Models: Approximate Inference	Theory Unit 5	(05.05. – 19.05.)
19.05. & 20.05.	**6 Bayesian Ranking**	**Theory Unit 6**	**Exercise 3**
26.05. & 27.05.	7 Linear Basis Function Models	Theory Unit 7	**(19.05. – 02.06.)**
02.06. & 03.06.	8 Bayesian Regression	Theory Unit 8	Exercise 4
09.06. & 10.06.	**no lecture**	9 Bayesian Classification	(02.06. – 23.06.)
16.06. & 17.06.	10 Non-Bayesian Classification Learning	Theory Unit 9 & 10	
23.06. & 24.06.	11 Gaussian Processes	Theory Unit 11	Exercise 5
30.06. & 01.07.	12 Information Theory	Theory Unit 12	(23.06. – 07.07.)
07.07. & 08.07.	13 Real-World Applications		

Overview

1. Questions and Updates
2. **Recap: Main Concepts of Unit 6**
3. Example: TrueSkill 1 vs 1
4. Hints for Exercise 3 (to be handed in Monday June 2)

Recap Unit 6: Overview of Concepts and Focus

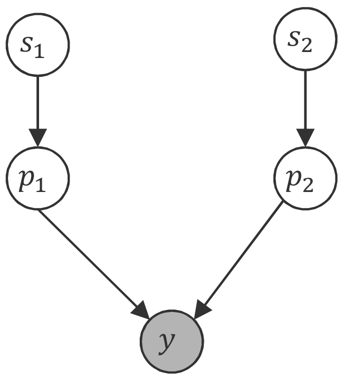
a) Bayesian Ranking: TrueSkill

b) Gaussian Messages in Factor Graphs

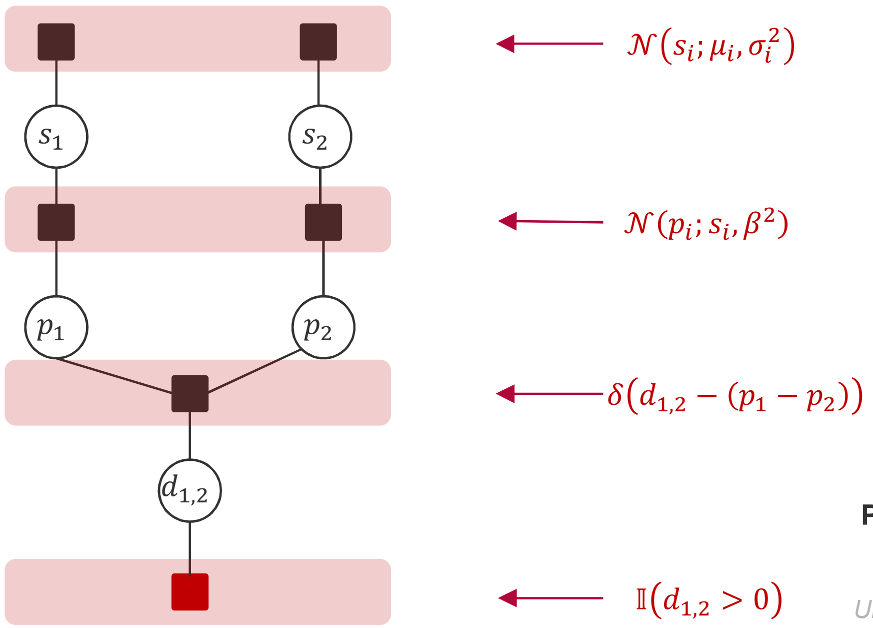
c) Approximated Messages (Truncated Gaussians)

TrueSkill Factor Graphs (Case 1 vs 1)

Bayesian Network

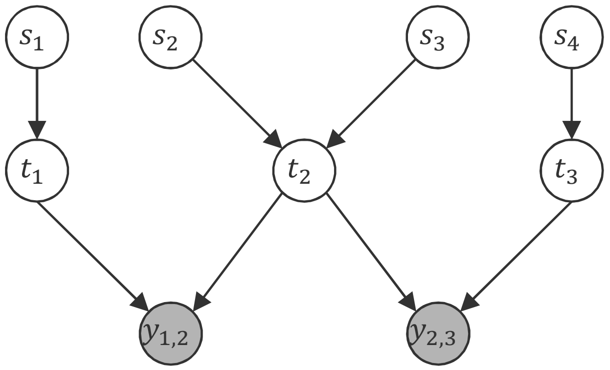


Factor Graph

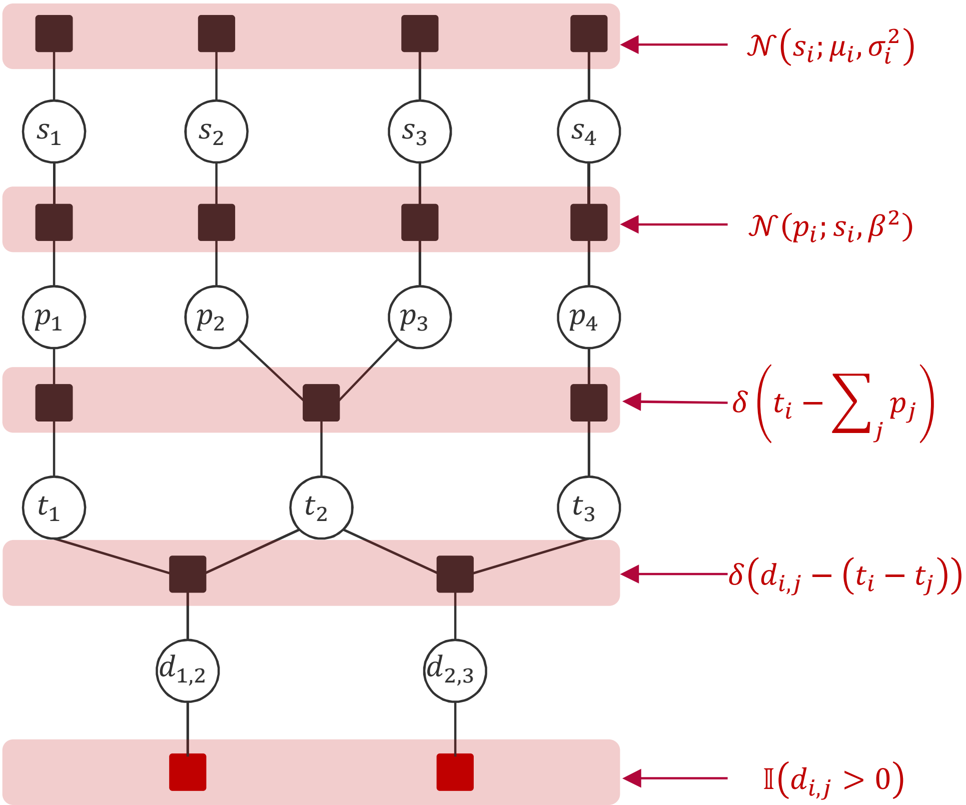


Generalized: TrueSkill Factor Graphs (Case 1 vs 2 vs 1)

Bayesian Network

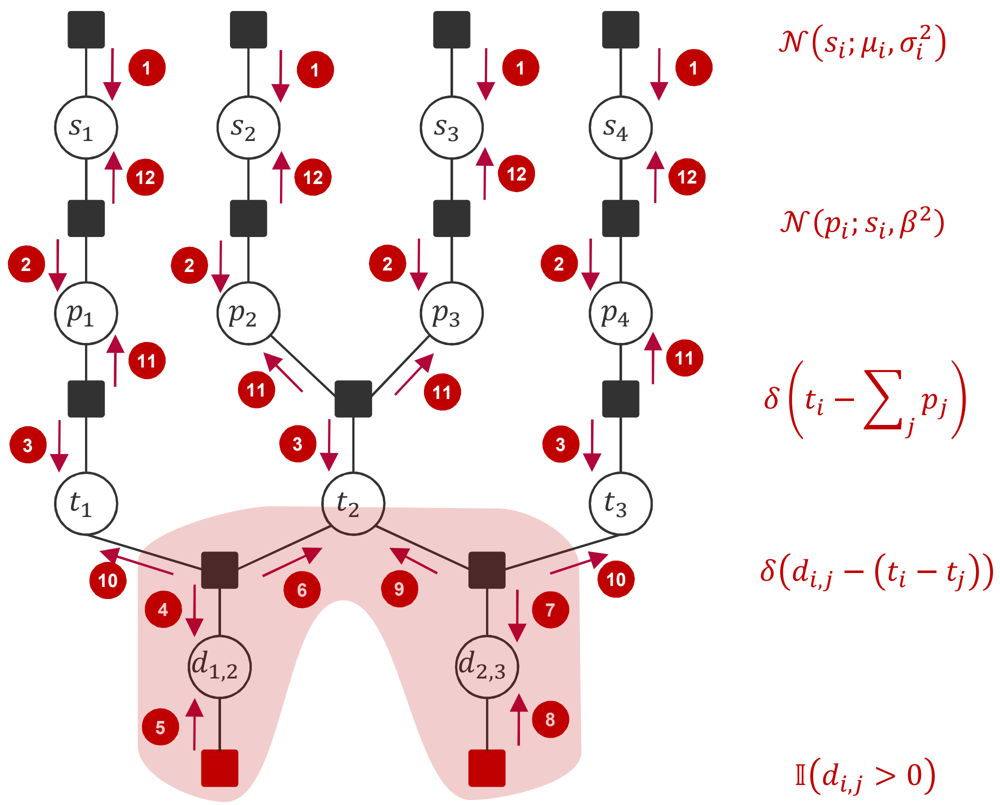


Factor Graph



(Approximate) Message Passing in TrueSkill Factor Graphs

TrueSkill Factor Graph

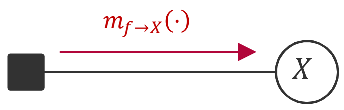


- Four Phases
  1. Pass prior messages (1)
  2. Pass messages *down* to the team performances (2 to 3)
  3. Iterate the approximate messages on the pairwise team differences (4 to 9)
  4. Pass messages back from *up* from team performances to player skill (10 – 12)

- Since this is a *tree,* the algorithm is guaranteed to converge!

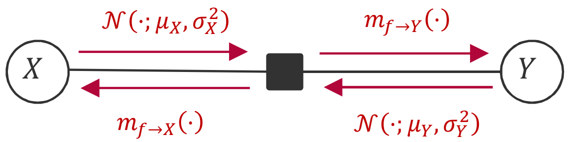
Message Update Equations (2025)

Gaussian Factor $\mathcal{N}(X;\mu,\sigma^2)$



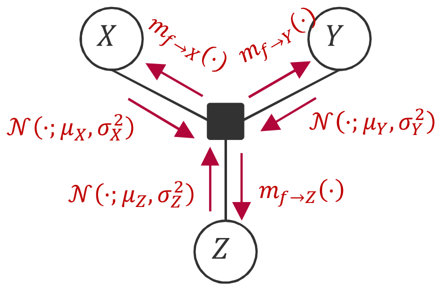
$$m_{f\to X}(x) = \mathcal{N}(x;\mu,\sigma^2)$$

Gaussian Mean Factor $\mathcal{N}(Y;X,\beta^2)$



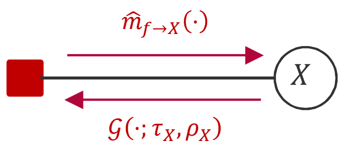
$$m_{f\to Y}(y) = \mathcal{N}(y;\mu_X,\sigma_X^2+\beta^2)$$
$$m_{f\to X}(x) = \mathcal{N}(x;\mu_Y,\sigma_Y^2+\beta^2)$$

Weighted Sum Factor $\delta\big(Z-(aX+bY)\big)$



$$m_{f\to Z}(z) = \mathcal{N}(z;a\mu_X+b\mu_Y,a^2\sigma_X^2+b^2\sigma_Y^2)$$
$$m_{f\to Y}(y) = \mathcal{N}(y;(\mu_Z-a\mu_X)/b,(\sigma_Z^2+a^2\sigma_X^2)/b^2)$$
$$m_{f\to X}(x) = \mathcal{N}(x;(\mu_Z-b\mu_Y)/a,(\sigma_Z^2+b^2\sigma_Y^2)/a^2)$$

Between Factor $\mathbb{I}(l\le X<u)$



Correction functions

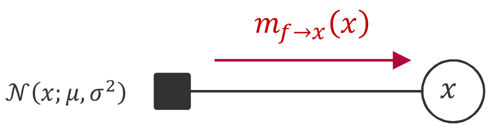
$$V := v_{l\sqrt{\rho_X},u\sqrt{\rho_X}}\left(\frac{\tau_X}{\sqrt{\rho_X}}\right)$$
$$W := w_{l\sqrt{\rho_X},u\sqrt{\rho_X}}\left(\frac{\tau_X}{\sqrt{\rho_X}}\right)$$

of doubly-truncated Gaussians

$$\widehat{m}_{f\to X}(x) = \mathcal{G}\left(x;\sqrt{\rho_X}\cdot\frac{V}{1-W}+\tau_X\cdot\frac{W}{1-W},\rho_X\cdot\frac{W}{1-W}\right)$$

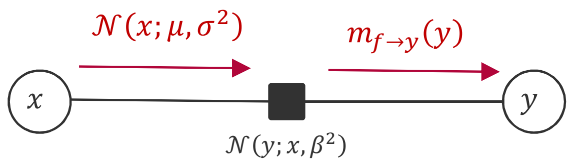
Message Update Equations (2024)

Gaussian Factor



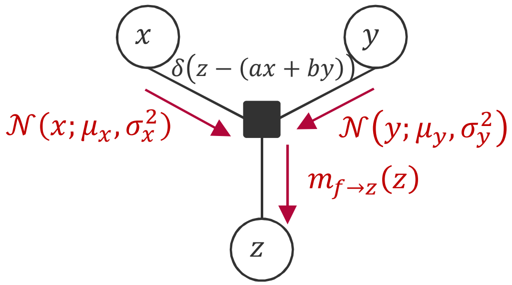
$$m_{f \to x}(x) = \mathcal{N}(x; \mu, \sigma^2)$$

Gaussian Mean Factor



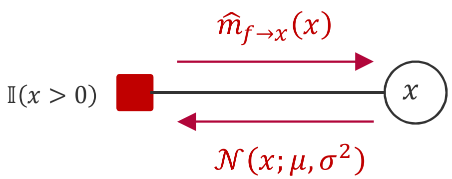
$$m_{f \to y}(y) = \int \mathcal{N}(y; x, \beta^2) \cdot \mathcal{N}(x; \mu, \sigma^2)\, dx = \mathcal{N}(y; \mu, \sigma^2 + \beta^2)$$

Weighted Sum Factor



$$m_{f \to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

Greater-Than Factor



$$\hat{m}_{f \to x}(x) = \frac{\hat{p}(x)}{m_{x \to f}(x)} = \frac{\mathcal{N}(x; \hat{\mu}, \hat{\sigma}^2)}{\mathcal{N}(x; \mu, \sigma^2)}$$

Mean and variance of a truncated Gaussian $\mathcal{N}(x; \mu, \sigma^2)$

Recap: Truncated Gaussians (2024)

- **Truncated Gaussians**. *A truncated Gaussian given by* $p(x) \propto \mathbb{I}(x > 0) \cdot \mathcal{N}(x; \mu, \sigma^2)$ *has the following three moments*

$$Z(\mu, \sigma^2) = \int_{-\infty}^{+\infty} p(x)\, dx = 1 - F(0; \mu, \sigma^2)$$

Follows from definition of F

$$E[X] = \int_{-\infty}^{+\infty} x \cdot p(x)\, dx = \mu + \sigma \cdot v\left(\frac{\mu}{\sigma}\right)$$

Additive update that goes to zero as $\frac{\mu}{\sigma} \to \infty$

$$\mathrm{var}[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 \cdot p(x)\, dx = \sigma^2 \cdot \left(1 - w\left(\frac{\mu}{\sigma}\right)\right)$$

Multiplicative update that goes to 1 as $\frac{\mu}{\sigma} \to \infty$

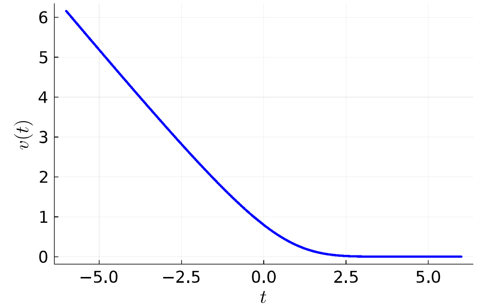
where the probit $F(t; \mu, \sigma^2) \coloneqq \int_{-\infty}^{t} \mathcal{N}(x; \mu, \sigma^2)\, dx$ *and*

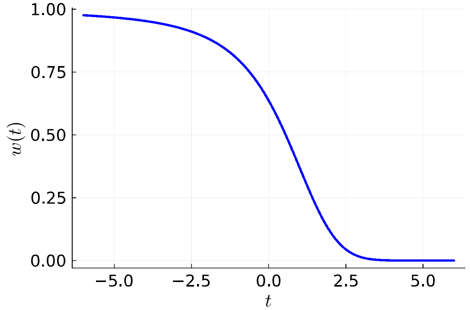
$$v(t) \coloneqq \frac{\mathcal{N}(t; 0,1)}{F(t; 0,1)}$$

Converges to $-t$ as $t \to -\infty$

$$w(t) \coloneqq v(t) \cdot [v(t) + t]$$

- This can be generalized to an arbitrary interval $[a, b]$ where the Gaussian is truncated!





Overview

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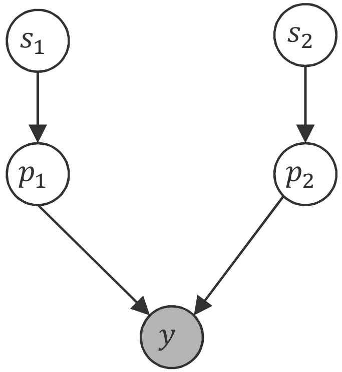
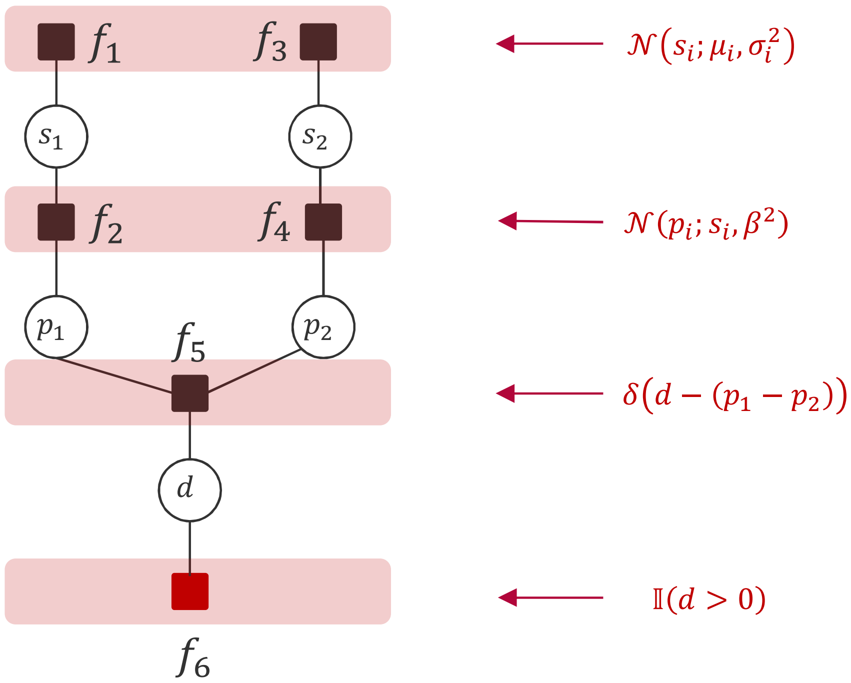
4. Hints for Exercise 3 (to be handed in Monday June 2)

TrueSkill Factor Graphs (Case 1 vs 1)

Bayesian Network

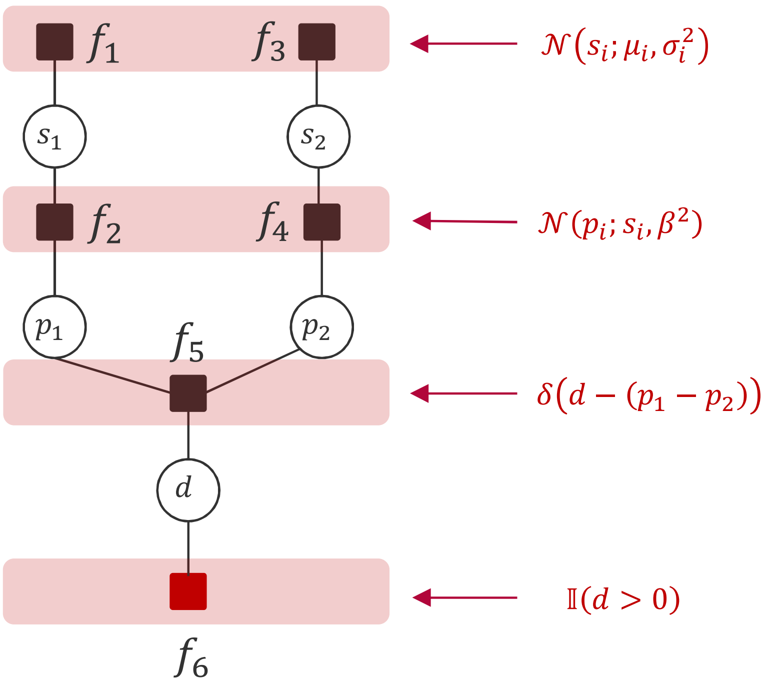


Factor Graph



TrueSkill Factor Graphs (Case 1 vs 1): Overall Idea?

Factor Graph

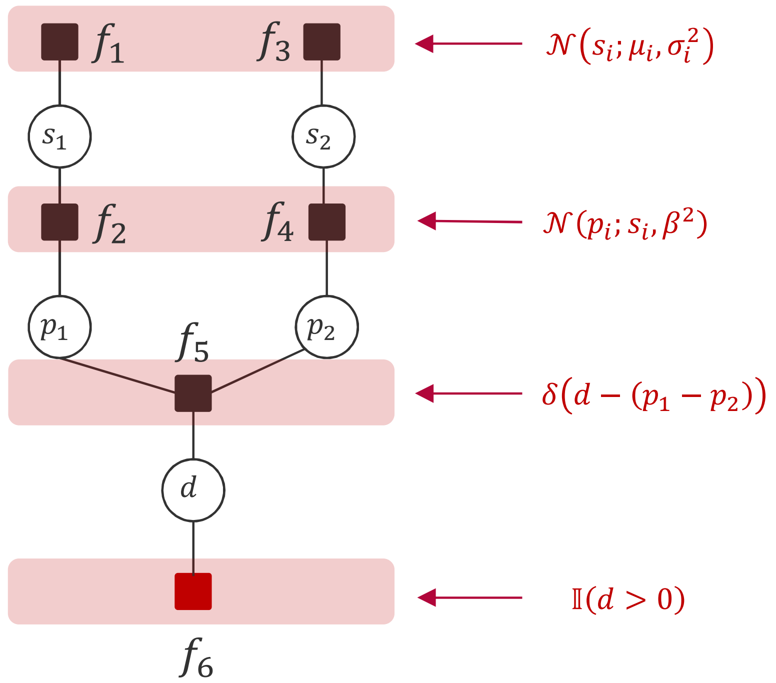


Tutorial 6

PML SS 2025

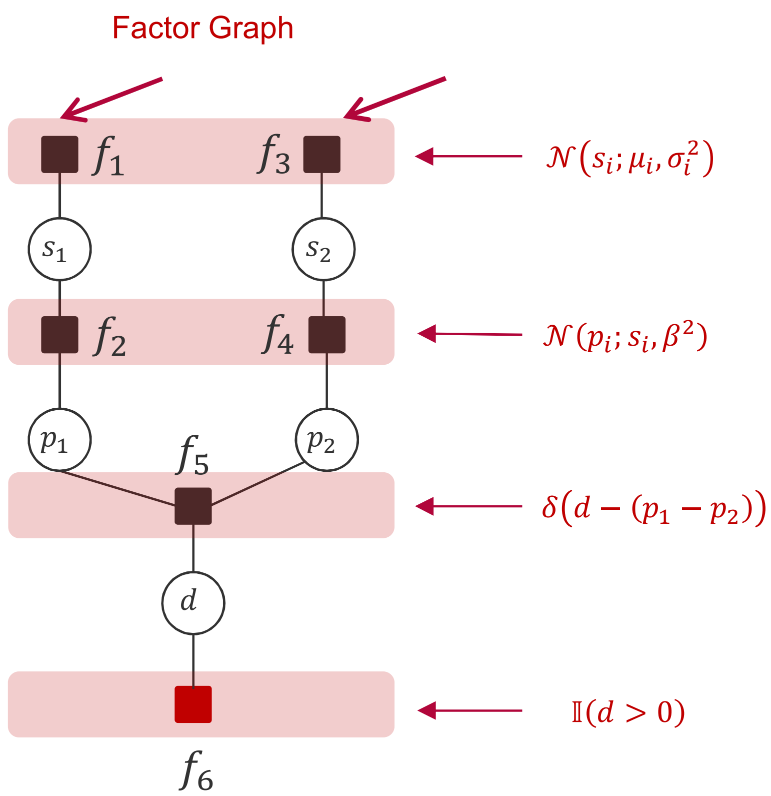
Play 1 Wins: How to Adapt the Skill Beliefs?

Factor Graph



$$Example: \ \mu_1 = 20, \ \mu_2 = 18, \ \sigma_1^2 = 4, \ \sigma_2^2 = 9, \ \beta^2 = 1, \ y = 1$$

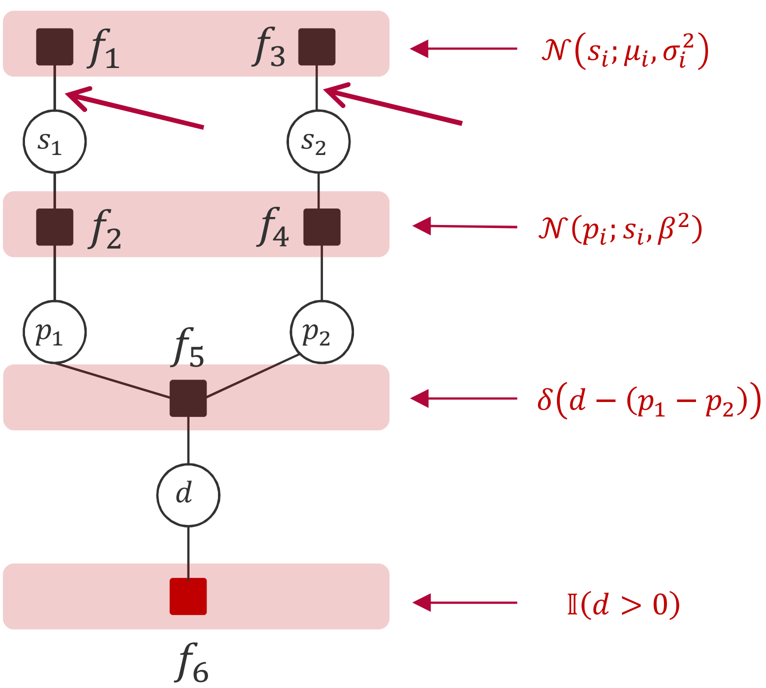
Let's go Through it: Prior Skill Distributions (Given)



$$f_1(s_1) = N(s_1; \mu_1, \sigma_1^2) \qquad f_3(s_2) = N(s_2; \mu_2, \sigma_2^2)$$

Prior Skill Distributions & Messages (Gaussian Factor)

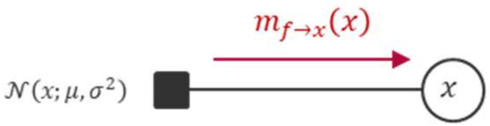
Factor Graph



$$f_1(s_1) = N(s_1; \mu_1, \sigma_1^2) \qquad f_3(s_2) = N(s_2; \mu_2, \sigma_2^2)$$

$$\boxed{m_{f_1 \to s_1}(s_1) = N(s_1; \mu_1, \sigma_1^2) \qquad m_{f_3 \to s_2}(s_2) = N(s_2; \mu_2, \sigma_2^2)}$$

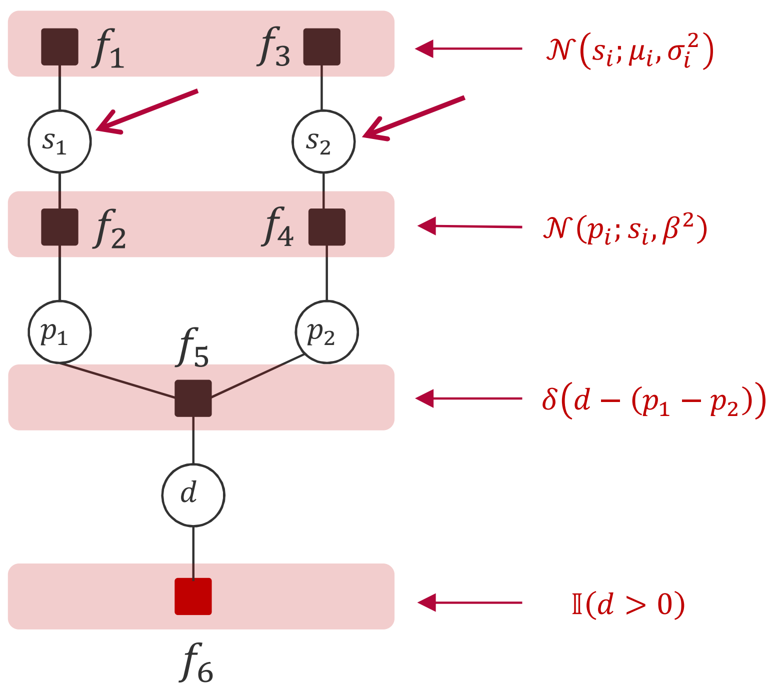
Gaussian Factor



$$\boxed{m_{f \to x}(x) = \mathcal{N}(x; \mu, \sigma^2)}$$

Marginals for Skills (Player 1 & 2)

Factor Graph



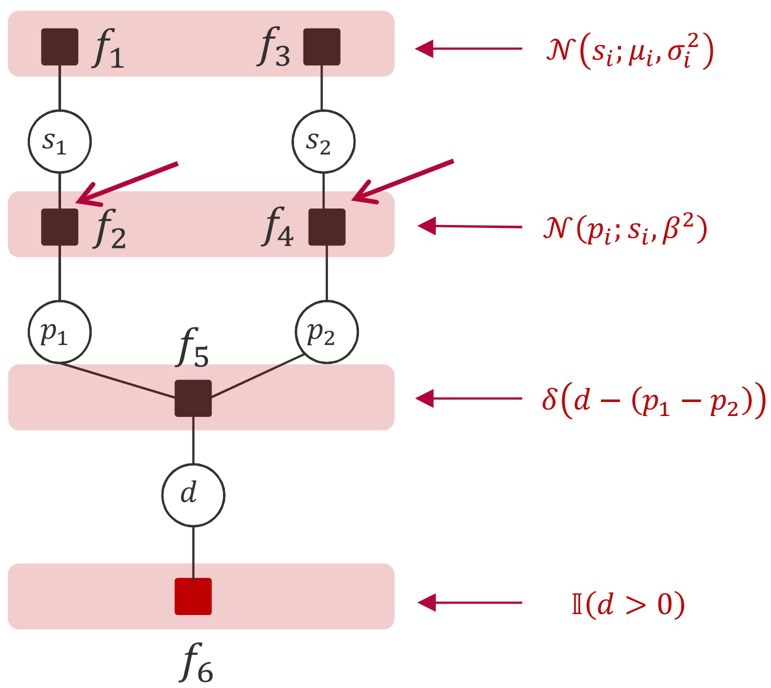
$$f_1(s_1) = N(s_1; \mu_1, \sigma_1^2) \qquad f_3(s_2) = N(s_2; \mu_2, \sigma_2^2)$$

$$m_{f_1 \to s_1}(s_1) = N(s_1; \mu_1, \sigma_1^2) \qquad m_{f_3 \to s_2}(s_2) = N(s_2; \mu_2, \sigma_2^2)$$

$$\boxed{\begin{aligned} p(s_1) &= m_{f_1 \to s_1}(s_1) \cdot \underbrace{m_{f_2 \to s_1}(s_1)}_{uniform} = m_{f_1 \to s_1}(s_1) = N(s_1; \mu_1, \sigma_1^2) \\ p(s_2) &= m_{f_3 \to s_2}(s_2) \cdot \underbrace{m_{f_4 \to s_2}(s_2)}_{uniform} = m_{f_3 \to s_2}(s_2) = N(s_2; \mu_2, \sigma_2^2) \end{aligned}}$$

Factors for Performance (Given)

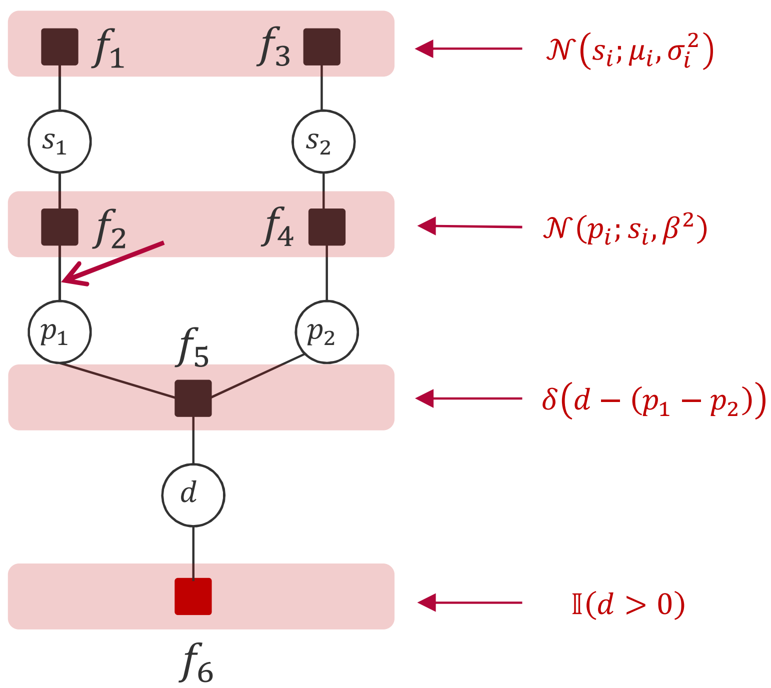
Factor Graph



$$f_2(s_1,p_1)=P(p_1\mid s_1)=N(p_1;s_1,\beta^2)$$

$$f_4(s_2,p_2)=P(p_2\mid s_2)=N(p_2;s_2,\beta^2)$$

Factor for Performance & Messages (Player 1)



$$f_2(s_1, p_1) = P(p_1 \mid s_1) = N(p_1; s_1, \beta^2)$$

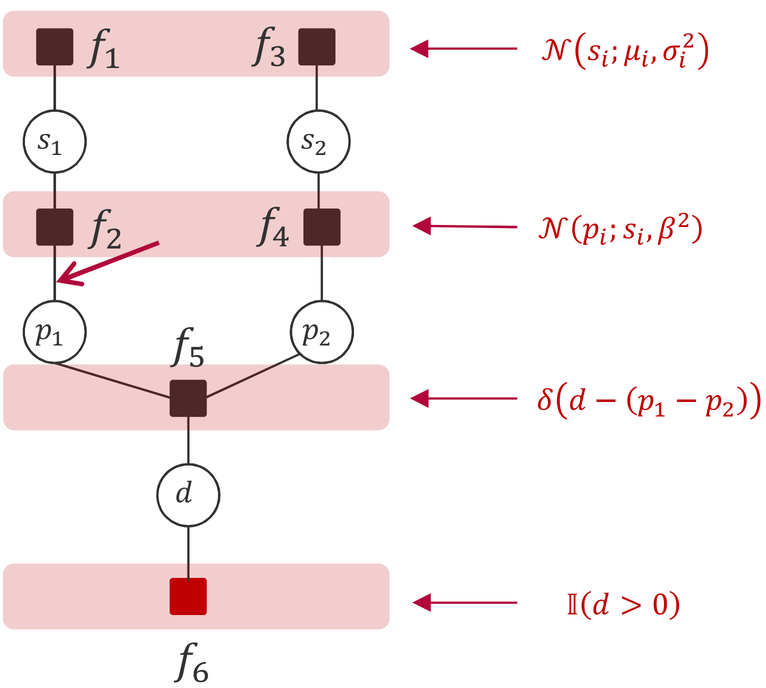
$$m_{f_2 \to p_1}(p_1) = \int_{s_1} f_2(s_1, p_1) \cdot \underbrace{m_{s_1 \to f_2}(s_1)}_{p(s_1)/\underbrace{m_{f_2 \to s_1}(s_1)}_{uniform}} \, ds_1 = \int_{s_1} f_2(s_1, p_1) \cdot p(s_1) \, ds_1$$

$$= \int_{s_1} \underbrace{N(p_1; s_1, \beta^2)}_{f_2(s_1, p_1)} \cdot \underbrace{N(s_1; \mu_1, \sigma_1^2)}_{p(s_1)} \, ds_1$$

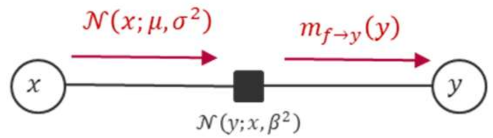
$$= ??$$

Factor for Performance & Message (Player 1)

Factor Graph



Gaussian Mean Factor



$$m_{f \to y}(y) = \int \mathcal{N}(y; x, \beta^2) \cdot \mathcal{N}(x; \mu, \sigma^2)\, dx = \mathcal{N}(y; \mu, \sigma^2 + \beta^2)$$

$$f_2(s_1, p_1) = P(p_1 \mid s_1) = N(p_1; s_1, \beta^2)$$

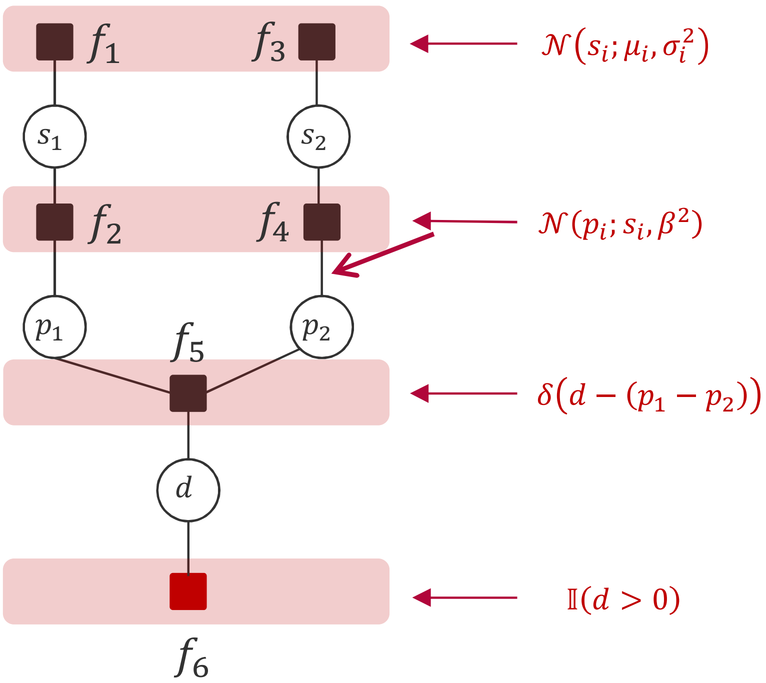
$$m_{f_2 \to p_1}(p_1) = \int_{s_1} f_2(s_1, p_1) \cdot \underbrace{m_{s_1 \to f_2}(s_1)}_{p(s_1)/\underbrace{m_{f_2 \to s_1}(s_1)}_{uniform}}\, ds_1 = \int_{s_1} f_2(s_1, p_1) \cdot p(s_1)\, ds_1$$

$$= \int_{s_1} \underbrace{N(p_1; s_1, \beta^2)}_{f_2(s_1, p_1)} \cdot \underbrace{N(s_1; \mu_1, \sigma_1^2)}_{p(s_1)}\, ds_1$$

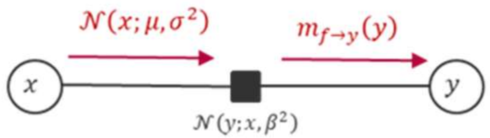
$$= N(p_1; \mu_1, \beta^2 + \sigma_1^2)$$

Factor for Performance & Message (Player 2)

Factor Graph



Gaussian Mean Factor



$$m_{f\to y}(y) = \int \mathcal{N}(y;x,\beta^2)\cdot\mathcal{N}(x;\mu,\sigma^2)\,dx = \mathcal{N}(y;\mu,\sigma^2+\beta^2)$$

$$f_4(s_2,p_2) = P(p_2\mid s_2) = N(p_2;s_2,\beta^2)$$

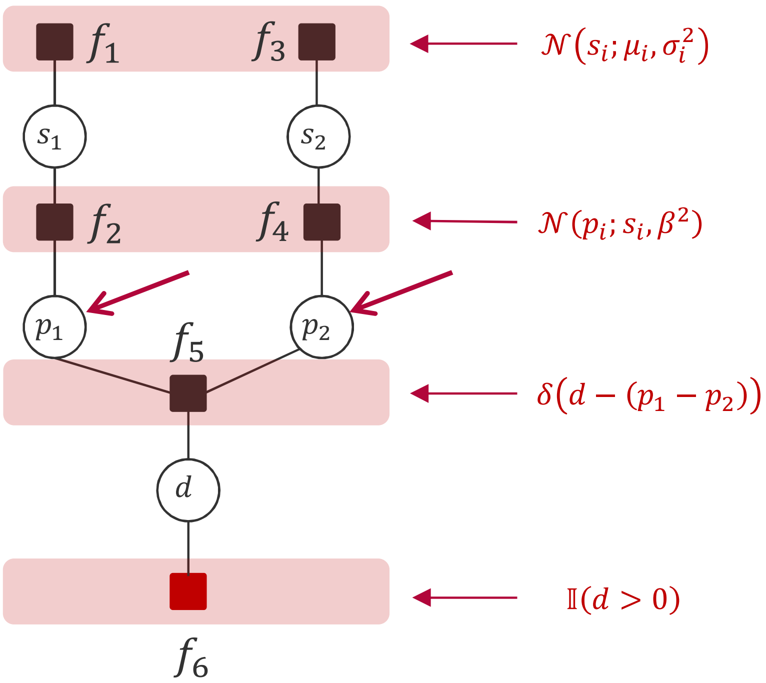
$$m_{f_4\to p_2}(p_2) = \int_{s_2} f_4(s_2,p_2)\cdot \underbrace{\underbrace{m_{s_2\to f_4}(s_2)}_{p(s_2)/m_{f_4\to s_2}(s_2)}}_{uniform}\; ds_2 = \int_{s_2} f_4(s_2,p_2)\cdot p(s_2)\; ds_2$$

$$= \int_{s_2} \underbrace{N(p_2;s_2,\beta^2)}_{f_4(s_2,p_2)}\cdot\underbrace{N(s_2;\mu_2,\sigma_2^2)}_{p(s_2)}\; ds_2$$

$$= N(p_2;\mu_2,\beta^2+\sigma_2^2)$$

Marginals for Performances (Player 1 & 2)

Factor Graph



$$m_{f_2 \to p_1}(p_1) = N(p_1; \mu_1, \beta^2 + \sigma_1^2)$$

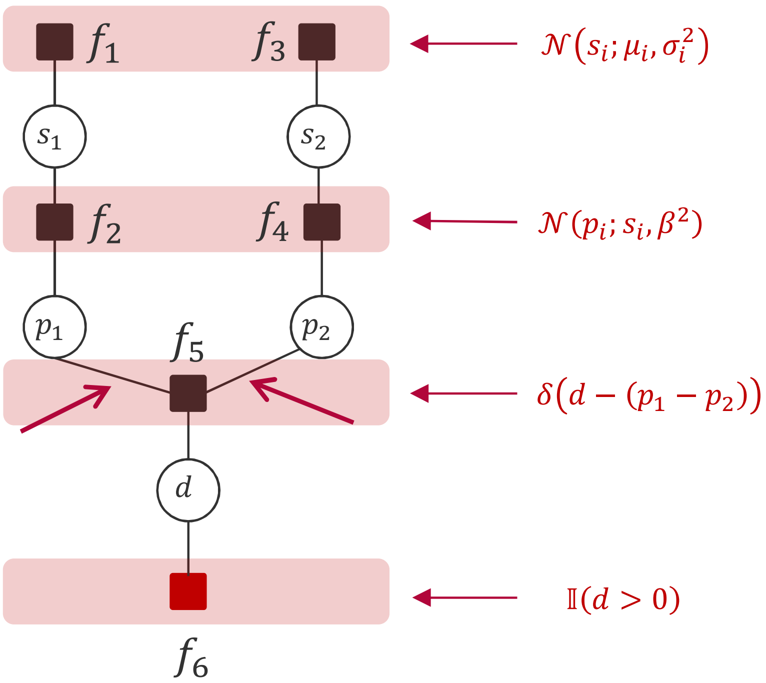
$$m_{f_4 \to p_2}(p_2) = N(p_2; \mu_2, \beta^2 + \sigma_2^2)$$

$$p(p_1) = m_{f_2 \to p_1}(p_1) \cdot \underbrace{m_{f_5 \to p_1}(p_1)}_{uniform} = m_{f_2 \to p_1}(p_1) = N(p_1; \mu_1, \beta^2 + \sigma_1^2)$$

$$p(p_2) = m_{f_4 \to p_2}(p_2) \cdot \underbrace{m_{f_5 \to p_2}(p_2)}_{uniform} = m_{f_4 \to p_2}(p_2) = N(p_2; \mu_2, \beta^2 + \sigma_2^2)$$

The Weighted Sum Factor

Factor Graph

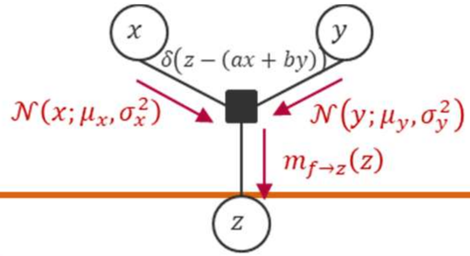


$$m_{p_1 \to f_5}(p_1) = p(p_1) / \underbrace{m_{f_5 \to p_1}(p_1)}_{uniform} = N(p_1; \mu_1, \beta^2 + \sigma_1^2)$$

$$m_{p_2 \to f_5}(p_2) = p(p_2) / \underbrace{m_{f_5 \to p_2}(p_2)}_{uniform} = N(p_2; \mu_2, \beta^2 + \sigma_2^2)$$

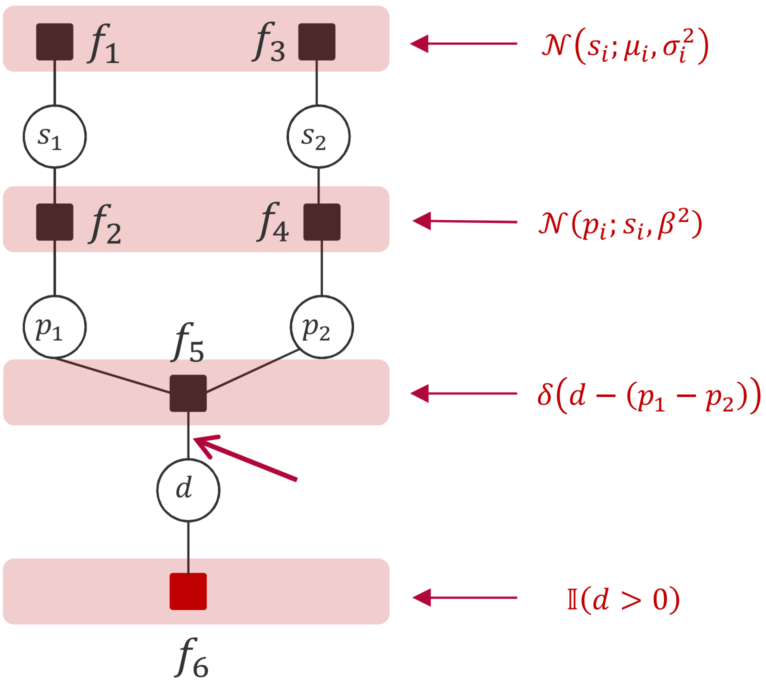
$$m_{f_5 \to d}(d) = ??$$

The Weighted Sum Factor



$$m_{f\to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

Factor Graph



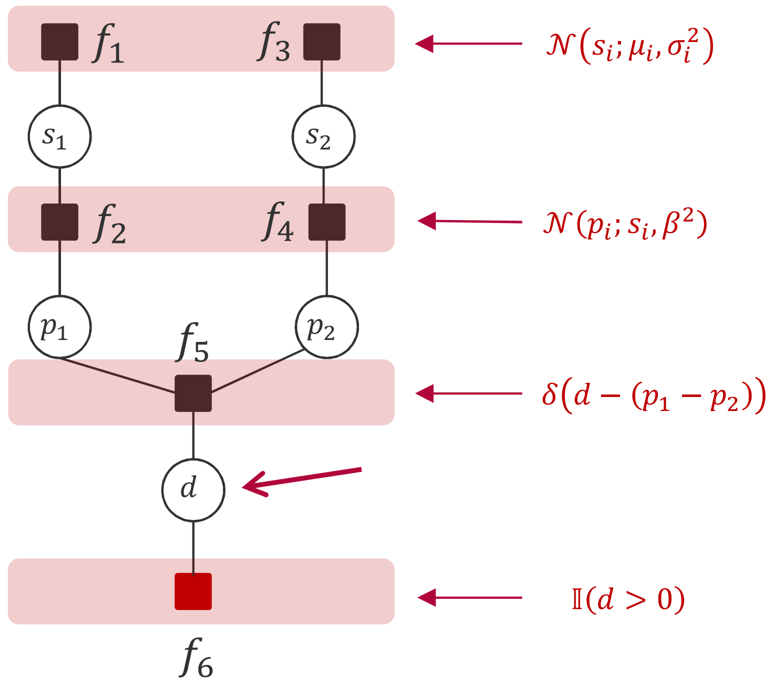
$$m_{p_1\to f_5}(p_1) = p(p_1)/\underbrace{m_{f_5\to p_1}(p_1)}_{uniform} = N(p_1;\mu_1,\beta^2+\sigma_1^2)$$

$$m_{p_2\to f_5}(p_2) = p(p_2)/\underbrace{m_{f_5\to p_2}(p_2)}_{uniform} = N(p_2;\mu_2,\beta^2+\sigma_2^2)$$

$$\boxed{\begin{aligned} &m_{f_5\to d}(d) \qquad (a=1, b=-1)\\ &= N(d;\mu_{m_{p_1\to f_5}(p_1)} - \mu_{m_{p_2\to f_5}(p_2)}, \sigma^2_{m_{p_1\to f_5}(p_1)} + \sigma^2_{m_{p_2\to f_5}(p_2)})\\ &= N(d;\mu_1-\mu_2,\beta^2+\sigma_1^2+\beta^2+\sigma_2^2)\end{aligned}}$$

Marginal for Weighted Sum / Performance Difference

Factor Graph



$$m_{f_5 \to d}(d) = N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)$$

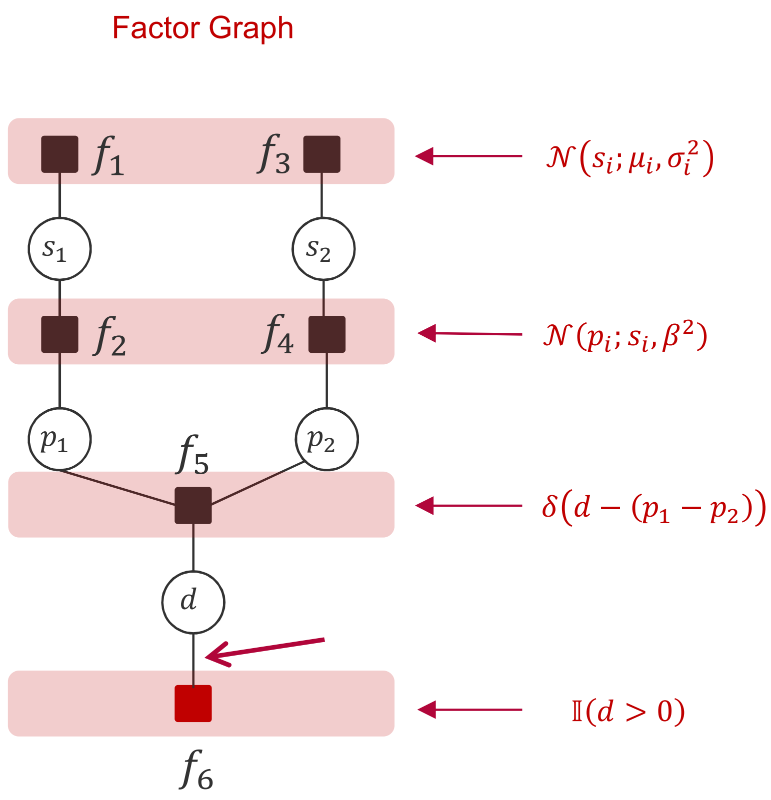
$$p(d) = m_{f_5 \to d}(d) \cdot \underbrace{m_{f_6 \to d}(d)}_{uniform} = m_{f_5 \to d}(d)$$

$$= N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)$$

Tutorial 6

PML SS 2025

Marginal for Weighted Sum / Performance Difference



$$m_{f_5 \to d}(d) = N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)$$

$$p(d) = m_{f_5 \to d}(d) \cdot \underbrace{m_{f_6 \to d}(d)}_{uniform} = m_{f_5 \to d}(d)$$

$$= N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)$$

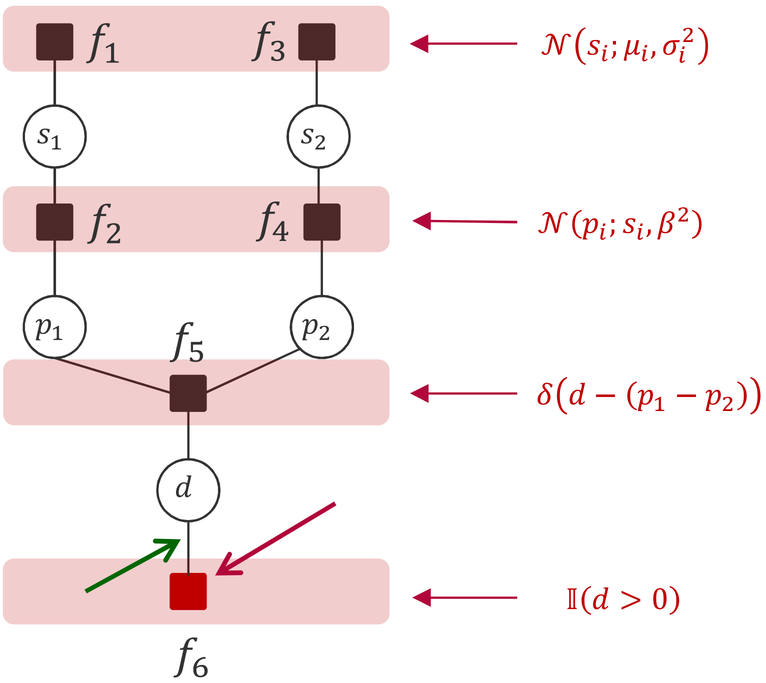
$$\boxed{\begin{aligned} m_{d \to f_6}(d) &= p(d) / \underbrace{m_{f_6 \to d}(d)}_{uniform} \\ &= N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2) \end{aligned}}$$

Tutorial 6

PML SS 2025

Factor of Observed Performance Difference (y=1)

Factor Graph



$$Definition: \; f_6(d) = 1_{\{d>0\}} \quad \Rightarrow \quad y(d) = 2 f_6(d) - 1$$

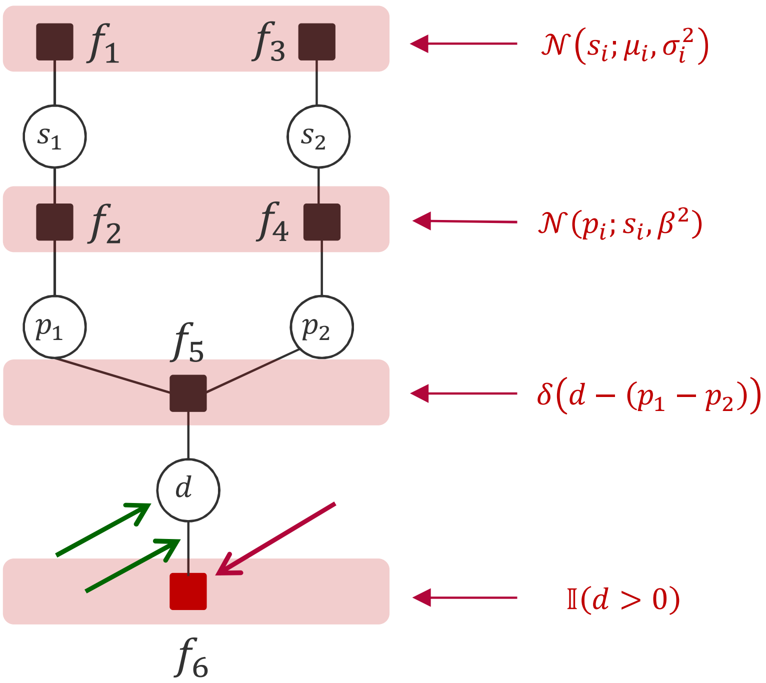
$$Observation: \; f_6(d) = 1_{\{d>0\}} = 1 \quad \Rightarrow \quad d > 0, y = 1$$

$$m_{f_6 \to d}(d) = 1_{\{d>0\}} \qquad m_{d \to f_6}(d) = N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)$$

$$p(d \mid d > 0) = ??$$

Factor of Observed Performance Difference (y=1)

Factor Graph



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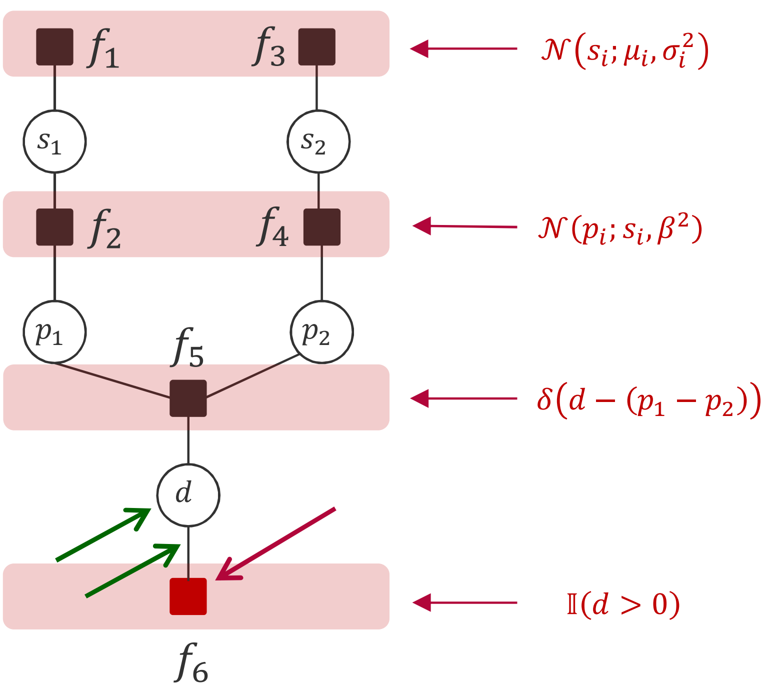
$$p(d \mid d > 0) = m_{f_6 \to d}(d) \cdot m_{f_5 \to d}(d)$$
$$= \underbrace{1_{\{d>0\}}}_{m_{f_6 \to d}(d)} \cdot \underbrace{N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)}_{m_{f_5 \to d}(d)}$$

Factor of Observed Performance Difference (y=1)

Factor Graph



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$$m_{f_6 \to d}(d) = 1_{\{d>0\}} \qquad m_{d \to f_6}(d) = N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)$$

$$\boxed{\begin{aligned} p(d \mid d>0) &= m_{f_6 \to d}(d) \cdot m_{f_5 \to d}(d) \\ &= \underbrace{1_{\{d>0\}}}_{m_{f_6 \to d}(d)} \cdot \underbrace{N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)}_{m_{f_5 \to d}(d)} \end{aligned}}$$

$$m_{f_6 \to d}(d) = \frac{p(d \mid d>0)}{m_{d \to f_6}(d)} = \frac{non-Gaussian}{Gaussian}$$

$$\hat{m}_{f_6 \to d}(d) = \frac{\hat{p}(d \mid d>0)}{m_{d \to f_6}(d)} = \frac{N(d; \mu_{p(d|d>0)}, \sigma^2_{p(d|d>0)})}{N(d; \mu_1 - \mu_2, \sigma_1^2 + 2\beta^2 + \sigma_2^2)}$$

Truncated Gaussian as Approximated Gaussian

- **Problem:** $p(d \mid d>0) = 1_{\{d>0\}} \cdot N(d; \underbrace{\mu_1 - \mu_2}_{\tilde{\mu}}, \underbrace{\sigma_1^2 + 2\beta^2 + \sigma_2^2}_{\tilde{\sigma}^2})$

- **Truncated Gaussians**. *A truncated Gaussian X given by* $p(x) \propto \mathbb{I}(x>0) \cdot \mathcal{N}(x;\mu,\sigma^2)$ *has the following three moments*

$$Z(\mu,\sigma^2) = \int_{-\infty}^{+\infty} p(x)\,dx = 1 - F(0;\mu,\sigma^2)$$

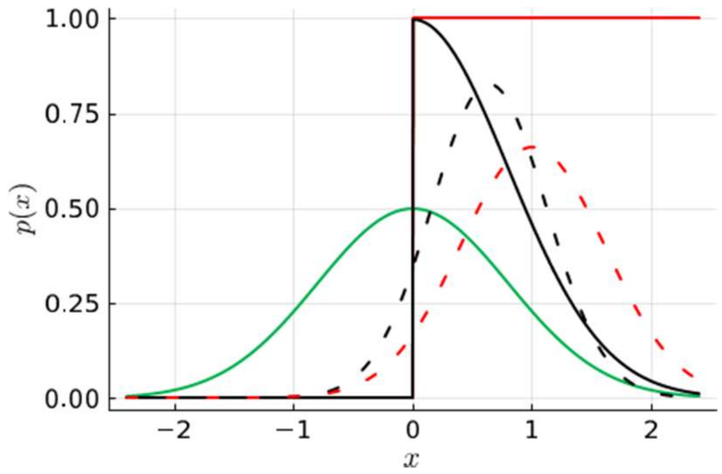
$$E[X] = \int_{-\infty}^{+\infty} x \cdot p(x)\,dx = \mu + \sigma \cdot v\left(\frac{\mu}{\sigma}\right)$$

$$\mathrm{var}[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 \cdot p(x)\,dx = \sigma^2 \cdot \left(1 - w\left(\frac{\mu}{\sigma}\right)\right)$$

where the probit $F(t;\mu,\sigma^2) \coloneqq \int_{-\infty}^{t} \mathcal{N}(x;\mu,\sigma^2)\,dx$ *and*

$$v(t) \coloneqq \frac{\mathcal{N}(t;0,1)}{F(t;0,1)} \longleftarrow$$

$$w(t) \coloneqq v(t) \cdot [v(t) + t]$$



$$Avoid : \frac{0}{0} \quad !!$$

Truncated Gaussian as Approximated Gaussian

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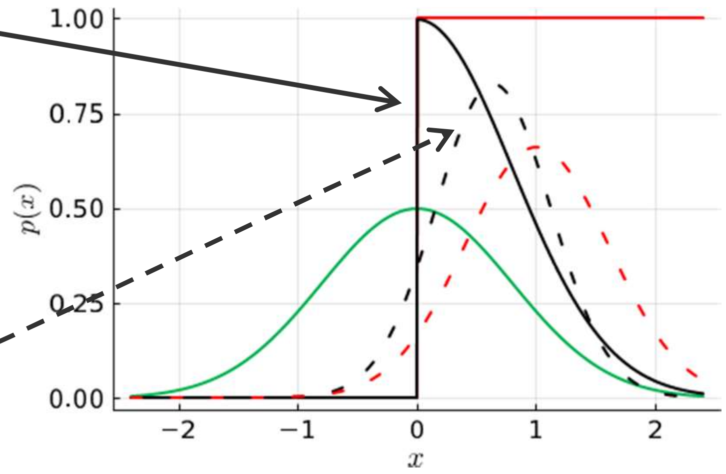

$$Avoid : \frac{0}{0} \quad !!$$

Recap: Convergence of v(t) and w(t) for small t

- **Problem:** $p(d \mid d>0) = 1_{\{d>0\}} \cdot N(d; \underbrace{\mu_1 - \mu_2}_{\tilde{\mu}}, \underbrace{\sigma_1^2 + 2\beta^2 + \sigma_2^2}_{\tilde{\sigma}^2})$

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$$v(t) \coloneqq \boxed{\frac{\mathcal{N}(t;0,1)}{F(t;0,1)}}$$

$$w(t) \coloneqq v(t) \cdot [v(t) + t]$$

$$v(t) = \frac{N(t)}{F(t)} \qquad v'(t) = \frac{-t \cdot N(t) \cdot F(t) - N(t) \cdot N(t)}{F(t)^2}$$

$$= \frac{-N(t)}{F(t)} \cdot \frac{t \cdot F(t) + N(t)}{F(t)}$$

$$= -v(t) \cdot \big(t + v(t)\big) = -w(t)$$

Recap: Convergence of v(t) and w(t) for small t

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$$v(t) := \frac{\mathcal{N}(t;0,1)}{F(t;0,1)}$$ ← Converges to $-t$ as $t\to-\infty$

$$w(t) := v(t)\cdot[v(t)+t]$$ ← Converges to 1 as $t\to-\infty$

$$v(t) = \frac{N(t)}{F(t)} \qquad v'(t) = \frac{-t\cdot N(t)\cdot F(t) - N(t)\cdot N(t)}{F(t)^2}$$

$$= \frac{-N(t)}{F(t)}\cdot\frac{t\cdot F(t)+N(t)}{F(t)}$$

$$= -v(t)\cdot\big(t+v(t)\big) = -w(t)$$

$$\lim_{t\to-\infty} = \frac{v(t)}{-t} = \lim_{t\to-\infty}\frac{N(t)/t}{-F(t)} = \frac{0}{0}$$

$$= \lim_{t\to-\infty}\frac{-t\cdot N(t)/t + N(t)\cdot(-t^{-2})}{-N(t)}$$

$$= 1 - \lim_{t\to-\infty}\frac{N(t)}{t^2} = 1-0 = 1$$

$$\Rightarrow \quad \lim_{t\to-\infty} -v'(t) = 1 = \lim_{t\to-\infty} w(t)$$

Recap: Convergence of v(t) and w(t) for small t

- **Problem:** $p(d \mid d>0) = 1_{\{d>0\}} \cdot N(d; \underbrace{\mu_1 - \mu_2}_{\tilde{\mu}}, \underbrace{\sigma_1^2 + 2\beta^2 + \sigma_2^2}_{\tilde{\sigma}^2})$

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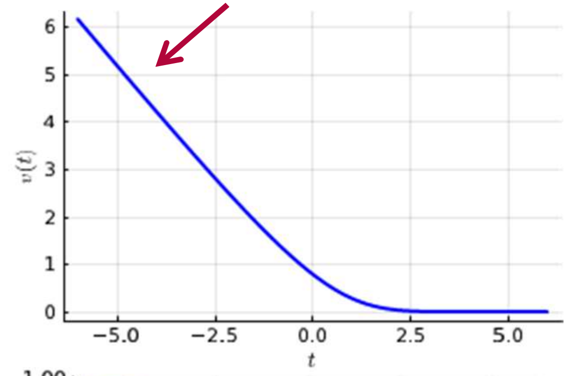
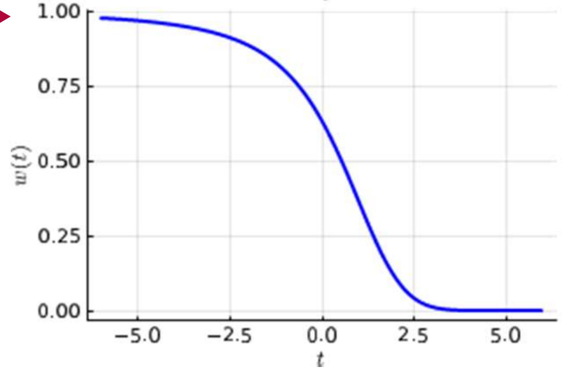
Converges to 1 as $t \to -\infty$

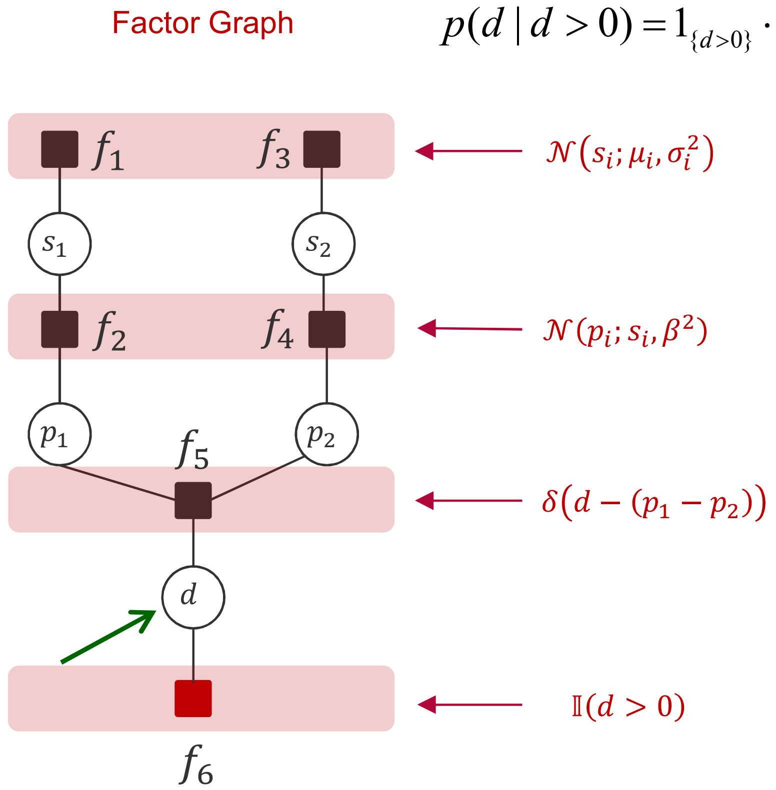




$$\Rightarrow \quad \lim_{t\to-\infty} -v'(t) = 1 = \lim_{t\to-\infty} w(t)$$

Introduction to Probabilistic Machine Learning

Backwards with Approximated Gaussian Message



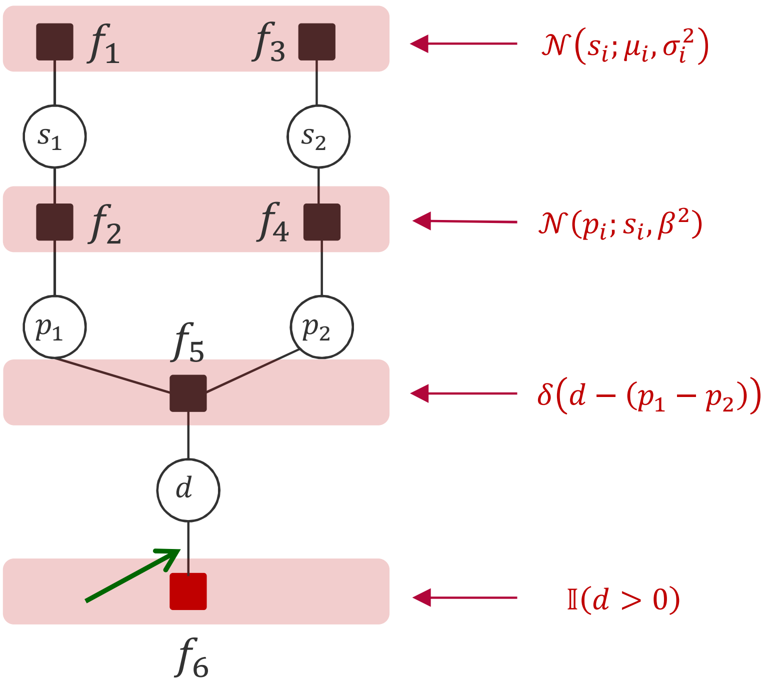
$$p(d \mid d > 0) = 1_{\{d>0\}} \cdot N(d; \underbrace{\mu_1 - \mu_2}_{\tilde{\mu}}, \underbrace{\sigma_1^2 + 2\beta^2 + \sigma_2^2}_{\tilde{\sigma}^2}) \qquad obtain: \; E[X] \quad and \quad V[X]$$

$$\hat{p}(d \mid d > 0) = N(d; E[X], V[X]) \;\approx\; p(d \mid d > 0)$$

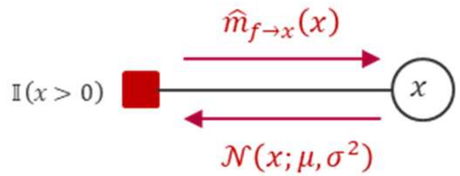
Backwards with Approximated Gaussian Message

Factor Graph

$$p(d \mid d>0) = 1_{\{d>0\}} \cdot N(d; \underbrace{\mu_1 - \mu_2}_{\tilde{\mu}}, \underbrace{\sigma_1^2 + 2\beta^2 + \sigma_2^2}_{\tilde{\sigma}^2}) \qquad obtain: \; E[X] \quad and \quad V[X]$$



$$\hat{p}(d \mid d>0) = N(d; E[X], V[X]) \approx p(d \mid d>0)$$



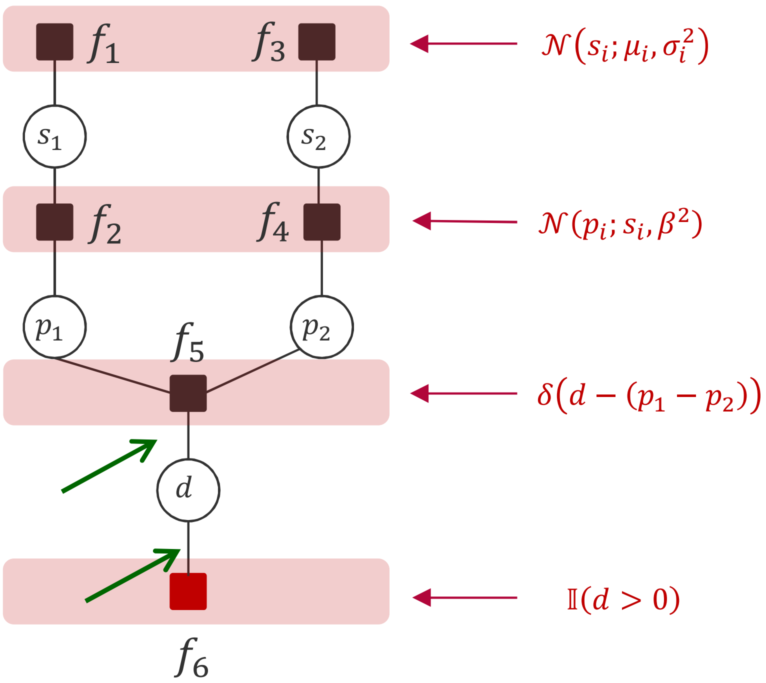
Mean and variance of a truncated Gaussian $\mathcal{N}(x; \mu, \sigma^2)$

$$\boxed{\hat{m}_{f \to x}(x) = \frac{\hat{p}(x)}{m_{x \to f}(x)} = \frac{\mathcal{N}(x; \hat{\mu}, \hat{\sigma}^2)}{\mathcal{N}(x; \mu, \sigma^2)}}$$

$$\hat{m}_{f_6 \to d}(d) = ??$$

Backwards with Approximated Gaussian Message

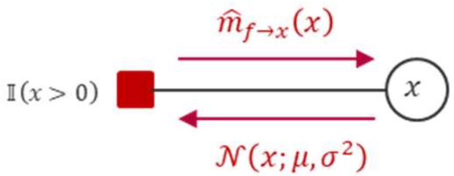
Factor Graph



$$p(d\,|\,d>0)=1_{\{d>0\}}\cdot N(d;\underbrace{\mu_1-\mu_2}_{\tilde{\mu}},\underbrace{\sigma_1^2+2\beta^2+\sigma_2^2}_{\tilde{\sigma}^2})$$

obtain: $E[X]$ *and* $V[X]$

$$\hat{p}(d\,|\,d>0)=N(d;E[X],V[X])\;\approx\;p(d\,|\,d>0)$$



$$\boxed{\hat{m}_{f\to x}(x)=\frac{\hat{p}(x)}{m_{x\to f}(x)}=\frac{\mathcal{N}(x;\hat{\mu},\hat{\sigma}^2)}{\mathcal{N}(x;\mu,\sigma^2)}}$$

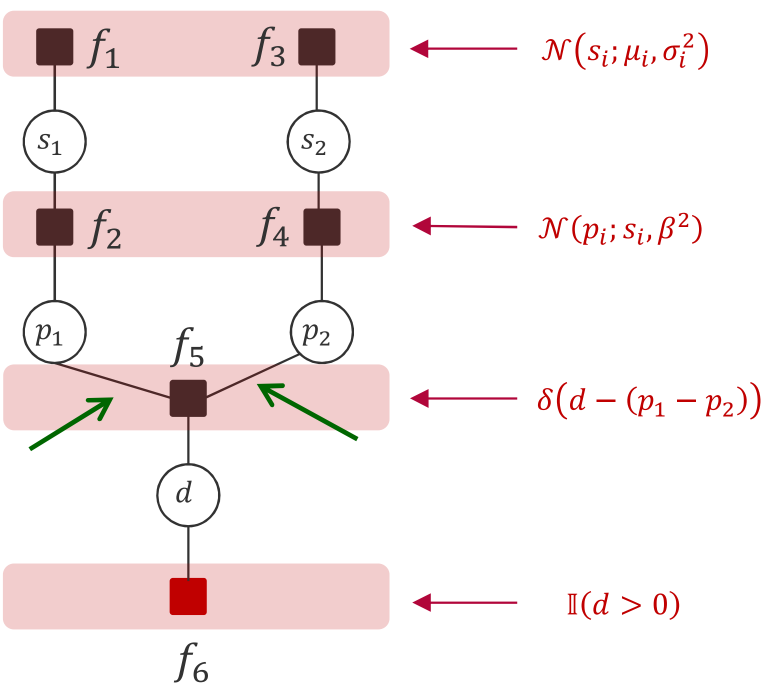
Mean and variance of a truncated Gaussian $\mathcal{N}(x;\mu,\sigma^2)$

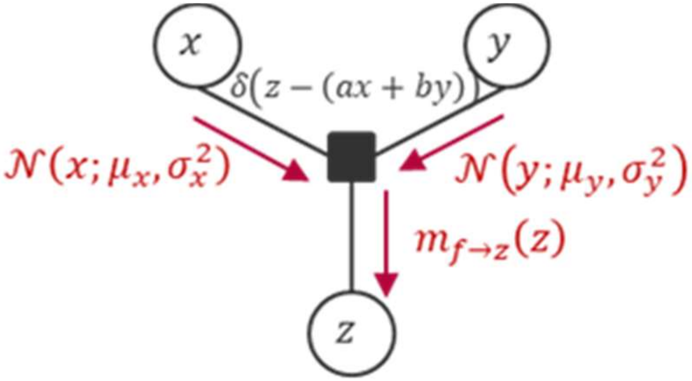
$$\hat{m}_{f_6\to d}(d)=\frac{\hat{p}(d\,|\,d>0)}{m_{d\to f_6}(d)}=\frac{N(d;E[X],V[X])}{N(d;\mu_1-\mu_2,\sigma_1^2+2\beta^2+\sigma_2^2)}$$

$$m_{d\to f_5}(d)=\frac{\hat{p}(d\,|\,d>0)}{m_{f_5\to d}(d)}=\frac{N(d;E[X],V[X])}{N(d;\mu_1-\mu_2,\sigma_1^2+2\beta^2+\sigma_2^2)}$$

Backwards: The Weighted Sum (Any Ideas?)

Factor Graph





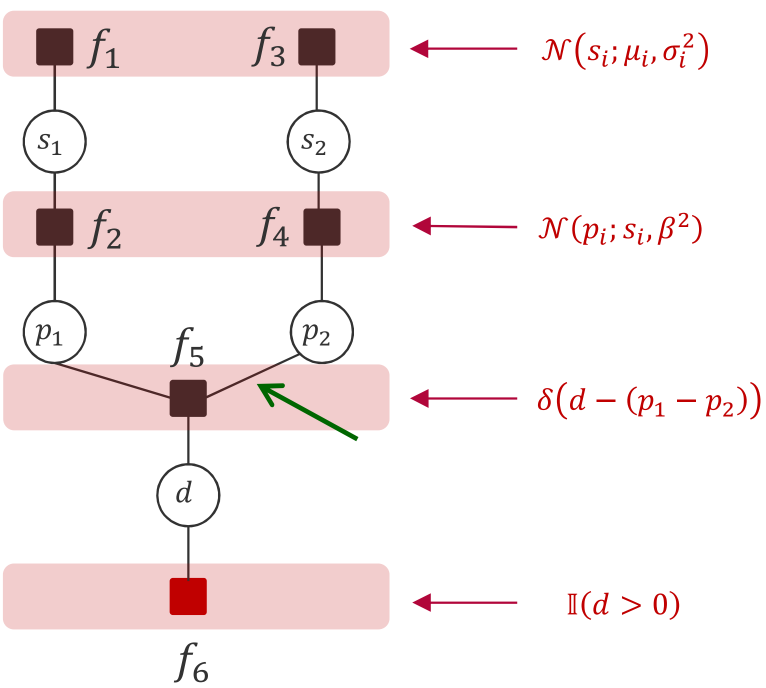
$$\boxed{m_{f \to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)}$$

$$m_{f_5 \to p_1}(p_1) = ??$$

$$m_{f_5 \to p_2}(p_2) = ??$$

Backwards: The Weighted Sum (Approach 1)

Factor Graph

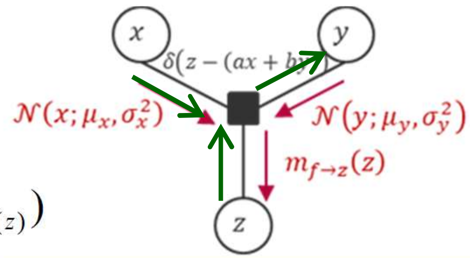


$$\text{given}: \; m_{z\to f}(z), m_{x\to f}(x)$$

$$z = x - y \quad \Leftrightarrow \quad \boxed{\begin{matrix} y = x - z \\ (a = 1, b = -1) \end{matrix}}$$

$$\text{then}: \; m_{f\to y}(y)$$

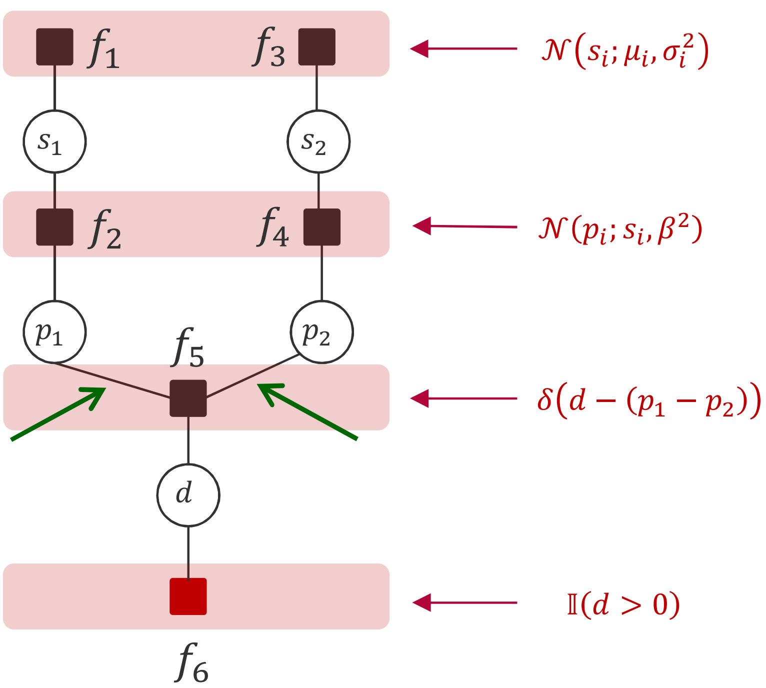
$$= N(y; \mu_{m_{x\to f}(x)} - \mu_{m_{z\to f}(z)}, \sigma^2_{m_{x\to f}(x)} + \sigma^2_{m_{z\to f}(z)})$$



$$m_{f\to z}(z) = \mathcal{N}(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2)$$

Backwards: The Weighted Sum (Approach 1)

Factor Graph

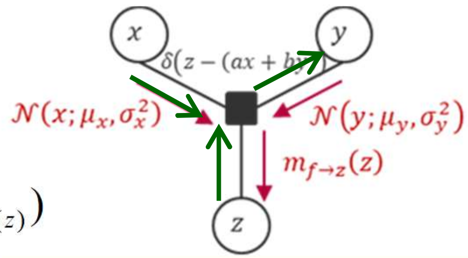


$$given: \; m_{z\to f}(z), m_{x\to f}(x)$$

$$z = x - y \quad \Leftrightarrow \quad \boxed{\begin{gathered} y = x - z \\ (a = 1, b = -1) \end{gathered}}$$

$$then: \; m_{f\to y}(y)$$

$$= N(y; \mu_{m_{x\to f}(x)} - \mu_{m_{z\to f}(z)}, \sigma^2_{m_{x\to f}(x)} + \sigma^2_{m_{z\to f}(z)})$$



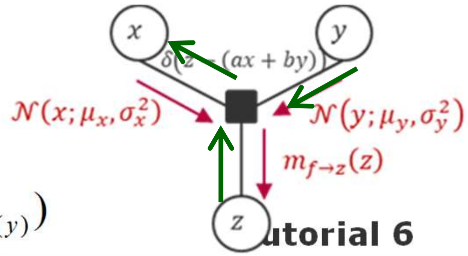
$$m_{f\to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

$$given: \; m_{z\to f}(z), m_{y\to f}(y)$$

$$z = x - y \quad \Leftrightarrow \quad \boxed{\begin{gathered} x = z + y \\ (a = 1, b = 1) \end{gathered}}$$

$$then: \; m_{f\to x}(x)$$

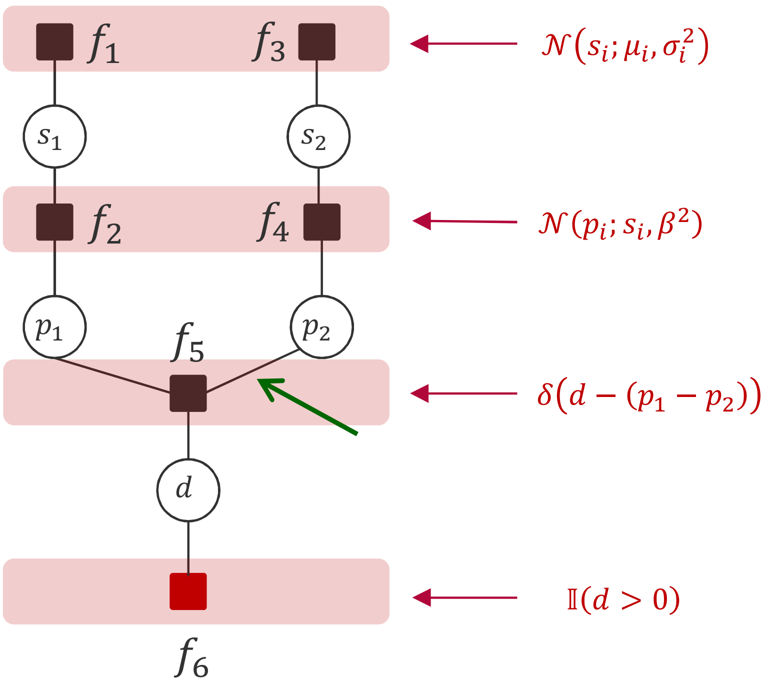
$$= N(x; \mu_{m_{z\to f}(z)} + \mu_{m_{y\to f}(y)}, \sigma^2_{m_{z\to f}(z)} + \sigma^2_{m_{y\to f}(y)})$$



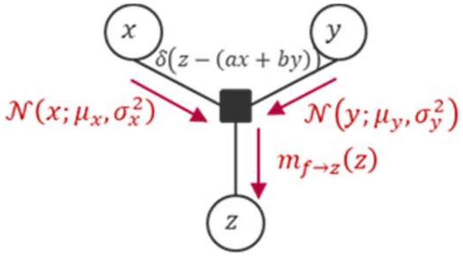
$$m_{f\to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

Backwards: The Weighted Sum (Approach 2)

Factor Graph



$$note: \; E[c \cdot X] = c \cdot E[X] \quad and \quad V[c \cdot X] = c^2 \cdot V[X]$$



$$m_{f \to z}(z) = \mathcal{N}(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2)$$

$given: \; m_{z \to f}(z), m_{x \to f}(x)$

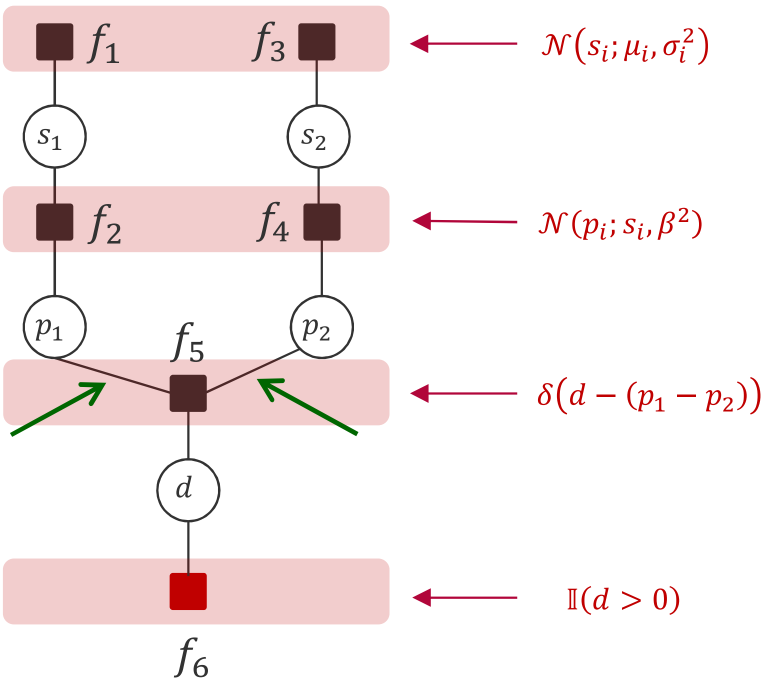
$z = a \cdot x + b \cdot y \quad \Leftrightarrow \quad \boxed{y = (z - a \cdot x) / b \quad (a = 1, b = -1)}$

$then: \; m_{f \to y}(y)$

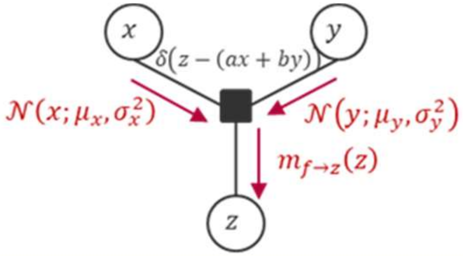
$$= N(y; (\mu_{m_{z \to f}(z)} - a \cdot \mu_{m_{x \to f}(x)}) / b, (\sigma^2_{m_{z \to f}(z)} + a^2 \cdot \sigma^2_{m_{x \to f}(x)}) / b^2)$$

Backwards: The Weighted Sum (Approach 2)

Factor Graph



$$note: \; E[c \cdot X] = c \cdot E[X] \quad and \quad V[c \cdot X] = c^2 \cdot V[X]$$



$$m_{f \to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

$$given: \; m_{z \to f}(z), m_{x \to f}(x)$$

$$z = a \cdot x + b \cdot y \quad \Leftrightarrow \quad \boxed{\begin{array}{c} y = (z - a \cdot x) / b \\ (a = 1, b = -1) \end{array}}$$

$$then: \; m_{f \to y}(y)$$

$$= N(y; (\mu_{m_{z \to f}(z)} - a \cdot \mu_{m_{x \to f}(x)}) / b, (\sigma^2_{m_{z \to f}(z)} + a^2 \cdot \sigma^2_{m_{x \to f}(x)}) / b^2)$$

$$given: \; m_{z \to f}(z), m_{y \to f}(y)$$

$$z = a \cdot x + b \cdot y \quad \Leftrightarrow \quad \boxed{\begin{array}{c} x = (z - b \cdot y) / a \\ (a = 1, b = -1) \end{array}}$$

$$then: \; m_{f \to x}(x)$$

$$= N(x; (\mu_{m_{z \to f}(z)} - b \cdot \mu_{m_{y \to f}(y)}) / a, (\sigma^2_{m_{z \to f}(z)} + b^2 \cdot \sigma^2_{m_{y \to f}(y)}) / a^2)$$

Tutorial 6

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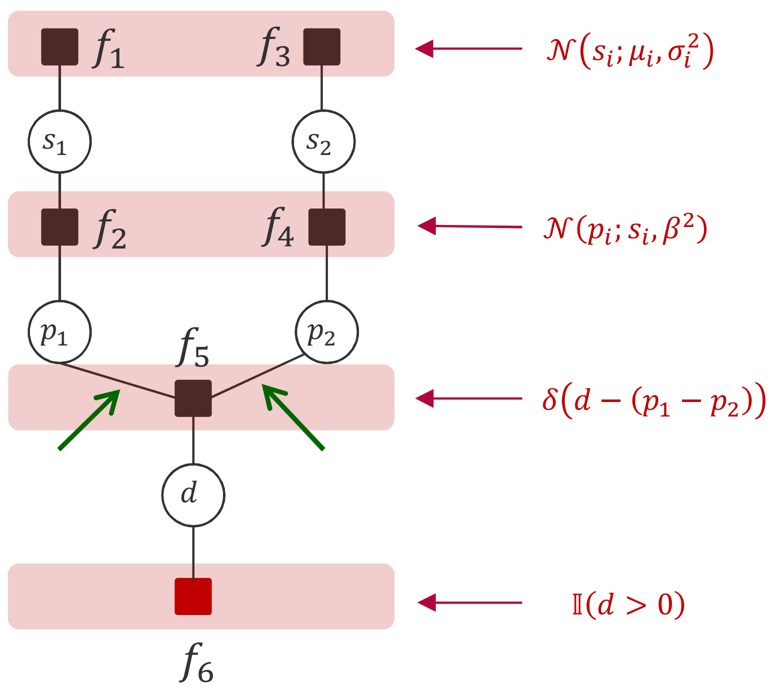
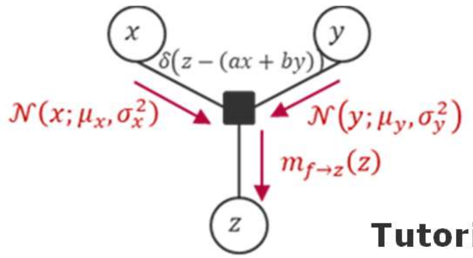
Backwards: The Weighted Sum (Application)

Factor Graph



$$m_{d\to f_5}(d) = \frac{\hat{p}(d \mid d>0)}{m_{f_5\to d}(d)} = \frac{N(d;E[X],V[X])}{N(d;\mu_1-\mu_2,\sigma_1^2+2\beta^2+\sigma_2^2)}$$

$$\boxed{\begin{aligned} m_{f_5\to p_1}(p_1) &= N\left(p_1;\mu_{m_{d\to f_5}(d)}+\mu_{m_{p_2\to f_5}(p_2)},\sigma^2_{m_{d\to f_5}(d)}+\sigma^2_{m_{p_2\to f_5}(p_2)}\right) \\ m_{f_5\to p_2}(p_2) &= N\left(p_2;\mu_{m_{p_1\to f_5}(p_1)}-\mu_{m_{d\to f_5}(d)},\sigma^2_{m_{d\to f_5}(d)}+\sigma^2_{m_{p_1\to f_5}(p_1)}\right) \end{aligned}}$$

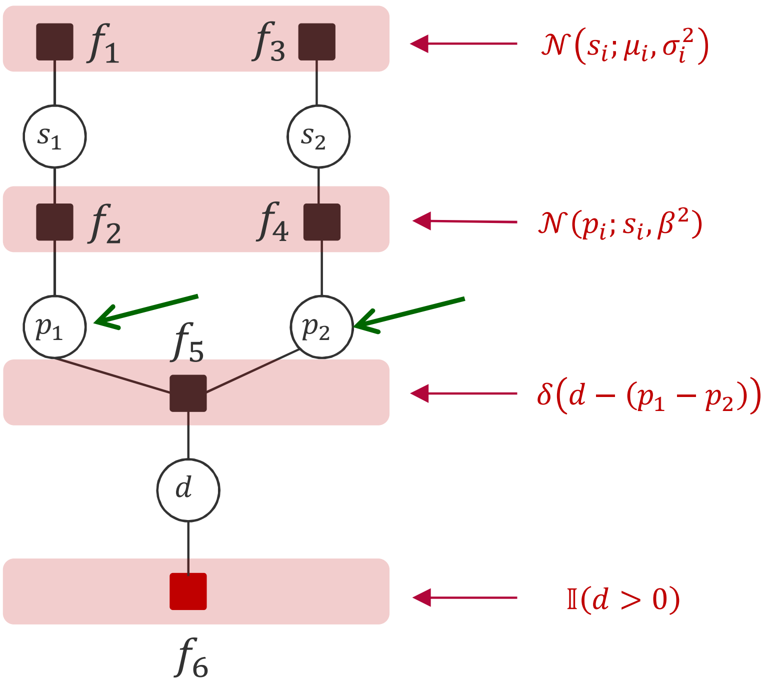


Tutorial 6

$$m_{f\to z}(z) = \mathcal{N}\left(z; a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2\right)$$

Backwards: Marginals of Performances (Player 1 & 2)

Factor Graph

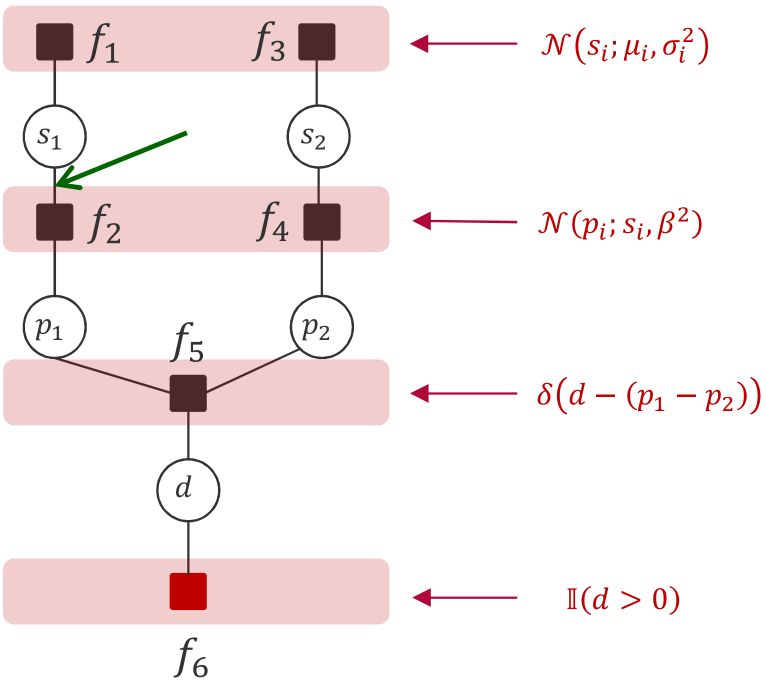


$$m_{f_5 \to p_1}(p_1) = \ldots \qquad m_{f_5 \to p_2}(p_2) = \ldots$$

$$\boxed{\begin{aligned} p(p_1) &= m_{f_2 \to p_1}(p_1) \cdot \underbrace{m_{f_5 \to p_1}(p_1)}_{\text{now Gaussian}} \\ p(p_2) &= m_{f_4 \to p_2}(p_2) \cdot \underbrace{m_{f_5 \to p_2}(p_2)}_{\text{now Gaussian}} \end{aligned}}$$

Backwards: Message to Skill (Player 1)

Factor Graph



$$p(p_1) = m_{f_2\to p_1}(p_1)\cdot \underbrace{m_{f_5\to p_1}(p_1)}_{\text{now Gaussian}}$$

$$m_{f_2\to s_1}(s_1) = \int_{p_1} f_2(s_1,p_1)\cdot \underbrace{m_{p_1\to f_2}(p_1)}_{p(p_1)/m_{f_2\to p_1}(p_1)}\ dp_1$$

$$= \int_{p_1} f_2(s_1,p_1)\cdot \underbrace{\frac{p(p_1)}{m_{f_2\to p_1}(p_1)}}_{\text{apply Division Theorem}}\ dp_1$$

$$= \int_{p_1} \underbrace{N(s_1;p_1,\beta^2)}_{f_2(s_1,p_1)}\cdot \underbrace{N(p_1;\mu_{m_{p_1\to f_2}(p_1)},\sigma^2_{m_{p_1\to f_2}(p_1)})}_{p(s_1)/m_{f_2\to p_1}(p_1)}\ dp_1$$

$$= N(s_1;\mu_{m_{p_1\to f_2}(p_1)},\beta^2+\sigma^2_{m_{p_1\to f_2}(p_1)})$$

Tutorial 6

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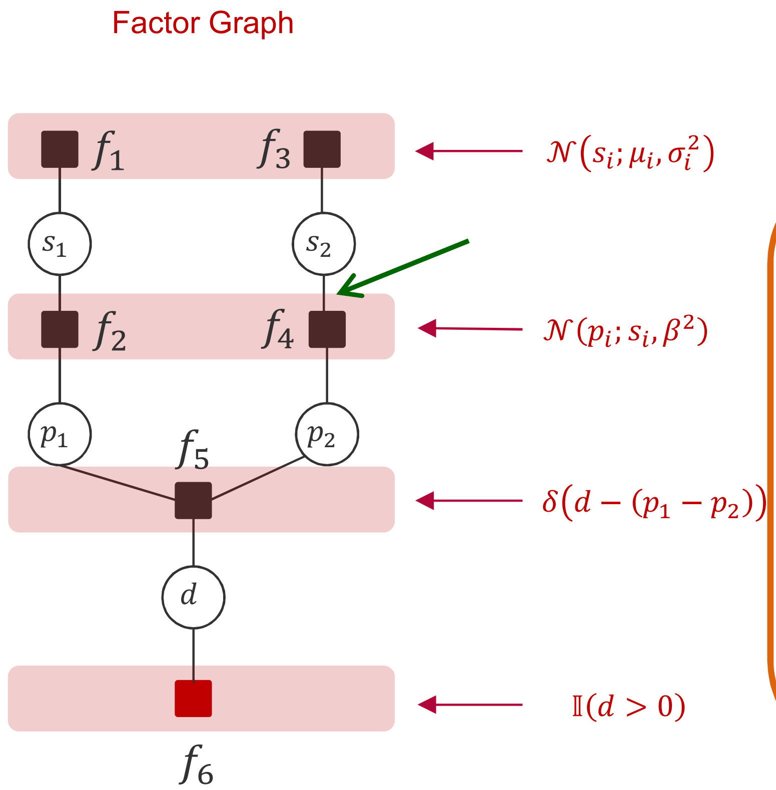
Backwards: Message to Skill (Player 2)



$$p(p_2) = m_{f_4\to p_2}(p_2)\cdot \underbrace{m_{f_5\to p_2}(p_2)}_{\text{now Gaussian}}$$

$$m_{f_4\to s_2}(s_2) = \int_{p_2} f_4(s_2,p_2)\cdot \underbrace{m_{p_2\to f_4}(p_2)}_{p(p_2)/m_{f_4\to p_2}(p_2)}\, dp_2$$

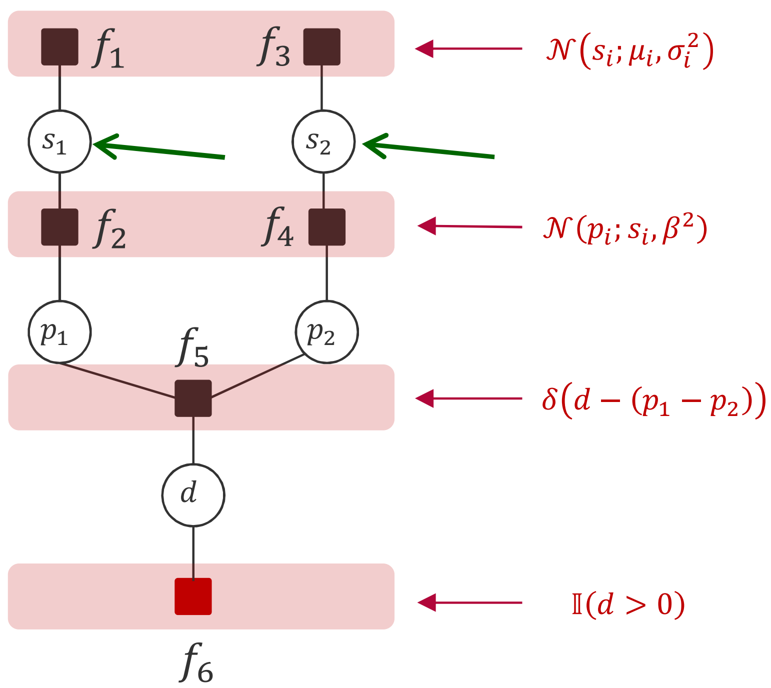
$$= \int_{p_2} f_4(s_2,p_2)\cdot \underbrace{\frac{p(p_2)}{m_{f_4\to p_2}(p_2)}}_{\text{apply Division Theorem}}\, dp_2$$

$$= \int_{p_2} \underbrace{N(s_2;p_2,\beta^2)}_{f_4(s_2,p_2)}\cdot \underbrace{N(p_2;\mu_{m_{p_2\to f_4}(p_2)},\sigma^2_{m_{p_2\to f_4}(p_2)})}_{p(s_2)/m_{f_4\to p_2}(p_2)}\, dp_2$$

$$= N(s_2;\mu_{m_{p_2\to f_4}(p_2)},\beta^2+\sigma^2_{m_{p_2\to f_4}(p_2)})$$

Backwards: Marginals of Skills (Player 1 & 2)

Factor Graph



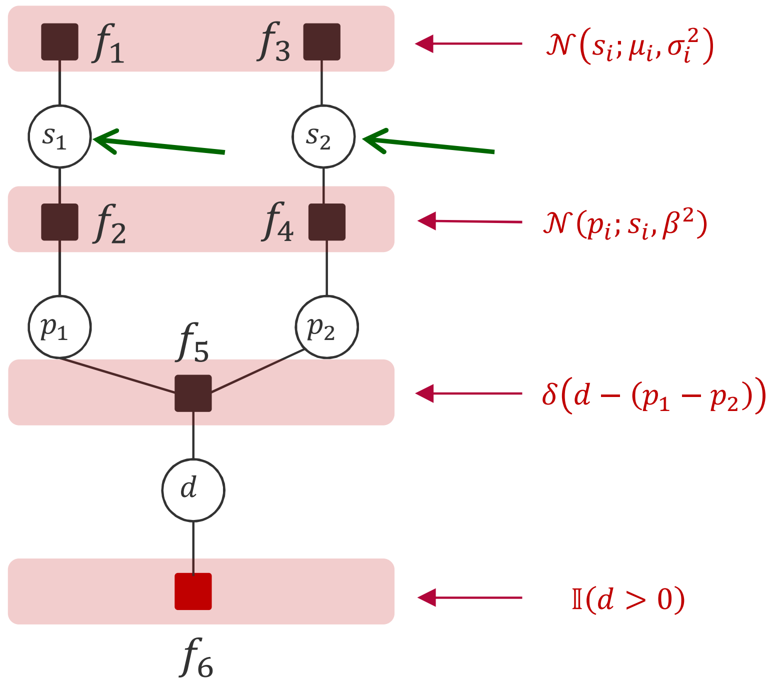
$m_{f_2 \to s_1}(s_1) = \ldots \qquad m_{f_4 \to s_2}(s_2) = \ldots$

$$p(s_1) = m_{f_1 \to s_1}(s_1) \cdot \underbrace{m_{f_2 \to s_1}(s_1)}_{\textit{now Gaussian}}$$

$$p(s_2) = m_{f_3 \to s_2}(s_2) \cdot \underbrace{m_{f_4 \to s_2}(s_2)}_{\textit{now Gaussian}}$$

Backwards: Marginals of Skills (Player 1 & 2)

Factor Graph



$m_{f_2 \to s_1}(s_1) = \ldots \qquad m_{f_4 \to s_2}(s_2) = \ldots$

$$p(s_1) = m_{f_1 \to s_1}(s_1) \cdot \underbrace{m_{f_2 \to s_1}(s_1)}_{\text{now Gaussian}}$$

$$p(s_2) = m_{f_3 \to s_2}(s_2) \cdot \underbrace{m_{f_4 \to s_2}(s_2)}_{\text{now Gaussian}}$$

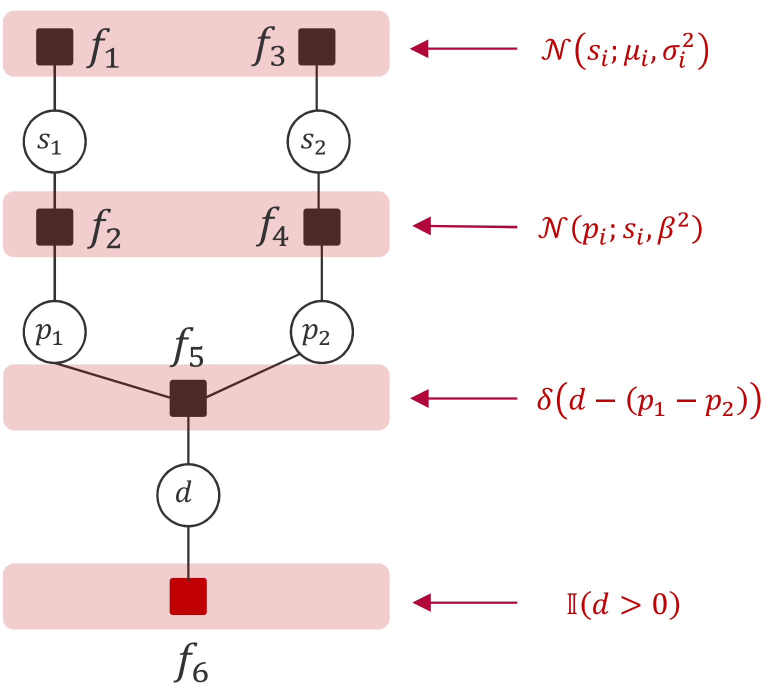
DONE!

Tutorial 6

PML SS 2025

Recap: Multiplication & Division of Gaussians

Factor Graph



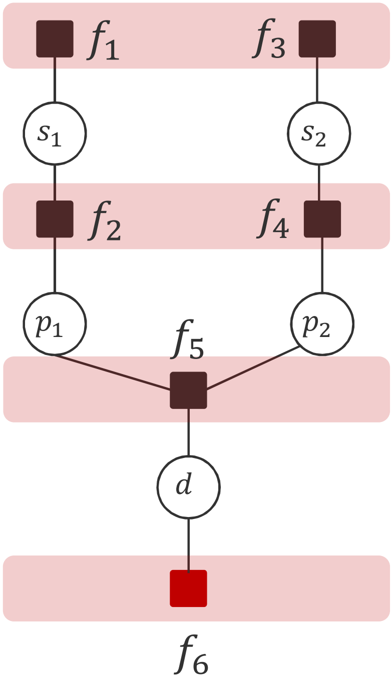
$$N(x;\mu,\sigma^2)=G\left(x;\underbrace{\frac{\mu}{\sigma^2}}_{\tau},\underbrace{\frac{1}{\sigma^2}}_{\rho}\right)\qquad G(x;\tau,\rho)=N\left(x;\underbrace{\frac{\tau}{\rho}}_{\mu},\underbrace{\frac{1}{\rho}}_{\sigma^2}\right)$$

$$N(x;\mu_1,\sigma_1^2)\cdot N(x;\mu_2,\sigma_2^2)=N\left(x;\frac{\mu_1\cdot\sigma_2^2+\mu_2\cdot\sigma_1^2}{\sigma_1^2+\sigma_2^2},\frac{\sigma_1^2\cdot\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)$$

$$\frac{N(x;\mu_1,\sigma_1^2)}{N(x;\mu_2,\sigma_2^2)}=N\left(x;\frac{\underbrace{\frac{\mu_1}{\sigma_1^2}}_{\tau_1}-\underbrace{\frac{\mu_2}{\sigma_2^2}}_{\tau_2}}{\underbrace{\frac{1}{\sigma_1^2}}_{\rho_1}-\underbrace{\frac{1}{\sigma_2^2}}_{\rho_2}},\frac{1}{\underbrace{\frac{1}{\sigma_1^2}}_{\rho_1}-\underbrace{\frac{1}{\sigma_2^2}}_{\rho_2}}\right)$$

Numerical Example for Comparison

Factor Graph

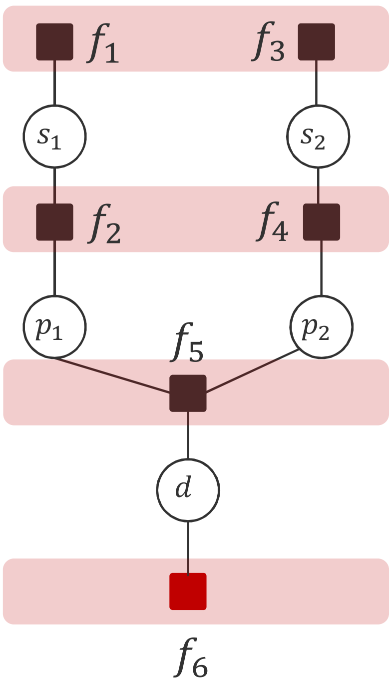


$$\mu_1 = 20, \mu_2 = 18, \sigma_1^2 = 4, \sigma_2^2 = 9, \beta^2 = 1$$

$$p(s_1) = N(\cdot; 20.52, 3.46)$$
$$p(s_2) = N(\cdot; 16.84, 6.25)$$
$$p(p_1) = N(\cdot; 20.65, 4.15)$$
$$p(p_2) = N(\cdot; 16.71, 6.60)$$
$$\hat{p}(d) = N(\cdot; 3.94, 7.36)$$
$$v\left(\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}^2}}\right) = v\left(\frac{2}{\sqrt{15}}\right) = 0.501$$
$$w\left(\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}^2}}\right) = 0.509$$
$$E[X] = 3.94$$
$$V[X] = 7.36$$

Numerical Example for Comparison

Factor Graph



$$\mu_1 = 20, \mu_2 = 18, \sigma_1^2 = 4, \sigma_2^2 = 9, \beta^2 = 1$$

$$p(s_1) = N(\cdot; 20.52, 3.46)$$
$$p(s_2) = N(\cdot; 16.84, 6.25)$$
$$p(p_1) = N(\cdot; 20.65, 4.15)$$
$$p(p_2) = N(\cdot; 16.71, 6.60)$$
$$\hat{p}(d) = N(\cdot; 3.94, 7.36)$$
$$v\left(\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}^2}}\right) = v\left(\frac{2}{\sqrt{15}}\right) = 0.501$$
$$w\left(\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}^2}}\right) = 0.509$$
$$E[X] = 3.94$$
$$V[X] = 7.36$$

$$m_{f_1 \to s_1}(s_1) = N(\cdot; 20, 4)$$
$$m_{f_2 \to p_1}(p_1) = N(\cdot; 20, 5)$$
$$m_{f_3 \to s_2}(s_2) = N(\cdot; 18, 9)$$
$$m_{f_4 \to p_2}(p_2) = N(\cdot; 18, 10)$$
$$m_{f_5 \to d}(p_2) = N(\cdot; 2, 15)$$
$$m_{f_6 \to d}(d) = N(\cdot; 5.81, 14.45)$$
$$m_{f_5 \to p_1}(p_1) = N(\cdot; 23.81, 24.45)$$
$$m_{f_5 \to p_2}(p_2) = N(\cdot; 14.19, 19.45)$$
$$m_{f_2 \to s_1}(s_2) = N(\cdot; 23.81, 25.45)$$
$$m_{f_4 \to s_2}(s_2) = N(\cdot; 14.19, 20.45)$$

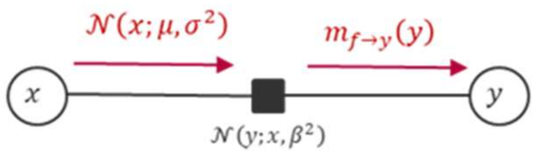
Overview

1. Questions and Updates
2. Recap: Main Concepts of Unit 5
3. Example: TrueSkill 1 vs 1
4. **Hints for Exercise 3 (to be handed in Monday June 2)**

Exercise 3 (until May 27)

- Part I: TrueSkill Models (1 vs 1 + Similar Extensions)
- Part II: Integral of a Product of 2 Gaussian Densities (Gaussian Mean Factor)

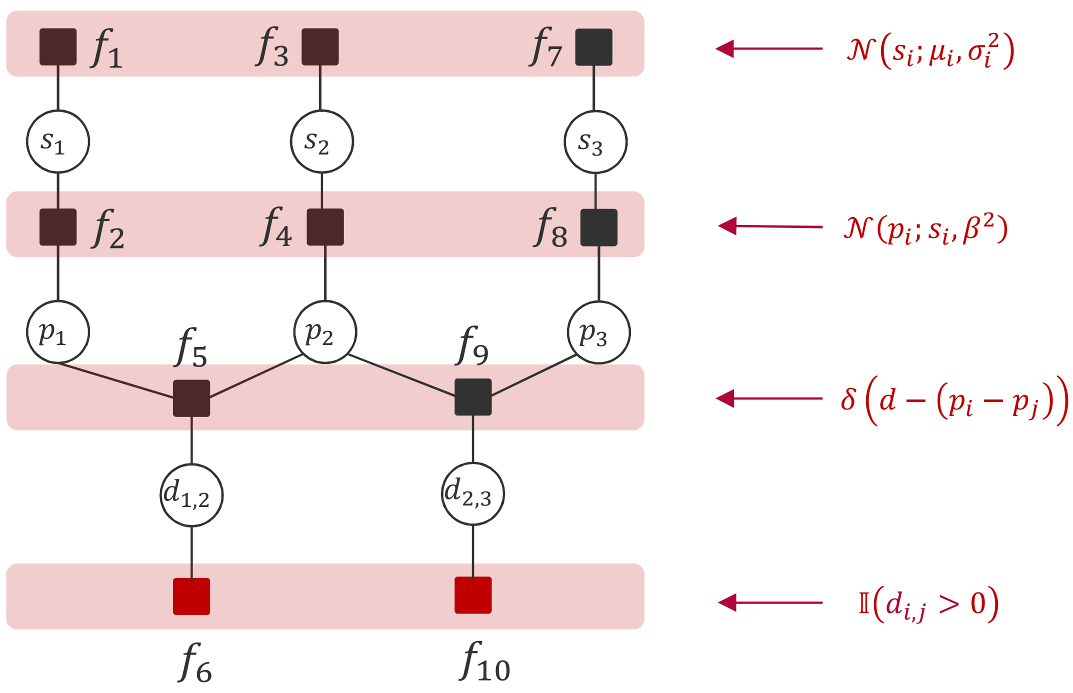
Gaussian Mean Factor



$$m_{f\to y}(y) = \int \mathcal{N}(y;x,\beta^2)\cdot\mathcal{N}(x;\mu,\sigma^2)\,dx = \mathcal{N}(y;\mu,\sigma^2+\beta^2)$$

Bonus Exercise 3 (Factor Graph Part I*, +1 Point)

Factor Graph



Summary

- Recap I: TrueSkill Factor Graph (1 vs 1)
- Recap II: Messages & Marginals (Forward)
- Recap III: Messages & Marginals (Backward)

Hasso Plattner Institut

See you next Week!