Representations

In this note, we will introduce and derive some properties of the *weighted empirical distribution*.

Definition 1 (Weighted empirical distribution). Given a sample $\boldsymbol{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and n weighting coefficients $\boldsymbol{w} := (w_1, \ldots, w_n) \in {\mathbb{R}^+}^n$, a one-dimensional random variable $X \in \mathbb{R}$ is distributed according to the weighted empirical distribution $\mathcal{E}(\cdot; \boldsymbol{x}, \boldsymbol{w})$ with

$$\mathcal{E}(x; \boldsymbol{x}, \boldsymbol{w}) := \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \delta(x - x_i), \quad Z := \sum_{j=1}^{n} w_j, \tag{1}$$

If $w_1 = w_2 = \cdots = w_n = 1$ we simply write

$$\mathcal{E}(x; \boldsymbol{x}) := \mathcal{E}(x; \boldsymbol{x}, \mathbf{1}) . \tag{2}$$

Cumulative Distribution Function

Theorem 1 (Cumlative distribution function). *Let $X \sim \mathcal{E}(\cdot; \boldsymbol{x}, \boldsymbol{w})$ be distributed according to a weighted empirical distribution with points $\boldsymbol{x} = (x_1, \ldots, x_n)$ and weights $\boldsymbol{w} = (w_1, \ldots, w_n)$. Then the cumulative distribution function $F_X(\cdot)$ of X is given by*

$$F_X(t) = \int_{-\infty}^{t} \mathcal{E}(x; \boldsymbol{x}, \boldsymbol{w}) \, \mathrm{d}x = \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \mathbb{I}(x_i \leq t), \quad Z := \sum_{j=1}^{n} w_j . \tag{3}$$

Proof. Using the definition of the Dirac delta, $\delta(x) := \lim_{\sigma^2 \to 0} \mathcal{N}\left(x; 0, \sigma^2\right)$, and (1) we see that

$$\begin{aligned}
F_X(t) &= \int_{-\infty}^{t} \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \lim_{\sigma^2 \to 0} \mathcal{N}\left(x - x_i; 0, \sigma^2\right) \, \mathrm{d}x \\
&= \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \lim_{\sigma^2 \to 0} \left[\int_{-\infty}^{t} \mathcal{N}\left(x; x_i, \sigma^2\right) \, \mathrm{d}x\right] \\
&= \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \lim_{\sigma^2 \to 0} \Phi\left(\frac{t - x_i}{\sigma}\right) \\
&= \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \mathbb{I}(x_i \leq t),
\end{aligned}$$

where the second line follows from exchanging the order or summation and integration, the third line uses the fact that $\Phi\left(t; \mu, \sigma^2\right) = \Phi\left(\frac{t-\mu}{\sigma}\right)$ and the last line exploits that $\lim_{z^2 \to \infty} \Phi(z) = 1$ and $\lim_{z^2 \to -\infty} \Phi(z) = 0$. □

Moments

Theorem 2 (Moments). *Let $X \sim \mathcal{E}(\cdot; \boldsymbol{x}, \boldsymbol{w})$ be distributed according to a weighted empirical distribution with points $\boldsymbol{x} = (x_1, \ldots, x_n)$ and weights $\boldsymbol{w} = (w_1, \ldots, w_n)$.Then we have for the k-th moment $\mathbb{E}\left[X^k\right]$ of X the following*

$$\mathbb{E}\left[X^k\right] = \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot x_i^k, \quad Z := \sum_{j=1}^{n} w_j . \tag{4}$$

Proof. Let $M_k\left(\mu,\sigma^2\right)$ be the k-th moment of the normal distribution with mean μ and variance σ^2. According to [1], we know that

$$M_k\left(\mu,\sigma^2\right) := \int_{-\infty}^{+\infty} x^k \cdot \mathcal{N}\left(x;\mu,\sigma^2\right)\, \mathrm{d}x = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{2i} \cdot (2i-1)!! \cdot \sigma^{2i} \cdot \mu^{k-2i}\,, \tag{5}$$

where $(2i-1)!!$ is the double factorial of $2i-1$ and the product of all odd numbers up to $2i-1$. Using the definition of the expectation, the Dirac delta, and (1), we see that

$$\begin{aligned}
\mathbb{E}\left[X^k\right] &= \int_{-\infty}^{+\infty} x^k \cdot \left[\frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \lim_{\sigma^2\to 0} \mathcal{N}\left(x - x_i; 0, \sigma^2\right)\right] \mathrm{d}x \\
&= \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \lim_{\sigma^2\to 0} \int_{-\infty}^{+\infty} x^k \cdot \mathcal{N}\left(x; x_i, \sigma^2\right)\, \mathrm{d}x \\
&= \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot \lim_{\sigma^2\to 0} M_k\left(x_i, \sigma^2\right) \\
&= \frac{1}{Z} \cdot \sum_{i=1}^{n} w_i \cdot x_i^k\,,
\end{aligned}$$

where we second line follows from exchanging the order of summation and integration, the third line uses the definition of the moment of a normal distribution, and the last line exploits that $\lim_{\sigma^2\to 0} M_k\left(x_i,\sigma^2\right) = M_k\left(x_i, 0\right) = x_i^k$ using (5). □

References

[1] Andreas Winkelbauer. Moments and absolute moments of the normal distribution. *arXiv preprint arXiv:1209.4340*, 2012.