WHY ARE THERE SO MANY RICH PEOPLE? ——A GEOMETRIC BROWNIAN MOTION MODEL TO EXPLAIN HEAVY TAIL IN WEALTH DISTRIBUTION

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ABSTRACT

Wealth distribution is widely observed by economists and policymakers. We often use it to infer the level of affluence and degree of fairness in a society. In most cases, scientists have found that the wealth distribution has a heavy tail compared to the normal distribution. This paper looks at the hidden dynamic of the formation of wealth distribution to explain this heavy tail phenomenon. Geometric Brownian Motion is used to portray the dynamic of personal wealth growth, and then combined with age distribution to derive the expression of wealth distribution. Further, the influence of the death rate, market rate of return, and market volatility on the distribution is considered, and some advice is provided to policymakers.

***K*eywords** Wealth distribution · Geometric Brownian Motion · Power-law distribution

1 Introduction

Normal distribution is widely used when we talk about certain physical quantities. The importance of normal distribution is mainly due to the central limited theorem which shows that the average of samples with finite mean and variance is a random variable that converges to a normal distribution as the number of samples increases. [1] Many physical quantities, such as the heights of a group of people or scores of a class of students, are expected to be the sum of many independent processes. Accordingly, they often have normal distributions.

However, wealth distribution is an exception. It has been demonstrated that there are more rich people than we expect [2]. Compared to the normal distribution (exponential), the tail of wealth distribution is heavier (FIG 1).

Since wealth distribution plays a crucial role in economics, many scientists have proposed hypotheses to explain the so-called heavy tail phenomenon. In this article, we combine the personal wealth growth dynamic and age distribution to provide a possible explanation.

2 Definition

A random variable X is said to have heavy-tailed distribution if its moment generating function is infinite for all $t > 0$, i.e.

$$\int_{-\infty}^{\infty} e^{tx} dF(x) = \infty, \forall t > 0$$

One common example of this is the fat-tailed distribution whose density function $p(x)$ satisfies the power law, i.e.

$$p(x) = Cx^{-\alpha} \, (x \geq x_{\min})$$

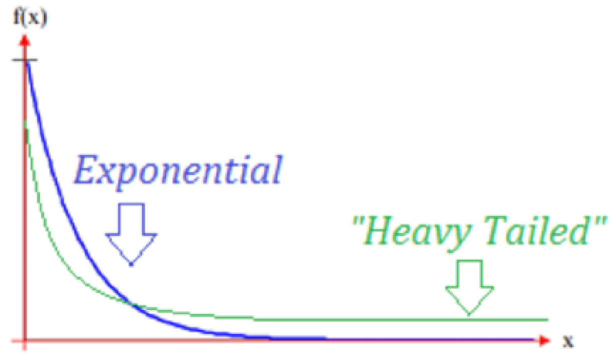

Figure 1: An illustration of the heavy-tailed distribution. The heavy-tailed distribution has a higher probability density than the normal distribution when x is large enough.

where alpha is the exponent, and C is the normalization constant. In this paper, we mainly focus on the power law distribution. We can test the power law by showing the probability density function in a log-log plot. (FIG 2) Since

$$\log p(x) = -\alpha \log x + \text{ Const}$$

the probability density function of power-law distribution in the log-log plot should be a straight line. The exponent α can be estimated by the minus slope of the straight line. In FIG 2, we can see there are lots of quantities that satisfy power-law distribution in nature, and the exponents are between one and three in most cases.

3 Our model

3.1 Hypothesis

Our model is based on two assumptions: one is the wealth distribution dynamic, and the other is the age distribution.

3.2 Wealth distribution dynamic

It is proper to assume that the personal wealth stochastic process $X(t)$ in our economy satisfies the following stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma \mathrm{X}(t)\mathrm{dB}(t)$$

where μ denotes the average rate of return in this economy, σ denotes the volatility, and $\mathrm{B(t)}$ denotes the standard Brownian motion.

To understand this SDE better, one can assume that there are two types of assets in this economy: bonds and stocks. Typical people invest β of their wealth in bonds and $1-\beta$ in stocks. The bonds provide them a certain riskfree rate of return μ_1, so the change of wealth from bonds is

$$\mathrm{d}X_1(t) = \mu_1 \beta X(t)\mathrm{d}t$$

Similarly, assuming the average rate of return and the volatility in the stock market is μ_2 and σ_2 respectively, we can obtain a dynamic of the stock market to obtain the change of wealth from stocks.

$$\mathrm{d}X_2(t) = \mu_2(1-\beta)X(t)\mathrm{d}t + \sigma_2(1-\beta)\mathrm{X}(t)\mathrm{dB}(t)$$

Therefore, we have

$$\mathrm{d}X(t) = \mathrm{d}X_1(t) + \mathrm{d}X_2(t) = (\mu_1\beta + \mu_2(1-\beta))\, X(t)\mathrm{d}t + \sigma_2(1-\beta)X(t)\mathrm{dB}(t)$$

Simply let $\mu = \mu_1\beta + \mu_2(1-\beta)$ and $\sigma = \sigma_2(1-\beta)$, then we can arrive at the previous SDE. Given that everyone has the same initial wealth x^*, we can formulate the Fokker-Planck equation corresponding to this SDE

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\{\mu x p(x,t)\} + \frac{\partial^2}{\partial x^2}\left\{\frac{\sigma^2 x^2}{2} p(x,t)\right\}$$

where $p(x,t)$ denotes the probability density function of personal wealth distribution at time t. With the initial condition $p(x,0) = \delta(x - x^*)$, we can achieve an analytical solution as follows:

$$p(x,t) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{\left\{\log x - \log x^* - \left(\mu - \frac{\sigma^2}{2}\right)t\right\}^2}{2\sigma^2 t}\right]$$

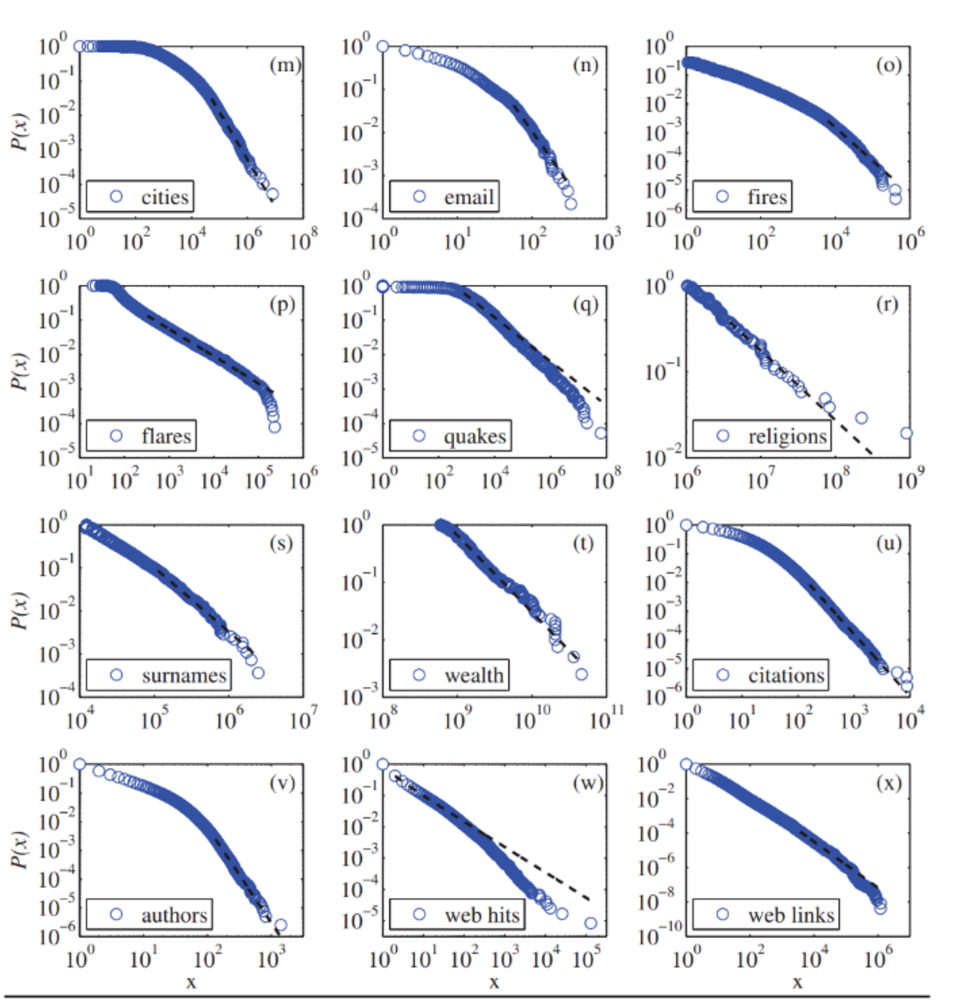

Figure 2: Tests of power law. The figure shows some physical quantities that satisfy the power-law distribution. In log-log plots, the data forms a straight line. The slope of line is the negative exponent of power law distribution. It is noticeable that the exponent is between 1 and 3 in most cases.

3.3 Age distribution

We assume the probability density function of age distribution $L(t)$ in our economy satisfies

$$L(t) = Ce^{-(t/\lambda)^k}$$

where λ is the scale parameter, k is the shape parameter, and C is the normalization constant. We can estimate k and λ using our empirical data. FIG 3 shows the shape of age distribution when $k = 5$. It is clear that the number of people stays stable at first and then experiences a slump when they become older. This is consistent with the real situation.

Another case we should notice is k=1. The distribution becomes an exponential distribution, which means that at any age the death rate is the same. It also provides us with a random death dynamic.

3.4 Numerical solution

Using the previous two assumptions, we can derive the stable distribution of wealth $p(x)$.

$$p(x) = \int_0^\infty L(t)p(x,t)\mathrm{d}t$$

This is the average of the personal wealth distribution weighted by the social age distribution. In general cases, we cannot obtain an analytical solution for this integral. We can use the numerical simulation to achieve the probability

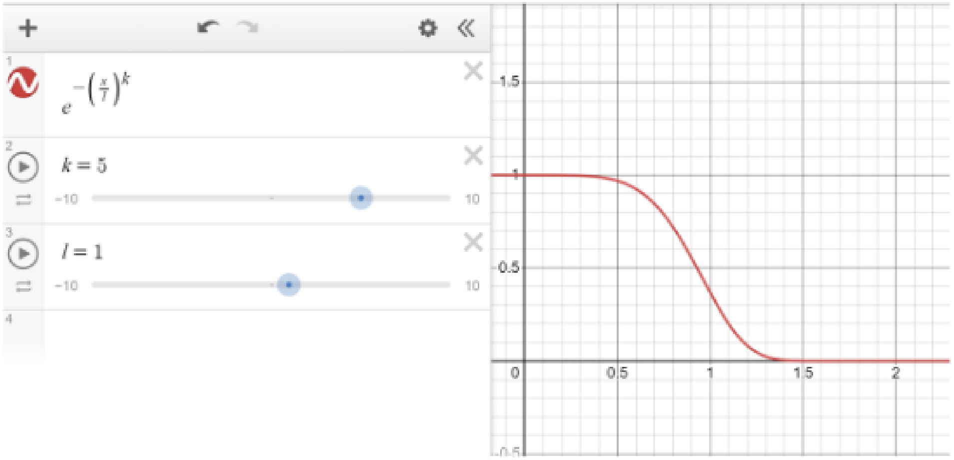

Figure 3: An illustration of the age distribution. The number of people below 70 years old stays stable. It has a dramatic decrease from 70 years old to 100 years old. Then, the percentage of people living more than 100 years old approaches zero. The figure shows that the form of the age distribuition L(t) previously mentioned is aligned with the real situation.

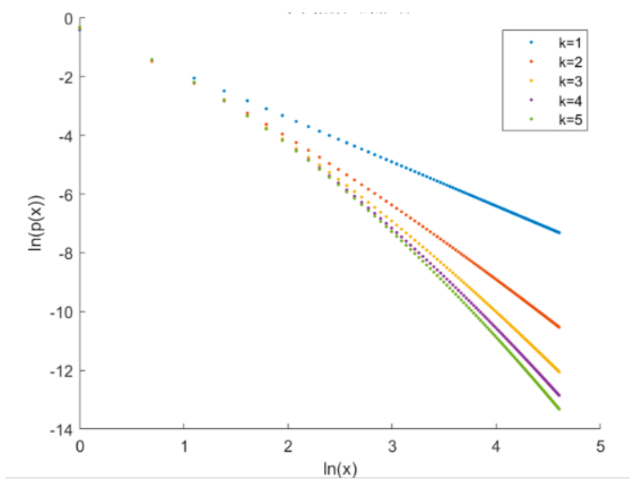

Figure 4: The numerical results of the model. When k=1, the probability density function is a straight line in log-log plot. When k>1, it also shows a heavy tail phenomenon compared to normal distribution.

density function in log-log plot (FIG 4). Although it is not a strictly straight line when $k > 1$, it still shows a heavy tail phenomenon compared to normal distribution.

3.5 An analytical solution in the case of random death

In this subsection, we can provide an analytical form of this integral when $k = 1$ to test the heavy tail phenomenon. When $k = 1$, the age distribution is

$$L(t) = Ce^{-t/\lambda}$$

This means that we have the same death rate at every age. Although this is different from reality, let's explore some intuitive conclusions first. When $\mathrm{k} = 1$, we can derive

$$p(x) = \begin{cases} \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (x^*)^{-\lambda_1} x^{\lambda_1 - 1} & (0 < x < x^*) \\ \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} (x^*)^{-\lambda_2} x^{\lambda_2 - 1} & (x^* \leq x) \end{cases}$$

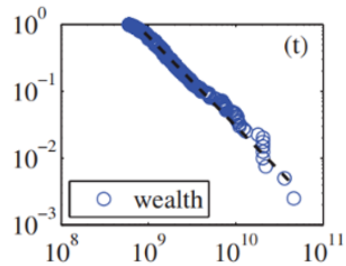

Figure 5: The empirical results of wealth distribution. The figure shows that the exponent of wealth distribution is approximately one.

where

$$\lambda_1 = \frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2} + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\eta}\right) > 0$$

$$\lambda_2 = \frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2} - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\eta}\right) < 0$$

$$\eta = 1/\lambda$$

This is a two side Pareto distribution. We mainly care about the situation when wealth x goes to infinity. We find that the wealth distribution deduced in this model exactly satisfies the power law distribution $p(x) = Cx^{-\alpha}$ with $\alpha = 1-\lambda_2$

3.6 The value of exponent in empirical datasets

Based on the previous discussion, we can establish a GBM model for wealth distribution with the exponent α.

$$\alpha = 1 - \lambda_2 = 1 - \frac{1}{\sigma^2}\left(\mu - \frac{\sigma^2}{2} - \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\eta}\right)$$

$$= \frac{2\frac{\eta}{\sigma^2}}{\frac{\mu}{\sigma^2} - \frac{1}{2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{\eta}{\sigma^2}}} + 1$$

According to results from empirical datasets (FIG 5), the exponent α in wealth distribution is approximately one. We further examine the conditions that guarantee the value of the exponent in our model.

Since $\sigma^2 \ll \mu$ and $\sigma^2 \ll \eta$, the following condition should be satisfied in order to obtain the empirical results.

$$\mu \gg \eta$$

The inequality above means that the average rate of wealth growth should be much larger than the rate of death. This is exactly consistent with reality where $\mu \approx 3\%$ and $\eta \approx 1.25\%$.

4 Potential applications for policymakers

Our model combines the personal wealth growth dynamic and the age distribution, which interprets the formation of the heavy tail phenomenon in wealth distribution. This is useful to give advice to governments and policymakers to control wealth distribution in society.

From the following expression of α, it is clear that α increases as $\frac{\eta}{\sigma^2}$ rises, and α decreases as $\frac{\mu}{\sigma^2}$ rises. This means that a higher death rate leads to a higher α, and a higher rate of return leads to a lower α. As for the volatility in the market, it has a double effect on wealth distribution. In a common situation, we can observe that $\mu \gg \eta$, which means $\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 \gg 2\frac{\eta}{\sigma^2}$. Consequently, the expression of α can be simplified as

$$\alpha = \frac{\eta}{\mu - \frac{1}{2}\sigma^2} + 1$$

Therefore, we can see that the increase in σ^2 will contribute to the increase of α in the general case.

For policymakers, if they want to boost the economy, they may wish to increase the percentage of people who are rich. Based on our model, they need to decrease the value of α to make the distribution flat. They can choose to increase the average rate of return and decrease the volatility in the market through focused policies. On the other hand, if they want to make everyone in society more equal in terms of wealth, it is suitable for them to increase α by lowering the rate of return and increasing volatility so that wealth distribution will be more centralized.

5 Conclusion

In this paper, we explore the hidden dynamic of wealth distribution. We first use GBM to model personal wealth growth. Then we add the age distribution to our model which ensures the stability of the wealth distribution. After that, we provide an analytical solution to our model in a special case. We further analyze the potential applications of our model and provide some suggestions to policymakers.

References

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