Bayesian Persuasion and Information Design

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Abstract: In this paper, we introduce several papers concerning signal sending behavior in a Bayesian game, including Bayesian persuasion with single receiver, Bayesian persuasion with multiple receivers and information design. We introduce the framework of these studies and the techniques authors adopt to solve the problems in the models.

1 Introduction

In this literature review, we summarize and compare the following three papers about the information design on games :Bayesian Persuasion, Bayesian persuasion with multiple senders and rich signal spaces,Information Design.

Consider a court involving a judge, an accuser and a defendant with their attorneys. The accuser's attorney wants the defendant to be convicted, while the defendant's attorney tries to avoid so. Meanwhile, the judge's payoff depends only on weather he makes the correct sentence. We assume that the judge and both attorneys are not sure weather the defendant is guilty, hence the judge will make sentence based on his own opinion about the case and the attorneys try to influence it by sending signals, say calling possible witness. However, as they do not know if the defendant is guilty, the witness may not provide advantageous testimony, so they will have to consider carefully about weather to call a witness, and, if possible, which witness to call.

This case motivates us to consider the role of signal sending in a strategic game. In this paper, we introduce three central theoretical studies concerning signal sending behavior. Kamenica and Gentzkow (2011) first constitute the model of Bayesian persuasion with a single sender choosing the information to reveal and a receiver who takes actions that will affect both agent's payoff. In yet another study, they extend this model to multiple receivers and rich signal spaces (Kamenica and Gentzkow, 2017). In a related study, Taneva (2019) extend the model to the case involving multiple interacting receivers. All these studies change the model crucially and reach some nontrivial new conclusions.

The models concerning signal sending have some important innovations compared with the former literature. In a classic Bayesian game, the agents try to find optimal strategy under incomplete information. However, the information itself is exogenous, in other words, each agent's belief about the world is considered as given and can not be changed in the game. Bayesian persuasion is an attempt to endogenize signal sending in a Bayesian game. Similarly, to reach a desirable result in a Bayesian game, classic mechanism provide different incentive to motivate the agents in a given information environment, while information design induce desirable action by providing optimal signal in a given mechanism.In this sense, we can also consider information design as a dual problem of mechanism design.

The remainder of the paper is organized as follows. We first introduce the studies listed above respectively. Because of the length of the article is limited, we will focus more on the framework of definitions and proofs rather than the technical details. At last, we will summarize briefly and prospect the possible application of these models.

2 Bayesian Persuasion[2]

In a Bayesian game, an individual may be able to influence others' belief in the state of the world by sending signal to them. To model this intuition, the very first problem is how to define belief and signal ,and how to derive the optimal signal, we will introduce how this study solves these problems in this section.

In this paper, the authors consider a Bayesian game with a receiver and a sender. Both agent has a continuous utility function, say $u(a, \omega)$ and $v(a, \omega)$ respectively. The receiver can take an action $a \in A$. The authors further assume $\omega \in \Omega$ is the state of the world and $\mu \in \Delta(\Omega)$ denotes receiver's belief in the state of the world, which is a distribution over Ω. At the beginning, the agents have common knowledge prior μ_0 .

One of the core techniques in the paper is the way to characterize information. Here, a signal consists of a realization space S and a family of conditional distributions $\{\pi(s|\omega)\}_{\omega\in\Omega}$, among which $s \in S$ is the realization of the signal the sender chooses. Characterizing information in this way is quite intuitive. The signal is a kind of observation in the world, which follows a given distribution conditional on different states of the world. The realization of the signal is a information and the sender can choose the signal to send. After the signal is realized, the sender and the receiver are able to share their posterior μ_s depend on information.

For we assume that all individuals are rational, the receiver will choose the action $\hat{a}$ such that $\hat{a}\left(\mu_s\right) = \arg\max_{a\in A} E_{\mu_s}[u(a, \omega)]$. This induces another question, how should the sender choose his signals π to maximize his own utility $\hat{v}(\mu) = E_{\mu_s}[v(\hat{a}, \omega)]$?

To analyze this problem, the authors first calculate μ_s by using Bayesian formula

$$\mu_s(\omega) = \frac{\pi(s \mid \omega)\mu_0(\omega)}{\sum_{\omega'\in\Omega} \pi\left(s \mid \omega'\right)\mu_0\left(\omega'\right)} \tag{1}$$

And they define

$$\tau(\mu) = \sum_{s:\mu_s=\mu} \sum_{\omega'\in\Omega} \pi\left(s \mid \omega'\right)\mu_0\left(\omega'\right) \tag{2}$$

It is clear that τ is induced by μ and represents a "distribution of distribution". Besides, each signal $\{\pi(s|\omega)\}_{\omega\in\Omega}$ induces a corresponding τ. Then we reach another important definition Bayes plausible:

Definition 2.1 *We call a distribution of posteriors is Bayes plausible, if:*

$$\sum_{\mathrm{Supp}(\tau)} \mu\tau(\mu) = \mu_0 \tag{3}$$

To simplify the situation, the authors assume that for any Bayes-plausible distribution of posteriors there is a signal inducing it, and we the sender's utility can be written by the expectation conditional on τ, i.e., $E_\tau \hat{v}(\mu)$. With these two assumptions, the authors use τ to represent signals' influence. Up till now, the authors have formulated the problem into an optimization problem, that is, to maximize $E_\tau \hat{v}(\mu)$, such that the distribution of posteriors is Bayes plausible.

The remaining problem is to find an easy way to derive the solution to such optimization problem. The authors adopt instruments from convex analysis to constitute such algorithm. Here we present the framework of their proof.

Definition 2.2 *We say V is the concave closure of v, if:*

$$V(\mu) \equiv \sup\{z \mid (\mu, z) \in \mathrm{co}(\hat{v})\} \tag{4}$$

where $\mathrm{co}(\hat{v})$ *denotes the convex hull of the graph of* $\hat{v}$

The authors prove that for a given μ_0 there is at least exists one τ such that $\hat{V}(\mu_0) = \hat{v}(\mu_s) > \hat{v}(\mu_0)$ and satisfies $E_\tau(\mu) = \mu_0$. Because τ can be induced by signals, sender can choose what signals to gather by the formula

$$\pi(s \mid \omega) = \mu_s(\omega)\tau\left(\mu_s\right)/\mu_0(\omega) \tag{5}$$

In other words, sender benefits from persuasion if and only if there exists a τ such that

$$E_\tau(\hat{v}(\mu)) > \hat{v}\left(E_\tau(\mu)\right) \tag{6}$$

3 Bayesian persuasion with multiple senders and rich signal spaces[1]

In this paper, Kamenica and Gentzkow extend their former model to a more general case. First, there are multiple senders in the game. Second, each sender can choose from a rich signal space. Third, the authors add more detailed comparison about the equilibrium in the discussion. In the following section, we will focus on these new aspects in the model and the nontrivial new properties, the meaning of other notes remain the same as the former paper.

In the baseline model, there are $n \geq 1$ senders indexed by i, each has continuous utility function $u_i(a, \omega)$. Π is a rich signal space, each $\pi_j \in \Pi$ is a partition of $\Omega \times [0, 1]$.

Suppose X is uniformly distributed over $[0,1]$, x is a value of X, then there is a one to one mapping between $\Omega \times [0,1]$ and π_j. We define this mapping as a realization of the signal. The intuition behind this definition is similar with the former model, given the state ω, there are several possible realization of the signal with its probability.

The authors introduce a lattice structure over Π to allow us to "add up" different signals. We say $\pi_1 \trianglerighteq \pi_2$ if $\forall x \in \pi_1, \exists y \in \pi_2, x \subset y$. Then $(\Pi, \trianglerighteq)$ is a lattice, on which the authors define join $\vee$ and meet $\wedge$.

Like the case with single receiver, each signal π induces a posterior distribution τ, let $< \pi >$ denotes the posterior distribution induced by π. The authors prove that, for all Bayes plausible posterior distributions, there is some π inducing it. This enables us to define informativeness over distributions of posteriors. We say τ is more informative that τ', if $\tau =< \pi >, \tau' =< \pi' >, \pi \trianglerighteq \pi'$, written $\tau \succ \tau'$.

Here the authors also use convex analysis to derive out the equilibrium of this game. Here, the equilibrium is also a BNE where each agent maximize its expected revenue. The authors prove that, a Bayes-plausible distribution of posterior is a equilibrium if and only if $\hat{v}_i(\mu) = V_i(\mu)$, where $V_i(\mu)$ is the cocavification of $\hat{v}_i$. Note that, this result implies that full revelation is always a equilibrium.

4 Information Design[3]

In the previous study, the authors mainly discuss how to analyze the information revelation in games involving single receiver and multiple senders. However, it is sometimes necessary to consider the case where there are more than one receivers with utility functions depending not only on their own behaviors and the true state of the world ,but also on other agents' actions. This case can be rather difficult to analyze, because we must take into account that different receivers may receive different information designed by senders. The choice of information become more complex as well, because apart from its direct influence on the receiver's action, it also influence the higher order beliefs due to the interaction of the multiple agents.

The authors use I to denote the set of players. Each player i has a finite set of actions A_i, and $A = A_1 \times \cdots \times A_N$ denotes the set of action profiles. Moreover, Θ denotes the set of possible world states. Agent i's utility function is $u_i : A \times \Theta \to \mathbb{R}$, while the designer(sender)'s is $V : A \times \Theta \to \mathbb{R}$. In the beginning, all agents and designer share a prior $\psi \in \mathrm{int}(\Delta(\Theta))$. We call $G = \left((A_i, u_i)_{i=1}^N, \psi\right)$ is a basic game.

Subsequently, the author defines the information structure in the case of multiple receivers. $S = \left((T_i)_{i=1}^N, \pi\right)$ is the information structure,where T_i is the set of signals of i and $T = T_1 \times \cdots \times T_N$. π is the signal designer can choose:$\pi : \Theta \to \Delta(T)$, which means that the sender can send different signals to different receivers, which will induce different actions of the agents. The tuple (G, S) is a game of incomplete information. The strategy of i given t_i is defined as $\beta_i : T_i \to \Delta(A_i)$.

The game goes as follows: First, all agents share a common knowledge about the prior distribution of the world ψ. Second, the sender choose to send π, the conditional distribution of signal realization is also common knowledge. Finally, according to received t_i, each agent chooses its best strategy $\beta_i(\cdot \mid t_i) \in \Delta(A_i)$.

Under this setting, the author define the Bayes Nash Equilibrium theoretically. The sets of $\beta_i(a_i \mid t_i)$ should satisfy the condition that any agent i does not have the motivation to change his strategy β_i given others' strategy.

BNE is the distribution $\nu : \Theta \rightarrow \Delta(A)$,which satisfies

$$\nu(a \mid \theta) := \sum_{t \in T} \pi(t \mid \theta) \left(\prod_{j=1}^{N} \beta_j(a_j \mid t_j) \right)$$

Intuitively, this result means a sender can reach a BNE by designing the π.

Then the author uses the previous conclusion $BCE(G) = \cup_S BNE(G, S)$ which is all BNE sender can choose . The optimal BNE is $\nu^* \in \arg\max_{\nu \in BCE(G)} E_\nu[V]$

Then the author shows that we can use a set of direct signal to replace all signals without loss of generality, which means $T_i = A_i$. Intuitively, designer sends recommendations as signals. Using it ,the author finally derives the optimal information structure $S^* = (A, \pi^*)$, where $\pi^*(a \mid \theta) = \nu^*(a \mid \theta)$.

5 Conclusions

In this paper, we summarize several central studies in game theory concerning signal sending. These three models involve single sender-receiver, multiple sender, multiple receiver, respectively. Though differ in detail, these papers all adopt a Bayesian framework to formalize signal sending. These models may apply to many fields in game theory, in economics, industrial organization, law economics and advertising all involve signal sending. We do believe these models have a promising future of application.

References

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