Def. 插值法.

插值节点，$[\min\{x\}_n, \max\{x\}_n]$ 为插值区间

$y=f(x)$，$\{x\}_n \in [a,b]$，$y_i = f(x_i)$，构造 $g(x)$：

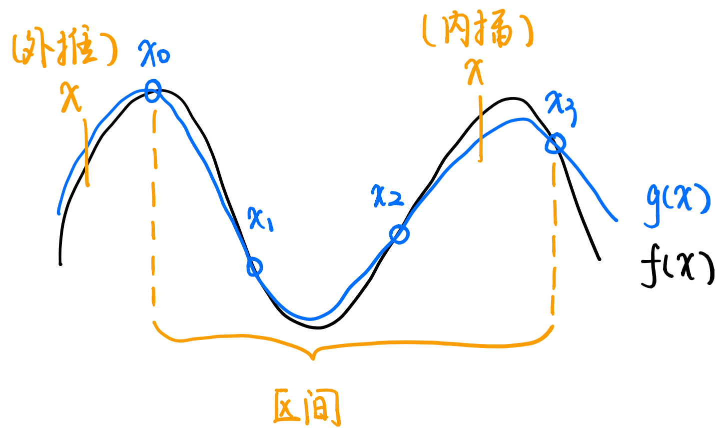
$$g(x_i) = y_i, \quad i=1,2,\cdots n.$$

* $x_0,\cdots x_n$ 有 $n+1$ 个点、

记误差 $r(x) = f(x) - g(x)$. 插值函数.

也叫余项

$f(x) = g(x) + r(x)$ 插值公式



* $g(x)$ 要易于计算，多用多项式，有理分式，三角多项式

本章重点、

§4.2. Lagrange 插值.

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

$$= f(x_0)\, l_0(x) + f(x_1)\, l_1(x) + \cdots + f(x_n)\, l_n(x)$$

其中 $$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}$$

即有 $$l_i(x_k) = \begin{cases} 0, & k \neq i \\ 1, & k = i \end{cases}$$

$P_n(x)$ 即为 Lagrange 插值多项式　　$f(x) \sim P_n(x)$.

* 记 $\omega_{n+1}(x) = (x-x_0)\cdots(x-x_n)$

$$\omega_{n+1}'(x_i) = (x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)$$

则 $P_n(x) = \dfrac{1}{(x-x_0)}\cdot\dfrac{\omega_{n+1}(x)}{\omega_{n+1}'(x_i)}$

e.g. 线性插值　$n=1$ 的 Lagrange 插值.

$$P_1(x) = f(x_0)\frac{(x-x_1)}{(x_0-x_1)} + f(x_1)\frac{(x-x_0)}{(x_1-x_0)}$$

$$= y_0 + \frac{y_1-y_0}{x_1-x_0}(x-x_0) \quad \text{呈线性.}$$

e.g. 二次插值　$n=2$ 的 Lagrange 插值

$$P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \cdots + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

The. $f(x)$ 在 $[a,b]$ n阶连续导，且 (a,b) $n+1$阶有界导.　(Page. 124)

则对 $\forall x\in[a,b]$, $\exists \xi \in (a,b)$:

x与ξ对应?

$$r_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$$

关于x的未知函数.

Prov. $g(t) = f(t) - p_n(t) - \underbrace{k(x)\,\omega_{n+1}(t)}_{r_n(x)}$

$$g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - k(x)\cdot(n+1)! = 0.$$

$$k(x) = \frac{r_n(x)}{\omega_{n+1}(x)}$$

$$\Rightarrow r_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$$

则有

$$f(x) = p_n(x) + r_n(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x).$$

* 带余项的Lagrange插值公式　　ξ与 $x_0,\cdots x_n$ 有关

e.g. 线性插值 $r_1(x) = \frac{1}{2} f''(\xi)(x-a)(x-b)$

$$|r_1(x)| \leq \frac{h^2}{8} |f''(\xi)|$$

e.g. 二次插值 $r_2(x) = \frac{1}{6} f'''(\xi)(x-x_0)(x-x_1)(x-x_2)$.

§4.3. 均差 & Newton 插值公式

基点 $x_0, x_1 \cdots x_n$, $f(x_0), f(x_1) \cdots f(x_n)$.

$$N_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x - x_0}(x - x_0) \quad \text{(线性插值)}$$

$\vdots$

$a_0 \cdots a_n$ 为待定系数

$$N_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

def. 差商 $f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}$ 关于 x_i, x_j 的一阶均差.

expecially. $f[x_i] = f(x_i)$

$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i}$ 关于 x_i, x_j, x_k 的二阶均差

$\vdots$

$$f[x_0, x_1 \cdots x_n] = \frac{f[x_1 \cdots x_n] - f[x_0 \cdots x_{n-1}]}{x_n - x_1} \quad n\text{阶均差}$$

$$= \sum_{i=0}^{n} \frac{f(x_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}$$

The. Newton 均差插值多项式

$$f(x) = N_n(x) + f[x, x_0, x_1 \cdots x_n]\,\omega_{n+1}(x).$$

其中 $N_n(x) = f(x_0) + f[x_0, x_1]\,\omega_1(x) + \cdots + f[x_0, x_1 \cdots x_n]\,\omega_n(x)$

其中 $\omega_k(x) = (x - x_0) \cdots (x - x_{k-1})$

$N_n(x) = P_n(x)$, $f[x, x_0 \cdots x_n] = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}$, $\xi \in (a, b)$.

*

则上述 $f(x) = f(x_0) + \cdots + f[x_0 \cdots x_n]\,\omega_n(x) + \quad \omega_{n+1}(x)$

(带余项的 Newton 插值)

§4.4. 有限点与等距点插值

$\Delta f(x) = f(x + \overset{\text{步长}}{h}) - f(x)$ 一阶前差

$\nabla f(x) = f(x) - f(x - h)$ 一阶后差

$\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$ 一阶中心差

（以上三者）一阶有限差

$\Delta^2 f(x) = \Delta[\Delta f(x)]$

$\Delta^n f(x) = \Delta[\Delta^{n-1} f(x)]$

∇ 与 δ 类似.

$$\Delta^n f(x) = \sum_{i=0}^{n} (-1)^i C_n^i f(x + (n-i)h) \qquad C_n^i = \frac{n(n-1)\cdots(n-i+1)}{i!}$$

⋮

(Page.134).

$y=f(x)$， $x_0<x_1<\cdots<x_n$ ，$f(x_0)\cdots f(x_n)$，有 $x_i-x_{i-1}=h$，

插值点 x： $x=x_0+sh$. 则有：

Newton 前差插值公式： x 位于 x_0 附近：$x=x_0+sh$ $(s>0)$

$$f(x)=f(x_0+sh)=f(x_0)+s\Delta f(x_0)+\frac{s(s-1)}{2!}\Delta^2 f(x_0)$$
$$+\cdots+\frac{s\cdot(s-1)\cdots(s-n+1)}{n!}\Delta^n f(x_0)$$
$$+\boxed{\frac{s(s-1)\cdots(s-n)}{(n+1)!}h^{n+1}f^{(n+1)}(\xi)}\ \text{余项}$$

Newton 后差插值公式： x 位于 x_n 附近 ： $x=x_n+th$ $(t<0)$

$$f(x)=f(x_n+th)=f(x_n)+t\nabla f(x_n)+\frac{t(t+1)}{2!}\nabla^2 f(x_n)$$
$$+\cdots+\frac{t(t+1)\cdots(t+n-1)}{n!}\nabla^n f(x_n)$$
$$+\boxed{\frac{t(t+1)\cdots(t+n)}{(n+1)!}h^{n+1}f^{(n+1)}(x_n)}\ \text{余项}$$

* 与 Newton 均差法相比.

$$\omega_k(x)=(\underset{\downarrow\,sh}{x-x_0})\cdots(\underset{\downarrow\,(s-k+1)h}{x-x_{k-1}})=s(s-1)\cdots(s-\overset{+1}{k})\cdot h^k$$

* 均差与有限差：

$$f[x_0\cdots x_n]=\frac{\Delta^n f(x_0)}{n!\,h^n}=\frac{\nabla^n f(x_n)}{n!\,h^n}$$ (归纳法证明)

Newton 前后差插值公式解题中需要列出前后差表

$$\Delta^2 f(x_i)=\Delta(\Delta f(x_i))=\Delta f(x_i+h)-\Delta f(x_i)$$
$$=\Delta f(x_{i+1})-\Delta f(x_i)$$

§4.5. Hermite 插值公式

在原有的要求上 ($f(x_i)=p(x_i)$), 进一步要求相应位置若干阶导数相等

$$\begin{cases} p'(x_0)=f'(x_0) \cdots p^{(m_0)}(x_0)=f^{(m_0)}(x_0) \\ \vdots \\ p'(x_n)=f'(x_n) \cdots p^{(m_n)}(x_n)=f^{(m_n)}(x_n) \end{cases}$$

讨论一次导的 Hermite 插值，即

$$H_{2n+1}(x): \quad H_{2n+1}(x_i)=f(x_i) \qquad i=0,\cdots n.$$
$$H_{2n+1}'(x_i)=f'(x_i)$$

构造讨论:

$$H_{2n+1}(x)=\sum_{i=0}^{n} f(x_i)A_i(x_i)+\sum_{i=0}^{n} f'(x_i)B_i(x_i)$$

$$\begin{cases} A_i(x_j)=1 \ (\text{当且仅当 } i=j), \quad A_i'(x_j)=0 \\ B_i(x_j)=0, \quad B_i'(x_j)=1 \ (\text{当且仅当 } i=j). \end{cases}$$

确保 $B_i'(x)\neq 0$

$$B_i(x)=\frac{(x-x_0)^2\cdots(x-x_{i-1})^2\boxed{(x-x_i)}(x-x_{i+1})^2\cdots(x-x_n)^2}{(x_i-x_0)^2\cdots(x_i-x_{i-1})^2(x_i-x_{i+1})^2\cdots(x_i-x_n)^2}$$

$$A_i(x)=\left\{1-2(x-x_i)\sum_{\substack{j=0\\j\neq i}}^{n}\frac{1}{x_i-x_j}\right\} l_i^2(x).$$

Hermite 插值多项式:

$$H_{2n+1}(x)=\sum_{i=0}^{n} f(x_i)\underbrace{\boxed{[1-2(x-x_i)l_i'(x_i)]\,l_i^2(x)}}_{A_i(x)}+\sum_{i=0}^{n} f'(x_i)\underbrace{\boxed{(x-x_i)\,l_i^2(x)}}_{B_i(x)}$$

其中： $$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x-x_{i+1})\cdots(x_i-x_n)}$$

$$= \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$l_i'(x_i) = \sum_{\substack{j=0\\ j\neq i}}^{n} \frac{1}{(x_i-x_j)}$$

余项： $$f(x) - H_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\,\omega_{n+1}^2(x).$$

§4.6. 样条插值.

* 前述方法当基点量较大时 高次逼近未必效果好.

↓

划分 $[a,b]$，在每个小区间上用低次插值.

$f(x)$： $a = x_1 < x_2 < \cdots < x_{n+1} = b$

$f(x_1), f(x_2) \cdots f(x_{n+1})$.

e.g. (分段线性) 在 $[x_i, x_{i+1}]$ 上线性插值

$$g_{1,i}(x) = f(x_i) + f[x_i, x_{i+1}](x-x_i)$$

误差： $\max\limits_{a\le x\le b} |f(x) - g_{1,i}(x)| \le \frac{h^2}{8} M_2$ （§4.2）.

$h = \max h_i$ $(h_i = x_{i+1} - x_i)$.

M_2: $\max\limits_{a\le x\le b} |f''(x)|$

e.g. (分段抛物线) 在 $[x_i, x_{i+2}]$ 上抛物线插值

$$g_{2,i}(x) = f(x_i) + f[x_i, x_{i+1}](x-x_i) + f[x_i, x_{i+1}, x_{i+2}](x-x_i)(x-x_{i+1})$$

e.g. (三次样条插值函数)

$a = x_1 < x_2 < \cdots < x_{n+1} = b$. $f(x)$ 2次连续可微

$$S(x)=\begin{cases} S_1(x), & x\in[x_1,x_2] \\ \vdots & \\ S_i(x), & x\in[x_i,x_{i+1}] \\ \vdots & \\ S_n(x), & x\in[x_n,x_{n+1}] \end{cases}$$

三次样条插值函数

不高于3次的多项式

且 $S(x_j)=f(x_j)$

推导过程.

$S''(x)$在每个子区间上都是线性函数:

$$S_i''(x)=m_i\frac{x_{i+1}-x_i}{h_i}+m_{i+1}\frac{x-x_i}{h_i}, \quad x\in[x_i,x_{i+1}].$$

$\int$

$$S_i'(x)=-m_i\frac{(x_{i+1}-x)^2}{2h_i}+m_{i+1}\frac{(x-x_i)^2}{2h_i}+A_i$$

$\int$

$$S_i(x)=m_i\frac{(x_{i+1}-x)^3}{6h_i}+m_{i+1}\frac{(x-x_i)^3}{6h_i}+A_i(x-x_i)+B_i$$

$$\begin{cases} B_i=f_i-m_i\dfrac{h_i^2}{6} \\ A_i=\dfrac{f_{i+1}-f_i}{h_i}-\dfrac{h_i}{6}(m_{i+1}-m_i) \end{cases}$$

未知量只有 m_i 与 m_{i+1} 了