章六. Def. 不定积分：原函数全体. ：$\int f(x)\,dx$

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$; $\int \frac{1}{1+x^2}\,dx = \arctan x + C$.

△ ex. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$.

$\int \sin^2 \frac{x}{2}\,dx = \int \frac{1-\cos x}{2}\,dx = \frac{1}{2}x - \frac{1}{2}\sin x + C$.

ex. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{\cos x}\,d\cos x = -\ln|\cos x| + C$. （换元积分 I）

△ $\int \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int \frac{1}{1+x}\,d\sqrt{x} = 2\arctan\sqrt{x} + C$.

△ $\int \sin mx \cos nx\,dx = \frac{1}{2}\int (\sin(m+n)x + \sin(m-n)x)\,dx$ （和差化积）

ex. $\int \sqrt{a^2-x^2}\,dx = \int a\sin x \cdot (-\sin x)\,dx = a\int \cos^2 x\,dx$ / $-a\int \sin^2 x\,dx$

$= \frac{a^2}{2}(t + \frac{\sin 2t}{2}) + C$. $t = \cos x$（代回）. （换元积分 II）

△ $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$. 替换 $x = a\sec t$, $x = a\tan t$.

ex. $\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$

$\Rightarrow \int e^x \sin x = \frac{1}{2}e^x(\sin x - \cos x) + C$ （分部积分）.

基本积分表. $\int \ln x\,dx = x(\ln x - 1) + C$ $\int \tan x\,dx = -\ln|\cos x| + C$.

（做练习. P260） $\int \operatorname{sh}\,dx = \operatorname{ch} + C$. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C$ $\int \frac{1}{\sqrt{x^2\pm a^2}} = \ln|x + \sqrt{x^2\pm a^2}| + C$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$ $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

Def. Riemann 可积：$f(x)$ 于 $[a,b]$ 有界，$\forall$ 划分 P：$\{x_i\}_{i=0}^{n}$ $a = x_0 < x_1 < \cdots < x_n = b$ ξ_i

$\xi = \max\{x_i - x_{i-1}\}_{i=1}^{n}$, $\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ 存在且与分划无关.（解释 Δx_i）

ex. $D(x)$ 可积性：$\xi_i \in I$ / $\xi_i \in Q$：$I = 0/1$ 不 Riemann 可积.

the 分划加细. 大和不增 小和不减. $\overline{S}(P)$, $\underline{S}(P)$.

the Darboux 定理：$\lim_{\lambda\to 0}\overline{S}(P) = L = \inf\{\overline{S}(P)\}$, $\lim_{\lambda\to 0}\underline{S}(P) = l = \sup\{\underline{S}(P)\}$.

□ P' : $\overline{S}(P') - L < \frac{\varepsilon}{2}$, $\delta = \min\{x'_i - x'_{i-1}\}_{i=1}^{n}$, $\frac{\varepsilon}{2(p-1)(M-m)}$

$\forall P\ (\lambda < \delta)$; P' 与 P 混合：P^* : $\overline{S}(P') \geq \overline{S}(P^*)$

$\overline{S}(P) - L \leq \overline{S}(P) - \overline{S}(P^*) + \overline{S}(P^*) - \overline{S}(P') + \overline{S}(P') - L \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ∎

the. Riemann可积 $\Leftrightarrow \lim\limits_{\lambda\to 0}\bar{S}(p)=\lim\limits_{\lambda\to 0}\underline{S}(p)$.

□. $\Rightarrow$ 记积分值为 I. 由可积：$\forall\varepsilon$, $\exists\delta$, $\lambda<\delta$, $P(\lambda)$有 $\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i - I < \frac{\varepsilon}{2}$

又由 $\bar{S}(p)$ 取法：$\bar{S}(p) - \sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i < \frac{\varepsilon}{2}$，则 $\bar{S}(p) - I < \varepsilon$ …

$\Leftarrow$ $\underline{S}(p) \le \sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i \le \bar{S}(p) \Rightarrow \lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n} f(\xi_i)\Delta x_i = I$ ▣

↳ (i) Riemann 可积 $\Leftrightarrow$ a.e.连续 $\Leftarrow$ 连续. (ii) $[a.b]$上单调必可积.

the. $f(x)$于$[a.b]$可积 $\Leftrightarrow \forall\varepsilon, \exists p$，振幅小于$\varepsilon$. □反证

△ ex. Riemann 函数可积 $R(x)=\{\frac{1}{p}, x=\frac{q}{p}; 1, x=0; 0, x$无理$\}$. ($R(x)$ a.e. 连续).

□. $R(x) > \frac{\varepsilon}{2}$ 有有限不连续点，可积且取 $\lambda < \frac{\varepsilon}{2k}$ (k是非c.点数)…

ex. 可积：振幅不能$\forall$小的区间长度可以$\forall$小.

$w_i \le M(w_i' + w_i'')$

thm. 定积分基本性质. (i) 线性，↳ 改变有限点值不变积分值 (ii) 乘积可积性.

(iii) 保序性 (iv) 绝对可积性 (v) 区间可加性 (vi) 第一中值定理 *

$f(x), g(x) \in R[a.b]$, g不变号，则 $\exists\eta$：$\int_a^b fg\,dx = \eta\int_a^b g\,dx$

△ ex. $f(x)\in C[a.b]$, $f(x)>0$，则 $\frac{1}{b-a}\int_a^b \ln f\,dx \le \ln\left(\frac{1}{b-a}\int_a^b f\,dx\right)$

□. $[a.b]$ n等分：$x_i = a+\frac{i}{n}(b-a)$. 则 $\Delta x_i = \frac{b-a}{n}$，由$\ln x$上凸性与Jensen不等式：

$\sum\limits_{i=1}^{n}\frac{1}{n}\ln f(x_i) \le \ln\left(\sum\limits_{i=1}^{n}\frac{1}{n}f(x_i)\right) \Rightarrow$ … 证毕. ▣ 最后要说明f与$\ln f$可积.

the. (Newton-Leibniz公式) $f(x)\in R[a.b]$, $F(x)=\int_a^x f(t)\,dt$.

(i) $F(x)\in C[a.b]$. (ii) 若 $f\in C[a,b]$，则 $F'(x)=f(x)$. (证明用中值定理)

ex. $\lim\limits_{x\to 0^+}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3} = \lim\limits_{x\to 0^+}\frac{2x\sin x}{3x^2} = \frac{2}{3}$

the 微积分基本定理：$f\in C[a.b]$. 则 $\int_a^b f\,dx = F(b)-F(a)$.

ex. $\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$; $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2$;

$\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) = \int_0^1 \frac{1}{1+x}dx = \ln(x+1)\Big|_0^1 = \ln 2$

the. 积分法：$\int_a^b u'v\,dx = uv - \int_a^b uv'\,dx$ $\left(\int_a^b v\,du = uv - \int_a^b u\,dv\right)$ (分部)

$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(x))\varphi'(x)\,dx$ (换元)

△ ex. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)\,dt \quad (x=\tan t)$

$= \int_0^{\frac{\pi}{4}} \ln\frac{\sin t+\cos t}{\cos t}\,dt = \int_0^{\frac{\pi}{4}} \ln\sqrt{2}\,dt + \int_0^{\frac{\pi}{4}} \ln\cos(\frac{\pi}{4}-t) - \int_0^{\frac{\pi}{4}} \ln\cos t\,dt$

$\int_0^{\frac{\pi}{4}} \ln\cos(\frac{\pi}{4}-t)\,dt = \int_{\frac{\pi}{4}}^{0} \ln\cos u(-du) = \int_0^{\frac{\pi}{4}} \ln\cos u\,du \quad (t=\frac{\pi}{4}-u)$.

△ ex. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin t+\cos t}dt$

$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x+\cos x}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\sin x+\cos x}dx = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\sin(x+\frac{\pi}{4})}dx$

$= \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{\sin x}dx \underset{\triangle}{=} \frac{1}{2\sqrt{2}}\left(\ln\frac{1-\cos x}{\sin x}\right)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})$

$*$

ex. $\{1, \sin x, \cos x, \sin 2x, \cos 2x, \cdots \sin nx, \cos nx, \cdots\}$ 是 $\forall\, 2\pi$ 区间上正交函数列.

Def. (正交函数列) $g_n(x)$. $\forall m,n$. $g_n(x), g_m(x) \in R[a,b]$. 且

$\int_a^b g_m(x)g_n(x)\,dx = \begin{cases} 0, & m\neq n \\ \int_a^b g_n^2(x)\,dx, & m=n. \end{cases}$ 代入证明即可.

(写P31练习)

ex. $\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$. $x=t+\frac{\pi}{2}$. (注意对称性).

$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$. 作差.

计算时的中间形式同定积分

Def. (反常积分) $\int_a^{\infty} f(x)\,dx$, $\cancel{\int_a^b f(x)\,dx \ (f(x)\text{无界})}$ (存在即收敛.不存在发散)

ex. $\int_1^{\infty} \frac{1}{x^p}dx$ 敛散性: $p>1$ 时 $\int_1^{\infty}\frac{1}{x^p}dx = \frac{1}{p-1}$; $p\leq 1$ 时发散

$\int_0^{\infty} \frac{1}{e^{ax}}dx$ 敛散性: $a>0$ 时 $\int_0^{\infty}\frac{1}{e^{ax}}dx = \frac{1}{a}$; $a\leq 0$ 时发散.

ex. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = 2\int_0^{\infty}\frac{1}{1+x^2}dx = \pi \quad (\arctan')$

Def. 瑕积分: $\int_a^b f\,dx = \lim_{\eta\to 0}\int_a^{b-\eta} f\,dx$ (f 在 b 左侧无界) $= \lim_{\xi\to b^-}\int_a^{\xi} f\,dx$

ex. $\int_a^{\infty} f(x)\,dx \xrightarrow{\times} f(x)\to 0\ (x\to\infty)$ 取 $f(x) = \begin{cases} n+1, & x\in[n, n+\frac{1}{n(n+1)^2}] \\ 0, & x\in(n+\frac{1}{n(n+1)^2}, n+1) \end{cases}$ $n=1,2,\cdots$

$\int_a^{\infty} f(x)\,dx = 1$, $f(x)$ 无界.

ex. $\int_0^{\infty} e^{-x}x^n\,dx$, 分部积分得 $n!$ ($n=0$: $\int_0^{\infty}\frac{1}{e^x}dx$, 向上迭代).

$\int_0^1 \ln x\,dx = (x\ln x)\Big|_0^1 - \int_0^1 dx = -1$ (或分部积分).

△ ex. $I_n = \int_0^{\infty} \frac{1}{(x^2+a^2)^n}dx$, ① $I_n = \frac{1}{a^2}\frac{2n-3}{2n-2}I_{n-1}$ ② $I_1 = \frac{\pi}{2a} \Rightarrow I_n = \frac{\pi}{2a^{2n-1}}\frac{(2n-3)!!}{(2n-2)!!}$

$\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = 2\int_0^{\frac{\pi}{4}} \ln\sin 2t\,dt = 2\int_0^{\frac{\pi}{4}} \ln 2\sin t\cos t\,dt$

$= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln\sin t\,dt + \underset{\sim\sim\sim}{2\int_0^{\frac{\pi}{4}} \ln\cos t\,dt} = -2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\sin t\,dt$

$\Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = \frac{\pi}{2}\ln 2 + 2\sim \quad \Rightarrow I = -\frac{\pi}{2}\ln 2.$

$$\int_0^{\infty} \frac{1}{(1+x^2)(1+x^{\alpha})}\,dx = \int_0^1 \sim dx + \underbrace{\int_1^{\infty} \sim dx} = \int_1^0 \frac{-t^{\alpha}}{(1+t^2)(1+t^{\alpha})}\,dt = \int_0^1 \frac{x^{\alpha}}{(1+x^2)(1+x^{\alpha})}\,dx$$

(P368 练习题)

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \arctan x \Big|_0^1 = \frac{\pi}{4}$$

thm. (Cauchy 收敛) $\int_a^{\infty} f\,dx$ 收 $\Leftrightarrow \forall \varepsilon>0, \exists A, \forall x_1, x_2 > A., \left|\int_{x_1}^{x_2} f\,dx\right| < \varepsilon.$

Def. $f \in R[a,\infty)$, $\int_a^{\infty} |f|\,dx$ 收即绝对收敛(绝对可积), else. 条件收敛.

thm. (比较判别法). $0 \le f \le K\varphi(x)$. $\varphi(x)$可积则f可积; f发散.则φ发散

thm. (Cauchy 判别法). $0 \le f \le \frac{K}{x^p}$ $(p>1)$, 则f收敛 (比较判别法特例).

thm. (积分第二中值定理) $f \in C[a.b]$, $g(x) \in \nearrow [a.b]$. 则$\exists \xi$:

$$\int_a^b fg\,dx = g(a)\int_a^{\xi} f\,dx + g(b)\int_{\xi}^b f\,dx$$

thm. (A-D判别法). A: $\int_a^{\infty} f\,dx$ 收敛. $g(x)$ 单调有界
D: $\int_a^{\infty} f\,dx$ 有界 $g(x)$ 单调趋于0. } 任一成立 $\int_a^{\infty} fg\,dx$ 收敛

ex. $\int_1^{\infty} \frac{\sin x}{x}\,dx$ 敛散性. $\frac{1}{x} \to 0$, $\int_a^{\infty} \sin x\,dx$ 有界. ✓

ex. $\int_a^{\infty} f\,dx$ 收敛, 且 $f(x)$在$[a,\infty)$一致连续, 则 $\lim\limits_{x\to\infty} f = 0$. (反证)