1 One Space Variable

1.1 Boundary value problem

The function domain is set to [0, 1]. Firstly we choose several partition points

$$x_i, i = 0, \cdots, N, x_0 = 0, x_N = 1$$

The number of partition points is $N + 1$. Distance between partition points can be fixed or not fixed.

Then we based on Gauss-Legendre quadrature rule choose collocation points. At beginning, for simple, you can ignore how to get these collocation points, just to know how many points in total and how many points each partition has is fine.

The number of collocation point within one partition is $r - 1$, r is the order of polynomial we are going to use. For example, here we choose $N = 4, r = 3$, the partition points and collocation points will look like this.

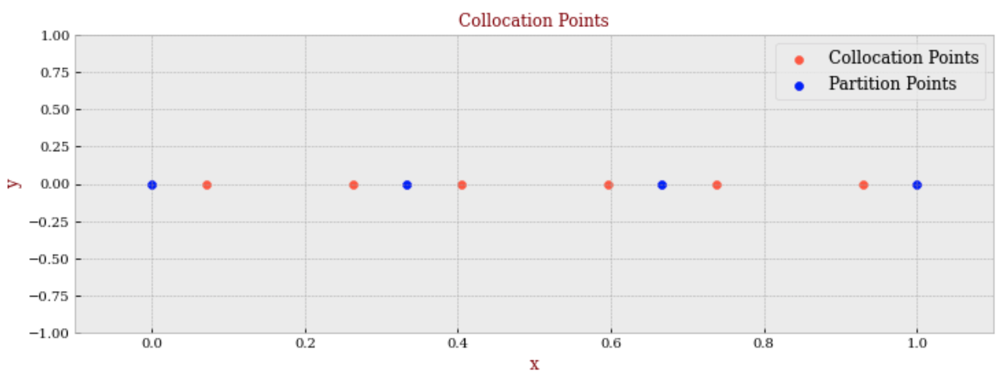

Figure 1: Partition Points and Collocation Points.

Actually as an example, $r = 3$ maybe too much, so the rest part we will use $N = 4, r = 2$.

After getting these collocation points, we can see we have N partitions, in each partition, we assign a r order polynomial. In this example, we have partition points x_0, x_1, x_2, x_3, collocation points $\xi_0, \cdots, \xi_5$ and polynomial in each partition

$$\begin{aligned} &a_{0,0} + a_{0,1}x + a_{0,2}x^2, x \in [x_0, x_1] \\ &a_{1,0} + a_{1,1}x + a_{1,2}x^2, x \in [x_1, x_2] \\ &a_{2,0} + a_{2,1}x + a_{2,2}x^2, x \in [x_2, x_3] \end{aligned} \tag{1}$$

These three polynomials should be C^1 continuous in the connect point, i.e. partition points within the domain. For example, first and second polynomials should be continuous at x_1, we can write two equations

$$\begin{cases} a_{0,0} + a_{0,1}x_1 + a_{0,2}x_1^2 & = a_{1,0} + a_{1,1}x_1 + a_{1,2}x_1^2 \\ 0 + a_{0,1} + 2a_{0,2}x_1 & = 0 + a_{1,1} + 2a_{1,2}x_1 \end{cases} \tag{2}$$

And for two boundary points

$$u(x) = \begin{cases} b_1, x = x_0 \\ b_2, x = x_3 \end{cases} \tag{3}$$

We also have two equations

$$\begin{cases} a_{0,0} + 0 + 0 & = b_1 \\ a_{1,0} + a_{1,1} + a_{1,2} & = b_2 \end{cases} \tag{4}$$

Let's sum up what we get so far. Firstly, we have $(r + 1) \times (N - 1) = 3 \times 3 = 9$ parameters. For boundary condition, we can get 2 equations. For C^1 condition, we can get $(N-2)\times(2) = 2\times 2 = 4$ equations. And now it's time for collocation points, we have $(N-1)\times(r-1) = 3\times 1 = 3$ collocation points. We can create the same number of equations for parameters, here is an example.

If the ODE is

$$\begin{cases} u(x) + u'(x) & = f(x), x \in [0, 1] \\ u(0) & = b_1 \\ u(1) & = b_2 \end{cases} \tag{5}$$

By substitute collocation point ξ_0 into the equation, we get

$$\begin{aligned} u(\xi_0) + u'(\xi_0) &= f(\xi_0) \\ \Longrightarrow a_{0,0} + a_{0,1}\xi_0 + a_{0,2}\xi_0^2 + a_{0,1} + 2a_{0,2}\xi_0 &= f(\xi_0) \\ \Longrightarrow a_{0,0} + a_{0,1}(\xi_0 + 1) + a_{0,2}(\xi_0^2 + 2\xi_0) &= f(\xi_0) \end{aligned} \tag{6}$$

Note that this collocation point is located in the first partition, so we use the polynomial in this partition to fit. Now we know each collocation points can generate one equation about parameters.

Sum everything up, we can get this equation

$$\overbrace{(r + 1) \times (N - 1)}^{\text{Parameters}} = \overbrace{2}^{\text{Boundary}} + \overbrace{(N - 2) \times 2}^{C^1\text{Continuous}} + \overbrace{(N - 1) \times (r - 1)}^{\text{Collocation}} \tag{7}$$

So let's finally write down the matrix of the algebra equation.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \xi_0 + 1 & \xi_0^2 + 2\xi_0 & 0 & 0 & 0 \\ 1 & x_1 & x_1^2 & -1 & -x_1 & -x_1^2 \\ 0 & 1 & 2x_1 & 0 & -1 & -2x_1 \\ 0 & 0 & 0 & 1 & \xi_1 + 1 & \xi_1^2 + 2\xi_1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{0,0} \\ a_{0,1} \\ a_{0,2} \\ a_{1,0} \\ a_{1,1} \\ a_{1,2} \end{bmatrix} = \begin{bmatrix} b_1 \\ f(\xi_0) \\ 0 \\ 0 \\ f(\xi_1) \\ b_2 \end{bmatrix} \tag{8}$$

Solve this, we can get the polynomial as a solution. One example solution is shown here.

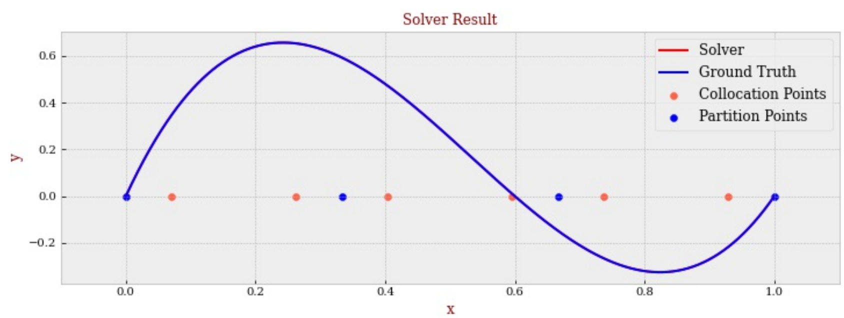

Figure 2: Linear PDE Simulation Solution.