

# Solutions to Vector Subspaces, Span, and Linear Independence

Christian B. Hughes

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## 1 Vector Subspaces

1. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_1 = -a_2 \right\}$$

a subspace of  $\mathbb{R}^2$ ? Justify your answer with rigorous reasoning.

**Solution.** The set  $S$  is a subspace of  $\mathbb{R}^2$ .

Recall that, in order to verify whether  $S$  is a subspace of  $\mathbb{R}^2$  or not, we need to verify whether  $S$  is closed under vector addition and scalar multiplication. Consider the following two arbitrary elements of  $S$ :  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$ . Because these vectors are in  $S$ , and all vectors in  $S$  have the property that  $a_1 = -a_2$ , we can rewrite these vectors as  $\begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} \in S$  and  $\begin{bmatrix} -b_2 \\ b_2 \end{bmatrix} \in S$ . We begin by verifying closure under vector addition:

$$\begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} + \begin{bmatrix} -b_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -a_2 - b_2 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} -(a_2 + b_2) \\ a_2 + b_2 \end{bmatrix},$$

so  $S$  is closed under vector addition. We now verify closure under scalar multiplication. Let  $c \in \mathbb{R}$ . Then

$$c \begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} c \cdot -a_2 \\ c \cdot a_2 \end{bmatrix} = \begin{bmatrix} -(ca_2) \\ ca_2 \end{bmatrix}.$$

Since  $S$  is closed under vector addition and scalar multiplication,  $S$  is a subspace of  $\mathbb{R}^2$ .

2. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = a_1^2 \right\}$$

a subspace of  $\mathbb{R}^2$ ? Justify your answer with rigorous reasoning.

**Solution.** The set  $S$  is *not* a subspace of  $\mathbb{R}^2$ .

Consider  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$ . We can rewrite these as  $\begin{bmatrix} a_1 \\ a_1^2 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_1^2 \end{bmatrix} \in S$  as a consequence of the constraint on members of  $S$ . First, we verify closure under vector addition:

$$\begin{bmatrix} a_1 \\ a_1^2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_1^2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1^2 + b_1^2 \end{bmatrix} \neq \begin{bmatrix} a_1 + b_1 \\ (a_1 + b_1)^2 \end{bmatrix},$$

thus  $S$  fails to be closed under vector addition and is consequently not a subspace of  $\mathbb{R}^2$ .

3. ( $\star$ ) Describe a nontrivial subspace of  $\mathbb{C}^2$  over the field  $\mathbb{R}$  that is not simply the restriction of some entries to 0. Prove it is a subspace.

**Solution.** The purpose of this exercise was primarily to draw attention to the fact that vector spaces of the form  $\mathbb{F}^n$  do not necessarily force the underlying field to be  $\mathbb{F}$ . There are many correct solutions to this problem, but we will share one of the solutions which we perceive to be the most straightforward. Set

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2 : a_2 = 5a_1 \right\}.$$

Taking arbitrary vectors  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$ , we can rewrite them as  $\begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix}$  and  $\begin{bmatrix} b_1 \\ 5b_1 \end{bmatrix}$ . First verifying closure under vector addition, we see

$$\begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ 5b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ 5a_1 + 5b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ 5(a_1 + b_1) \end{bmatrix}.$$

Let  $c \in \mathbb{R}$ . Then

$$c \begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot 5a_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ 5(ca_1) \end{bmatrix}.$$

Thus,  $S$  is a subspace of  $\mathbb{C}^2$  over the field  $\mathbb{R}$ . It is also nontrivial.

4. Choose your favorite vector

$$v = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2.$$

Can you describe a nontrivial vector subspace of  $\mathbb{R}^2$  containing  $v$ ?

**Solution.** Let  $v = \begin{bmatrix} 25 \\ 4 \end{bmatrix}^1$ . The line passing through the origin containing  $v$  is a subspace (one can feel this intuitively, later formally justifying it in this exercise. Alternatively, one could simply Exercise 5.) We can write this line as

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : (a_2 - 4) = \frac{4}{25}(a_1 - 25) \right\}.$$

Simplifying, we have

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = \frac{4}{25}a_1 \right\}.$$

We verify closure under vector addition; take arbitrary vectors  $\begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ \frac{4}{25}b_1 \end{bmatrix} \in S$ . We compute the following:

$$\begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ \frac{4}{25}b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \frac{4}{25}a_1 + \frac{4}{25}b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \frac{4}{25}(a_1 + b_1) \end{bmatrix}.$$

Thus, we have closure under vector addition. Let  $c \in \mathbb{R}$ . Then

$$c \begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot \frac{4}{25}a_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ \frac{4}{25}(ca_1) \end{bmatrix}.$$

Consequently,  $S$  is closed under vector addition and scalar multiplication, and  $S$  contains  $\begin{bmatrix} 25 \\ 4 \end{bmatrix}$ .

5. (★) Show that every line passing through the origin of  $\mathbb{R}^2$  (i.e., containing the zero vector) is a subspace of  $\mathbb{R}^2$ .

**Solution.** First, note that the horizontal axis (i.e., all vectors with zero in their second component) and the vertical axis (i.e., all vectors with zero in their first component) are subspaces of  $\mathbb{R}^2$ . This can be verified the standard way. In all other cases, lines have the form

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = ka_1 \right\}$$

where  $k$  is a fixed integer in  $\mathbb{R} \setminus \{0\}$ . Note that there is no constant term in  $a_2 = ka_1$ , since subspaces necessarily contain the origin. Selecting  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$ , we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ ka_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ kb_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ ka_1 + kb_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ k(a_1 + b_1) \end{bmatrix}.$$

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<sup>1</sup>This procedure will work with any vector in  $\mathbb{R}^2$  with the exception of vectors with 0 in their first component, in which case your subspace is simply the vertical axis in  $\mathbb{R}^2$ .

Thus,  $S$  is closed under vector addition. Let  $c \in \mathbb{R}$ . Then

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cka_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ k(ca_1) \end{bmatrix}.$$

Consequently,  $S$  is closed under both vector addition and scalar multiplication, and is thus a subspace of  $\mathbb{R}^2$ .

6. Why is  $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{Z}_5^2 : a_1, a_2 \text{ are odd modulo } 5 \right\}$  not a subspace of  $\mathbb{Z}_5^2$ ?

**Solution.** A trivial justification would be to take  $\begin{bmatrix} 3 \\ 3 \end{bmatrix} \in S$ . Choosing  $0 \in \mathbb{Z}_5$ , we have

$$0 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \cdot 3 \\ 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S.$$

7. Let  $\mathbb{F}$  be a field (and thus a vector space). What are the subspaces of  $\mathbb{F}$ ?

**Solution.** Given a vector space  $V$  over a field  $\mathbb{F}$ ,  $V$  and  $\{0\}$  are always subspaces of  $V$ . In this case,  $\mathbb{F}$  and  $\{0\}$ ,  $0 \in \mathbb{F}$  are subspaces of  $\mathbb{F}$ . But are there any other subspaces of  $\mathbb{F}$ ? Suppose such a subspace  $U \neq \{0\} \neq \mathbb{F}$  exists. Choose a nonzero  $u \in U$ . For any  $x \in \mathbb{F}$ , we have  $x = (xu)u^{-1}$ . Then, since  $\mathbb{F}$  is closed under scalar multiplication by  $\mathbb{F}$ ,  $x = (xu)u^{-1} \in U$  for all  $x \in \mathbb{F}$ , and thus  $U = \mathbb{F}$ , a contradiction. Consequently, the only two subspaces of a field  $\mathbb{F}$  are  $\{0\}$  and  $\mathbb{F}$  itself.

8. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_3 = a_1 - a_2 \right\}$$

a subspace of  $\mathbb{R}^3$ ? Justify your answer with rigorous reasoning.

**Solution.** The set  $S$  is a vector subspace of  $\mathbb{R}^3$ .

Let  $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$ . We check for closure under vector addition first:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_1 - a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_1 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_1 + b_1 - a_2 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ (a_1 + b_1) - (a_2 + b_2) \end{bmatrix}.$$

Letting  $c \in \mathbb{R}$ , we have

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_1 - a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ c(a_1 - a_2) \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ (ca_1) - (ca_2) \end{bmatrix}.$$

Consequently,  $S$  is closed under both vector addition and scalar multiplication, and is thus a subspace of  $\mathbb{R}^3$ .

9. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_2 = 4a_3, a_1 = -2a_2 \right\}$$

a subspace of  $\mathbb{R}^3$ ? Justify your answer with rigorous reasoning.

**Solution.** The set  $S$  is a vector subspace of  $\mathbb{R}^3$ .

Let  $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$  and  $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$ . We first check for closure under vector addition.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -8a_3 \\ 4a_3 \\ a_3 \end{bmatrix} + \begin{bmatrix} -8b_3 \\ 4b_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} -8(a_3 + b_3) \\ 4(a_3 + b_3) \\ (a_3 + b_3) \end{bmatrix} \in S.$$

Thus, we have closure under vector addition. Let  $c \in \mathbb{R}$ . Then

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \cdot \begin{bmatrix} -8a_3 \\ 4a_3 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8(ca_3) \\ 4(ca_3) \\ (ca_3) \end{bmatrix} \in S$$

.

This gives us closure under scalar multiplication. Since  $S \subset \mathbb{R}^3$  is closed under both vector addition and scalar multiplication, it is a subspace of  $\mathbb{R}^3$ .

10. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_1 + 2a_2 + 4a_3 = 5 \right\}$$

a subspace of  $\mathbb{R}^3$ ? Justify your answer with rigorous reasoning.

**Solution.** The set  $S$  is *not* a subspace of  $\mathbb{R}^3$ .

Let  $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$ . Then choosing  $0 \in \mathbb{R}$ , we have

$$0 \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \cdot a_1 \\ 0 \cdot a_2 \\ 0 \cdot a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

.

However,  $0 + 2 \cdot 0 + 4 \cdot 0 \neq 5$ , so the zero vector is not in  $S$ , and thus  $S$  is not closed under scalar multiplication. Thus,  $S$  is not a subspace of  $\mathbb{R}^3$ .

**Remark 1.1.** Notice that those sets which were subspaces of their respective super-spaces had linear constraints at most (i.e., no conditions on squares, cubes, and so on). This hints at the idea that having at most linear constraints on a set is crucial to that set being a subspace.

Even the linear constraints, however, were not sufficient; our constraints also had to ensure that the relevant geometric figure formed by the subset (i.e., line, plane) passed through the origin.

## 2 Span and Linear Independence

1. Is the list of vectors

$$\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 4 \end{bmatrix} \right\}$$

in  $\mathbb{R}^3$  linearly independent? Justify your response.

**Solution.** The list *is* linearly independent in  $\mathbb{R}^3$ .

Recall that asking whether a list of vectors  $(v_1, \dots, v_n)$  is linearly independent is the same as asking whether the equation

$$(v_1 \quad \cdots \quad v_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution  $c_1, \dots, c_n$  such that at least one of the  $c_i$ ,  $1 \leq i \leq n$  is nonzero. We can determine whether this is the case via Gaussian elimination on the matrix of the  $v_i$ ,  $1 \leq i \leq n$  augmented (appended as the last column) with the zero column vector. In our case, we obtain the following matrix as our augmented matrix:

$$\left( \begin{array}{ccc|c} 5 & 2 & 7 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right).$$

We now apply Gaussian elimination to this matrix:

$$R_1 \leftarrow \frac{R_1}{5} : \left( \begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right)$$

$$R_2 \leftarrow R_2 + 2R_1 : \left( \begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 4/5 & 14/5 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - R_1 : \left( \begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 4/5 & 14/5 & 0 \\ 0 & 13/5 & 13/5 & 0 \end{array} \right)$$

$$R_2 \leftarrow \frac{5R_2}{4} : \left( \begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 13/5 & 13/5 & 0 \end{array} \right)$$

$$R_1 \leftarrow R_1 - \frac{2R_2}{5} : \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 13/5 & 13/5 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - \frac{13R_2}{5} : \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 0 & -13/2 & 0 \end{array} \right)$$

$$R_3 \leftarrow \frac{-2R_3}{13} : \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

$$R_2 \leftarrow R_2 - \frac{7R_3}{2} : \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right).$$

Thus, the only solution to our system of equations is

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Consequently, the list is independent in  $\mathbb{R}^3$ .

2. Is the list of vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$$

in  $\mathbb{Z}_5^3$  linearly independent? Justify your response.

**Solution.** The list is *not* linearly independent (i.e, the list is linearly *dependent*).

Recall that all row operations done here are computed modulo 5. We first form our augmented matrix:

$$\left( \begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array} \right).$$

We apply Gaussian elimination to this augmented matrix:

$$R_1 \leftrightarrow R_2 : \left( \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 1 & 4 & 0 \end{array} \right)$$

$$R_2 \leftarrow R_2 - 2R_1 : \left( \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - 3R_1 : \left( \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - 3R_2 : \left( \begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$R_1 \leftarrow R_1 - 2R_2 : \left( \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Thus, we obtain the solution

$$c_1 + 2c_2 = 0 \pmod{5}, \quad c_3 = 0 \pmod{5}$$

This implies that

$$c_1 = -2c_2 \pmod{5}, \quad c_3 = 0 \pmod{5}.$$

Letting  $c_2 = 2$ ,  $c_1 = -2 \cdot 2 = 3 \cdot 2 = 1 \pmod{5}$ , and  $c_3 = 0$ , so we obtain the solution

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consequently, the list is not linearly independent.

3. Describe the set

$$\text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

What geometric figure in  $\mathbb{R}^3$  does it form?

**Solution.** Recall that the span of a list of vectors is the set of all linear combinations of those vectors. In our case:

$$\text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_1 \cdot \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, c_1, c_2 \in \mathbb{R} \right\}.$$

Thus, all vectors in this span have the form

$$\begin{bmatrix} -2c_1 \\ c_1 \\ -3c_1 \end{bmatrix} + \begin{bmatrix} 5c_2 \\ c_2 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2c_1 + 5c_2 \\ c_1 + c_2 \\ -3c_1 + 2c_2 \end{bmatrix}, c_1, c_2 \in \mathbb{R}.$$

This set forms a plane through the origin, since the two vectors in the list are linearly independent (and thus are not scalar multiples of each other).

4. Is

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

where all vectors are in  $\mathbb{Z}_5^3$ ? Justify your response.

**Solution.** The vector *is* in the span of the given list.

Let  $V$  be a vector space over the field  $\mathbb{F}$ . Recall that asking if a vector  $v \in \text{span}\{v_1, \dots, v_n\}$ , where  $v_i \in V$ ,  $1 \leq i \leq n$ , is equivalent to asking whether there exist scalars  $c_i \in \mathbb{F}$ ,  $1 \leq i \leq n$  such that

$$v = c_1 v_1 + \dots + c_n v_n.$$

We can express this as an equation involving matrices.

$$\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = v.$$

Once again, Gaussian elimination proves to be an excellent tool to determine whether such solutions exist, and the values of such solutions should they exist. We form our augmented matrix:

$$\left( \begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 3 & 4 \\ 4 & 2 & 1 \end{array} \right).$$

We now apply Gaussian elimination. Recall that all computations are done modulo 5.

$$R_1 \leftrightarrow R_2 : \left( \begin{array}{cc|c} 1 & 3 & 4 \\ 3 & 3 & 0 \\ 4 & 2 & 1 \end{array} \right).$$

$$R_2 \leftarrow R_2 - 3R_1 : \left( \begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 4 & 3 \\ 4 & 2 & 1 \end{array} \right).$$

$$R_3 \leftarrow R_3 - 4R_1 : \left( \begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{array} \right).$$

$$R_2 \leftarrow 4R_2 : \left( \begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right).$$

$$R_1 \leftarrow R_1 - 3R_2 : \left( \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right).$$

So indeed,

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

5. Identify a vector  $v$  in  $\mathbb{R}^3$  such that when  $v$  is added to the list

$$\left\{ \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} \right\},$$

the resultant list is linearly independent.

**Solution.** An easy choice is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This is because linear dependence captures the relationship of proportionality, but

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is not proportional to } \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix}.$$

That is, there is no pair of scalars  $c_1, c_2 \in \mathbb{R}$  such that

$$c_1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

6. Is

$$\begin{bmatrix} 4 \\ 3 \\ -5 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} -2 \\ -6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} \right\},$$

where all vectors are in  $\mathbb{R}^4$ ? Justify your response.

**Solution.** The given vector is *not* in the span of the given list of vectors.

We first form the corresponding augmented matrix:

$$\left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ -6 & 1 & 5 & 3 \\ 2 & -9 & -3 & -5 \\ -2 & 2 & 1 & 1 \end{array} \right).$$

We now apply Gaussian elimination to this matrix:

$$R_2 \leftarrow R_2 - R_3 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 2 & -9 & -3 & -5 \\ -2 & 2 & 1 & 1 \end{array} \right)$$

$$R_3 \leftarrow R_3 - R_1 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & -9 & -1 & -1 \\ -2 & 2 & 1 & 1 \end{array} \right)$$

$$R_4 \leftarrow R_4 - R_1 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & -9 & -1 & -1 \\ 0 & 2 & -1 & -3 \end{array} \right)$$

$$R_3 \leftarrow R_3 + 9R_2 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -10 & -82 \\ 0 & 2 & -1 & -3 \end{array} \right)$$

$$R_4 \leftarrow R_4 - 2R_2 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -10 & -82 \\ 0 & 0 & 1 & 15 \end{array} \right)$$

$$R_3 \leftrightarrow R_4 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array} \right)$$

$$R_2 \leftarrow R_2 + R_3 : \left( \begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array} \right)$$

$$R_1 \leftarrow R_1 - 2R_3 : \left( \begin{array}{ccc|c} -2 & 0 & 0 & -26 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array} \right)$$

$$R_4 \leftarrow R_4 + 10R_3 : \left( \begin{array}{ccc|c} -2 & 0 & 0 & -26 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 68 \end{array} \right).$$

The fourth row of the last matrix gives us  $0 = 68$ , which implies that the system is inconsistent. Thus

$$\begin{bmatrix} 4 \\ 3 \\ -5 \\ 1 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} -2 \\ -6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

7. (★) Show that, for any list  $\{v_1, \dots, v_n\}$  of vectors in a vector space  $V$ ,  $\text{span}\{v_1, \dots, v_n\}$  is the smallest (inclusion-wise) subspace of  $V$  containing  $\{v_1, \dots, v_n\}$ .

**Solution.** Let  $U \subset V$  be a subspace with  $v_1, \dots, v_n \in U$ . Then  $cv_i \in U$ ,  $1 \leq i \leq n$  for all  $c \in \mathbb{F}$ , where  $\mathbb{F}$  is the underlying field of  $V$ , since subspaces are closed under scalar multiplication. Subspaces are also closed under addition, so

$$c_1v_1 + \dots + c_nv_n \in U, \quad c_i \in \mathbb{R}, \quad 1 \leq i \leq n.$$

Consequently,  $\text{span}\{v_1, \dots, v_n\} \subset U$ . Thus, any subspace of  $V$  containing  $\{v_1, \dots, v_n\}$  contains  $\text{span}\{v_1, \dots, v_n\}$ .

8. Give an example of a list of vectors in  $\mathbb{R}^3$  which is linearly *dependent*. Find some  $v$  in the span of the list such that there are two distinct linear combinations of the list equal to  $v$ .

**Solution.** Consider the list

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

9. Let  $V$  be a vector space over the field  $\mathbb{F}$ . Show that every list of vectors in the trivial subspace of  $V$  is both linearly independent and spanning (i.e., spans the entire trivial subspace).

**Solution.** The statement is actually not true! Any list containing  $0$  is linearly dependent, so  $\{0\}$  is not linearly independent.

10.  $(\star)$  Show that every spanning list of vectors  $L = \{v_1, \dots, v_n\}$  in a vector space  $V$  can be made into a linearly independent list  $L'$  such that  $\text{span} L = \text{span} L'$ .

**Solution.** Let  $L = \{v_1, \dots, v_n\}$  span  $V$ . If  $L$  is already linearly independent, then set  $L' = L$ , and we are done. If  $L$  is not linearly independent (i.e., linearly dependent), proceed as follows:

Begin with  $L_0 = \emptyset$  (the empty list). For  $i = 1, \dots, n$ , if  $v_i \notin \text{span}(L_{i-1})$ , set  $L_i = L_{i-1} \cup \{v_i\}$ . Otherwise, set  $L_i = L_{i-1}$  (that is, drop  $v_i$  from the list). Let  $L' = L_n$ . Because the list is finite, this process ends.

Evidently,  $L' \subset L$ , so  $\text{span} L' \subset \text{span} L$ . Conversely, every dropped  $v_i$  lies in  $\text{span} L'$  by definition. Every dropped  $v_i$  was in  $\text{span} L_{i-1} \subset \text{span} L'$  at the moment it was dropped. Hence, each  $v_i \in \text{span} L'$ , so  $\text{span} L \subset \text{span} L'$ , so  $\text{span} L' = \text{span} L$ .

The list  $L'$  is linearly independent by construction, since whenever we add a new vector, it is not in the span of the vectors already kept. This is exactly the criterion for linear independence.

Thus,  $L'$  is a linearly independent list with  $\text{span}L' = \text{span}L$ , and we are done.