Solutions to Test Two Supplementary Practice Test

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BIE-LA1 - Winter 2025

1. Let
$$M = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 2 & 0 & 1 & 5 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
with entries in $\mathbb{Z}_7$. Is M invertible? If so, compute its inverse.

Solution. A matrix M is invertible if and only if it can be row-reduced to the identity. Furthermore, the matrix obtained from the application of these row-reductions of the identity matrix is the inverse of M. Thus, we augment M with the identity matrix, forming the matrix

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 5 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

We now apply Gaussian elimination to row-reduce the left half of this augmented matrix to the identity.

$$R_4 \leftarrow 6R_1 + R_4 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 5 & 0 & 0 & 1 & 0 \\ 0 & 6 & 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$R_3 \leftarrow 5R_1 + R_3 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3 & 5 & 0 & 1 & 0 \\ 0 & 6 & 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$R_3 \leftarrow 4R_2 + R_3 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 5 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right)$$

$$R_4 \leftarrow R_2 + R_4 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 & 5 & 4 & 1 & 0 \\ 0 & 0 & 4 & 4 & 6 & 1 & 0 & 1 \end{array}\right)$$

$$R_3 \leftarrow 6R_3 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 3 & 6 & 0 \\ 0 & 0 & 4 & 4 & 6 & 1 & 0 & 1 \end{array}\right)$$

$$R_2 \leftarrow 4R_3 + R_2 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 6 & 3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 3 & 6 & 0 \\ 0 & 0 & 4 & 4 & 6 & 1 & 0 & 1 \end{array}\right)$$

$$R_4 \leftarrow 3R_3 + R_4 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 6 & 3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 3 & 5 & 3 & 4 & 1 \end{array}\right)$$

$$R_4 \leftarrow 5R_4 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 1 & 6 & 3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 6 & 5 \end{array}\right)$$

$$R_1 \leftarrow 6R_4 + R_1 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 4 & 6 & 1 & 2 \\ 0 & 1 & 0 & 5 & 1 & 6 & 3 & 0 \\ 0 & 0 & 1 & 2 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 6 & 5 \end{array}\right)$$

$$R_2 \leftarrow 2R_4 + R_2 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 4 & 6 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 2 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 6 & 5 \end{array}\right)$$

$$R_3 \leftarrow 5R_4 + R_3 : \left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 4 & 6 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 4 & 1 & 6 & 5 \end{array}\right)$$

$$R_1 \leftarrow 5R_2 + R_1 : \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 4 & 6 & 3 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 4 & 1 & 6 & 5 \end{array}\right).$$

The left block is the identity, so M is invertible, and its inverse is

$$M^{-1} = \begin{pmatrix} 0 & 4 & 6 & 3 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \\ 4 & 1 & 6 & 5 \end{pmatrix}.$$

2. Consider the lists of vectors $L_1 = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$ and $L_2 = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$ of vectors in $\mathbb{R}^4$.

   (a) Is it true that $\mathrm{span} L_1 = \mathrm{span} L_2$?

Solution. We place all four vectors as columns of a single matrix and apply Gaussian elimination:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 3 & 1 \end{pmatrix}.$$

$$R_2 \leftarrow R_2 - R_1 : \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 3 & 1 \end{pmatrix}$$

$$R_4 \leftarrow R_4 - 2R_1 : \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - R_2 : \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$R_4 \leftarrow R_4 - R_2 : \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is a row-echelon form of A. There are pivots in the first and second columns, and such pivots would also occur if we augmented taking L_2 first. We observe that the span does not change when the two lists are taken together. Thus the spans are the same.

   (b) Is L_1 a basis for $\mathbb{R}^4$? If not, extend it to a basis for $\mathbb{R}^4$.

Solution. The list L_1 has only two vectors in $\mathbb{R}^4$. Since any spanning set for $\mathbb{R}^4$ must contain at least four linearly independent vectors, L_1 cannot be a basis for $\mathbb{R}^4$.

We now extend L_1 to a basis by augmenting with the identity and applying Gaussian elimination.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$R_2 \leftarrow R_2 - R_1 : \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_4 \leftarrow R_4 - 2R_1 : \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - R_2 : \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{pmatrix}$$

$$R_4 \leftarrow R_4 - R_2 : \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

$$R_4 \leftarrow R_4 + R_3 : \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}$$

This matrix is in row-echelon form. The pivot columns are the first, second, third, and fourth columns. The first two pivot columns correspond to the vectors in the original list, and the last two correspond to the standard basis vectors e_1 and e_2.

Thus, L_1 is not a basis for $\mathbb{R}^4$, but

$$\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

is a basis for $\mathbb{R}^4$ extending L_1.

(c) Denote the basis of part (*b*) by $\mathcal{B}$. Compute $[I]_{\mathcal{E}\leftarrow\mathcal{B}}$.

Solution. By definition, the change-of-basis matrix $[I]_{\mathcal{E}\leftarrow\mathcal{B}}$ has as columns the vectors of $\mathcal{B}$ written in the standard basis. Thus

$$[I]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}.$$

3. Consider the basis $\mathcal{B} = \left(\left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ -1 \\ 1 \end{smallmatrix}\right] \right)$.

(a) Compute $[I]_{\mathcal{E}\leftarrow\mathcal{B}}$ and $[I]_{\mathcal{B}\leftarrow\mathcal{E}}$.

Solution. Writing the vectors of $\mathcal{B}$ as columns gives the change of basis from $\mathcal{B}$–coordinates to the standard basis:

$$[I]_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 0 & 3 & 1 \end{pmatrix}.$$

Thus $[I]_{\mathcal{B}\leftarrow\mathcal{E}}$ is its inverse. We find it by Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \leftarrow R_2 - 2R_1 : \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_3 \leftarrow R_3 - 3R_2 : \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 16 & 6 & -3 & 1 \end{array}\right)$$

$$R_3 \leftarrow \tfrac{1}{16} R_3 : \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{16} & \frac{1}{16} \end{array}\right)$$

$$R_1 \leftarrow R_1 - 2R_3 : \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{16} & \frac{1}{16} \end{array}\right)$$

$$R_2 \leftarrow R_2 + 5R_3 : \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{8} & \frac{1}{16} & \frac{5}{16} \\ 0 & 0 & 1 & \frac{3}{8} & -\frac{3}{16} & \frac{1}{16} \end{array}\right)$$

Hence

$$[I]_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{16} & \frac{5}{16} \\ \frac{3}{8} & -\frac{3}{16} & \frac{1}{16} \end{pmatrix}.$$

(b) Let $\mathcal{B}'$ be the basis $\left(\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 6 \end{bmatrix}\right)$. Compute $[I]_{\mathcal{B}' \leftarrow \mathcal{B}}$ and $[I]_{\mathcal{B} \leftarrow \mathcal{B}'}$.

Solution. First write the matrices whose columns are the basis vectors:

$$[I]_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 0 & 3 & 1 \end{pmatrix}, \qquad [I]_{\mathcal{E} \leftarrow \mathcal{B}'} = \begin{pmatrix} 3 & 2 & 6 \\ 2 & 3 & -2 \\ 4 & -3 & 6 \end{pmatrix}.$$

To get $[I]_{\mathcal{B} \leftarrow \mathcal{B}'}$, we solve

$$[I]_{\mathcal{E} \leftarrow \mathcal{B}} \, [I]_{\mathcal{B} \leftarrow \mathcal{B}'} = [I]_{\mathcal{E} \leftarrow \mathcal{B}'},$$

i.e., we row-reduce the augmented matrix $\left([I]_{\mathcal{E} \leftarrow \mathcal{B}} \mid [I]_{\mathcal{E} \leftarrow \mathcal{B}'}\right)$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 2 & 6 \\ 2 & 1 & -1 & 2 & 3 & -2 \\ 0 & 3 & 1 & 4 & -3 & 6 \end{array}\right)$$

$$R_2 \leftarrow R_2 - 2R_1 : \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 2 & 6 \\ 0 & 1 & -5 & -4 & -1 & -14 \\ 0 & 3 & 1 & 4 & -3 & 6 \end{array}\right)$$

$$R_3 \leftarrow R_3 - 3R_2 : \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 2 & 6 \\ 0 & 1 & -5 & -4 & -1 & -14 \\ 0 & 0 & 16 & 16 & 0 & 48 \end{array}\right)$$

$$R_3 \leftarrow \tfrac{1}{16} R_3 : \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 2 & 6 \\ 0 & 1 & -5 & -4 & -1 & -14 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{array}\right)$$

$$R_1 \leftarrow R_1 - 2R_3 : \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & -5 & -4 & -1 & -14 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{array}\right)$$

$$R_2 \leftarrow R_2 + 5R_3 : \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{array}\right)$$

Thus the left block is the identity and the right block is the change-of-basis matrix

$$[I]_{\mathcal{B} \leftarrow \mathcal{B}'} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Since change-of-basis matrices are inverses of each other, we obtain

$$[I]_{\mathcal{B}' \leftarrow \mathcal{B}} = \left([I]_{\mathcal{B} \leftarrow \mathcal{B}'}\right)^{-1} = \begin{pmatrix} \frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & -\frac{2}{7} & \frac{3}{7} \end{pmatrix}.$$

4. Let

$$M = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 1 & -1 & 2 & 0 \end{pmatrix}.$$

(a) Compute $\ker(M)$ and $\operatorname{ran}(M)$.

Solution. We would like to understand the system of linear equations induced by this matrix. We first apply Gaussian elimination to reduce the matrix to row-echelon form.

$$M = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 1 & -1 & 2 & 0 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - R_1 : \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & -3 & 2 & -1 \end{pmatrix}$$

$$R_3 \leftarrow R_3 + 3R_2 : \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 11 & 11 \end{pmatrix}$$

$$R_3 \leftarrow \frac{1}{11} R_3 : \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus, we are left with

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Recall that the kernel of a matrix A is the set of all vectors v such that $Av = 0$. Recall that Gaussian elimination does not alter the solution set of the original system of linear equations. Thus, we would like to solve

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

For scalars $x_1, x_2, x_3, x_4 \in \mathbb{R}$. We obtain the following system of linear equations

$$x_1 + 2x_2 + x_4 = 0, \quad x_2 + 3x_3 + 4x_4 = 0, \quad x_3 + x_4 = 0.$$

We now apply these constraints to the vector of the x_i, obtaining

$$\begin{pmatrix} x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} x_4 \\ -x_4 \\ -x_4 \\ x_4 \end{pmatrix} = \operatorname{span} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \ker M.$$

Observing the row-reduced form of M, we find a pivot in every column, and consequently its range is $\mathbb{R}^3$.

(b) What is the dimension of $\ker M$ and what is the rank of M? Justify your response.

Solution. Clearly the dimension of the range of M, $\mathbb{R}^3$, is 3. By the rank theorem, $\dim(\ker(M)) = 4 - 3 = 1$. Thus, the dimension of the kernel of M is 1.

(c) Describe the range of M^{T}. What is the rank of M^{T}?

Solution. Recall that $\operatorname{ran}(M^{\mathrm{T}})$ is the row space of M, viewed as a subspace of $\mathbb{R}^4$. From the row-echelon form of M, we saw that there is a pivot in each of the three rows, so the three rows of M are linearly independent and form a basis of the row space. Thus

$$\operatorname{ran}(M^{\mathrm{T}}) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

Using the rank theorem, we also have

$$\operatorname{rank}(M^{\mathrm{T}}) = 3.$$

5. (Disclaimer: according to updated information, you are unlikely to have true/false questions on your second test. I am leaving these here because I think they are good practice.)

(a) **True/False:** There is a spanning list of length 3 in $\mathbb{Z}_{13}^2$.

Solution. True. A trivial example is $\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right)$, since for any vector $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] \in \mathbb{Z}_{13}^2$, we have

$$\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] = a_1 \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] + a_2 \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right].$$

(b) **True/False:** There is a linearly independent list of length 7 in $\mathbb{R}^6$.

Solution. False. Any linearly independent list in a finite-dimensional vector space of dimension n has length at most n. $\dim(\mathbb{R}^6) = 6 < 7$, so no list of length 7 in $\mathbb{R}^6$ can be linearly independent.

(c) **True/False:** There is a matrix $A \in M_{5,7}(\mathbb{R})$ with $\dim(\ker A) = 2$ and $\operatorname{rank}(A) = 4$.

Solution. False. By the rank theorem, we have

$$7 = \dim(\ker(A)) + \operatorname{rank}(A)$$

for each $A \in M_{5,7}(\mathbb{R})$. Thus, we obtain

$$7 \neq 2 + 4,$$

so no such matrix A exists in $M_{5,7}(\mathbb{R})$.

(d) **True/False:** Each matrix $A \in M_{8,8}(\mathbb{R})$ with $\operatorname{rank}(A) = 0$ has linearly independent columns.

Solution. True. Recall that a matrix is invertible if and only if its rank is 0. Thus, A is invertible, and every invertible matrix necessarily has linearly independent columns.