Solutions to Vector Subspaces, Span, and Linear Independence

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1 Vector Subspaces

1. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_1 = -a_2 \right\}$$
a subspace of $\mathbb{R}^2$? Justify your answer with rigorous reasoning.

Solution. The set S *is* a subspace of $\mathbb{R}^2$.

Recall that, in order to verify whether S is a subspace of $\mathbb{R}^2$ or not, we need to verify whether S is closed under vector addition and scalar multiplication. Consider the following two arbitrary elements of S: $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \in S$. Because these vectors are in S, and all vectors in S have the property that $a_1 = -a_2$, we can rewrite these vectors as $\left[\begin{smallmatrix} -a_2 \\ a_2 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} -b_2 \\ b_2 \end{smallmatrix}\right] \in S$. We begin by verifying closure under vector addition:

$$\begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} + \begin{bmatrix} -b_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -a_2 - b_2 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} -(a_2 + b_2) \\ a_2 + b_2 \end{bmatrix},$$

so S is closed under vector addition. We now verify closure under scalar multiplication. Let $c \in \mathbb{R}$. Then

$$c \begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} c \cdot -a_2 \\ c \cdot a_2 \end{bmatrix} = \begin{bmatrix} -(ca_2) \\ ca_2 \end{bmatrix}.$$

Since S is closed under vector addition and scalar multiplication, S is a subspace of $\mathbb{R}^2$.

2. Is
$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = a_1^2 \right\}$$
a subspace of $\mathbb{R}^2$? Justify your answer with rigorous reasoning.

Solution. The set S is *not* a subspace of $\mathbb{R}^2$.

Consider $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \in S$. We can rewrite these as $\left[\begin{smallmatrix} a_1 \\ a_1^2 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ b_1^2 \end{smallmatrix}\right] \in S$ as a consequence of the constraint on members of S. First, we verify closure under vector addition:

$$\begin{bmatrix} a_1 \\ a_1^2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_1^2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1^2 + b_1^2 \end{bmatrix} \neq \begin{bmatrix} a_1 + b_1 \\ (a_1 + b_1)^2 \end{bmatrix},$$

thus S fails to be closed under vector addition and is consequently not a subspace of $\mathbb{R}^2$.

3. ($\star$) Describe a nontrivial subspace of $\mathbb{C}^2$ over the field $\mathbb{R}$ that is not simply the restriction of some entries to 0. Prove it is a subspace.

Solution. The purpose of this exercise was primarily to draw attention to the fact that vector spaces of the form $\mathbb{F}^n$ do not necessarily force the underlying field to be $\mathbb{F}$. There are many correct solutions to this problem, but we will share one of the solutions which we perceive to be the most straightforward. Set

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2 : a_2 = 5a_1 \right\}.$$

Taking arbitrary vectors $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \in S$, we can rewrite them as $\left[\begin{smallmatrix} a_1 \\ 5a_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} b_1 \\ 5b_1 \end{smallmatrix}\right]$. First verifying closure under vector addition, we see

$$\begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ 5b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ 5a_1 + 5b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ 5(a_1 + b_1) \end{bmatrix}.$$

Let $c \in \mathbb{R}$. Then

$$c \begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot 5a_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ 5(ca_1) \end{bmatrix}.$$

Thus, S is a subspace of $\mathbb{C}^2$ over the field $\mathbb{R}$. It is also nontrivial.

4. Choose your favorite vector

$$v = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2.$$

Can you describe a nontrivial vector subspace of $\mathbb{R}^2$ containing v?

Solution. Let $v = \left[\begin{smallmatrix} 25 \\ 4 \end{smallmatrix}\right]$[1]. The line passing through the origin containing v is a subspace (one can feel this intuitively, later formally justifying it in this exercise. Alternatively, one could simply Exercise 5.) We can write this line as

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : (a_2 - 4) = \frac{4}{25}(a_1 - 25) \right\}.$$

Simplifying, we have

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = \frac{4}{25}a_1 \right\}.$$

We verify closure under vector addition; take arbitrary vectors $\left[\begin{smallmatrix} a_1 \\ \frac{4}{25}a_1 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ \frac{4}{25}b_1 \end{smallmatrix}\right] \in S$. We compute the following:

$$\begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ \frac{4}{25}b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \frac{4}{25}a_1 + \frac{4}{25}b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \frac{4}{25}(a_1 + b_1) \end{bmatrix}.$$

Thus, we have closure under vector addition. Let $c \in \mathbb{R}$. Then

$$c\begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot \frac{4}{25}a_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ \frac{4}{25}(ca_1) \end{bmatrix}.$$

Consequently, S is closed under vector addition and scalar multiplication, and S contains $\left[\begin{smallmatrix} 25 \\ 4 \end{smallmatrix}\right]$.

5. ($\star$) Show that every line passing through the origin of $\mathbb{R}^2$ (i.e., containing the zero vector) is a subspace of $\mathbb{R}^2$.

Solution. First, note that the horizontal axis (i.e., all vectors with zero in their second component) and the vertical axis (i.e., all vectors with zero in their first component) are subspaces of $\mathbb{R}^2$. This can be verified the standard way. In all other cases, lines have the form

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = ka_1 \right\}$$

where k is a fixed integer in $\mathbb{R} \setminus \{0\}$. Note that there is no constant term in $a_2 = ka_1$, since subspaces necessarily contain the origin. Selecting $\left[\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] \in S$, we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ ka_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ kb_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ ka_1 + kb_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ k(a_1 + b_1) \end{bmatrix}.$$

[1] This procedure will work with any vector in $\mathbb{R}^2$ with the exception of vectors with 0 in their second component, in which case your subspace is simply the vertical axis in $\mathbb{R}^2$.

Thus, S is closed under vector addition. Let $c \in \mathbb{R}$. Then

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cka_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ k(ca_1) \end{bmatrix}.$$

Consequently, S is closed under both vector addition and scalar multiplication, and is thus a subspace of $\mathbb{R}^2$.

6. Why is $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{Z}_5^2 : a_1, a_2 \text{ are odd modulo } 5 \right\}$ not a subspace of $\mathbb{Z}_5^2$?

Solution. A trivial justification would be to take $\left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right] \in S$. Choosing $0 \in \mathbb{Z}_5$, we have

$$0 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \cdot 3 \\ 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S.$$

7. Let $\mathbb{F}$ be a field (and thus a vector space). What are the subspaces of $\mathbb{F}$?

Solution. Given a vector space V over a field $\mathbb{F}$, V and $\{0\}$ are always subspaces of V. In this case, $\mathbb{F}$ and $\{0\}$, $0 \in \mathbb{F}$ are subspaces of $\mathbb{F}$. But are there any other subspaces of $\mathbb{F}$? Suppose such a subspace $U \neq \{0\} \neq \mathbb{F}$ exists. Choose a nonzero $u \in U$. For any $x \in \mathbb{F}$, we have $x = (xu)u^{-1}$. Then, since $\mathbb{F}$ is closed under scalar multiplication by $\mathbb{F}$, $x = (xu)u^{-1} \in \mathbb{F}$ for all $x \in \mathbb{F}$, and thus $U = \mathbb{F}$, a contradiction. Consequently, the only two subspaces of a field $\mathbb{F}$ are $\{0\}$ and $\mathbb{F}$ itself.

8. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_3 = a_1 - a_2 \right\}$$

a subspace of $\mathbb{R}^3$? Justify your answer with rigorous reasoning.

Solution. The set S *is* a vector subspace of $\mathbb{R}^3$.

Let $\left[\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right] \in S$ and $\left[\begin{smallmatrix} b_1 \\ b_2 \\ b_3 \end{smallmatrix}\right] \in S$. We check for closure under vector addition first:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_1 - a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_1 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_1 + b_1 - a_2 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ (a_1 + b_1) - (a_2 + b_2) \end{bmatrix}.$$

Letting $c \in \mathbb{R}$, we have

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_1 - a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ c(a_1 - a_2) \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ (ca_1) - (ca_2) \end{bmatrix}.$$

Consequently, S is closed under both vector addition and scalar multiplication, and is thus a subspace of $\mathbb{R}^3$.

9. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_2 = 4a_3, a_1 = -2a_2 \right\}$$

a subspace of $\mathbb{R}^3$? Justify your answer with rigorous reasoning.

Solution. The set S *is* a vector subspace of $\mathbb{R}^3$.

Let $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$. We first check for closure under vector addition.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -8a_3 \\ 4a_3 \\ a_3 \end{bmatrix} + \begin{bmatrix} -8b_3 \\ 4b_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} -8(a_3 + b_3) \\ 4(a_3 + b_3) \\ (a_3 + b_3) \end{bmatrix} \in S.$$

Thus, we have closure under vector addition. Let $c \in \mathbb{R}$. Then

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \cdot \begin{bmatrix} -8a_3 \\ 4a_3 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8(ca_3) \\ 4(ca_3) \\ (ca_3) \end{bmatrix} \in S$$

.

This gives us closure under scalar multiplication. Since $S \subset \mathbb{R}^3$ is closed under both vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^3$.

10. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_1 + 2a_2 + 4a_3 = 5 \right\}$$

a subspace of $\mathbb{R}^3$? Justify your answer with rigorous reasoning.

Solution. The set S is *not* a subspace of $\mathbb{R}^3$.

Let $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$. Then choosing $0 \in \mathbb{R}$, we have

$$0 \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \cdot a_1 \\ 0 \cdot a_2 \\ 0 \cdot a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

.

However, $0 + 2 \cdot 0 + 4 \cdot 0 \neq 5$, so the zero vector is not in S, and thus S is not closed under scalar multiplication. Thus, S is not a subspace of $\mathbb{R}^3$.

Remark 1.1. *Notice that those sets which were subspaces of their respective super-spaces had linear constraints at most (i.e., no conditions on squares, cubes, and so on). This hints at the idea that having at most linear constraints on a set is crucial to that set being a subspace.*

Even the linear constraints, however, were not sufficient; our constraints also had to ensure that the relevant geometric figure formed by the subset (i.e., line, plane) passed through the origin.

2 Span and Linear Independence

1. Is the list of vectors

$$\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 4 \end{bmatrix} \right\}$$

in $\mathbb{R}^3$ linearly independent? Justify your response.

Solution. The list *is* linearly independent in $\mathbb{R}^3$.

Recall that asking whether a list of vectors $(v_1, \ldots, v_n)$ is linearly independent is the same as asking whether the equation

$$\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution $c_1, \ldots, c_n$ such that at least one of the c_i, $1 \le i \le n$ is nonzero. We can determine whether this is the case via Gaussian elimination on the matrix of the v_i, $1 \le i \le n$ augmented (appended as the last column) with the zero column vector. In our case, we obtain the following matrix as our augmented matrix:

$$\left(\begin{array}{ccc|c} 5 & 2 & 7 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right).$$

We now apply Gaussian elimination to this matrix:

$$R_1 \leftarrow \frac{R_1}{5} : \left(\begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right)$$

$$R_2 \leftarrow R_2 + 2R_1 : \left(\begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 4/5 & 14/5 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - R_1 : \left(\begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 4/5 & 14/5 & 0 \\ 0 & 13/5 & 13/5 & 0 \end{array}\right)$$

$$R_2 \leftarrow \frac{5R_2}{4} : \left(\begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 13/5 & 13/5 & 0 \end{array}\right)$$

$$R_1 \leftarrow R_1 - \frac{2R_2}{5} : \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 13/5 & 13/5 & 0 \end{array}\right)$$

$$R_3 \leftarrow R_3 - \frac{13R_2}{5} : \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 0 & -13/2 & 0 \end{array}\right)$$

$$R_3 \leftarrow \frac{-2R_3}{13} : \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$R_2 \leftarrow R_2 - \frac{7R_3}{2} : \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Thus, the only solution to our system of equations is

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Consequently, the list is independent in $\mathbb{R}^3$.

2. Is the list of vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$$

in $\mathbb{Z}_5^3$ linearly independent? Justify your response.

Solution. The list is *not* linearly independent (i.e, the list is linearly *dependent*).

Recall that all row operations done here are computed modulo 5. We first form our augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right).$$

We apply Gaussian elimination to this augmented matrix:

$$R_1 \leftrightarrow R_2 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right)$$

$$R_2 \leftarrow R_2 - 2R_1 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right)$$

$$R_3 \leftarrow R_3 - 3R_1 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$R_3 \leftarrow R_3 - 3R_2 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1 \leftarrow R_1 - 2R_2 : \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, we obtain the solution

$$c_1 + 2c_2 = 0 \mod 5, \quad c_3 = 0 \mod 5$$

This implies that

$$c_1 = -2c_2 \mod 5, \quad c_3 = 0 \mod 5.$$

Letting $c_2 = 2$, $c_1 = -2 \cdot 2 = 3 \cdot 2 = 1 \mod 5$, and $c_3 = 0$, so we obtain the solution

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consequently, the list is not linearly independent.

3. Describe the set

$$\operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}\right\}.$$

What geometric figure in $\mathbb{R}^3$ does it form?

Solution. Recall that the span of a list of vectors is the set of all linear combinations of those vectors. In our case:

$$\operatorname{span}\left\{\begin{bmatrix}-2\\1\\-3\end{bmatrix},\begin{bmatrix}5\\1\\2\end{bmatrix}\right\}=\left\{\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\in\mathbb{R}^3:\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=c_1\cdot\begin{bmatrix}-2\\1\\-3\end{bmatrix}+c_2\cdot\begin{bmatrix}5\\1\\2\end{bmatrix},\ c_1,c_2\in\mathbb{R}\right\}.$$

Thus, all vectors in this span have the form

$$\begin{bmatrix}-2c_1\\c_1\\-3c_1\end{bmatrix}+\begin{bmatrix}5c_2\\c_2\\2c_2\end{bmatrix}=\begin{bmatrix}-2c_1+5c_2\\c_1+c_2\\-3c_1+2c_2\end{bmatrix},\ c_1,c_2\in\mathbb{R}.$$

This set forms a plane through the origin, since the two vectors in the list are linearly dependent (and thus are not scalar multiples of each other).

4. Is

$$\begin{bmatrix}0\\4\\1\end{bmatrix}\in\operatorname{span}\left\{\begin{bmatrix}-2\\1\\4\end{bmatrix},\begin{bmatrix}3\\3\\2\end{bmatrix}\right\},$$

where all vectors are in $\mathbb{Z}_5^3$? Justify your response.

Solution. The vector *is* in the span of the given list.

Let V be a vector space over the field $\mathbb{F}$. Recall that asking if a vector $v\in\operatorname{span}\{v_1,\dots,v_n\}$, where $v_i\in V$, $1\le i\le n$, is equivalent to asking whether there exist scalars $c_i\in\mathbb{F}$, $1\le i\le n$ such that

$$v=c_1v_1+\cdots+c_nv_n.$$

We can express this as an equation involving matrices.

$$\begin{pmatrix}v_1&\cdots&v_n\end{pmatrix}\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}=v.$$

Once again, Gaussian elimination proves to be an excellent tool to determine whether such solutions exist, and the values of such solutions should they exist. We form our augmented matrix:

$$\left(\begin{array}{cc|c}3&3&0\\1&3&4\\4&2&1\end{array}\right).$$

We now apply Gaussian elimination. Recall that all computations are done modulo 5.

$$R_1 \leftrightarrow R_2 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 3 & 3 & 0 \\ 4 & 2 & 1 \end{array}\right).$$

$$R_2 \leftarrow R_2 - 3R_1 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 4 & 3 \\ 4 & 2 & 1 \end{array}\right).$$

$$R_3 \leftarrow R_3 - 4R_1 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{array}\right).$$

$$R_2 \leftarrow 4R_2 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

$$R_1 \leftarrow R_1 - 3R_2 : \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

So indeed,

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

5. Identify a vector v in $\mathbb{R}^3$ such that when v is added to the list

$$\left\{ \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} \right\},$$

the resultant list is linearly independent.

Solution. An easy choice is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This is because linear dependence captures the relationship of proportionality, but

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is not proportional to } \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix}.$$

That is, there is no pair of scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

6. Is

$$\begin{bmatrix} 4 \\ 3 \\ -5 \\ 1 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} \right\},$$

where all vectors are in $\mathbb{R}^4$? Justify your response.

Solution. The given vector is *not* in the span of the given list of vectors.

We first form the corresponding augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ -6 & 1 & 5 & 3 \\ 2 & -9 & -3 & -5 \\ -2 & 2 & 1 & 1 \end{array}\right).$$

We now apply Gaussian elimination to this matrix:

$$R_2 \leftarrow R_2 - R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 2 & -9 & -3 & -5 \\ -2 & 2 & 1 & 1 \end{array}\right)$$

$$R_3 \leftarrow R_3 - R_1 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & -9 & -1 & -1 \\ -2 & 2 & 1 & 1 \end{array}\right)$$

$$R_4 \leftarrow R_4 - R_1 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & -9 & -1 & -1 \\ 0 & 2 & -1 & -3 \end{array}\right)$$

$$R_3 \leftarrow R_3 + 9R_2 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -10 & -82 \\ 0 & 2 & -1 & -3 \end{array}\right)$$

$$R_4 \leftarrow R_4 - 2R_2 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -10 & -82 \\ 0 & 0 & 1 & 15 \end{array}\right)$$

$$R_3 \leftrightarrow R_4 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array}\right)$$

$$R_2 \leftarrow R_2 + R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array}\right)$$

$$R_1 \leftarrow R_1 - 2R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 0 & -26 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array}\right)$$

$$R_1 \leftarrow R_1 - 2R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 0 & -26 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 68 \end{array}\right).$$

The fourth row of the last matrix gives us $0 = 68$, which implies that the system is inconsistent. Thus

$$\begin{bmatrix} 4 \\ 3 \\ -5 \\ 1 \end{bmatrix} \notin \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

7. ($\star$) Show that, for any list $\{v_1, \dots v_n\}$ of vectors in a vector space V, $\operatorname{span}\{v_1, \dots v_n\}$ is the smallest (inclusion-wise) subspace of V containing $\{v_1, \dots v_n\}$.

Solution. Let $U \subset V$ be a subspace with $v_1, \dots, v_n \in U$. Then $cv_i \in U$, $1 \le i \le n$ for all $c \in \mathbb{F}$, where $\mathbb{F}$ is the underlying field of V, since subspaces are closed under scalar multiplication. Subspaces are also closed under addition, so

$$c_1 v_1 + \dots + c_n v_n \in U, \quad c_i \in \mathbb{R}, \quad 1 \le i \le n.$$

Consequently, $\operatorname{span}\{v_1, \dots, v_n\} \subset U$. Thus, any subspace of V containing $\{v_1, \dots, v_n\}$ contains $\operatorname{span}\{v_1, \dots, v_n\}$.

8. Give an example of a list of vectors in $\mathbb{R}^3$ which is linearly *dependent*. Find some v in the span of the list such that there are two distinct linear combinations of the list equal to v.

Solution. Consider the list

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

9. Let V be a vector space over the field $\mathbb{F}$. Show that every list of vectors in the trivial subspace of V is both linearly independent and spanning (i.e., spans the entire trivial subspace).

Solution. The statement is actually not true! Recall that the empty list $\varnothing$ is considered to be linearly independent, but it certainly does not span the space $\{0\}$.

10. ($\star$) Show that every spanning list of vectors $L = \{v_1, \dots v_n\}$ in a vector space V can be made into a linearly independent list L' such that $\text{span}L = \text{span}L'$.

Solution. Let $L = \{v_1, \dots v_n\}$ span V. If L is already linearly independent, then set $L' = L$, and we are done. If L is not linearly independent (i.e., linearly dependent), proceed as follows:

Begin with $L_0 = \varnothing$ (the empty list). For $i = 1, \dots, n$, if $v_i \notin \text{span}(L_{i-1})$, set $L_i = L_{i-1} \cup \{v_i\}$. Otherwise, set $L_i = L_{i-1}$ (that is, drop v_i from the list). Let $L' = L_n$. Because the list is finite, this process ends.

Evidently, $L' \subset L$, so $\text{span}L' \subset \text{span}L$. Conversely, every dropped v_i lies in $\text{span}L'$ by definition. Every dropped v_i was in $\text{span}L_{i-1} \subset \text{span}L'$ at the moment it was dropped. Hence, each $v_i \in \text{span}L'$, so $\text{span}L \subset \text{span}L'$, so $\text{span}L' = \text{span}L$.

The list L' is linearly independent by construction, since whenever we add a new vector, it is not in the span of the vectors already kept. This is exactly the criterion for linear independence.

Thus, L' is a linearly independent list with $\mathrm{span} L' = \mathrm{span} L$, and we are done.