

Solutions to Vector Subspaces, Span, and Linear Independence

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1 Vector Subspaces

1. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_1 = -a_2 \right\}$$

a subspace of \mathbb{R}^2 ? Justify your answer with rigorous reasoning.

Solution. The set S is a subspace of \mathbb{R}^2 .

Recall that, in order to verify whether S is a subspace of \mathbb{R}^2 or not, we need to verify whether S is closed under vector addition and scalar multiplication. Consider the following two arbitrary elements of S : $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$. Because these vectors are in S , and all vectors in S have the property that $a_1 = -a_2$, we can rewrite these vectors as $\begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} \in S$ and $\begin{bmatrix} -b_2 \\ b_2 \end{bmatrix} \in S$. We begin by verifying closure under vector addition:

$$\begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} + \begin{bmatrix} -b_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -a_2 - b_2 \\ a_2 + b_2 \end{bmatrix} = \begin{bmatrix} -(a_2 + b_2) \\ a_2 + b_2 \end{bmatrix},$$

so S is closed under vector addition. We now verify closure under scalar multiplication. Let $c \in \mathbb{R}$. Then

$$c \begin{bmatrix} -a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} c \cdot -a_2 \\ c \cdot a_2 \end{bmatrix} = \begin{bmatrix} -(ca_2) \\ ca_2 \end{bmatrix}.$$

Since S is closed under vector addition and scalar multiplication, S is a subspace of \mathbb{R}^2 .

2. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = a_1^2 \right\}$$

a subspace of \mathbb{R}^2 ? Justify your answer with rigorous reasoning.

Solution. The set S is *not* a subspace of \mathbb{R}^2 .

Consider $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$. We can rewrite these as $\begin{bmatrix} a_1 \\ a_1^2 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_1^2 \end{bmatrix} \in S$ as a consequence of the constraint on members of S . First, we verify closure under vector addition:

$$\begin{bmatrix} a_1 \\ a_1^2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_1^2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_1^2 + b_1^2 \end{bmatrix} \neq \begin{bmatrix} a_1 + b_1 \\ (a_1 + b_1)^2 \end{bmatrix},$$

thus S fails to be closed under vector addition and is consequently not a subspace of \mathbb{R}^2 .

3. (*) Describe a nontrivial subspace of \mathbb{C}^2 over the field \mathbb{R} that is not simply the restriction of some entries to 0. Prove it is a subspace.

Solution. The purpose of this exercise was primarily to draw attention to the fact that vector spaces of the form \mathbb{F}^n do not necessarily force the underlying field to be \mathbb{F} . There are many correct solutions to this problem, but we will share one of the solutions which we perceive to be the most straightforward. Set

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2 : a_2 = 5a_1 \right\}.$$

Taking arbitrary vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$, we can rewrite them as $\begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ 5b_1 \end{bmatrix}$. First verifying closure under vector addition, we see

$$\begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ 5b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ 5a_1 + 5b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ 5(a_1 + b_1) \end{bmatrix}.$$

Let $c \in \mathbb{R}$. Then

$$c \begin{bmatrix} a_1 \\ 5a_1 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot 5a_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ 5(ca_1) \end{bmatrix}.$$

Thus, S is a subspace of \mathbb{C}^2 over the field \mathbb{R} . It is also nontrivial.

4. Choose your favorite vector

$$v = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2.$$

Can you describe a nontrivial vector subspace of \mathbb{R}^2 containing v ?

Solution. Let $v = \begin{bmatrix} 25 \\ 4 \end{bmatrix}$ ¹. The line passing through the origin containing v is a subspace (one can feel this intuitively, later formally justifying it in this exercise. Alternatively, one could simply Exercise 5.) We can write this line as

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : (a_2 - 4) = \frac{4}{25}(a_1 - 25) \right\}.$$

Simplifying, we have

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = \frac{4}{25}a_1 \right\}.$$

We verify closure under vector addition; take arbitrary vectors $\begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ \frac{4}{25}b_1 \end{bmatrix} \in S$. We compute the following:

$$\begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ \frac{4}{25}b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \frac{4}{25}a_1 + \frac{4}{25}b_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \frac{4}{25}(a_1 + b_1) \end{bmatrix}.$$

Thus, we have closure under vector addition. Let $c \in \mathbb{R}$. Then

$$c \begin{bmatrix} a_1 \\ \frac{4}{25}a_1 \end{bmatrix} = \begin{bmatrix} c \cdot a_1 \\ c \cdot \frac{4}{25}a_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ \frac{4}{25}(ca_1) \end{bmatrix}.$$

Consequently, S is closed under vector addition and scalar multiplication, and S contains $\begin{bmatrix} 25 \\ 4 \end{bmatrix}$.

5. (\star) Show that every line passing through the origin of \mathbb{R}^2 (i.e., containing the zero vector) is a subspace of \mathbb{R}^2 .

Solution. First, note that the horizontal axis (i.e., all vectors with zero in their second component) and the vertical axis (i.e., all vectors with zero in their first component) are subspaces of \mathbb{R}^2 . This can be verified the standard way. In all other cases, lines have the form

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2 : a_2 = ka_1 \right\}$$

where k is a fixed integer in $\mathbb{R} \setminus \{0\}$. Note that there is no constant term in $a_2 = ka_1$, since subspaces necessarily contain the origin. Selecting $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in S$, we have

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ ka_1 \end{bmatrix} + \begin{bmatrix} b_1 \\ kb_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ ka_1 + kb_1 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ k(a_1 + b_1) \end{bmatrix}.$$

¹This procedure will work with any vector in \mathbb{R}^2 with the exception of vectors with 0 in their first component, in which case your subspace is simply the vertical axis in \mathbb{R}^2 .

Thus, S is closed under vector addition. Let $c \in \mathbb{R}$. Then

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cka_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ k(ca_1) \end{bmatrix}.$$

Consequently, S is closed under both vector addition and scalar multiplication, and is thus a subspace of \mathbb{R}^2 .

6. Why is $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{Z}_5^2 : a_1, a_2 \text{ are odd modulo 5} \right\}$ not a subspace of \mathbb{Z}_5^2 ?

Solution. A trivial justification would be to take $\begin{bmatrix} 3 \\ 3 \end{bmatrix} \in S$. Choosing $0 \in \mathbb{Z}_5$, we have

$$0 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \cdot 3 \\ 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S.$$

7. Let \mathbb{F} be a field (and thus a vector space). What are the subspaces of \mathbb{F} ?

Solution. Given a vector space V over a field \mathbb{F} , V and $\{0\}$ are always subspaces of V . In this case, \mathbb{F} and $\{0\}$, $0 \in \mathbb{F}$ are subspaces of \mathbb{F} . But are there any other subspaces of \mathbb{F} ? Suppose such a subspace $U \neq \{0\} \neq \mathbb{F}$ exists. Choose a nonzero $u \in U$. For any $x \in \mathbb{F}$, we have $x = (xu)u^{-1}$. Then, since \mathbb{F} is closed under scalar multiplication by \mathbb{F} , $x = (xu)u^{-1} \in U$ for all $x \in \mathbb{F}$, and thus $U = \mathbb{F}$, a contradiction. Consequently, the only two subspaces of a field \mathbb{F} are $\{0\}$ and \mathbb{F} itself.

8. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_3 = a_1 - a_2 \right\}$$

a subspace of \mathbb{R}^3 ? Justify your answer with rigorous reasoning.

Solution. The set S is a vector subspace of \mathbb{R}^3 .

Let $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$. We check for closure under vector addition first:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_1 - a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_1 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_1 + b_1 - a_2 - b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ (a_1 + b_1) - (a_2 + b_2) \end{bmatrix}.$$

Letting $c \in \mathbb{R}$, we have

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_1 - a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ c(a_1 - a_2) \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ (ca_1) - (ca_2) \end{bmatrix}.$$

Consequently, S is closed under both vector addition and scalar multiplication, and is thus a subspace of \mathbb{R}^3 .

9. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_2 = 4a_3, a_1 = -2a_2 \right\}$$

a subspace of \mathbb{R}^3 ? Justify your answer with rigorous reasoning.

Solution. The set S is a vector subspace of \mathbb{R}^3 .

Let $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$. We first check for closure under vector addition.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -8a_3 \\ 4a_3 \\ a_3 \end{bmatrix} + \begin{bmatrix} -8b_3 \\ 4b_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} -8(a_3 + b_3) \\ 4(a_3 + b_3) \\ (a_3 + b_3) \end{bmatrix} \in S.$$

Thus, we have closure under vector addition. Let $c \in \mathbb{R}$. Then

$$c \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c \cdot \begin{bmatrix} -8a_3 \\ 4a_3 \\ a_3 \end{bmatrix} = \begin{bmatrix} -8(ca_3) \\ 4(ca_3) \\ (ca_3) \end{bmatrix} \in S$$

This gives us closure under scalar multiplication. Since $S \subset \mathbb{R}^3$ is closed under both vector addition and scalar multiplication, it is a subspace of \mathbb{R}^3 .

10. Is

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : a_1 + 2a_2 + 4a_3 = 5 \right\}$$

a subspace of \mathbb{R}^3 ? Justify your answer with rigorous reasoning.

Solution. The set S is *not* a subspace of \mathbb{R}^3 .

Let $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in S$. Then choosing $0 \in \mathbb{R}$, we have

$$0 \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \cdot a_1 \\ 0 \cdot a_2 \\ 0 \cdot a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

However, $0 + 2 \cdot 0 + 4 \cdot 0 \neq 5$, so the zero vector is not in S , and thus S is not closed under scalar multiplication. Thus, S is not a subspace of \mathbb{R}^3 .

Remark 1.1. Notice that those sets which were subspaces of their respective super-spaces had linear constraints at most (i.e., no conditions on squares, cubes, and so on). This hints at the idea that having at most linear constraints on a set is crucial to that set being a subspace.

Even the linear constraints, however, were not sufficient; our constraints also had to ensure that the relevant geometric figure formed by the subset (i.e., line, plane) passed through the origin.

2 Span and Linear Independence

1. Is the list of vectors

$$\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 4 \end{bmatrix} \right\}$$

in \mathbb{R}^3 linearly independent? Justify your response.

Solution. The list is linearly independent in \mathbb{R}^3 .

Recall that asking whether a list of vectors (v_1, \dots, v_n) is linearly independent is the same as asking whether the equation

$$(v_1 \ \dots \ v_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution c_1, \dots, c_n such that at least one of the c_i , $1 \leq i \leq n$ is nonzero. We can determine whether this is the case via Gaussian elimination on the matrix of the v_i , $1 \leq i \leq n$ augmented (appended as the last column) with the zero column vector. In our case, we obtain the following matrix as our augmented matrix:

$$\left(\begin{array}{ccc|c} 5 & 2 & 7 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right).$$

We now apply Gaussian elimination to this matrix:

$$\begin{aligned} R_1 &\leftarrow \frac{R_1}{5} : \left(\begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right) \\ R_2 &\leftarrow R_2 + 2R_1 : \left(\begin{array}{ccc|c} 1 & 2/5 & 7/5 & 0 \\ 0 & 4/5 & 14/5 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right) \end{aligned}$$

$$R_3 \leftarrow R_3 - R_1 : \begin{pmatrix} 1 & 2/5 & 7/5 & | & 0 \\ 0 & 4/5 & 14/5 & | & 0 \\ 0 & 13/5 & 13/5 & | & 0 \end{pmatrix}$$

$$R_2 \leftarrow \frac{5R_2}{4} : \begin{pmatrix} 1 & 2/5 & 7/5 & | & 0 \\ 0 & 1 & 7/2 & | & 0 \\ 0 & 13/5 & 13/5 & | & 0 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - \frac{2R_2}{5} : \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 7/2 & | & 0 \\ 0 & 13/5 & 13/5 & | & 0 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - \frac{13R_2}{5} : \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 7/2 & | & 0 \\ 0 & 0 & -13/2 & | & 0 \end{pmatrix}$$

$$R_3 \leftarrow \frac{-2R_3}{13} : \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 7/2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

$$R_2 \leftarrow R_2 - \frac{7R_3}{2} : \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}.$$

Thus, the only solution to our system of equations is

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Consequently, the list is independent in \mathbb{R}^3 .

2. Is the list of vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right\}$$

in \mathbb{Z}_5^3 linearly independent? Justify your response.

Solution. The list is *not* linearly independent (i.e, the list is linearly *dependent*).

Recall that all row operations done here are computed modulo 5. We first form our augmented matrix:

$$\begin{pmatrix} 2 & 4 & 0 & | & 0 \\ 1 & 2 & 2 & | & 0 \\ 3 & 1 & 4 & | & 0 \end{pmatrix}.$$

We apply Gaussian elimination to this augmented matrix:

$$R_1 \leftrightarrow R_2 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 1 & 4 & 0 \end{array} \right)$$

$$R_2 \leftarrow R_2 - 2R_1 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - 3R_1 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

$$R_3 \leftarrow R_3 - 3R_2 : \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$R_1 \leftarrow R_1 - 2R_2 : \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Thus, we obtain the solution

$$c_1 + 2c_2 = 0 \pmod{5}, \quad c_3 = 0 \pmod{5}$$

This implies that

$$c_1 = -2c_2 \pmod{5}, \quad c_3 = 0 \pmod{5}.$$

Letting $c_2 = 2$, $c_1 = -2 \cdot 2 = 3 \cdot 2 = 1 \pmod{5}$, and $c_3 = 0$, so we obtain the solution

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Consequently, the list is not linearly independent.

3. Describe the set

$$\text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

What geometric figure in \mathbb{R}^3 does it form?

Solution. Recall that the span of a list of vectors is the set of all linear combinations of those vectors. In our case:

$$\text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c_1 \cdot \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, c_1, c_2 \in \mathbb{R} \right\}.$$

Thus, all vectors in this span have the form

$$\begin{bmatrix} -2c_1 \\ c_1 \\ -3c_1 \end{bmatrix} + \begin{bmatrix} 5c_2 \\ c_2 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2c_1 + 5c_2 \\ c_1 + c_2 \\ -3c_1 + 2c_2 \end{bmatrix}, c_1, c_2 \in \mathbb{R}.$$

This set forms a plane through the origin, since the two vectors in the list are linearly independent (and thus are not scalar multiples of each other).

4. Is

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

where all vectors are in \mathbb{Z}_5^3 ? Justify your response.

Solution. The vector *is* in the span of the given list.

Let V be a vector space over the field \mathbb{F} . Recall that asking if a vector $v \in \text{span}\{v_1, \dots, v_n\}$, where $v_i \in V$, $1 \leq i \leq n$, is equivalent to asking whether there exist scalars $c_i \in \mathbb{F}$, $1 \leq i \leq n$ such that

$$v = c_1 v_1 + \dots + c_n v_n.$$

We can express this as an equation involving matrices.

$$(v_1 \quad \cdots \quad v_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = v.$$

Once again, Gaussian elimination proves to be an excellent tool to determine whether such solutions exist, and the values of such solutions should they exist. We form our augmented matrix:

$$\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 3 & 4 \\ 4 & 2 & 1 \end{array} \right).$$

We now apply Gaussian elimination. Recall that all computations are done modulo 5.

$$R_1 \leftrightarrow R_2 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 3 & 3 & 0 \\ 4 & 2 & 1 \end{array} \right).$$

$$R_2 \leftarrow R_2 - 3R_1 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 4 & 3 \\ 4 & 2 & 1 \end{array} \right).$$

$$R_3 \leftarrow R_3 - 4R_1 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{array} \right).$$

$$R_2 \leftarrow 4R_2 : \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right).$$

$$R_1 \leftarrow R_1 - 3R_2 : \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right).$$

So indeed,

$$\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

5. Identify a vector v in \mathbb{R}^3 such that when v is added to the list

$$\left\{ \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} \right\},$$

the resultant list is linearly independent.

Solution. An easy choice is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This is because linear dependence captures the relationship of proportionality, but

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is not proportional to } \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix}.$$

That is, there is no pair of scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 6 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

6. Is

$$\begin{bmatrix} 4 \\ 3 \\ -5 \\ 1 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} -2 \\ -6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} \right\},$$

where all vectors are in \mathbb{R}^4 ? Justify your response.

Solution. The given vector is *not* in the span of the given list of vectors.

We first form the corresponding augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ -6 & 1 & 5 & 3 \\ 2 & -9 & -3 & -5 \\ -2 & 2 & 1 & 1 \end{array} \right).$$

We now apply Gaussian elimination to this matrix:

$$R_2 \leftarrow R_2 - R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 2 & -9 & -3 & -5 \\ -2 & 2 & 1 & 1 \end{array} \right)$$

$$R_3 \leftarrow R_3 - R_1 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & -9 & -1 & -1 \\ -2 & 2 & 1 & 1 \end{array} \right)$$

$$R_4 \leftarrow R_4 - R_1 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & -9 & -1 & -1 \\ 0 & 2 & -1 & -3 \end{array} \right)$$

$$R_3 \leftarrow R_3 + 9R_2 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -10 & -82 \\ 0 & 2 & -1 & -3 \end{array} \right)$$

$$R_4 \leftarrow R_4 - 2R_2 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -10 & -82 \\ 0 & 0 & 1 & 15 \end{array} \right)$$

$$R_3 \leftrightarrow R_4 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array} \right)$$

$$R_2 \leftarrow R_2 + R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array} \right)$$

$$R_1 \leftarrow R_1 - 2R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 0 & -26 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & -10 & -82 \end{array} \right)$$

$$R_4 \leftarrow R_4 + 10R_3 : \left(\begin{array}{ccc|c} -2 & 0 & 0 & -26 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 15 \\ 0 & 0 & 0 & 68 \end{array} \right).$$

The fourth row of the last matrix gives us $0 = 68$, which implies that the system is inconsistent. Thus

$$\begin{bmatrix} 4 \\ 3 \\ -5 \\ 1 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} -2 \\ -6 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -9 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

7. (\star) Show that, for any list $\{v_1, \dots, v_n\}$ of vectors in a vector space V , $\text{span}\{v_1, \dots, v_n\}$ is the smallest (inclusion-wise) subspace of V containing $\{v_1, \dots, v_n\}$.

Solution. Let $U \subset V$ be a subspace with $v_1, \dots, v_n \in U$. Then $cv_i \in U$, $1 \leq i \leq n$ for all $c \in \mathbb{F}$, where \mathbb{F} is the underlying field of V , since subspaces are closed under scalar multiplication. Subspaces are also closed under addition, so

$$c_1v_1 + \dots + c_nv_n \in U, \quad c_i \in \mathbb{R}, \quad 1 \leq i \leq n.$$

Consequently, $\text{span}\{v_1, \dots, v_n\} \subset U$. Thus, any subspace of V containing $\{v_1, \dots, v_n\}$ contains $\text{span}\{v_1, \dots, v_n\}$.

8. Give an example of a list of vectors in \mathbb{R}^3 which is linearly *dependent*. Find some v in the span of the list such that there are two distinct linear combinations of the list equal to v .

Solution. Consider the list

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We have

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

9. Let V be a vector space over the field \mathbb{F} . Show that every list of vectors in the trivial subspace of V is both linearly independent and spanning (i.e., spans the entire trivial subspace).

Solution. The statement is actually not true! Any list containing 0 is linearly dependent, so $\{0\}$ is not linearly independent.

10. (*) Show that every spanning list of vectors $L = \{v_1, \dots, v_n\}$ in a vector space V can be made into a linearly independent list L' such that $\text{span } L = \text{span } L'$.

Solution. Let $L = \{v_1, \dots, v_n\}$ span V . If L is already linearly independent, then set $L' = L$, and we are done. If L is not linearly independent (i.e., linearly dependent), proceed as follows:

Begin with $L_0 = \emptyset$ (the empty list). For $i = 1, \dots, n$, if $v_i \notin \text{span}(L_{i-1})$, set $L_i = L_{i-1} \cup \{v_i\}$. Otherwise, set $L_i = L_{i-1}$ (that is, drop v_i from the list). Let $L' = L_n$. Because the list is finite, this process ends.

Evidently, $L' \subset L$, so $\text{span } L' \subset \text{span } L$. Conversely, every dropped v_i lies in $\text{span } L'$ by definition. Every dropped v_i was in $\text{span } L_{i-1} \subset \text{span } L'$ at the moment it was dropped. Hence, each $v_i \in \text{span } L'$, so $\text{span } L \subset \text{span } L'$, so $\text{span } L' = \text{span } L$.

The list L' is linearly independent by construction, since whenever we add a new vector, it is not in the span of the vectors already kept. This is exactly the criterion for linear independence.

Thus, L' is a linearly independent list with $\text{span}L' = \text{span}L$, and we are done.