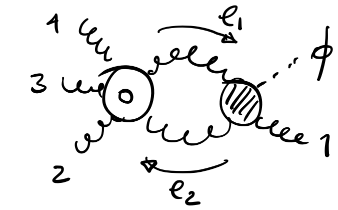


$-\ell_2=\ell_1+P\,,\quad P=2+3+4$

$$\frac{1}{3}\sum_x \frac{\mathrm{tr}_-(x)}{\langle 23\rangle\cdots\langle \ell_1\ell_2\rangle}\, A^{\text{tree}}(\phi,\, 1^+,\, (-\ell_1)^-,\, \ell_2^-)$$

$$=\frac{1}{3}\,\frac{\sum_x \mathrm{tr}_-(x)}{\langle 23\rangle\langle 34\rangle\langle 4\ell_1\rangle\cancel{\langle \ell_1\ell_2\rangle}\langle \ell_2 2\rangle}\;\frac{-\langle \ell_2\ell_1\rangle^{\cancel{3}\,2}}{\langle \ell_1 1\rangle\langle 1\ell_2\rangle}$$

$$=\frac{1}{3}\left(\sum_x \mathrm{tr}_-(x)\right)\frac{1}{\langle 23\rangle\langle 34\rangle}\;\frac{\langle \ell_2\ell_1\rangle^2}{\langle 4\ell_1\rangle\langle 1\ell_1\rangle\langle 1\ell_2\rangle\langle \ell_2 2\rangle}$$

$$\sum_x \mathrm{tr}_-(x) = \mathrm{tr}_-(34\ell_1 2) + \boxed{\mathrm{tr}_-(34\ell_1(-\ell_2))} + \mathrm{tr}_-(3\ell_1(-\ell_2)2)$$
$$+\,\mathrm{tr}_-(34(-\ell_2)2) + \mathrm{tr}_-(4\ell_1(-\ell_2)2)$$
$$= \mathrm{tr}_-(34(\ell_2-3)2) + \mathrm{tr}_-(3(-2-4)(-\ell_2)2) +$$
$$+\,\mathrm{tr}_-(34(-\ell_2)2) + \mathrm{tr}_-(4(-2-3)(-\ell_2)2) + \boxed{\mathrm{tr}_-(34\ell_1(-\ell_2))}$$

$$\boxed{\mathrm{tr}_-(34\ell_1(-\ell_2))} = \mathrm{tr}_-((-\ell_1-2)4\,\ell_1(-\ell_2))$$
$$= +\,\mathrm{tr}_-(\ell_1 4\ell_1\ell_2) + \mathrm{tr}_-(24\ell_1(+\ell_2))$$
$$= -\,\mathrm{tr}_-(\ell_1 4(2+3)\ell_2) + \mathrm{tr}_-(2(\ell_2-3)\ell_1\ell_2)$$
$$= -\,\mathrm{tr}_-(\ell_1 4 2\ell_2) - \mathrm{tr}_-(\ell_1 4 3\ell_2)$$
$$+\,\mathrm{tr}_-(2\ell_2\ell_1\ell_2) + \mathrm{tr}_-(23(-4-2)\ell_2)$$
$$= -\,\mathrm{tr}_-(\ell_1 4 2(3+4)) - \mathrm{tr}_-(\ell_1 4 3(2+4))$$
$$+\,\mathrm{tr}_-(2\ell_2\ell_1(3+4)) + \mathrm{tr}_-(234\ell_2) + \mathrm{tr}_-(232\ell_2)$$
$$= -\,\mathrm{tr}_-(\ell_1 4 2(3+4)) - \mathrm{tr}_-(\ell_1 4 3(2+4))$$
$$+\,\mathrm{tr}_-(2\ell_2(-2-4)3) + \mathrm{tr}_-(2\ell_2(-2-3)4)$$
$$+\,\mathrm{tr}_-(234(\ell_1+3)) + \mathrm{tr}_-(232\ell_2)$$

Mathematica

$$\to\ \text{amp} = \underbrace{\sum \#\,\frac{\langle \ell_1\ell_2\rangle^2[\#,\ell_2]}{\langle \#\,\ell_1\rangle\langle \#\,1\rangle\langle \#\,\ell_2\rangle}}_{(I)} + \underbrace{\sum \#\,\frac{\langle \ell_1\ell_2\rangle^2[\#,\ell_1]}{\langle \#\,\ell_1\rangle\langle \#\,\ell_2\rangle\langle \#\,\ell_2\rangle}}_{(II)}$$

Now we use Schoten id.

(I) $$\frac{\langle e_1 e_2\rangle}{\langle a e_1\rangle\langle b e_1\rangle} \to ?$$

$$\langle e_1 e_2\rangle\langle ab\rangle + \langle e_1 a\rangle\langle b e_2\rangle + \langle e_1 b\rangle\langle e_2 a\rangle = 0$$

$$\langle e_1 e_2\rangle = \frac{1}{\langle ab\rangle}\left(-\langle e_1 a\rangle\langle b e_2\rangle - \langle e_1 b\rangle\langle e_2 a\rangle\right)$$

$$\frac{\langle e_1 e_2\rangle}{\langle a e_1\rangle\langle b e_1\rangle} = \frac{1}{\langle ab\rangle}\left(\frac{\langle b e_2\rangle}{\langle b e_1\rangle} - \frac{\langle a e_2\rangle}{\langle a e_1\rangle}\right)$$

$$\frac{\langle e_1 e_2\rangle[c e_2]}{\langle a e_1\rangle\langle b e_1\rangle} = \frac{[c e_2]}{\langle ab\rangle}\left(\frac{\langle b e_2\rangle}{\langle b e_1\rangle} - \frac{\langle a e_2\rangle}{\langle a e_1\rangle}\right)$$

(II) $$\frac{-\langle e_1 e_2\rangle}{\langle a e_2\rangle\langle b e_2\rangle} \to ?$$

$$\langle e_2 e_1\rangle\langle ab\rangle + \langle e_2 a\rangle\langle b e_1\rangle + \langle e_2 b\rangle\langle e_1 a\rangle = 0$$

$$\langle e_2 e_1\rangle\langle ab\rangle = -\langle e_2 a\rangle\langle b e_1\rangle - \langle e_2 b\rangle\langle e_1 a\rangle$$

$$\langle e_2 e_1\rangle = \frac{1}{\langle ab\rangle}\left(-\langle e_2 a\rangle\langle b e_1\rangle - \langle e_2 b\rangle\langle e_1 a\rangle\right)$$

$$\frac{\langle e_2 e_1\rangle}{\langle a e_2\rangle\langle b e_2\rangle} = \frac{1}{\langle ab\rangle}\left(\frac{\langle b e_1\rangle}{\langle b e_2\rangle} - \frac{\langle a e_1\rangle}{\langle a e_2\rangle}\right)$$

$$\frac{\langle e_1 e_2\rangle}{\langle a e_2\rangle\langle b e_2\rangle} = \frac{1}{\langle ba\rangle}\left(\frac{\langle b e_1\rangle}{\langle b e_2\rangle} - \frac{\langle a e_1\rangle}{\langle a e_2\rangle}\right)$$

$$\frac{\langle e_1 e_2\rangle[c e_1]}{\langle a e_2\rangle\langle b e_2\rangle} = \frac{[c e_1]}{\langle ba\rangle}\left(\frac{\langle b e_1\rangle}{\langle b e_2\rangle} - \frac{\langle a e_1\rangle}{\langle a e_2\rangle}\right)$$

→ Mathematica

Substitution to simplify the structure:

$$\# \quad \frac{\langle 4\,\ell_2\rangle\langle \ell_1\ell_2\rangle[2\ell_2]}{\langle 1\,\ell_2\rangle\langle 4\ell_1\rangle} \longrightarrow \frac{\langle \ell_1\ell_2\rangle\left([2\ell_1]\langle 4\ell_1\rangle + [23]\langle 43\rangle\right)}{\langle 1\,\ell_2\rangle\langle 4\,\ell_1\rangle}$$

Now we use momentum conservation to obtain some simplifications and we obtain an expression of terms with the following structure

$$\#\,\frac{\langle a\,\ell_1\rangle}{\langle b\,\ell_1\rangle} \;,\quad \#\,\frac{\langle a\,\ell_2\rangle}{\langle b\,\ell_2\rangle}$$

$$\#\,\frac{\langle \ell_1\ell_2\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle} + \#\,\frac{\langle a\,\ell_2\rangle\langle b\,\ell_1\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle}$$

$Tr_-(ba\ell_1\ell_2) = \frac{1}{2}Tr$

- $$\frac{\langle \ell_1\ell_2\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle} = \frac{-[a\,\ell_1\ell_2 b]}{\langle a\ell_1 a]\langle b\ell_2 b]} = \frac{1}{\langle ab\rangle}\,\frac{\langle b\,a\,\ell_1\ell_2\,b]}{\langle a\ell_1 a]\langle b\ell_2 b]}$$

$$= \frac{1}{\langle ab\rangle}\,\frac{2(b\cdot a)(\ell_1\cdot\ell_2) - 2(b\cdot\ell_1)(a\cdot\ell_2) + 2(b\cdot\ell_2)(a\cdot\ell_1)}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

$$= \frac{1}{\langle ab\rangle}\Big(-P^2(b\cdot a) - 2(b\cdot\ell_2 - b\cdot P)(a\cdot\ell_1 + a\cdot P) + 2(b\cdot\ell_2)(a\cdot\ell_1)\Big)\frac{1}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

$$= \frac{1}{\langle ab\rangle}\Big(-P^2(b\cdot a) + 2\,b\cdot P\;a\cdot\ell_1 - 2\,b\cdot\ell_2\;a\cdot P + 2\,b\cdot P\;a\cdot P\Big)\frac{1}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

$$= \frac{1}{\langle ab\rangle}\left(\frac{2\,b\cdot P\;a\cdot P - P^2\,b\cdot a}{(2a\cdot\ell_1)(2b\cdot\ell_2)} + \frac{b\cdot P}{2b\cdot\ell_2} - \frac{a\cdot P}{2a\cdot\ell_1}\right)$$

- $$\frac{\langle a\,\ell_2\rangle\langle b\,\ell_1\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle} = \frac{\langle a\ell_2 b]\langle b\ell_1 a]}{\langle a\ell_1 a]\langle b\ell_2 b]} = \frac{\frac{1}{2}Tr(a\ell_2 b\ell_1)}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

$$= \frac{2(a\cdot\ell_2)(b\cdot\ell_1) - 2(a\cdot b)(\ell_2\cdot\ell_1) + 2(a\cdot\ell_1)(b\cdot\ell_2)}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

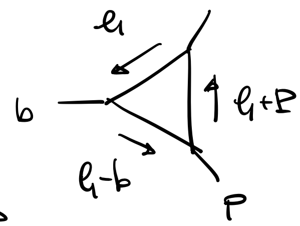
$$= \frac{2(a\cdot\ell_1 + a\cdot P)(b\cdot\ell_2 - b\cdot P) + P^2(a\cdot b) - 2(a\cdot\ell_1)(b\cdot\ell_2)}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

$$= \frac{-2(a\cdot\ell_1)\,b\cdot P + 2\,a\cdot P\,(b\cdot\ell_2) - 2\,a\cdot P\;b\cdot P + P^2 a\cdot b}{(2a\cdot\ell_1)(2b\cdot\ell_2)}$$

$$= \frac{-2a.P\, b.P + P^2 a.b}{(2a.\ell_1)(2b.\ell_2)} - \frac{b.P}{2b.\ell_2} + \frac{a.P}{2a.\ell_1}$$

- $$\frac{\langle a\ell_1\rangle}{\langle b\ell_1\rangle} = \frac{\langle a\ell_1 b]}{\langle b\ell_1 b]} = \frac{\langle a\ell_1 b]}{(2\ell_1\cdot b)} \cdot \underbrace{\frac{1}{\ell_1^2}\,\frac{1}{(\ell_1+P)^2}}_{\text{implicit dependence that comes from the cut propagators}}$$

$$\ell_1^\mu = \alpha P^\mu + \beta b^\mu$$



$$\begin{cases} \ell_1\cdot P = \alpha P^2 + \beta\, b\cdot P \\ \ell_1\cdot b = \alpha\, P\cdot b \end{cases}$$

$$\begin{cases} \frac{1}{2}\left((\ell_1+P)^2 - P^2\right) = \alpha P^2 + \beta\, b\cdot P \\ \frac{1}{2}\left(-(\ell_1-b)^2\right) = \alpha\, P\cdot b \end{cases}$$

$$\begin{cases} \alpha = -\frac{1}{2P\cdot b}(\ell_1-b)^2 \\ \beta\, b.P = \frac{1}{2}\left(\underbrace{\cancel{(\ell_1+P)^2}}_{\text{on-shell}} - P^2\right) - \alpha P^2 \end{cases}$$

$$\beta\, b.P = -\frac{1}{2}P^2 + \frac{P^2}{2P\cdot b}(\ell_1-b)^2$$

$$\begin{cases} \beta = \frac{P^2}{-2b.P}\left(1 - \frac{(\ell_1-b)^2}{P.b}\right) \\ \alpha = -\frac{1}{2P.b}(\ell_1-b)^2 \end{cases}$$

$$\frac{\ell_1^\mu}{2\ell_1\cdot b} = \frac{\alpha P^\mu + \beta b^\mu}{-(\ell_1-b)^2} = \frac{1}{2P.b}P^\mu + b^\mu(\ldots)$$

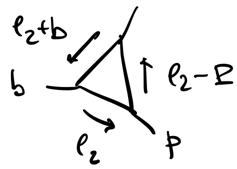
$$\frac{\langle a\ell_1 b]}{(2\ell_1\cdot b)} \longrightarrow \frac{1}{2P.b}\langle aPb] + (\ldots)\underbrace{\langle abb]}_{=0}$$

$$\boxed{\frac{\langle a\ell_1\rangle}{\langle b\ell_1\rangle} = \frac{\langle aPb]}{2P.b} = \frac{\langle aPb]}{\langle bPb]}}$$

- A similar computation holds for

$$\frac{\langle a \ell_2\rangle}{\langle b \ell_2\rangle} = \frac{\langle a \ell_2 b]}{\langle b \ell_2 b]} = \frac{\langle a \ell_2 b]}{2\,\ell_2 \cdot b} \quad \frac{1}{\ell_2^2}\,\frac{1}{(\ell_2-P)^2}$$

$$\rightarrow \quad \ell_2^\mu = \alpha P^\mu + \beta b^\mu$$



$$\ell_2\cdot b = \alpha\, b\cdot P$$

$$\tfrac{1}{2}(\ell_2+b)^2 = \alpha\, b\cdot P$$

$$\alpha = \frac{1}{2b\cdot P}(\ell_2+b)^2$$

$$\rightarrow \quad \frac{\ell_2^\mu}{(\ell_2+b)^2} = \frac{\alpha P^\mu}{(\ell_2+b)^2} + (\dots)b^\mu$$

$$= \frac{1}{2b\cdot P} P^\mu + (\dots) b^\mu$$

$$\frac{\langle a \ell_2 b]}{(\ell_2+b)^2} = \frac{1}{2b\cdot P}\langle a P b] + (\dots)\,\cancel{\langle a b b]}^{=0}$$

$$\rightarrow \quad \frac{\langle a \ell_2\rangle}{\langle b \ell_2\rangle} = \frac{\langle a P b]}{\langle b P b]}$$