Chapter 5

# Cut constructible pieces (2L $\phi + 4g^+$) from 4d cuts

## 5.1 Normalization of the color decomposition

In the self-dual Higgs model, we used the following color organization for one-loop amplitudes:

$$\mathcal{A}^{1L}(\phi;1,2,\ldots,n) = g^n C\, c_\Gamma \sum_{c=1}^{\lfloor n/2 \rfloor+1} \sum_{\sigma} Gr_{n;c} A^{1L}_{n,c}\left(\phi;\sigma(1,2,\ldots,n)\right).$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to $x$. Due to the coupling between $\phi$ and gluons, we explicitly extracted the effective constant $C$. Furthermore we organized the trace-based color decompositions inserting in the definition the standard loop factor

$$c_\Gamma := \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{(4\pi)^2} + \mathcal{O}(\epsilon).$$

In section [3] we used the following definition for one-loop QCD amplitudes as consider:

$$\mathcal{A}^{1L}(1,2,\ldots,n) = g^n c_\Gamma \sum_{c=1}^{\lfloor n/2 \rfloor+1} \sum_{\sigma} Gr_{n;c} A^{1L}_{n,c}\left(\sigma(1,2,\ldots,n)\right)$$

with the following expression for the partial amplitudes in the all-plus sector

$$A^{1L}_{n,1}(1^+,2^+,\ldots,n^+) = \frac{1}{3} \sum_{1\le k_1<k_2<k_3<k_4\le n} \frac{\langle k_1 k_2 k_3 k_4 k_1]}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} + \mathcal{O}(\epsilon). \tag{5.1}$$

The decomposition of the two-loop $\phi$+gluon amplitude will be

$$\mathcal{A}^{2L}(\phi,1,2,\ldots,n) = g^{n+2} c_\Gamma^2 N_C \sum_{c=1}^{\lfloor n/2 \rfloor+1} \sum_{\sigma} Gr_{n;c} A^{2L}_{n,c}\left(\phi;\sigma(1,2,\ldots,n)\right)$$

Keeping in mind that

$$Gr_{n;1}(1) = N_C \,\mathrm{Tr}\left(T^{a_1} T^{a_2} \ldots T^{a_n}\right),$$

the unrenormalized amplitude at leading color becomes

$$\mathcal{A}^{2L}(\phi,1,2,\ldots,n)|_{\text{leading color}} = g^{n+2} c_\Gamma^2 N_C^2 C \sum_{\sigma} \mathrm{Tr}\left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\right) A^{2L}_{n,1}\left(\phi;\sigma(1,2,\ldots,n)\right)$$

In this chapter, we compute the cut-constructible pieces of $A^{2L}_{n,1}\left(\phi;1^+,2^+,3^+,4^+\right)$ which represent the main purpose of the present project.

## 5.2 Structure of the two-loop amplitude

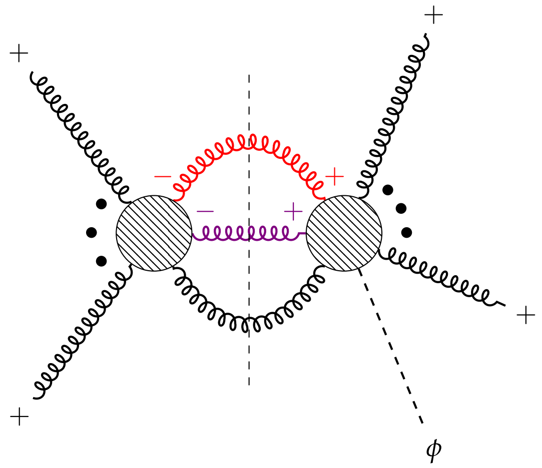
The unitarity condition of the S-matrix can be expanded perturbatively to find the discontinuity of the two-loop amplitude (2.17).

We observe the presence of double cuts in which we impose two on-shell constraints and consider the product of two sub-amplitudes respectively at tree and one-loop level. In our case with one self-dual field, we can separate the double cuts into two sectors. In the first one we consider a tree-level YM vertex and a one-loop sub-amplitude involving $\phi$, while in the second sector we will study the contributions from double cuts with a pure gluon interaction at one-loop level and a tree $\phi$+gluon sub-amplitude. To obtain the discontinuity, we have to integrate over the two-particle phase-space

$$\int \mathrm{d}\Phi_2 = \int \mathrm{d}^4\ell_1 \mathrm{d}^4\ell_2\, \delta^{(+)}(\ell_1^2)\delta^{(+)}(\ell_1^2)\delta^{(4)}(\ell_2 - \ell_1 - P_a), \tag{5.2}$$

and we will describe this contributions in terms of scalar integrals.

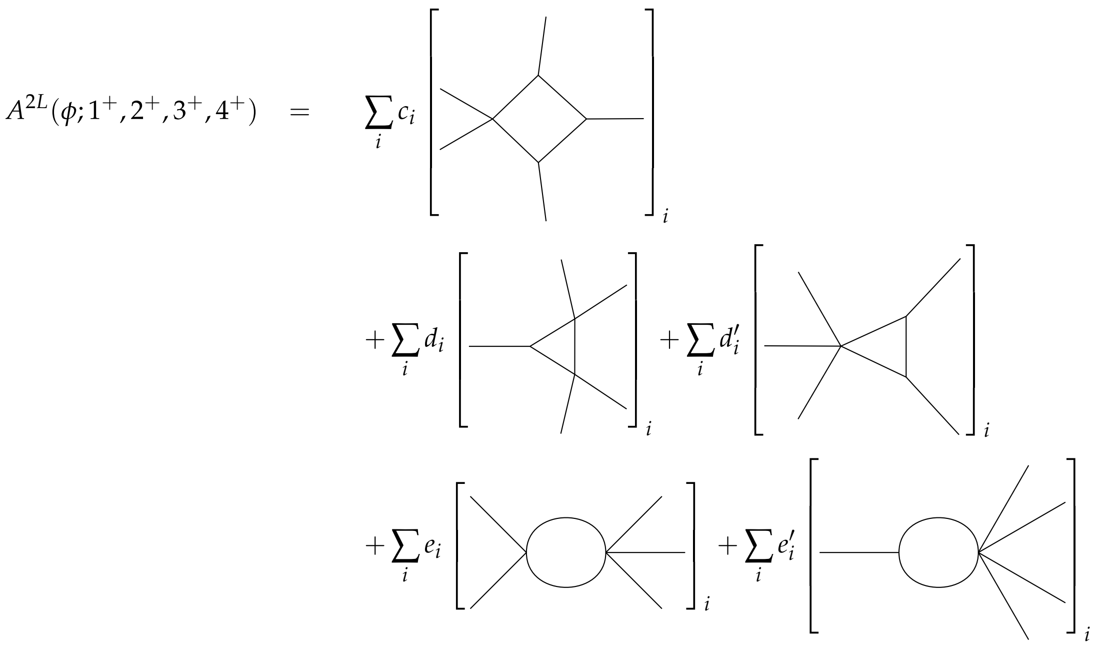
In principle, at two-loop level we can also have contributions from three-particle cuts with tree-level sub-amplitudes. In the all-plus configuration, the three-particle cuts vanish, thanks to the behavior of tree-level amplitudes. We can consider the following helicity configuration of the three-double cut in order to understand the absence of this contribution.




(5.3)

We have to consider at least two inner gluons with negative helicity to obtain a non-trivial result from the vertex involving only gluons. This causes the vanishing behavior of the sub-amplitude with the self-dual Higgs, indeed $A^{tree}(\phi; +, +, \ldots, +, \pm) = 0$.

The absence of three-particle cuts shows that the reduction of the two-loop amplitude corresponds to a one-loop integral decomposition. Then the discontinuities of the amplitude can be expressed in terms of scalar boxes, triangles and bubbles. The structure of the amplitude can be expressed pictorially in following way.

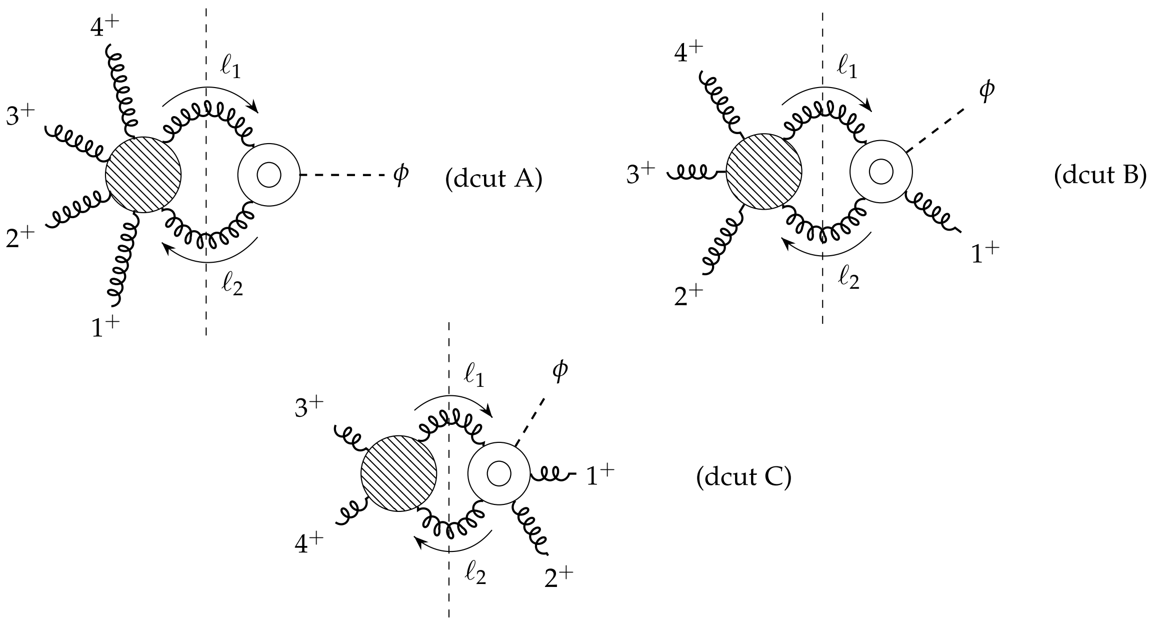


$$+\left[(d = 4-2\epsilon) \text{ finite contributions}\right] + \mathcal{O}(\epsilon)$$

Besides contributions proportional to the scalar one-loop integrals, in the amplitude we have also an effect due to the dimensional regularization which can produce a finite contribution. In the decomposition of the one-loop amplitude, this remainder part is rational.
In this chapter we will focus the attention on the contributions proportional to the scalar integrals. We will find the coefficient in terms of external kinematics using on-shell methods, separating the discontinuities into two sectors with different perturbative levels of YM and self-dual Higgs sub-amplitudes.

## 5.3 Double cuts with 1L $\phi$ amplitudes and tree YM amplitudes

In this section, we study the double cuts which factorize into the product of a one-loop $\phi$+gluon amplitude and a tree level gluon amplitude. Preliminary, we computed the quadruple cuts in order to isolate and extract the contributions proportional to the four-point integrals using simple calculations (see Appendix B).
Computing the double cuts, we will find all the discontinuities of the amplitude. We will compare the box contributions obtained with the two independent computations in order to have a partial check for the correctness of double cuts.
We have to compute three different double cuts to capture the discontinuities in the three channels $s_\phi, s_{\phi 1}, s_{34}$.




Due to the colorless of the self-dual Higgs, we are also interested in configurations obtained through a cyclic permutation of gluons. The last possible channel with a different structure is characterized by a single external gluon (for example, the fourth gluon) on the left-hand side and it does not contribute to the discontinuities of the amplitude (indeed $s_4$ vanishes).
In each double cut, we will reduce the product of the sub-amplitudes in terms of scalar integrals considering the possible helicity configuration of the inner gluons. With these computations, we will extract the coefficients of boxes, triangles and bubbles which characterized the discontinuities of the two-loop amplitude in this sector with a 1L $\phi$+gluon sub-amplitude.

### 5.3.1 Double cut in $s_\phi$ channel

Let us study the double cut represented by the diagram (dcut A).
We consider only gluons circulating in the loop: the other possible particles in our theory, the complex scalars $\phi$ and $\phi^\dagger$, can be coupled in a diagram only with an effective vertex which implies a top quark loop, then diagrams with inner scalars are associated to higher perturbative corrections.

In order to evaluate the double cut, we have to compute the following object,

$$
\begin{aligned}
A^{2L}_{int}|_{\text{dcut A}} &= \sum_{\lambda_1=\pm}\sum_{\lambda_2=\pm} A^{tree}(1^+,2^+,3^+,4^+,\ell_1^{\lambda_1},(-\ell_2)^{\lambda_2})A^{1L}(\phi;\ell_2^{-\lambda_2},(-\ell_1)^{-\lambda_1}) \\
&= A^{tree}(1^+,2^+,3^+,4^+,\ell_1^-,(-\ell_2)^-)A^{1L}(\phi;\ell_2^+,(-\ell_1)^+) \\
&= A^{1L}(\phi;1^+,2^+,3^+,4^+)\frac{\langle\ell_1\ell_2\rangle\langle 41\rangle}{\langle\ell_2 1\rangle\langle\ell_1 4\rangle}.
\end{aligned}
$$

We reconstruct the propagators at the denominator intending to write it in terms of scalar integrals,

$$
\begin{aligned}
\frac{A^{2L}_{int}|_{\text{dcut A}}}{A^{1L}(\phi;1^+,2^+,3^+,4^+)} &= \frac{\langle\ell_1\ell_2\rangle\langle 41\rangle}{\langle\ell_2 1\rangle\langle\ell_1 4\rangle} = \frac{\langle\ell_1\ell_2\rangle\langle 41\rangle}{\langle\ell_2 1\rangle\langle\ell_1 4\rangle}\frac{[1\ell_2][4\ell_1]}{[1\ell_2][4\ell_1]} = \frac{-\langle 14\ell_1\ell_2 1]}{(\ell_2-p_1)^2(\ell_1+p_4)^2} \\
&= -\frac{\mathrm{tr}_-(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)}{(\ell_2-p_1)^2(\ell_1+p_4)^2} = \frac{-\frac{1}{2}\,\mathrm{tr}(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)+\frac{1}{2}\,\mathrm{tr}_5(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)}{(\ell_2-p_1)^2(\ell_1+p_4)^2}.
\end{aligned}
$$

Let us focus our attention to the following term:

$$
\frac{\mathrm{tr}_5(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)}{(\ell_2-p_1)^2(\ell_1+p_4)^2} = \frac{\mathrm{tr}_5(\not{p}_1\not{p}_4\not{\ell}_1\not{P}_a)}{(\ell_1+P-p_1)^2(\ell_1+p_4)^2} \tag{5.4}
$$

where we introduce the momentum $P_a = p_1+p_2+p_3+p_4 = -p_\phi$.
The unitarity procedure requires the integration over the phase-space (5.2), then we can perform the substitution $\ell_1 \to -\ell_1 - p_4 - P_a + p_1$ in the integrand (5.4) obtaining the following integral expression:

$$
\begin{aligned}
\int \mathrm{d}\Phi_2 \frac{\mathrm{tr}_5(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)}{(\ell_2-p_1)^2(\ell_1+p_4)^2} &= \int \mathrm{d}\Phi_2 \frac{\mathrm{tr}_5(\not{p}_1\not{p}_4(-\not{\ell}_1-\not{p}_4-\not{P}_a+\not{p}_1)\not{P}_a)}{(-\ell_1-p_4)^2(-\ell_1-P_a+p_1)^2} \\
&= \int \mathrm{d}\Phi_2 \frac{-\mathrm{tr}_5(\not{p}_1\not{p}_4\not{\ell}_1\not{P}_a)}{(\ell_1+p_4)^2(\ell_1+P_a-p_1)^2} = -\int \mathrm{d}\Phi_2 \frac{\mathrm{tr}_5(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)}{(\ell_2-p_1)^2(\ell_1+p_4)^2}.
\end{aligned}
$$

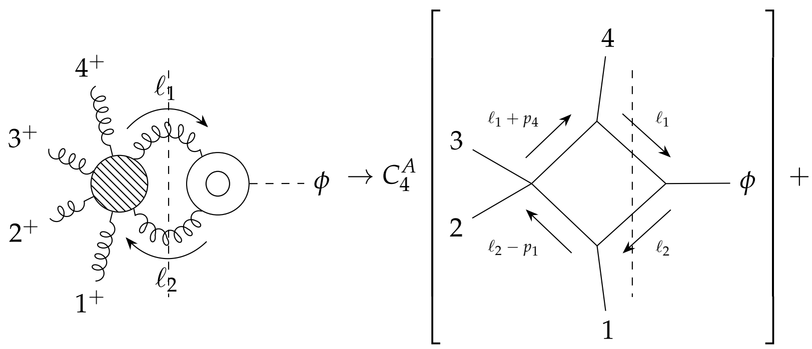
We used the antisymmetry property of $\gamma_5$ to show that

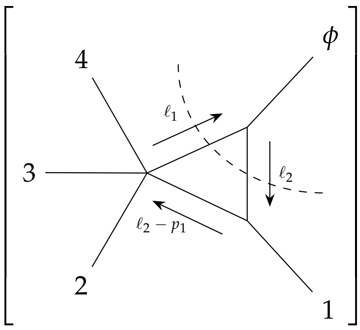
$$
\mathrm{tr}_5(\not{p}_1\not{p}_4\not{q}\not{P}_a) = 0 \quad \text{for} \quad q = p_1, p_4 \text{ or } P_a.
$$

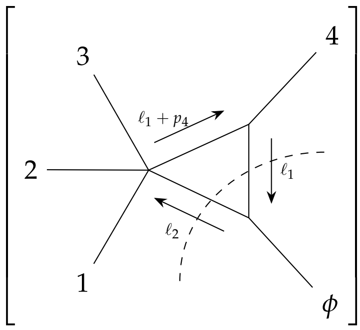
This computation shows that the contribution proportional to $\mathrm{tr}_5$ represents a spurious term which vanishes after the phase-space integration. We are left with the following contribution

$$
\begin{aligned}
\frac{A^{2L}_{int}|_{\text{dcut A}}}{A^{1L}(\phi;1^+,2^+,3^+,4^+)} &= \frac{-\frac{1}{2}\,\mathrm{tr}(\not{p}_1\not{p}_4\not{\ell}_1\not{\ell}_2)}{(\ell_2-p_1)^2(\ell_1+p_4)^2} + \text{s.t.} = \frac{-\frac{1}{2}\,\mathrm{tr}(\not{p}_1\not{p}_4\not{\ell}_1\not{P}_a)}{(\ell_2-p_1)^2(\ell_1+p_4)^2} + \text{s.t.} = \\
&= \frac{(p_1\cdot p_4)P_a^2 - 2(p_1\cdot P_a)(p_4\cdot P_a)}{(\ell_2-p_1)^2(\ell_1+p_4)^2} - \frac{p_4\cdot P_a}{(\ell_1+p_4)^2} - \frac{p_1\cdot P_a}{(\ell_2-p_1)^2} + \text{s.t.}
\end{aligned}
$$

where "s.t." implies the presence of spurious terms which vanish after the phase-space integration.
We represent our result diagrammatically



$+C^A_{3,1}$ 

$+\,C^A_{3,2}$ 


where the coefficients in front of scalar integrals are

$$
\begin{aligned}
C_4^A &:= A^{1L}(\phi;1^+,2^+,3^+,4^+)\left[(p_1\cdot p_4)P_a^2 - 2(p_1\cdot P)(p_4\cdot P_a)\right]\\
&= \frac{1}{2}A^{1L}(\phi;1^+,2^+,3^+,4^+)\left[s_{14}s_\phi - (s_{1\phi}-s_\phi)(s_{\phi 4}-s_\phi)\right]\\
C_{3,1}^A &:= A^{1L}(\phi;1^+,2^+,3^+,4^+)\left[-p_1\cdot P_a\right] = \frac{1}{2}(s_{\phi 1}-s_\phi)A^{1L}(\phi;1^+,2^+,3^+,4^+)\\
C_{3,2}^A &:= A^{1L}(\phi;1^+,2^+,3^+,4^+)\left[-p_4\cdot P_a\right] = \frac{1}{2}(s_{4\phi}-s_\phi)A^{1L}(\phi;1^+,2^+,3^+,4^+).
\end{aligned}
$$

In a more compact form, we have

$$
\begin{aligned}
A^{2L}|_{\text{dcut A}} \equiv \int \mathrm{d}\Phi_2\, A^{2L}_{int}|_{\text{dcut A}} =& C_4^A\, I_4^{2me}(s_{\phi 4}, s_{1\phi}; m_1^2 = s_{23}, m_2^2 = s_\phi)\Big|_{s_\phi\text{-cut}}\\
&+ C^A_{3,1} I_3^{2m}(s_{1\phi}, s_\phi)|_{s_\phi\text{-cut}} + C^A_{3,2} I_3^{2m}(s_{\phi 4}, s_\phi)|_{s_\phi\text{-cut}}
\end{aligned}
$$

where we have introduced the easy two-mass box and the three-point integral with two massive vertices which expressions can be found in App. (A).
The coefficient $C_4^A$ is consistent with the result from quadruple cuts (B.9). Investigating the amplitude using the double cut, we were able to extract more information: we obtained the coefficients of three-point two-mass integrals and the absence of other triangles or bubbles.

### 5.3.2 Double cut in $s_{\phi 1}$ channel

The next double cut concerns the $s_{\phi 1}$ channel (dcut B). The discontinuities along this channel are related to the phase-space integration of the following quantity.

$$
\begin{aligned}
A^{2L}_{int}|_{\text{dcut B}} &= \sum_{\lambda_1=\pm}\sum_{\lambda_2=\pm} A^{tree}(2^+,3^+,4^+,\ell_1^{\lambda_1},(-\ell_2)^{\lambda_2})A^{1L}(\phi;1^+,\ell_2^{-\lambda_2},(-\ell_1)^{-\lambda_1})\\
&= A^{tree}(2^+,3^+,4^+,\ell_1^-,(-\ell_2)^-)A^{1L}(\phi;1^+,\ell_2^+,(-\ell_1)^+)\\
&= \frac{-2m_H^4}{\langle 23\rangle\langle 34\rangle}\frac{\langle \ell_1\ell_2\rangle^2}{\langle 1\ell_1\rangle\langle 1\ell_2\rangle\langle 2\ell_2\rangle\langle 4\ell_1\rangle}
\end{aligned}
$$

In this step of the calculation, we find the Schouten identity very useful for reducing the complexity of the object. In our case we apply the substitutions

$$
\begin{aligned}
\frac{\langle \ell_2\ell_1\rangle}{\langle 1\ell_2\rangle\langle 2\ell_2\rangle} &= \frac{1}{\langle 12\rangle}\left(\frac{\langle 1\ell_1\rangle}{\langle \ell_2 1\rangle} + \frac{\langle \ell_1 2\rangle}{\langle \ell_2 2\rangle}\right),\\
\frac{\langle \ell_1\ell_2\rangle}{\langle 1\ell_1\rangle\langle 4\ell_1\rangle} &= \frac{1}{\langle 14\rangle}\left(\frac{\langle 1\ell_2\rangle}{\langle \ell_1 1\rangle} + \frac{\langle \ell_2 4\rangle}{\langle \ell_1 4\rangle}\right).
\end{aligned}
$$

and we obtain

$$
A^{2L}_{int}|_{\text{dcut B}} = A^{1L}(\phi;1^+,2^+,3^+,4^+)(-1+\Sigma_1+\Sigma_2+\Sigma_3)
$$

where

$$
\Sigma_1 := \frac{\langle 1\ell_1\rangle\langle 4\ell_2\rangle}{\langle \ell_2 1\rangle\langle \ell_1 4\rangle},\quad \Sigma_2 := \frac{\langle \ell_1 2\rangle\langle \ell_2 1\rangle}{\langle \ell_2 2\rangle\langle \ell_1 1\rangle},\quad \Sigma_3 := \frac{\langle \ell_1 2\rangle\langle 4\ell_2\rangle}{\langle \ell_2 2\rangle\langle \ell_1 4\rangle}.
$$

We can easily simplify every addend reconstructing the propagators at the denominator. For example, let us focus on the term

$$\Sigma_1 = \frac{\langle 1\ell_1 4] \langle 4\ell_2 1]}{\langle 1\ell_2 1] \langle 4\ell_1 4]} = \frac{\mathrm{tr}_-\left(\not{p}_1 \not{\ell}_1 \not{p}_4 \not{\ell}_2\right)}{(\ell_2+p_1)^2(\ell_1+p_4)^2}.$$

Avoiding terms which vanish after the integration, we obtain

$$\begin{aligned}\Sigma_1 &= \frac{\frac{1}{2}\,\mathrm{tr}\left(\not{p}_1 \not{\ell}_1 \not{p}_4 \not{\ell}_2\right)}{(\ell_2+p_1)^2(\ell_1+p_4)^2} + \text{spurious terms} \\ &= 1 - \frac{\frac{1}{2}\,\mathrm{tr}\left(\not{p}_1 \not{p}_4 \not{\ell}_1 \not{\ell}_2\right)}{(\ell_2+p_1)^2(\ell_1+p_4)^2} + \text{s.t.} = 1 - \frac{\frac{1}{2}\,\mathrm{tr}\left(\not{p}_1 \not{p}_4 \not{\ell}_1 \not{P}_b\right)}{(\ell_2+p_1)^2(\ell_1+p_4)^2} + \text{s.t.}\end{aligned}$$

where in the last passage we introduced $P_b = p_2 + p_3 + p_4 = -p_1 - p_\phi$ and we used the on-shell property $\ell_2^2 = 0$.
Expanding the trace of four Gamma matrices, we have

$$\Sigma_1 = 1 - \frac{2(p_1\cdot P_b)(p_4\cdot P_b) - (p_1 \cdot_4)P_b^2}{(\ell_2+p_1)^2(\ell_1+p_4)^2} + \frac{p_4\cdot P_b}{(\ell_1+p_4)^2} - \frac{p_1\cdot P_b}{(\ell_2+p_1)^2}.$$

Similarly, one can compute the other two addends obtaining

$$\begin{aligned}\Sigma_2 &= 1 - \frac{2(p_2\cdot P_b)(p_1\cdot P_b) - (p_1\cdot p_2)P_b^2}{(\ell_2-p_2)^2(\ell_1-p_1)^2} - \frac{p_1\cdot P_b}{(\ell_1-p_1)^2} + \frac{p_2\cdot P_b}{(\ell_2-p_2)^2} + \text{s.t.}, \\ \Sigma_3 &= -1 - \frac{2(p_2\cdot P_b)(p_4\cdot P_b) - (p_2\cdot p_4)P_b^2}{(\ell_2-p_2)^2(\ell_1+p_4)^2} - \frac{p_4\cdot P_b}{(\ell_1+p_4)^2} - \frac{p_2\cdot P_b}{(\ell_2-p_2)^2} + \text{s.t.}\end{aligned}$$

Summing the contributions, we find

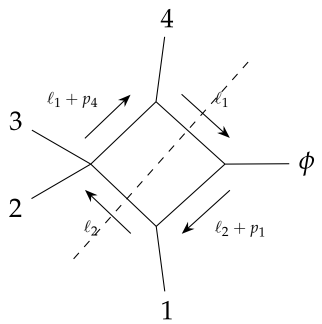
$$\begin{aligned}A^{2L}|_{\text{dcut B}} \int \mathrm{d}\Phi_2 A^{2L}_{int}|_{\text{dcut B}} =& C^B_{4,1} I_4^{2me}(s_{\phi 4}, s_{1\phi}; s_{23}, s_\phi)|_{s_{\phi 1}\text{-cut}} + C^B_{4,2} I_4^{2me}(s_{1\phi}, s_{\phi 2}; s_{34}, s_\phi)|_{s_{\phi 1}\text{-cut}} + \\ & C^B_{4,3} I_4^{1m}(s_{23}, s_{34}; s_{1\phi})|_{s_{\phi 1}\text{-cut}} + 2C^B_3 I_3^{2m}(s_{1\phi}, s_\phi)|_{s_{\phi 1}\text{-cut}}\end{aligned} \tag{5.5}$$

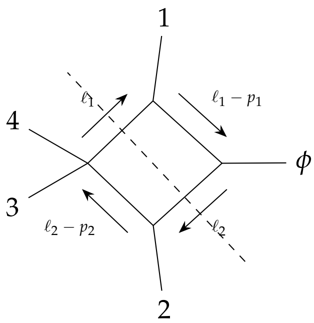
with the coefficients

$$\begin{aligned}C^B_{4,1} &:= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[(p_1\cdot p_4)P_b^2 - 2(p_1\cdot P_b)(p_4\cdot P_b)\right] \\ &= \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[s_{14}s_\phi - (s_{1\phi}-s_\phi)(s_{4\phi}-s_\phi)\right], \\ C^B_{4,2} &:= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[(p_1\cdot p_2)P_b^2 - 2(p_1\cdot P_b)(p_2\cdot P_b)\right] \\ &= \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[s_{12}s_\phi - (s_{1\phi}-s_\phi)(s_{2\phi}-s_\phi)\right], \\ C^B_{4,3} &:= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[(p_2\cdot p_4)P_b^2 - 2(p_2\cdot P_b)(p_4\cdot P_b)\right], \\ C^B_3 &:= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[-p_1\cdot P_b\right] = \frac{1}{2}(s_{\phi 1}-s_\phi)A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+).\end{aligned}$$

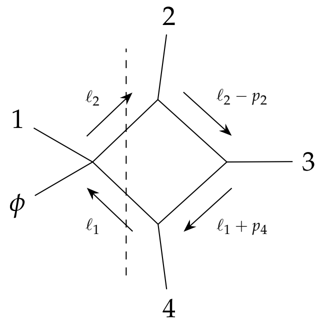
We have two easy boxes with only a different organization of legs, in fact in the amplitude $\phi$ is unordered, therefore investigating the $\phi 1$ channel we find two four-point contributions with different orders but with the same relative positions between gluons.

$$I_4^{2me}(s_{\phi 4}, s_{1\phi}; s_{23}, s_\phi)|_{s_{\phi 1}\text{-cut}} =$$



$I_4^{2me}(s_{1\phi}, s_{\phi 2}; s_{34}, s_\phi)|_{s_{\phi 1}\text{-cut}} =$ 


The coefficients $C^B_{4,1}$ and $C^B_{4,2}$ are consistent with the results from quadruple cuts (B.9).
In this cut, we also observe the presence of a contribution proportional to the one-mass box

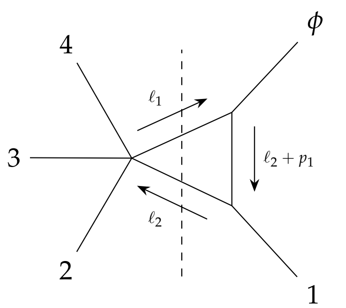
$I_4^{1m}(s_{23}, s_{34}; s_{1\phi})|_{s_{\phi 1}\text{-cut}} =$ 

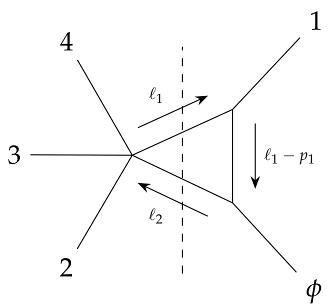

which cannot be seen studying the double cut in the $s_\phi$ channel.
The coefficient is

$$
\begin{aligned}
C^B_{4,3} &= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left[(p_2 \cdot p_4)(p_2 + p_3 + p_4)^2 - 2(p_2 \cdot (p_3 + p_4))(p_4 \cdot (p_2 + p_3))\right] \\
&= \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\,[s_{24}(s_{23} + s_{24} + s_{34}) - (s_{23} + s_{24})(s_{24} + s_{34})] \\
&= -\frac{1}{2} s_{23} s_{34} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)
\end{aligned}
$$

and it is equal to $d_1^{1m}$ computed using quadruple cuts (B.16).

From this double cuts we are able to extract information about triangles and bubbles. We observe that the only non-vanishing three-point contribution in this channel is $I_3^{2m}(s_{1\phi}, s_\phi)$ which comes from the following two diagrams.

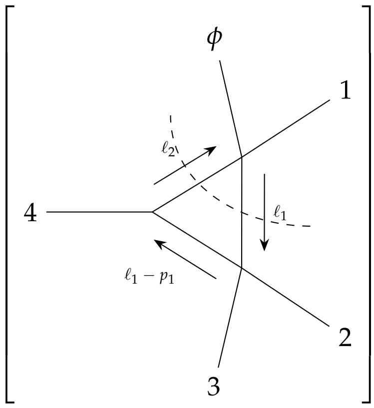



These two equivalent contributions proportional to the three-point integral are related to the switch of the legs $\phi$ and 1. Although the topology of these contributions is not the same, the scalar integrals can be mapped in each other through a simple change of variables: for this reason we collect together the two objects.
Obviously the other triangle $I_3^{2m}(s_{4\phi}, s_\phi)$ detected using the first double cut cannot be obtained in the $s_{\phi 1}$ channel. But using the present double cut, we could expect other three-point contributions, for example in the intermediate steps we observe the presence of terms like the following which emerges

in $\Sigma_1$:

$$\frac{p_4 \cdot P_b}{(\ell_1 + p_4)^2} \to (p_4 \cdot P_b)$$




The same contribution with an opposite kinematical coefficient comes from $\Sigma_3$, so there is no contribution proportional to $I_3^{2m}(s_{23}, s_{\phi 1})$ in the amplitude. The same fact holds for the integral $I_3^{2m}(s_{34}, s_{\phi 1})$ which appears in intermediate steps but vanishes in the sum. In conclusion we observe simplifications such that in the result (5.5) three-point contributions with a $s_{\phi 1}$ vertex are absent.
We finish pointing out that in $s_{\phi 1}$ channel the double cut does not show bubble contributions.

### 5.3.3 Double cut in $s_{34}$ channel

Using Schouten identity in the first step of the double cut in $s_{\phi 1}$ channel, we were able to reduce the amplitude in terms of boxes, triangles and (absent) bubbles.
We want to use the same method for the double cut in $s_{34}$ channel (dcut C). We have to compute

$$\begin{aligned}
A_{int}^{2L}|_{\text{dcut C}} &= \sum_{\lambda_1=\pm} \sum_{\lambda_2=\pm} A^{tree}(3^+, 4^+, \ell_1^{\lambda_1}, (-\ell_2)^{\lambda_2}) A^{1L}(\phi; 1^+, 2^+, \ell_2^{-\lambda_2}, (-\ell_1)^{-\lambda_1}) \\
&= A^{tree}(3^+, 4^+, \ell_1^-, (-\ell_2)^-) A^{1L}(\phi; 1^+, 2^+, \ell_2^+, (-\ell_1)^+) \\
&= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \left( \Sigma_1' + \Sigma_2' + \Sigma_3' + \Sigma_4' \right)
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1' &= \frac{\langle 2\ell_1 \rangle \langle 4\ell_2 \rangle}{\langle \ell_2 2 \rangle \langle \ell_1 4 \rangle}, \qquad \Sigma_2' = \frac{\langle 2\ell_1 \rangle \ell_2 1 \rangle}{\langle \ell_2 2 \rangle \langle \ell_1 1 \rangle}, \\
\Sigma_3' &= \frac{\langle \ell_1 3 \rangle \langle 4\ell_2 \rangle}{\langle \ell_2 3 \rangle \langle \ell_1 4 \rangle}, \qquad \Sigma_4' = \frac{\langle \ell_1 3 \rangle \langle \ell_2 1 \rangle}{\langle \ell_2 3 \rangle \langle \ell_1 1 \rangle}.
\end{aligned}$$

Reconstructing the propagators and using trace identities, we can expand $\Sigma_i'$,

$$\begin{aligned}
\Sigma_1' &= 1 - \frac{2(p_2 \cdot P_c)(p_4 \cdot P_c) - (p_2 \cdot p_4) P_c^2}{(\ell_2 + p_2)^2 (\ell_1 + p_4)^2} + \frac{p_4 \cdot P_c}{(\ell_1 + p_4)^2} - \frac{p_2 \cdot P_c}{(\ell_2 + p_2)^2} + \text{s.t.}, \\
\Sigma_2' &= -1 - \frac{2(p_2 \cdot P_c)(p_1 \cdot P_c) - (p_1 \cdot p_2) P_c^2}{(\ell_2 + p_2)^2 (\ell_1 - p_1)^2} + \frac{p_1 \cdot P_c}{(\ell_1 - p_1)^2} + \frac{p_2 \cdot P_c}{(\ell_2 + p_2)^2} + \text{s.t.}, \\
\Sigma_3' &= -1 - \frac{2 p_3 \cdot p_4}{(\ell_2 - p_3)^2}, \\
\Sigma_4' &= 1 - \frac{2(p_1 \cdot P_c)(p_3 \cdot P_c) - (p_1 \cdot p_3) P_c^2}{(\ell_2 - p_3)^2 (\ell_1 - p_1)^2} - \frac{p_1 \cdot P_c}{(\ell_1 - p_1)^2} + \frac{p_3 \cdot P_c}{(\ell_2 - p_3)^2} + \text{s.t.},
\end{aligned}$$

where we introduced $P_c = p_3 + p_4 = -p_\phi - p_1 - p_2$.
$\Sigma_1'$ and $\Sigma_4'$ produce one-mass four-point contributions and triangles, while $\Sigma_2'$ shows the $s_{34}$-cut of an easy two-mass box $I_4^{2me}(s_{\phi 1}, s_{\phi 2}; s_{34}, s_\phi)$ in addition to three-point integrals with one uncut propagator. The addend $\Sigma_3'$ brings out a different structure. For this reason, let us show the calculation explicitly to demonstrate the interesting simplification. After some simple spinor algebra, it yields

$$\Sigma_3' = -\frac{\langle 3\ell_1 4] \langle 4\ell_2 3]}{\langle 3\ell_2 3] \langle 4\ell_1 4]} = -1 + \frac{2(p_3 \cdot P_c)(p_4 \cdot P_c) - (p_3 \cdot p_4) P_c^2}{(\ell_2 + p_3)^2 (\ell_1 + p_4)^2} - \frac{p_4 \cdot P_c}{(\ell_1 + p_4)^2} + \frac{p_3 \cdot P_c}{(\ell_2 + p_3)^2}$$

Remembering that $P_c = p_3 + p_4 = \ell_2 - \ell_1$, the second term vanishes and the other two triangular contributions sum together:

$$\Sigma_3' = -1 - \frac{p_4 \cdot p_3}{(\ell_2 - P_c + p_4)^2} + \frac{p_3 \cdot p_4}{(\ell_2 + p_3)^2} = -1 - \frac{p_4 \cdot p_3}{(\ell_2 - p_3)^2} + \frac{p_3 \cdot p_4}{(\ell_2 + p_3)^2} = -1 - \frac{p_3 \cdot p_4}{(\ell_2 - p_3)^2}.$$

Knowing the expression for $\Sigma_i'$, we can compute the resulting discontinuities in this channel:

$$A^{2L}|_{\text{dcut C}} = \int d\Phi_2\, A^{2L}_{int}|_{\text{dcut C}} = C^C_{4,1} I_4^{1m}(s_{23}, s_{34}; s_{1\phi})|_{s_{34}\text{-cut}} + C^C_{4,2} I_4^{2me}(s_{1\phi}, s_{\phi 2}; s_{34}, s_\phi)|_{s_{34}\text{-cut}}$$
$$+ C^C_{4,3} I_4^{1m}(s_{34}, s_{41}; s_{2\phi})|_{s_{34}\text{-cut}} \tag{5.6}$$

where

$$\begin{aligned}
C^C_{4,1} &= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \left[ (p_2 \cdot p_4) P_c^2 - 2(p_2 \cdot P_c)(p_4 \cdot P_c) \right] \\
&= -\frac{1}{2} s_{23} s_{34} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+), \\
C^C_{4,2} &= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \left[ (p_1 \cdot p_2) P_c^2 - 2(p_2 \cdot P - c)(p_1 \cdot P_c)) \right] \\
&= \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \left[ s_{12} s_\phi - (s_{1\phi} - s_\phi)(s_{4\phi} - s_\phi) \right], \\
C^C_{4,3} &= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \left[ (p_3 \cdot p_1) P_c^2 - 2(p_3 \cdot P_c)(p_1 \cdot P_c) \right] \\
&= -\frac{1}{2} s_{34} s_{41} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+).
\end{aligned}$$

These coefficients are consistent with the results from quadruple cut investigations.
Looking at the equation (5.6), we observe a curious simplicity due to large cancellations in the sum $\Sigma_1' + \Sigma_2' + \Sigma_3' + \Sigma_4'$. We observe the absence of constant terms, which represent bubbles in a double cut computation, and we also see that some three-point structures which emerge in the intermediate steps vanishing in the final result.

### 5.3.4 Summary of the results

Computing the double cut in the $s_\phi$ channel, we found the following terms,

$$\frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \Big[ \big( s_{14} s_\phi - (s_{1\phi} - s_\phi)(s_{\phi 4} - s_\phi) \big)\, I_4^{2me}(s_{\phi 4}, s_{1\phi}; s_{23}, s_\phi)$$
$$+ (s_{\phi 1} - s_\phi) I_3^{2m}(s_{1\phi}, s_\phi) + (s_{4\phi} - s_\phi) I_3^{2m}(s_{4\phi}, s_\phi) \Big].$$

Studying the second channel $s_{\phi 1}$, we were able to investigate the following contributions for the two-loop amplitude,

$$\frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \Big[ \big( s_{14} s_\phi - (s_{1\phi} - s_\phi)(s_{4\phi} - s_\phi) \big)\, I_4^{2me}(s_{\phi 4, s_{1\phi}}; s_{23}, s_\phi)$$
$$+ \big( s_{12} s_\phi - (s_{1\phi} - s_\phi)(s_{2\phi} - s_\phi) \big)\, I_4^{2me}(s_{\phi 2}, s_{1\phi}; s_{34}, s_\phi)$$
$$- s_{23} s_{34} I_4^{1m}(s_{23}, s_{34}; s_{1\phi}) + 2(s_{\phi 1} - s_\phi) I_3^{2m}(s_{1\phi}, s_\phi) \Big].$$

While in the last double cut we obtained the following discontinuities written in terms of scalar integrals,

$$\frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+) \Big[ \big( s_{12} s_\phi - (s_{1\phi} - s_\phi)(s_{4\phi} - s_\phi) \big)\, I_4^{2me}(s_{1\phi}, s_{\phi 2}; s_{34}, s_\phi)$$
$$- s_{23} s_{34} I_4^{1m}(s_{23}, s_{34}; s_{1\phi}) - s_{34} s_{41} I_4^{1m}(s_{34}, s_{41}; s_{2\phi}) \Big]. \tag{5.7}$$

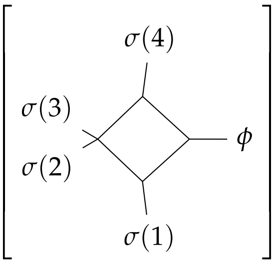
We checked the consistency of the box coefficients found in the double cuts with the results obtained using the quadruple cuts. We saw the absence of bubbles in every computation and the only non-vanishing three-point coefficients are observed in triangles with the $\phi$ field present alone in a vertex.

### 5.3.5 IR structure and the remainder finite part

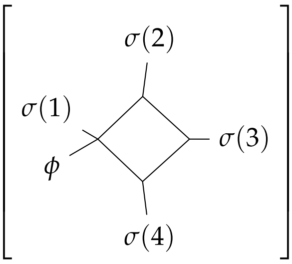
We want to check the IR structure of the cut-constructible part of the amplitude that can be predicted in term of the one-loop result. Subtracting these poles, we will be able to express the finite remainder contribution in this sector.

We can summarize the discontinuities of the two-loop amplitude with a one-loop $\phi$+gluon sub-amplitude keeping in mind the possible cyclic permutation of gluons. Computing the double cuts, we have already observed similar contributions connected by a $\mathbb{Z}_4$ transformation. For example, in the expression (5.7) we observed two one-mass boxes which corresponds to a cyclic permutation of gluons. If we consider the $s_{41}$ channel, we will obtain the contribution proportional to $I_4^{1m}(s_{41}, s_{12}; s_{3\phi})$ in addition to the term $I_4^{1m}(s_{34}, s_{41}; s_{2\phi})$. Considering all the possible position of the $\phi$ field, we can obtain all the four-point contributions with a single external mass and similar observations hold for the other topologies. Then the unrenormalized cut-constructible part of the two-loop amplitude in this sector is

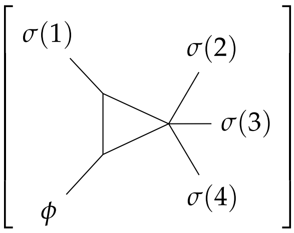
$$A^{2L}_{cc(I)} = \sum_{\sigma\in\mathbb{Z}_4} d^{2me}_{\sigma(1)}$$



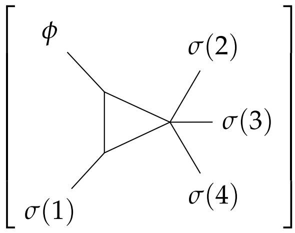

$$+ \sum_{\sigma\in\mathbb{Z}_4} d^{1m}_{\sigma(1)}$$




$$+ \sum_{\sigma\in\mathbb{Z}_4} c^{2m}_{\sigma(1)}$$




$$+ \sum_{\sigma\in\mathbb{Z}_4} c^{2mr}_{\sigma(1)}$$




where

$$d^{2me}_{\sigma(1)} = \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left(s_{\sigma(1)\sigma(4)} s_\phi - (s_{\sigma(4)\phi} - s_\phi)(s_{\sigma(1)\phi} - s_\phi)\right)$$
$$d^{1m}_{\sigma(1)} = \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)(-s_{\sigma(2)\sigma(3)} s_{\sigma(3)\sigma(4)})$$
$$c^{2m}_{\sigma(1)} = c^{2mr}_{\sigma(1)} = \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)(s_{\phi\sigma(1)} - s_\phi).$$

We can extract the poles in $\epsilon$ from the scalar integrals to check the universal infrared structure of the amplitude.

$$\left[\frac{A^{2L}_{cc(I)}}{A^{1L}}\right]_{\text{poles}} = \sum_{\sigma\in\mathbb{Z}_4}\Big\{ d^{2me}_{\sigma(1)}\left[I_4^{2me}(s_{\phi\sigma(4)}, s_{\sigma(1)\phi}; s_{\sigma(2)\sigma(3)}, s_\phi)\right]_{IR} + d^{1m}_{\sigma(1)}\left[I_4^{1m}(s_{\sigma(2)\sigma(3)}, s_{\sigma(3)\sigma(4)}; s_{\sigma(1)\phi})\right]_{IR}$$
$$+ \left(c^{2m}_{\sigma(1)} + c^{2mr}_{\sigma(1)}\right)\left[I_3^{2m}(s_{\phi\sigma(1)}, s_\phi)\right]_{IR}\Big\} = -\sum_{i=1}^{4} \frac{1}{\epsilon^2}(-s_{i,i+1})^{-\epsilon} \tag{5.8}$$

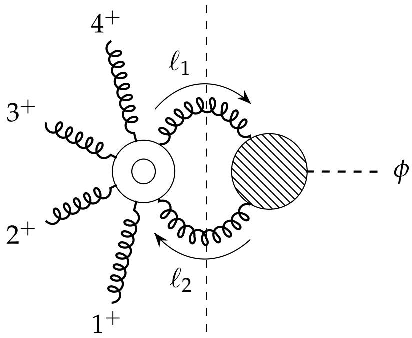
After the cancellation of all the poles which involve the self-dual Higgs, we remain with a divergent structure depending only by the Mandelstam variables associated to massless external legs. This is exactly the expectation from the universal expression for the IR structure of amplitudes.
Subtracting the divergences, we obtain the remainder part expressed in terms of weight-two functions,

$$\left[A^{2L}_{cc(I)}\right]_{\text{finite}} = A^{1L}(\phi; 1^+2^+3^+4^+)\sum_{\sigma\in\mathbb{Z}_4}\Bigg[2\,\mathrm{Li}_2\left(1 - \frac{s_\phi}{s_{\sigma(1)\phi}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{\sigma(2)\sigma(3)}}{s_{\sigma(1)\phi}}\right)$$
$$+ \mathrm{Li}_2\left(1 - \frac{s_{\sigma(2)\sigma(3)}}{s_{\sigma(4)\phi}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{\sigma(4)\phi}}{s_{\sigma(1)\sigma(2)}}\right) + \mathrm{Li}_2\left(1 - \frac{s_{\sigma(4)\phi}}{s_{\sigma(2)\sigma(3)}}\right)$$

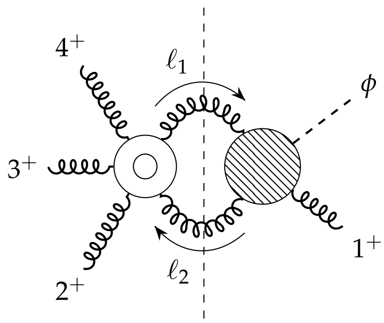
$$-\mathrm{Li}_2\left(1-\frac{s_{\phi}s_{\sigma(2)\sigma(3)}}{s_{\sigma(1)\phi}s_{\sigma(4)\phi}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{\sigma(1)\phi}}{s_{\sigma(4)\phi}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{\sigma(1)\sigma(2)}}{s_{\sigma(2)\sigma(3)}}\right)+\frac{\pi^2}{6}\Bigg]. \qquad (5.9)$$

## 5.4 Double cuts with tree $\phi$ amplitudes and 1L YM amplitudes

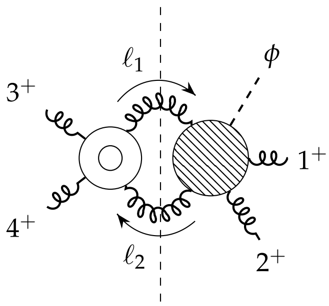
The aim of this section is the computation of the double cuts which factorize in a product of a tree level $\phi$+gluon amplitude and a one-loop gluon amplitude. We will follow the same procedure as in the other sector. In [App. B] the quadruple cuts were computed and we proceed with the double cut computations.
As already observed in the first sector, we have to study three different channels.




(dcut D)




(dcut E)




(dcut F)

We will need to use QCD partial amplitudes at one-loop level in the all-plus sector. In (dcut F) we will compute the product between the tree-level $\phi$+gluon and the one-loop four gluon amplitude which presents only one contribution

$$A^{1L}_{n,1}(\ell_1^+,(-\ell_2)^+,3^+,4^+)=\frac{1}{3}\frac{\langle\ell_1(-\ell_2)34]}{\langle\ell_1(-\ell_2)\rangle\langle(-\ell_2)3\rangle\langle34\rangle\langle\ell_14\rangle}.$$

This double cut computation is simpler than the other two calculations. Indeed, if we consider (dcut E) and (dcut D), we have to study respectively five and fifteen contributions as one can see considering the expression (5.1).
For simplicity, we will start to reduce the product of the two sub-amplitudes in $s_{34}$ channel in order to write it in terms of scalar integrals and capture the coefficient of the boxes, triangles and bubbles. After it, we will procede with the most complex double cuts.

### 5.4.1 Double cut in $s_{34}$ channel

In $s_{34}$ channel, we have to consider the product between a tree $\phi$+gluon vertex and a sub-amplitude involving four gluons. We want to reduce this object in terms of scalar integrands and we can achieve it with a simple use of spinor algebra and gamma properties. Explicitly, we have to compute the product:

$$\begin{aligned}A^{2L}_{int}|_{\text{dcut F}}&=A^{1L}(3^+,4^+,\ell_1^+,(-\ell_2)^+)A^{tree}(\phi;1^+,2^+,\ell_2^-,(-\ell_1)^-)\\&=\frac{1}{3}\frac{[\ell_1\ell_2][34]}{-\langle\ell_1\ell_2\rangle\langle34\rangle}\frac{\langle\ell_1\ell_2\rangle^4}{\langle12\rangle\langle2\ell_2\rangle\langle\ell_2\ell_1\rangle\langle\ell_11\rangle}\end{aligned}$$

where we used the analytic continuation of the spinors (2.21).
Now we used the property

$$[\ell_1 \ell_2]\langle \ell_2 \ell_1 \rangle = 2\ell_1 \cdot \ell_2 = -(\ell_2 - \ell_1)^2 = -(p_3 + p_4)^2 = -s_{34}$$

and we obtain

$$A^{2L}_{int}|_{\text{dcut F}} = -\frac{1}{3}\frac{[34]^2}{\langle 12 \rangle}\frac{[1\ell_1\ell_2 2]}{\langle 2\ell_2 2]\langle 1\ell_1 1]} \tag{5.10}$$

Neglecting spurious terms, we obtain

$$A^{2L}_{int}|_{\text{dcut F}} = \frac{1}{3}\frac{[34]^2}{\langle 12 \rangle^2}\left[\frac{p_1 \cdot p_2\, P_d^2 - 2(p_1 \cdot P_d)(p_2 \cdot P_d)}{(\ell_2 + p_2)^2(\ell_1 - p_1)^2} + \frac{p_1 \cdot P_d}{(\ell_1 - p_1)^2} + \frac{p_2 \cdot P_d}{(\ell_2 + p_2)^2}\right].$$

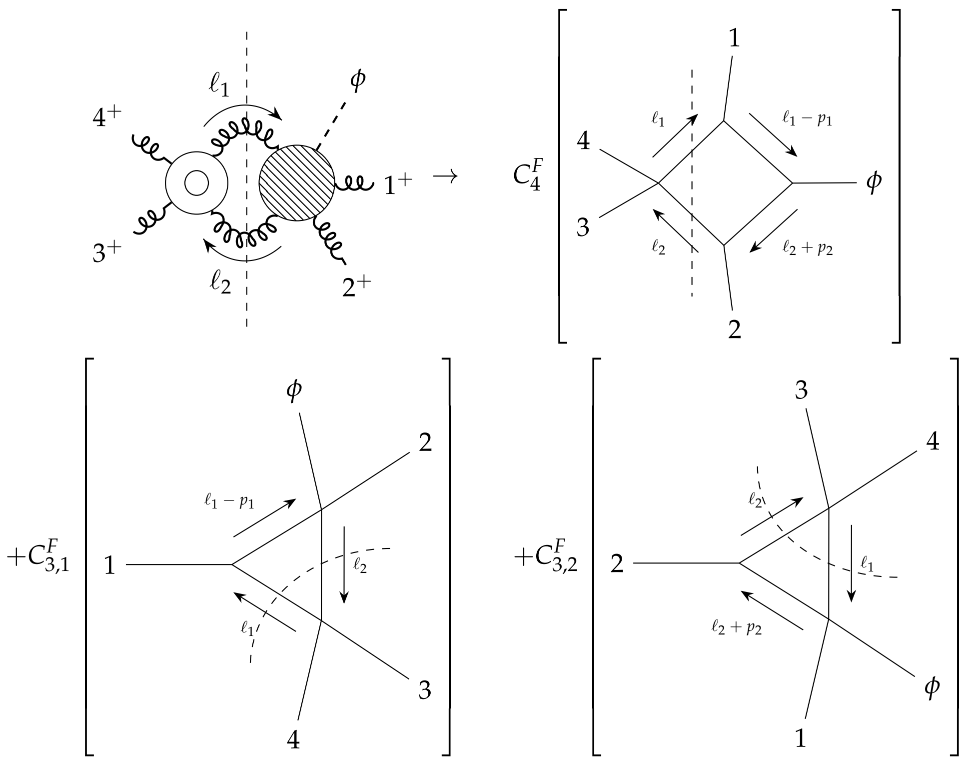
After the phase-space integration, we have

$$A^{2L}|_{\text{dcut F}} = \int d\Phi_2\, A^{2L}|_{\text{dcut D}} = C_4^F\, I_4^{2me}(s_{1\phi}, s_{2\phi}; s_{34}, s_\phi)|_{s_{34}\text{-cut}} + C_{3,1}^F\, I_3^{2m}(s_{34}, s_{2\phi})|_{s_{34}\text{-cut}} + C_{3,2}^F\, I_3^{2m}(s_{34}, s_{1\phi})|_{s_{34}\text{-cut}}$$

with the coefficients

$$C_4^F = \frac{1}{6}\frac{[34]^2}{\langle 12 \rangle^2}(-s_{1\phi}s_{2\phi} + s_\phi s_{34}),$$
$$C_{3,1}^F = \frac{1}{6}\frac{[34]^2}{\langle 12 \rangle^2}(s_{13} + s_{14}) = \frac{1}{6}\frac{[34]^2}{\langle 12 \rangle^2}(s_{2\phi} - s_{34}),$$
$$C_{3,2}^F = \frac{1}{6}\frac{[34]^2}{\langle 12 \rangle^2}(s_{23} + s_{24}) = \frac{1}{6}\frac{[34]^2}{\langle 12 \rangle^2}(s_{1\phi} - s_{34}).$$

The coefficient of the four-point integral is consistent with the result from the quadruple cuts.
We show in the following picture the result of the reduction process for the double cut already computed.



### 5.4.2 Double cut in $s_{\phi 1}$ channel

Considering the double cut (dcut E), we have to simplify the following integrand,

$$\sum_{\lambda_1=\pm 1}\sum_{\lambda_2=\pm 1} A^{tree}\left(\phi;1^+,\ell_2^{\lambda_2},(-\ell_1)^{\lambda_1}\right) A^{1L}\left(\ell_1^{-\lambda_1},(-\ell_2)^{-\lambda_2},2^+,3^+,4^+\right).$$

To have a non-vanishing result from the tree-level vertex, we select the helicity of the inner gluons, then we have to reduce the following expression:

$$\begin{aligned} A^{2L}_{int}|_{\text{dcut E}} &= A^{tree}\left(\phi;1^+,\ell_2^-,(-\ell_1)^-\right) A^{1L}\left(\ell_1^+,(-\ell_2)^+,2^+,3^+,4^+\right) \\ &= \frac{1}{3}\frac{1}{\langle 23\rangle\langle 34\rangle}\left(\sum_{i=1}^{5}\text{tr}_-(x_i)\right)\frac{\langle \ell_2\ell_1\rangle^2}{\langle 4\ell_1\rangle\langle 1\ell_1\rangle\langle 1\ell_2\rangle\langle \ell_2 2\rangle} \end{aligned} \tag{5.11}$$

where the possible arguments of the trace correspond to the allowed combinations of four gluons in the one-loop sub-amplitude which respect the color order. Precisely, we have to consider the sum

$$\begin{aligned} \sum_{i=1}^{5}\text{tr}_-(x_i) =& \text{Tr}_-(\not{3}\not{4}\not{\ell}_1\not{2}) + \text{tr}_-(\not{3}\not{\ell}_1(-\not{\ell}_2)\not{2}) + \text{tr}_-(\not{3}\not{4}(-\not{\ell}_2)\not{2}) \\ &+ \text{tr}_-(\not{4}\not{\ell}_1(-\not{\ell}_2)\not{2}) + \text{tr}_-(\not{3}\not{4}\not{\ell}_1(-\not{\ell}_2)) \end{aligned}$$

We can reorganize the sum using the momentum conservation in order to have at numerator spinor angle products with $\ell_1$ or $\ell_2$ which can be simplified with the denominator. In other words, we rewrite the sum in order to reduce directly as much as possible the number of loop momenta. We consider an alternative form of 5.4.2,

$$\begin{aligned} \sum_i \text{tr}_-(x_i) =& \text{tr}_-(\not{3}\not{4}(\not{\ell}_2-\not{3})\not{2}) + \text{tr}_-(\not{3}(-\not{2}-\not{4})(-\not{\ell}_2)\not{2}) + \text{tr}_-(\not{3}\not{4}(-\not{\ell}_2)\not{2}) \\ &+ \text{tr}_-(\not{4}(-\not{2}-\not{3})(-\not{\ell}_2)\not{2}) - \text{tr}_-(\not{\ell}_1\not{4}\not{2}(\not{3}+\not{4})) - \text{tr}_-(\not{\ell}_1\not{4}\not{3}(\not{2}+\not{4})) \\ &- \text{tr}_-(\not{2}\not{\ell}_2(\not{2}+\not{4})\not{3}) - \text{tr}_-(\not{2}\not{\ell}_2(\not{2}+\not{3})\not{4}) + \text{tr}_-(\not{2}\not{3}\not{4}(\not{\ell}_1+\not{3})) + \text{tr}_-(\not{2}\not{3}\not{2}\not{\ell}_2). \end{aligned}$$

The next step is the reduction of the number spinor products with loop momenta using Schouten identities. We applied the following substitution,

$$\frac{\langle \ell_1\ell_2\rangle}{\langle a\ell_1\rangle\langle b\ell_1\rangle} = \frac{1}{\langle ab\rangle}\left(\frac{\langle b\ell_2\rangle}{\langle b\ell_1\rangle} - \frac{\langle a\ell_2\rangle}{\langle a\ell_1\rangle}\right)$$

where $a$ and $b$ are generic labels for gluons. Similar expressions arise for the contributions which involve spinors associated to $\ell_2$.

In conclusion, we can also use momentum conservation to simplify some spinor products. Using Mathematica to apply each step, we obtained an expression which contains a linear combination of terms with the following structure:

$$\left\{\frac{\langle \ell_1\ell_2\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle}, \frac{\langle a\ell_2\rangle\langle b\ell_1\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle}, \frac{\langle a\ell_1\rangle}{\langle b\ell_1\rangle}, \frac{\langle a\ell_2\rangle}{\langle b\ell_2\rangle}\right\}$$

Using simple properties for traces of Gamma matrices and avoiding spurious terms, we can reduce the first two terms.

$$\begin{aligned} \frac{\langle \ell_1\ell_2\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle} &= \frac{1}{\langle ab\rangle}\frac{\langle ba\ell_1\ell_2 b]}{\langle a\ell_1 a]\langle b\ell_2 b]} = \frac{1}{\langle ab\rangle}\frac{\frac{1}{2}\text{Tr}\left(\not{b}\not{a}\not{\ell}_1\not{\ell}_2 b\right)}{(2\ell_1\cdot a)(2\ell_2\cdot b)} + \text{spurious terms} \\ &= \frac{1}{\langle ab\rangle}\left(\frac{2(b\cdot P_e)(a\cdot P_e) - P_c^2(b\cdot a)}{(2a\cdot\ell_1)(2b\cdot\ell_2)} + \frac{b\cdot P_e}{2b\cdot\ell_2} - \frac{a\cdot P_c}{2a\cdot\ell_1}\right) + \text{s.t.} \end{aligned} \tag{5.12}$$

$$\begin{aligned} \frac{\langle a\ell_2\rangle\langle b\ell_1\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle} &= \frac{\langle a\ell_2 b\ell_1 a]}{\langle a\ell_1 a]\langle b\ell_2 b]} = \frac{\frac{1}{2}\text{Tr}\left(\not{a}\not{\ell}_2\not{b}\not{\ell}_1\right)}{(2a\cdot\ell_1)(2b\cdot\ell_2)} + \text{s.t.} \\ &= \frac{-2(a\cdot P_e)(b\cdot P_e) + P_e^2(a\cdot b)}{(2a\cdot\ell_1)(2b\cdot\ell_2)} - \frac{b\cdot P_e}{2b\cdot\ell_2} + \frac{a\cdot P_e}{2a\cdot\ell_1} + \text{s.t.} \end{aligned} \tag{5.13}$$

where $P_c = p_2 + p_3 + p_4 = -p_1 - p_\phi$.

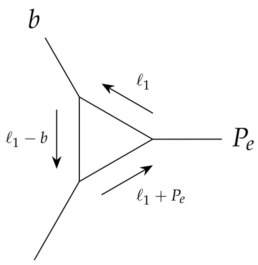
For the last structures which involve only single spinor products at denominator, we can apply the reduction method at integrand level to write the tensor tree-point integrand in terms of the arguments of scalar triangles and bubbles. Studying the term

$$\frac{\langle a\ell_1\rangle}{\langle b\ell_1\rangle} = \frac{\ell_1 b]}{\langle b\ell_1 b]} = \langle a|\frac{\ell_1^\mu}{2\ell_1\cdot b}\gamma_\mu|b], \tag{5.14}$$

we can restore the implicit presence of cut propagators at denominator and we can try to reduce the integrand

$$\frac{\ell_1^\mu}{-(\ell_1-b)^2\ell_1^2(\ell_1+P_e)^2}$$

of a tensor three-point integral whose kinematics is described diagrammatically as follow.




Then we can decompose the loop momenta in a basis composed by the external momenta $b^\mu$ and $P_e^\mu$ and two vectors which span the orthogonal space:

$$\ell_1^\mu = \alpha P_e^\mu + \beta b^\mu + \gamma\omega_1^\mu + \delta\omega_2^\mu$$

with $\omega_i \cdot P_e^\mu = \omega_i \cdot b^\mu = 0$ for $i = 1, 2$. The contributions proportional to the vectors $\omega_i^\mu$ vanish after the integration, for this reason we will neglect their spurious presence at the integrand level. We can write the coefficients $\alpha$ and $\beta$ in terms of external kinematics and propagators solving the following system.

$$\begin{cases} \ell_1 \cdot b = \alpha P \cdot b \\ \ell_1 \cdot P_e = \alpha P_e^2 + \beta b \cdot P_e \end{cases} \qquad \begin{cases} \alpha = -\frac{1}{2P_e\cdot b}(\ell_1 - b)^2 \\ \beta = -\frac{P_e^2}{2b\cdot P_e}\left(1 - \frac{(\ell_1-b)^2}{P_e\cdot b}\right) \end{cases}$$

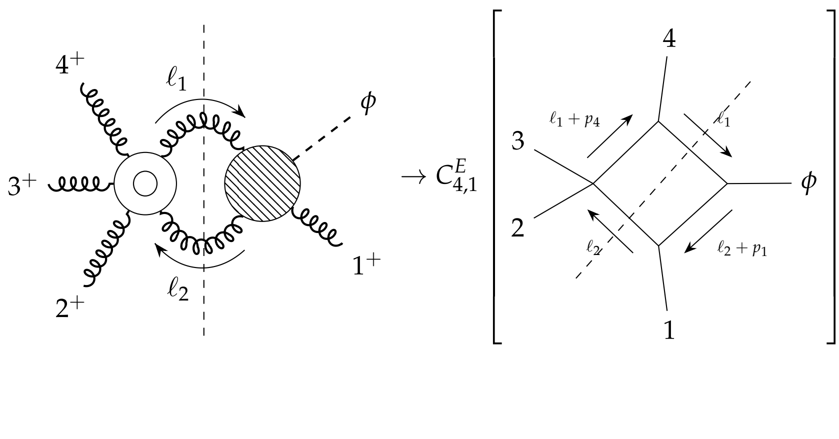
In the last passage we used the on-shell constraint $(\ell_1 + P)^2 = \ell_1^2 = 0$ for the loop momenta in the cut. We are interested in the computation of (5.14), then we will contract the reduced integrand with a spinor of the momentum $b$. Using Dirac equation, we can observe the vanishing behavior of the contribute proportional to $\beta$. Consequently, the result only depends on $\alpha$ which is proportional to the uncut propagator and gives a bubble contribution,

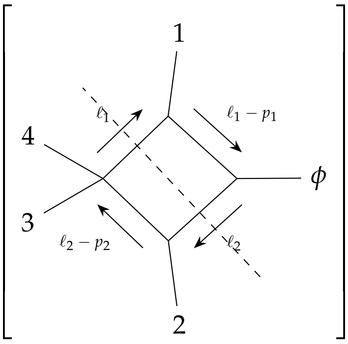
$$\frac{\langle a\ell_1\rangle}{\langle b\ell_1\rangle} = \frac{\langle a(\alpha P_e + \beta b)b]}{2\ell_1 \cdot b} = \frac{\alpha\langle aP_e b]}{2\ell_1\cdot b} = \frac{\langle aP_e b]}{2P_c \cdot b} = \frac{\langle aP_e b]}{\langle bP_e b]}. \tag{5.15}$$

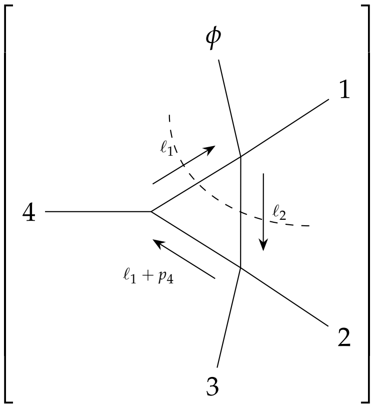
A similar computation holds for the last structure,

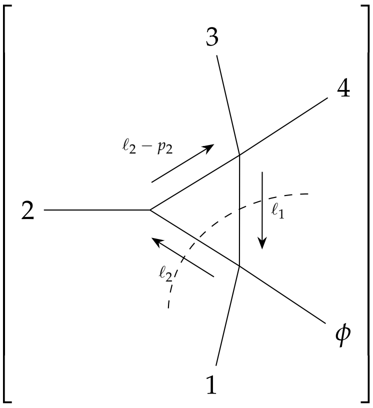
$$\frac{\langle a\ell_1\rangle}{\langle b\ell_2\rangle} = \frac{\langle a\ell_1 b]}{\langle b\ell_2 b]} = \frac{\langle aP_e b]}{\langle bP_e b]}. \tag{5.16}$$

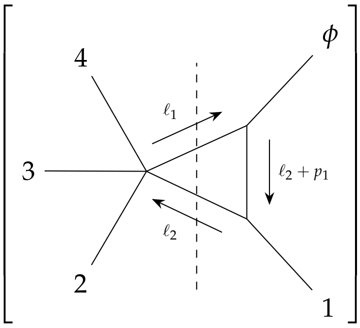
Applying the substitutions (5.12, 5.13, 5.15, 5.16), we can write the double cut of the amplitude in terms of scalar integrals as shown pictorially.



$+ C^E_{4,2}$ 

$+ C^E_{3,1}$ 


$+ C^E_{3,2}$ 

$+ C^E_{3,3}$ 


with the coefficients

$$
\begin{aligned}
C^E_{4,1} &= \frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}(-s_{1\phi}s_{4\phi} + s_\phi s_{23}),\\
C^E_{4,2} &= \frac{1}{6}\frac{[34]^2}{\langle 12\rangle^2}(-s_{2\phi}s_{1\phi} + s_\phi s_{34}),\\
C^E_{3,1} &= \frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}(s_{1\phi} - s_{23}),\\
C^E_{3,2} &= \frac{1}{6}\frac{[34]^2}{\langle 12\rangle^2}(s_{1\phi} - s_{34}),\\
C^E_{3,3} &= \frac{1}{6}\left(\frac{[23]^2}{\langle 14\rangle^2} + \frac{[34]^2}{\langle 12\rangle^2}\right)(s_{1\phi} - s_\phi).
\end{aligned}
$$

In the intermediate steps some bubbles emerge but, summing together these contributions, we observe the cancellation of terms proportional to the two-point scalar integrals.
This result was checked by an independent computation with a different approach. Starting from (5.11), we can apply Schouten identities to avoid spurious poles in the expression

$$
\frac{\langle \ell_1 \ell_2\rangle}{\langle 1\ell_1\rangle\langle 1\ell_2\rangle}.
$$

We can reconstruct the propagators at denominator to obtain a sum of tensor integrals. We reduced these contributions in a sum of scalar boxes, triangles and bubbles using a general approach at the integrand level. Indeed we decompose the loop momentum at numerator in terms of external kinematics and coefficients written as a function of propagators. We used momentum twistor parametrization to reduce the expression making explicit the momentum conservation.
During the computation we have to treat carefully the contribution which comes from the spurious direction. For example, if we consider a four-point tensor integral with propagators $D_i$, we can decompose the loop momenta at numerator in terms of three external vectors and an orthogonal direction $\omega$. In the reduction process, we have to consider that

$$
\int_\ell \frac{\ell\cdot\omega}{D_0D_1D_2D_3} = 0, \text{ but } \int_\ell \frac{(\ell\cdot\omega)^2}{D_0D_1D_2D_3} \neq 0.
$$

For this reason, we cannot neglect the contributions proportional to the orthogonal directions in the decomposition of the numerator because their product could generate a non-vanishing result. In other words, they do not represent spurious contributions. Until now, we have not see similar effects because we only reduced integrands with a single loop momentum at numerator in each addend.
Keeping in mind this technical observation, we wrote the double cut in terms of scalar integrals and we checked the consistency of the result from the first less general reduction method.

### 5.4.3 Double cut in $s_\phi$ channel

The last double cut (dcut D) investigates the discontinuity in $s_\phi$ channel. The non-vanishing condition of the tree level $\phi$+gluon vertex fixes the helicity of the inner gluons, then we have to consider the all-pus six gluon sub-amplitude which is the sum of fifteen traces (5.1).

$$\begin{aligned} A^{2L}_{int}|_{\text{dcut D}} &= A^{tree}(\phi; \ell_2^-, (-\ell_1)^-) A^{1L}(\ell_1^+, (-\ell_2)^+, 1^+, 2^+, 3^+, 4^+) \\ &= \frac{1}{3} \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle} \left( \sum_{i=1}^{15} \text{tr}_-(x_i) \right) \frac{\langle \ell_1 \ell_2\rangle}{\langle 4\ell_1\rangle\langle 1\ell_2\rangle} \end{aligned}$$

Before the expansion of traces in terms of spinor products, we use momentum conservation to have atmost one loop momentum ($\ell_1$ or $\ell_2$) in every trace. For example,

$$\text{tr}_-(\not{1}\not{4}\not{\ell}_1\not{\ell}_2) = \text{tr}_-(\not{1}\not{4}\not{\ell}_1\not{2}) + \text{tr}_-(\not{1}\not{4}\not{\ell}_1\not{3}) + \text{tr}_-(\not{1}\not{4}\not{\ell}_1\not{4}).$$

We immediately use the property $[\ell_1\ell_2]\langle\ell_2\ell_1\rangle = 2\ell_1\cdot\ell_2 = -(\ell_2-\ell_1)^2 = -s_\phi$. Moreover we consider an additional use of momentum conservation for the numerators which present the following structure,

$$[a\ell_1]\langle\ell_1\ell_2\rangle = [a(\ell_2 - 1 - 2 - 3 - 4)\ell_2\rangle = -[a(1+2+3+4)\ell_2\rangle,$$

where $a = 1,2,3,4$ represent a generic external gluon. A similar reduction holds for $[a\ell_2]\langle\ell_1\ell_2\rangle$.
The next step is the use of Schouten identities in order to reduce the number of loop momenta at denominator for the addends with more than two angle brackets involving $\ell_1$ or $\ell_2$. We obtain contributions which present only one of the following structures,

$$\left\{ \frac{\langle\ell_1\ell_2\rangle}{\langle a\ell_1\rangle\langle b\ell_2\rangle}, \frac{\langle a\ell_1\rangle}{\langle b\ell_1\rangle}, \frac{\langle a\ell_2\rangle}{\langle b\ell_2\rangle} \right\}.$$

We have already computed the explicit expansion of these objects in terms of scalar integrals (5.12, 5.15, 5.16). Obviously, we only have to perform the substitution of $P_e$ with the loop momenta $P_d = p_1 + p_2 + p_3 + p_4 = -p_\phi$ in this channel. The result is

$$\begin{aligned} A^{2L}|_{\text{dcut D}} = \int d\Phi_2 A^{2L}_{int}|_{\text{dcut D}} =& C_4^D I_4^{2me}(s_{1\phi}, s_{4\phi}; s_\phi, s_{23})|_{s_\phi\text{-cut}} + C_{3,1}^D I_3^{2m}(s_\phi, s_{1\phi})|_{s_\phi\text{-cut}} \\ &+ C_{3,2}^D I_3^{2m}(s_\phi, s_{4\phi})_{s_\phi\text{-cut}} + C_2^D I_2(s_\phi)|_{s_\phi\text{-cut}} \end{aligned}$$

with the coefficients

$$\begin{aligned} C_4^D &= C_{4,1}^E = \frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}(-s_{1\phi}s_{4\phi} + s_\phi s_{23}), \\ C_{3,1}^D &= \frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}(s_{1\phi} - s_\phi), \\ C_{3,2}^D &= \frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}(s_{4\phi} - s_\phi), \\ C_2^D &= \frac{1}{3}\frac{s_\phi}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \left[ \frac{s_{12}s_{14} + s_{13}s_{14} + s_{14}^2 - s_{13}s_{23} - s_{14}s_\phi - [13241\rangle}{s_{1\phi} - s_\phi} \right. \\ &\qquad \left. + \frac{s_{23}s_{34} - s_{14}s_\phi + [12341\rangle}{s_{4\phi} - s_\phi} \right]. \end{aligned}$$

We observe the presence of bubbles in $s_\phi$ channel which represent the only new information from this last double cut. The coefficient for the two-point scalar integral shows a strange structure with

unphysical poles.
This contribution can be simplified because we have to consider all the possible permutations $\sigma$ of gluons which are in correspondence with the elements of the group $\mathbb{Z}_4$. The bubble integral $I_2(s_\phi)$ does not depend on the order of gluons, then in order to obtain the resulting bubble contribution, we have to sum directly the coefficients

$$\sum_{\sigma\in\mathbb{Z}_4} \sigma\left\{C_2^D\right\} = \frac{2}{3}\frac{s_\phi^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -\frac{1}{3}A^{1L}(\phi;1^+2^+3^+4^+),$$

where $\sigma$ cyclically shifts all indices of gluons. In conclusion, the UV divergence generated by the bubble is proportional to the one-loop amplitude.

### 5.4.4 Summary of the result

The simplest double cut in this sector permits us to extract the discontinuities in $s_{34}$-channel written in terms of scalar integrals in following way:

$$\frac{1}{6}\frac{[34]^2}{\langle 12\rangle^2}\left[(-s_{1\phi}s_{2\phi}+s_\phi s_{34})I_4^{2me}(s_{1\phi},s_{2\phi};s_{34},s_\phi)+(s_{2\phi}-s_{34})I_3^{2m}(s_{34},s_{2\phi})+(s_{1\phi}-s_{34})I_3^{2m}(s_{34},s_{1\phi})\right].$$

In the $s_{\phi 1}$-channel we obtained two different structures related to different topologies due to the presence of the unordered $\phi$. Moreover we observed the presence of a new three-point contribution $I_3^{2m}(s_\phi,s_{1\phi})$,

$$\frac{1}{6}\frac{[34]^2}{\langle 12\rangle}\left[(-s_{1\phi}s_{2\phi}+s_\phi s_{34})I_4^{2me}(s_{1\phi},s_{2\phi};s_{34},s_\phi)+(s_{1\phi}-s_{34})I_3^{2m}(s_{34},s_{1\phi})+(s_{1\phi}-s\phi)I_3^{2m}(s_\phi,s_{1\phi})\right]+$$
$$\frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}\left[(-s_{1\phi}s_{4\phi}+s_\phi s_{23})I_4^{2me}(s_{1\phi},s_{4\phi};s_{23},s_\phi)+(s_{1\phi}-s_{23})I_3^{2m}(s_{23},s_{1\phi})+(s_{1\phi}-s\phi)I_3^{2m}(s_\phi,s_{1\phi})\right].$$

In $s_\phi$-channel, we obtained contributions already observed in the previous channels,

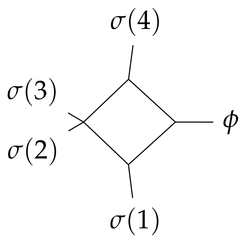
$$\frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}\left[(-s_{1\phi}s_{4\phi}+s_\phi s_{23})I_4^{2me}(s_{1\phi},s_{2\phi},s_{34},s_\phi)+(s_{1\phi}-s_\phi)I_3^{2m}(s_\phi,s_{1\phi})+(s_{4\phi}-s_\phi)I_3^{2m}(s_\phi,s_{4\phi})\right],$$

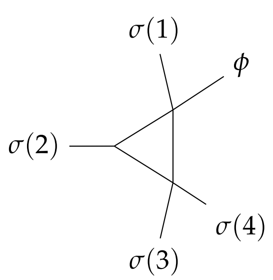
and we also found a term proportional to the bubble integral $I_2(s_\phi)$ which assumes the following expression after the cyclic sum of gluons,

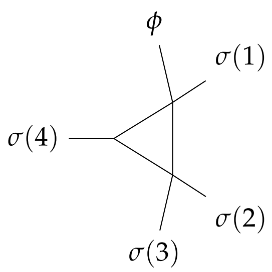
$$-\frac{1}{3}A^{1L}(\phi;1^+2^+3^+4^+)I_2(s_\phi).$$

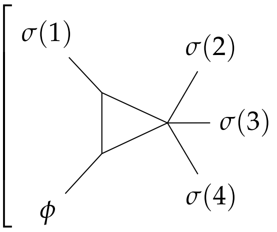
### 5.4.5 IR structure and the remainder finite part

We can summarize the unrenormalized cut-constructible part of the two-loop amplitude in the sector in which we consider a one-loop sub-amplitude without the self-dual Higgs.

$$A^{2L}_{cc(II)} = \sum_{\sigma\in\mathbb{Z}_4} d^{2me(II)}_{\sigma(1)}\Big[$$


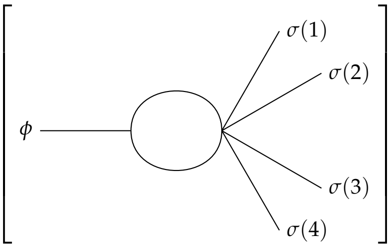
$$\Big] + \sum_{\sigma\in\mathbb{Z}_4} c^{2m(IIa)}_{\sigma(1)}\Big[$$


$$\Big] + \sum_{\sigma\in\mathbb{Z}_4} c^{2m(IIb)}_{\sigma(1)}\Big[$$


$$\Big] + \sum_{\sigma\in\mathbb{Z}_4} c^{2m(IIc)}_{\sigma(1)}\Big[$$


$$\Big]$$

$$-\frac{1}{3}A^{1L}(\phi;1^{+}2^{+}3^{+}4^{+})$$




where

$$\begin{aligned}
d^{2me(II)}_{\sigma(1)} &= \frac{1}{6}\frac{[\sigma(2)\sigma(3)]^2}{\langle\sigma(1)\sigma(4)\rangle^2}\left(-s_{\sigma(1)\phi}s_{\sigma(4)\phi}+s_{\phi}s_{\sigma(2)\sigma(3)}\right),\\
c^{2m(IIa)}_{\sigma(1)} &= \frac{1}{6}\frac{[\sigma(3)\sigma(4)]^2}{\langle\sigma(1)\sigma(2)\rangle^2}\left(s_{\sigma(1)\phi}-s_{\sigma(3)\sigma(4)}\right),\\
c^{2m(IIb)}_{\sigma(1)} &= \frac{1}{6}\frac{[\sigma(2)\sigma(3)]^2}{\langle\sigma(1)\sigma(4)\rangle^2}\left(s_{\sigma(1)\phi}-s_{\sigma(2)\sigma(3)}\right),\\
c^{2m(IIc)}_{\sigma(1)} &= \frac{1}{6}\left(\frac{[\sigma(2)\sigma(3)]^2}{\langle\sigma(1)\sigma(4)\rangle^2}+\frac{[\sigma(3)\sigma(4)]^2}{\langle\sigma(1)\sigma(2)\rangle^2}\right)\left(s_{\sigma(1)\phi}-s_{\phi}\right).
\end{aligned}$$

Using the structure of divergences in the scalar integrals, we can extract the poles of the amplitude in this sector. We observe the cancellations of the whole divergent contribution of boxes and triangles. The sum of four-point and three-point integrals gives a finite contribution and the only divergent term comes from the bubble:

$$A^{2L}_{cc(II)} = -\frac{1}{3}A^{1L}(\phi;1^{+}2^{+}3^{+}4^{+})\frac{1}{\epsilon}+\mathcal{O}(1). \tag{5.17}$$

The finite contribution that comes from the sum of boxes and triangles can be interpreted as the result of a scalar integral in higher dimensions. We can show that an $n$-point scalar integral can be expressed as a sum of $(n-1)$-point functions and an $n$-point integral evaluated in six dimensions [23]:

$$I_n^{\{D=4-2\epsilon\}} = \frac{1}{2}\left[-\sum_{i=1}^{n}c_i I^{(i)}_{n-1}+(n-5+2\epsilon)c_0 I_n^{\{D=6-2\epsilon\}}\right]$$

where $I^{(i)}_{n-1}$ is an $(n-1)$-point integral in $4-2\epsilon$ dimensions obtained from $I_n$ by removing the propagator between legs $(i-1)$ and $i$. The factor $c_0$ corresponds to the sum of the coefficients $c_i$ which can be related to the elements of the modified Cayley matrix.
In our case we can decompose the 2m easy box in a sum of triangles and a remainder addend which is proportional to the four-point integral close to six dimensions. The divergent structure of the four-dimensional box is encoded in the three-point integrals while the six-dimensional box represents the finite remainder part without poles,

$$\begin{aligned}
I_4^{2me\{D=4-2\epsilon\}}(s_{1\phi},s_{4\phi};s_{23},s_{\phi}) &= -\sum_{i=1}^{4}\frac{c_i}{2}I_3^{(i)}-\frac{c_0}{2}I_4^{2me\{D=6\}}+\mathcal{O}(\epsilon)\\
&= -\frac{c_1}{2}I_3^{2m\{D=4-2\epsilon\}}(s_{1\phi},s_{23})-\frac{c_2}{2}I_3^{2m\{D=4-2\epsilon\}}(s_{\phi},s_{\phi 4})\\
&\quad -\frac{c_3}{2}I_3^{2m\{D=4-2\epsilon\}}(s_{\phi},s_{1\phi})-\frac{c_4}{2}I_3^{2m\{D=4-2\epsilon\}}(s_{4\phi},s_{23})\\
&\quad -\frac{c_0}{2}I_4^{2me\{D=6\}}(s_{1\phi},s_{4\phi};s_{23},s_{\phi})+\mathcal{O}(\epsilon).
\end{aligned}$$

The coefficients $c_i$ can be set in order to capture the divergences. They are the ratio of the Gram determinants, for instance

$$\frac{c_1}{2} = \frac{s_{1\phi}-s_{23}}{-s_{1\phi}s_{4\phi}+s_{\phi}s_{23}}.$$

Then we can compute the coefficient of the six-dimensional easy box,

$$c_0 = \sum_{i=1}^{4}c_i = -\frac{2\,s_{14}}{-s_{1\phi}s_{4\phi}+s_{\phi}s_{23}},$$

which allows us to write the finite part in a compact way,

$$(-s_{1\phi}s_{4\phi}+s_\phi s_{23})\left(I_4^{2me\{D=4-2\epsilon\}}+\sum_{i=1}^{4}\frac{c_i}{2}I_3^{(i)}\right)=s_{14}I_4^{2me\{D=6\}}+\mathcal{O}(\epsilon).$$

In conclusion the finite remainder part can be obtained considering the contribution at $\mathcal{O}(1)$ from the boxes and the contribution proportional to the two-point integral. Explicitly, subtracting the divergence (5.17) we remain with the following terms:

$$\begin{aligned}\left[A^{2L}_{cc(I)}\right]_{\text{finite}} =& A^{2L}_{cc(I)}-\left[A^{2L}_{cc(I)}\right]_{1/\epsilon\text{ poles}}\\ =&\sum_{\sigma\in\mathbb{Z}_4}\frac{1}{3}\frac{[\sigma(2)\sigma(3)]^2}{\langle\sigma(1)\sigma(4)\rangle^2}\left[\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{\sigma(1)\phi}}\right)+\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{\sigma(4)\phi}}\right)+\mathrm{Li}_2\left(1-\frac{s_{\sigma(2)\sigma(3)}}{s_{\sigma(1)\phi}}\right)\right.\\ &\left.+\mathrm{Li}_2\left(1-\frac{s_{\sigma(2)\sigma(3)}}{s_{\sigma(4)\phi}}\right)-\mathrm{Li}_2\left(1-\frac{s_\phi s_{\sigma(2)\sigma(3)}}{s_{\sigma(1)\phi}s_{\sigma(4)\phi}}\right)+\frac{1}{2}\log^2\left(\frac{s_{\sigma(1)\phi}}{s_{\sigma(4)\phi}}\right)\right]\\ &-\frac{1}{3}A^{1L}(\phi;1^+,2^+,3^+,4^+)\left[2+\log\left(\frac{\mu_R^2}{-s_\phi}\right)\right]+\mathcal{O}(\epsilon).\end{aligned}$$

## 5.5 Results for the cut-constructible part

We can sum the contributions found using double cuts in the two sectors.

### 5.5.1 Divergences

Using (5.8) and (5.17), the divergent contribution for the cut-constructible part of the unrenomalized amplitude is

$$A^{2L}_{cc}=-\sum_{i=1}^{4}\frac{1}{\epsilon^2}(-s_{i,i+1})^{-\epsilon}-\frac{1}{3}A^{1L}(\phi;1^+2^+3^+4^+)\frac{1}{\epsilon}+\mathcal{O}(1).$$

From the universal factorization of infrared and ultraviolet terms in scattering amplitude, at two-loops, the structure of $\epsilon$-poles is predicted in terms of the one-loop [24]. Due to the vanishing behavior of the tree-level amplitude, the IR structure of the UV renormalized two-loop amplitude is

$$A^{2L}_{ren}(\phi;1^+2^+\ldots n^+)=\mathcal{I}^{(1)}_{\phi+n}A^{1L}(\phi;1^+2^+\ldots n^+)+\mathcal{O}(1)$$

indeed soft and collinear factorization properties guarantee that all the IR poles can be absorbed in the operator $\mathcal{I}^{(1)}$. We already studied the IR structure in the all-plus sector for a pure QCD theory (**??**) and we observed the cancelation of $1/\epsilon$ poles at two-loop level. In our case, the operator shows the same structure

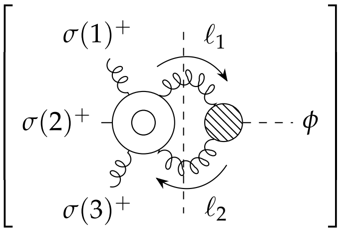
$$\mathcal{I}^{(1)}_{\phi+n}=-\left[\frac{1}{\epsilon^2}\sum_{i=1}^{n}(-s_{i,i+1})^{-\epsilon}+n\frac{\gamma_g}{\epsilon}\right]=-\left[\frac{1}{\epsilon^2}\sum_{i=1}^{n}(-s_{i,i+1})^{-\epsilon}+n\frac{\beta_0}{2\epsilon}\right]$$

where $\gamma_g$ is the gluon anomalous dimension and $\beta_0$ is the leading term of QCD Callan-Symanzik function. The difference between the pure QCD case and the self-dual Higgs model is the presence of and effective coupling which scales as two powers of the gluon coupling. Then, the structure of the unrenormalized amplitude should be

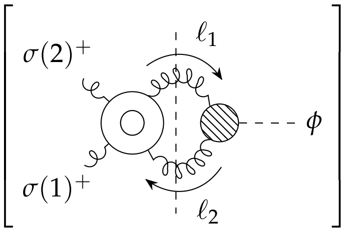
$$\begin{aligned}A^{2L}&=\mathcal{I}^{(1)}_{\phi+n}A^{1L}(\phi;1^+2^+\ldots n^+)+(n+2)\frac{\beta_0}{2\epsilon}A^{1L}(\phi;1^+2^+\ldots n^+)+\mathcal{O}(\epsilon^0)\\&=\left[-\frac{1}{\epsilon^2}\sum_{i=1}^{n}(-s_{i,i+1})^{-\epsilon}+\frac{\beta_0}{\epsilon}\right]A^{1L}(\phi;1^+2^+\ldots n^+)+\mathcal{O}(\epsilon^0).\end{aligned}\tag{5.18}$$

This argument for the unrenormalized amplitude yields a simple pole which does not depend on the number $n$ of gluons coupled with the self-dual Higgs. We considered the double cuts in $s_\phi$ channel at lower multiplicity in order to check this statement,

$\sum_{\sigma\in\mathbb{Z}_3}$




, $\sum_{\sigma\in\mathbb{Z}_2}$




.

In these simple computations, we observed the same UV divergence found in the unitarity computation for the $\phi$+4g amplitude.

Before checking the correctness of the prefactor for the $1/\epsilon$ pole, we have to carefully organized the contributions in the amplitude. We can treat the number $d_s$ of dimensions in which we allow the polarizations directions of internal gluons as a free parameter. Then we can expand the amplitude,

$$\begin{aligned} A^{2L}(\phi;1^+2^+\dots n^+) =& A^{2L[0]}(\phi;1^+2^+\dots n^+) + (d_s-2)A^{2L[1]}(\phi;1^+2^+\dots n^+) \\ &+ (d_s-2)^2 A^{2L[2]}(\phi;1^+2^+\dots n^+). \end{aligned}$$

We observe the presence of a UV divergence computing a double cut in a sector with a one-loop YM sub-amplitude which is proportional to $(d_s-2)$. For pure gluon corrections, the Callan-Symanzik function is

$$\beta_0 = 4 - \frac{(d_s-2)}{6},$$

then using (5.18-5.5.1) we can predict the structure of poles for $A^{2L[1]}$,

$$(d_s-2)A^{2L[1]}(\phi;1^+2^+\dots n^+) = -\frac{(d_s-2)}{6}\frac{1}{\epsilon}A^{1L}(\phi;1^+2^+\dots n^+).$$

This is exaclty the divergent contribution observed in the second sector due to the presence of the bubble integral in $s_\phi$-channel.

### 5.5.2 Collinear factorization of divergences

### 5.5.3 Finite remainder part

Subtracting the divergences, the finite remainder expression of the full cut-constructible part for the two-loop $\phi$+four gluon amplitude is

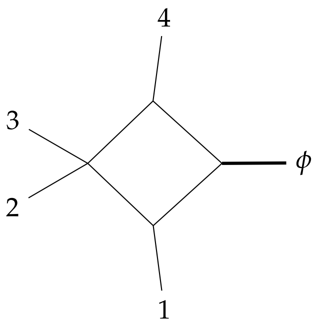
$$\begin{aligned} \left[A^{2L}_{cc}\right]_{finite} =& A^{1L}(\phi;1^+2^+3^+4^+)\Bigg[2\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{1\phi}}\right) + 2\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{2\phi}}\right) + 2\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{3\phi}}\right) + 2\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{4\phi}}\right) \\ &-\mathrm{Li}_2\left(1-\frac{s_\phi s_{34}}{s_{1\phi}s_{2\phi}}\right) - \mathrm{Li}_2\left(1-\frac{s_\phi s_{41}}{s_{2\phi}s_{3\phi}}\right) - \mathrm{Li}_2\left(1-\frac{s_\phi s_{23}}{s_{1\phi}s_{4\phi}}\right) - \mathrm{Li}_2\left(1-\frac{s_\phi s_{12}}{s_{3\phi}s_{4\phi}}\right) \\ &+\frac{1}{2}\log^2\left(\frac{s_{12}}{s_{23}}\right) + \frac{1}{2}\log^2\left(\frac{s_{1\phi}}{s_{23}}\right) + \frac{1}{2}\log^2\left(\frac{s_{2\phi}}{s_{1\phi}}\right) + \frac{1}{2}\log^2\left(\frac{s_{1\phi}}{s_{34}}\right) + \frac{1}{2}\log^2\left(\frac{s_{23}}{s_{34}}\right) \\ &+\frac{1}{2}\log^2\left(\frac{s_{2\phi}}{s_{34}}\right) + \frac{1}{2}\log^2\left(\frac{s_{12}}{s_{3\phi}}\right) + \frac{1}{2}\log^2\left(\frac{s_{3\phi}}{s_{2\phi}}\right) + \frac{1}{2}\log^2\left(\frac{s_{2\phi}}{s_{41}}\right) + \frac{1}{2}\log^2\left(\frac{s_{34}}{s_{41}}\right) \\ &+\frac{1}{2}\log^2\left(\frac{s_{3\phi}}{s_{41}}\right) + \frac{1}{2}\log^2\left(\frac{s_{41}}{s_{12}}\right) + \frac{1}{2}\log^2\left(\frac{s_{12}}{s_{4\phi}}\right) + \frac{1}{2}\log^2\left(\frac{s_{1\phi}}{s_{4\phi}}\right) + \frac{1}{2}\log^2\left(\frac{s_{23}}{s_{4\phi}}\right) \\ &+\frac{1}{2}\log^2\left(\frac{s_{4\phi}}{s_{3\phi}}\right) + \frac{2\pi^2}{3}\Bigg] + \sum_{\sigma\in\mathbb{Z}_4}\frac{1}{3}\frac{[\sigma(2)\sigma(3)]^2}{\langle\sigma(1)\sigma(4)\rangle^2}\Bigg[\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{\sigma(1)\phi}}\right) \\ &+\mathrm{Li}_2\left(1-\frac{s_\phi}{s_{\sigma(4)\phi}}\right) + \mathrm{Li}_2\left(1-\frac{s_{\sigma(2)\sigma(3)}}{s_{\sigma(1)\phi}}\right) + \mathrm{Li}_2\left(1-\frac{s_{\sigma(2)\sigma(3)}}{s_{\sigma(4)\phi}}\right) - \mathrm{Li}_2\left(1-\frac{s_\phi s_{\sigma(2)\sigma(3)}}{s_{\sigma(1)\phi}s_{\sigma(4)\phi}}\right) \\ &+\frac{1}{2}\log^2\left(\frac{s_{\sigma(1)\phi}}{s_{\sigma(4)\phi}}\right)\Bigg] - \frac{1}{3}A^{1L}(\phi;1^+,2^+,3^+,4^+)\left[2+\log\left(\frac{\mu_R^2}{-s_\phi}\right)\right] + \mathcal{O}(\epsilon). \end{aligned}$$

Appendix B

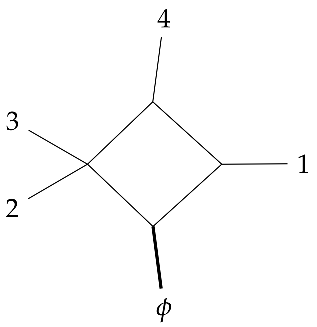
# Quadruple cuts

We explicitly compute the coefficients in front of the four-point scalar integrals in the decomposition of the two-loop amplitude. We will refer to these integrals indicating Mandelstam variables $s$ and $t$ and the masses. The three possible scalar boxes potentially present in our amplitude are diagrammatically described as follows. Obviously, similar contributions can be also present if they differ from the following ones only for cyclic permutations of the gluons.

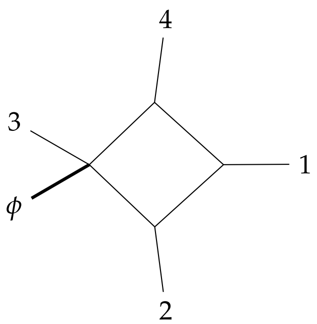
$$I_4^{2me}(s_{\phi 4}, s_{\phi 1}; m_1^2 = s_{23}, m_3^2 = s_{\phi}) = \qquad \text{(B.1)}$$




$$I_4^{2mh}(s_{14}, s_{\phi 1}; m_1^2 = s_{23}, m_2^2 = s_{\phi}) = \qquad \text{(B.2)}$$




$$I_4^{1m}(s_{14}, s_{12}; m^2 = s_{3\phi}) = \qquad \text{(B.3)}$$



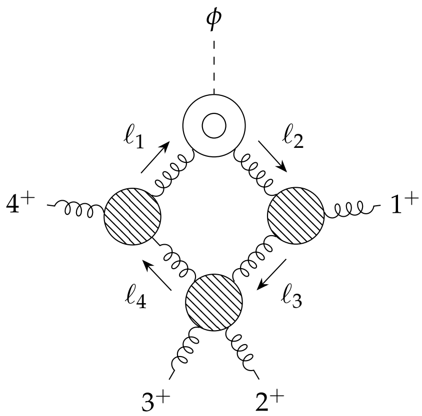

The first configuration (B.1) is called two-mass easy box and presents two external masses ad diagonally opposite corners. The second one (B.2) is identified as the hard configuration because of its greater complexity in its calculation and it is characterized by the presence of two external masses at adjacent corners. Lastly, in the third scalar box ( B.3) we have a single massive point. These four-point integrals can be computed solving differential equations for boxes with massless internal legs at $\mathcal{O}(\epsilon^0)$ [25].

## B.1 First sector with a 1L $\phi$+gluon subamplitude

We have to compute the quadruple cuts for the three possible four-point configurations imposing the four on-shell constraints for the inner gluons and studying all the possible helicity organizations of the sub-amplitudes. We start considering the first sector in which the $\phi$+gluon sub-amplitude is considered at one-loop level.

### Easy two-mass box

We start considering the following quadruple cut in the two-mass easy configuration.




The loop momenta are

$$\{\ell_1,\ \ell_2,\ \ell_3,\ \ell_4\} = \{\ell_1,\ \ell_1 - p_\phi,\ \ell_1 - p_\phi - p_1,\ \ell_1 + p_4\}\,.$$

We impose the on-shell conditions $\ell_i^2 = 0$ and we obtain

$$\left\{\ell_1^2 = 0,\ \ell_1 \cdot p_\phi = \frac{m_H^2}{2},\ \ell_1 \cdot p_4 = 0,\ \ell_1 \cdot p_{\phi 1} = \frac{s_{\phi 1}}{2}\right\},$$
$$\left\{\ell_1^2 = 0,\ \ell_1 \cdot p_\phi = \frac{m_H^2}{2},\ \ell_1 \cdot p_4 = 0,\ \ell_1 \cdot p_1 = \frac{s_{\phi 1} - m_H^2}{2}\right\}. \tag{B.4}$$

We consider the following decomposition of the loop momentum

$$\ell_1^\mu = a_1 p_1^\mu + a_4 p_4^\mu + d_1 \frac{\langle 1\sigma^\mu 4]}{2} + d_4 \frac{\langle 4\sigma^\mu 1]}{2}$$

where $a_1, a_4, d_1, d_4$ are coefficients we determine using on-shell conditions (B.4).
We start considering the constraints:

$$\ell_1 \cdot p_4 = a_1 p_1 \cdot p_4 = 0 \Rightarrow a_1 = 0$$
$$\ell_1 \cdot p_1 = a_4 p_4 \cdot p_1 = \frac{s_{\phi 1} - m_H^2}{2} \Rightarrow a_4 = \frac{s_{\phi 1} - m_H^2}{s_{41}}.$$

Now we want to impose the condition $\ell_1^2 = 0$:

$$a_1 \ell_1 \cdot p_1 + a_4 \ell_1 \cdot p_4 + d_1 \frac{\langle 1\ell_1 4]}{2} + d_4 \frac{\langle 4\ell_1 1]}{2} = 0$$
$$2a_1 a_4 (\ell_1 \cdot p_4) - 2d_1 d_4 p_4 \cdot p_1 = 0$$
$$d_1 d_4 = a_1 a_2 = 0 \Rightarrow d_1 = 0 \text{ or } d_4 = 0.$$

We have two possible solutions and the last constraint on the product $\ell_1 \cdot p_\phi$ fixes the last non-vanishing coefficient. In conclusion we have the solutions:

$$\ell_1^{(1)} = a_4 p_4^\mu + d_1 \frac{\langle 1\sigma^\mu 4]}{2} \text{ with } \begin{cases} a_4 = \frac{s_{\phi 1} - s_\phi}{s_{41}} \\ d_1 = \frac{1}{\langle 1\phi 4]}\left(s_\phi - \frac{(s_{\phi 1} - s_\phi)(s_{\phi 4} - s_\phi)}{s_{41}}\right) \end{cases},$$

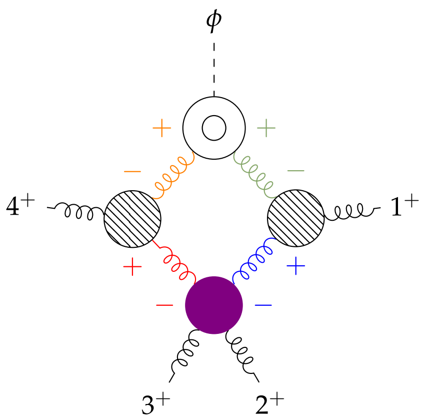
$$\ell_1^{(2)} = a_4 p_4^\mu + d_4 \frac{\langle 4\sigma^\mu 1]}{2} \text{ with } \begin{cases} a_4 = \frac{s_{\phi 1} - s_\phi}{s_{41}} \\ d_4 = \frac{1}{\langle 4\phi 1]}\left(s_\phi - \frac{(s_{\phi 1}-s_\phi)(s_{\phi 4}-s_\phi)}{s_{41}}\right) \end{cases} . \tag{B.5}$$

Using the property $2\ell_1^\mu = \langle \ell_1|\sigma^\mu|\ell_1]$, we can write the spinors of the solutions in the following way:

$$\begin{cases} |\ell_1^{(1)}\rangle = a_4|4\rangle + d_1|1\rangle \\ |\ell_1^{(1)}] = |4] \end{cases} , \tag{B.6}$$

$$\begin{cases} |\ell_1^{(2)}\rangle = |4\rangle \\ |\ell_1^{(2)}] = a_4|4] + d_4|1] \end{cases} . \tag{B.7}$$

After the kinematic considerations, we can study the quadruple cut of the two-loop amplitude which corresponds to the sum over the allowed helicity configurations of the products with trees and one-loop factors. There is only one allowed helicity configuration which has a non-vanishing tree-level four gluon amplitude.




We only have to compute the following contribution:

$$\begin{aligned} c_{2me}(\ell_1) :=& A^{tree}(2^+, 3^+, \ell_4^-, (-\ell_3)^-) A^{tree}(4^+, (-\ell_4)^+, \ell_1^-) \\ & A^{1L}(\phi; (-\ell_1)^+ \ell_2^+) A^{tree}(1^+, (-\ell_2)^- \ell_3^+) \\ =& \frac{\langle \ell_4 \ell_3\rangle^3}{\langle 23\rangle\langle 3\ell_4\rangle\langle \ell_3 2\rangle} \frac{[\ell_4 4]^3}{[4\ell_1][\ell_1\ell_4]} \frac{-2m_H^4}{\langle \ell_1\ell_2\rangle\langle \ell_2\ell_1\rangle} \frac{[1\ell_3]^3}{[\ell_3\ell_2][\ell_2 1]} \\ =& \frac{2m_H^4}{\langle 23\rangle} \frac{[4\ell_4\ell_3 1]^3}{[4\ell_1\ell_2 1]\langle 3\ell_4\ell_1\ell_2\ell_3 2\rangle} = \frac{-2m_H^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\langle 14\ell_1\phi 1] \end{aligned}$$

We immediately observe that $c_{2me}(\ell_1^{(1)}) = 0$: this is due to the relation $|\ell_1^{(1)}] = |4]$.
Using the second solution and in particular the spinors (B.7), we can compute the only non-trivial contribution:

$$\begin{aligned} c_{2me}(\ell_1^{(2)}) &= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\langle 14\ell_1^{(2)}]\langle \ell_1^{(2)}\phi 1] \\ &= A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left(s_{14}s_\phi - (s_{\phi 1} - s_\phi)(s_{\phi 4} - s_\phi)\right). \end{aligned}$$

Now we are able to extract the coefficient in front of the box integral with massless inner lines and two external masses $s_\phi$ and $s_{23}$ at diagonally opposite corners. To connect the desired coefficients with the generalized unitarity cuts, it is useful to introduce the following vector, orthogonal to three given Lorentz vectors $p_a, p_b, p_c$:

$$\omega^\mu(p_a, p_b, p_c) = N_\omega\left(\langle a\sigma^\mu b]\langle bca] - \langle b\sigma^\mu a]\langle acb]\right),$$

where $N_\omega$ represents a normalization constant.
We know that a general decomposition of a scalar box integral is:

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\mathcal{D}(k)}{p_1 p_2 p_3 p_4}, \quad \begin{cases} \mathcal{D}(k) = d_1 + d_2(k \cdot n) \\ p_i = \text{ propagators} \end{cases}$$

The contribution proportional to $(k \cdot n)$ is a spurious term that vanishes after the integration. If we fix the generic direction $n$ in order to be equal to the vector $\omega^\mu(p_1, p_4, p_\phi)$, we have

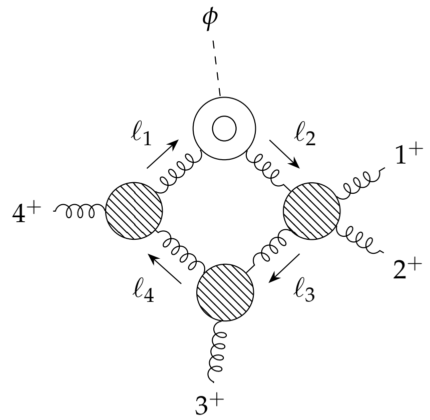
$$\mathcal{D}_{2me}(\ell_1^{(1)}) = d_1^{(2me)} + d_2^{(2me)}(\ell_1^{(1)} \cdot \omega) \equiv c_{2me}(\ell_1^{(1)})$$
$$\mathcal{D}_{2me}(\ell_1^{(2)}) = d_1^{(2me)} + d_2^{(2me)}(\ell_1^{(2)} \cdot \omega) = d_1^{(2me)} - d_2^{(2me)}(\ell_1^{(1)} \cdot \omega) \equiv c_{2me}(\ell_1^{(2)}).$$

Therefore the coefficient in front of the integrated object $I_4^{2me}(s_{\phi 4}, s_{1\phi}; s_{23}, s_\phi)$ is

$$d_1^{(2me)} = \frac{1}{2}\left(c_{2me}(\ell_1^{(1)}) + c_{2me}(\ell_1^{(2)})\right) \tag{B.8}$$
$$= \frac{1}{2} A^{1L}(\phi; 1^+, 2^+, 3^+, 4^+)\left(s_{14}s_\phi - (s_{\phi 1} - s_\phi)(s_{\phi 4} - s_\phi)\right). \tag{B.9}$$

**Hard two-mass box**

An other possible configuration, that we can study through quadruple cuts, includes two external masses at adjacent corners.




This configuration is known as *hard* two-mass box because the calculation of the scalar box integral with two adjacent masses ($I_4^{2mh}$) is more complicated than the previous integral $I_4^{2me}$.
In this case, the loop momenta are:

$$\{\ell_1,\ \ell_2,\ \ell_3,\ \ell_4\} = \{\ell_1,\ \ell_1 - p_\phi,\ \ell_1 + p_4 + p_3,\ \ell_1 + p_4\}.$$

We choose the following parametrization of the loop momentum $\ell_1$:

$$\ell_1^\mu = a_3 p_3^\mu + a_4 p_4^\mu + d_3 \frac{\langle 3\sigma^\mu 4]}{2} + d_4 \frac{\langle 4\sigma^\mu 3]}{2}.$$

The quadruple cut requires the on-shell conditions $\ell_i^2 = 0$ and, using these constraints, we find the solutions:

$$\ell_1^{(1)} = -p_4^\mu + d_3 \frac{\langle 3\sigma^\mu 4]}{2} \quad \text{with} \quad d_3 = \frac{s_{4\phi}}{\langle 3\phi 4]}, \tag{B.10}$$
$$\ell_1^{(2)} = -p_4^\mu + d_4 \frac{\langle 4\sigma^\mu 3]}{2} \quad \text{with} \quad d_4 = \frac{s_{4\phi}}{\langle 4\phi 3]}. \tag{B.11}$$

Cutting the two-loop amplitude, we have only the following helicity contribution due to the fact that the tree-level four gluon amplitude with two external positive gluons requires the negative helicity for the other two particles.

$$\begin{aligned} c_{2mh}(\ell_1) :=& A^{tree}(3^+, (-\ell_3)^+, \ell_4^-) A^{tree}(4^+, (-\ell_4)^+, \ell_1^-) \\ & A^{1L}(\phi; (-\ell_1)^+ \ell_2^+) A^{tree}(1^+, 2^+, \ell_3^-, (-\ell_2)^-) \\ =& \frac{[3\ell_3]^3}{[\ell_3 \ell_4][\ell_4 3]]} \frac{[4\ell_4]^3}{[\ell_4 \ell_1][\ell_1 4]} \frac{-2m_H^4}{\langle \ell_1 \ell_2\rangle\langle \ell_2 \ell_1\rangle} \frac{\langle \ell_3 \ell_2\rangle^3}{\langle 12\rangle\langle 2\ell_3\rangle\langle \ell_2 1\rangle} \end{aligned}$$

Remembering that $\ell_4 = \ell_1 + p_4$, from the solution (B.10) we find the expressions for the spinors associated to the gluon with momenta $\ell_1^{(1)}$ and $\ell_4^{(1)}$:

$$\begin{cases} |\ell_1^{(1)}\rangle = -|4\rangle + d_3|3\rangle \\ |\ell_1^{(1)}] = |4] \end{cases} \Rightarrow \begin{cases} |\ell_4^{(1)}\rangle = d_3|3\rangle \\ |\ell_4^{(1)}] = |4] \end{cases} . \tag{B.12}$$

The relation $|\ell_4^{(1)}] = |4]$ shows us that the anti-MHV amplitude $A^{tree}(4^+, (-\ell_4)^+, \ell_1^-)$ vanishes if we consider the first solution, therefore $c_{2mh}(\ell_1^{(1)}) = 0$.
Similar considerations hold for the second solution, in fact the spinors associated to the momenta $\ell_1^{(2)}$ and $\ell_3^{(2)} = \ell_1^{(2)} + p_3 + p_4$ are

$$\begin{cases} |\ell_1^{(2)}\rangle = |4\rangle \\ |\ell_1^{(2)}] = -|4] + d_4|3] \end{cases} \Rightarrow \begin{cases} |\ell_3^{(2)}\rangle = |4\rangle + \frac{1}{d_4}|3\rangle \\ |\ell_3^{(2)}] = d_4|4] \end{cases} . \tag{B.13}$$

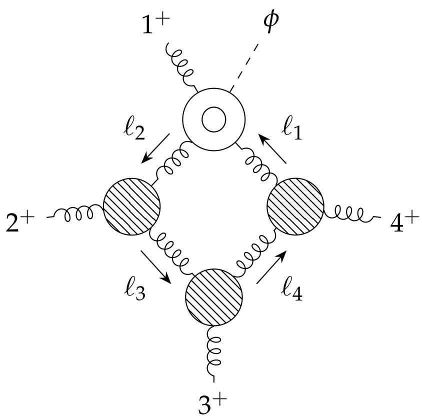
The last relation implies that

$$A^{tree}(3^+, (-\ell_3)^+, \ell_4^-) = 0 \;\Rightarrow\; c_{2mh}(\ell_3^{(2)}) = 0.$$

In conclusion the coefficient investigated using this quadruple cut vanishes and therefore in this sector we observe the absence of a contribution proportional to the integral $I_4^{2mh}(s_{34}, s_{4\phi}; s_{12}, s_\phi)$.

#### One-mass box

The last configuration we have to consider is the following one-mass box.




If we express $\ell_1$ in the decomposition

$$\ell_1^\mu = a_3 p_3^\mu + a_4 p_4^\mu + d_3 \frac{\langle 3\sigma^\mu 4]}{2} + d_4 \frac{\langle 4\sigma^\mu 3]}{2},$$

we can determine the coefficients to satisfy the constraints:

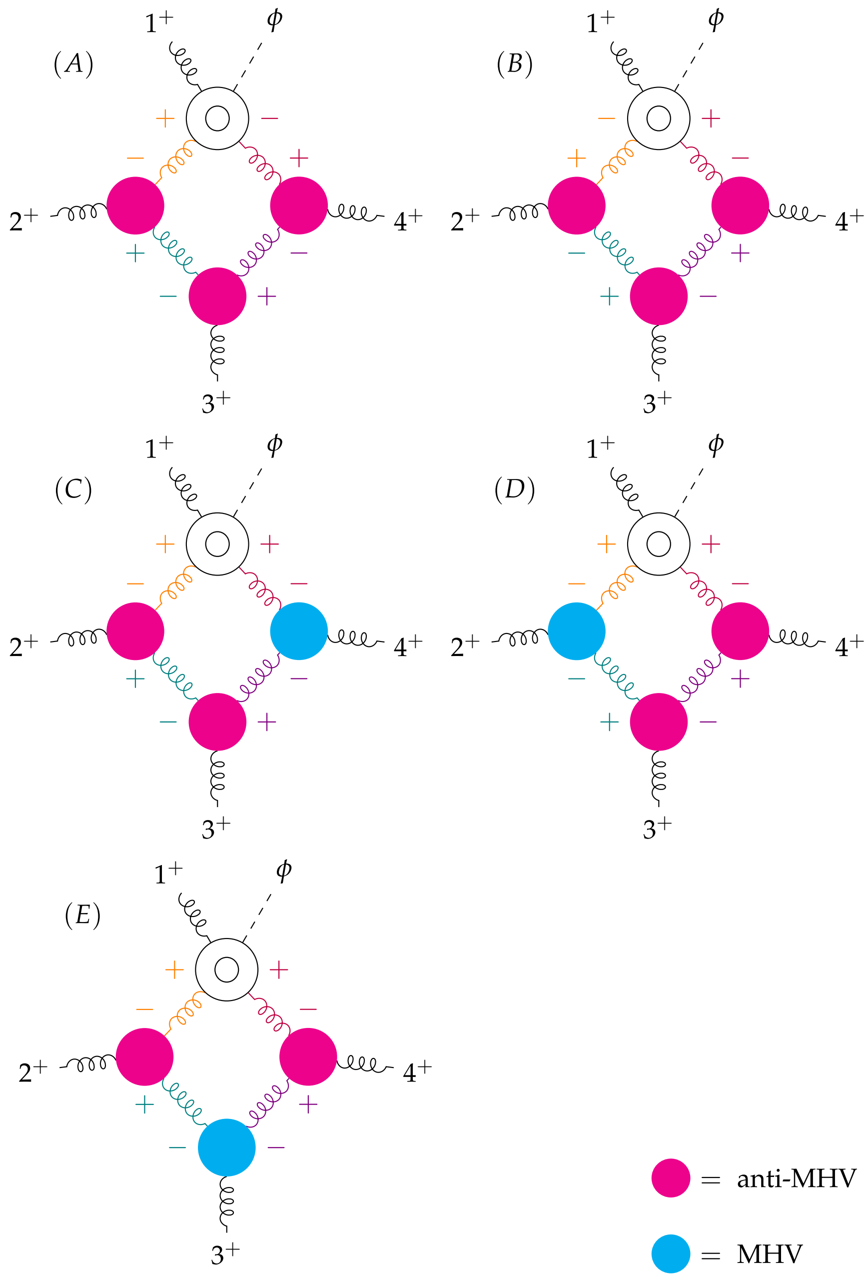
$$\{\ell_1,\ \ell_2,\ \ell_3,\ \ell_4\} = \{\ell_1,\ \ell_1 - p_\phi - p_1,\ \ell_1 + p_3 + p_4,\ \ell_1 + p_4\} = \{0,0,0,0\}.$$

We find two solutions:

$$\ell_1^{(1)} = -p_4^\mu + d_3 \frac{\langle 3\sigma^\mu 4]}{2} \text{ with } d_3 = \frac{s_{\phi 1} - s_{24} - s_{34}}{\langle 324]}, \tag{B.14}$$

$$\ell_1^{(2)} = -p_4^\mu + d_4 \frac{\langle 4\sigma^\mu 3]}{2} \text{ with } d_4 = \frac{s_{\phi 1} - s_{24} - s_{34}}{\langle 423]}. \tag{B.15}$$

We have to study the possible helicity configurations: the non-trivial possibilities are represented in the following diagrams.



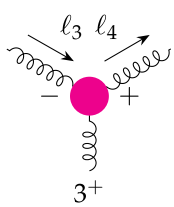

If we consider the first solution in which the spinors have the following relations

$$\begin{cases} |\ell_1^{(1)}] \propto |\ell_4^{(1)}] \propto |4] \\ |\ell_4^{(1)}\rangle \propto |\ell_3^{(1)}\rangle \propto |3\rangle \end{cases},$$

we immediately observe that:

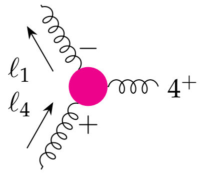
1. the diagrams $(A)$ and $(C)$ vanish, in fact

$$A^{tree}(3^+,(-\ell_3)^-,\ell_4^+) =$$



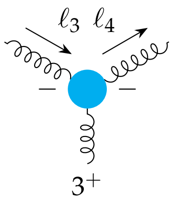

$$\propto [3\ell_4^{(1)}]^3 = 0;$$

2. the diagrams $(B)$ and $(D)$ are zero due to the relation

$$A^{tree}(4^+,(-\ell_4)^+,\ell_1^-) =$$


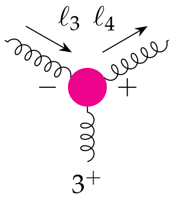
$$\propto [\ell_4^{(1)}4]^3 = 0;$$

3. the diagram $(E)$ vanishes because of the presence of the following factor:

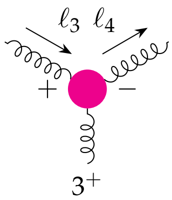
$$A^{tree}(3^+,(-\ell_3)^-,\ell_4^-) =$$


$$\propto \langle \ell_3^{(1)}\ell_4^{(1)}\rangle^3 = 0.$$

Therefore we have no contributions considering the first solution $\ell_1^{(1)}$.
For the second solution, we observe that $|\ell_4^{(2)}] \propto |\ell_3^{(2)}] \propto |3]$. These relations simplify the calculation because four diagrams are zero due to the vanishing sub-amplitudes:



$$\propto [3\ell_4^{(2)}]^3 = 0 \Rightarrow \begin{cases} (A) = 0 \\ (C) = 0 \end{cases}$$



$$\propto [3\ell_3^{(2)}]^3 = 0 \Rightarrow \begin{cases} (B) = 0 \\ (D) = 0 \end{cases}$$

The only non-trivial contribution is represented by the diagram with the gluon $3^+$ associated to a MHV vertex $(E)$. The quadruple cut of this diagram is described by the following product of sub-amplitudes:

$$\begin{aligned} c_{1m}(\ell_1) &= A^{tree}(3^+,\ell_4^-,(-\ell_3)^-)A^{tree}((-\ell_4)^+,4^+,\ell_1^-)A^{1L}(\phi;1^+,(-\ell_1)^+,\ell_2^+)A^{tree}((-\ell_2)^-,2^+,\ell_3^+) \\ &= \frac{\langle \ell_4\ell_3\rangle^3}{\langle \ell_3 3\rangle\langle 3\ell_4\rangle}\frac{[\ell_4 4]^3}{[4\ell_1][\ell_1\ell_4]}\frac{-2m_H^4}{\langle \ell_2\ell_1\rangle\langle \ell_1 1\rangle\langle 1\ell_2\rangle}\frac{[2\ell_3]^3}{[\ell_3\ell_2][\ell_2 2]}. \end{aligned}$$

Using the momentum conservation laws together with some simple spinor algebra, we obtain

$$c_{1m}(\ell_1) = \frac{-2m_H^4}{\langle 21\rangle}\frac{[34]\langle \ell_4\ell_3\rangle^2[\ell_4 4][2\ell_3]}{\langle 3\ell_4\rangle\langle 1\ell_4\rangle[\ell_1\ell_4]\langle \ell_1\ell_3\rangle}.$$

Now we explicitly use the spinors of the second solution $\ell_1^{(2)}$ and we find

$$c_{1m}(\ell_1^{(2)}) = \frac{-2m_H^4}{\langle 21\rangle}\frac{[34][23]}{\langle 41\rangle} = -s_{34}s_{23}A^{1L}(\phi;1^+2^+3^+4^+).$$

As shown in the easy two-mass case (B.8), the desired coefficient is the arithmetic average of the results from the two solutions of the loop momentum:

$$d_1^{1m} = \frac{1}{2}\left(c_{1m}(\ell_1^{(1)}) + c_{1m}(\ell_1^{(2)})\right) = -\frac{1}{2}s_{34}s_{23}A^{1L}(\phi;1^+,2^+,3^+,4^+) \qquad \text{(B.16)}$$

This is the coefficient of the integral function $I_4^{1m}(s_{23}, s_{34}; s_{1\phi})$ where

$$I_4^{1m}(s,t;m^2) = \frac{2}{st}\left[\frac{1}{\epsilon^2}\left((-s)^{-\epsilon}+(-t)^{-\epsilon}+(-m^2)^{-\epsilon}\right)\right.$$
$$\left. -\mathrm{Li}_2\left(1-\frac{m^2}{s}\right)-\mathrm{Li}_2\left(1-\frac{m^2}{t}\right)-\frac{1}{2}\ln^2\left(\frac{-s}{-t}\right)-\frac{\pi^2}{6}\right].$$

#### Summary of the results

We have finished to compute the possible quadruple cuts in this sector with a one-loop $\phi$ amplitude. We obtain two non-vanishing contributions in the amplitude proportional to the one-mass box and the two-mass easy four-point integral. In the two-loop amplitude, the following terms are present considering the cyclic permutation of the external gluons,

$$\sum_{\sigma\in\mathbb{Z}_4} A^{1L}(\phi;\sigma(1^+),\sigma(2^+),\sigma(3^+),\sigma(4^+))\left[-\frac{1}{2}s_{\sigma(3)\sigma(4)}s_{\sigma(2)\sigma(3)}\,I_4^{1m}(s_{\sigma(2)\sigma(3)},s_{\sigma(3)\sigma(4)};s_{\sigma(1)\phi})\right.$$
$$\left.+\frac{1}{2}\left(s_{\sigma(1)\sigma(4)}s_{\phi}-(s_{\phi\sigma(1)}-s_{\phi})(s_{\phi\sigma(4)}-s_{\phi})\right)I_4^{2me}(s_{\phi\sigma(4)},s_{\sigma(1)\phi};s_{\sigma(2)\sigma(3)},s_{\phi})\right].$$

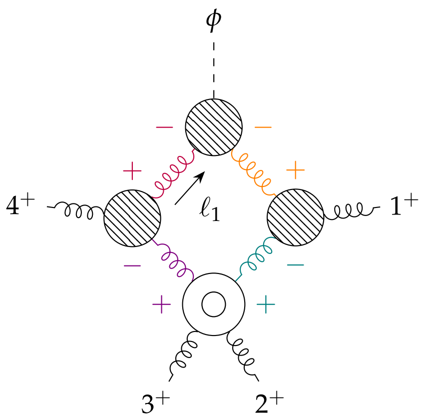
On the contrary, we observed the absence of two-mass hard scalar boxes in our amplitude in this sector with a one-loop self-dual Higgs sub-amplitude.

## B.2 Second sector with a 1L YM subamplitude

We impose four on-shell constraint in order to investigate the contributions proportional to the four-point scalar integrals. As observe in the first sector, in principle we can have three configurations (B.1, B.2, B.3) due to the presence of five asymptotic states with one external massive scale. We will consider the possible helicity configurations of the inner gluons and compute the product of the sub-amplitudes. Connecting these expressions with the coefficient in front of the scalar boxes, we will deduce the weight of the two-mass easy four-point integral in the two-loop amplitude and we will demonstrate the absence of other boxes in this sector with a one-loop YM sub-amplitude.

#### Easy two-mass box

Let us start computing the quadruple cut in the configuration of two external masses at opposite corners. In principle, one can consider three different possibilities because there are three subamplitudes with only gluons and each of these can be considered at one-loop level, but the only non-trivial configuration is the following one.




In fact, we represent the only allowed helicity configuration: the only non vanishing $\phi + 2g$ amplitude requires negative helicity (see **??**) and, according to the non-vanishing condition of three gluon amplitudes, this completely fixes the helicity of all internal gluons. If we choose a different position for the one-loop gluon sub-amplitude, we can see the presence of the tree four gluon amplitude $A^{tree}(2^+,3^+,+,\pm)$ which nullifies the contribution.

Established what the only relevant loop configuration is for this quadruple cut, we can compute the discontinuity keeping in mind the definition of loop momenta:

$$\{\ell_1,\ \ell_2,\ \ell_3,\ \ell_4\} = \{\ell_1,\ \ell_1 - p_\phi,\ \ell_1 - p_\phi - p_1,\ \ell_1 + p_4\}\,,$$

and the on-shell solutions already computed [B.5]. Doing the quadruple cut, we need to consider the following product of sub-amplitudes:

$$\begin{aligned}
c'_{2me}(\ell_1) :=& A^{1L}(2^+,3^+,\ell_4^+,(-\ell_3)^-)A^{tree}(4^+,\ell_1^+,(-\ell_4)^-)\\
& A^{tree}(\phi;\ell_2^-,(-\ell_1)^-)A^{tree}(1^+,\ell_3^-,(-\ell_2)^+)\\
=& \frac{1}{3}\frac{[\ell_4(-\ell_3)][23]}{\langle\ell_4(-\ell_3)\rangle\langle 23\rangle}\frac{[4\ell_1]^3}{[\ell_1\ell_4][\ell_4 4]}\left(-\langle\ell_2\ell_1\rangle^2\right)\frac{[\ell_2 1]^3}{[\ell_3\ell_2][1\ell_3]}\\
=& \frac{1}{3}\frac{[\ell_4\ell_3][23]}{\langle\ell_4\ell_3\rangle\langle 23\rangle}\frac{[4\ell_1]^3}{[\ell_1\ell_4][\ell_4 4]}\langle\ell_2\ell_1\rangle^2\frac{[\ell_2 1]^3}{[\ell_3\ell_2][1\ell_3]}
\end{aligned}$$

where in the last passage we carefully used the analytic continuation of spinors.
Using a simple spinor algebra, we can do the following simplifications:

$$\begin{aligned}
c'_{2me}(\ell_1) &= \frac{[\ell_4\ell_3][23][4\ell_1]^3\langle\ell_1\ell_2 1]^2[\ell_2 1]}{3\langle\ell_4(\ell_2-1)\ell_2]\langle 23\rangle[\ell_1\ell_4][\ell_4 4][1\ell_3]}\\
&= \frac{[\ell_4\ell_3][23][4\ell_1]^3\langle\ell_1(\ell_1-\phi)1]\langle\ell_1(\ell_3+1)1][\ell_2 1]}{-3\langle\ell_4 1\rangle[1\ell_2]\langle 23\rangle[\ell_1\ell_4][\ell_4 4][1\ell_3]}\\
&= \frac{-[\ell_4\ell_3][23][4\ell_1]^3\langle\ell_1\phi 1]\langle\ell_1\ell_3\rangle[\ell_3 1]}{3\langle\ell_4 1\rangle\langle 23\rangle[\ell_1\ell_4][\ell_4 4][1\ell_3]}\\
&= \frac{[23][4\ell_1]^3\langle\ell_1\phi 1]\langle\ell_1(\ell_4+p_{23})\ell_4][\ell_3 1]}{-3\langle 1(\ell_1+4)4]\langle 23\rangle[\ell_1\ell_4][1\ell_3]}\\
&= \frac{[23][4\ell_1]^2\langle\ell_1\phi 1]\langle\ell_1 p_{23}\ell_4]}{3\langle 1\ell_1\rangle\langle 23\rangle[\ell_1\ell_4]} = \frac{[23][4\ell_1\ell_1\phi 1][4\ell_1 p_{23}\ell_4]}{3\langle 23\rangle\langle 14\rangle[4\ell_4]}\\
&= \frac{[23][4\ell_4\ell_1\phi 1][4\ell_4 p_{23}\ell_4]}{3\langle 23\rangle\langle 14\rangle[4\ell_4]} = \frac{[23][4\ell_4]}{3\langle 23\rangle\langle 14\rangle}\langle\ell_4\phi 1]\langle\ell_4 p_{23}\ell_4].
\end{aligned}$$

Now using the property

$$\langle\ell_4 p_{23}\ell_4] = (\ell_4+p_{23})^2 - s_{23} = \ell_3^2 - s_{23} = -s_{23},$$

we obtain

$$c'_{2me}(\ell_1) = \frac{1}{3}\frac{[23]^2}{\langle 14\rangle}[4\ell_4]\langle\ell_4\phi 1].$$

For the first solution $\ell_1^{(1)}$, we have $[4\ell_4] = 0$, then $c'_{2me}(\ell_1^{(1)})$ vanishes. A non-trivial contribution comes from the second solution:

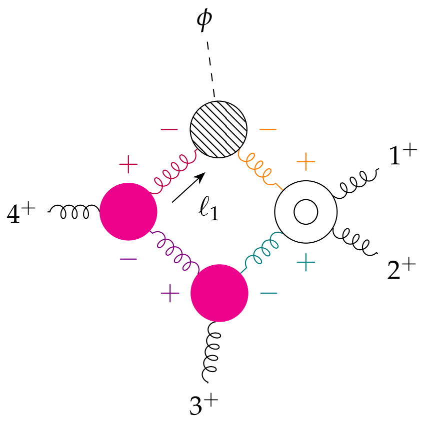
$$\begin{aligned}
c'_{2me}(\ell_1^{(2)}) &= \frac{1}{3}\frac{[23]^2}{\langle 14\rangle}d_4[41]\langle 4\phi 1] = \frac{1}{3}\frac{[23]^2}{\langle 14\rangle}s_{14}\left(s_\phi - \frac{(s_{\phi 1}-s_\phi)(s_{\phi 4}-s_\phi)}{s_{14}}\right)\\
&= \frac{1}{3}\frac{[23]^2}{\langle 14\rangle^2}\left(-s_{1\phi}s_{4\phi} + s_\phi(s_{14}+s_{\phi 1}+s_{4\phi}-s_\phi)\right) = \frac{1}{3}\frac{[23]^2}{\langle 14\rangle^2}(-s_{1\phi}s_{4\phi}+s_\phi s_{23}).
\end{aligned}$$

In conclusion in this sector with a one-loop YM sub-amplitude, the coefficient of the easy two-mass four-point integral $I_4^{2me}(s_{\phi 4}, s_{1\phi}; s_{23}, s_\phi)$ is:

$$d_1'^{(2me)} = \frac{1}{2}\left(c'_{2me}(\ell_1^{(1)}) + c'_{2me}(\ell_1^{(2)})\right) = \frac{1}{6}\frac{[23]^2}{\langle 14\rangle^2}(-s_{1\phi}s_{4\phi}+s_\phi s_{23}). \tag{B.17}$$

#### Hard two-mass box

Now we are interested in the quadruple cut acting on a configuration with two massive adjacent corners in the sector with a one-loop Yang-Mills sub-amplitude. In the easy two-mass box, we observed the presence of only one allowed configuration; similar considerations hold for the harder case: the tree-level $\phi$+gluon amplitude requires a negative helicity for the inner gluons, then the vertex with four gluon must have at least three positive gluons and therefore we have to consider the one-loop level for this four gluon sub-amplitude in order to obtain a non-trivial result.



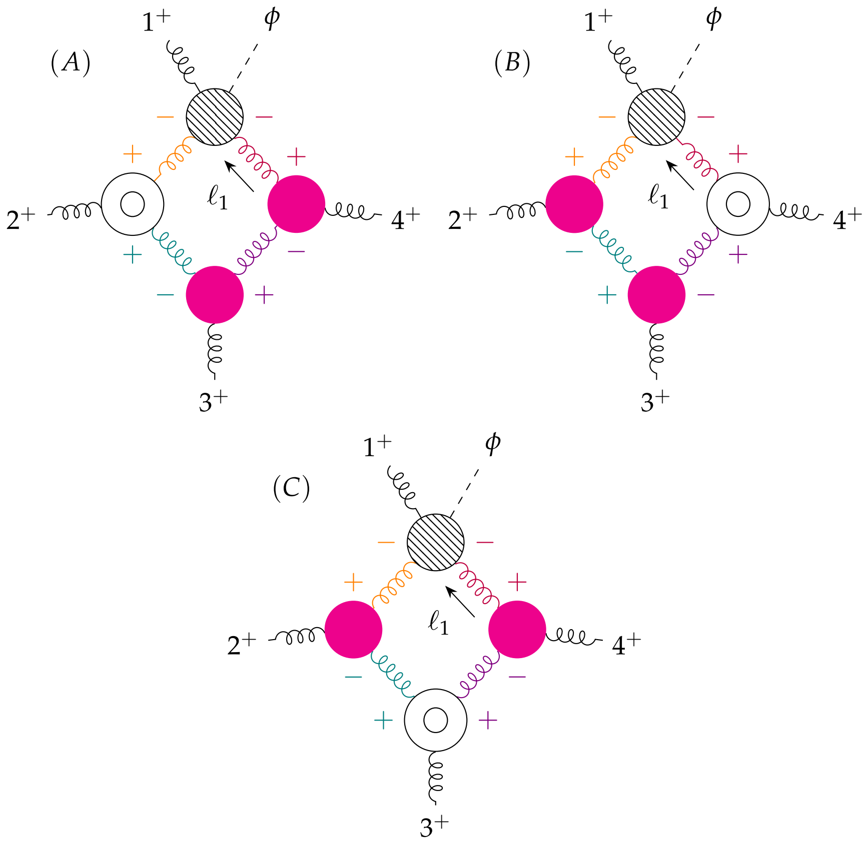
In the diagram, we explicitly represent the adjacent anti-MHV vertexes, then the contribution vanishes. In fact, we can consider the two kinematical solutions for the loop momentum $\ell_1$ using the on-shell conditions dictated by the quadruple cut [B.12, B.13] and we can observe that:

1. for the first solution $\ell_1^{(1)}$,
$$A^{tree}(4^+,(-\ell_4)^+,\ell_1^-) \propto [\ell_4^{(1)}4]^3 = 0;$$
2. considering the second solution $\ell_1^{(2)}$,
$$A^{tree}(3^+,(-\ell_3)^-,\ell_4^+) \propto [3\ell_4^{(2)}]^3 = 0.$$

This explicitly shows us that, also in the sector with a one-loop YM sub-amplitude, the contribution proportional to the hard two-mass four-point integral is absent.

### One-mass box

The last quadruple cut we need to consider is applied to the one-mass box configuration. Also in this case, we need to consider a one-loop pure gluon sub-amplitude and in principle we have three possible contributions. The self-dual Higgs is coupled with four gluons: one external with positive helicity and two inner gluons which must have negative helicity in order to obtain a non-vanishing tree amplitude. This completely fixes the helicity of the other inner legs, therefore we have to consider only three helicity configurations, one for each possible position of the one-loop sub-amplitude.




We have represented the only possible configurations which do not have tree-level three-gluon amplitudes with all-plus particles. One can show that the contributions $(A)$ and $(B)$ vanish when we consider the two on-shell solutions $\ell_1^{(1)}$ and $\ell_1^{(2)}$, in fact they present two adjacent anti-MHV vertexes.

But there is a general observation that immediately proves the absence of one-mass boxes in our amplitude in this sector: the three diagrams required a one-loop all-plus three gluon vertex, but we can show that $A^{nL}(a^+, b^+, c^+) = 0$.
To prove this statement we use special three-point kinematics and little group scaling [3]. If three light-like vectors satisfy the momentum conservation $p_a^\mu + p_b^\mu + p_c^\mu = 0$, then one product between $[ab]$ and $\langle ab\rangle$ must vanish due to the relation

$$\langle ab\rangle[ba] = 2p_a \cdot p_b = (p_a + p_b)^2 = p_c^2 = 0.$$

Supposing $[ab]$ different to zero, the condition

$$[abc\rangle = [a(-a-c)c\rangle = 0$$

implies $\langle bc\rangle = 0$ and a similar observation holds to the square product $\langle ac\rangle = 0$ considering the spinor structure $[bac\rangle$. This shows that an on-shell three-point amplitude with massless particles can only depend on either angle or square brackets. If we suppose that the amplitude can be written in terms of square brackets, we can write the result in the following form

$$A(a^{h_a}, b^{h_b}, c^{h_c}) = \xi\ [ab]^{x_{ab}}[ac]^{x_{ac}}[bc]^{x_{bc}}.$$

If we consider the little group scaling

$$|p\rangle \to t|p\rangle, \quad |p] \to t^{-1}|p]$$

which does not change the momentum $p^\mu = \frac{1}{2}\langle p|\sigma^\mu|p]$, an amplitude with massless spin-1 particles transforms homogeneously with weight $-2h_i$ where $h_i = \pm 1$ is the helicity of the $i-$th particle. In fact this changing is inherited from the beaviour of polarization vectors $\epsilon_\pm^\mu(p;q)$. Applying the little group scaling separately for the three momenta, one can find the relation between the exponents and the helicity of the external gluons obtaining

$$A(a^{h_a}, b^{h_b}, c^{h_c}) = \xi\ [ab]^{h_a+h_b-h_c}[ac]^{h_a+h_c-h_b}[bc]^{h_b+h_c-h_a}.$$

In our case with all-plus gluons, using basic principles and independently from the loop-level we have

$$A(a^+, b^+, c^+) = \xi\,[ab][ac][bc]$$

Using dimensional analysis, the color-ordered three-gluon amplitude must have mass dimension 1, therefore the parameter $\xi$ must have a mass-dimension $-2$. We need to understand if in our theory a constant $\xi$ with this dimensional property can emerge. We observe that Bose-symmetry requires that the coupling must be associated with antisymmetric structure constants, therefore the natural term which can produce this amplitude is $\mathrm{Tr}\left(G^\mu{}_\nu G^\nu{}_\lambda G^\lambda{}_\mu\right)$ which is a dimension-6 operator. However we do not have this object in our Lagrangian which only contains the YM structure and the effective 5-dimension operators which describes the couplings between gluons and scalars.
This shows that at any loop level the all-plus three gluon amplitude is zero: as a consequence, all the possible contributions in this quadruple cut vanish and the coefficient of the one-mass four-point integral is zero in this sector, contrary to what observed in the cut-constructible pieces with a one-loop $\phi$ amplitude.

### Summary of the results

From the quadruple cuts, we found a four-point contribution in the cut-constructible part of this sector which is proportional to the two-mass easy box. We obtained

$$\frac{1}{3}\frac{[23]^2}{\langle 14\rangle^2}(-s_{1\phi}s_{4\phi} + s_\phi s_{23})I_4^{2me}(s_{\phi 4}, s_{1\phi}; s_{23}, s_\phi)$$

and similar contributions can be observed applying a cyclic permutation of the gluons.
Studying the allowed helicity configurations for the sub-amplitudes involved in the computation of the two-mass hard contribution and the one-mass four-point coefficient, we demonstrated the absence of other boxes in this sector. Although the lack of the hard configuration was already seen in the previous sector, we also observed the absence of contributions proportional to the one-mass four-point integral in the current sector. This fact is due to the vanishing behavior of the one-loop three gluon amplitude in the all-plus configuration.