Lab 2: Probability Theory

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1. Meanwhile, at the Unfair Coin Factory

It is given that there is a bucket that contains 100 coins. 99 of those are fair coins, but one of them is a trick coin that always comes up heads. Let T be the event that we select the trick coin and that means P(T)=0.01. Probability of getting heads in all k times is given by $P(H_k)$ which is $1/2^k$.

a. $P(T|H_k)$:

From the the given data, we know that $P(T) = 0.01$ and $P(H) = 0.99$ Also, if the the coin selected is a tricked one, then the probability of getting a head is 1 i,e. $P(H_k|T) = 1$.

From Baye's theorem we know that:

$$P(H_k) = P(H_k|T).P(T) + P(H_k|!T).P(!T) = (1 * 1/100) + (1/2^k).(99/100) = 1/100.(1 + 99/2^k)$$

$$P(H_k) = 1/100.(1 + 99/2^k)$$

$$P(T|H_k) = P(T).P(H_k|T)/P(H_k) = (0.01 * 1)/(1/100.(1 + 99/2^k)) = 1/(1 + 99/2^k)$$

$$P(T|H_k) = 1/(1 + 99/2^k)$$

b. For $P(T|H_k) > 0.99$, K value:

$$P(T|H_k) > 0.99 \implies 1/(1 + 99/2^k) > 0.99 \implies k > log_2(99 * 99)$$

$$\implies k > log_2(9801) = 13.25871 = 14$$

So, when k = 14, the conditional probability that we have the trick coin is higher than 0.99

2. Wise Investments

We are given two companies and the probability that they reach unicorn status is given as $P(S) = 3/4$. Let random variable X be the total number of companies that reach unicorn status. X can take on the values 0, 1, and 2. From the given data, X is what we call a binomial random variable with parameters n =2 and p =3/4.

X	P(X)
0	$1/4 * 1/4$
1	$1/4 * 3/4$
2	$3/4 * 3/4$

a.

Complete expression for the probability mass function of X, cmf(X): From binomial probability distribution, we know that an experiment that consists of n trials and results in x successes and the probability of success on an individual trial is p and failure is $q = (1 - p)$, then the binomial probability is as follows. Which is also, cmf(X).

$$pmf(x) = P(X = x) = \begin{cases} \binom{n}{x} \cdot p^x \cdot q^{n-x}, x = 0, 1, ...n \\ 0, otherwise \end{cases}$$

b.

Cumulative probability function of X, $F(X)$: is the summation of individual probability functions until x.

$$CPF(x) = F(X = x) = \sum_{n=i}^{x} \binom{n}{i} \cdot p^i \cdot q^{n-i}$$

c.

$E(X)$:

For Binomial probability distribution, $E(X)$ is $n.p$

$$E(X) = n.p = 2 * 3/4 = 3/2 = 1.5$$

d.

$var(X)$:

For Binomial probability distribution, $var(X)$ is $n.p.q$, where $q = (1 - p)$

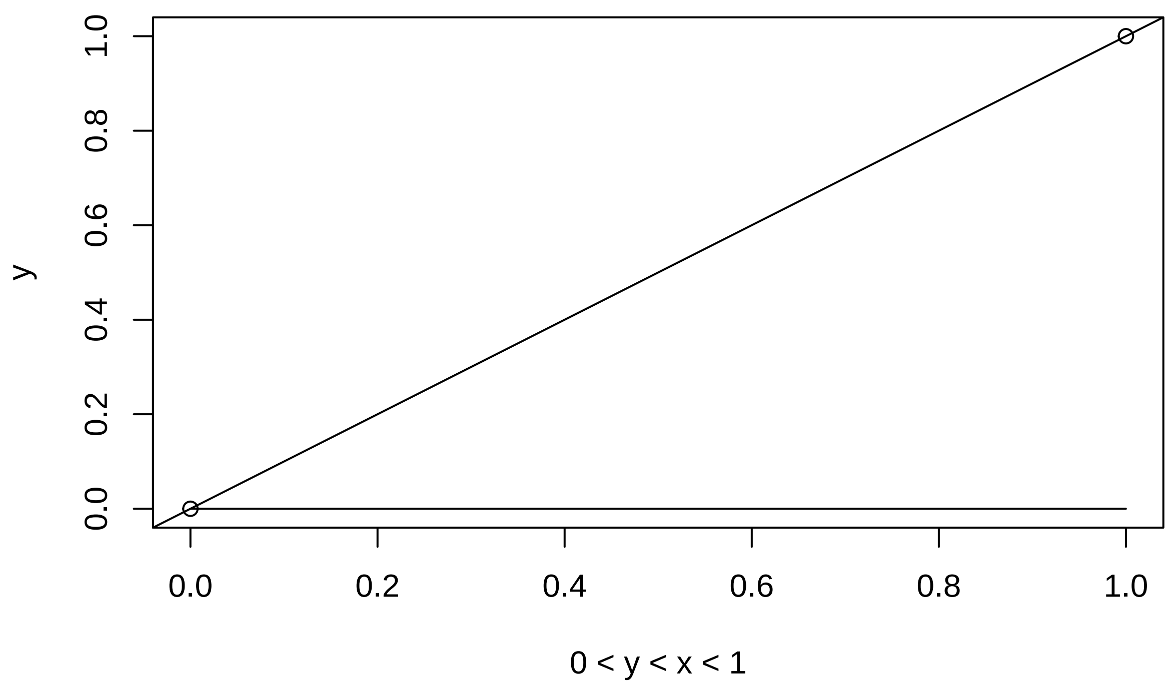
$$var(X) = n.p.q = 2 * 3/4 * 1/4 = 3/8$$

3. Relating Min and Max

It is given that continuous random variables X and Y have a joint distribution with probability density function $f(x, y)$.

$$f(x, y) = \begin{cases} 2, \ 0 < \text{y} < \text{x} < 1 \\ 0, \ \text{otherwise.} \end{cases}$$

a.



b.

The marginal probability density function of X, $f_X(x)$:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 2.dy = 2[y]_0^x = 2x$$

$\implies f_X(x) = 2x$

c.

Unconditional expectation of X, $E(X)$:

$$E(X) = \int_0^1 x.f_X(x)dx = \int_0^1 x.2x.dx = 2.\int_0^1 x^2.dx = 2[x^3/3]_0^1 = 2/3$$

$\implies E(X) = 2/3$

d.

The conditional probability density function of Y, conditional on X, $f_{Y|X}(y|x)$:

$$f_{Y|X} = f(x,y)/f_X(x) = 2/2x = 1/x$$

$\implies f_{Y|X}(y|x) = 1/x.$

e.

The conditional expectation of Y, conditional on X, $E(Y|X)$:

$$E(Y|X) = \int_0^x y.f_{Y|X}(Y|X)dy = \int_0^x (y/x).dy = 1/x.[y^2/2]_0^x = x/2$$

$\implies E(Y|X) = x/2$

f.

$E(XY)$:

$$E(XY) = E[E[XY|X]] = E[X.E[Y|X]] = E[x.x/2] = 1/2.E[X^2]$$
$$\text{Calculating } E[X^2] = \int_0^1 x^2.f_X(x)dx = \int_0^1 (x^2.2x)dx = 2.\int_0^1 (x^3)dx = 2.[x^4/4]_0^1 = 1/2$$

$\implies E(XY) = 1/2.E[X^2] = 1/2 * 1/2 = 1/4$

g.

$cov(X, Y)$:

$$\text{Calculating } E[Y] = E[E[Y|X]] = E[X/2] = 1/2.E[X] = 1/2 * 2/3 = 1/3 \implies E[Y] = 1/3$$

$$cov(X, Y) = E[XY] - E[X].E[Y] = 1/4 - (2/3 * 1/3) = 1/4 - 2/9 = 1/36$$

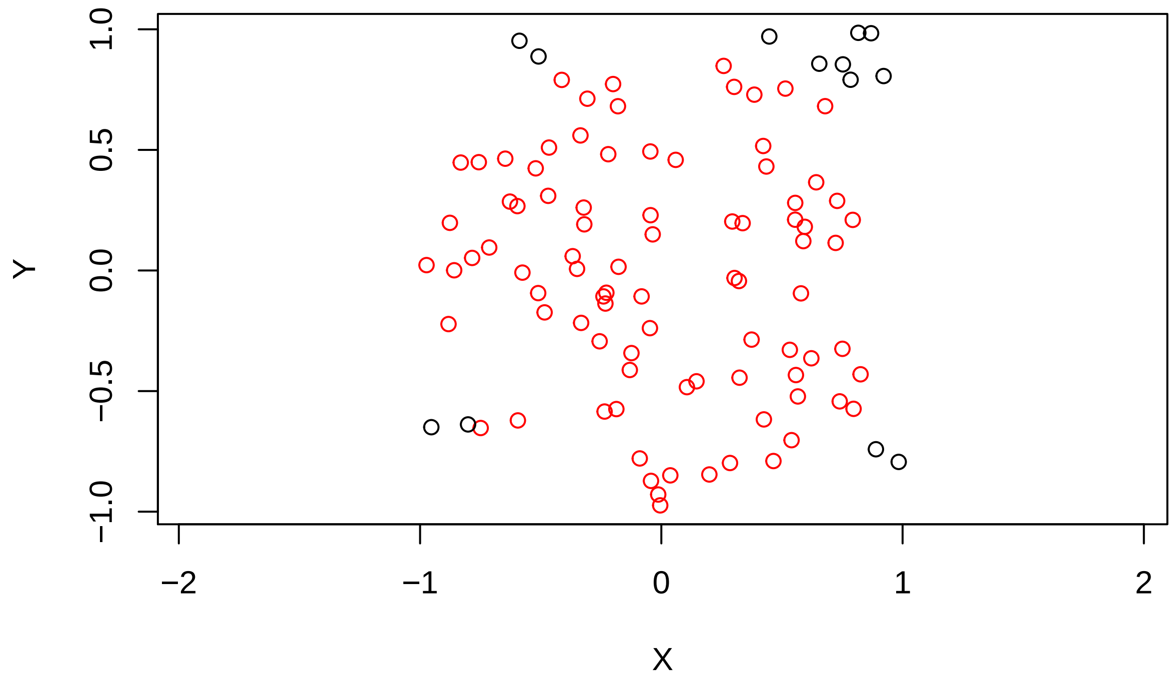
$\implies cov(X, Y) = 1/36$

4. Circles, Random Samples, and the Central Limit Theorem

e

```
set.seed(0)
n = 100
X <- runif(n, min=-1, max=1)
Y <- runif(n, min=-1, max=1)

D <- (X**2+Y**2) < 1
plot(X,Y, col=D+1, asp=1)
```




```r
(mean(D))
```

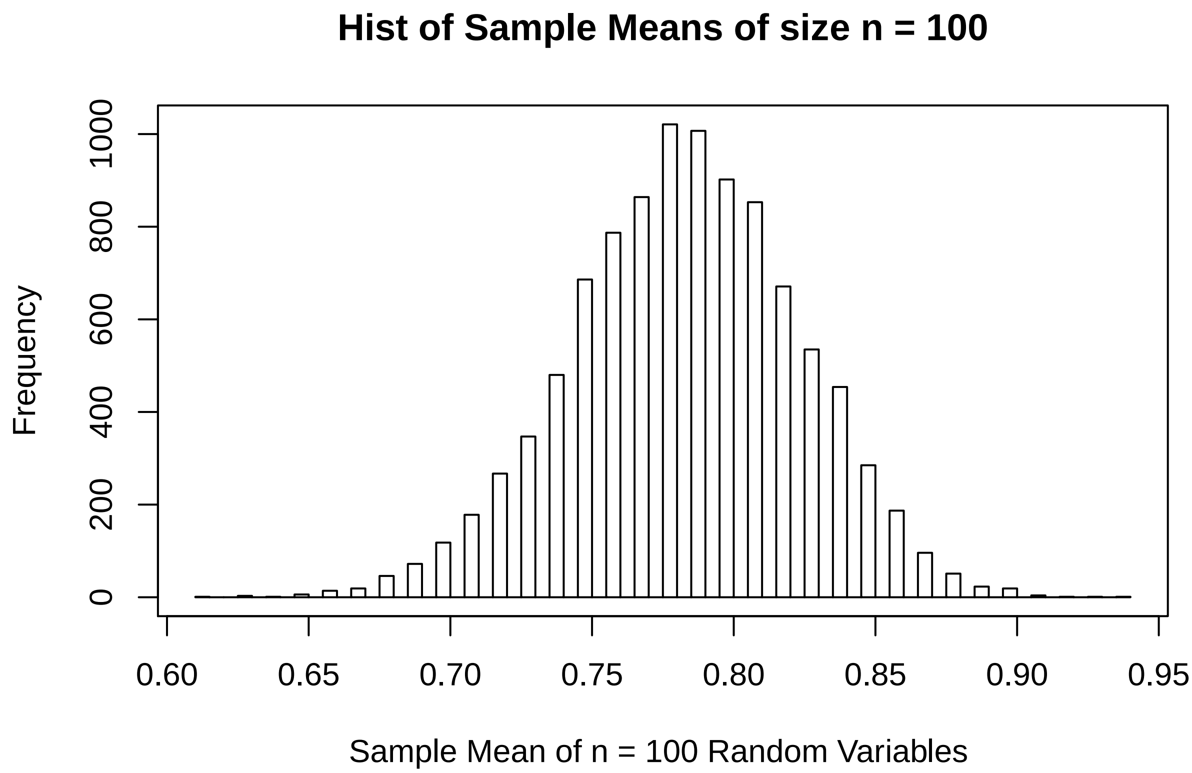
```
## [1] 0.87
```

f

```r
execute_study = function(n){
Xi <- runif(n, min=-1, max=1)
Yi <- runif(n, min=-1, max=1)
Di <- (Xi**2+Yi**2) < 1
mean(Di)
}
```

```r
l <- c()
for (i in 1:10000) {
l[i] <- execute_study(n)
}

hist(l, xlab= paste("Sample Mean of n =", n, "Random Variables "), ylab="Frequency",
     main= paste("Hist of Sample Means of size n =", n), breaks="FD")
box()
```




```r
paste("Standard Deviation of 10,000 trials of size n =", n, "is", sd(l))
```

```
## [1] "Standard Deviation of 10,000 trials of size n = 100 is 0.0406163313537457"
```

i

```r
Z <- sum(l>0.75)/10000
Z
```

```
## [1] 0.7762
```